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On the spectral representation of Lévy and multivariate CARMA processes and related mixing properties

Diplomarbeit

von

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Hiermit erkläre ich, dass ich die Diplomarbeit selbstständig angefertigt und nur die angegebenen Quellen verwendet habe.

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Contents

List of abbreviations

a.e.	almost everywhere
a.s.	almost surely
cf.	confer
e.g.	for example (exempli gratia)
etc.	et cetera
i.e.	that is (id est)
iff	if and only if
i.i.d.	independent and identically distributed
resp.	respectively
s.t.	such that
w.l.o.g.	without loss of generality
w.r.t.	with respect to

Chapter 1

Introduction

Continuous time autoregressive moving average (CARMA) processes are the continuous time analogue of the well-known (discrete time) ARMA processes (see e.g. [13]). The advantage of continuous time modelling is that it allows handling irregularly spaced time series and in particular high frequency data often appearing in finance. In practice multivariate models are necessary in many applications in order to take account of the joint behavior of several time series. The multivariate version of the CARMA process (MCARMA) was introduced by Marquardt and Stelzer [41].

Originally, driving processes of CARMA models were restricted to Brownian motion; however, Brockwell [11] allowed for Lévy processes which have a finite r-th moment for some $r > 0$.

It was not only the past two years where the worst worldwide financial and economic crisis since 1929 has demonstrated impressively that there is a necessity for new models incorporating more of the so-called stylized facts (for instance heavy tails, i.e. very high/low values are far more likely than in the normal distribution) which one can observe in real financial observation data. Bargel and Wenzel [5] explained in a press release of the Deutsche Mathematiker-Vereinigung from 1st April 2009 that new models are formed using Lévy processes but that these are not established as a business standard yet.

In order to illustrate the basic difference between a Gaussian distribution and a distribution with a long tail, Montroll and Shlesinger [43] proposed to compare the distribution of heights with the distribution of annual incomes for American adult males. An average individual who seeks a friend twice his height would fail. On the other hand, one who has an average income will have no trouble to discover a richer person, who, with a little diligence, may locate a third person with twice his income, etc. The income distribution in its upper range has a Pareto inverse power tail; however, most of the income distributions follow a log-normal curve, but the last few percent have a stable tail with exponent $\alpha = 1.6$ (cf. Badgar [4]), i.e., the mean is finite but the variance of the corresponding 1.6-stable distribution diverges.

Mandelbrot [40] and Fama [25] proposed the α-stable distribution for modelling stock returns. However, in the case of financial data these models are extreme in the sense that stable distributions (excluding the Gaussian) do not have a finite variance. But in contrast to "classical" finance (stocks, bonds, currencies, storable commodities, etc.) there are many other fields of application where it is reasonable to assume infinite variance for the data. García, Klüppelberg and Müller [28] for instance fitted a stable CARMA$(2, 1)$ model to spot prices from the Singapore New Electricity Market. Other examples where α-stable stochastic modelling seems to be appropriate are given in Chapter 7 in Janicki and Weron [32].

Figure 1.1 depicts a simulation of (univariate) driving Lévy processes which are symmetric α-stable with α-values of 2 (Brownian Motion) and 1.6 (heavy-tailed). Furthermore, we plotted for each of the two driving processes the simulated values of a corresponding stationary CARMA$(3,1)$ process with parameters $p(z) = z^3 + 4.5z^2 + 6.5z + 3$ and $q(z) = z$ and a CARMA$(2,1)$ process with parameters $p(z) = z^2 + 5z + 6$ and $q(z) = 2z + 4$. One can see nicely, how the tail behavior of the driving Lévy process determines the tail behavior of the corresponding CARMA processes. Moreover recall that in the case of Brownian Motion the sample paths of the related CARMA processes are continuous whereas in the case $\alpha = 1.6$ only the corresponding CARMA$(3,1)$ process possesses continuous sample paths, i.e. the heavy tails come from the jumps of the underlying Lévy process, but the paths exhibit only continuous shocks. The CARMA$(2,1)$ process in that case has a jump whenever the underlying Lévy process has one (cf. [41, Proposition 3.32]).

The main topic of this thesis is the analysis of the spectral representation of Lévy and Lévy-driven CARMA processes, which in the Brownian case is well-understood. In the Lévy-case with finite second moments this representation has been briefly studied in [41]. We extend the representation to the cases when the driving Lévy process has infinite second moment, but is regularly varying with index between one and two. Furthermore, we study in detail properties of the resulting random measure. Additionally we provide new conditions for multivariate infinitely divisible processes to be mixing and apply them to CARMA processes.

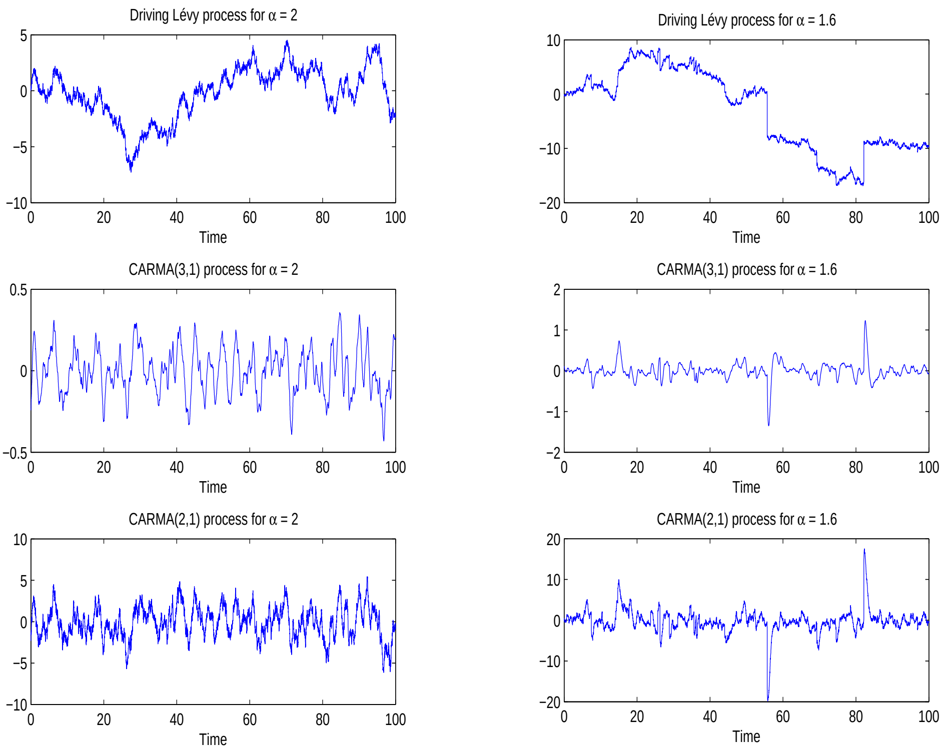

Figure 1.1: CARMA$(3,1)$, CARMA$(2,1)$ and driving Lévy processes

Outline of the thesis

In Chapter 2, we begin with a summary of employed notation. In Section 2.2 we provide a brief overview of important results concerning multivariate Lévy processes and the last section is dedicated to Fourier transformations on the real line. We focus on the L^p, $1 \leq p \leq 2$, theory which is rather standard. However, since we consider complex matrix-valued functions on the real line we give the "multivariate" proofs of well-known results for complex-valued functions on the real line.

The third chapter considers multivariate CARMA processes where the underlying Lévy processes are supposed to have finite second moments. In that case one can use so-called random orthogonal measures, which we introduce in Section 3.1, in order to construct a spectral representation of square-integrable Lévy processes. This representation is recalled in Section 3.2. Thereafter we propose further properties of the "Fourier transform" of a square-integrable Lévy process, i.e. we show some general results concerning the relation between properties of a Lévy process and its associated random orthogonal measure. Chapter 3 finishes with the definition of multivariate CARMA processes.

Symmetric α-stable MCARMA processes are then considered in the fourth chapter. Sections 4.1 and 4.2 recall basic results about symmetric stable random variables, processes and integrals. A general procedure how one can extend symmetric α-stable random measures from semi-rings to the Borel sets with finite control measure is described in the third section and will be used for the proof of a spectral representation, in the summability sense, of symmetric α-stable Lévy processes in Section 4.4. The last part of this chapter gives both, a spectral and a moving average representation of symmetric α-stable MCARMA processes.

Chapter 5 is an extension of the results obtained in Chapter 4 to regularly varying Lévy and MCARMA processes. The proofs are similar to those of the preceding chapter, however, one has to find conditions such that the introduced stochastic integrals depend continuously on their integrands.

The last chapter of this thesis recalls well-known mixing conditions for univariate infinitely divisible processes and then establishes their multivariate generalizations. Finally, we apply these conditions in order to show that the MCARMA processes of Chapter 4 and 5 are always mixing and thus in particular ergodic.

Chapter 2

Preliminaries

2.1 General notation

We denote by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{R}$ and $\mathbb{C}$ the set of all natural numbers, integers, real and complex numbers, resp., and $\mathbb{R}_+ := [0, \infty)$. For the minimum of two real numbers $a, b \in \mathbb{R}$ we write shortly $a \wedge b := \min(a, b)$ and for the maximum $a \vee b := \max(a, b)$. The real and imaginary part of a complex number $z \in \mathbb{C}$ is written as $\mathrm{Re}(z)$ and $\mathrm{Im}(z)$, respectively. The set of $n \times d$ matrices over the field $\mathbb{K}$ is denoted by $M_{n\times d}(\mathbb{K})$, where $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. We set $M_d(\mathbb{K}) := M_{d\times d}(\mathbb{K})$ and define $\mathbb{S}_d(\mathbb{K})$ as the linear subspace of symmetric and Hermitian matrices in the real and complex case, respectively. The positive semidefinite cone is denoted by $\mathbb{S}_d^+(\mathbb{K})$ and the strictly positive definite matrices by $\mathbb{S}_d^{++}(\mathbb{K})$. For a positive definite and positive semidefinite matrix $A \in \mathbb{S}_d(\mathbb{K})$ we also write $A > 0$ and $A \geq 0$, respectively. The transpose of $A \in M_{n\times d}(\mathbb{R})$ is written as A', the complex conjugate transpose of $A \in M_{n\times d}(\mathbb{C})$ as A^*. The identity matrix in $M_d(\mathbb{K})$ shall be denoted by I_d.

On $\mathbb{R}^d$ and $\mathbb{C}^d$, resp., we denote by $|\cdot|$ the Euclidean norm, whereas $\|\cdot\|$ stands for any other arbitrary norm. Recall the fact that two norms on a finite dimensional linear space are always equivalent which is why the concrete type of a norm in that case shall not be important for our results. A scalar product on linear spaces is written as $\langle \cdot, \cdot \rangle$; in $\mathbb{R}^d$ and $\mathbb{C}^d$, we usually take the Euclidean one. If X and Y are normed linear spaces, let $B(X, Y)$ be the set of bounded linear operators from X into Y. On $B(X, Y)$ we will usually use the operator norm which, in the case of Y being a Banach space, makes $B(X, Y)$ itself to a Banach space. If X is a topological space, we denote by $\mathscr{B}(X)$ the Borel σ-algebra on X, that is the smallest σ-algebra on X containing all open subsets of X. The Lebesgue measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is written as λ^d.

For a random variable X defined on some probability space $(\Omega, \mathscr{F}, \mathbb{P})$ we call the image measure $X(\mathbb{P})$ distribution of X which is written as $\mathscr{L}(X)$. For two random variables X and Y the notation $X \stackrel{\mathscr{D}}{=} Y$ means equality in distribution. For a d-dimensional $\mathbb{K}$-valued, $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$, random variable X, we say that $X \in L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{K}^d)$, $1 \leq p < \infty$, if $\mathbb{E}[|X|^p]$ is finite. If we define the norm $\|X\|_{L^p} := (\mathbb{E}[|X|^p])^{1/p}$ and, as usual, do not distinguish between random variables and equivalence classes of random variables, $L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{K}^d)$ becomes a Banach space. If we consider a sequence of random variables $(X_n)_{n\in\mathbb{N}}$, we shall denote almost sure convergence of the sequence to some random variable X by $X_n \stackrel{\text{a.s.}}{\to} X$. Convergence in probability will be denoted by $X_n \stackrel{\mathbb{P}}{\to} X$, convergence in L^p by $X_n \stackrel{L^p}{\to} X$ and convergence in distribution by $X_n \stackrel{w}{\to} X$.

2.2 Multivariate Lévy processes

We give a summary of elementary properties of multivariate Lévy processes that will be needed throughout this thesis. For a more general treatment and proofs we refer the reader to the standard books on this topic, e.g. Sato [54], Applebaum [1] and Bertoin [8].

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space. An increasing family $\mathbb{F} := (\mathscr{F}_t)_{t\in\mathbb{R}_+}$ of sub-σ-algebras of $\mathscr{F}$ is called **filtration**. $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F})$ is referred to as a **filtered probability space**. A family $X = (X_t)_{t\in\mathbb{R}_+}$ of $\mathbb{R}^d$-valued random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$ is called a d-dimensional **stochastic process**. We say that X is **adapted** to $\mathbb{F}$ if X_t is $\mathscr{F}_t$-measurable for all $t \in \mathbb{R}_+$.

Definition 2.1. *An adapted $\mathbb{R}^d$-valued stochastic process $L = (L_t)_{t\in\mathbb{R}_+}$ is called **Lévy process** with respect to $\mathbb{F}$, if*

(i) $L_0 = 0$ *a.s.,*

(ii) $L_t - L_s$ *is independent of* $\mathscr{F}_s$ *for all* $s, t \in \mathbb{R}_+$ *with* $s \le t$*,*

(iii) $L_{t+h} - L_t \overset{\mathscr{D}}{=} L_{s+h} - L_s$ *for all* $s, t, h \in \mathbb{R}_+$*,*

(iv) L *is continuous in probability, i.e. for all* $s \in \mathbb{R}_+$ *we have* $L_t - L_s \xrightarrow{\mathbb{P}} 0$ *as* $t \to s$*.*

We call a process just Lévy process if it is a Lévy process w.r.t. its own natural filtration $(\mathscr{F}_t^L := \sigma(L_s : 0 \le s \le t))_{t\in\mathbb{R}_+}$*.*

Property (ii) says in particular that L has independent increments and property (iii) says that L has stationary increments.

2.2.1 Infinitely divisible distributions

Recall that for two probability measures μ_1 and μ_2 on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ the convolution is defined by

$$\mu_1 * \mu_2(A) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathbb{1}_A(x+y)\,\mu_1(dx)\,\mu_2(dy), \quad A \in \mathscr{B}(\mathbb{R}^d).$$

The convolution operation is commutative and associative. If X_1 and X_2 are independent random variables on $\mathbb{R}^d$ with distribution μ_1 and μ_2, resp., then $X_1 + X_2$ has distribution $\mu_1 * \mu_2$.

The n-fold convolution of a probability measure μ with itself will be denoted by

$$\mu^{n*} := \underbrace{\mu * \mu * \ldots * \mu}_{n \text{ times}}.$$

Definition 2.2. *A probability measure μ on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ is called **infinitely divisible** if, for any $n \in \mathbb{N}$, there is a probability measure μ_n on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ such that $\mu = \mu_n^{n*}$.*

Or equivalently, μ is infinitely divisible iff its characteristic function $\Phi_\mu : \mathbb{R}^d \to \mathbb{C}$, $\Phi_\mu(z) := \int_{\mathbb{R}^d} e^{i\langle z,x\rangle}\mu(dx)$, has n-th roots, i.e. for any $n \in \mathbb{N}$ there is a characteristic function $\Phi_n : \mathbb{R}^d \to \mathbb{C}$ such that $\Phi_\mu(z) = (\Phi_n(z))^n$ for all $z \in \mathbb{R}^d$.

It can be shown that there is a close connection between Lévy processes and infinitely divisible distributions. Recall that two stochastic processes X and Y (not necessarily defined on a common probability space) are identical in law, written as $X \stackrel{\mathscr{D}}{=} Y$, if the systems of their finite dimensional distributions are identical.

Lemma 2.3 (cf. [54], Lemma 7.9)**.**
If μ is infinitely divisible, then, for every $t \in \mathbb{R}_+$, μ^{t*} is definable and infinitely divisible.

Theorem 2.4 (cf. [54], Theorem 7.10)**.**

(i) Let $(L_t)_{t\in\mathbb{R}_+}$ be a Lévy process on $\mathbb{R}^d$. Then, for any $t \in \mathbb{R}_+$, the distribution of L_t is infinitely divisible and, letting $\mu = \mathscr{L}(L_1)$, we have $\mathscr{L}(L_t) = \mu^{t*}$.

(ii) Conversely, if μ is an infinitely divisible distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, then there is a Lévy process $(L_t)_{t\in\mathbb{R}_+}$ with $\mathscr{L}(L_1) = \mu$.

(iii) If $L^{(1)} = (L_t^{(1)})_{t\in\mathbb{R}_+}$ and $L^{(2)} = (L_t^{(2)})_{t\in\mathbb{R}_+}$ are Lévy processes on $\mathbb{R}^d$ such that $\mathscr{L}(L_1^{(1)}) = \mathscr{L}(L_1^{(2)})$, then $L^{(1)}$ and $L^{(2)}$ are identical in law.

Hence the collection of all infinitely divisible distributions is in one-to-one correspondence with the collection of all Lévy processes, when two processes identical in law are considered as the same.

We recall the well-known Lévy-Khintchine representation of characteristic functions of all infinitely divisible distributions.

Theorem 2.5 (cf. [54], Theorem 8.1)**.**

(i) Let μ be an infinitely divisible distribution on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$, then there exist $\gamma \in \mathbb{R}^d$, $\Sigma \in \mathbb{S}_d^+(\mathbb{R})$ and a measure ν on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ satisfying

$$\nu(\{0\}) = 0 \qquad \textit{and} \qquad \int_{\mathbb{R}^d} \left(1 \wedge \|x\|^2\right) \nu(dx) < \infty \tag{2.1}$$

such that the characteristic function of μ is given by

$$\Phi_\mu(z) = \exp\left\{ i\langle\gamma, z\rangle - \frac{1}{2}\langle z, \Sigma z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\right) \nu(dx) \right\}, \tag{2.2}$$

for each $z \in \mathbb{R}^d$.
Moreover, the representation of $\Phi_\mu(z)$ in (2.2) by γ, Σ and ν is unique.

(ii) Conversely, if $\Sigma \in \mathbb{S}_d^+(\mathbb{R})$, ν a measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$ satisfying (2.1) and $\gamma \in \mathbb{R}^d$, then there exists an infinitely divisible distribution μ whose characteristic function is given by (2.2).

We call (γ, Σ, ν) the **generating triplet** of μ. The matrix Σ and the measure ν are called, respectively, the **Gaussian covariance matrix** and the **Lévy measure** of μ.

Corollary 2.6 (cf. [54], Corollary 8.3)**.**
If μ has generating triplet (γ, Σ, ν), then, for every $t \in \mathbb{R}_+$, μ^{t*} has generating triplet $(t\gamma, t\Sigma, t\nu)$.

Henceforth we can characterize due to the Theorems 2.4 and 2.5 every Lévy process L in a unique way by the generating triplet of the distribution of L_1.

Remark 2.7.

(i) The function $\mathbb{1}_{[0,1]}(\|x\|)$ is called **truncation function**. It makes the integrand of the integral in the right-hand side of (2.2) integrable with respect to ν. There are many other ways of getting an integrable integrand incorporating only a change in γ (cf. [54, Remark 8.4] for examples).

(ii) If ν satisfies the additional condition $\int_{\{\|x\|\le 1\}} \|x\| \, \nu(dx) < \infty$, then the truncation function can be omitted and we get

$$\Phi_\mu(z) = \exp\left\{ i\langle \gamma_0, z\rangle - \frac{1}{2}\langle z, \Sigma z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1\right) \nu(dx) \right\}$$

with $\gamma_0 := \gamma - \int_{\mathbb{R}^d} x \mathbb{1}_{[0,1]}(\|x\|) \, \nu(dx)$. The constant γ_0 is called **drift** of μ and we may also call (γ_0, Σ, ν) generating triplet of μ (w.r.t. the truncation function zero).

(iii) If ν satisfies $\int_{\{\|x\|\ge 1\}} \|x\| \, \nu(dx) < \infty$, which is equivalent to finiteness of $\int_{\mathbb{R}^d} \|x\| \, \mu(dx)$ (cf. upcoming Theorem 2.20), we can write the characteristic function as

$$\Phi_\mu(z) = \exp\left\{ i\langle \gamma_1, z\rangle - \frac{1}{2}\langle z, \Sigma z\rangle + \int_{\mathbb{R}^d} \left(e^{i\langle z,x\rangle} - 1 - i\langle z, x\rangle\right) \nu(dx) \right\}$$

with $\gamma_1 := \gamma + \int_{\{\|x\|>1\}} x \, \nu(dx)$. The constant γ_1 is called **center** of μ and is equal to the mean of μ (cf. [54, Example 25.12]).

2.2.2 Sample paths and the Lévy-Itô decomposition

Definition 2.8. *Let $X = (X_t)_{t\in\mathbb{R}_+}$ be a stochastic process. X is said to be a* ***càdlàg process*** *(**c**ontinu **à** **d**roite, **l**imité **à** **g**auche) if almost all paths are right continuous and have existing limits from the left.*

For a càdlàg process X we define the process X_- by $X_{t-} := \lim_{s\nearrow t} X_s$ for $t > 0$ and $X_{0-} := X_0$ and the process of jumps ΔX by $\Delta X_t := X_t - X_{t-}$ and $\Delta X_0 := 0$. Note that, a.s., $\Delta X \neq 0$ at most countably often.

Definition 2.9.
Let X and Y be two stochastic processes on a common probability space $(\Omega, \mathscr{F}, \mathbb{P})$.

(i) X and Y are said to be ***modifications*** *of each other if $\mathbb{P}(X_t = Y_t) = 1$ for all $t \in \mathbb{R}_+$.*

(ii) X and Y are called ***indistinguishable*** *if $\mathbb{P}(X_t = Y_t \text{ for all } t \in \mathbb{R}_+) = 1$.*

Obviously, two indistinguishable processes are also modifications of each other. If two càdlàg processes are modifications of each other, then they are indistinguishable.

Theorem 2.10 (cf. [54], Theorem 11.5 and [47], Theorem I.30)**.**
Let L be a Lévy process. Then there exists a modification $\widetilde{L}$ of L which is a càdlàg process and in particular also a Lévy process. $\widetilde{L}$ is unique up to indistinguishability.

We will henceforth always assume that a given Lévy process has a.s. càdlàg paths.

Since it is important for stochastic integration purposes, we make the following definition.

Definition 2.11. *A filtered probability space $(\Omega, \mathscr{F}, \mathbb{P}, \mathbb{F} = (\mathscr{F}_t)_{t\in\mathbb{R}_+})$ is said to satisfy the* ***usual hypotheses*** *if*

(i) the filtration $\mathbb{F}$ is ***right continuous****, i.e. $\mathscr{F}_t = \bigcap_{s>t} \mathscr{F}_s$ for all $t \in \mathbb{R}_+$,*

(ii) $\mathbb{F}$ is ***complete****, i.e. $\mathscr{F}_0$ contains all the $\mathbb{P}$-null sets of $\mathscr{F}$.*

Given a filtration $(\mathscr{F}_t)_{t\in\mathbb{R}_+}$ we can always enlarge it to satisfy the completeness property (ii) in the following way: let $\mathscr{N}$ denote the collection of all $\mathbb{P}$-null sets in $\mathscr{F}$ and define $\mathscr{G}_t := \mathscr{F}_t \vee \mathscr{N}$ for each $t \in \mathbb{R}_+$, then $(\mathscr{G}_t)_{t\in\mathbb{R}_+}$ is another filtration of $\mathscr{F}$, called the **augmented filtration**. Lévy processes provide us with examples of filtrations that satisfy the usual hypotheses. Recall that the natural filtration of a Lévy process L is given by $\big(\mathscr{F}_t^L := \sigma(L_s : 0 \le s \le t)\big)_{t\in\mathbb{R}_+}$.

Theorem 2.12 (cf. [1], Theorem 2.1.9 and [47], Theorem I.31)**.**
If L is the càdlàg version of a Lévy process, then its augmented natural filtration is right continuous.

As we shall make use of two-sided Lévy processes, i.e. Lévy processes starting in the infinite past, let us briefly introduce them. Taking two independent copies $(L_t^{(1)})_{t\in\mathbb{R}_+}$ and $(L_t^{(2)})_{t\in\mathbb{R}_+}$ of a one-sided Lévy process, we set

$$L_t := \begin{cases} L_t^{(1)} & \text{if } t \ge 0 \\ -L_{(-t)-}^{(2)} & \text{if } t < 0 \end{cases}$$

and call $L = (L_t)_{t\in\mathbb{R}}$ a two-sided Lévy process. Obviously L also has a.s. càdlàg paths.

Finally, we conclude this subsection by the Lévy-Itô decomposition which shows that every Lévy process can be decomposed pathwise into the independent sum of the so-called **continuous part** (composed of a deterministic drift γt and a Brownian motion with covariance matrix Σ) and the so-called **jump part** (which consists itself of two parts – the sum of all large jumps and the sum of small jumps compensated by an infinite drift). We define the **punctured Euclidean space** by $\mathbb{R}_*^d := \mathbb{R}^d \setminus \{0\}$.

Definition 2.13. *Let ρ be a σ-finite measure on $(\mathbb{R}_+ \times \mathbb{R}_*^d, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d))$. A mapping $J : \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d) \times \Omega \to \overline{\mathbb{Z}}_+ := \{0, 1, 2, \ldots\} \cup \{+\infty\}$ is called* ***Poisson random measure*** *on $(\mathbb{R}_+ \times \mathbb{R}_*^d, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d))$ with intensity measure ρ if*

(i) $J(A, \cdot) \sim \mathrm{Poisson}(\rho(A))$ for all $A \in \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d)$,

(ii) the random variables $J(A_1, \cdot), \ldots, J(A_n, \cdot)$ are independent whenever $A_1, \ldots, A_n \in \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d)$ are pairwise disjoint,

(iii) $J(\cdot, \omega)$ is a measure on $(\mathbb{R}_+ \times \mathbb{R}_*^d, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}_*^d))$ for all $\omega \in \Omega$.

For integration w.r.t. Poisson random measures we refer to Sato [54, Proposition 19.5].

Theorem 2.14 (cf. [54], Theorem 19.2 and [1], Theorem 2.4.16)**.**
Let L be a Lévy process on $\mathbb{R}^d$ with generating triplet (γ, Σ, ν) defined on a probability space $(\Omega, \mathscr{F}, \mathbb{P})$. Assume that L has càdlàg paths for all $\omega \in \Omega$. For $A \in \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}^d_*)$ we set

$$J(A, \omega) := \# \{s \in \mathbb{R}_+ : (s, \Delta L_s(\omega)) \in A\} .$$

Then the following hold:

(i) J is a Poisson random measure on $(\mathbb{R}_+ \times \mathbb{R}^d_*, \mathscr{B}(\mathbb{R}_+ \times \mathbb{R}^d_*))$ with intensity measure $\lambda^1|_{\mathbb{R}_+} \otimes \nu|_{\mathbb{R}^d_*}$ (where $\lambda^1|_{\mathbb{R}_+}$ is the Lebesgue measure on $\mathbb{R}_+$).

(ii) There is $\Omega_1 \in \mathscr{F}$ with $\mathbb{P}(\Omega_1) = 1$ s.t. for all $\omega \in \Omega_1$

$$\begin{aligned} L^{(1)}_t(\omega) = \lim_{\varepsilon \searrow 0} \int_{(0,t] \times \{x :\, \|x\| \in (\varepsilon, 1]\}} & x\, (J(d(s, x), \omega) - ds\, \nu(dx)) \\ + \int_{(0,t] \times \{x :\, \|x\| > 1\}} & x\, J(d(s, x), \omega) \end{aligned}$$

is definable for every $t \in \mathbb{R}_+$ and the first term converges uniformly on compact intervals of t. $L^{(1)}$ is a Lévy process on $(\Omega, \mathscr{F}, \mathbb{P})$ with generating triplet $(0, 0, \nu)$.

(iii) Define

$$L^{(2)}_t(\omega) = L_t(\omega) - L^{(1)}_t(\omega) \quad \text{for } \omega \in \Omega_1.$$

Then there is $\Omega_2 \in \mathscr{F}$ with $\mathbb{P}(\Omega_2) = 1$ such that $L^{(2)}_t(\omega)$ is continuous in t for all $\omega \in \Omega_2$. $L^{(2)}$ is a Lévy process with generating triplet $(\gamma, \Sigma, 0)$ (i.e. a Brownian motion with drift γ and covariance matrix Σ).

(iv) The two processes $L^{(1)}$ and $L^{(2)}$ are independent.

Remark 2.15.

(i) Since $\nu(\{x : \|x\| > \varepsilon\}) < \infty$ for every $\varepsilon > 0$, a Lévy process has only finitely many jumps bigger in norm than any $\varepsilon > 0$ on bounded time intervals.
However, if $\nu(\mathbb{R}^d) = \infty$ it has infinitely (but countably) many arbitrarily small jumps on any finite time interval. We shall speak of infinite activity of the Lévy process in that case.

(ii) In general neither

$$\int_{(0,t] \times \{x :\, \|x\| \in (\varepsilon, 1]\}} x\, J(d(s, x)) \qquad \text{nor} \qquad \int_{(0,t] \times \{x :\, \|x\| \in (\varepsilon, 1]\}} x\, ds\, \nu(dx)$$

converges for $\varepsilon \to 0$. That is "the first addend of $L^{(1)}$ is given by the sum of infinitely many small jumps, which are not summable, compensated by an infinite drift".

(iii) $\int_{(0,t] \times \{x :\, \|x\| > 1\}} x\, J(d(s, x))$ is simply the sum of all large jumps.

(iv) If the small jumps are summable (ensured by the condition that $\int_{\{\|x\| \le 1\}} \|x\|\, \nu(dx)$ is finite), then the compensation and limiting procedure is not needed, see Sato [54, Theorem 19.3].

2.2.3 δ-variation of sample paths

From the Lévy-Itô decomposition we can deduce many important path properties. One of them is the so-called δ-variation of sample paths.
Let $D(\mathbb{R}_+, \mathbb{R}^d)$ denote the set of all càdlàg functions on $\mathbb{R}_+$ with values in $\mathbb{R}^d$.

Definition 2.16. *Let $f \in D(\mathbb{R}_+, \mathbb{R}^d)$ and $\delta > 0$, then*

$$V_f^\delta(t) := \sup_{\Delta_t} \sum_{j=1}^{n} \|f(s_j) - f(s_{j-1})\|^\delta$$

*with the supremum taken over all finite partitions $\Delta_t : 0 = s_0 < s_1 < \ldots < s_n = t$, $n \in \mathbb{N}$, of $[0,t]$, is called **δ-variation** of f.*
*If $V_f^\delta(t) < \infty$ for all $t \in \mathbb{R}_+$, then f is said to be of **finite δ-variation**.*

For $\delta = 1$, we shall speak of variation instead of 1-variation. For many important processes, conditions for the almost sure boundedness of δ-variation of sample paths on bounded time intervals are already known: Lévy [36] showed that, for Brownian motion, the δ-variation is a.s. bounded if and only if $\delta > 2$, a result refined by Taylor [59]. The δ-variation of Gaussian processes was investigated by Jain and Monrad [31]. The same problem for Lévy processes was addressed in the papers by Fristedt and Taylor [27] and Bretagnolle [10]. We recall the following results:

Proposition 2.17 (cf. [54], Theorem 21.9 and [10], Theorem IIIb)**.**
Suppose $1 \le \delta < 2$ and let L be a Lévy process with generating triplet (γ, Σ, ν). Then L has a.s. finite δ-variation iff $\Sigma = 0$ and $\int_{\{\|x\| \le 1\}} \|x\|^\delta \, \nu(dx) < \infty$.

Remark 2.18.
One must not confound the notions of 2-variation and quadratic variation, the latter used in stochastic analysis in order to define for instance stochastic integration w.r.t. Brownian motion (see e.g. Protter [47]). In the case of Brownian motion, the quadratic variation (which uses a refining sequence of interval partitions) is finite whereas the 2-variation is infinite.

We conclude this part by introducing the Blumenthal-Getoor-Index, defined for the first time in [9].

Definition 2.19. *Let ν be a Lévy measure on $(\mathbb{R}^d, \mathscr{B}(\mathbb{R}^d))$. Then the **Blumenthal-Getoor-Index** of ν is given by*

$$\beta(\nu) := \inf \left\{ \alpha > 0 : \int_{\{\|x\| \le 1\}} \|x\|^\alpha \, \nu(dx) < \infty \right\}.$$

Obviously, $0 \le \beta(\nu) \le 2$ holds for every Lévy measure ν.

2.2.4 Moments of Lévy processes

We discuss the question whether moments of L_t for a Lévy process $(L_t)_{t \in \mathbb{R}_+}$ are finite or not. One has the following central theorem:

Theorem 2.20 (cf. [54], Corollary 25.8)**.**
Let $(L_t)_{t\in\mathbb{R}_+}$ be a d-dimensional Lévy process with generating triplet (γ, Σ, ν) and $a > 0$, $0 < b \leq 1$, $c \geq 0$. If $g : \mathbb{R}^d \to \mathbb{R}_+$ is of one of the forms

(i) $g(x) = \|x\|^a$

(ii) $g(x) = (0 \vee \log \|x\|)^a$

(iii) $g(x) = \|x\|^c \cdot e^{a\|x\|^b}$,

then $\mathbb{E}[g(L_t)] < \infty$ for all $t \in \mathbb{R}_+$ if and only if $\mathbb{E}[g(L_1)] < \infty$ and $\mathbb{E}[g(L_1)] < \infty$ if and only if $\int_{\{\|x\|\geq 1\}} g(x)\,\nu(dx) < \infty$.

This means that the property of finite moments is time independent in the class of Lévy processes.

2.2.5 Stochastic integrals with respect to Lévy processes

We consider the stochastic integral process $(X_t)_{t\in\mathbb{R}}$ given by

$$X_t := \int_{\mathbb{R}} f_t(s)\,L(ds),\ t \in \mathbb{R}, \tag{2.3}$$

where $f : \mathbb{R} \times \mathbb{R} \to M_{k\times d}(\mathbb{R})$, $(t, s) \mapsto f_t(s)$, is a measurable function and $L = (L_t)_{t\in\mathbb{R}}$ is a two-sided d-dimensional Lévy process.
We first recall an existence result. Let therefore (γ, Σ, ν) be the generating triplet of L. Necessary and sufficient conditions for the integral in (2.3) to exist are (cf. [48, Theorem 2.7])

$$\int_{\mathbb{R}} \left\| f_t(s)\gamma + \int_{\mathbb{R}^d} f_t(s)x \left(\mathbb{1}_{\{\|f_t(s)x\|\leq 1\}} - \mathbb{1}_{\{\|x\|\leq 1\}}\right) \nu(dx) \right\| ds < \infty \tag{2.4}$$

$$\int_{\mathbb{R}} \|f_t(s)\Sigma f_t(s)'\|\,ds < \infty \tag{2.5}$$

and

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f_t(s)x\|^2\right) \nu(dx)\,ds < \infty. \tag{2.6}$$

Moreover, it is possible to characterize the distribution of every X_t. Namely, if the integral is well-defined, then the distribution of X_t is infinitely divisible with generating triplet $(\gamma_X^t, \Sigma_X^t, \nu_X^t)$ given by (cf. [48, Theorem 2.7] and [55, Corollary 2.19 and Proposition 5.5])

$$\gamma_X^t = \int_{\mathbb{R}} \left(f_t(s)\gamma + \int_{\mathbb{R}^d} f_t(s)x \left(\mathbb{1}_{\{\|f_t(s)x\|\leq 1\}} - \mathbb{1}_{\{\|x\|\leq 1\}}\right) \nu(dx) \right) ds \tag{2.7}$$

$$\Sigma_X^t = \int_{\mathbb{R}} f_t(s)\Sigma f_t(s)' ds \tag{2.8}$$

and

$$\nu_X^t(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A(f_t(s)x)\,\nu(dx)\,ds,\ A \in \mathscr{B}(\mathbb{R}_*^k). \tag{2.9}$$

2.3 Fourier transforms on the real line

In this section we summarize important properties of Fourier transforms on the real line. The theory for complex-valued functions in L^1 and L^2, resp., is rather standard. Also the extension to complex-valued functions in L^p with $p \in (1,2)$, for which the Fourier transforms can be defined by continuity as functions in L^q with $q = p/(p-1)$, is quite common. For good expositions we refer the reader to Katznelson [33] or Stein and Weiss [58].

Since we shall consider complex matrix-valued functions on the real line and, to the best of our knowledge, general results cannot be found elsewhere yet, we found it useful to give an outline of the "multivariate" case, although it is obvious that one can generalize the well-known univariate theory by a componentwise perception to the multivariate case. We will essentially follow Chapter VI of [33] to derive the following results.

2.3.1 The L^1 theory

Let us denote by

$$L^1(M_d(\mathbb{C})) := \left\{ f : \mathbb{R} \to M_d(\mathbb{C}) \text{ measurable, } \int_{\mathbb{R}} \|f(t)\|\, dt < \infty \right\}$$

the space of all Lebesgue integrable complex matrix-valued functions on the real line. It is independent of the norm $\|\cdot\|$ on $M_d(\mathbb{C})$ (since $M_d(\mathbb{C})$ is a finite dimensional linear space) and it is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_d(\mathbb{C})$ where all components f_{ij}, $i, j = 1, \ldots, d$, are in the usual space $L^1(\mathbb{C})$. We assume throughout the whole section that the norm on $M_d(\mathbb{C})$ is given by

$$\|A\| = \sqrt{\operatorname{tr}(AA^*)}, \quad A \in M_d(\mathbb{C}),$$

where $\operatorname{tr}(\,\cdot\,)$ denotes the trace of a matrix. Note that $\langle A, B\rangle_{M_d(\mathbb{C})} := \operatorname{tr}(AB^*)$ defines a scalar product on the Hilbert space $M_d(\mathbb{C})$. On $L^1(M_d(\mathbb{C}))$ we define the norm

$$\|f\|_{L^1} := \int_{\mathbb{R}} \|f(t)\|\, dt$$

and $L^1(M_d(\mathbb{C}))$ becomes a Banach space with respect to $\|\cdot\|_{L^1}$.

The Fourier transform $\widehat{f}$ of $f \in L^1(M_d(\mathbb{C}))$ is defined by

$$\widehat{f}(\xi) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} f(x)\, dx, \quad \xi \in \mathbb{R}. \tag{2.10}$$

Clearly, $\widehat{f} : \mathbb{R} \to M_d(\mathbb{C})$ and $\widehat{f}$ can be interpreted as the componentwise Fourier transformation of f_{ij}, $i, j = 1, \ldots, d$, i.e. $\widehat{f} = (\widehat{f_{ij}})$. That is why standard results for Fourier transforms in the space $L^1(\mathbb{C})$ (cf. for example [33]) can be generalized:

Theorem 2.21. *Let $f, g \in L^1(M_d(\mathbb{C}))$ and $\alpha \in M_d(\mathbb{C})$. Then:*

(i) $\widehat{f+g} = \widehat{f} + \widehat{g}$ *and* $\widehat{\alpha f} = \alpha \widehat{f}$ *(i.e. the Fourier transformation is a linear operator).*

(ii) For $f_y(x) := f(x-y)$ with arbitrary $y \in \mathbb{R}$, we have

$$\widehat{f_y}(\xi) = e^{-i\xi y} \cdot \widehat{f}(\xi), \; \xi \in \mathbb{R}.$$

(iii) For $f_c(x) := c \cdot f(cx)$ *with arbitrary* $c \in \mathbb{R}\setminus\{0\}$, *we have*

$$\widehat{f_c}(\xi) = \widehat{f}\left(\frac{\xi}{c}\right), \ \xi \in \mathbb{R}.$$

(iv) For every $\xi \in \mathbb{R}$, *we have*

$$\left\|\widehat{f}(\xi)\right\| \leq \|f\|_{L^1}.$$

Proof. The theorem follows immediately from the definition (2.10). □

Theorem 2.22. *Assume that we have given* $f \in L^1(\mathbb{C})$ *and* $g \in L^1(M_d(\mathbb{C}))$. *For almost all* $x \in \mathbb{R}$, $f(x-y)g(y)$ *is integrable as a function of* y *and, if we write*

$$h(x) := \int_{\mathbb{R}} f(x-y)g(y)\,dy,$$

then $h \in L^1(M_d(\mathbb{C}))$ *and*

$$\|h\|_{L^1} \leq \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

We denote $h = f * g$ *and call* h *the* ***convolution*** *of* f *and* g. *Moreover,*

$$\widehat{h}(\xi) = \sqrt{2\pi}\widehat{f}(\xi) \cdot \widehat{g}(\xi)$$

for any $\xi \in \mathbb{R}$.

Proof. The function $F(x,y) := f(x-y)g(y)$ is obviously measurable. For almost all $y \in \mathbb{R}$, the components of the function $x \mapsto F(x,y)$ are constant multiples of $f(\cdot - y)$, hence integrable and

$$\int_{\mathbb{R}} \left(\int_{\mathbb{R}} \|F(x,y)\|\,dx \right) dy = \int_{\mathbb{R}} \|f\|_{L^1} \cdot \|g(y)\|\,dy = \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

Thus, due to the Theorems of Tonelli and Fubini, $y \mapsto F(x,y)$ is integrable for almost all $x \in \mathbb{R}$ and

$$\int_{\mathbb{R}} \|h(x)\|\,dx = \int_{\mathbb{R}} \left\| \int_{\mathbb{R}} F(x,y)\,dy \right\| dx \leq \int_{\mathbb{R}} \int_{\mathbb{R}} \|F(x,y)\|\,dx\,dy = \|f\|_{L^1} \cdot \|g\|_{L^1}.$$

Finally, for $\xi \in \mathbb{R}$ we obtain

$$\begin{aligned}\widehat{h}(\xi) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} h(x)\,dx = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi x} \left(\int_{\mathbb{R}} f(x-y)g(y)\,dy \right) dx \\ &= \int_{\mathbb{R}} \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi(x-y)} f(x-y)\,dx \right) e^{-i\xi y} g(y)\,dy = \widehat{f}(\xi) \cdot \int_{\mathbb{R}} e^{-i\xi y} g(y)\,dy \\ &= \sqrt{2\pi}\widehat{f}(\xi) \cdot \widehat{g}(\xi),\end{aligned}$$

where the change in the order of integration is again justified by Fubini's Theorem. □

Corollary 2.23. *Let $f \in L^1(\mathbb{C}),\ g \in L^1(M_d(\mathbb{C}))$,*

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} F(\xi)\, d\xi$$

for some $F \in L^1(\mathbb{C})$. Then, for every $x \in \mathbb{R}$,

$$(f * g)(x) = \int_{\mathbb{R}} e^{i\xi x} F(\xi) \widehat{g}(\xi)\, d\xi.$$

Proof. For almost all $x \in \mathbb{R}$

$$(f * g)(x) = \int_{\mathbb{R}} f(x-y) g(y) dy = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \int_{\mathbb{R}} e^{i\xi(x-y)} F(\xi) g(y)\, d\xi\, dy.$$

The function $F(\xi)g(y)$ is integrable in (ξ, y), hence the Theorem of Fubini yields

$$(f * g)(x) = \int_{\mathbb{R}} e^{i\xi x} F(\xi) \left(\frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{-i\xi y} g(y)\, dy \right) d\xi = \int_{\mathbb{R}} e^{i\xi x} F(\xi) \widehat{g}(\xi)\, d\xi.$$

□

It is obvious that one can also understand the convolution operation componentwise. Since $L^1(\mathbb{C})$ with $*$ as multiplication is a Banach algebra without unit element, our next objective is to approximate the missing "convolution one" by an appropriate sequence of functions. We make the following definition:

Definition 2.24. *A **summability kernel** on the real line is a family of continuous functions $(k_\lambda)_{\lambda \in (0,\infty)}$ on $\mathbb{R}$ satisfying the following:*

(i) $\int_{\mathbb{R}} k_\lambda(x)\, dx = 1$ *for all* $\lambda \in (0, \infty)$,

(ii) $\int_{\mathbb{R}} |k_\lambda(x)|\, dx = O(1)$ *as* $\lambda \to \infty$,

(iii) $\lim\limits_{\lambda \to \infty} \int_{\{|x|>\delta\}} |k_\lambda(x)|\, dx = 0$ *for all* $\delta > 0$.

Example 2.25. Probably the best known summability kernel is the so-called **Féjer kernel** on $\mathbb{R}$ defined by

$$F_\lambda(x) := \lambda \cdot F(\lambda x),\ \lambda \in (0, \infty),$$

where

$$F(x) := \frac{1}{2\pi} \left(\frac{\sin x/2}{x/2} \right)^2 = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\xi x} (1 - |\xi|) \mathbb{1}_{[-1,1]}(\xi)\, d\xi. \tag{2.11}$$

The second equality in (2.11) is obtained directly by integration and using the identity $\cos 2x = 1 - 2\sin^2 x,\ x \in \mathbb{R}$.
We only have to prove $\int_{\mathbb{R}} F(x)\, dx = 1$, since this implies, introducing the change of variable $y = \lambda x$,

$$\int_{\mathbb{R}} F_\lambda(x)\, dx = \int_{\mathbb{R}} F(y)\, dy = 1,\ \lambda \in (0, \infty),$$

$$\int_{\mathbb{R}} |F_\lambda(x)|\, dx = \int_{\mathbb{R}} |F(y)|\, dy = 1,\ \lambda \in (0, \infty)$$

and

$$\int_{\{|x|>\delta\}} |F_\lambda(x)|\, dx = \int_{\{|y|>\lambda\delta\}} |F(y)|\, dy \to 0 \text{ as } \lambda \to \infty,$$

which shows that $(F_\lambda)_{\lambda\in(0,\infty)}$ is indeed a summability kernel on the real line.
For the proof of $\int_{\mathbb{R}} F(x)\, dx = 1$ we refer to [33, p. 125].

Using the Féjer kernel we get an approximative convolution one in the following sense:

Theorem 2.26. *Let $f \in L^1(M_d(\mathbb{C}))$ and $(F_\lambda)_{\lambda\in(0,\infty)}$ the Féjer summability kernel on $\mathbb{R}$, then, for any $x \in \mathbb{R}$,*

$$(F_\lambda * f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\lambda}^{\lambda} e^{i\xi x} \left(1 - \frac{|\xi|}{\lambda}\right) \widehat{f}(\xi)\, d\xi \tag{2.12}$$

and

$$\lim_{\lambda\to\infty} \|f - F_\lambda * f\|_{L^1} = 0. \tag{2.13}$$

Proof. We first show (2.12). Setting

$$\Delta(x) := \begin{cases} \frac{1}{\sqrt{2\pi}}(1-|x|), & |x| \le 1 \\ 0, & \text{otherwise} \end{cases}$$

we obtain, due to (2.11), $\widehat{\Delta}(\xi) = F(-\xi) = F(\xi)$ for all $\xi \in \mathbb{R}$. Theorem 2.21 (iii) yields, for all $\xi \in \mathbb{R}$, $F_\lambda(\xi) = \widehat{\Delta_\lambda}(\xi)$ with

$$\Delta_\lambda(x) := \begin{cases} \frac{1}{\sqrt{2\pi}}\left(1-\frac{|x|}{\lambda}\right), & |x| \le \lambda \\ 0, & \text{otherwise.} \end{cases} \tag{2.14}$$

Hence,

$$F_\lambda(x) = \frac{1}{2\pi} \int_{-\lambda}^{\lambda} e^{i\xi x} \left(1 - \frac{|\xi|}{\lambda}\right) d\xi$$

using again the symmetry of $F_\lambda(\,\cdot\,)$. Then Corollary 2.23 implies (2.12).

As to (2.13), let $f \in L^1(M_d(\mathbb{C}))$ and define $\varphi : \mathbb{R} \to L^1(M_d(\mathbb{C}))$, $\varphi(\tau) := f(\,\cdot - \tau)$. Note that φ is a continuous $L^1(M_d(\mathbb{C}))$-valued function on the real line. If $\varepsilon > 0$, then there is a continuous function g with compact support on $\mathbb{R}$ such that $\|f - g\|_{L^1} < \varepsilon$ and thus, due to the Dominated Convergence Theorem,

$$\begin{aligned} \|\varphi(\tau) - \varphi(\tau_0)\|_{L^1} = \|f(\,\cdot-\tau) - f(\,\cdot-\tau_0)\|_{L^1} &\le \|f(\,\cdot-\tau) - g(\,\cdot-\tau)\|_{L^1} \\ &+ \|g(\,\cdot-\tau) - g(\,\cdot-\tau_0)\|_{L^1} + \|g(\,\cdot-\tau_0) - f(\,\cdot-\tau_0)\|_{L^1} \\ &\le \varepsilon + \underbrace{\|g(\,\cdot-\tau) - g(\,\cdot-\tau_0)\|_{L^1}}_{\to 0 \text{ as } \tau\to\tau_0} + \varepsilon \end{aligned}$$

which proves the continuity of φ since ε can be taken arbitrarily small.

We next show that

$$\int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\, d\tau \xrightarrow{L^1} \varphi(0) = f \text{ as } \lambda \to \infty. \tag{2.15}$$

Note that the left-hand side in (2.15) is stated in terms of vector-valued integrals, i.e. they have to be understood as improper Riemann integrals of Banach space-valued (here $L^1(M_d(\mathbb{C}))$-valued) functions, defined and continuous on the real line. The existence is assured since $\int_{-\infty}^{\infty} \|F_\lambda(\tau)\varphi(\tau)\|_{L^1}\, d\tau < \infty$. Now, due to $\int_{-\infty}^{\infty} F_\lambda(\tau)\, d\tau = 1$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\, d\tau - \varphi(0) &= \int_{-\infty}^{\infty} F_\lambda(\tau)\left(\varphi(\tau) - \varphi(0)\right) d\tau \\ &= \int_{-\delta}^{\delta} F_\lambda(\tau)\left(\varphi(\tau) - \varphi(0)\right) d\tau + \int_{\{|\tau|>\delta\}} F_\lambda(\tau)\left(\varphi(\tau) - \varphi(0)\right) d\tau\end{aligned}$$

for arbitrary $\delta > 0$. Then

$$\begin{aligned}\left\|\int_{-\delta}^{\delta} F_\lambda(\tau)\left(\varphi(\tau) - \varphi(0)\right) d\tau\right\|_{L^1} &\le \max_{|\tau|\le\delta} \|\varphi(\tau) - \varphi(0)\|_{L^1} \cdot \int_{-\infty}^{\infty} |F_\lambda(\tau)|\, d\tau \\ &= \max_{|\tau|\le\delta} \|\varphi(\tau) - \varphi(0)\|_{L^1} \qquad (2.16)\end{aligned}$$

since (F_λ) is a nonnegative summability kernel and

$$\begin{aligned}\left\|\int_{\{|\tau|>\delta\}} F_\lambda(\tau)\left(\varphi(\tau) - \varphi(0)\right) d\tau\right\|_{L^1} &\le \max_{\tau} \|\varphi(\tau) - \varphi(0)\|_{L^1} \cdot \int_{\{|\tau|>\delta\}} |F_\lambda(\tau)|\, d\tau \\ &\le 2\, \|f\|_{L^1} \cdot \int_{\{|\tau|>\delta\}} |F_\lambda(\tau)|\, d\tau. \qquad (2.17)\end{aligned}$$

Given $\varepsilon > 0$, we can choose, due to the continuity of $\varphi(\tau)$ at $\tau = 0$, a $\delta > 0$ such that (2.16) is bounded by ε and keeping that δ, (2.17) is bounded by ε for large λ since $\int_{\{|\tau|>\delta\}} |F_\lambda(\tau)|\, d\tau \to 0$ as $\lambda \to \infty$. Thus (2.15) is shown.

Although the integrals in (2.15) have the formal appearance of a convolution, the operation involved (i.e. vector integration) is different from the convolution as defined in Theorem 2.22. The ambiguity, however, is harmless, since

$$\int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\, d\tau = F_\lambda * f \qquad (2.18)$$

for all $f \in L^1(M_d(\mathbb{C}))$. To establish (2.18), assume first that f is continuous with compact support on $\mathbb{R}$. Then, for any $n \in \mathbb{N}$,

$$\int_{-n}^{n} F_\lambda(\tau)\varphi(\tau)\, d\tau = \lim_{N\to\infty} \sum_{j=0}^{N} (\tau_{j+1} - \tau_j) F_\lambda(\tau_j) f(\cdot - \tau_j)$$

where the limit is taken in the $L^1(M_d(\mathbb{C}))$ norm as the subdivision (τ_j) of $[-n, n]$ becomes finer and finer. On the other hand,

$$\lim_{N\to\infty} \sum_{j=0}^{N} (\tau_{j+1} - \tau_j) F_\lambda(\tau_j) f(t - \tau_j) = \left(F_\lambda \mathbb{1}_{[-n,n]} * f\right)(t)$$

for any $t \in \mathbb{R}$. Hence, for all $n \in \mathbb{N}$,

$$\int_{-n}^{n} F_\lambda(\tau)\varphi(\tau)\, d\tau = F_\lambda \mathbb{1}_{[-n,n]} * f. \qquad (2.19)$$

Now, the left-hand side in (2.19) converges in $L^1(M_d(\mathbb{C}))$ to $\int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\,d\tau$ as $n \to \infty$ by definition of the improper vector-valued integral. For the right-hand side in (2.19) we observe

$$\begin{aligned}\left\|F_\lambda \mathbb{1}_{[-n,n]} * f - F_\lambda * f\right\|_{L^1} &= \left\|\left(F_\lambda \mathbb{1}_{[-n,n]} - F_\lambda\right) * f\right\|_{L^1} \\ &\leq \left\|F_\lambda \mathbb{1}_{[-n,n]} - F_\lambda\right\|_{L^1} \cdot \|f\|_{L^1} \to 0\end{aligned}$$

as $n \to \infty$ by the Dominated Convergence Theorem. Thus (2.18) holds for continuous f with compact support. For arbitrary $f \in L^1(M_d(\mathbb{C}))$, let $\varepsilon > 0$ and let g be continuous with compact support on $\mathbb{R}$ such that $\|f-g\|_{L^1} < \varepsilon$. Then

$$\int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\,d\tau - F_\lambda * f = \int_{-\infty}^{\infty} F_\lambda(\tau)(f-g)(\cdot - \tau)\,d\tau + F_\lambda * (g-f)$$

and thus $\left\| \int_{-\infty}^{\infty} F_\lambda(\tau)\varphi(\tau)\,d\tau - F_\lambda * f\right\|_{L^1} \leq 2\varepsilon$. This completes the proof of (2.18).

Finally, (2.15) together with (2.18) yields (2.13) which concludes the proof. $\square$

Corollary 2.27 (Inversion formula)**.**
Let $f \in L^1(M_d(\mathbb{C}))$ and $\widehat{f} \in L^1(M_d(\mathbb{C}))$. Then, for almost all $x \in \mathbb{R}$,

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x} \widehat{f}(\xi)\,d\xi.$$

Proof. The function $\xi \mapsto \mathbb{1}_{[-\lambda,\lambda]}(\xi)e^{i\xi x}\left(1 - |\xi|/\lambda\right)\widehat{f}(\xi)$ converges pointwise for all $x \in \mathbb{R}$ to the function $\xi \mapsto e^{i\xi x}\widehat{f}(\xi)$ as $\lambda \to \infty$. Since $\widehat{f} \in L^1(M_d(\mathbb{C}))$ we can use the Dominated Convergence Theorem to deduce for all $x \in \mathbb{R}$

$$\begin{aligned}\left(F_\lambda * f\right)(x) &\overset{(2.12)}{=} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbb{1}_{[-\lambda,\lambda]}(\xi)e^{i\xi x}\left(1 - \frac{|\xi|}{\lambda}\right)\widehat{f}(\xi)\,d\xi \\ &\overset{\lambda\to\infty}{\to} \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\xi x}\widehat{f}(\xi)\,d\xi.\end{aligned}$$

Theorem 2.26 shows that $\|F_\lambda * f - f\|_{L^1} \to 0$ as $\lambda \to \infty$, hence there is a subsequence $(F_{\lambda_k})_{k\in\mathbb{N}}$ such that $F_{\lambda_k} * f \overset{k\to\infty}{\to} f$ almost everywhere. $\square$

Before extending the Fourier transformation theory to L^p, $1 < p \leq 2$, we finish the L^1 theory by the following two observations which we will use several times in Chapter 4 and 5 where we consider Fourier transformations of α-stable and regularly varying Lévy processes, respectively.
Due to Corollary 2.27 we have, for almost all $\xi \in \mathbb{R}$,

$$\widehat{F_\lambda}(\xi) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{|\xi|}{\lambda}\right)\mathbb{1}_{[-\lambda,\lambda]}(\xi) \overset{(2.14)}{=} \Delta_\lambda(\xi) \tag{2.20}$$

and by virtue of Theorem 2.22, we obtain for all $f \in L^1(M_d(\mathbb{C}))$,

$$\widehat{F_\lambda * f}(\xi) = \sqrt{2\pi}\widehat{F_\lambda}(\xi) \cdot \widehat{f}(\xi) = \sqrt{2\pi}\Delta_\lambda(\xi) \cdot \widehat{f}(\xi) \tag{2.21}$$

for almost all $\xi \in \mathbb{R}$.

2.3.2 The L^p, $1 < p \leq 2$, theory

For $p \in (1, 2]$ we set

$$L^p(M_d(\mathbb{C})) := \left\{ f : \mathbb{R} \to M_d(\mathbb{C}) \text{ measurable}, \ \int_{\mathbb{R}} \|f(t)\|^p \, dt < \infty \right\}.$$

We equip $L^p(M_d(\mathbb{C}))$ with the norm $\|f\|_{L^p} := \left(\int_{\mathbb{R}} \|f(t)\|^p \, dt\right)^{1/p}$. The fact that the Lebesgue measure of $\mathbb{R}$ is infinite implies that $L^p(M_d(\mathbb{C})) \subseteq L^1(M_d(\mathbb{C}))$ does not hold if $p > 1$. Thus we have to find a new way to define the Fourier transform for the spaces $L^p(M_d(\mathbb{C}))$, $1 < p \leq 2$.

We first consider the case $p = 2$. Let

$$C_0(M_d(\mathbb{C})) := \{ f : \mathbb{R} \to M_d(\mathbb{C}), \ f \text{ continuous with compact support} \}$$

which is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_d(\mathbb{C})$ where all components f_{ij} are continuous complex-valued functions with compact support on $\mathbb{R}$. We obtain the following result:

Lemma 2.28. *For $f \in C_0(M_d(\mathbb{C}))$,*

$$\int_{\mathbb{R}} \left\| \widehat{f}(\xi) \right\|^2 d\xi = \int_{\mathbb{R}} \|f(x)\|^2 \, dx.$$

Proof. Recall that, if $f = (f_{ij})$, we have $\widehat{f} = (\widehat{f_{ij}})$ since the Fourier transformation can be interpreted componentwise (cf. Section 2.3.1). The proof of the lemma in the one-dimensional case can be found in [33, Chapter VI.3]. Thus

$$\int_{\mathbb{R}} \left\| \widehat{f}(\xi) \right\|^2 d\xi = \sum_{i,j=1}^{d} \int_{\mathbb{R}} \left| \widehat{f_{ij}}(\xi) \right|^2 d\xi = \sum_{i,j=1}^{d} \int_{\mathbb{R}} |f_{ij}(x)|^2 \, dx = \int_{\mathbb{R}} \|f(x)\|^2 \, dx.$$

□

The next theorem is central in the L^2 theory:

Theorem 2.29 (Plancherel)**.**
There exists a unique linear bijective operator

$$\mathcal{F} : L^2(M_d(\mathbb{C})) \to L^2(M_d(\mathbb{C}))$$

satisfying

$$\mathcal{F}f = \widehat{f} \text{ for all } f \in L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C})) \tag{2.22}$$

$$\|\mathcal{F}f\|_{L^2} = \|f\|_{L^2} \text{ for all } f \in L^2(M_d(\mathbb{C})). \tag{2.23}$$

Proof. We observe first that $L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$ is dense in $L^2(M_d(\mathbb{C}))$, consequently any continuous operator defined on $L^2(M_d(\mathbb{C}))$ is determined by its values on $L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$. This shows the uniqueness claim.

By virtue of Lemma 2.28, (2.23) holds for all $f \in C_0(M_d(\mathbb{C}))$ and since $C_0(M_d(\mathbb{C}))$ is dense in $L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$ (w.r.t. the norm $\|\cdot\|_{L^1} + \|\cdot\|_{L^2}$), (2.23) also holds for all

$f \in L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$. The mapping $f \mapsto \widehat{f}$ clearly can be extended by continuity to an isometry from $L^2(M_d(\mathbb{C}))$ into $L^2(M_d(\mathbb{C}))$.

It remains to show that $\mathcal{F}$ is onto. Note that $\langle f, g\rangle_{L^2(M_d(\mathbb{C}))} := \int_{\mathbb{R}} \operatorname{tr}(f(t)g^*(t))\,dt$ defines a scalar product on $L^2(M_d(\mathbb{C}))$. One can then establish easily Parseval's formula

$$\langle f, g\rangle_{L^2(M_d(\mathbb{C}))} = \langle \mathcal{F}f, \mathcal{F}g\rangle_{L^2(M_d(\mathbb{C}))} \tag{2.24}$$

for $f, g \in L^2(M_d(\mathbb{C}))$ which is in fact equivalent to (2.23). We have moreover

$$\langle \mathcal{F}f, g^*\rangle_{L^2(M_d(\mathbb{C}))} = \langle \mathcal{F}g, f^*\rangle_{L^2(M_d(\mathbb{C}))} \tag{2.25}$$

for all $f, g \in L^2(M_d(\mathbb{C}))$. For, if $f, g \in C_0(M_d(\mathbb{C}))$, the Theorem of Fubini yields

$$\begin{aligned}
\langle \mathcal{F}f, g^*\rangle_{L^2(M_d(\mathbb{C}))} &= \int_{\mathbb{R}} \operatorname{tr}(\mathcal{F}f(t)g(t))\,dt = \operatorname{tr}\int_{\mathbb{R}}\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t} f(\xi)\,d\xi\right) g(t)\,dt \\
&= \operatorname{tr}\int_{\mathbb{R}} f(\xi)\left(\frac{1}{\sqrt{2\pi}}\int_{\mathbb{R}} e^{-i\xi t} g(t)\,dt\right) d\xi = \int_{\mathbb{R}} \operatorname{tr}(f(\xi)\mathcal{F}g(\xi))\,d\xi \\
&= \langle \mathcal{F}g, f^*\rangle_{L^2(M_d(\mathbb{C}))}
\end{aligned}$$

where we used the fact that $\operatorname{tr}(AB) = \operatorname{tr}(BA)$ for all $A, B \in M_d(\mathbb{C})$.
For general $f, g \in L^2(M_d(\mathbb{C}))$, one can choose sequences $(f_n), (g_n) \subseteq C_0(M_d(\mathbb{C}))$ such that $f_n \xrightarrow{L^2} f$, $g_n \xrightarrow{L^2} g$ as $n \to \infty$. With the continuity of the scalar product, (2.25) also holds for the limits f and g.
Hence, for arbitrary $f \in L^2(M_d(\mathbb{C}))$, setting $g := (\mathcal{F}f)^*$ and $h := (\mathcal{F}g)^*$, we deduce

$$\|f-h\|_{L^2}^2 = \|f\|_{L^2}^2 + \underbrace{\|h\|_{L^2}^2}_{=\|f\|_{L^2}^2} - \int_{\mathbb{R}} \operatorname{tr}(h(t)f^*(t) + f(t)h^*(t))\,dt.$$

Since

$$\begin{aligned}
\langle f, h\rangle_{L^2(M_d(\mathbb{C}))} &= \langle h^*, f^*\rangle_{L^2(M_d(\mathbb{C}))} = \langle \mathcal{F}g, f^*\rangle_{L^2(M_d(\mathbb{C}))} \\
&\overset{(2.25)}{=} \langle \mathcal{F}f, g^*\rangle_{L^2(M_d(\mathbb{C}))} = \langle \mathcal{F}f, \mathcal{F}f\rangle_{L^2(M_d(\mathbb{C}))} \\
&\overset{(2.24)}{=} \langle f, f\rangle_{L^2(M_d(\mathbb{C}))} = \|f\|_{L^2}^2
\end{aligned}$$

and

$$\langle h, f\rangle_{L^2(M_d(\mathbb{C}))} = \overline{\langle f, h\rangle}_{L^2(M_d(\mathbb{C}))} = \|f\|_{L^2}^2$$

we finally obtain $\|f-h\|_{L^2}^2 = 0$ which proves that $\mathcal{F}$ is surjective. □

In view of (2.22) we shall write henceforth $\widehat{f}$ instead of $\mathcal{F}f$ which simplifies notation.

Remark 2.30.
For $f \in L^2(M_d(\mathbb{C}))$ we can define $f_n := f\mathbb{1}_{[-n,n]} \in L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$. Obviously $f_n \xrightarrow{L^2} f$ as $n \to \infty$ and we obtain the following form of Plancherel's Theorem: the sequence $\widehat{f_n}(\cdot) = \frac{1}{\sqrt{2\pi}}\int_{-n}^{n} e^{-ix\cdot} f(x)\,dx$ converges in $L^2(M_d(\mathbb{C}))$ to a function, denoted by $\widehat{f}$, for which (2.23) is valid.

We now turn our attention to $p \in (1, 2)$. Using the Riesz-Thorin Theorem (cf. [33, p. 97]), one can establish the well-known Hausdorff-Young Theorem:

Theorem 2.31 (cf. [33], p. 142 and [56], p. 6)**.**
Let $1 < p < 2$ and $q = \frac{p}{p-1}$ (i.e. $\frac{1}{p} + \frac{1}{q} = 1$). Then there is a constant $C(p)$ such that for all $f \in L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$,

$$\left(\int_{\mathbb{R}} \left\| \widehat{f}(\xi) \right\|^q d\xi \right)^{\frac{1}{q}} \leq C(p) \left(\int_{\mathbb{R}} \|f(x)\|^p \, dx \right)^{\frac{1}{p}} . \tag{2.26}$$

Using this result, we extend the Fourier transform to the space $L^p(M_d(\mathbb{C}))$, $1 < p < 2$, by continuity in the following way:

Definition 2.32.
Let $f \in L^p(M_d(\mathbb{C}))$, $1 < p < 2$, and $q = \frac{p}{p-1}$. Then the ***Fourier transform*** *$\widehat{f}$ of f is defined as the limit in $L^q(M_d(\mathbb{C}))$ of the sequence $\frac{1}{\sqrt{2\pi}} \int_{-n}^{n} e^{-ix\cdot} f(x) \, dx$ as $n \to \infty$.*

Note that $\widehat{f}$ is well-defined, since $L^1(M_d(\mathbb{C})) \cap L^2(M_d(\mathbb{C}))$ is dense in $L^1(M_d(\mathbb{C})) \cap L^p(M_d(\mathbb{C}))$ (w.r.t. the norm $\|\cdot\|_{L^1} + \|\cdot\|_{L^p}$) which implies that (2.26) holds for all $f \in L^1(M_d(\mathbb{C})) \cap L^p(M_d(\mathbb{C}))$.
The mapping $f \mapsto \widehat{f}$ so-defined is a linear continuous operator from $L^p(M_d(\mathbb{C}))$ into $L^q(M_d(\mathbb{C}))$, however, it is no longer an isometry and the range is not the whole of $L^q(M_d(\mathbb{C}))$.

We want to generalize Theorem 2.26 to the L^p case for $p \in (1, 2]$. To this end, we first generalize Theorem 2.22 and get

Theorem 2.33. *Let $f \in L^1(\mathbb{C})$ and $g \in L^p(M_d(\mathbb{C}))$, $1 < p \leq 2$. For almost all $x \in \mathbb{R}$, the convolution $h(x) := (f * g)(x)$ is well-defined, $h \in L^p(M_d(\mathbb{C}))$ and*

$$\|h\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p} .$$

Moreover, $\widehat{h} = \sqrt{2\pi} \widehat{f} \cdot \widehat{g}$.

Proof. Consider the function $F(x, y) := |f(x - y)| \cdot \|g(y)\|^p$ which is obviously measurable. Similar to the proof of Theorem 2.22 we obtain that, for almost all $x \in \mathbb{R}$, the integral $\int_{\mathbb{R}} F(x, y) \, dy$ exists and

$$\int_{\mathbb{R}} \int_{\mathbb{R}} F(x, y) \, dy \, dx = \|f\|_{L^1} \cdot \|g\|_{L^p}^p .$$

Hence, with $\frac{1}{p} + \frac{1}{q} = 1$, we have for almost all $x \in \mathbb{R}$,

$$\begin{aligned} \|h(x)\| = \|(f * g)(x)\| &\leq \int_{\mathbb{R}} |f(x - y)| \cdot \|g(y)\| \, dy \\ &\leq \left(\int_{\mathbb{R}} |f(x - y)| \cdot \|g(y)\|^p \, dy \right)^{1/p} \cdot \left(\int_{\mathbb{R}} |f(x - y)| \, dy \right)^{1/q} < \infty \end{aligned}$$

where we used Hölder's Inequality. Thus the convolution is almost everywhere defined and

$$\int_{\mathbb{R}} \|h(x)\|^p \, dx \leq \|f\|_{L^1} \cdot \|g\|_{L^p}^p \cdot \|f\|_{L^1}^{p/q} , \quad \text{i.e. } \|h\|_{L^p} \leq \|f\|_{L^1} \cdot \|g\|_{L^p} . \tag{2.27}$$

Finally, note that $\widehat{h}$ as well as $\sqrt{2\pi}\widehat{f}\cdot\widehat{g}$ are elements in $L^q(M_d(\mathbb{C}))$ with $q = p/(p-1)$. Let $(g_n)_{n\in\mathbb{N}} \subseteq C_0(M_d(\mathbb{C}))$ be a sequence of continuous functions with compact support on $\mathbb{R}$ such that $g_n \to g$ as $n \to \infty$ in $L^p(M_d(\mathbb{C}))$. Then, using Theorem 2.22 and 2.31 together with Definition 2.32, we obtain

$$\begin{aligned}
\left\|\widehat{h} - \sqrt{2\pi}\widehat{f}\cdot\widehat{g}\right\|_{L^q} &\le \left\|\widehat{h} - \widehat{f * g_n}\right\|_{L^q} + \left\|\widehat{f * g_n} - \sqrt{2\pi}\widehat{f}\cdot\widehat{g}\right\|_{L^q} \\
&= \left\|f * \widehat{(g - g_n)}\right\|_{L^q} + \left\|\sqrt{2\pi}\widehat{f}\cdot\widehat{g_n - g}\right\|_{L^q} \\
&\le C(p)\cdot\|f * (g - g_n)\|_{L^p} + \sqrt{2\pi}\,\|f\|_{L^1}\cdot\left\|\widehat{g_n - g}\right\|_{L^q} \\
&\overset{(2.27)}{\le} C(p)\cdot\|f\|_{L^1}\cdot\|g - g_n\|_{L^p} + \sqrt{2\pi}\,\|f\|_{L^1}\cdot C(p)\cdot\|g_n - g\|_{L^p} \\
&\overset{n\to\infty}{\to} 0.
\end{aligned}$$

This shows that $\widehat{h} = \sqrt{2\pi}\widehat{f}\cdot\widehat{g}$ and concludes the proof. □

As a consequence, if we take for f the Féjer kernel F_λ, we have

$$\widehat{F_\lambda * g} = \sqrt{2\pi}\widehat{F_\lambda}\cdot\widehat{g} \overset{(2.20)}{=} \sqrt{2\pi}\Delta_\lambda\cdot\widehat{g} \tag{2.28}$$

for all $g \in L^p(M_d(\mathbb{C}))$, $1 < p \le 2$. Since $F_\lambda * g$ is bounded (namely, for all $x \in \mathbb{R}$, $\|(F_\lambda * g)(x)\| \le \|F_\lambda\|_{L^q}\cdot\|g\|_{L^p}$ with $1/p + 1/q = 1$) and in $L^p(M_d(\mathbb{C}))$ (cf. Theorem 2.33) we also obtain that $F_\lambda * g \in L^2(M_d(\mathbb{C}))$. Hence, by virtue of Theorem 2.29 and its proof (in particular the "surjective" part), we deduce

$$F_\lambda * g = \frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda} e^{i\xi\,\cdot}\left(1 - \frac{|\xi|}{\lambda}\right)\widehat{g}(\xi)\,d\xi \tag{2.29}$$

for all $g \in L^p(M_d(\mathbb{C}))$, $1 < p \le 2$.
The last point which we have to show for the generalization of Theorem 2.26 to the L^p case is that $\|g - F_\lambda * g\|_{L^p} \to 0$ as $\lambda \to \infty$ for all $g \in L^p(M_d(\mathbb{C}))$. In fact, this can be established in a completely analogous way to the proof of (2.13). We only have to replace $L^1(M_d(\mathbb{C}))$ by $L^p(M_d(\mathbb{C}))$, $1 \le p < \infty$. We thus have

Theorem 2.34. *Let $f \in L^p(M_d(\mathbb{C}))$ with $1 \le p \le 2$. Then*

$$f = \lim_{\lambda\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\lambda}^{\lambda} e^{i\xi\,\cdot}\left(1 - \frac{|\xi|}{\lambda}\right)\widehat{f}(\xi)\,d\xi$$

in the $L^p(M_d(\mathbb{C}))$ norm.

Chapter 3

Multivariate CARMA processes

In this chapter we introduce multivariate Lévy-driven continuous time autoregressive moving average (CARMA) processes. They extend the well-known univariate CARMA models, dating back to [20], which have been extensively studied over the recent years (see e.g. [11, 12, 14, 60] and references therein). Originally, driving processes of CARMA models were restricted to Brownian motion, however, [11] allowed for Lévy processes which have a finite r-th moment for some $r > 0$. The multivariate CARMA model (MCARMA) was introduced in [41] where an explicit construction using a state space representation and a spectral representation of the driving Lévy process is given.

The chapter is organized as follows. In the first section we start with a brief summary of notions and results concerning random orthogonal measures. We then recall the construction of such a random orthogonal measure allowing for a spectral representation of square-integrable Lévy processes before we shall give some examples and study further properties of that measure in the third section. Multivariate CARMA and causal MCARMA processes are introduced in the last section.

3.1 Random orthogonal measures

As we shall make use of spectral representations of stationary processes (see [21, 29, 52] for comprehensive treatments), we need the notion of random orthogonal measures. In this section we briefly introduce these measures and integration w.r.t. them.

Let S be a non-empty set. Recall that a subset $\mathscr{R}$ of the power set $\mathscr{P}(S)$ of S is called **ring** if $\emptyset \in \mathscr{R}$ and $A, B \in \mathscr{R}$ implies $A \cup B \in \mathscr{R}$ and $A \backslash B \in \mathscr{R}$. If in addition $S \in \mathscr{R}$, then $\mathscr{R}$ is called **algebra**.

Definition 3.1. *Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space and let S be a non-empty set with a ring $\mathscr{R}$. If $\zeta : \mathscr{R} \to L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ is a map satisfying*

(i) $\zeta(\emptyset) = 0$,

(ii) for all sequences $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{R}$ pairwise disjoint with $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{R}$

$$\zeta\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \zeta(A_n)$$

(i.e. the series converges in L^2 to $\zeta(\bigcup_{n\in\mathbb{N}} A_n)$),

(iii) for all $A_1, A_2 \in \mathscr{R}$ with $A_1 \cap A_2 = \emptyset$

$$\mathbb{E}[\zeta(A_1)\zeta(A_2)^*] = 0,$$

then ζ is called a d-dimensional ***random orthogonal measure*** *on $\mathscr{R}$.*

The map $F : \mathscr{R} \to M_d(\mathbb{C})$, $F(A) := \mathbb{E}[\zeta(A)\zeta(A)^*]$ is called ***spectral function*** *w.r.t. the random orthogonal measure ζ.*

One often only has that $\zeta : \mathscr{R} \to L^2$ is additive which is less than (ii) in Definition 3.1. In this case ζ is referred to as a random orthogonal content.

F assumes values in the positive semidefinite matrices. From the point of view of functional analysis, the positive semidefinite matrices form the cone of positive bounded operators in the C^*-algebra $B(\mathbb{C}^d) := B(\mathbb{C}^d, \mathbb{C}^d)$. For every sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{R}$ pair-wise disjoint with $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{R}$, one has

$$F\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \mathbb{E}\Big[\zeta\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)\zeta\Big(\bigcup_{n\in\mathbb{N}} A_n\Big)^*\Big] = \lim_{k\to\infty}\sum_{n=1}^{k} \mathbb{E}[\zeta(A_n)\zeta(A_n)^*] = \sum_{n\in\mathbb{N}} F(A_n),$$

where the convergence of the series holds for any (operator) norm on $B(\mathbb{C}^d)$. Obviously $F(A) \geq F(\emptyset) = 0$ for every $A \in \mathscr{R}$, where, for $X, Y \in \mathbb{S}_d^+(\mathbb{C})$ positive semidefinite, $X \geq Y$ if and only if $X - Y \geq 0$, i.e. $X - Y \in \mathbb{S}_d^+(\mathbb{C})$. Thus F is a positive-operator-valued (POV) measure on $\mathscr{R}$ (cf. for instance [57] for the notion of a POV measure).

Let us denote by $\sigma(\mathscr{R})$ the σ-ring generated by $\mathscr{R}$ (i.e. $\sigma(\mathscr{R})$ is the smallest σ-ring containing $\mathscr{R}$). In the following we will assume throughout that $S = \mathbb{R}$, $\mathscr{R} \subseteq \mathscr{B}_0(\mathbb{R})$ and $\sigma(\mathscr{R}) = \mathscr{B}(\mathbb{R})$ where $\mathscr{B}_0(\mathbb{R})$ is the ring of all Borel sets with finite Lebesgue measure. We only encounter random orthogonal measures whose associated spectral functions have constant density w.r.t. the Lebesgue measure λ^1 on $\mathbb{R}$, i.e. $F(A) = \lambda^1(A)C$ for some $C \in \mathbb{S}_d^+(\mathbb{C})$ and all $A \in \mathscr{R}$, which simplifies the upcoming integration theory considerably.

Due to our assumption that F has constant density w.r.t. to λ^1, one can immediately extend F to a POV measure $\widetilde{F}$ on $\mathscr{B}_0(\mathbb{R})$ by setting $\widetilde{F}(A) := \lambda^1(A)C$ for all $A \in \mathscr{B}_0(\mathbb{R})$. By abuse of notation we denote that extension again by F. It is referred to as the **spectral measure** of the random orthogonal measure ζ.

Our next aim is to extend also ζ to the ring of all Borel sets with finite Lebesgue measure $\mathscr{B}_0(\mathbb{R})$. To this end, consider an elementary function $\varphi : \mathbb{R} \to M_d(\mathbb{C})$, $\varphi = \sum_{k=1}^n \varphi_k \mathbb{1}_{A_k}$, where $A_k \in \mathscr{R}$ are pairwise disjoint and $\varphi_k \in M_d(\mathbb{C})$, $k = 1, \ldots, n$, $n \in \mathbb{N}$. We define the stochastic integral of φ with respect to ζ by

$$J(\varphi) := \sum_{k=1}^{n} \varphi_k \zeta(A_k) \in L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d).$$

It is standard to show that $J(\varphi)$ is independent of the concrete representation of the elementary function φ. Moreover, it is easy to deduce that

$$\mathbb{E}[J(\varphi)J(\psi)^*] = \int_{\mathbb{R}} \varphi(t)F(dt)\psi(t)^* = \int_{\mathbb{R}} \varphi(t)C\psi(t)^* dt \tag{3.1}$$

for all elementary functions φ and ψ. Indeed there exist representations of φ and ψ with

$\varphi = \sum\limits_{k=1}^{n} \varphi_k \mathbb{1}_{A_k}$ and $\psi = \sum\limits_{k=1}^{n} \psi_k \mathbb{1}_{A_k}$ where $A_j \cap A_k = \emptyset$ for all $j \neq k$. Then

$$\begin{aligned}\mathbb{E}[J(\varphi)J(\psi)^*] &= \sum_{k=1}^{n} \mathbb{E}[\varphi_k \zeta(A_k)\zeta(A_k)^* \psi_k^*] = \sum_{k=1}^{n} \varphi_k \underbrace{\mathbb{E}[\zeta(A_k)\zeta(A_k)^*]}_{=F(A_k)} \psi_k^* \\ &= \int_{\mathbb{R}} \varphi(t)F(dt)\psi(t)^* = \int_{\mathbb{R}} \varphi(t)C\psi(t)^* dt.\end{aligned}$$

We want to extend integration w.r.t. ζ to all functions in $L^2(M_d(\mathbb{C}))$ which is defined as (cf. Section 2.3)

$$L^2(M_d(\mathbb{C})) := \left\{ f : \mathbb{R} \to M_d(\mathbb{C}) \text{ measurable}, \int_{\mathbb{R}} \|f(t)\|^2 \, dt < \infty \right\}$$

with $\|\cdot\|$ being any norm on $M_d(\mathbb{C})$. Note that $\int_{\mathbb{R}} \|f(t)\|^2 \, dt < \infty$ implies the existence of $\int_{\mathbb{R}} f(t)F(dt)f(t)^*$. Recall that the space $L^2(M_d(\mathbb{C}))$ is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_d(\mathbb{C})$ where all components are in the usual space $L^2(\mathbb{C})$. With the ordinary norm $\|f\|_{L^2} := (\int_{\mathbb{R}} \|f(t)\|^2 \, dt)^{1/2}$, the space $L^2(M_d(\mathbb{C}))$ becomes a Banach space and thus it is in particular complete.

Proposition 3.2. *Let $\zeta : \mathscr{R} \to L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ be a random orthogonal measure on the ring $\mathscr{R} \subseteq \mathscr{B}_0(\mathbb{R})$ such that $\sigma(\mathscr{R}) = \mathscr{B}(\mathbb{R})$.*
Define $T(\mathscr{R}) := \{\varphi = \sum_{k=1}^{n} \varphi_k \mathbb{1}_{A_k},\ A_k \in \mathscr{R}$ pairwise disjoint, $\varphi_k \in M_d(\mathbb{C})\}$ the linear space of all $\mathscr{R}$-elementary functions. Then $T(\mathscr{R})$ is dense in $L^2(M_d(\mathbb{C}))$.

Proof. It is a well-known result that the linear space of $\mathscr{B}_0(\mathbb{R})$-elementary functions is dense in $L^2(M_d(\mathbb{C}))$. Recall the simple fact that for $A \in \sigma(\mathscr{R}) = \mathscr{B}(\mathbb{R})$ with $\lambda^1(A) < \infty$ there exists for every $\varepsilon > 0$ a set $A_\varepsilon = \bigcup_{i=1}^{n} A_i$ with $A_i \in \mathscr{R}$ pairwise disjoint such that $\lambda^1(A \triangle A_\varepsilon) < \varepsilon$ (where $A \triangle B := (A \backslash B) \cup (B \backslash A)$ is the symmetric difference of the sets A and B). Thus every $\mathscr{B}_0(\mathbb{R})$-elementary function can be approximated by an $\mathscr{R}$-elementary function. $\square$

If now $f \in L^2(M_d(\mathbb{C}))$, let $(\varphi_n)_{n \in \mathbb{N}} \subseteq T(\mathscr{R})$ be a sequence s.t. $\varphi_n \to f$ in $L^2(M_d(\mathbb{C}))$. Hence, using (3.1), we have

$$\begin{aligned}\|J(\varphi_n) - J(\varphi_m)\|_{L^2}^2 &= \mathbb{E}[|J(\varphi_n - \varphi_m)|^2] \leq c \cdot \|\mathbb{E}[J(\varphi_n - \varphi_m)J(\varphi_n - \varphi_m)^*]\| \\ &\leq c' \cdot \|C\| \cdot \|\varphi_n - \varphi_m\|_{L^2}^2 \end{aligned} \tag{3.2}$$

for some constants $c, c' > 0$. This yields that $J(\varphi_n)$ converges in mean square since it is a Cauchy sequence in $L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$. We set

$$\int_{\mathbb{R}} f \, d\zeta := J(f) := \underset{n\to\infty}{\text{l.i.m.}}\, J(\varphi_n) \in L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$$

where l.i.m. means the L^2-limit.

Proposition 3.3. *Suppose the settings of Proposition 3.2 and assume that the corresponding spectral measure is given by $F : \mathscr{B}_0(\mathbb{R}) \to M_d(\mathbb{C})$, $F(dt) = C\,dt$ for some $C \in \mathbb{S}_d^+(\mathbb{C})$. Then the following properties hold for the introduced integral:*

(i) *For all* $f \in L^2(M_d(\mathbb{C}))$, *the integral* $\int_{\mathbb{R}} f\,d\zeta$ *is independent of the approximating sequence of* $\mathscr{R}$*-elementary functions* φ_n.

(ii) $\mathbb{E}[(\int_{\mathbb{R}} f\,d\zeta)(\int_{\mathbb{R}} g\,d\zeta)^*] = \int_{\mathbb{R}} f(t)Cg(t)^* dt$ *for all* $f, g \in L^2(M_d(\mathbb{C}))$.

(iii) $\int_{\mathbb{R}}(\alpha f + \beta g)\,d\zeta = \alpha \int_{\mathbb{R}} f\,d\zeta + \beta \int_{\mathbb{R}} g\,d\zeta$ *for all* $\alpha, \beta \in M_d(\mathbb{C})$ *and* $f, g \in L^2(M_d(\mathbb{C}))$.

(iv) *If* $(f_n)_{n\in\mathbb{N}}$ *is a sequence in* $L^2(M_d(\mathbb{C}))$ *and* $f \in L^2(M_d(\mathbb{C}))$ *with* $\|f_n - f\|_{L^2} \to 0$ *as* $n \to \infty$, *then* $\int_{\mathbb{R}} f_n\,d\zeta \overset{n\to\infty}{\to} \int_{\mathbb{R}} f\,d\zeta$ *in* $L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$.

Proof. These properties are an easy consequence of the definition of the integral and (3.1) and (3.2), respectively. □

Now we are able to extend $\zeta : \mathscr{R} \to L^2$ to a random orthogonal measure defined on the ring of all Borel sets with finite Lebesgue measure $\mathscr{B}_0(\mathbb{R})$. More precisely, for all $A \in \mathscr{B}_0(\mathbb{R})$, define

$$\widetilde{\zeta}(A) := J(\mathbb{1}_A) = \int_{\mathbb{R}} \mathbb{1}_A\,d\zeta.$$

Being completely correct one should write $\mathrm{I}_d \mathbb{1}_A$ with I_d being the identity matrix in $M_d(\mathbb{C})$ instead of $\mathbb{1}_A$ for the integrand above. However, we forego this correctness to simplify notation in the upcoming proof of Theorem 3.4. Obviously $\widetilde{\zeta}|_{\mathscr{R}} = \zeta$. Moreover we have:

Theorem 3.4. *Let* $\widetilde{\zeta} : \mathscr{B}_0(\mathbb{R}) \to L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ *be the extension of a random orthogonal measure to the ring of all Borel sets with finite Lebesgue measure as constructed above. Then:*

(i) $\mathbb{E}\left[\widetilde{\zeta}(A_1)\widetilde{\zeta}(A_2)^*\right] = F(A_1 \cap A_2)$ *for all* $A_1, A_2 \in \mathscr{B}_0(\mathbb{R})$.

(ii) $\widetilde{\zeta}(A_1 \cup A_2) = \widetilde{\zeta}(A_1) + \widetilde{\zeta}(A_2)$ *for all* $A_1, A_2 \in \mathscr{B}_0(\mathbb{R})$ *with* $A_1 \cap A_2 = \emptyset$.

(iii) *If* $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{B}_0(\mathbb{R})$ *are pairwise disjoint with* $\bigcup_{n\in\mathbb{N}} A_n \in \mathscr{B}_0(\mathbb{R})$ *then* $\widetilde{\zeta}\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} \widetilde{\zeta}(A_n)$ *in* $L^2(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$.

Proof.
(i) For arbitrary $A_1, A_2 \in \mathscr{B}_0(\mathbb{R})$, we deduce

$$\begin{aligned}
\mathbb{E}\left[\widetilde{\zeta}(A_1)\widetilde{\zeta}(A_2)^*\right] &= \mathbb{E}\left[\left(\int_{\mathbb{R}} \mathbb{1}_{A_1}\,d\zeta\right)\left(\int_{\mathbb{R}} \mathbb{1}_{A_2}\,d\zeta\right)^*\right] \\
&= \int_{\mathbb{R}} \mathbb{1}_{A_1}(t) C \mathbb{1}_{A_2}(t)^* dt \\
&= \int_{A_1\cap A_2} C\,dt = \lambda^1(A_1 \cap A_2)C = F(A_1 \cap A_2).
\end{aligned}$$

(ii) Let $A_1, A_2 \in \mathscr{B}_0(\mathbb{R})$ be two disjoint sets, then

$$\begin{aligned}
\widetilde{\zeta}(A_1 \cup A_2) &= \int_{\mathbb{R}} \mathbb{1}_{A_1\cup A_2}\,d\zeta = \int_{\mathbb{R}} (\mathbb{1}_{A_1} + \mathbb{1}_{A_2})\,d\zeta \\
&= \int_{\mathbb{R}} \mathbb{1}_{A_1}\,d\zeta + \int_{\mathbb{R}} \mathbb{1}_{A_2}\,d\zeta = \widetilde{\zeta}(A_1) + \widetilde{\zeta}(A_2).
\end{aligned}$$

(iii) Let $A := \bigcup_{n\in\mathbb{N}} A_n$ and $f_n := \mathbb{1}_A - \sum_{i=1}^{n} \mathbb{1}_{A_i}$. Due to (3.2) and Proposition 3.3 (ii), we obtain (with the same c' as in (3.2))

$$\mathbb{E}\left[\left|\widetilde{\zeta}(A) - \sum_{i=1}^{n} \widetilde{\zeta}(A_i)\right|^2\right] \le c' \cdot \|C\| \cdot \|f_n\|_{L^2}^2 \to 0 \text{ as } n \to \infty.$$

□

Theorem 3.4 shows that $\widetilde{\zeta}$ is indeed a random orthogonal measure on $\mathscr{B}_0(\mathbb{R})$. Of course it is possible to define integration of functions $f \in L^2(M_d(\mathbb{C}))$ w.r.t. the extended random orthogonal measure $\widetilde{\zeta}$ in the same way as it has been done above for ζ. It is easy to see that Proposition 3.3 remains valid.

3.2 Spectral representation of square-integrable Lévy processes

We recall the central theorem in [41] establishing a spectral representation for square-integrable Lévy processes using random orthogonal measures.

Theorem 3.5. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided square-integrable Lévy process in $\mathbb{R}^d$ with $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1 L_1^*] = \Sigma_L$. Then there is a d-dimensional random orthogonal measure Φ_L defined on $\mathscr{B}_0(\mathbb{R})$ with spectral measure F_L such that $\mathbb{E}[\Phi_L(\Delta)] = 0$ for any bounded Borel set Δ,*

$$F_L(dt) = \frac{\Sigma_L}{2\pi}\, dt$$

and

$$L_t = \int_{-\infty}^{\infty} \frac{e^{i\mu t} - 1}{i\mu}\, \Phi_L(d\mu).$$

The random measure Φ_L is uniquely determined by

$$\Phi_L\left([a,b)\right) = \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu}\, L(d\mu)$$

for all $-\infty < a < b < \infty$.

Proof. See [41, Theorem 3.5]. □

Remark 3.6. The proof of [41, Theorem 3.5] starts with the observation that $\widetilde{\Phi}\left([a,b)\right) := L_b - L_a$ defines a random orthogonal measure on the semi-ring of intervals $[a,b)$ with $-\infty < a < b < \infty$. The associated spectral function $\widetilde{F}_L$ satisfies $\widetilde{F}_L(dt) = \Sigma_L\, dt$. In Section 3.1 we described the general procedure how one can extend a random orthogonal measure from a ring that generates the Borel σ-algebra to $\mathscr{B}_0(\mathbb{R})$. The definition on a semi-ring (that generates $\mathscr{B}(\mathbb{R})$) is not a constraint for that procedure since it is well-known how a measure can be extended from a semi-ring to the generated ring therefrom. Namely, if $\mathscr{S}\mathscr{R}$ is a semi-ring, the generated ring therefrom has the form $\{\bigcup_{i=1}^n A_i :\ n \in \mathbb{N},\, A_i \in \mathscr{S}\mathscr{R},\, i = 1,\dots,n\}$ where the sets A_i can be chosen w.l.o.g. pairwise disjoint.

3.3 Properties of the associated random orthogonal measure

In this section we study further properties of the random orthogonal measure Φ_L in Theorem 3.5. Therefore recall that, if the integral $X_t = \int_{\mathbb{R}} f(t,s)\, L(ds)$ with $f : \mathbb{R} \times \mathbb{R} \to M_{k\times d}(\mathbb{R})$ measurable and $(L_t)_{t\in\mathbb{R}}$ a two-sided d-dimensional Lévy process with generating triplet $(\gamma, 0, \nu)$ is well-defined, then the distribution of X_t is infinitely divisible with generating triplet $(\gamma_X^t, 0, \nu_X^t)$ where

$$\gamma_X^t = \int_{\mathbb{R}} \left(f(t,s)\gamma + \int_{\mathbb{R}^d} f(t,s)x \left(\mathbb{1}_{\{\|f(t,s)x\|\le 1\}} - \mathbb{1}_{\{\|x\|\le 1\}} \right) \nu(dx) \right) ds$$

and

$$\nu_X^t(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A(f(t,s)x)\, \nu(dx)\, ds,\ \ A \in \mathscr{B}(\mathbb{R}_*^k)$$

(cf. Section 2.2.5).

Example 3.7. We start with the simple example of a (compensated) Poisson process in $\mathbb{R}$. Let $(N_t)_{t\in\mathbb{R}}$ be a Poisson process with rate κ and consider $L = (L_t)_{t\in\mathbb{R}}$, $L_t := N_t - \kappa t$. Then $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = \mathrm{Var}(L_1) = \kappa$ and hence the requirements of Theorem 3.5 are fulfilled.

The original Lévy measure is given by $\nu = \kappa \cdot \delta_1$ where δ_1 denotes the Dirac measure in 1. Thus it is obvious that L has finite activity, a.s. finite variation and finite exponential moments (i.e. $\mathbb{E}[\exp\{\alpha\, |L_1|\}] < \infty$ for every $\alpha > 0$).

We fix $t > 0$ and consider $Z_t := \Phi_L([0,t)) = \int_{-\infty}^{\infty} \frac{1-e^{-i\mu t}}{2\pi i\mu} L(d\mu)$. Then the distribution of Z_t is infinitely divisible and we obtain for its Lévy measure, identifying $\mathbb{C}$ with $\mathbb{R}^2$,

$$\begin{aligned}\nu_{Z_t}(A) &= \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_A \left(\frac{1-e^{-i\mu t}}{2\pi i\mu} x \right) \nu(dx)\, d\mu \\ &= \kappa \cdot \int_{\mathbb{R}} \mathbb{1}_A \left(\frac{1-e^{-i\mu t}}{2\pi i\mu} \right) d\mu = \kappa \cdot \lambda^1 \left(\left\{ \mu \in \mathbb{R} :\ \frac{1-e^{-i\mu t}}{2\pi i\mu} \in A \right\} \right)\end{aligned}$$

for every $A \in \mathscr{B}(\mathbb{C}_*)$, denoting the Lebesgue measure on $\mathbb{R}$ by λ^1 and the punctured complex plane by $\mathbb{C}_* := \mathbb{C}\setminus\{0\}$. Hence, ν_{Z_t} is an infinite measure on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$. That is, Z_t has not a compound Poisson distribution.

We define $f : \mathbb{R} \to \mathbb{C}$, $f(\mu) := \frac{1-e^{-i\mu t}}{2\pi i\mu}$ and denote by $f(\lambda^1)$ the image measure of λ^1 by f on $(\mathbb{C}, \mathscr{B}(\mathbb{C}))$. Then one has

$$\begin{aligned}\int_{\{|x|\le 1\}} |x|^\delta\, \nu_{Z_t}(dx) &= \kappa \cdot \int_{\{|x|\le 1\}} |x|^\delta\, f(\lambda^1)(dx) = \kappa \cdot \int_{f^{-1}(\{|x|\le 1\})} |f(y)|^\delta\, \lambda^1(dy) \\ &= \frac{\kappa}{2^{\frac{\delta}{2}}\pi^\delta} \cdot \int_{\left\{\frac{\sqrt{1-\cos(yt)}}{|y|} \le \sqrt{2}\pi\right\}} \frac{(1-\cos(yt))^{\frac{\delta}{2}}}{|y|^\delta}\, \lambda^1(dy).\end{aligned}$$

Since

$$\lim_{y\to 0} \frac{\sqrt{1-\cos(yt)}}{|y|} = \frac{t}{\sqrt{2}} \quad \text{and} \quad \lim_{y\to\infty} \frac{\sqrt{1-\cos(yt)}}{|y|} = 0$$

we observe that

$$\{|y| \ge c\} \subseteq \left\{ \frac{\sqrt{1-\cos(yt)}}{|y|} \le \sqrt{2}\pi \right\}$$

for a sufficiently large $c > 0$.
Setting $A := \{|y| \geq c\}$ and $B := \{|y| < c\} \cap \left\{ \frac{\sqrt{1-\cos(yt)}}{|y|} \leq \sqrt{2}\pi \right\}$ yields

$$\int_{\{|x|\leq 1\}} |x|^\delta \, \nu_{Z_t}(dx) = \frac{\kappa}{2^{\frac{\delta}{2}}\pi^\delta} \cdot \left(\int_A \frac{(1-\cos(yt))^{\frac{\delta}{2}}}{|y|^\delta} \lambda^1(dy) + \int_B \frac{(1-\cos(yt))^{\frac{\delta}{2}}}{|y|^\delta} \lambda^1(dy) \right).$$

The second integral on the right-hand side is nonnegative and bounded by $(\sqrt{2}\pi)^\delta 2c$ and the first integral is finite if and only if $\delta > 1$. Hence, ν_{Z_t} integrates $|x|^\delta$ locally at zero iff $\delta > 1$.

Analogously, for any $\alpha > 0$ and c' sufficiently large s.t.

$$\{|y| > c'\} \subseteq \left\{ \frac{\sqrt{1-\cos(yt)}}{|y|} < \sqrt{2}\pi \right\},$$

we obtain

$$\begin{aligned} \int_{\{|x|\geq 1\}} \exp\{\alpha\,|x|\}\, \nu_{Z_t}(dx) &\leq \kappa \cdot \int_{\{|y|\leq c'\}} \exp\left\{ \alpha \cdot \frac{\sqrt{1-\cos(yt)}}{\sqrt{2}\pi\,|y|} \right\} \lambda^1(dy) \\ &\leq \kappa \cdot 2c' \cdot c'' \end{aligned}$$

for some $c'' \in \mathbb{R}_+$, since the integrand converges to $\exp\left\{\alpha \cdot \frac{t}{2\pi}\right\}$ as $y \to 0$ and thus it is in particular bounded on $\{|y| \leq c'\}$. This implies that Z_t has finite exponential moments and in particular finite moments of all orders.

This example shows that although the original process has finite activity and a.s. finite variation, ν_{Z_t} is not finite and does not integrate $|x|$ locally at zero. However, the "transformation" preserves moments. Indeed, this can be shown in general.

Theorem 3.8. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided square-integrable Lévy process in $\mathbb{R}^d$ without a Brownian component such that $\mathbb{E}[L_1] = 0$ and let Φ_L be the corresponding random orthogonal measure of Theorem 3.5. Moreover, letting $(\gamma, 0, \nu)$ the generating triplet of L, we assume that $\nu \neq 0$. Then, for all $t > 0$, we have the following results for $Z_t := \Phi_L([0,t)) = \int_{-\infty}^{\infty} \frac{1-e^{-i\mu t}}{2\pi i\mu} L(d\mu)$:*

(i) The distribution of Z_t is infinitely divisible, its corresponding Lévy measure ν_{Z_t} is infinite and $\int_{\{\|x\|\leq 1\}} \|x\|\, \nu_{Z_t}(dx) = \infty$.

(ii) For any $\delta \in (1,2)$, the Lévy process L has a.s. finite δ-variation if and only if $\int_{\{\|x\|\leq 1\}} \|x\|^\delta\, \nu_{Z_t}(dx)$ is finite.

(iii) The Blumenthal-Getoor-Index of ν_{Z_t} is given by $\beta(\nu_{Z_t}) = 1 \vee \beta(\nu)$.

(iv) If $\alpha > 2$, then $\mathbb{E}[\|L_1\|^\alpha] < \infty$ if and only if $\mathbb{E}[\|Z_t\|^\alpha] < \infty$.

(v) If $\mathbb{E}[\exp\{\alpha\,\|L_1\|\}] < \infty$ for some $\alpha > 0$, then $\mathbb{E}[\exp\{\alpha(t)\,\|Z_t\|\}] < \infty$ with

$$\alpha(t) = \frac{\pi}{\sqrt{2}c(t)}\alpha \quad \text{and} \quad c(t) = \sup_{\mu\in\mathbb{R}} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|} \in (0,\infty).$$

Proof.
(i) Obviously, $Z_t = \Phi_L([0,t)) = \int_{-\infty}^{\infty} \frac{1-e^{-i\mu t}}{2\pi i\mu} L(d\mu)$ has an infinitely divisible distribution for every $t > 0$.

Its Lévy measure, identifying again $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{C}^d$ with $(\mathbb{R}^2)^d$, is given by

$$\nu_{Z_t}(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A \left(\frac{1-e^{-i\mu t}}{2\pi i\mu} x \right) \nu(dx)\, d\mu = f(\lambda^1 \otimes \nu)(A),\ A \in \mathscr{B}\left(\mathbb{C}_*^d\right),$$

where $f : \mathbb{R} \times \mathbb{R}^d \to \mathbb{C}^d$, $f(\mu, x) := \frac{1-e^{-i\mu t}}{2\pi i\mu} x$ and λ^1 denotes again the one-dimensional Lebesgue measure. Since $\nu_{Z_t}(\{0\}) = 0$, we obtain

$$\begin{aligned}
\nu_{Z_t}\left(\mathbb{C}^d\right) = \nu_{Z_t}\left(\mathbb{C}_*^d\right) &= \lambda^1 \otimes \nu \left(\left\{ (\mu, x) \in \mathbb{R} \times \mathbb{R}^d :\ \frac{1-e^{-i\mu t}}{2\pi i\mu} x \neq 0 \right\} \right) \\
&= \lambda^1 \otimes \nu \left(\left\{ \mu \in \mathbb{R} :\ \frac{1-e^{-i\mu t}}{2\pi i\mu} \neq 0 \right\} \times \left\{ x \in \mathbb{R}^d :\ x \neq 0 \right\} \right) \\
&= \underbrace{\lambda^1 \left(\left\{ \mu \in \mathbb{R} :\ \frac{1-e^{-i\mu t}}{2\pi i\mu} \neq 0 \right\} \right)}_{=\infty} \cdot \underbrace{\nu\left(\mathbb{R}^d\right)}_{>0} = \infty.
\end{aligned}$$

Now, for any $\delta \in [1, 2)$, we observe

$$\begin{aligned}
&\int_{\{\|x\| \le 1\}} \|x\|^\delta \nu_{Z_t}(dx) \\
&= \int_{\left\{(\mu,y)\in\mathbb{R}\times\mathbb{R}^d:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\| \le \sqrt{2}\pi\right\}} \left(\frac{\|y\|}{\sqrt{2}\pi} \right)^\delta \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} (\lambda^1 \otimes \nu)(d(\mu, y)) \\
&= \left(\frac{1}{\sqrt{2}\pi} \right)^\delta \cdot \int_{\mathbb{R}^d \setminus \{0\}} \|y\|^\delta \left(\int_{\left\{\mu\in\mathbb{R}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\| \le \sqrt{2}\pi\right\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} d\mu \right) \nu(dy)
\end{aligned} \tag{3.3}$$

due to the Theorem of Fubini.
Since the inner integral $\int_{\left\{\mu\in\mathbb{R}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\| \le \sqrt{2}\pi\right\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} d\mu = \infty$ for all $y \in \mathbb{R}^d \setminus \{0\}$ if $\delta = 1$ (cf. Example 3.7), we deduce

$$\int_{\{\|x\| \le 1\}} \|x\|\, \nu_{Z_t}(dx) = \infty, \tag{3.4}$$

and (i) is shown.
(ii) Let now $\delta \in (1, 2)$ and assume that L has a.s. finite δ-variation. Note that the inner integral in (3.3) satisfies

$$\int_{\left\{\mu\in\mathbb{R}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\| \le \sqrt{2}\pi\right\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} d\mu \le \int_{\mathbb{R}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} d\mu =: C(\delta) < \infty \tag{3.5}$$

and thus (3.3) becomes

$$\begin{aligned}
\int_{\{\|x\| \le 1\}} \|x\|^\delta \nu_{Z_t}(dx) &\le C(\delta) \cdot \left(\frac{1}{\sqrt{2}\pi} \right)^\delta \cdot \int_{\mathbb{R}^d \setminus \{0\}} \|y\|^\delta \nu(dy) \\
&\le C(\delta) \cdot \left(\frac{1}{\sqrt{2}\pi} \right)^\delta \cdot \left(\int_{\{\|y\| \le 1\}} \|y\|^\delta \nu(dy) + \int_{\{\|y\| > 1\}} \|y\|^2 \nu(dy) \right).
\end{aligned}$$

The first integral on the right-hand side is finite since we assume that L has a.s. finite δ-variation (cf. Proposition 2.17) and the second integral is finite as well since L is square-integrable (cf. Theorem 2.20). Hence, the integral $\int_{\{\|x\|\leq 1\}} \|x\|^\delta \nu_{Z_t}(dx)$ is finite.

Conversely, assume that $\int_{\{\|x\|\leq 1\}} \|x\|^\delta \nu_{Z_t}(dx) < \infty$. Then (cf. (3.3))

$$\begin{aligned}
&\int_{\{\|x\|\leq 1\}} \|x\|^\delta \nu_{Z_t}(dx) \\
&\quad= \left(\frac{1}{\sqrt{2}\pi}\right)^\delta \cdot \int_{\mathbb{R}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} \left(\int_{\left\{y\in\mathbb{R}^d\setminus\{0\}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|\leq\sqrt{2}\pi\right\}} \|y\|^\delta \nu(dy)\right) d\mu
\end{aligned}$$

and due to

$$\left\{y\in\mathbb{R}^d\setminus\{0\}:\ \|y\|\leq \pi\cdot|\mu|\right\} \subseteq \left\{y\in\mathbb{R}^d\setminus\{0\}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|\leq\sqrt{2}\pi\right\}$$

for all $\mu\neq 0$, we deduce

$$\begin{aligned}
&\int_{\{\|x\|\leq 1\}} \|x\|^\delta \nu_{Z_t}(dx) \\
&\qquad\geq \left(\frac{1}{\sqrt{2}\pi}\right)^\delta \cdot \int_{\{|\mu|\geq 1\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} \left(\int_{\{\|y\|\leq\pi|\mu|\}} \|y\|^\delta \nu(dy)\right) d\mu \\
&\qquad\geq \left(\frac{1}{\sqrt{2}\pi}\right)^\delta \cdot \int_{\{|\mu|\geq 1\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} \left(\int_{\{\|y\|\leq\pi\}} \|y\|^\delta \nu(dy)\right) d\mu \\
&\qquad= \left(\frac{1}{\sqrt{2}\pi}\right)^\delta \cdot \int_{\{|\mu|\geq 1\}} \frac{(1-\cos(\mu t))^{\frac{\delta}{2}}}{|\mu|^\delta} d\mu \cdot \int_{\{\|y\|\leq\pi\}} \|y\|^\delta \nu(dy).
\end{aligned}$$

The first integral on the right-hand side is strictly positive and finite since $\delta > 1$. This implies $\int_{\{\|y\|\leq\pi\}} \|y\|^\delta \nu(dy) < \infty$ and thus L has a.s. finite δ-variation which completes the proof of (ii).
(iii) To establish (iii) combine (ii) and (3.4).
(iv) We still have to prove that moments are preserved by the "transformation". Let $\alpha > 2$ and assume that $\mathbb{E}[\|L_1\|^\alpha] < \infty$. Then, analogously to (3.3) and (3.5),

$$\begin{aligned}
&\int_{\{\|x\|\geq 1\}} \|x\|^\alpha \nu_{Z_t}(dx) \\
&\quad= \left(\frac{1}{\sqrt{2}\pi}\right)^\alpha \cdot \int_{\mathbb{R}^d\setminus\{0\}} \|y\|^\alpha \left(\int_{\left\{\mu\in\mathbb{R}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|\geq\sqrt{2}\pi\right\}} \frac{(1-\cos(\mu t))^{\frac{\alpha}{2}}}{|\mu|^\alpha} d\mu\right) \nu(dy) \\
&\quad\leq C(\alpha)\cdot\left(\frac{1}{\sqrt{2}\pi}\right)^\alpha \cdot \left(\int_{\mathbb{R}^d} \left(1\wedge\|y\|^2\right)\nu(dy) + \int_{\{\|y\|>1\}} \|y\|^\alpha \nu(dy)\right).
\end{aligned}$$

The first integral on the right-hand side is finite since ν is a Lévy measure (cf. (2.1)) and the second integral is finite since we assume that L_1 has a finite α-th moment. Hence, $\int_{\{\|x\|\geq 1\}} \|x\|^\alpha \nu_{Z_t}(dx) < \infty$ which implies (cf. Theorem 2.20) that $\mathbb{E}[\|Z_t\|^\alpha] < \infty$.

Conversely, let $\mathbb{E}[\|L_1\|^\alpha] = \infty$, i.e. $\int_{\{\|y\|\geq 1\}} \|y\|^\alpha \nu(dy) = \infty$. Since $\lim_{\mu\to 0} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|} = \frac{t}{\sqrt{2}}$ (cf. Example 3.7), there is a positive constant $c > 0$ s.t. $\frac{\sqrt{1-\cos(\mu t)}}{|\mu|} \geq \frac{t}{2}$ for all $|\mu| \leq c$. By virtue of

$$\left\{y \in \mathbb{R}^d \setminus \{0\} : \ \|y\| \geq \frac{2\sqrt{2}}{t}\pi\right\} \subseteq \left\{y \in \mathbb{R}^d \setminus \{0\} : \ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|} \|y\| \geq \sqrt{2}\pi\right\}$$

for any $|\mu| \leq c$, we obtain

$$\begin{aligned}
&\int_{\{\|x\|\geq 1\}} \|x\|^\alpha \nu_{Z_t}(dx) \\
&\quad = \left(\frac{1}{\sqrt{2}\pi}\right)^\alpha \cdot \int_{\mathbb{R}} \frac{(1-\cos(\mu t))^{\frac{\alpha}{2}}}{|\mu|^\alpha} \left(\int_{\left\{y\in\mathbb{R}^d\setminus\{0\}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|\geq\sqrt{2}\pi\right\}} \|y\|^\alpha \nu(dy)\right) d\mu \\
&\quad \geq \left(\frac{1}{\sqrt{2}\pi}\right)^\alpha \cdot \int_{\{|\mu|\leq c\}} \frac{(1-\cos(\mu t))^{\frac{\alpha}{2}}}{|\mu|^\alpha} \left(\int_{\left\{\|y\|\geq\frac{2\sqrt{2}}{t}\pi\right\}} \|y\|^\alpha \nu(dy)\right) d\mu \\
&\quad \geq \left(\frac{t}{2\sqrt{2}\pi}\right)^\alpha \cdot 2c \cdot \int_{\left\{\|y\|\geq\frac{2\sqrt{2}}{t}\pi\right\}} \|y\|^\alpha \nu(dy) = \infty.
\end{aligned}$$

This implies $\int_{\{\|x\|\geq 1\}} \|x\|^\alpha \nu_{Z_t}(dx) = \infty$, i.e. $\mathbb{E}[\|Z_t\|^\alpha] = \infty$ and thus (iv) is shown.
(v) We know by virtue of Theorem 2.20 that $\mathbb{E}[\exp\{\alpha' \|Z_t\|\}] < \infty$ if and only if $\int_{\{\|x\|\geq 1\}} \exp\{\alpha' \|x\|\} \nu_{Z_t}(dx)$ is finite which is obviously the case if and only if the integral $\int_{\{\|x\|>1\}} \exp\{\alpha' \|x\|\} \nu_{Z_t}(dx)$ if finite. We have

$$\begin{aligned}
&\int_{\{\|x\|>1\}} \exp\{\alpha' \|x\|\} \nu_{Z_t}(dx) \\
&\quad = \int_{\mathbb{R}^d\setminus\{0\}} \int_{\left\{\mu\in\mathbb{R}:\ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|>\sqrt{2}\pi\right\}} \exp\left\{\alpha' \frac{\|y\|}{\sqrt{2}\pi} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\right\} d\mu\, \nu(dy). \qquad (3.6)
\end{aligned}$$

Setting $c(t) := \sup_{\mu\in\mathbb{R}} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}$, observe first that $c(t) \in (0,\infty)$ for any $t > 0$. Thus we obtain for any $y \in \mathbb{R}^d \setminus \{0\}$ with $\|y\| \leq \frac{\sqrt{2}\pi}{c(t)}$ that $\frac{\sqrt{1-\cos(\mu t)}}{|\mu|} \|y\| \leq \sqrt{2}\pi$ for all $\mu \in \mathbb{R}$. This gives that the inner integral in (3.6) vanishes for all $y \in \mathbb{R}^d \setminus \{0\}$ with $\|y\| \leq \frac{\sqrt{2}\pi}{c(t)}$ and hence, due to

$$\left\{\mu \in \mathbb{R} : \ \frac{\sqrt{1-\cos(\mu t)}}{|\mu|} \|y\| > \sqrt{2}\pi\right\} \subseteq \left\{\mu \in \mathbb{R} : \ |\mu| < \frac{\|y\|}{\pi}\right\}$$

for any $y \in \mathbb{R}^d \setminus \{0\}$,

$$
\begin{aligned}
&\int_{\{\|x\|>1\}} \exp\{\alpha' \|x\|\} \, \nu_{Z_t}(dx) \\
&\qquad = \int_{\left\{\|y\|>\frac{\sqrt{2}\pi}{c(t)}\right\}} \int_{\left\{\frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\|y\|>\sqrt{2}\pi\right\}} \exp\left\{\alpha' \frac{\|y\|}{\sqrt{2}\pi} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\right\} d\mu \, \nu(dy) \\
&\qquad \leq \int_{\left\{\|y\|>\frac{\sqrt{2}\pi}{c(t)}\right\}} \int_{\left\{|\mu|<\frac{\|y\|}{\pi}\right\}} \exp\left\{\alpha' \frac{\|y\|}{\sqrt{2}\pi} \frac{\sqrt{1-\cos(\mu t)}}{|\mu|}\right\} d\mu \, \nu(dy) \\
&\qquad \leq \frac{2}{\pi} \cdot \int_{\left\{\|y\|>\frac{\sqrt{2}\pi}{c(t)}\right\}} \|y\| \cdot \exp\left\{\alpha' \frac{c(t) \cdot \|y\|}{\sqrt{2}\pi}\right\} \nu(dy).
\end{aligned}
\tag{3.7}
$$

Since

$$
\mathbb{E}\left[\|L_1\| \cdot \exp\left\{\frac{\alpha}{2}\|L_1\|\right\}\right] \leq \mathbb{E}\left[\|L_1\|^2\right]^{\frac{1}{2}} \cdot \mathbb{E}\left[\exp\{\alpha\|L_1\|\}\right]^{\frac{1}{2}} < \infty
$$

by assumption, Theorem 2.20 (iii) shows that the right-hand side of (3.7) is finite for $\alpha' = \alpha(t) := \frac{\pi}{\sqrt{2}c(t)}\alpha > 0$. This proves (v). $\square$

Example 3.9. We consider a real-valued (compensated) compound Poisson process, i.e. $L_t := \sum_{i=1}^{N_t} U_i - \kappa t \mathbb{E}[U_1]$, $t \in \mathbb{R}$, where $(N_t)_{t\in\mathbb{R}}$ is a Poisson process with rate κ, independent of $(U_i)_{i\geq 1}$ which is an i.i.d. sequence of real-valued random variables with distribution μ_U. Suppose moreover that $\mu_U \in L^2$, i.e. $\mathbb{E}[U_1^2] = \int_{\mathbb{R}} x^2 \, \mu_U(dx) < \infty$. Then $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1^2] = \kappa \mathbb{E}[U_1^2]$ and thus Theorem 3.8 applies. In particular, for any $\alpha > 2$, the process $Z_t = \Phi_L([0,t))$, $t \in \mathbb{R}_+$, has finite α-th moments if and only if $\mu_U \in L^\alpha$ (i.e. $\int_{\mathbb{R}} |x|^\alpha \, \mu_U(dx) < \infty$).

Example 3.10. If we take a Gamma process with corresponding Lévy measure $\nu_\Gamma(dx) = c \cdot \frac{e^{-\alpha x}}{x} \mathbb{1}_{(0,\infty)}(x) \, \lambda^1(dx)$ for some $\alpha, c > 0$, then the original process has already infinite activity but a.s. finite variation and finite moments of any order. Theorem 3.8 shows that the associated random orthogonal measure of the compensated Gamma process has also finite moments of any order and ν_{Z_t} integrates $|x|^\delta$ locally at zero for every $\delta > 1$, i.e. the Blumenthal-Getoor-Index of ν_{Z_t} equals 1.

Example 3.11. Finally, we consider a normal inverse Gaussian process which has itself a.s. infinite variation but finite moments of any order. Due to Theorem 3.8 the corresponding random orthogonal measure of the compensated NIG process has again finite moments of any order and ν_{Z_t} (like ν) does not integrate $|x|$ locally at zero. Moreover, the Blumenthal-Getoor-Index of ν_{Z_t} coincides with the index of ν. Since the Lévy density f of ν satisfies $f(x) \sim \pi^{-1}\delta x^{-2}$ as $x \to 0$ for some constant $\delta > 0$ (cf. [6, (3.18)]) one immediately obtains that these indices are equal to 1.

Remark 3.12. The fact that we assume in Theorem 3.8 the Lévy process to have no Brownian component is not really a constraint. It is a well-known result that

$$
W_t = \int_{-\infty}^{\infty} \frac{e^{i\mu t} - 1}{i\mu} \, \Phi(d\mu), \quad t \in \mathbb{R},
$$

is a d-dimensional standard Wiener process if Φ is a d-dimensional Gaussian random orthogonal measure satisfying $\mathbb{E}[\Phi(A)] = 0$ and $\mathbb{E}[\Phi(A)\Phi(A)^*] = \lambda^1(A)\frac{\mathrm{I}_d}{2\pi}$ for all $A \in$

$\mathscr{B}_0(\mathbb{R})$ (cf. for instance [2, Section 2.1, Lemma 5]). Conversely, every standard Wiener process $(W_t)_{t\in\mathbb{R}}$ yields such a Gaussian random orthogonal measure by setting

$$\Phi\left([a,b)\right) = \int_{-\infty}^{\infty} \frac{e^{-i\mu a} - e^{-i\mu b}}{2\pi i\mu} \, W(d\mu)$$

for $-\infty < a < b < \infty$ and extending that measure to $\mathscr{B}_0(\mathbb{R})$ (cf. Section 3.1 and Theorem 3.5). In other words Brownian components are "preserved" by the transition to their associated random orthogonal measures.
Since we can decompose every Lévy process in a Brownian component and another Lévy process without a Brownian component (Lévy-Itô decomposition) we deduce that the Brownian part is "preserved" whereas Theorem 3.8 applies for the other part.

3.4 Definition of multivariate CARMA and causal MCARMA processes

We finish this chapter with the definition of a d-dimensional CARMA (MCARMA) process using the spectral representation for square-integrable Lévy processes. In addition we recall an extension of that definition, called causal MCARMA processes, allowing for driving Lévy processes with finite logarithmic moment.

Definition 3.13 (cf. [41], Definition 3.18)**.**
*Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided square-integrable d-dimensional Lévy process with $\mathbb{E}[L_1] = 0$ and $\mathbb{E}[L_1 L_1^*] = \Sigma_L$. A d-dimensional Lévy-driven continuous time autoregressive moving average process $(Y_t)_{t\in\mathbb{R}}$ of order (p,q) with $p,q \in \mathbb{N}_0$, $p > q$ (**MCARMA**(p,q) **process**) is defined as*

$$\begin{aligned} Y_t :&= \int_{-\infty}^{\infty} e^{i\lambda t} P(i\lambda)^{-1} Q(i\lambda)\, \Phi(d\lambda), \quad t \in \mathbb{R}, \quad \textit{where} \\ P(z) :&= \mathrm{I}_d z^p + A_1 z^{p-1} + \ldots + A_p, \\ Q(z) :&= B_0 z^q + B_1 z^{q-1} + \ldots + B_q \end{aligned}$$

and Φ is the Lévy orthogonal random measure of Theorem 3.5 satisfying $\mathbb{E}[\Phi(d\lambda)] = 0$ and $\mathbb{E}[\Phi(d\lambda)\Phi(d\lambda)^*] = \frac{\Sigma_L}{2\pi}\, d\lambda$. Here $A_i \in M_d(\mathbb{R})$, $i = 1,\ldots,p$, and $B_j \in M_d(\mathbb{R})$, $j = 1,\ldots,q$, are real matrices satisfying $B_0 \neq 0$ and $\mathscr{N}(P) := \{z \in \mathbb{C} : \det(P(z)) = 0\} \subseteq \mathbb{R}\setminus\{0\} + i\mathbb{R}$.

Remark 3.14.

(i) An MCARMA process Y can be interpreted as a solution to the p-th order d-dimensional differential equation

$$P(D)Y_t = Q(D)DL_t, \quad t \in \mathbb{R},$$

where D denotes the differentiation operator w.r.t. t (cf. [41, Remark 3.19 (a)]).

(ii) The well-definedness is ensured by [41, Lemmata 3.10 and 3.11] which show that the integrand is in $L^2(M_d(\mathbb{C}))$ and thus integrable with respect to Φ (cf. Section 3.1).

(iii) Assuming $\mathbb{E}[L_1] = 0$ is actually no restriction. If $\mathbb{E}[L_1] = \mu_L \neq 0$, one simply observes that the compensated Lévy process $\widetilde{L}_t = L_t - \mu_L t$ has zero expectation and

$$P(D)^{-1}Q(D)DL_t = P(D)^{-1}Q(D)D\widetilde{L}_t + P(D)^{-1}Q(D)\mu_L.$$

The first term on the right-hand side is the MCARMA process driven by the compensated Lévy process $\widetilde{L}$ and the second an ordinary differential equation having the unique solution $-A_p^{-1}B_q\mu_L$, as simple calculations show. Thus, the definition can be immediately extended to $\mathbb{E}[L_1] \neq 0$ (cf. [41, Remark 3.19 (c)]).

Definition 3.15 (cf. [41], Definition 3.20)**.**
Let $L = (L_t)_{t\in\mathbb{R}}$ be a d-dimensional Lévy process with associated Lévy measure ν satisfying $\int_{\{\|x\|\geq 1\}} \log\|x\|\,\nu(dx) < \infty$, and $p, q \in \mathbb{N}_0$ with $p > q$. We suppose $A_1, A_2, \dots, A_p \in M_d(\mathbb{R})$, $B_0, B_1, \dots, B_q \in M_d(\mathbb{R})$, $B_0 \neq 0$, and define the polynomial P as in Definition 3.13. Let $A \in M_{dp}(\mathbb{R})$ be the matrix

$$A := \begin{pmatrix} 0 & \mathrm{I}_d & 0 & \dots & 0 \\ 0 & 0 & \mathrm{I}_d & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \mathrm{I}_d \\ -A_p & -A_{p-1} & \dots & \dots & -A_1 \end{pmatrix}$$

and define another matrix $\beta = (\beta_1', \dots, \beta_p')' \in M_{dp\times d}(\mathbb{R})$ by

$$\beta_{p-j} := -\sum_{i=1}^{p-j-1} A_i\beta_{p-j-i} + B_{q-j}, \quad j = 0, 1, \dots, p-1, \tag{3.8}$$

setting $B_i = 0$ for $i < 0$. Denote the spectrum of A by $\sigma(A)$ and assume that $\sigma(A) = \mathscr{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$. Let $(G_t)_{t\in\mathbb{R}}$ be the unique strictly stationary solution to the stochastic differential equation

$$dG_t = AG_tdt + \beta dL_t, \quad t \in \mathbb{R}. \tag{3.9}$$

Then the d-dimensional process $Y_t := \left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \dots, 0_{M_d(\mathbb{C})}\right) G_t$ is said to be a ***causal Lévy-driven MCARMA**(p,q) **process*** *with state space representation G.*

Remark 3.16.

(i) The unique strictly stationary solution to (3.9) is given by

$$G_t = \int_{-\infty}^t e^{(t-s)A}\beta\, L(ds), \quad t \in \mathbb{R}$$

(cf. [41, Theorem 3.12 and remark after Theorem 3.12]). From the same theorem it follows that Definition 3.15 coincides with Definition 3.13, when L is square-integrable.

(ii) The fact $\sigma(A) = \mathscr{N}(P)$ has been shown in [41, Lemma 3.8].

(iii) An MCARMA process Y can be represented as the moving average process

$$Y_t = \int_{-\infty}^{\infty} g(t-s)\, L(ds), \quad t \in \mathbb{R},$$

with the kernel matrix function

$$g(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu)\, d\mu.$$

An analogous result holds for causal MCARMA processes with the kernel function g replaced by

$$\widetilde{g}(t) = \left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{tA} \beta \mathbb{1}_{[0,\infty)}(t)$$

(cf. [41, Theorem 3.22 and Remark 3.23]).

Chapter 4

Multivariate symmetric α-stable CARMA processes

In this chapter we consider multivariate symmetric α-stable ($S\alpha S$) CARMA processes. We are going to derive both, a spectral and a moving average representation for them. Our results extend those of Chapter 3 and we shall prove that in the causal case they are in line with the extended definition of MCARMA processes proposed by Definition 3.15. We focus on the symmetric stable case with index of stability $\alpha \in (1, 2)$. The case $\alpha = 2$ corresponds to the well-known Gaussian case and thus it is already contained in the preceding chapter.

The relationship between harmonizable $S\alpha S$ processes (i.e. Fourier transforms of possibly dependently scattered $S\alpha S$ noises) and moving averages of stationarily and independently scattered $S\alpha S$ noises has been studied for a long time. It was first investigated by Cambanis and Soltani in [18] where it has been proven that, if $\alpha \in (1, 2)$, no $S\alpha S$ process having a moving average representation is strongly harmonizable (i.e. a Fourier transform of some independently scatterd $S\alpha S$ noise). Moreover, they stated that every $S\alpha S$ process having such a moving average representation was instead harmonizable. A few years later this claim was corrected and it has been shown that they are not even harmonizable (cf. [39] and also [17]). Again three years later Cambanis and Houdré [16] have shown that the crux of the dichotomy between Fourier transforms and moving averages is a "boundedness" property of the stable noise in the Fourier transformation. Furthermore, they derived the result that a large class of moving averages of $S\alpha S$ Lévy processes are Fourier transforms, in the summability sense, of dependently scattered $S\alpha S$ noises. We will basically follow the ideas of this paper to develop our results for $S\alpha S$ MCARMA and their driving Lévy processes. Since we are obliged to consider Fourier transforms in the summability sense, we shall make heavily use of the theory in Section 2.3.

At this point a word of caution is necessary. The Fourier transforms considered in [16, 18, 39] and here are "usual" Fourier transforms of random noises and rely on norm properties of the noise and the process. With a weaker notion of Fourier transformation via random Schwartz distributions, stable moving averages are Fourier transforms of random distributions (cf. [22, 23, 39]).

The outline of the chapter is as follows: in the first section we introduce real and complex $S\alpha S$ random variables and define the notion of covariation, a concept which has analogous properties to those of the covariance, although it is not as powerful because it lacks some of the nice properties of the covariance. In Section 2 we will explain $S\alpha S$ random measures and integration w.r.t. them before we describe in the third section a general procedure how one can extend such a random measure from the semi-ring of intervals $[a, b)$ with $-\infty < a < b < \infty$ to the ring of all Borel sets with finite control measure where the latter will be defined in combination with stable integrals. For more details concerning these concepts we refer the reader to standard textbooks on this topic like for instance Samorodnitsky and Taqqu [53] or Janicki and Weron [32]. A good exposition for the complex case is also [15]. Thereafter we construct a (dependently scattered) $S\alpha S$ noise allowing for a spectral representation (in the summability sense) of $S\alpha S$ Lévy processes and state some further properties of this noise in Section 4. Using these results we deduce a spectral and a moving average representation of $S\alpha S$ MCARMA processes in the last section which we finish with the proof of consistency to Definition 3.15 in the causal case.

4.1 Symmetric stable random variables

We start with the univariate real case.

Definition 4.1. *A real-valued random variable X is called **symmetric α-stable** ($S\alpha S$) with index of stability $\alpha \in (0, 2]$, if its characteristic function is of the form*

$$\Phi_X(z) = \mathbb{E}\left[\exp\{izX\}\right] = \exp\{-\sigma^\alpha |z|^\alpha\}, \quad z \in \mathbb{R},$$

for some $\sigma \geq 0$. We shall write $X \sim S\alpha S(\sigma)$.

We recall well-known properties of real $S\alpha S$ random variables (cf. [53, Chapter 1.2]):

Proposition 4.2.

(i) Let X_1 and X_2 be independent random variables with $X_i \sim S\alpha S(\sigma_i)$, $i = 1, 2$. Then $X_1 + X_2 \sim S\alpha S(\sigma)$ with $\sigma = (\sigma_1^\alpha + \sigma_2^\alpha)^{1/\alpha}$.

(ii) Let $X \sim S\alpha S(\sigma)$, $\alpha \in (0, 2)$, then

$$\begin{aligned} \mathbb{E}\left[|X|^p\right] &< \infty \quad \textit{for any} \quad p \in (0, \alpha), \\ \mathbb{E}\left[|X|^p\right] &= \infty \quad \textit{for any} \quad p \geq \alpha. \end{aligned}$$

Let now $X = (X_1, \ldots, X_d)'$ be an $\mathbb{R}^d$-valued random variable.

Definition 4.3. *X is said to be a **symmetric α-stable** random vector in $\mathbb{R}^d$ with index of stability $\alpha \in (0, 2]$ if all linear combinations $\sum_{i=1}^d a_i X_i$, $a_i \in \mathbb{R}$, $i = 1, \ldots, d$, are $S\alpha S$.*

We shall also say that $X_1, \ldots, X_d$ are **jointly** $S\alpha S$.

Remark 4.4. In contrast to what one might expect, a random vector $X = (X_1, \ldots, X_d)'$ is in general not $S\alpha S$ if all its components are $S\alpha S$, i.e. the symmetry of an α-stable random vector cannot be regarded as a componentwise property. Indeed, there are non-symmetric α-stable random vectors X whose components $X_1, X_2, \ldots, X_d$ are all $S\alpha S$ (cf. [53, Remark 2 in Section 2.4] for a counterexample and [53, Definition 2.1.1] for the general definition of α-stable random vectors).

Theorem 4.5 (cf. [53], Theorem 2.3.1 and 2.4.3).
X is a symmetric α-stable random vector in $\mathbb{R}^d$ with $\alpha \in (0,2)$ iff there is a symmetric finite measure Γ_X on the unit sphere $S_d := \left\{x \in \mathbb{R}^d : |x| = 1\right\}$ (i.e. $\Gamma_X(A) = \Gamma_X(-A)$ for any Borel set A of S_d) s.t. the joint characteristic function of X is of the form

$$\Phi_X(z) = \mathbb{E}\left[\exp\left\{i\langle z, X\rangle\right\}\right] = \exp\left\{-\int_{S_d} |\langle z, s\rangle|^\alpha \, \Gamma_X(ds)\right\}, \quad z \in \mathbb{R}^d. \tag{4.1}$$

*Moreover, the measure Γ_X is unique. It is referred to as the **spectral measure** of the $S\alpha S$ random vector X.*

Note that in the Gaussian case $\alpha = 2$, expression (4.1) also holds but Γ_X is not unique any longer (cf. [53, Remark 3 in Section 2.4]).

Let us now introduce the covariation of jointly $S\alpha S$ random variables.

Definition 4.6 (cf. [53], Definition 2.7.1).
*If X and Y are real jointly $S\alpha S$ random variables with $\alpha \in (1,2]$ and $\Gamma_{X,Y}$ the corresponding spectral measure of (X,Y), then we define the **covariation of** X **on** Y by*

$$[X,Y]_\alpha := \int_{S_2} x \cdot y^{\langle \alpha-1\rangle} \, \Gamma_{X,Y}(d(x,y))$$

where, for all $y \in \mathbb{R}$ and $\beta > 0$,

$$y^{\langle\beta\rangle} := |y|^\beta \cdot \operatorname{sign}(y) = \begin{cases} y^\beta & \text{if } y \geq 0, \\ -|y|^\beta & \text{if } y < 0. \end{cases}$$

Note that Definition 4.6 reduces to the well-known covariance in the case $\alpha = 2$, where the exact relationship is $[X,Y]_2 = 1/2 \cdot \operatorname{Cov}(X,Y)$ (cf. [53, Example 2.7.2]). The covariation has analogous properties to those of the covariance, however, it lacks some of the nice properties of the covariance like for example symmetry or linearity in the second argument. We summarize the properties in the following proposition:

Proposition 4.7. *Let $X_1, \ldots, X_d, Y$ be real jointly $S\alpha S$ random variables with $\alpha \in (1,2]$. Denote by Γ_X the corresponding spectral measure of $X = (X_1, \ldots, X_d)'$. Then:*

(i) $[X_1, X_2]_\alpha = \int_{S_d} s_1 s_2^{\langle\alpha-1\rangle} \, \Gamma_X(ds)$ *and* $[X_1, X_1]_\alpha = \int_{S_d} |s_1|^\alpha \, \Gamma_X(ds)$.

(ii) $[X_1 + X_2, Y]_\alpha = [X_1, Y]_\alpha + [X_2, Y]_\alpha$ *(Additivity in the first argument).*

(iii) $[aX_1, bX_2]_\alpha = ab^{\langle\alpha-1\rangle} [X_1, X_2]_\alpha$ *for all $a, b \in \mathbb{R}$ (Scaling).*

(iv) The covariation is in general neither symmetric in its arguments nor linear in its second argument.

(v) If X_1 and X_2 are independent, then $[X_1, X_2]_\alpha = 0$.

(vi) If X_1 and X_2 are independent, then $[Y, X_1 + X_2]_\alpha = [Y, X_1]_\alpha + [Y, X_2]_\alpha$.

(vii) $\|Y\|_\alpha = [Y,Y]_\alpha^{1/\alpha}$ *defines a norm on any linear space of jointly $S\alpha S$ random variables. Convergence in $\|\cdot\|_\alpha$ is equivalent to convergence in probability and convergence in $L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{R})$ for any $p < \alpha$. If $X_1, \ldots, X_d$ are independent, then* $\left\|\sum_{k=1}^d X_k\right\|_\alpha^\alpha = \sum_{k=1}^d \|X_k\|_\alpha^\alpha$.

(viii) $|[X_1, X_2]_\alpha| \leq \|X_1\|_\alpha \cdot \|X_2\|_\alpha^{\alpha-1}$.

Proof. See [53, Lemma 2.7.5, Corollary 2.7.9 and 2.7.10, Property 2.7.11 and 2.7.15, Proposition 2.8.3 and Property 2.8.4]. □

Now we pass on to complex $S\alpha S$ random variables which we define in the following way:

Definition 4.8. *A complex random variable $X = X_1 + iX_2$ is **symmetric α-stable**, or $S\alpha S$, if X_1 and X_2 are jointly $S\alpha S$.*

*Several complex random variables $X_1, \dots, X_d$ are **jointly** $S\alpha S$ (or equivalently the $\mathbb{C}^d$-valued random vector $X = (X_1^*, \dots, X_d^*)^*$ is said to be $S\alpha S$) if their real and imaginary parts* $\mathrm{Re}(X_1)$, $\mathrm{Im}(X_1), \dots,$ $\mathrm{Re}(X_d)$, $\mathrm{Im}(X_d)$ *are jointly $S\alpha S$.*

It is easy to show that $X = (X_1^*, \dots, X_d^*)^*$ is a $S\alpha S$ random vector in $\mathbb{C}^d$ if and only if all linear combinations $\sum_{i=1}^d a_i X_i$, $a_i \in \mathbb{C}$, $i = 1, \dots, d$, are $S\alpha S$.
The joint characteristic function of a complex $S\alpha S$ random variable $X = X_1 + iX_2$ can be written, with $z = z_1 + iz_2$, as

$$\begin{aligned}\mathbb{E}\left[\exp\left\{i\mathrm{Re}(zX^*)\right\}\right] &= \mathbb{E}\left[\exp\left\{i(z_1X_1 + z_2X_2)\right\}\right]\\ &= \exp\left\{-\int_{S_2} |z_1s_1 + z_2s_2|^\alpha \, \Gamma_{X_1,X_2}(d(s_1,s_2))\right\}\end{aligned}$$

where Γ_{X_1,X_2} is the spectral measure of (X_1, X_2).

We modify the notion of covariation and obtain the following extension of the real case (cf. [15]).

Definition 4.9. *If $X = X_1+iX_2$ and $Y = Y_1+iY_2$ are complex jointly $S\alpha S$ with $\alpha \in (1, 2]$ and Γ_{X_1,X_2,Y_1,Y_2} the corresponding spectral measure of (X_1, X_2, Y_1, Y_2), then we define the **covariation of** X **on** Y by*

$$[X, Y]_\alpha := \int_{S_4} (x_1 + ix_2)\,(y_1 + iy_2)^{\langle \alpha-1\rangle}\, \Gamma_{X_1,X_2,Y_1,Y_2}(d(x_1, x_2, y_1, y_2))$$

where, for all $z \in \mathbb{C}$ and $\beta > 0$, we use the convention

$$z^{\langle\beta\rangle} := \begin{cases} |z|^{\beta-1} \cdot z^* & \text{if } z \neq 0,\\ 0 & \text{if } z = 0.\end{cases}$$

One can generalize Proposition 4.7 to the complex case with the following modifications: (i) becomes

$$[X_1, X_2]_\alpha = \int_{S_{2d}} (s_1 + is_2)\,(s_3 + is_4)^{\langle\alpha-1\rangle}\, \Gamma_X(ds) \quad \text{and}$$

$$[X_1, X_1]_\alpha = \int_{S_{2d}} |s_1 + is_2|^\alpha \, \Gamma_X(ds)$$

with Γ_X being the spectral measure of $(\mathrm{Re}(X_1), \mathrm{Im}(X_1), \dots, \mathrm{Re}(X_d), \mathrm{Im}(X_d))$ and the terms symmetric and linear in (iv) have to be replaced by Hermitian and sesquilinear, respectively. For the proofs we refer to [15, Theorem 2.1 and Lemma 2.2].
There it also has been shown that Definition 4.9 is indeed an extension of the real case, i.e.

one has $[X,Y]_\alpha = [X_1,Y_1]_\alpha$ if $X_2 = Y_2 = 0$. However, there is one remarkable difference between the real and complex case, namely, $\|X\|_\alpha$ determines the distribution of X via $\Phi_X(z) = \exp\{-\|X\|_\alpha^\alpha \cdot |z|^\alpha\}$ in the real case whereas this is no longer valid in the complex case where for $X = X_1 + iX_2$, only $\|X\|_\alpha^\alpha = \Gamma_{X_1,X_2}(S_2)$ holds, i.e. the covariation norm only determines the total mass of the spectral measure.

Concerning moments of complex $S\alpha S$ random variables, we recall the following:

Corollary 4.10. *Let X be a complex $S\alpha S$ random variable with $\alpha \in (0,2)$, then*

$$\mathbb{E}[|X|^p] < \infty \Leftrightarrow 0 < p < \alpha.$$

If $X_1,\ldots,X_d$ are complex jointly $S\alpha S$ random variables with $\alpha \in (0,2)$ and d-fold dependent, i.e.

$$\Gamma\left(\left\{u \in S_{2d} : (u_1^2+u_2^2)\cdots(u_{2d-1}^2+u_{2d}^2) \neq 0\right\}\right) > 0$$

where Γ is the spectral measure of $(\mathrm{Re}(X_1),\mathrm{Im}(X_1),\ldots,\mathrm{Re}(X_d),\mathrm{Im}(X_d))$, *then*

$$\mathbb{E}[|X_1|^{p_1}\cdots|X_d|^{p_d}] < \infty \Leftrightarrow 0 < p_1+\ldots+p_d < \alpha.$$

Proof. See for instance [15, Section 4]. □

4.2 Symmetric stable processes and integrals

Throughout this section we shall assume that $\alpha \in (1,2)$. A real or complex stochastic process is said to be $S\alpha S$ if all finite dimensional margins are $S\alpha S$.

In the following we introduce complex-valued $S\alpha S$ random measures and complex stable stochastic integrals. Let $L^0(\Omega,\mathscr{F},\mathbb{P};\mathbb{C})$ denote the set of all complex random variables defined on $(\Omega,\mathscr{F},\mathbb{P})$ and let k be a measure on the product space $(\mathbb{R}\times S_2, \mathscr{B}(\mathbb{R})\times\mathscr{B}(S_2))$, where S_2 denotes again the unit circle in $\mathbb{R}^2$, such that

$$\text{for every } A \in \mathscr{B}(\mathbb{R}) \text{ with } k(A\times S_2) < \infty$$

$$k(A\times\cdot\,) \text{ is a (finite) symmetric measure on } (S_2,\mathscr{B}(S_2)). \tag{4.2}$$

We define $\mathscr{B}_0(\mathbb{R}) := \{A \in \mathscr{B}(\mathbb{R}) : k(A\times S_2) < \infty\}$.

Definition 4.11. *A complex-valued $S\alpha S$* ***random measure*** *on $(\mathbb{R},\mathscr{B}(\mathbb{R}))$ with* ***circular control measure*** *k is a set function*

$$M : \mathscr{B}_0(\mathbb{R}) \to L^0(\Omega,\mathscr{F},\mathbb{P};\mathbb{C})$$

satisfying

(i) $M(A)$ is complex $S\alpha S$ with spectral measure $k(A\times\cdot\,)$ for all $A \in \mathscr{B}_0(\mathbb{R})$,

(ii) $M(A_1)$, $M(A_2),\ldots,$ $M(A_n)$ are independent whenever A_1, $A_2,\ldots,$ $A_n \in \mathscr{B}_0(\mathbb{R})$ are pairwise disjoint,

(iii) $M\Big(\bigcup_{j=1}^\infty A_j\Big) = \sum_{j=1}^\infty M(A_j)$ a.s. for all A_1, $A_2,\ldots \in \mathscr{B}_0(\mathbb{R})$ that are mutually disjoint with $\bigcup_{j=1}^\infty A_j \in \mathscr{B}_0(\mathbb{R})$.

A $S\alpha S$ random measure is an **independently scattered** (property (ii)) and **σ-additive** (property (iii)) set function.

Definition 4.12. *The measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ defined by $m(A) := k(A \times S_2)$ is called **control measure** of the complex-valued $S\alpha S$ random measure M.*

In order to define integration w.r.t. complex $S\alpha S$ random measures, the first step is to define such an integral for simple functions $f = \sum_{i=1}^n f_i \mathbb{1}_{A_i}$ where $f_i \in \mathbb{C}$, $i = 1, \dots, n$, and $A_i \in \mathscr{B}_0(\mathbb{R})$, $i = 1, \dots, n$, are pairwise disjoint sets. As usual, for such an f we set

$$\int_{\mathbb{R}} f\, dM := \sum_{i=1}^n f_i\, M(A_i). \tag{4.3}$$

It is easy to check that for simple functions the integral is linear and one can show that its real and imaginary part are jointly $S\alpha S$. Thereafter we can extend integration to the space

$$L^\alpha(\mathbb{C}; dm) := \left\{ f : \mathbb{R} \to \mathbb{C} \text{ measurable}, \int_{\mathbb{R}} |f(x)|^\alpha\, m(dx) < \infty \right\}$$

by choosing a sequence of simple functions $(f_n)_{n\in\mathbb{N}}$ such that

$$\int_{\mathbb{R}} |f_n(x) - f(x)|^\alpha\, m(dx) \overset{n\to\infty}{\longrightarrow} 0 \quad (\text{i.e. } f_n \to f \text{ as } n \to \infty \text{ in } L^\alpha(\mathbb{C}; dm))$$

and then proving that $(\int_{\mathbb{R}} f_n\, dM)_{n\in\mathbb{N}}$ converges in probability. We define

$$\int_{\mathbb{R}} f\, dM := \mathbb{P} - \lim_{n\to\infty} \int_{\mathbb{R}} f_n\, dM \tag{4.4}$$

where $\mathbb{P}-\lim$ means the limit in probability. For more details about that construction, see [53, Section 3.4 and Section 6.2]. Note in particular, that the integral does not depend on the concrete choice of the approximating sequence $(f_n)_{n\in\mathbb{N}}$. We summarize its properties in the following proposition:

Proposition 4.13.

(i) For any $f \in L^\alpha(\mathbb{C}; dm)$, the integral $\int_{\mathbb{R}} f\, dM$ is a complex $S\alpha S$ random variable with joint characteristic function

$$\begin{aligned} &\mathbb{E}\left[\exp\left\{ i \left(z_1 \operatorname{Re} \int_{\mathbb{R}} f\, dM + z_2 \operatorname{Im} \int_{\mathbb{R}} f\, dM \right) \right\} \right] \\ &\qquad = \exp\Bigg\{ - \int_{\mathbb{R}} \int_{S_2} |z_1(s_1 \operatorname{Re} f(x) - s_2 \operatorname{Im} f(x)) \\ &\qquad\qquad + z_2(s_1 \operatorname{Im} f(x) + s_2 \operatorname{Re} f(x))|^\alpha\, k(dx, ds) \Bigg\}, \quad z_1, z_2 \in \mathbb{R}. \end{aligned}$$

(ii) For all $f, g \in L^\alpha(\mathbb{C}; dm)$, $a, b \in \mathbb{C}$,

$$\int_{\mathbb{R}} (af + bg)\, dM = a \cdot \int_{\mathbb{R}} f\, dM + b \cdot \int_{\mathbb{R}} g\, dM.$$

(iii) Let $f : \mathbb{R} \to \mathbb{C}$ be measurable s.t. there is a sequence $(f_n)_{n\in\mathbb{N}}$ of measurable simple functions satisfying

$$f_n(x) \overset{n\to\infty}{\to} f(x) \quad \text{for } m\text{-almost every } x \in \mathbb{R}$$

and s.t. the sequence of integrals $(\int_{\mathbb{R}} f_n\, dM)_{n\in\mathbb{N}}$ defined by (4.3) converges in distribution, then $f \in L^\alpha(\mathbb{C}; dm)$.

(iv) Let $X_j = \int_{\mathbb{R}} f_j\, dM$, $j = 1, 2, \ldots$, and $X = \int_{\mathbb{R}} f\, dM$ with f, f_j, $j = 1, 2, \ldots$, being in $L^\alpha(\mathbb{C}; dm)$. Then

$$\mathbb{P} - \lim_{j\to\infty} X_j = X \Leftrightarrow \lim_{j\to\infty} \int_{\mathbb{R}} |f_j(x) - f(x)|^\alpha\, m(dx) = 0. \tag{4.5}$$

Proof. See for instance [53, Proposition 6.2.1, 6.2.2 and 6.2.3]. □

Remark 4.14.

(i) It is easy to deduce that the convergence in probability of the sequence $(\int_{\mathbb{R}} f_n\, dM)_{n\in\mathbb{N}}$ in (4.4) and $(\int_{\mathbb{R}} f_j\, dM)_{j\in\mathbb{N}}$ in (4.5), respectively, is equivalent to convergence in $L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C})$ for any $p < \alpha$ (cf. also Proposition 4.7 (vii)).

(ii) The assumption that there is a sequence $(f_n)_{n\in\mathbb{N}}$ s.t. $f_n \to f$ m-a.e. in Proposition 4.13 (iii) can be replaced by the assumption that there is a sequence $(f_n)_{n\in\mathbb{N}}$ s.t. $f_n \to f$ in $L^\alpha(\mathbb{C}; dm)$. Note that the latter is a stronger assumption, since convergence in $L^\alpha(\mathbb{C}; dm)$ implies convergence m-a.e. for some subsequence $(f_{n_j})_{j\in\mathbb{N}}$.

We conclude this section by extending the notion of stable integrals to the multivariate case for which we will use a componentwise perception. Let $L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ denote the collection of all $\mathbb{C}^d$-valued random variables defined on $(\Omega, \mathscr{F}, \mathbb{P})$ and let k be a measure on the product space $(\mathbb{R} \times S_{2d}, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(S_{2d}))$, where S_{2d} stands for the unit sphere in $\mathbb{R}^{2d}$, such that, analogously to (4.2),

$$\text{for every } A \in \mathscr{B}(\mathbb{R}) \text{ with } k(A \times S_{2d}) < \infty$$

$$k(A \times \cdot) \text{ is a (finite) symmetric measure on } (S_{2d}, \mathscr{B}(S_{2d})). \tag{4.6}$$

Again the control measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is defined by $m(A) := k(A \times S_{2d})$ and $\mathscr{B}_0(\mathbb{R}) := \{A \in \mathscr{B}(\mathbb{R}) : m(A) < \infty\}$.

Definition 4.15.
*A complex-valued d-**dimensional** $S\alpha S$ **random measure** on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with circular control measure k is an independently scattered, σ-additive set function*

$$M : \mathscr{B}_0(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$$

s.t. $M(A) = (M_1(A), \ldots, M_d(A))$ is a complex $S\alpha S$ random vector with spectral measure $k(A\times \cdot)$ for all $A \in \mathscr{B}_0(\mathbb{R})$, i.e. $k(A\times \cdot)$ is the spectral measure of the real random vector $(\operatorname{Re} M_1(A), \operatorname{Im} M_1(A), \ldots, \operatorname{Re} M_d(A), \operatorname{Im} M_d(A))$.

Now we take the same approach as in the one-dimensional case to define multivariate symmetric stable integrals. First we set

$$\int_{\mathbb{R}} f\,dM := \sum_{i=1}^{n} f_i\,M(A_i) \tag{4.7}$$

for any simple function $f = \sum_{i=1}^{n} f_i \mathbb{1}_{A_i}$, where $f_i \in M_d(\mathbb{C})$, $i = 1,\dots,n$, and $A_i \in \mathscr{B}_0(\mathbb{R})$, $i = 1,\dots,n$, are pairwise disjoint. Obviously, the integral is linear for simple functions and it is a complex d-dimensional $S\alpha S$ random vector due to the following two facts:

Corollary 4.16.

(i) If $X = (X_1^*,\dots,X_d^*)^*$ is a $\mathbb{C}^d$-valued $S\alpha S$ random variable and $A \in M_d(\mathbb{C})$, then AX is also jointly $S\alpha S$.

(ii) If $X = (X_1^*,\dots,X_d^*)^*$ and $Y = (Y_1^*,\dots,Y_d^*)^*$ are $S\alpha S$ random vectors and independent, then $X+Y$ is jointly $S\alpha S$.

Proof.
(i) Due to the remark after Definition 4.8, AX is jointly $S\alpha S$ if and only if all linear combinations $\sum_{i=1}^{d} b_i(AX)_i$, $b_i \in \mathbb{C}$, $i = 1,\dots,d$, are $S\alpha S$ random variables. Now, letting $A = (a_{ij})$,

$$\sum_{i=1}^{d} b_i(AX)_i = \sum_{i=1}^{d} b_i\left(\sum_{j=1}^{d} a_{ij}X_j\right) = \sum_{j=1}^{d}\left(\sum_{i=1}^{d} b_i a_{ij}\right)X_j$$

is a linear combination of the random variables X_j and thus $S\alpha S$.
(ii) Due to Definition 4.8, we assume w.l.o.g. that X and Y are real random vectors. Due to the independence assumption, we obtain for the characteristic function of $X+Y$

$$\begin{aligned}-\log\Phi_{X+Y}(z) &= -\log\mathbb{E}[\exp\{i\langle z,X\rangle\}] - \log\mathbb{E}[\exp\{i\langle z,Y\rangle\}]\\ &\overset{(4.1)}{=} \int_{S_d} |\langle z,s\rangle|^{\alpha}\,\Gamma_X(ds) + \int_{S_d} |\langle z,s\rangle|^{\alpha}\,\Gamma_Y(ds)\\ &= \int_{S_d} |\langle z,s\rangle|^{\alpha}\,(\Gamma_X+\Gamma_Y)(ds)\end{aligned}$$

with Γ_X and Γ_Y being the spectral measures of X and Y, respectively. Since $\Gamma := \Gamma_X + \Gamma_Y$ is a symmetric finite measure on S_d, we apply once more Theorem 4.5 to conclude. □

Note that the integral in (4.7) can be understood componentwise: denoting the coordinates of M by $M = (M_1^*,\dots,M_d^*)^*$, the i-th element of $\int_{\mathbb{R}} f\,dM$ is given by $\sum_{j=1}^{d}\int_{\mathbb{R}} f_{ij}\,dM_j$ where the integrals are one-dimensional stable integrals introduced previously and f_{ij} denotes the component function of f in the i-th row and j-th column. Therefore we can extend integration to

$$L^{\alpha}(M_d(\mathbb{C});dm) := \left\{f:\mathbb{R}\to M_d(\mathbb{C}) \text{ measurable}, \int_{\mathbb{R}} \|f(x)\|^{\alpha}\,m(dx) < \infty\right\}$$

where $\|\cdot\|$ denotes an arbitrary norm on $M_d(\mathbb{C})$. The space $L^{\alpha}(M_d(\mathbb{C});dm)$ is equal to the space of functions $f = (f_{ij}) : \mathbb{R} \to M_d(\mathbb{C})$ with all components f_{ij} being in $L^{\alpha}(\mathbb{C};dm)$ (cf.

also Section 2.3 and 3.1). Hence, for any $f = (f_{ij}) \in L^\alpha(M_d(\mathbb{C}); dm)$, it is clearly possible to choose a sequence of simple functions $(f^{(n)})_{n\in\mathbb{N}}$ s.t.

$$\int_{\mathbb{R}} \left| f_{ij}^{(n)}(x) - f_{ij}(x) \right|^\alpha m(dx) \overset{n\to\infty}{\to} 0 \quad \text{for all} \quad i,j = 1,\ldots,d. \tag{4.8}$$

Now note that every coordinate M_j, $j = 1,\ldots,d$, of the $S\alpha S$ random measure M is itself a $S\alpha S$ random measure according to Definition 4.11 with control measure m_j satisfying

$$\begin{aligned} m_j(A) &= k_j(A \times S_2) = \|M_j(A)\|_\alpha^\alpha = [M_j(A), M_j(A)]_\alpha \\ &= \int_{S_{2d}} |s_{2j-1} + is_{2j}|^\alpha \, k(A \times \cdot)(ds) \le k(A \times S_{2d}) \\ &= m(A), \quad A \in \mathscr{B}(\mathbb{R}), \end{aligned}$$

where we used the complex version of Proposition 4.7 (i) and (vii). Hence

$$f_{ij} \in L^\alpha(\mathbb{C}; dm_k) \quad \text{for all} \quad i,j,k = 1,\ldots,d$$

and (4.8) holds as well with m replaced by m_k. Thus, for all $i,j = 1,\ldots,d$, the integrals $\int_{\mathbb{R}} f_{ij}^{(n)} dM_j$ converge in probability to $\int_{\mathbb{R}} f_{ij}\, dM_j$, respectively. This implies that $\sum_{j=1}^d \int_{\mathbb{R}} f_{ij}^{(n)} dM_j$ converges in probability to $\sum_{j=1}^d \int_{\mathbb{R}} f_{ij}\, dM_j$ for all $i = 1,\ldots,d$. Consequently, the integrals $\int_{\mathbb{R}} f^{(n)} dM = \big(\sum_{j=1}^d \int_{\mathbb{R}} f_{ij}^{(n)} dM_j\big)_{i=1,\ldots,d}$ converge in probability and we define

$$\int_{\mathbb{R}} f\, dM := \mathbb{P} - \lim_{n\to\infty} \int_{\mathbb{R}} f^{(n)} dM = \left(\sum_{j=1}^d \int_{\mathbb{R}} f_{ij}\, dM_j \right)_{i=1,\ldots,d}. \tag{4.9}$$

As in the univariate case this definition does not depend on the approximating sequence. Since we used a componentwise perception and since convergence in probability implies convergence in distribution, we get the following proposition in analogy to Proposition 4.13:

Proposition 4.17.

(i) *For any $f \in L^\alpha(M_d(\mathbb{C}); dm)$ the integral $\int_{\mathbb{R}} f\, dM$ is a $\mathbb{C}^d$-valued $S\alpha S$ random variable.*

(ii) *For all $f, g \in L^\alpha(M_d(\mathbb{C}); dm)$, $A, B \in M_d(\mathbb{C})$,*

$$\int_{\mathbb{R}} (Af + Bg)\, dM = A \cdot \int_{\mathbb{R}} f\, dM + B \cdot \int_{\mathbb{R}} g\, dM.$$

(iii) *Let $f : \mathbb{R} \to M_d(\mathbb{C})$ be measurable s.t. there is a sequence $(f^{(n)})_{n\in\mathbb{N}}$ of measurable simple functions satisfying*

$$\|f_n(x) - f(x)\| \overset{n\to\infty}{\to} 0 \quad \text{for } m\text{-almost every } x \in \mathbb{R}$$

and s.t. the sequences of integrals $\int_{\mathbb{R}} f_{ij}^{(n)} dM_k$ converge in distribution for all $i,j,k = 1,\ldots,d$, then $f \in L^\alpha(M_d(\mathbb{C}); dm)$.

(iv) *Let* $X_j = \int_{\mathbb{R}} f_j \, dM$, $j = 1, 2, \ldots$, *and* $X = \int_{\mathbb{R}} f \, dM$ *with* f, f_j, $j = 1, 2, \ldots$, *being in* $L^\alpha(M_d(\mathbb{C}); dm)$. *Then*

$$\lim_{j\to\infty} \int_{\mathbb{R}} \|f_j(x) - f(x)\|^\alpha \, m(dx) = 0 \Rightarrow \mathbb{P} - \lim_{j\to\infty} X_j = X. \tag{4.10}$$

Proof. Use the univariate case (Proposition 4.13) and the fact that the set of complex d-dimensional $S\alpha S$ random variables is closed under convergence in distribution in order to show (i) and (iv). The statement (ii) follows immediately by the definition of the integral and its linearity for simple functions. As to (iii), Proposition 4.13 gives that for all $i, j, k = 1, \ldots, d$ the component function f_{ij} is in $L^\alpha(\mathbb{C}; dm_k)$. Since

$$\begin{aligned} \sum_{k=1}^d m_k(A) &= \int_{S_{2d}} \underbrace{\sum_{k=1}^d |s_{2k-1} + i s_{2k}|^\alpha}_{\geq c \text{ for some } c>0 \text{ and all } s \in S_{2d}} \quad k(A \times \cdot)(ds) \\ &\geq c \cdot m(A), \quad A \in \mathscr{B}(\mathbb{R}), \end{aligned}$$

this implies $f_{ij} \in L^\alpha(\mathbb{C}; dm)$ for any $i, j = 1, \ldots, d$ and thus $f \in L^\alpha(M_d(\mathbb{C}); dm)$. □

Remark 4.18. As in the univariate case the convergence in probability in (4.9) and (4.10), resp., can be replaced by convergence in $L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ for any $p < \alpha$.

4.3 Extension of $S\alpha S$ random measures from semi-rings to $\mathscr{B}_0(\mathbb{R})$

In general, it is convenient to define a $S\alpha S$ random measure on the semi-ring of intervals $[a, b)$ with $-\infty < a < b < \infty$, denoted henceforth by $\mathscr{S}\mathscr{R}(\mathbb{R})$, and then to extend it to $\mathscr{B}_0(\mathbb{R})$. As announced at the beginning of this chapter we are going to use this section for the description of a general procedure how one can construct such an extension.

We assume that k is a measure on $(\mathbb{R} \times S_{2d}, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(S_{2d}))$ satisfying (4.6). Suppose moreover that we have given a complex-valued d-dimensional $S\alpha S$ random measure on $\mathscr{S}\mathscr{R}(\mathbb{R})$, i.e. an independently scattered, σ-additive set function $M : \mathscr{S}\mathscr{R}(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ such that $M(A)$ is jointly $S\alpha S$ with spectral measure $k(A \times \cdot)$ for all $A \in \mathscr{S}\mathscr{R}(\mathbb{R})$. We set $m(A) := k(A \times S_{2d})$ and assume that $\mathscr{S}\mathscr{R}(\mathbb{R}) \subseteq \mathscr{B}_0(\mathbb{R}) = \{A \in \mathscr{B}(\mathbb{R}) : m(A) < \infty\}$.

Now the first step is to extend M to the ring $\mathscr{E}(\mathbb{R})$ generated by $\mathscr{S}\mathscr{R}(\mathbb{R})$. Note that $\mathscr{E}(\mathbb{R})$ contains all elementary subsets of $\mathbb{R}$, i.e. all finite unions of the half-open bounded intervals in $\mathscr{S}\mathscr{R}(\mathbb{R})$, where w.l.o.g. the union can always be taken over mutually disjoint intervals. Thus an appropriate extension is given by

$$\widetilde{M}(A) := \sum_{i=1}^n M([a_i, b_i)) \tag{4.11}$$

for $A = \bigcup_{i=1}^n [a_i, b_i) \in \mathscr{E}(\mathbb{R})$ with pairwise disjoint intervals $[a_i, b_i)$, $i = 1, \ldots, n$. It is easy to check that this is indeed a $S\alpha S$ random measure on $\mathscr{E}(\mathbb{R})$ and that $\widetilde{M}(A)$ is jointly $S\alpha S$ with spectral measure $k(A \times \cdot)$ for all $A \in \mathscr{E}(\mathbb{R})$ (cf. Corollary 4.16 (ii)). We shall denote the extension $\widetilde{M}$ also by M.

The second step is the extension to $\mathscr{B}_0(\mathbb{R})$. Therefore, note that the same procedure as in Section 4.2 is possible in order to introduce integration of functions $f \in L^\alpha(M_d(\mathbb{C}); dm)$ w.r.t. the $S\alpha S$ random measure M given by (4.11). In fact, since for all $A \in \mathscr{B}_0(\mathbb{R})$ and all $\varepsilon > 0$ there exists $A_\varepsilon \in \mathscr{E}(\mathbb{R})$ such that $m(A \triangle A_\varepsilon) < \varepsilon$, where $A \triangle A_\varepsilon := (A \backslash A_\varepsilon) \cup (A_\varepsilon \backslash A)$ is the symmetric difference of the sets A and A_ε, we can approximate every $\mathscr{B}_0(\mathbb{R})$-simple function by an $\mathscr{E}(\mathbb{R})$-simple function. That is why the sequence $(f^{(n)})_{n\in\mathbb{N}}$ in (4.8) can be assumed to be $\mathscr{E}(\mathbb{R})$-simple. Thus one can define

$$\int_{\mathbb{R}} f \, dM := \mathbb{P} - \lim_{n\to\infty} \int_{\mathbb{R}} f^{(n)} dM$$

analogously to Section 4.2, where $(f^{(n)})_{n\in\mathbb{N}}$ is now an arbitrary sequence of $\mathscr{E}(\mathbb{R})$-simple functions such that $\int_{\mathbb{R}} \left\| f^{(n)}(x) - f(x) \right\|^\alpha m(dx) \to 0$ as $n \to \infty$. The extension of the $S\alpha S$ random measure M to $\mathscr{B}_0(\mathbb{R})$ is then defined by

$$\widetilde{M}(A) := \int_{\mathbb{R}} \mathbb{1}_A \, dM, \quad A \in \mathscr{B}_0(\mathbb{R}), \tag{4.12}$$

where

$$\mathbb{1}_A(x) := \begin{cases} \mathrm{I}_d, & x \in A \\ 0_{M_d(\mathbb{C})}, & \text{otherwise.} \end{cases}$$

The definition is well-posed since $\mathbb{1}_A \in L^\alpha(M_d(\mathbb{C}); dm)$ for all $A \in \mathscr{B}_0(\mathbb{R})$. $\widetilde{M}$ is now a $S\alpha S$ random measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ according to Definition 4.15 and $\widetilde{M}(A)$ is jointly $S\alpha S$ with spectral measure $k(A \times \cdot)$ for all $A \in \mathscr{B}_0(\mathbb{R})$.

For, if $A \in \mathscr{B}_0(\mathbb{R})$, we choose a sequence $(A_n)_{n\in\mathbb{N}} \subseteq \mathscr{E}(\mathbb{R})$ such that $m(A_n \triangle A) < n^{-1}$ for all $n \in \mathbb{N}$. This implies that

$$\int_{\mathbb{R}} \left\| \mathbb{1}_{A_n}(x) - \mathbb{1}_A(x) \right\|^\alpha m(dx) \to 0 \qquad \text{as } n \to \infty$$

and thus $M(A_n) \xrightarrow{\mathbb{P}} \widetilde{M}(A)$ as $n \to \infty$. Since convergence in probability implies convergence in distribution, we obtain for the characteristic function of $\widetilde{M}(A)$ at the point $z \in \mathbb{R}^{2d}$, due to Lévy's Continuity Theorem,

$$\begin{aligned} \Phi_{\widetilde{M}(A)}(z) &= \mathbb{E}\left[\exp\left\{ i \sum_{j=1}^{d} \left(z_{2j-1} \operatorname{Re} \widetilde{M}_j(A) + z_{2j} \operatorname{Im} \widetilde{M}_j(A) \right) \right\}\right] \\ &= \lim_{n\to\infty} \Phi_{M(A_n)}(z) = \lim_{n\to\infty} \exp\left\{ - \int_{S_{2d}} \left| \langle z, s \rangle \right|^\alpha k(A_n \times ds) \right\}. \end{aligned}$$

Setting $\mu_n(\cdot) := k(A_n \times \cdot) - k(A \times \cdot)$, we obtain

$$\begin{aligned} &\left| \int_{S_{2d}} \left| \langle z, s \rangle \right|^\alpha k(A_n \times ds) - \int_{S_{2d}} \left| \langle z, s \rangle \right|^\alpha k(A \times ds) \right| \\ &\qquad = \left| \int_{S_{2d}} \left| \langle z, s \rangle \right|^\alpha \mu_n(ds) \right| \\ &\qquad \le \int_{S_{2d}} \left| \langle z, s \rangle \right|^\alpha \left| \mu_n \right|(ds) \le |z|^\alpha \cdot \left| \mu_n \right|(S_{2d}) \end{aligned}$$

where $|\mu_n|$ denotes the total variation of the signed measure μ_n and since

$$\begin{aligned}|\mu_n|(S_{2d}) &= \sup\left\{\sum_{i=1}^{\infty}\underbrace{|\mu_n(B_i)|}_{\le k((A_n\triangle A)\times B_i)}\;:\;(B_i)_{i\in\mathbb{N}}\text{ a partition of } S_{2d}\right\}\\ &\le k((A_n\triangle A)\times S_{2d}) = m(A_n\triangle A) < \frac{1}{n}\end{aligned}$$

we deduce that

$$\Phi_{\widetilde{M}(A)}(z) = \exp\left\{-\int_{S_{2d}}|\langle z,s\rangle|^{\alpha}\,k(A\times ds)\right\},\quad z\in\mathbb{R}^{2d}.$$

This shows that $\widetilde{M}(A)$ is jointly $S\alpha S$ with spectral measure $k(A\times\cdot)$ for all $A\in\mathscr{B}_0(\mathbb{R})$.

To prove that $\widetilde{M}$ is independently scattered, let $A_1,\ldots,A_k\in\mathscr{E}(\mathbb{R})$ be mutually disjoint. Then the joint characteristic function of $M(A_1),\ldots,M(A_k)$ is given, due to their independence, by

$$\begin{aligned}&\mathbb{E}\Bigg[\exp\Bigg\{i\sum_{j=1}^{k}\Big(z_j^{(1)}\operatorname{Re}M_1(A_j)+z_j^{(2)}\operatorname{Im}M_1(A_j)+\ldots\\ &\qquad\qquad+z_j^{(2d-1)}\operatorname{Re}M_d(A_j)+z_j^{(2d)}\operatorname{Im}M_d(A_j)\Big)\Bigg\}\Bigg]\\ &=\prod_{j=1}^{k}\mathbb{E}\Big[\exp\Big\{i\Big(z_j^{(1)}\operatorname{Re}M_1(A_j)+z_j^{(2)}\operatorname{Im}M_1(A_j)+\ldots\\ &\qquad\qquad+z_j^{(2d-1)}\operatorname{Re}M_d(A_j)+z_j^{(2d)}\operatorname{Im}M_d(A_j)\Big)\Big\}\Big]\\ &=\prod_{j=1}^{k}\exp\left\{-\int_{S_{2d}}\left|z_j^{(1)}s_1+z_j^{(2)}s_2+\ldots+z_j^{(2d)}s_{2d}\right|^{\alpha}k(A_j\times ds)\right\}\\ &=\prod_{j=1}^{k}\exp\left\{-\int_{\mathbb{R}}\int_{S_{2d}}\left|\Big(z_j^{(1)}s_1+z_j^{(2)}s_2+\ldots+z_j^{(2d)}s_{2d}\Big)\mathbb{1}_{A_j}(x)\right|^{\alpha}k(dx\times ds)\right\}\\ &=\exp\left\{-\int_{\mathbb{R}}\int_{S_{2d}}\left|\sum_{j=1}^{k}\Big(z_j^{(1)}s_1+z_j^{(2)}s_2+\ldots+z_j^{(2d)}s_{2d}\Big)\mathbb{1}_{A_j}(x)\right|^{\alpha}k(dx\times ds)\right\}\end{aligned}\tag{4.13}$$

which is actually true for arbitrary, not necessarily disjoint sets $A_1,\ldots,A_k\in\mathscr{E}(\mathbb{R})$ (cf. also [53, p. 273]). Again Lévy's Continuity Theorem and the argumentation from above (using this time the Dominated Convergence Theorem) yield (4.13) for arbitrary $A_1,\ldots,A_k\in\mathscr{B}_0(\mathbb{R})$ and the joint characteristic function of $\widetilde{M}(A_1),\ldots,\widetilde{M}(A_k)$. In particular, since the joint characteristic function can be factorized for pairwise disjoint $A_1,\ldots,A_k\in\mathscr{B}_0(\mathbb{R})$, we obtain that $\widetilde{M}$ is independently scattered.

Finally, we have to show the σ-additivity. Additivity of $\widetilde{M}$ is clear by definition. Thus we have to prove that, for all $A_1,A_2,\ldots\in\mathscr{B}_0(\mathbb{R})$ with $A:=\bigcup_{i=1}^{\infty}A_i\in\mathscr{B}_0(\mathbb{R})$,

$$\widetilde{M}(A)=\sum_{i=1}^{\infty}\widetilde{M}(A_i)\quad\text{a.s.},$$

$$\text{i.e.}\quad \widetilde{M}(A) = \lim_{n\to\infty} \sum_{i=1}^{n} \widetilde{M}(A_i) = \lim_{n\to\infty} \widetilde{M}\left(\bigcup_{i=1}^{n} A_i\right) \quad \text{a.s.}$$

$$\text{or}\quad \widetilde{M}(A) = \lim_{n\to\infty} \widetilde{M}\left(\bigcup_{i=1}^{n} A_i\right) \quad \text{in probability}$$

since the series $\sum_{i=1}^{n} \widetilde{M}(A_i)$ has independent summands. Due to Proposition 4.17 (iv) (which is also true for the $S\alpha S$ random measure M on $\mathscr{E}(\mathbb{R})$) and the fact that $\mathbb{1}_{\cup_{i=1}^{n} A_i} \to \mathbb{1}_A$ as $n \to \infty$ in $L^\alpha(M_d(\mathbb{C}); dm)$, this is indeed the case.

Remark 4.19. Our point of view of extending a $S\alpha S$ random measure from the semi-ring of half-open bounded intervals on the real line to $\mathscr{B}_0(\mathbb{R})$ seems to be, to the best of our knowledge, new. However, in [53, Chapter 3] several definitions of univariate (real) stable integrals are mentioned. The first definition there as an α-stable stochastic process parameterized by their integrands (which are also supposed to be in L^α) is used in order to show that an α-stable random measure exists at all. This verification of existence seems to be in some kind related to our approach, since they define, for any $A \in \mathscr{B}_0(\mathbb{R})$, the measure $M(A)$ as the stable integral of $\mathbb{1}_A$ (understood as the stochastic process at "time" $\mathbb{1}_A$). In the following they describe using this existence of an α-stable random measure the constructive definition of stable integrals which can be found in our Section 4.2 and verify that both definitions actually coincide. Nevertheless, only the univariate case is studied in that different ways and they do not consider stable integrals w.r.t. an α-stable random measure only defined on $\mathscr{E}(\mathbb{R})$. Note also that Section 4.3 and 3.1 are very similar.

4.4 Spectral representation of $S\alpha S$ Lévy processes

In this section we will establish a spectral representation for symmetric α-stable Lévy processes, where again $1 < \alpha < 2$ is assumed throughout the whole section. Let $\mathscr{E}(\mathbb{R})$ denote the collection of all elementary subsets of $\mathbb{R}$ (cf. Section 4.3). As already explained in the introduction to this chapter (cf. also upcoming Remark 4.28) it is only possible to derive spectral representations in the summability sense.

It will turn out that we have to integrate w.r.t. what we shall call $S\alpha S$ random contents, i.e. possibly dependently scattered, additive $S\alpha S$ noises defined on $\mathscr{E}(\mathbb{R})$.

Definition 4.20. *For $\alpha \in (1, 2)$ a d-dimensional $S\alpha S$* ***random content*** *is a set function*

$$M : \mathscr{E}(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$$

satisfying

(i) $M(A)$ is a d-dimensional $S\alpha S$ random vector for all $A \in \mathscr{E}(\mathbb{R})$,

(ii) $M(\bigcup_{i=1}^{n} A_i) = \sum_{i=1}^{n} M(A_i)$ a.s. whenever $A_1, \ldots, A_n \in \mathscr{E}(\mathbb{R})$ are pairwise disjoint (i.e. M is additive).

Analogously to Section 4.2 and 4.3, integration of simple functions $f = \sum_{i=1}^{n} f_i \mathbb{1}_{A_i}$ (with $f_i \in M_d(\mathbb{C})$, $i = 1, \ldots, n$, $n \in \mathbb{N}$, and $A_i \in \mathscr{E}(\mathbb{R})$ mutually disjoint) w.r.t. a $S\alpha S$ random content M is defined by

$$\int_{\mathbb{R}} f\, dM := \sum_{i=1}^{n} f_i\, M(A_i).$$

This integral is jointly $S\alpha S$ given that the random vector $(M(A_1), \dots, M(A_n))$ is jointly $S\alpha S$ whenever $A_i \in \mathscr{E}(\mathbb{R})$ are mutually disjoint. Note that, in general, $\int_{\mathbb{R}} f\, dM$ is not symmetric and thus only a complex d-dimensional α-stable random vector (cf. Remark 4.4). It is obvious that the integral is linear for simple functions and that it is well-defined due to the additivity of M. Integration of more general functions is difficult, but we have the following:

Theorem 4.21. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a $S\alpha S$ Lévy process in $\mathbb{R}^d$ with $\alpha \in (1, 2)$. Then there is a $S\alpha S$ random content $M : \mathscr{E}(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ s.t.*

$$L_t = \mathbb{P} - \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} \frac{e^{i\mu t} - 1}{i\mu} \cdot \left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}. \tag{4.14}$$

The $S\alpha S$ random content M is given by

$$M(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\mathbb{1}_A}(\mu)\, L(d\mu), \quad A \in \mathscr{E}(\mathbb{R}),$$

where $\widehat{\mathbb{1}_A}(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\mu x} \mathbb{1}_A(x)\, dx$ is the Fourier transform of $\mathbb{1}_A$ (cf. Section 2.3). Moreover, the random content M is dependently scattered, i.e. the increments of M are not independent.

Proof.
Step 1: We first show that M is well-defined and a $S\alpha S$ random content on $\mathscr{E}(\mathbb{R})$. For $-\infty < a < b < \infty$ we obtain

$$\widehat{\mathbb{1}_{[a,b)}}(\mu) = \frac{1}{\sqrt{2\pi}} \cdot \frac{e^{-i\mu a} - e^{-i\mu b}}{i\mu}, \quad \mu \in \mathbb{R},$$

which is obviously an element of $L^\alpha(\mathbb{C})$ since α is supposed to be greater than 1. Hence, for any $A \in \mathscr{E}(\mathbb{R})$, the Fourier transform $\widehat{\mathbb{1}_A}$ is in $L^\alpha(\mathbb{C})$.

Given the Lévy process $(L_t)_{t\in\mathbb{R}}$, we define $\widetilde{M}([a, b)) := L_b - L_a$ for $-\infty < a < b < \infty$ which is clearly a $S\alpha S$ random measure on the semi-ring $\mathscr{SR}(\mathbb{R})$ of half-open bounded intervals in $\mathbb{R}$. The circular control measure k on $(\mathbb{R} \times S_d, \mathscr{B}(\mathbb{R}) \times \mathscr{B}(S_d))$ is given by the product measure $\lambda^1 \otimes \Gamma$ where λ^1 denotes the one-dimensional Lebesgue measure and Γ is the spectral measure of L_1. We extend this $S\alpha S$ random measure to $\mathscr{B}_0(\mathbb{R})$ using the general procedure of Section 4.3 (note that the assumption $\mathscr{SR}(\mathbb{R}) \subseteq \mathscr{B}_0(\mathbb{R})$ is fulfilled since the control measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ is given by $\Gamma(S_d) \cdot \lambda^1$). The extension shall also be denoted by $\widetilde{M}$. It is clear that integration w.r.t. $\widetilde{M}$ is the same as integration w.r.t. the Lévy process L. Thus M is well-defined (cf. Section 4.2) and obviously a $S\alpha S$ random content according to Definition 4.20 (cf. Proposition 4.17 (i), (ii) and Section 2.3).
Step 2: Next we want to study integration of more general than simple functions w.r.t. M. For simple functions $f = \sum_{i=1}^n f_i \mathbb{1}_{A_i}$ we deduce, using the linearity of the Fourier transformation, the identity

$$\int_{-\infty}^{\infty} f\, dM = \sum_{i=1}^{n} f_i\, M(A_i) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{i=1}^{n} f_i \widehat{\mathbb{1}_{A_i}}\, dL = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f}\, dL. \tag{4.15}$$

If now $f : \mathbb{R} \to M_d(\mathbb{C})$ is a measurable function s.t. there is a sequence of $\mathscr{E}(\mathbb{R})$-simple functions f_n satisfying

$$\widehat{f_n} \overset{L^\alpha(M_d(\mathbb{C}))}{\to} \widehat{f} \quad \text{as } n \to \infty, \tag{4.16}$$

then we can define the integral $\int_{-\infty}^{\infty} f\,dM$ as the limit in probability of the sequence of simple integrals $\int_{-\infty}^{\infty} f_n\,dM$ which is well-defined since every f_n is $\mathscr{E}(\mathbb{R})$-simple. Due to Proposition 4.17 (iv), this limit exists and satisfies

$$\int_{-\infty}^{\infty} f\,dM = \mathbb{P}-\lim_{n\to\infty}\int_{-\infty}^{\infty} f_n\,dM \overset{(4.15)}{=} \mathbb{P}-\lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}_n\,dL = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\widehat{f}\,dL. \quad (4.17)$$

We shall call measurable functions $f:\mathbb{R}\to M_d(\mathbb{C})$ satisfying (4.16) M-integrable.
Step 3: Let us now define, for any $-\infty < a < b < \infty$,

$$f(\mu) := \frac{e^{i\mu b}-e^{i\mu a}}{i\mu}, \quad \mu\in\mathbb{R}.$$

Then $f\cdot\Phi_\lambda:\mathbb{R}\to\mathbb{C}$ with

$$\Phi_\lambda(\mu) := \left(1-\frac{|\mu|}{\lambda}\right)\mathbb{1}_{[-\lambda,\lambda]}(\mu), \quad \mu\in\mathbb{R},$$

is continuous with compact support on $\mathbb{R}$. Moreover, note that $f\cdot\Phi_\lambda$ is M-integrable.
For, writing

$$f(t)\cdot\Phi_\lambda(t) = \left(\frac{\sin(tb)-\sin(ta)}{t} - i\frac{\cos(tb)-\cos(ta)}{t}\right)\cdot\left(1-\frac{|t|}{\lambda}\right)\mathbb{1}_{[-\lambda,\lambda]}(t), \quad t\in\mathbb{R},$$

one immediately verifies that there is a lower sequence of $\mathscr{E}(\mathbb{R})$-simple functions s.t.

$$|f_n| \le |f\Phi_\lambda|$$

$$f_n \overset{L^1(\mathbb{C})}{\to} f\Phi_\lambda \quad \text{as } n\to\infty \quad \text{and}$$
$$\mathrm{Var}(f_n) \le \mathrm{Var}(f\Phi_\lambda) < \infty$$

where $\mathrm{Var}(\cdot)$ denotes the total variation (cf. [16, proof of Theorem 3.1]). We show that $\widehat{f_n}\to\widehat{f\Phi_\lambda}$ in $L^\alpha(\mathbb{C})$ as $n\to\infty$. We have

$$\begin{aligned}\left\|\widehat{f_n}-\widehat{f\Phi_\lambda}\right\|_{L^\alpha}^\alpha &= \int_{-1}^{1}\underbrace{\left|\widehat{f_n}(\mu)-\widehat{f\Phi_\lambda}(\mu)\right|^\alpha}_{\le\|f_n-f\Phi_\lambda\|_{L^1}^\alpha} d\mu + \int_{\{|\mu|>1\}}\left|\widehat{f_n}(\mu)-\widehat{f\Phi_\lambda}(\mu)\right|^\alpha d\mu \\ &\le 2\cdot\|f_n-f\Phi_\lambda\|_{L^1}^\alpha + \int_{\{|\mu|>1\}}\left|\widehat{f_n}(\mu)-\widehat{f\Phi_\lambda}(\mu)\right|^\alpha d\mu \qquad (4.18)\end{aligned}$$

where the first addend vanishes as $n\to\infty$. Integration by parts yields

$$\begin{aligned}\left|\widehat{f_n}(\mu)\right| &= \left|\int_{\mathbb{R}} f_n(t)e^{-i\mu t}\,dt\right| = \left|\int_{\mathbb{R}} f_n(t)\,d\left(-\frac{1}{i\mu}e^{-i\mu t}\right)(t)\right| \le \frac{1}{|\mu|}\left(2\sup_t|f_n(t)| + \mathrm{Var}(f_n)\right) \\ &\le \frac{1}{|\mu|}\left(2\sup_t|f(t)\Phi_\lambda(t)| + \mathrm{Var}(f\Phi_\lambda)\right) \in L^\alpha((-\infty,-1)\cup(1,\infty))\end{aligned}$$

and since $\left|\widehat{f_n}(\mu)-\widehat{f\Phi_\lambda}(\mu)\right| \le \|f_n-f\Phi_\lambda\|_{L^1} \overset{n\to\infty}{\to} 0$ for all $\mu\in\mathbb{R}$, we obtain, due to the Dominated Convergence Theorem, that the second term in (4.18) vanishes as well as $n\to\infty$. Hence $f\Phi_\lambda$ is M-integrable.

We set $g(\mu) := \sqrt{2\pi}\mathbb{1}_{[a,b)}(\mu)$ and $h(\mu) := g(-\mu)$, $\mu \in \mathbb{R}$. Then

$$\widehat{h}(t) = \frac{e^{itb} - e^{ita}}{it} = f(t)$$

and hence, due to Corollary 2.27,

$$\widehat{f\Phi_\lambda} = \widehat{\Phi_\lambda \widehat{h}} \underset{(2.21)}{\overset{(2.14)}{=}} \widehat{\widehat{F_\lambda * h}} = (F_\lambda * h)(-\,\cdot\,) = F_\lambda * g \overset{L^\alpha(\mathbb{C})}{\to} g \quad \text{as } \lambda \to \infty$$

where the convergence follows from (2.29) together with Theorem 2.34. Thus

$$\begin{aligned} \int_{-\infty}^{\infty} f\Phi_\lambda \, dM &\overset{(4.17)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f\Phi_\lambda} \, dL = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (F_\lambda * g) \, dL \\ &\overset{\mathbb{P}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g \, dL = L_b - L_a \quad \text{as } \lambda \to \infty \end{aligned}$$

by virtue of Proposition 4.17 (iv) and the claimed spectral representation of the $S\alpha S$ Lévy process is shown.

Step 4: We still have to show that the increments of M are not independent. Assume that they were independent s.t., in particular, $\operatorname{Re} M_1([a,b))$ and $\operatorname{Re} M_1([c,d))$ have to be independent for disjoint intervals $[a,b)$ and $[c,d)$. Since $\operatorname{Re} M_1([a,b)) = \int_{-\infty}^{\infty} \operatorname{Re}\big(\widehat{\mathbb{1}_{[a,b)}}\big)\, dL^{(1)}$ where $L^{(1)}$ denotes the first component of L, we can use [53, Theorem 3.5.3] to deduce that

$$\operatorname{Re}\left(\widehat{\mathbb{1}_{[a,b)}}(\mu)\right) \cdot \operatorname{Re}\left(\widehat{\mathbb{1}_{[c,d)}}(\mu)\right) = \frac{(\sin(\mu b) - \sin(\mu a)) \cdot (\sin(\mu d) - \sin(\mu c))}{\mu^2} = 0$$

Lebesgue-a.e. on $\mathbb{R}$, which is obviously not true. Hence the increments of M cannot be independent. $\square$

Remark 4.22.

(i) Again the limit in probability which occurs in the spectral representation of the Lévy process in (4.14) can be replaced by a limit in $L^p(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ for any $p < \alpha$ (cf. Remark 4.14 (i) and 4.18).

(ii) The assumption (4.16), used in the second step of the proof above for the extension of integration w.r.t. M to more general integrands, is strong. However, it holds for any continuous function f with compact support on $\mathbb{R}$ if f is in addition of bounded variation as one can observe from Step 3 in the preceding proof. All the functions appearing in the next section where we consider multivariate symmetric α-stable CARMA processes shall be of this type.

(iii) One might ask whether the random content M in Theorem 4.21 can be extended to $\mathscr{B}_0(\mathbb{R})$, i.e. to a dependently scattered, σ-additive $S\alpha S$ noise on $\mathscr{B}_0(\mathbb{R})$. Although it is not necessary for the developed spectral representation of the Lévy process L, it seems to be an interesting question. We know that, for any $A_1, A_2, \ldots \in \mathscr{B}_0(\mathbb{R})$ pairwise disjoint with $A := \bigcup_{i\in\mathbb{N}} A_i \in \mathscr{B}_0(\mathbb{R})$, the sequence $\widehat{\mathbb{1}_{\cup_{i=1}^n A_i}}$ converges pointwise and in $L^2(\mathbb{C})$ to $\widehat{\mathbb{1}_A}$, but we could not establish the convergence in $L^\alpha(\mathbb{C})$ to this day which, however, seems to be necessary (cf. Proposition 4.17 (iii)). We could not find a solution to this problem in literature either. Finally, note that the general procedure of Section 4.3 cannot be applied, since M is not independently scattered.

The following properties for the $S\alpha S$ random content in Theorem 4.21 are easy to deduce:

Proposition 4.23. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided $S\alpha S$ Lévy process in $\mathbb{R}^d$ with $\alpha \in (1,2)$ and let M be its corresponding $S\alpha S$ random content. Moreover, letting $(\gamma, 0, \nu)$ the generating triplet of L, we assume that $\nu \neq 0$ (i.e. L is non-trivial). Then $(Z_t)_{t\in\mathbb{R}_+}$ with*

$$Z_t := M([0,t)) = \int_{-\infty}^{\infty} \frac{1-e^{-i\mu t}}{2\pi i\mu}\, L(d\mu), \quad t \in \mathbb{R}_+,$$

is a $S\alpha S$ stochastic process.

Additionally, for any $t > 0$, we have the following results for Z_t and its corresponding Lévy measure ν_{Z_t}:

(i) $\int_{\{\|x\|\le 1\}} \|x\|\, \nu_{Z_t}(dx) = \infty$ which in particular implies that ν_{Z_t} is infinite.

(ii) $\mathbb{E}[\|Z_t\|^p] < \infty$ if and only if $0 < p < \alpha$.

(iii) For any $\delta \in (0,2)$, we have $\int_{\{\|x\|\le 1\}} \|x\|^\delta\, \nu(dx) < \infty$ iff $\int_{\{\|x\|\le 1\}} \|x\|^\delta\, \nu_{Z_t}(dx) < \infty$ which is in turn the case iff $\delta > \alpha$. Consequently, the Blumenthal-Getoor-Indices of ν_{Z_t} and ν are both equal to α.

Proof. We first show that $(Z_t)_{t\in\mathbb{R}_+}$ is a $S\alpha S$ process, i.e. all finite dimensional margins are (complex) $S\alpha S$. Let $(t_1,\ldots,t_m)' \in \mathbb{R}_+^m$, then

$$\begin{pmatrix} Z_{t_1} \\ \vdots \\ Z_{t_m} \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \int_{-\infty}^{\infty} \widehat{\mathbb{1}_{[0,t_1)}}(\mu)\, L(d\mu) \\ \vdots \\ \int_{-\infty}^{\infty} \widehat{\mathbb{1}_{[0,t_m)}}(\mu)\, L(d\mu) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_{t_1,\ldots,t_m}(\mu)\, L(d\mu) \tag{4.19}$$

where $g_{t_1,\ldots,t_m} : \mathbb{R} \to M_{md\times d}(\mathbb{C})$ is defined by

$$g_{t_1,\ldots,t_m}(\mu) := \begin{pmatrix} \widehat{\mathbb{1}_{[0,t_1)}}(\mu) \\ \vdots \\ \widehat{\mathbb{1}_{[0,t_m)}}(\mu) \end{pmatrix}$$

(note that we can understand $\widehat{\mathbb{1}_A}$ for any $A \in \mathscr{E}(\mathbb{R})$ as a function mapping from $\mathbb{R}$ to $M_d(\mathbb{C})$ by $\widehat{\mathbb{1}_A}(\mu) = \mathrm{diag}(\widehat{\mathbb{1}_A}(\mu), \ldots, \widehat{\mathbb{1}_A}(\mu))$).

It is not hard to verify that all results for stable integrals in Section 4.2 remain valid if we take integrands mapping into the complex $md \times d$ matrices with any $m \in \mathbb{N}$ (where m equals 1 in Section 4.2). We want to use Proposition 4.17 (i) to deduce that (4.19) is a $S\alpha S$ random vector in $\mathbb{C}^{md}$. Therefore, we only have to verify that $g_{t_1,\ldots,t_m} \in L^\alpha(M_{md\times d}(\mathbb{C}))$ which is actually true since there is $c > 0$ s.t.

$$\int_{\mathbb{R}} \|g_{t_1,\ldots,t_m}(s)\|^\alpha\, ds \le c \cdot \sum_{k=1}^{m} \int_{\mathbb{R}} \left|\widehat{\mathbb{1}_{[0,t_k)}}(s)\right|^\alpha ds < \infty.$$

Thus $(Z_t)_{t\in\mathbb{R}_+}$ is a $S\alpha S$ process.

Since, for any $t > 0$, ν_{Z_t} is obviously non-trivial (cf. (2.9) and Theorem 3.8), [54, Proposition 14.5] immediately implies statement (i) and Corollary 4.10 shows statement (ii).

As to (iii), [54, Theorem 14.3] implies that there is a finite measure κ on the unit sphere $S_d = \{x \in \mathbb{R}^d : \|x\| = 1\}$ s.t.

$$\nu(A) = \int_{S_d} \kappa(d\xi) \int_0^\infty \mathbb{1}_A(r\xi) \frac{dr}{r^{1+\alpha}}$$

for all $A \in \mathscr{B}(\mathbb{R}^d)$. Hence, simple calculations show that

$$\int_{\{\|x\|\le 1\}} \|x\|^\delta \, \nu(dx) = \int_{S_d} \kappa(d\xi) \int_0^1 \frac{1}{r^{1+\alpha-\delta}} \, dr$$

where the right-hand side is finite iff $\delta > \alpha$. Since $(Z_t)_{t\in\mathbb{R}_+}$ is a $S\alpha S$ process the same is true with ν replaced by ν_{Z_t} and (iii) is shown. □

4.5 Symmetric α-stable MCARMA processes

In this section we use the results obtained in Section 4.4 to derive a spectral representation (in the summability sense) and a moving average representation of $S\alpha S$ MCARMA processes. The proof of consistency to Definition 3.15 in the causal case forms the conclusion of the chapter.

Before turning to these representations we are going to state three lemmata. The first one establishes a spectral representation for moving averages of $S\alpha S$ Lévy processes having kernel functions that are in $L^1 \cap L^\alpha$ and satisfy in addition that their Fourier transforms are of bounded variation on compacta.

Lemma 4.24. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a d-dimensional $S\alpha S$ Lévy process with $1 < \alpha < 2$ and M the corresponding $S\alpha S$ random content of Theorem 4.21. Assume that we have given some $h \in L^1(M_d(\mathbb{C})) \cap L^\alpha(M_d(\mathbb{C}))$ s.t. $\widehat{h}$ is of bounded variation on compacta.*
Define

$$G_t := \mathbb{P} - \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{i\mu t} \widehat{h}(\mu) \left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}.$$

Then $(G_t)_{t\in\mathbb{R}}$ is a $S\alpha S$ process and, for all $t \in \mathbb{R}$,

$$G_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t-\mu) \, L(d\mu).$$

Proof. Since $h \in L^1(M_d(\mathbb{C}))$, the Fourier transform $\widehat{h}$ is obviously continuous and thus the function

$$f_{\lambda,t}(\mu) := e^{i\mu t} \widehat{h}(\mu) \left(1 - \frac{|\mu|}{\lambda}\right) \mathbb{1}_{[-\lambda,\lambda]}(\mu)$$

is continuous with compact support on $\mathbb{R}$ and has bounded variation by assumption. Consequently, it can be approximated in the $L^1(M_d(\mathbb{C}))$-norm by a sequence of $\mathscr{E}(\mathbb{R})$-simple functions f_n satisfying in addition $\widehat{f_n} \to \widehat{f_{\lambda,t}}$ in $L^\alpha(M_d(\mathbb{C}))$ as $n \to \infty$ (this can be shown in the same way as in Step 3 of the proof of Theorem 4.21; cf. also Remark 4.22 (ii)). This gives that $\int_{-\infty}^{\infty} f_{\lambda,t}(\mu) \, M(d\mu)$ is well-defined for any $\lambda > 0$ and $t \in \mathbb{R}$.

Then

$$G_t = \mathbb{P} - \lim_{\lambda\to\infty} \int_{-\infty}^{\infty} f_{\lambda,t}(\mu) \, M(d\mu) \overset{(4.17)}{=} \mathbb{P} - \lim_{\lambda\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f_{\lambda,t}}(\mu) \, L(d\mu).$$

Setting $\Phi_\lambda(\mu) := \big(1 - \frac{|\mu|}{\lambda}\big)\mathbb{1}_{[-\lambda,\lambda]}(\mu)$ and $h_t(\mu) := h(\mu + t)$, we have

$$f_{\lambda,t} = \Phi_\lambda e^{it\,\cdot}\,\widehat{h} \overset{(2.14)}{=} \sqrt{2\pi}\Delta_\lambda e^{it\,\cdot}\,\widehat{h} = \sqrt{2\pi}\Delta_\lambda \widehat{h_t} \overset{(2.28)}{=} \widehat{F_\lambda * h_t}$$

and thus, due to Corollary 2.27,

$$\widehat{f_{\lambda,t}} = \widehat{\widehat{F_\lambda * h_t}} = (F_\lambda * h_t)(-\,\cdot\,) = F_\lambda * (h_t(-\,\cdot\,)) \overset{L^\alpha(M_d(\mathbb{C}))}{\longrightarrow} h_t(-\,\cdot\,)$$

as $\lambda \to \infty$ where the convergence holds by virtue of (2.29) together with Theorem 2.34. Hence, due to Proposition 4.17 (iv),

$$G_t = \mathbb{P} - \lim_{\lambda\to\infty} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\lambda * (h_t(-\,\cdot\,))\,dL = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h_t(-\,\cdot\,)\,dL$$

for all $t \in \mathbb{R}$.

We still have to show that $(G_t)_{t\in\mathbb{R}}$ is a $S\alpha S$ process, i.e. all finite dimensional distributions are symmetric α-stable. For $(t_1, \dots, t_m)' \in \mathbb{R}^m$ we have (cf. proof of Proposition 4.23)

$$\begin{pmatrix} X_{t_1} \\ \vdots \\ X_{t_m} \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_{t_1,\dots,t_m}(\mu)\,L(d\mu) \tag{4.20}$$

where $g_{t_1,\dots,t_m} : \mathbb{R} \to M_{md\times d}(\mathbb{C})$ is defined by

$$g_{t_1,\dots,t_m}(\mu) := \begin{pmatrix} h(t_1 - \mu) \\ \vdots \\ h(t_m - \mu) \end{pmatrix}.$$

In order to show that (4.20) is a $S\alpha S$ random vector in $\mathbb{C}^{md}$, we only have to verify that $g_{t_1,\dots,t_m} \in L^\alpha(M_{md\times d}(\mathbb{C}))$ (cf. Proposition 4.17 (i)). This is actually true since there are $c_1, c_2 > 0$ s.t.

$$\begin{aligned}
\int_{\mathbb{R}} \|g_{t_1,\dots,t_m}(s)\|^\alpha\,ds &\le c_1 \cdot \int_{\mathbb{R}} \Bigg(\max_{\substack{i=1,\dots,md\\ j=1,\dots,d}} \Big| g^{(ij)}_{t_1,\dots,t_m}(s) \Big| \Bigg)^\alpha ds \\
&\le c_1 \cdot \int_{\mathbb{R}} \sum_{k=1}^{m} \Big(\max_{i,j=1,\dots,d} \big| h^{(ij)}(t_k - s) \big| \Big)^\alpha ds \\
&\le c_1 \cdot c_2 \cdot \sum_{k=1}^{m} \int_{\mathbb{R}} \|h(t_k - s)\|^\alpha\,ds < \infty
\end{aligned}$$

where $g^{(ij)}_{t_1,\dots,t_m}$ and $h^{(ij)}$ denote the (i,j)-th component function of $g_{t_1,\dots,t_m}$ and h, respectively. $\square$

The next lemma relates the zeros of what is to become the autoregressive polynomial of an MCARMA process to the spectrum of a particular matrix A (cf. also Section 3.4).

Lemma 4.25. *Let $A_1, \dots, A_p \in M_d(\mathbb{C})$, $p \in \mathbb{N}$, and define*

$$P : \mathbb{C} \to M_d(\mathbb{C}),\ z \mapsto \mathrm{I}_d z^p + A_1 z^{p-1} + \ldots + A_p.$$

Setting $\mathscr{N}(P) := \{z \in \mathbb{C} :\ \det(P(z)) = 0\}$ and

$$A := \begin{pmatrix} 0 & \mathrm{I}_d & 0 & \dots & 0 \\ 0 & 0 & \mathrm{I}_d & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & 0 \\ 0 & 0 & \dots & 0 & \mathrm{I}_d \\ -A_p & -A_{p-1} & \dots & \dots & -A_1 \end{pmatrix} \in M_{dp}(\mathbb{C})$$

one has $\mathscr{N}(P) = \sigma(A)$ where $\sigma(A)$ denotes the spectrum of A.

Moreover, if $\mathscr{N}(P) \subseteq \mathbb{R}\setminus\{0\} + i\mathbb{R}$, then $P(iz) \in \mathrm{GL}_d(\mathbb{C})$ for all $z \in \mathbb{R}$ (i.e. $P(iz)$ is an invertible $d \times d$ matrix for all $z \in \mathbb{R}$).

Proof. See [41, Lemma 3.8 and 3.10]. □

The last lemma verifies the assumptions of Lemma 4.24 in the case of $S\alpha S$ MCARMA processes.

Lemma 4.26. *Let $p, q \in \mathbb{N}_0$ with $p > q$ and $B_0, B_1, \dots, B_q \in M_d(\mathbb{C})$ with $B_0 \neq 0$. Define the matrix A and the polynomial P as in Lemma 4.25 and assume that $\mathscr{N}(P) \subseteq \mathbb{R}\setminus\{0\} + i\mathbb{R}$. Setting $Q : \mathbb{C} \to M_d(\mathbb{C})$, $z \mapsto B_0 z^q + B_1 z^{q-1} \ldots + B_q$, the function*

$$g : \mathbb{R} \to M_d(\mathbb{C}),\ g(\mu) := P(i\mu)^{-1} Q(i\mu)$$

is continuous and of bounded variation on compacta. Moreover $g(\mu) = \widehat{h}(\mu)$ for almost every $\mu \in \mathbb{R}$ with

$$\begin{aligned} h(\mu) := \widehat{g}(-\mu) &= \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{i\mu s} P(is)^{-1} Q(is)\, ds \\ &= \sqrt{2\pi} \sum_{\lambda} \sum_{s=0}^{m(\lambda)-1} \left(\mu^s e^{\lambda\mu} \mathbb{1}_{\{\mathrm{Re}(\lambda)\cdot\mu<0\}} C_{\lambda s} \right) \in L^1(M_d(\mathbb{C})) \cap L^\alpha(M_d(\mathbb{C})) \end{aligned} \tag{4.21}$$

where $\sum_{\lambda}$ denotes the sum over all distinct zeros in $\mathscr{N}(P)$, the multiplicity of the zero λ is written as $m(\lambda)$ and $C_{\lambda s}$ are constant complex-valued $d \times d$ matrices.

Proof. We need the following consequence of the residue theorem from complex analysis (cf., for instance, [35, Section VI.2, Theorem 2.2] or [26, Section III.7, Theorem 7.11]): let q and $p : \mathbb{C} \to \mathbb{C}$ be polynomials where p is of higher degree than q. Assume that p has no zeros on the real line. Then

$$\int_{-\infty}^{\infty} e^{i\mu t} \frac{q(t)}{p(t)}\, dt = \left\{ \begin{array}{ll} 2\pi i \cdot \sum\limits_{\substack{z\in\mathbb{C}:\, \mathrm{Im}(z)>0,\\ p(z)=0}} \mathrm{Res}_z f, & \mu > 0 \\ -2\pi i \cdot \sum\limits_{\substack{z\in\mathbb{C}:\, \mathrm{Im}(z)<0,\\ p(z)=0}} \mathrm{Res}_z f, & \mu < 0 \end{array} \right\}$$

with $f : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^{i\mu z} \frac{q(z)}{p(z)}$ and $\mathrm{Res}_z f$ denoting the residual of the function f at the point z.

Turning now to our function g, note first that it is well-defined by virtue of Lemma 4.25, it is clearly continuous and we have from elementary matrix theory that

$$g(\mu) = P(i\mu)^{-1}Q(i\mu) = \frac{S(i\mu)}{\det(P(i\mu))}$$

where $S : \mathbb{C} \to M_d(\mathbb{C})$ is some matrix-valued polynomial. Due to [41, Lemma 3.11] it is not hard to see that the complex-valued polynomial $\det(P(i\mu))$ in μ is of higher degree than $S(i\mu)$. Since all zeros of P are assumed to have non-vanishing real part, the zeros of $P(i\,\cdot)$ have non-vanishing imaginary part. This implies on the one hand that all components of the function g are continuously differentiable and hence of bounded variation on compacta which shows that g is of bounded variation on compacta as well.

On the other hand this enables us to apply the above stated results from complex function theory componentwise and we deduce for all $j, k = 1, \ldots, d$ and $\mu \in \mathbb{R}$, $\mu \neq 0$,

$$\begin{aligned}
\left(\sqrt{2\pi}h(\mu)\right)_{jk} &= \left(\int_{-\infty}^{\infty} e^{i\mu t} g(t)\, dt\right)_{jk} \\
&= 2\pi i \Bigg(\mathbb{1}_{\{\mu>0\}} \cdot \sum_{\substack{z\in\mathbb{C}:\, \mathrm{Im}(z)>0,\\ \det(P(iz))=0}} \mathrm{Res}_z f_{jk} - \mathbb{1}_{\{\mu<0\}} \cdot \sum_{\substack{z\in\mathbb{C}:\, \mathrm{Im}(z)<0,\\ \det(P(iz))=0}} \mathrm{Res}_z f_{jk} \Bigg) \\
&= 2\pi i \Bigg(\mathbb{1}_{\{\mu>0\}} \cdot \sum_{\substack{z\in\mathbb{C}:\, \mathrm{Re}(z)<0,\\ \det(P(z))=0}} \mathrm{Res}_{-iz} f_{jk} - \mathbb{1}_{\{\mu<0\}} \cdot \sum_{\substack{z\in\mathbb{C}:\, \mathrm{Re}(z)>0,\\ \det(P(z))=0}} \mathrm{Res}_{-iz} f_{jk} \Bigg)
\end{aligned} \tag{4.22}$$

where $f_{jk} : \mathbb{C} \to \mathbb{C}$, $z \mapsto e^{i\mu z} \frac{S_{jk}(iz)}{\det(P(iz))}$.

Let λ denote the distinct zeros of $\det(P(z))$ (i.e. the distinct eigenvalues of A) and $m(\lambda)$ the multiplicity of the zero λ. Since it is well-known that the residual of any meromorphic function $f : \mathbb{C} \to \mathbb{C}$ at a pole a of order $n \in \mathbb{N}$ is given by

$$\mathrm{Res}_a f = \frac{1}{(n-1)!} \left[\frac{d^{n-1}}{dz^{n-1}} (z-a)^n f(z) \right]_{z=a}$$

(cf. [26, Section III.6, Remark 6.4.1]), the residual of f_{jk} at the point $-i\lambda$, with λ being any zero of $\det(P(z))$, can be written as

$$\begin{aligned}
\mathrm{Res}_{-i\lambda} f_{jk} &= \frac{1}{(m(\lambda)-1)!} \left[\frac{d^{m(\lambda)-1}}{dz^{m(\lambda)-1}} (z+i\lambda)^{m(\lambda)} e^{i\mu z} \frac{S_{jk}(iz)}{\det(P(iz))} \right]_{z=-i\lambda} \\
&= -i \cdot \sum_{s=0}^{m(\lambda)-1} c_{\lambda s}^{jk} \mu^s e^{\lambda\mu}
\end{aligned} \tag{4.23}$$

for some complex constants $c_{\lambda s}^{jk}$ (where the sum reduces to $-i \cdot S_{jk}(\lambda) / \left[\frac{d}{dz}\det(P(z))\right]_{z=\lambda} e^{\lambda\mu}$ if $m(\lambda) = 1$). Thus (4.22) becomes

$$\begin{aligned}
h(\mu)_{jk} &= \sqrt{2\pi} \Bigg(\mathbb{1}_{\{\mu>0\}} \cdot \sum_{\lambda:\, \mathrm{Re}(\lambda)<0} \sum_{s=0}^{m(\lambda)-1} c_{\lambda s}^{jk} \mu^s e^{\lambda\mu} - \mathbb{1}_{\{\mu<0\}} \cdot \sum_{\lambda:\, \mathrm{Re}(\lambda)>0} \sum_{s=0}^{m(\lambda)-1} c_{\lambda s}^{jk} \mu^s e^{\lambda\mu} \Bigg) \\
&= \sqrt{2\pi} \sum_{\lambda} \sum_{s=0}^{m(\lambda)-1} \left(\widetilde{c}_{\lambda s}^{jk} \mu^s e^{\lambda\mu} \mathbb{1}_{\{\mathrm{Re}(\lambda)\cdot\mu<0\}} \right)
\end{aligned} \tag{4.24}$$

with $\widetilde{c}_{\lambda s}^{jk} := c_{\lambda s}^{jk}$ if $\mathrm{Re}(\lambda) < 0$ and $\widetilde{c}_{\lambda s}^{jk} := -c_{\lambda s}^{jk}$ if $\mathrm{Re}(\lambda) > 0$.

Hence h is obviously in $L^1(M_d(\mathbb{C})) \cap L^\alpha(M_d(\mathbb{C}))$ for all $\alpha \in (1, 2]$ and by virtue of the inversion formula (cf. the "surjective part" of the proof of Theorem 2.29) we obtain $\widehat{h} = g$ in $L^2(M_d(\mathbb{C}))$ and thus for almost every $\mu \in \mathbb{R}$ we have $\widehat{h}(\mu) = g(\mu)$.

Finally, defining the $d \times d$ matrices $C_{\lambda s} := (\widetilde{c}_{\lambda s}^{jk})_{jk}$, we obtain the claimed representation of h in (4.21). □

Now, due to the Lemmata 4.24, 4.25 and 4.26, we are able to give a definition of symmetric α-stable CARMA processes which is at the same time what we shall call their spectral representation.

Definition 4.27. *Let $L = (L_t)_{t \in \mathbb{R}}$ be a d-dimensional $S\alpha S$ Lévy process with $\alpha \in (1, 2)$ and M the corresponding $S\alpha S$ random content of Theorem 4.21. A d-dimensional symmetric α-stable Lévy-driven continuous time autoregressive moving average process $(Y_t)_{t \in \mathbb{R}}$ of order (p, q) with $p, q \in \mathbb{N}_0$, $p > q$ ($S\alpha S$ **MCARMA**(p, q) **process**) is defined as the $S\alpha S$ process*

$$\begin{aligned}
Y_t :&= \mathbb{P} - \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}, \quad \textit{where} \\
P(z) :&= \mathrm{I}_d z^p + A_1 z^{p-1} + \ldots + A_p \quad \textit{and} \\
Q(z) :&= B_0 z^q + B_1 z^{q-1} + \ldots + B_q
\end{aligned}$$

are the autoregressive and moving average polynomial, respectively.
Here $A_i \in M_d(\mathbb{R})$, $i = 1, \ldots, p$, and $B_j \in M_d(\mathbb{R})$, $j = 1, \ldots, q$, are real matrices satisfying $B_0 \neq 0$ and $\mathscr{N}(P) = \{z \in \mathbb{C} : \det(P(z)) = 0\} \subseteq \mathbb{R} \setminus \{0\} + i\mathbb{R}$.

Remark 4.28. One might think that $S\alpha S$ MCARMA processes also have a bona fide spectral representation of the form

$$Y_t = \int_{-\infty}^{\infty} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \, \widetilde{M}(d\mu), \quad t \in \mathbb{R},$$

for some appropriate extension $\widetilde{M}$ of the $S\alpha S$ random content of Theorem 4.21 to $\mathscr{B}_0(\mathbb{R})$. However, we have already mentioned in the introduction to this chapter that this is not possible given that $(Y_t)_{t \in \mathbb{R}}$ is non-trivial. Namely, [39, Proposition 1.9] implies that if (Y_t) were representable in that way, $Y_t \equiv 0$ for all $t \in \mathbb{R}$.

Lemmata 4.24 and 4.26 yield the following moving average representation as well:

Corollary 4.29. *Let $Y = (Y_t)_{t \in \mathbb{R}}$ be a $S\alpha S$ MCARMA(p, q) process, then Y has the moving average representation*

$$Y_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t - \mu) \, L(d\mu)$$

for all $t \in \mathbb{R}$, where h is given by (4.21).

Remark 4.30. Alternatively, the $S\alpha S$ MCARMA(p, q) process can be written as

$$Y_t = \frac{1}{2\pi} \int_{-\infty}^{\infty} \widetilde{h}(t - \mu) \, L(d\mu)$$

with kernel function
$$\widetilde{h}(t) = \int_{\mathbb{R}} e^{its} P(is)^{-1} Q(is)\, ds, \quad t \in \mathbb{R}.$$
This is in line with [44, Example 3.7] where the same moving average representation has also been established for $\alpha \leq 1$.

The question which may be asked at the end of this chapter is whether our definition (spectral representation) and moving average representation of a symmetric α-stable MCARMA process coincides in the causal case where $\sigma(A) = \mathcal{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$ with the definition of causal MCARMA processes made in [41] (cf. Definition 3.15). Indeed, the following (more general) lemma gives the consistency.

Lemma 4.31. *Let the polynomials P and Q be defined as in Definition 4.27 and the matrices A and β as in Definition 3.15. Then, for any $t \in \mathbb{R}$,*
$$\left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{tA}\beta = \frac{1}{2\pi i}\int_\rho P(z)^{-1} Q(z) e^{tz}\, dz \tag{4.25}$$
where ρ is a simple closed curve in the complex plane that encircles all eigenvalues of the matrix A.

Proof. By virtue of [7, Proposition 11.2.1] we have, for any $t \in \mathbb{R}$,
$$e^{tA} = \frac{1}{2\pi i}\int_\rho (z\mathrm{I}_{dp} - A)^{-1} e^{tz}\, dz$$
where ρ is a simple closed curve in the complex plane enclosing the spectrum of A. Setting
$$h_{k,p}(z) := \sum_{u=0}^{p-k} A_{p-k-u} z^u, \quad k = 1, \ldots, p, \tag{4.26}$$
with $A_0 := \mathrm{I}_d$, and
$$r_k(z) := -\sum_{u=0}^{k} A_{p-u} z^u, \quad k = 0, 1, \ldots, p-2,$$
one easily verifies that, for all $z \notin \sigma(A)$, the $d \times d$ blocks $c_{ij}(z)$, $i, j = 1, \ldots, p$, of the matrix $(z\mathrm{I}_{dp} - A)^{-1} \in M_{dp}(\mathbb{C})$ are given by
$$c_{ij}(z) = P(z)^{-1} \begin{cases} h_{j,p}(z) z^{i-1}, & \text{if } i \leq j, \\ r_{j-1}(z) z^{i-j-1}, & \text{if } i > j. \end{cases}$$
Indeed, one can show by simple calculations that this matrix is a left inverse for $z\mathrm{I}_{dp} - A$ and thus, due to [7, Corollary 2.6.4], it is the unique inverse of $z\mathrm{I}_{dp} - A$. Hence
$$\left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{tA} = \frac{1}{2\pi i}\int_\rho P(z)^{-1} \cdot \left(h_{1,p}(z), h_{2,p}(z), \ldots, h_{p,p}(z)\right) e^{tz}\, dz$$
which implies
$$\left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{tA}\beta = \frac{1}{2\pi i}\int_\rho P(z)^{-1} \cdot \left[\sum_{j=1}^{p} h_{j,p}(z)\beta_j\right] e^{tz}\, dz.$$

Since $B_i = 0$ for all $i < 0$, we obtain

$$\begin{aligned}
\sum_{j=1}^{p} h_{j,p}(z)\beta_j &\overset{(4.26)}{=} \sum_{j=1}^{p}\sum_{u=0}^{p-j} A_{p-j-u}\beta_j z^u = \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p-u} A_{p-j-u}\beta_j\right) z^u \\
&= \sum_{u=0}^{p-1}\left(\sum_{j=1}^{p-u-1} A_j\beta_{p-j-u} + \beta_{p-u}\right) z^u \\
&\overset{(3.8)}{=} \sum_{u=0}^{p-1} B_{q-u} z^u = \sum_{u=0}^{q} B_{q-u} z^u \\
&= Q(z)
\end{aligned}$$

and thus (4.25) is shown. □

Now observe that the causal MCARMA process in Definition 3.15 can be represented as

$$Y_t = \int_{-\infty}^{t} \left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{(t-s)A}\beta\, L(ds), \quad t \in \mathbb{R}.$$

Since $\sigma(A) = \mathscr{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$, we obtain for all $s < t$, due to Lemma 4.31 and the Residue Theorem,

$$\begin{aligned}
\left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{(t-s)A}\beta &= \frac{1}{2\pi i}\int_\rho \underbrace{P(z)^{-1}Q(z)e^{(t-s)z}}_{=:f(z)}\, dz \\
&= \sum_{\substack{z\in\mathbb{C}:\,\mathrm{Re}(z)<0,\\ \det(P(z))=0}} \left(\mathrm{Res}_z f_{jk}\right)_{j,k=1,\ldots,d} \\
&= i \cdot \sum_{\substack{z\in\mathbb{C}:\,\mathrm{Re}(z)<0,\\ \det(P(z))=0}} \left(\mathrm{Res}_{-iz} \widetilde{f}_{jk}\right)_{j,k=1,\ldots,d}
\end{aligned}$$

with $\widetilde{f}(z) := P(iz)^{-1}Q(iz)e^{i(t-s)z}$, $z \in \mathbb{C}$. Using then (4.23) and (4.24), we deduce

$$\left(\mathrm{I}_d, 0_{M_d(\mathbb{C})}, \ldots, 0_{M_d(\mathbb{C})}\right) e^{(t-s)A}\beta \cdot \mathbb{1}_{\{s<t\}} = \frac{1}{\sqrt{2\pi}} h(t-s)$$

and the claimed consistency follows from Corollary 4.29.

Remark 4.32. The same results in the univariate case can be found in [14, Lemma 2.3]. However, the first claim in the proof of this lemma concerning the elements of the matrix e^{tA} seems to be in contradiction to our proof of Lemma 4.31. Since at least the upper triangle of e^{tA} is correct in [14], the result (2.10) there remains true and is in line with the multivariate version (4.25) in our Lemma 4.31.

Chapter 5

Multivariate regularly varying CARMA processes

In this chapter we will generalize the foregoing results obtained for $S\alpha S$ Lévy and CARMA processes to the setting of regularly varying processes. Since every α-stable Lévy process is also regularly varying with index α, this is actually an extension of Chapter 4. We shall concentrate on the case where the index of variation satisfies $\alpha \in (1, 2]$ since the case $\alpha > 2$ implies that the driving Lévy process possesses finite second moments and thus it is already contained in Chapter 3.

The chapter is organized as follows: first we recall the notion of multivariate regular variation and give a useful connection to Lévy measures in the case of infinitely divisible distributions. In Section 2 we then prove a spectral representation for multivariate regularly varying Lévy processes. Our approach therefore will be similar to Theorem 4.21, but we have to specify suitable conditions in Step 2 of the proof of that theorem ensuring that integration can again be extended to more general than simple functions. Having shown the spectral representation for the driving Lévy process, we will be able to state analogously to Section 4.5 a spectral and a moving average representation of multivariate regularly varying CARMA processes in the last section.

5.1 Multivariate regular variation

As announced we first recall the notion of multivariate regular variation which can be used for the analysis of the tail behavior of multivariate stochastic processes. However, there is not only one single definition of multivariate regular variation, but many different equivalent ones. For detailed and good introductions into the different approaches, we refer the reader to Resnick [49] and Lindskog [37]. We start with a definition from [30].

Let therefore $\xrightarrow{v}$ denote vague convergence. It is defined on the one-point uncompactification $\overline{\mathbb{R}}^d \setminus \{0\}$ (where $\overline{\mathbb{R}} := [-\infty, \infty]$), which assures that the Borel sets of $\mathbb{R}^d$ that are bounded away form the origin can be referred to as the relatively compact sets in the vague topology.

Definition 5.1. *An $\mathbb{R}^d$-valued random vector X with unbounded support is called* ***regularly varying*** *with index $\alpha > 0$, if there exist a function $l : \mathbb{R} \to \mathbb{R}$ which is slowly varying at infinity and a non-zero Radon measure κ defined on $\mathscr{B}(\overline{\mathbb{R}}^d \setminus \{0\})$ with $\kappa(\overline{\mathbb{R}}^d \setminus \mathbb{R}^d) = 0$ such that, as $u \to \infty$,*

$$u^\alpha l(u)\mathbb{P}(u^{-1}X \in \cdot\,) \xrightarrow{v} \kappa(\cdot\,)$$

on $\mathscr{B}(\overline{\mathbb{R}}^d \setminus \{0\})$. We write $X \in RV(\alpha, l, \kappa)$.

Similarly, we call a Radon measure ν regularly varying, if α, l and κ exist as above such that

$$u^\alpha l(u)\nu(u\,\cdot\,) \xrightarrow{v} \kappa(\,\cdot\,)$$

as $u \to \infty$ and we write $\nu \in RV(\alpha, l, \kappa)$.

A d-dimensional real-valued stochastic process $(X_t)_{t\in\mathbb{R}}$ is called regularly varying with index α if all its finite dimensional distributions are regularly varying with index α.

We say that a complex random vector $X = (X_1, \dots, X_d)$ is regularly varying if and only if the real random vector $(\mathrm{Re}(X_1), \mathrm{Im}(X_1), \dots, \mathrm{Re}(X_d), \mathrm{Im}(X_d))$ is regularly varying.

We want to recall another definition which is widely used and gives a nice interpretation of multivariate regular variation. The following theorem shows their equivalence. Let therefore $S_d = \left\{x \in \mathbb{R}^d : \|x\| = 1\right\}$ denote the unit sphere in $\mathbb{R}^d$.

Theorem 5.2 (cf. [44], Theorem A.1)**.**
Let X be a random vector in $\mathbb{R}^d$. Then the following characterizations are equivalent:

(i) X is regularly varying with index α in the sense of Definition 5.1.

(ii) There exists an S_d-valued random variable Θ s.t. for every $t > 0$

$$\frac{\mathbb{P}\left(\|X\| > tu, \frac{X}{\|X\|} \in \cdot\,\right)}{\mathbb{P}(\|X\| > u)} \xrightarrow{v} t^{-\alpha}\mathbb{P}(\Theta \in \,\cdot\,)$$

on $\mathscr{B}(S_d)$ as $u \to \infty$.

The second characterization in particular implies that $\lim_{u\to\infty} \frac{\mathbb{P}(\|X\|>tu)}{\mathbb{P}(\|X\|>u)} = t^{-\alpha}$ for any $t > 0$, i.e. whenever a random variable X is (multivariate) regularly varying of index α, the univariate random variable $\|X\|$ is also regularly varying with the same index. Thus we immediately infer, using the well-known results for the univariate case (cf. e.g. [24]):

Proposition 5.3.
Let X be an $\mathbb{R}^d$-valued regularly varying random variable with index α. Then

$$\begin{aligned} \mathbb{E}[\|X\|^p] &< \infty \quad \textit{for any} \quad 0 < p < \alpha, \\ \mathbb{E}[\|X\|^p] &= \infty \quad \textit{for any} \quad p > \alpha. \end{aligned}$$

For infinitely divisible random variables, the following very useful connection between regular variation of the random variable and its Lévy measure exists.

Theorem 5.4 (cf. [30], Proposition 3.1)**.**
Let X be an infinitely divisible $\mathbb{R}^d$-valued random vector with Lévy measure ν. Then $X \in RV(\alpha, l, \kappa)$ if and only if $\nu \in RV(\alpha, l, \kappa)$.

One can show that a d-dimensional Lévy process $L = (L_t)_{t\in\mathbb{R}}$ is regularly varying of index α iff its characteristic Lévy measure ν (i.e. the Lévy measure of L_1) is regularly varying of index α. Strictly speaking the Lévy process $(L_t)_{t\in\mathbb{R}}$ is not regularly varying since $L_0 \equiv 0$ a.s., but, as all other finite dimensional margins are regularly varying, we neglect that inaccuracy.

5.2 Spectral representation of regularly varying Lévy processes

In analogy to Definition 4.20 we define:

Definition 5.5. *For $\alpha \in (1, 2]$ a d-dimensional* ***regularly varying random content*** *with index α is a set function*

$$M : \mathscr{E}(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$$

satisfying

(i) $M(A)$ is a complex d-dimensional random vector that is regularly varying with index α for all $A \in \mathscr{E}(\mathbb{R})$,

(ii) $M(\bigcup_{i=1}^n A_i) = \sum_{i=1}^n M(A_i)$ a.s. whenever $A_1, \dots, A_n \in \mathscr{E}(\mathbb{R})$ are pairwise disjoint (i.e. M is additive).

Integration of simple functions $f = \sum_{i=1}^n f_i \mathbb{1}_{A_i}$ (with $f_i \in M_d(\mathbb{C})$, $i = 1, \dots, n$, $n \in \mathbb{N}$, and $A_i \in \mathscr{E}(\mathbb{R})$ mutually disjoint) w.r.t. M is again defined by

$$\int_{\mathbb{R}} f \, dM := \sum_{i=1}^n f_i \, M(A_i)$$

which is obviously a complex d-dimensional random vector. The integral is linear for simple functions and it is well-defined due to the additivity of M. We have the following theorem in analogy to Theorem 4.21:

Theorem 5.6. *Let $L = (L_t)_{t \in \mathbb{R}}$ be a Lévy process in $\mathbb{R}^d$, regularly varying of index $\alpha \in (1, 2]$ and suppose $\mathbb{E}[L_1] = 0$. Then there is a regularly varying random content $M : \mathscr{E}(\mathbb{R}) \to L^0(\Omega, \mathscr{F}, \mathbb{P}; \mathbb{C}^d)$ with index α such that*

$$L_t = \mathbb{P} - \lim_{\lambda \to \infty} \int_{-\lambda}^{\lambda} \frac{e^{i\mu t} - 1}{i\mu} \cdot \left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}.$$

The random content M is given by

$$M(A) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{\mathbb{1}_A}(\mu) \, L(d\mu), \quad A \in \mathscr{E}(\mathbb{R}),$$

where $\widehat{\mathbb{1}_A}$ is the Fourier transform of $\mathbb{1}_A$.

Before we pass on to the proof of Theorem 5.6, we recall two results from [44] and establish one more lemma which will be used in the proof.

Theorem 5.7 (cf. [44], Theorem 2.5)**.**
Let $L = (L_t)_{t \in \mathbb{R}}$ be a d-dimensional Lévy process with generating triplet (γ, Σ, ν), let ν be regularly varying with index $\alpha \in (1, 2]$ and let $f : \mathbb{R} \to M_{k \times d}(\mathbb{R})$ be measurable. Then f is L-integrable if it is bounded, $\mathbb{E}[L_1] = 0$ and $f \in L^\delta(M_{k \times d}(\mathbb{R}))$ for some $\delta < \alpha$.

Theorem 5.8 (cf. [44], Theorem 3.2).
Let $L = (L_t)_{t\in\mathbb{R}}$ be a d-dimensional Lévy process with generating triplet (γ, Σ, ν) and let $\nu \in RV(\alpha, l, \kappa_\nu)$. If $f : \mathbb{R} \to M_{k\times d}(\mathbb{R})$ is L-integrable, $f \in L^\alpha(M_{k\times d}(\mathbb{R}))$ and $\kappa_\nu\left(f^{-1}(s)(\mathbb{R}^k \setminus \{0\})\right) = 0$ does not hold for almost every s, then $X := \int_{\mathbb{R}} f(s)\, L(ds) \in RV(\alpha, l, \kappa_X)$ with

$$\kappa_X(A) := \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A(f(s)x)\, \kappa_\nu(dx)\, ds, \quad A \in \mathscr{B}(\overline{\mathbb{R}}^k \setminus \{0\}).$$

The following lemma will be central in the proof of Theorem 5.6 for the extension of integration to more general integrands.

Lemma 5.9. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a d-dimensional Lévy process with $\mathbb{E}[L_1] = 0$ and generating triplet (γ, Σ, ν) where ν is supposed to be regularly varying with index $\alpha \in (1, 2]$. Let $f : \mathbb{R} \to M_{k\times d}(\mathbb{R})$ be measurable and $f_n : \mathbb{R} \to M_{k\times d}(\mathbb{R})$ be a sequence of measurable functions s.t. $f_n \to f$ as $n \to \infty$ in $L^\delta(M_{k\times d}(\mathbb{R}))$ for some $\delta < \alpha$. Moreover, assume that $\|f_n(s) - f(s)\| + \|f(s)\| \leq C$ for all $n \in \mathbb{N}$, $s \in \mathbb{R}$ and some constant $C > 0$. Then the sequence of integrals $\int_{\mathbb{R}} f_n\, dL$ converges in probability to $\int_{\mathbb{R}} f\, dL$ as $n \to \infty$.*

We use the following general result on the relation between convergence of characteristic triplets and weak convergence of probability distributions.

Lemma 5.10 (cf. [54], Theorem 8.7).
Let $c : \mathbb{R}^d \to \mathbb{R}$ be bounded and continuous s.t.

$$c(x) = 1 + o(\|x\|) \quad \text{as } \|x\| \to 0 \quad \text{and}$$

$$\|x\|\, c(x) \to 0 \quad \text{as } \|x\| \to \infty.$$

Assume that μ_n for $n \in \mathbb{N}$ are infinitely divisible distributions on $\mathbb{R}^d$ with generating triplets $(\gamma_n, \Sigma_n, \nu_n)_c$ (i.e. (2.2) holds with the truncation function $c(x)$ in place of $\mathbb{1}_{[0,1]}(\|x\|)$) and let μ be another probability distribution on $\mathbb{R}^d$. Then $\mu_n \xrightarrow{w} \mu$ if and only if μ is infinitely divisible with characteristic triplet $(\gamma, \Sigma, \nu)_c$ satisfying the following three conditions:

(1) If f is a bounded, continuous function from $\mathbb{R}^d$ to $\mathbb{R}$ vanishing in a neighborhood of 0, then

$$\lim_{n\to\infty} \int_{\mathbb{R}^d} f(x)\, \nu_n(dx) = \int_{\mathbb{R}^d} f(x)\, \nu(dx)$$

(i.e. ν_n converges to ν vaguely as measures on $\overline{\mathbb{R}}^d \setminus \{0\}$).

(2) Define the symmetric positive semidefinite matrices $\Sigma_{n,\varepsilon}$ by

$$\langle z, \Sigma_{n,\varepsilon} z\rangle = \langle z, \Sigma_n z\rangle + \int_{\{\|x\|\leq\varepsilon\}} \langle z, x\rangle^2\, \nu_n(dx).$$

Then

$$\lim_{\varepsilon\searrow 0} \limsup_{n\to\infty} |\langle z, \Sigma_{n,\varepsilon} z\rangle - \langle z, \Sigma z\rangle| = 0$$

for all $z \in \mathbb{R}^d$.

(3) $\gamma_n \to \gamma$ *as* $n \to \infty$.

Proof of Lemma 5.9.
Note first that the integrals $\int_{\mathbb{R}} f_n\,dL$ and $\int_{\mathbb{R}} f\,dL$ are well-defined due to Theorem 5.7. Letting $g_n := f_n - f$, we have to show that $\int_{\mathbb{R}} g_n\,dL \xrightarrow{\mathbb{P}} 0$ as $n \to \infty$ which is equivalent to $\int_{\mathbb{R}} g_n\,dL \xrightarrow{w} 0$ as $n \to \infty$.

The distribution of every $\int_{\mathbb{R}} g_n\,dL$ is infinitely divisible and possesses the generating triplet $(\gamma_n, \Sigma_n, \nu_n)$ given by (cf. Section 2.2.5 (2.7)-(2.9))

$$\gamma_n = \int_{\mathbb{R}} \left(g_n(s)\gamma + \int_{\mathbb{R}^d} g_n(s)x \left(\mathbb{1}_{[0,1]}(\|g_n(s)x\|) - \mathbb{1}_{[0,1]}(\|x\|) \right) \nu(dx) \right) ds,$$

$$\Sigma_n = \int_{\mathbb{R}} g_n(s)\Sigma g_n(s)' ds,$$

$$\nu_n(B) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_B(g_n(s)x)\,\nu(dx)\,ds, \quad B \in \mathscr{B}(\mathbb{R}^k_*).$$

Since we want to use Lemma 5.10, we change the truncation function from $\mathbb{1}_{[0,1]}(\|x\|)$ to $c(x) := \mathbb{1}_{[0,1]}(\|x\|) + \mathbb{1}_{(1,2]}(\|x\|)(2 - \|x\|)$ which satisfies the assumptions of Lemma 5.10. Concerning the generating triplet of $\int_{\mathbb{R}} g_n\,dL$, only γ_n is affected, namely, using c as truncation function we obtain the generating triplet $(\gamma_{n,c}, \Sigma_n, \nu_n)_c$ where

$$\gamma_{n,c} = \gamma_n + \int_{\mathbb{R}^k} x \left(c(x) - \mathbb{1}_{[0,1]}(\|x\|) \right) \nu_n(dx) = \gamma_n + \int_{\{\|x\| \in (1,2]\}} x\,(2 - \|x\|)\,\nu_n(dx).$$

The remainder of the proof is dedicated to the verification of conditions (1) - (3) in Lemma 5.10.

We first show that

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|g_n(s)x\|^2\right) \nu(dx)\,ds \overset{n\to\infty}{\to} 0. \tag{5.1}$$

We get

$$\begin{aligned} &\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|g_n(s)x\|^2\right)\,\nu(dx)\,ds \\ &\quad = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{\{\|g_n(s)x\|>1\}}\,\nu(dx)\,ds + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|g_n(s)x\|^2\,\mathbb{1}_{\{\|g_n(s)x\|\le 1\}}\,\nu(dx)\,ds. \end{aligned} \tag{5.2}$$

If we assume w.l.o.g. that our norm on $M_{k\times d}(\mathbb{R})$ is compatible with the vector norm (which is e.g. the case for any operator norm), we can use the inequality

$$\|g_n(s)\|\,\|x\| \ge \|g_n(s)x\| > 1$$

which implies $\|x\| \ge 1/\|g_n(s)\|$. Thus the first term on the right-hand side of (5.2) can be bounded by

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_{\{\|g_n(s)x\|>1\}}\,\nu(dx)\,ds \le \int_{\mathbb{R}} \nu\left(\left\{\|x\| \ge \frac{1}{\|g_n(s)\|}\right\}\right) ds.$$

Now we can apply the Potter bounds (cf. [49, Proposition 2.6]), giving the existence of some t_0 s.t. for all $t \ge t_0$ a regularly varying function (in our case ν) can be bounded.

Therefore, we distinguish the cases $1/\|g_n(s)\| > t_0$ and $1/\|g_n(s)\| \leq t_0$.
For the first case we set $\widetilde{C} := \sup\{\|g_n(s)\| : \|g_n(s)\| < 1/t_0\} \leq 1/t_0$. Then we can apply the Potter bounds for $t = 1/\widetilde{C} \geq t_0$ in order to get

$$\begin{aligned}
&\int_{\mathbb{R}} \mathbb{1}_{\{1/\|g_n(s)\|>t_0\}}\, \nu\left(\left\{\|x\| \geq \frac{1}{\|g_n(s)\|}\right\}\right) ds \\
&\quad \leq (1+\alpha-\delta) \int_{\mathbb{R}} \mathbb{1}_{\{1/\|g_n(s)\|>t_0\}}\, \nu\left(\left\{\|x\| \geq \frac{1}{\widetilde{C}}\right\}\right) \left(\frac{\|g_n(s)\|}{\widetilde{C}}\right)^{\delta} ds \overset{n\to\infty}{\to} 0
\end{aligned}$$

since $g_n \to 0$ as $n \to \infty$ in $L^\delta(M_{k\times d}(\mathbb{R}))$.
In the other case we note that $\sup\|g_n(s)\| \leq C < \infty$ and obtain

$$\begin{aligned}
\int_{\mathbb{R}} \mathbb{1}_{\{1/\|g_n(s)\|\leq t_0\}}\, \nu\left(\left\{\|x\| \geq \frac{1}{\|g_n(s)\|}\right\}\right) ds &\leq \int_{\mathbb{R}} \mathbb{1}_{\{1/\|g_n(s)\|\leq t_0\}}\, \nu\left(\left\{\|x\| \geq \frac{1}{C}\right\}\right) ds \\
&= \nu\left(\left\{\|x\| \geq \frac{1}{C}\right\}\right) \cdot \lambda^1\left(\left\{s : \|g_n(s)\| \geq \frac{1}{t_0}\right\}\right) \\
&\leq \nu\left(\left\{\|x\| \geq \frac{1}{C}\right\}\right) \cdot t_0^\delta \cdot \int_{\mathbb{R}} \|g_n(s)\|^\delta ds \to 0
\end{aligned}$$

as $n \to \infty$, using again the assumption that g_n converges to 0 in $L^\delta(M_{k\times d}(\mathbb{R}))$.

The second term on the right hand side of (5.2) can be bounded by

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d} \|g_n(s)x\|^2\, \mathbb{1}_{\{\|g_n(s)x\|\leq 1\}}\, \nu(dx)\, ds \\
&\quad = \int_{\mathbb{R}}\int_{\{\|x\|<1\}} \|g_n(s)x\|^2\, \mathbb{1}_{\{\|g_n(s)x\|\leq 1\}}\, \nu(dx)\, ds \\
&\qquad\qquad + \int_{\mathbb{R}}\int_{\{\|x\|\geq 1\}} \|g_n(s)x\|^2\, \mathbb{1}_{\{\|g_n(s)x\|\leq 1\}}\, \nu(dx)\, ds \\
&\quad \leq \int_{\mathbb{R}} \|g_n(s)\|^2 ds \int_{\{\|x\|<1\}} \|x\|^2\, \nu(dx) \\
&\qquad\qquad + \int_{\mathbb{R}} \|g_n(s)\|^\delta ds \int_{\{\|x\|\geq 1\}} \|x\|^\delta\, \nu(dx) \\
&\quad \overset{n\to\infty}{\to} 0
\end{aligned}$$

where we used the fact that the boundedness of the sequence g_n together with the convergence to 0 in $L^\delta(M_{k\times d}(\mathbb{R}))$ implies that g_n converges to 0 also in $L^2(M_{k\times d}(\mathbb{R}))$. Moreover, note that $\int_{\{\|x\|\geq 1\}} \|x\|^\delta\, \nu(dx) < \infty$ due to Theorem 2.20, since $0 < \delta < \alpha$ and hence the underlying Lévy process has a finite δ-th moment. Putting these results together we have shown that (5.1) holds.

Let us now verify condition (1) of Lemma 5.10. We show that ν_n converges in total variation to the zero measure outside of any fixed neighborhood of 0. Let therefore $U = \{\|x\| < \varepsilon\}$, $\varepsilon > 0$, be such a neighborhood. We obtain

$$\begin{aligned}
\nu_n(\mathbb{R}^k \backslash U) &= \lambda^1 \otimes \nu\left(\{(s,x) : \|g_n(s)x\| \geq \varepsilon\}\right) \\
&\leq \frac{1}{1\wedge\varepsilon^2} \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left(1 \wedge \|g_n(s)x\|^2\right) \mathbb{1}_{\{\|g_n(s)x\|\geq\varepsilon\}}\, \nu(dx)\, ds \\
&\leq \frac{1}{1\wedge\varepsilon^2} \int_{\mathbb{R}}\int_{\mathbb{R}^d} \left(1 \wedge \|g_n(s)x\|^2\right) \nu(dx)\, ds \overset{n\to\infty}{\to} 0
\end{aligned}$$

by virtue of (5.1). This immediately implies condition (1) of Lemma 5.10.

As to condition (2), note first that, for some $c_1 > 0$,

$$\|\Sigma_n\| \leq c_1 \cdot \|\Sigma\| \cdot \int_{\mathbb{R}} \|g_n(s)\|^2 \, ds \overset{n\to\infty}{\to} 0$$

since g_n converges to 0 in $L^2(M_{k\times d}(\mathbb{R}))$ as previously noted. Hence, using again (5.1) we obtain for any $\varepsilon \in (0,1)$,

$$\begin{aligned}
|\langle z, \Sigma_{n,\varepsilon} z\rangle| &\leq |\langle z, \Sigma_n z\rangle| + \int_{\{\|y\|\leq\varepsilon\}} \langle z, y\rangle^2 \, \nu_n(dy) \\
&= |\langle z, \Sigma_n z\rangle| + \int_{\{(s,x)\colon \|g_n(s)x\|\leq\varepsilon\}} \langle z, g_n(s)x\rangle^2 \, (\lambda^1 \otimes \nu)(d(s,x)) \\
&\leq c_2 \cdot \|z\|^2 \left(\|\Sigma_n\| + \int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|g_n(s)x\|^2\right) \nu(dx)\, ds \right) \overset{n\to\infty}{\to} 0
\end{aligned}$$

where c_2 is a positive constant. This in particular yields $\lim\limits_{\varepsilon\searrow 0} \limsup\limits_{n\to\infty} |\langle z, \Sigma_{n,\varepsilon} z\rangle| = 0$ for all $z \in \mathbb{R}^k$.

Finally we show condition (3), i.e. $\gamma_{n,c} \to 0$ as $n \to \infty$. Recall that

$$\gamma_{n,c} = \gamma_n + \int_{\{\|x\|\in(1,2]\}} x\,(2 - \|x\|)\, \nu_n(dx).$$

We immediately obtain that

$$\int_{\{\|x\|\in(1,2]\}} \|x\|\,(2 - \|x\|)\, \nu_n(dx) \leq \nu_n\left(\{\|x\| \in (1,2]\}\right) \overset{n\to\infty}{\to} 0$$

since ν_n converges in total variation to the zero measure outside of any fixed neighborhood of 0. For γ_n we have

$$\begin{aligned}
\|\gamma_n\| &\leq \int_{\mathbb{R}} \left\| g_n(s)\gamma + \int_{\mathbb{R}^d} g_n(s)x \left(\mathbb{1}_{[0,1]}(\|g_n(s)x\|) - \mathbb{1}_{[0,1]}(\|x\|)\right) \nu(dx) \right\| ds \\
&= \int_{\mathbb{R}} \left\| g_n(s)\gamma + \int_{\{\|x\|>1\}} g_n(s)x\, \mathbb{1}_{\{\|g_n(s)x\|\leq 1\}}\, \nu(dx) \right. \\
&\qquad\qquad \left. - \int_{\{\|x\|\leq 1\}} g_n(s)x\, \mathbb{1}_{\{\|g_n(s)x\|>1\}}\, \nu(dx) \right\| ds.
\end{aligned} \tag{5.3}$$

Since $\mathbb{E}[L_1] = 0$, we know $\gamma = -\int_{\{\|x\|>1\}} x\,\nu(dx)$ (cf. Remark 2.7 (iii)). Choose any $\xi \in (\delta, \alpha)$, $\xi > 1$. Then (5.3) becomes

$$\begin{aligned}
&\int_{\mathbb{R}} \left\| -\int_{\{\|x\|>1\}} g_n(s)x\,\nu(dx) + \int_{\{\|x\|>1\}} g_n(s)x\, \mathbb{1}_{\{\|g_n(s)x\|\leq 1\}}\, \nu(dx) \right. \\
&\qquad\qquad \left. - \int_{\{\|x\|\leq 1\}} g_n(s)x\, \mathbb{1}_{\{\|g_n(s)x\|>1\}}\, \nu(dx) \right\| ds \\
&\leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|g_n(s)x\|\, \mathbb{1}_{\{\|g_n(s)x\|>1\}}\, \nu(dx)\, ds \leq \int_{\mathbb{R}} \int_{\mathbb{R}^d} \|g_n(s)x\|^\xi\, \mathbb{1}_{\left\{\|x\|\geq \frac{1}{C}\right\}}\, \nu(dx)\, ds \\
&\leq C^{\xi-\delta} \int_{\mathbb{R}} \|g_n(s)\|^\delta\, ds \int_{\left\{\|x\|\geq\frac{1}{C}\right\}} \|x\|^\xi\, \nu(dx) \overset{n\to\infty}{\to} 0
\end{aligned}$$

since $g_n \to 0$ in $L^\delta(M_{k\times d}(\mathbb{R}))$. Note again that $\int_{\{\|x\|\geq \frac{1}{C}\}} \|x\|^\xi \, \nu(dx)$ is finite since $0 < \xi < \alpha$ and hence the underlying Lévy process has a finite ξ-th moment. Together this shows $\gamma_{n,c} \to 0$ as $n \to \infty$.

We conclude the proof by using Lemma 5.10. □

Now we are able to give a proof of Theorem 5.6. Since it is similar to the proof of Theorem 4.21, we shall only sketch it.

Sketch of the proof of Theorem 5.6.

- Due to Theorem 5.4, it follows immediately that the characteristic Lévy measure ν of L is regularly varying with index $\alpha \in (1,2]$. For any $A \in \mathscr{E}(\mathbb{R})$, we split the Fourier transform $\widehat{\mathbb{1}_A}$ up into its real and imaginary part and Theorem 5.7 can be applied in order to show that M is well-defined. Using then Theorem 5.8 we obtain that M is a regularly varying random content.

- For simple functions $f = \sum_{i=1}^n f_i \mathbb{1}_{A_i}$ one can deduce the same identity as in (4.15), namely
$$\int_{-\infty}^{\infty} f \, dM = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f} \, dL. \tag{5.4}$$

- If now $f : \mathbb{R} \to M_d(\mathbb{C})$ is a measurable function s.t. there is a sequence of $\mathscr{E}(\mathbb{R})$-simple functions f_n satisfying
$$\widehat{f_n} \overset{L^\delta(M_d(\mathbb{C}))}{\to} \widehat{f} \quad \text{as } n \to \infty \quad \text{for some } \delta < \alpha \quad \text{and}$$
$$\left\| \widehat{f_n}(\mu) - \widehat{f}(\mu) \right\| + \left\| \widehat{f}(\mu) \right\| \leq C \quad \text{for all } n \in \mathbb{N},\ \mu \in \mathbb{R} \text{ and some constant } C > 0,$$
then we define the integral $\int_{-\infty}^{\infty} f \, dM$ as the limit in probability of the sequence of simple integrals $\int_{-\infty}^{\infty} f_n \, dM$. Note that this sequence of integrals is well-defined since every f_n is $\mathscr{E}(\mathbb{R})$-simple. Since we can always identify $\mathbb{C}$ with $\mathbb{R}^2$ and $\mathbb{C}^d$ with $(\mathbb{R}^2)^d$ and since the multiplication of two complex numbers $x = x_1 + ix_2$ and $y = y_1 + iy_2$ can be regarded as the (real) matrix-vector multiplication
$$\begin{pmatrix} x_1 & -x_2 \\ x_2 & x_1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \\ y_2 \end{pmatrix},$$
it is easy to see that Lemma 5.9 holds with functions that take values in the complex $k \times d$ matrices as well. Thus, due to that Lemma, we have
$$\int_{-\infty}^{\infty} \widehat{f_n} \, dL \overset{\mathbb{P}}{\to} \int_{-\infty}^{\infty} \widehat{f} \, dL \tag{5.5}$$
as $n \to \infty$. Using (5.4), we know that
$$\int_{-\infty}^{\infty} f_n \, dM = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f_n} \, dL$$
and hence the sequence of simple integrals $\int_{-\infty}^{\infty} f_n \, dM$ converges in probability which shows that $\int_{-\infty}^{\infty} f \, dM$ is well-defined. Moreover, (5.5) immediately yields
$$\int_{-\infty}^{\infty} f \, dM = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f} \, dL. \tag{5.6}$$
We shall call such functions, as in Chapter 4, M-integrable.

- Now define, for any $-\infty < a < b < \infty$,

$$f(\mu) := \frac{e^{i\mu b} - e^{i\mu a}}{i\mu} \quad \text{and} \quad \Phi_\lambda(\mu) := \left(1 - \frac{|\mu|}{\lambda}\right) \mathbb{1}_{[-\lambda,\lambda]}(\mu), \quad \mu \in \mathbb{R}.$$

Completely analogous as in the proof of Theorem 4.21 one can then show that $f \cdot \Phi_\lambda$ is M-integrable (note that the additional boundedness assumption follows directly from Theorem 2.21 (iv)).

- Finally, setting $g(\mu) := \sqrt{2\pi}\mathbb{1}_{[a,b)}(\mu)$ and $h(\mu) := g(-\mu)$, $\mu \in \mathbb{R}$, we obtain $\widehat{h} = f$ and hence (cf. proof of Theorem 4.21)

$$\widehat{f\Phi_\lambda} = F_\lambda * g \overset{L^\delta(\mathbb{C})}{\longrightarrow} g \quad \text{as } \lambda \to \infty$$

for any $1 \leq \delta < \alpha$. Thus, applying once more Lemma 5.9, we deduce

$$\begin{aligned} \int_{-\infty}^{\infty} f\Phi_\lambda \, dM &\overset{(5.6)}{=} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f\Phi_\lambda} \, dL = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} (F_\lambda * g) \, dL \\ &\overset{\mathbb{P}}{\to} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g \, dL = L_b - L_a \quad \text{as } \lambda \to \infty. \end{aligned}$$

□

We get the following properties for the regularly varying random content M (cf. Section 3.3 and 4.4).

Proposition 5.11. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a two-sided Lévy process in $\mathbb{R}^d$ with $\mathbb{E}[L_1] = 0$ and generating triplet (γ, Σ, ν). Assume moreover that $\nu \in RV(\alpha, l, \kappa_\nu)$ for some $\alpha \in (1, 2]$ and let M be the associated regularly varying random content of Theorem 5.6. Then the process*

$$Z_t := M\left([0, t)\right) = \int_{-\infty}^{\infty} \frac{1 - e^{-i\mu t}}{2\pi i\mu} \, L(d\mu), \quad t > 0,$$

is regularly varying with index α.

Furthermore we have, for any $t > 0$, the following results for Z_t and its corresponding Lévy measure ν_{Z_t}:

(i) $\int_{\{\|x\|\leq 1\}} \|x\| \, \nu_{Z_t}(dx) = \infty$ *and thus ν_{Z_t} is in particular infinite.*

(ii) $\mathbb{E}[\|Z_t\|^p] < \infty$ *for any $0 < p < \alpha$ and $\mathbb{E}[\|Z_t\|^p] = \infty$ for any $p > \alpha$.*

(iii) For any $\delta \in (1, \alpha)$ the integral $\int_{\{\|x\|\leq 1\}} \|x\|^\delta \, \nu(dx)$ is finite iff $\int_{\{\|x\|\leq 1\}} \|x\|^\delta \, \nu_{Z_t}(dx)$ is finite, i.e. the Lévy process L has a.s. finite δ-variation if and only if $\Sigma = 0$ and $\int_{\{\|x\|\leq 1\}} \|x\|^\delta \, \nu_{Z_t}(dx) < \infty$. Moreover, if the Lévy process satisfies in addition $\mathbb{E}[\|L_1\|^\alpha] < \infty$, then the statement is also true for $\delta = \alpha$.

(iv) The implication

$$\int_{\{\|x\|\leq 1\}} \|x\|^\delta \, \nu_{Z_t}(dx) < \infty \Rightarrow \int_{\{\|x\|\leq 1\}} \|x\|^\delta \, \nu(dx) < \infty$$

is valid for every $\delta \in (1, 2)$.

Proof. Let $(t_1,\dots,t_m)' \in (0,\infty)^m$ and observe that

$$\begin{pmatrix} Z_{t_1} \\ \vdots \\ Z_{t_m} \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \begin{pmatrix} \int_{-\infty}^{\infty} \widehat{\mathbb{1}_{[0,t_1)}}(\mu)\, L(d\mu) \\ \vdots \\ \int_{-\infty}^{\infty} \widehat{\mathbb{1}_{[0,t_m)}}(\mu)\, L(d\mu) \end{pmatrix} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} g_{t_1,\dots,t_m}(\mu)\, L(d\mu)$$

where $g_{t_1,\dots,t_m} : \mathbb{R} \to M_{md\times d}(\mathbb{C})$ is defined by

$$g_{t_1,\dots,t_m}(\mu) := \begin{pmatrix} \widehat{\mathbb{1}_{[0,t_1)}}(\mu) \\ \vdots \\ \widehat{\mathbb{1}_{[0,t_m)}}(\mu) \end{pmatrix}$$

(cf. proof of Proposition 4.23). Since we obviously have $g_{t_1,\dots,t_m} \in L^\alpha(M_{md\times d}(\mathbb{C}))$ and $\kappa_\nu\left(g^{-1}_{t_1,\dots,t_m}(\mu)(\mathbb{C}^{md}\setminus\{0\})\right) = 0$ does not hold for almost every μ, a simple application of [44, Theorem 3.2] shows that the process $(Z_t)_{t>0}$ is regularly varying of index α which also implies (ii) by virtue of Proposition 5.3. In order to establish the other statements, repeat the proof of Theorem 3.8 with some evident modifications. □

5.3 Spectral representation of multivariate regularly varying CARMA processes

The next lemma provides the analogous results of Lemma 4.24 in the regularly varying setting. It gives more insight into the spectral representation of moving averages of regularly varying Lévy processes and will enable us to state a spectral representation for multivariate regularly varying CARMA processes (in the summability sense).

Lemma 5.12. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process in $\mathbb{R}^d$ with generating triplet (γ,Σ,ν) where $\nu \in RV(\alpha, l, \kappa_\nu)$ with $\alpha \in (1,2]$ and suppose $\mathbb{E}[L_1] = 0$. Let M be the corresponding random content of Theorem 5.6 and assume that $h \in L^1(M_d(\mathbb{C})) \cap L^\alpha(M_d(\mathbb{C}))$ s.t. in addition h is bounded and its Fourier transformation $\widehat{h}$ is of bounded variation on compacta. Define*

$$G_t := \mathbb{P} - \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{i\mu t}\,\widehat{h}(\mu)\left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}.$$

Then, for all $t \in \mathbb{R}$,

$$G_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t-\mu)\, L(d\mu).$$

If $\kappa_\nu\left(h^{-1}(s)(\mathbb{C}^d\setminus\{0\})\right) = 0$ does not hold for almost every s, then the process $(G_t)_{t\in\mathbb{R}}$ is also regularly varying of index α.

Proof. The proof is similar to that of Lemma 4.24. Setting

$$f_{\lambda,t}(\mu) := e^{i\mu t}\,\widehat{h}(\mu)\left(1 - \frac{|\mu|}{\lambda}\right) \mathbb{1}_{[-\lambda,\lambda]}(\mu)$$

one notes that $f_{\lambda,t} \in L^1(M_d(\mathbb{C}))$ and can be approximated in the $L^1(M_d(\mathbb{C}))$-norm by a sequence of $\mathscr{E}(\mathbb{R})$-simple functions f_n satisfying in addition $\widehat{f_n} \to \widehat{f_{\lambda,t}}$ in $L^\delta(M_d(\mathbb{C}))$ as

$n \to \infty$ for some $\delta < \alpha$ (this can be shown in the same way as in the proof of Theorem 4.21 and Theorem 5.6, respectively). Thus $f_{\lambda,t}$ is M-integrable for any $\lambda > 0$ and $t \in \mathbb{R}$.

Then, analogously to the proof of Lemma 4.24, we obtain

$$\int_{-\infty}^{\infty} f_{\lambda,t}\, dM = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \widehat{f_{\lambda,t}}\, dL = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F_\lambda * h(t - \cdot\,)\, dL \xrightarrow{\mathbb{P}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t - \cdot\,)\, dL$$

as $n \to \infty$ for all $t \in \mathbb{R}$, since $F_\lambda * h(t - \cdot\,) \to h(t - \cdot\,)$ in $L^\delta(M_d(\mathbb{C}))$ for any $1 \le \delta < \alpha$ which implies together with the boundedness of $F_\lambda * h(t - \cdot\,)$ and $h(t - \cdot\,)$ the claimed stochastic convergence by virtue of Lemma 5.9.

The additional statement follows from [44, Corollary 3.5]. □

Let us now give both, a spectral representation and a moving average representation for regularly varying MCARMA processes with index $\alpha \in (1, 2]$. We begin with a definition which is at the same time their spectral representation. Note that the well-definedness is ensured by the Lemmata 4.25, 4.26 and 5.12.

Definition 5.13. *Let $L = (L_t)_{t\in\mathbb{R}}$ be a d-dimensional Lévy process with generating triplet (γ, Σ, ν) where $\nu \in RV(\alpha, l, \kappa_\nu)$ with index $\alpha \in (1, 2]$ and let M be the corresponding random content of Theorem 5.6. A d-dimensional regularly varying Lévy-driven continuous time autoregressive moving average process $(Y_t)_{t\in\mathbb{R}}$ of order (p, q) with $p, q \in \mathbb{N}_0$, $p > q$ (**regularly varying MCARMA**(p, q) **process**) of index α is defined as the regularly varying process*

$$Y_t := \mathbb{P} - \lim_{\lambda\to\infty} \int_{-\lambda}^{\lambda} e^{i\mu t} P(i\mu)^{-1} Q(i\mu) \left(1 - \frac{|\mu|}{\lambda}\right) M(d\mu), \quad t \in \mathbb{R}, \quad \textit{where}$$
$$P(z) := \mathrm{I}_d z^p + A_1 z^{p-1} + \ldots + A_p \quad \textit{and}$$
$$Q(z) := B_0 z^q + B_1 z^{q-1} + \ldots + B_q$$

are the autoregressive and moving average polynomial, respectively.
Here $A_i \in M_d(\mathbb{R})$, $i = 1, \ldots, p$, and $B_j \in M_d(\mathbb{R})$, $j = 1, \ldots, q$, are real matrices satisfying $B_0 \neq 0$ and $\mathscr{N}(P) = \{z \in \mathbb{C} : \det(P(z)) = 0\} \subseteq \mathbb{R}\setminus\{0\} + i\mathbb{R}$ and κ_ν is a Radon measure s.t. $\kappa_\nu\left(h^{-1}(s)(\mathbb{C}^d\setminus\{0\})\right) = 0$ does not hold for almost every s, where $h = \widehat{g}(-\cdot\,)$, letting $g = P(i\,\cdot\,)^{-1}Q(i\,\cdot\,)$ (cf. (4.21)).

Lemmata 4.26 and 5.12 together yield immediately the following moving average representation:

Corollary 5.14. *Let $Y = (Y_t)_{t\in\mathbb{R}}$ be a regularly varying MCARMA(p, q) process of index $\alpha \in (1, 2]$, then Y has the moving average representation*

$$Y_t = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} h(t - \mu)\, L(d\mu)$$

for all $t \in \mathbb{R}$, where $h(\mu) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i\mu s} P(is)^{-1} Q(is)\, ds$ is given by (4.21).

Remark 5.15.

(i) One can ask the same question as in Chapter 4 whether this definition (spectral representation) and moving average representation of a regularly varying MCARMA process coincides in the causal case where $\sigma(A) = \mathscr{N}(P) \subseteq (-\infty, 0) + i\mathbb{R}$ with the definition of causal MCARMA processes made in [41] (cf. Definition 3.15). This is indeed still the case and can be established in a completely analogous way as in Chapter 4.

(ii) For $\alpha = 2$ we can distinguish the following two cases: if $\mathbb{E}[\|L_1\|^2] < \infty$, then we are in the setting of Chapter 3 and one can derive a bona fide spectral representation for the driving Lévy and the associated MCARMA process. If L_1 has infinite variance, then the L^2-theory of Chapter 3 is not applicable but we get a spectral representation (in the summability sense) for the driving Lévy and the associated MCARMA process according to Theorem 5.6 and Definition 5.13, respectively.

Chapter 6

Mixing of multivariate CARMA processes

In this chapter we will study mixing properties of the multivariate CARMA processes considered in Chapter 4 and 5. Let us first recall the notions mixing, weakly mixing and ergodic.

Let $(X_t)_{t\in\mathbb{R}}$ be a real-valued strictly stationary process defined on the canonical space $(\mathbb{R}^{\mathbb{R}}, \mathscr{F}, \mathbb{P})$, i.e. $\mathscr{F} = \mathscr{B}(\mathbb{R}^{\mathbb{R}})$. The process $(X_t)_{t\in\mathbb{R}}$ is said to be **ergodic** if

$$\frac{1}{T}\int_0^T \mathbb{P}(A\cap S^t B)\,dt \overset{T\to\infty}{\rightarrow} \mathbb{P}(A)\mathbb{P}(B),$$

weakly mixing if

$$\frac{1}{T}\int_0^T \big|\mathbb{P}(A\cap S^t B) - \mathbb{P}(A)\mathbb{P}(B)\big|\,dt \overset{T\to\infty}{\rightarrow} 0$$

and **mixing** if

$$\mathbb{P}(A\cap S^t B) \overset{t\to\infty}{\rightarrow} \mathbb{P}(A)\mathbb{P}(B) \tag{6.1}$$

where $(S^t)_{t\in\mathbb{R}}$ is the induced group of shift transformations on $\mathbb{R}^{\mathbb{R}}$ (i.e. $S^t(x_s)_{s\in\mathbb{R}} = (x_{s-t})_{s\in\mathbb{R}}$ for any $(x_s)_{s\in\mathbb{R}} \in \mathbb{R}^{\mathbb{R}}$ and $t \in \mathbb{R}$) and $A, B \in \mathscr{F}$. It is obvious that mixing implies weakly mixing and weakly mixing implies ergodic, respectively. However, the inverse implications are not true in general since in ergodic theory there are examples of flows $(S^t)_{t\in\mathbb{R}}$ that are weakly mixing, but not mixing and ergodic, but not weakly mixing, respectively (cf. [19], [45]).

In the same way as in the univariate case we shall call an $\mathbb{R}^d$-valued strictly stationary process $(X_t)_{t\in\mathbb{R}}$ defined on the canonical space $\big((\mathbb{R}^d)^{\mathbb{R}}, \mathscr{F} = \mathscr{B}((\mathbb{R}^d)^{\mathbb{R}}), \mathbb{P}\big)$ mixing if (6.1) holds for any $A, B \in \mathscr{F}$ where $(S^t)_{t\in\mathbb{R}}$ is now the group of shift transformations on $(\mathbb{R}^d)^{\mathbb{R}}$ defined again by $S^t(x_s)_{s\in\mathbb{R}} = (x_{s-t})_{s\in\mathbb{R}}$ for any $t \in \mathbb{R}$ and $(x_s)_{s\in\mathbb{R}} \in (\mathbb{R}^d)^{\mathbb{R}}$.

Usually, the weak mixing property is much closer to mixing than to ergodicity (cf. [51, Proposition 1]). However, in the case of stationary Gaussian processes, weak mixing and ergodicity coincide (see [19, Chapter 14 §2]). Podgórski [46] has shown that the same is true for symmetric stable processes, a result extended by Kokoszka and Podgórski [34] to symmetric semistable processes. Five years later Rosiński and Żak [51, Theorem 1] have even proven the equivalence of weak mixing and ergodicity for general stationary infinitely divisible processes.

The chapter is structured as follows: Section 1 gives a brief summary of important results concerning mixing and ergodic properties of stationary infinitely divisible processes. In particular we recall the characterization of mixing via their Lévy characteristics which dates back to the fundamental paper by Maruyama [42]. In the second section we shall generalize these mixing conditions in terms of Lévy characteristics to multivariate infinitely divisible processes, which is not yet to be found in the literature to the best of our knowledge. The last section will apply these results in order to prove mixing for the multivariate $S\alpha S$ and regularly varying CARMA processes of Chapter 4 and 5, respectively. For causal MCARMA processes with finite r-th moment for some $r > 0$ strong mixing can already be found in [41, Proposition 3.34] (note that strongly mixing implies mixing).

6.1 Results for infinitely divisible processes

This section summarizes some results concerning the properties mixing, weak mixing and ergodicity for real-valued strictly stationary infinitely divisible processes. Recall that a stochastic process is said to be infinitely divisible if all its finite dimensional margins are infinitely divisible.

As mentioned in the introduction to this chapter, the description of the mixing property for such processes can be characterized in terms of their Lévy characteristics, namely, Maruyama [42, Theorem 6] showed that an infinitely divisible strictly stationary process $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

(M1) the covariance function $r(t)$ of its Gaussian part tends to 0 as $t \to \infty$,

(M2) $\lim_{t\to\infty} \nu_{0t}(|xy| > \delta) = 0$ for every $\delta > 0$ and

(M3) $\lim_{t\to\infty} \int_{\{0<x^2+y^2\leq 1\}} xy\, \nu_{0t}(dx, dy) = 0$

where ν_{0t} is the Lévy measure of $\mathscr{L}(X_0, X_t)$.

This result has been improved by [38], where the implication (M2)$\Rightarrow$(M3) has been established.

However, condition (M2) is not very easy to verify even for symmetric stable processes as mentioned in [50]. Recall that the density of a subset D of the positive half-line is defined by $\lim_{C\to\infty} |D \cap [0, C]| /C$ if the limit exists, where $|\cdot|$ denotes the one-dimensional Lebesgue measure. We also recall the following well-known characterization of weak mixing: a stationary stochastic process is weakly mixing iff for any $A, B \in \mathscr{F}$ there is some D, a subset of the density one in $\mathbb{R}_+$, such that $\lim_{t\to\infty,\, t\in D} \mathbb{P}(A \cap S^t B) = \mathbb{P}(A)\mathbb{P}(B)$ (cf. [51, Proposition 1]). The following theorem provides another useful criterion for (weak) mixing of infinitely divisible processes:

Theorem 6.1 (cf. [50], Theorem 1 and 3)**.**
Let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary infinitely divisible process s.t. ν_0, the Lévy measure of $\mathscr{L}(X_0)$, has no atoms in $2\pi\mathbb{Z}$. Then $(X_t)_{t\in\mathbb{R}}$

(i) is mixing if and only if $\lim_{t\to\infty} \mathbb{E}[e^{i(X_t - X_0)}] = \left|\mathbb{E}[e^{iX_0}]\right|^2$ and

(ii) weakly mixing if and only if there is some set D of density one in $\mathbb{R}_+$ such that $\lim_{t\to\infty,\, t\in D} \mathbb{E}[e^{i(X_t - X_0)}] = \left|\mathbb{E}[e^{iX_0}]\right|^2$.

Rosiński and Żak reformulated Theorem 6.1 in terms of the codifference. The latter is defined for jointly infinitely divisible real random variables X_1 and X_2 by

$$\tau(X_1, X_2) := \log \mathbb{E}[e^{i(X_1 - X_2)}] - \log \mathbb{E}[e^{iX_1}] - \log \mathbb{E}[e^{-iX_2}].$$

Hence, the following version of Theorem 6.1 is easily obtained:

Proposition 6.2 (cf. [51], Proposition 4)**.**
Let $(X_t)_{t\in\mathbb{R}}$ be a strictly stationary infinitely divisible process s.t. ν_0, the Lévy measure of $\mathscr{L}(X_0)$, has no atoms in $2\pi\mathbb{Z}$. Then $(X_t)_{t\in\mathbb{R}}$

(i) is mixing if and only if $\tau(t) := \tau(X_0, X_t) \to 0$ as $t \to \infty$ and

(ii) weakly mixing if and only if there exists a set D of the density one in $\mathbb{R}_+$ such that $\tau(t) \to 0$ as $t \to \infty$, $t \in D$.

Since $S\alpha S$ and regularly varying MCARMA processes always possess a moving average representation w.r.t. their driving Lévy processes, we want to recall mixing conditions for stationary infinitely divisible processes of the form

$$\int_{\mathbb{R}} f_t(s)\, L(ds), \quad t \in \mathbb{R},$$

with $f_t(s) : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(s,t) \mapsto f_t(s)$, measurable and $L = (L_t)_{t\in\mathbb{R}}$ a Lévy process in $\mathbb{R}$. We have the following results:

Theorem 6.3 (cf. [50], Theorem 4, 5 and Remark 4)**.**
Let $L = (L_t)_{t\in\mathbb{R}}$ be a Lévy process in $\mathbb{R}$ with generating triplet (γ, σ^2, ν) and let $f_t(s)$: $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$, $(s,t) \mapsto f_t(s)$, be measurable. Assume that

$$X_t := \int_{\mathbb{R}} f_t(s)\, L(ds), \quad t \in \mathbb{R},$$

is a well-defined and strictly stationary infinitely divisible process.

(i) $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

$$\lim_{t\to\infty} \left\{ \left| \int_{\mathbb{R}} f_0(s) f_t(s) \sigma^2\, ds \right| + \int_{\mathbb{R}} \int_{\mathbb{R}} \big(1 \wedge (|f_0(s)x| \cdot |f_t(s)x|)\big)\, \nu(dx)\, ds \right\} = 0.$$

(ii) If in addition $(X_t)_{t\in\mathbb{R}}$ has no Gaussian part, then it is mixing if and only if, for any $\varepsilon > 0$,

$$\lambda^1 \left(\{s \in \mathbb{R} : |f_0(s)| > \varepsilon, |f_t(s)| > \varepsilon\}\right) \to 0 \quad \text{as } t \to \infty.$$

6.2 Generalization to multivariate infinitely divisible processes

This section is dedicated to the generalization of the results in Section 6.1 to multivariate infinitely divisible processes. The extension to the multivariate setting is straightforward, however, since it is not yet to be found in the literature to the best of our knowledge we shall also give the proofs.

We denote the j-th component of an $\mathbb{R}^d$-valued stochastic process $(X_t)_{t\in\mathbb{R}}$ by $\big(X_t^{(j)}\big)_{t\in\mathbb{R}}$. In analogy to Theorem 6.1 we will show the following:

Theorem 6.4. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary infinitely divisible process s.t. ν_0, the Lévy measure of $\mathscr{L}(X_0)$, satisfies*

$$\nu_0\left(\left\{x=(x_1,\dots,x_d)'\in\mathbb{R}^d:\ \exists j\in\{1,\dots,d\}\,,\,x_j\in 2\pi\mathbb{Z}\right\}\right)=0. \tag{6.2}$$

Then $(X_t)_{t\in\mathbb{R}}$ is mixing if and only if

$$\lim_{t\to\infty}\mathbb{E}\left[e^{i\left(X_t^{(j)}-X_0^{(k)}\right)}\right]=\mathbb{E}\left[e^{iX_0^{(j)}}\right]\cdot\mathbb{E}\left[e^{-iX_0^{(k)}}\right] \tag{6.3}$$

for every $j,k=1,\dots,d$.

Proof. We follow the proof of [50, Theorem 1].
"$\Rightarrow$": Let $(X_t)_{t\in\mathbb{R}}$ be mixing which implies

$$\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle+i\langle\theta_2,X_t\rangle}\right]\overset{t\to\infty}{\longrightarrow}\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle}\right]\cdot\mathbb{E}\left[e^{i\langle\theta_2,X_0\rangle}\right]$$

for any $\theta_1,\theta_2\in\mathbb{R}^d$ (see e.g. [19] or [45]) and in particular, setting $(\theta_1,\theta_2)=(-e_k,e_j),\ j,k=1,\dots,d$, with e_j the j-th unit vector in $\mathbb{R}^d$, (6.3) holds.
"$\Leftarrow$": We will prove the converse. Assume that (6.3) holds for every $j,k=1,\dots,d$. Note first that then

$$\mathbb{E}\left[e^{i\left(X_t^{(j)}+X_0^{(k)}\right)}\right]\overset{t\to\infty}{\longrightarrow}\mathbb{E}\left[e^{iX_0^{(j)}}\right]\cdot\mathbb{E}\left[e^{iX_0^{(k)}}\right] \tag{6.4}$$

holds for every $j,k=1,\dots,d$ as well (cf. [50, Theorem 1, Step 1]).

We shall prove the following two conditions:

(M1) the covariance matrix function $\Sigma(t)$ of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$ tends to 0 as $t\to\infty$ and

(M2) $\lim_{t\to\infty}\nu_{0t}(\|x\|\cdot\|y\|>\delta)=0$ for every $\delta>0$

where ν_{0t} is the Lévy measure of $\mathscr{L}(X_0,X_t)$ on $(\mathbb{R}^{2d},\mathscr{B}(\mathbb{R}^{2d}))$. Having established (M1) and (M2), we will conclude with the upcoming Lemma 6.6 which shows that these two conditions imply mixing.

As to (M1), since (X_0,X_t) has a $2d$-dimensional infinitely divisible distribution, its characteristic function can be written, due to the Lévy-Khintchine formula, for every $(\theta_1,\theta_2)\in\mathbb{R}^d\times\mathbb{R}^d$, as

$$\begin{aligned}&\mathbb{E}\left[e^{i\langle\theta_1,X_0\rangle+i\langle\theta_2,X_t\rangle}\right]\\&\quad=\exp\left\{i\langle\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix},\begin{pmatrix}\gamma_1\\\gamma_2\end{pmatrix}\rangle-\frac{1}{2}\langle\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix},\Sigma\begin{pmatrix}\theta_1\\\theta_2\end{pmatrix}\rangle\right.\\&\qquad\left.+\int_{\mathbb{R}^{2d}}e^{i\langle\theta_1,x\rangle+i\langle\theta_2,y\rangle}-1-\left(i\langle\theta_1,x\rangle+i\langle\theta_2,y\rangle\right)\mathbb{1}_{[0,1]}\left(\left\|\left(x',y'\right)'\right\|\right)\nu_{0t}(d(x,y))\right\}\end{aligned} \tag{6.5}$$

where $\gamma_1,\gamma_2\in\mathbb{R}^d$, $\Sigma\in\mathbb{S}_{2d}^+(\mathbb{R})$ and ν_{0t} is the Lévy measure of $\mathscr{L}(X_0,X_t)$ on $(\mathbb{R}^{2d},\mathscr{B}(\mathbb{R}^{2d}))$. Since $\mathscr{L}(X_0)=\mathscr{L}(X_t)$, observe that

$$\Sigma=\begin{pmatrix}\Sigma(0)&\Sigma(t)\\\Sigma(t)'&\Sigma(0)\end{pmatrix} \tag{6.6}$$

with $\Sigma(t)$ being the covariance matrix function of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$. If we denote the generating triplet of $\mathscr{L}(X_0)$ by $(\gamma, \Sigma(0), \nu_0)$, we can use [54, Proposition 11.10] in order to deduce

$$\gamma_1 = \gamma - \int_{\mathbb{R}^{2d}} x\left(\mathbb{1}_{[0,1]}\left(\|x\|\right) - \mathbb{1}_{[0,1]}\left(\left\|\left(x', y'\right)'\right\|\right)\right)\nu_{0t}(d(x,y)) \tag{6.7}$$

and

$$\gamma_2 = \gamma - \int_{\mathbb{R}^{2d}} y\left(\mathbb{1}_{[0,1]}\left(\|y\|\right) - \mathbb{1}_{[0,1]}\left(\left\|\left(x', y'\right)'\right\|\right)\right)\nu_{0t}(d(x,y)). \tag{6.8}$$

Putting the results (6.5)-(6.8) together, the characteristic function of (X_0, X_t) at the point $(\theta_1, \theta_2) \in \mathbb{R}^d \times \mathbb{R}^d$ can be written as

$$\begin{aligned}
&\mathbb{E}\left[e^{i\langle\theta_1, X_0\rangle + i\langle\theta_2, X_t\rangle}\right] \\
&\quad = \exp\Bigg\{ i\langle\theta_1+\theta_2, \gamma\rangle - \frac{1}{2}\Big(\langle\theta_1, \Sigma(0)\theta_1\rangle + 2\langle\theta_1, \Sigma(t)\theta_2\rangle + \langle\theta_2, \Sigma(0)\theta_2\rangle\Big) \\
&\qquad + \int_{\mathbb{R}^{2d}} e^{i\langle\theta_1, x\rangle + i\langle\theta_2, y\rangle} - 1 - i\langle\theta_1, x\rangle\mathbb{1}_{[0,1]}(\|x\|) - i\langle\theta_2, y\rangle\mathbb{1}_{[0,1]}(\|y\|)\,\nu_{0t}(d(x,y))\Bigg\}.
\end{aligned} \tag{6.9}$$

By substituting $(-e_k, e_j)$, $(0, e_j)$ and $(-e_k, 0)$, $j, k = 1, \ldots, d$, for (θ_1, θ_2) in (6.9) we get the description of (6.3) in terms of the covariance matrix function of the Gaussian part and the Lévy measure ν_{0t}, namely

$$\begin{aligned}
&\lim_{t\to\infty} \mathbb{E}\left[e^{i\left(X_t^{(j)} - X_0^{(k)}\right)}\right]\cdot\left(\mathbb{E}\left[e^{iX_t^{(j)}}\right]\cdot\mathbb{E}\left[e^{-iX_0^{(k)}}\right]\right)^{-1} \\
&\qquad = \lim_{t\to\infty}\exp\left\{\sigma_{jk}(t) + \int_{\mathbb{R}^{2d}}\left(e^{i(y^{(j)} - x^{(k)})} - e^{iy^{(j)}} - e^{-ix^{(k)}} + 1\right)\nu_{0t}(d(x,y))\right\} = 1
\end{aligned}$$

for arbitrary $j, k = 1, \ldots, d$, where $\sigma_{jk}(t)$ is the (j,k)-th element of $\Sigma(t)$ and $x^{(k)}$ and $y^{(j)}$ denote the k-th and j-th element of x and y, respectively. Next, taking logarithms of both sides and using the identity

$$\operatorname{Re}\left(e^{i(y-x)} - e^{iy} - e^{-ix} + 1\right) = (\cos x - 1)(\cos y - 1) + \sin x \sin y$$

we obtain

$$\lim_{t\to\infty}\sigma_{jk}(t) + \int_{\mathbb{R}^{2d}}\left((\cos x^{(k)} - 1)(\cos y^{(j)} - 1) + \sin x^{(k)}\sin y^{(j)}\right)\nu_{0t}(d(x,y)) = 0 \tag{6.10}$$

for any $j, k = 1, \ldots, d$.

Starting with (6.4) and using the same arguments as above, we get

$$\lim_{t\to\infty} -\sigma_{jk}(t) + \int_{\mathbb{R}^{2d}}\left((\cos x^{(k)} - 1)(\cos y^{(j)} - 1) - \sin x^{(k)}\sin y^{(j)}\right)\nu_{0t}(d(x,y)) = 0 \tag{6.11}$$

for every $j, k = 1, \ldots, d$.

Adding (6.10) and (6.11) yields, due to the consistency of Lévy measures (cf. [54, Proposition 11.10]),

$$\begin{aligned}
&\lim_{t\to\infty}\int_{\mathbb{R}^{2d}}(\cos x^{(k)} - 1)(\cos y^{(j)} - 1)\,\nu_{0t}(d(x,y)) \\
&\qquad\qquad = \lim_{t\to\infty}\int_{\mathbb{R}^2}(\cos x - 1)(\cos y - 1)\,\nu_{0t}^{(jk)}(dx, dy) = 0
\end{aligned} \tag{6.12}$$

for every $j,k = 1,\dots,d$, where $\nu_{0t}^{(jk)}$ denotes the Lévy measure of $\mathscr{L}\big(X_0^{(k)}, X_t^{(j)}\big)$ on $(\mathbb{R}^2, \mathscr{B}(\mathbb{R}^2))$.

Now we use a tightness argument. Fix $j,k \in \{1,\dots,d\}$ and observe first that the family $\big\{\mathscr{L}\big(X_0^{(k)}, X_t^{(j)}\big)\big\}_{t\in\mathbb{R}}$ is tight. Indeed, letting $B_r := \{(x,y)\in\mathbb{R}^2 : x^2+y^2 \le r^2\}$, we have by stationarity

$$\begin{aligned}\mathbb{P}\left(\left(X_0^{(k)}, X_t^{(j)}\right) \notin B_r\right) &\le \mathbb{P}\left(\left|X_0^{(k)}\right|^2 > \frac{r^2}{2}\right) + \mathbb{P}\left(\left|X_t^{(j)}\right|^2 > \frac{r^2}{2}\right)\\ &= \mathbb{P}\left(\left|X_0^{(k)}\right|^2 > \frac{r^2}{2}\right) + \mathbb{P}\left(\left|X_0^{(j)}\right|^2 > \frac{r^2}{2}\right)\end{aligned}$$

and hence $\lim\limits_{r\to\infty}\sup\limits_{t\in\mathbb{R}} \mathbb{P}\left(\left(X_0^{(k)}, X_t^{(j)}\right) \notin B_r\right) = 0$. Thus, due to the Theorem of Prohorov, the family is relatively compact (in the topology of weak convergence). Choose any sequence $\tau_n \to \infty$, $\tau_n \in \mathbb{R}$, and let F_{jk} be an accumulation point of $\big\{\mathscr{L}\big(X_0^{(k)}, X_{\tau_n}^{(j)}\big)\big\}_{n\in\mathbb{N}}$. Then F_{jk} is an infinitely divisible distribution on $\mathbb{R}^2$ with some Lévy measure ν_{jk} (cf. [54, Lemma 7.8]). Now let $(t_n)_{n\in\mathbb{N}}$ be a subsequence of $(\tau_n)_{n\in\mathbb{N}}$ s.t.

$$\mathscr{L}\left(X_0^{(k)}, X_{t_n}^{(j)}\right) \xrightarrow{w} F_{jk} \quad \text{as } n\to\infty. \tag{6.13}$$

Then, for every $\delta > 0$ with $\nu_{jk}(\partial B_\delta) = 0$,

$$\nu_{0t_n}^{(jk)}\Big|_{B_\delta^c} \xrightarrow{w} \nu_{jk}\big|_{B_\delta^c} \quad \text{as } n\to\infty \tag{6.14}$$

(see for instance Lemma 5.10 or [3]). Since $(\cos x - 1)(\cos y - 1) \ge 0$, we deduce

$$\begin{aligned}0 &\le \int_{B_\delta^c} (\cos x-1)(\cos y-1)\,\nu_{jk}(dx,dy) \overset{(6.14)}{=} \lim_{n\to\infty}\int_{B_\delta^c}(\cos x-1)(\cos y-1)\,\nu_{0t_n}^{(jk)}(dx,dy)\\ &\le \lim_{n\to\infty}\int_{\mathbb{R}^2}(\cos x-1)(\cos y-1)\,\nu_{0t_n}^{(jk)}(dx,dy) \overset{(6.12)}{=} 0.\end{aligned}$$

Since δ can be taken arbitrarily small we infer that every Lévy measure ν_{jk} is concentrated on $\{(x,y)\in\mathbb{R}^2 : x\in 2\pi\mathbb{Z} \text{ or } y \in 2\pi\mathbb{Z}\}$.

By the stationarity of the process and (6.13), the projection of ν_{jk} onto the first and second axis coincide with $\nu_0^{(k)}$ and $\nu_0^{(j)}$, respectively, on the complement of every neighborhood of zero. Hence, by our assumption (6.2) on ν_0, for every $m\in\mathbb{Z}$, $m\neq 0$,

$$\begin{aligned}\nu_{jk}\left(\{2\pi m\}\times\mathbb{R}\right) &= \nu_0^{(k)}\left(\{2\pi m\}\right) = \nu_0(\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{k-1}\times\{2\pi m\}\times\underbrace{\mathbb{R}\times\ldots\times\mathbb{R}}_{d-k})\\ &\le \nu_0\left(\left\{x\in\mathbb{R}^d : \exists l\in\{1,\dots,d\},\, x_l\in 2\pi\mathbb{Z}\right\}\right) = 0\end{aligned}$$

and similarly $\nu_{jk}\left(\mathbb{R}\times\{2\pi m\}\right) = 0$. This shows that every ν_{jk}, $j,k = 1,\dots,d$, is actually concentrated on the axes of $\mathbb{R}^2$ and on each of them coincides with $\nu_0^{(k)}$ and $\nu_0^{(j)}$, respectively.

Now, observe that, for every $t\in\mathbb{R}$,

$$\begin{aligned}\int_{B_\delta}|xy|\,\nu_{0t}^{(jk)}(dx,dy) &\le \frac{1}{2}\int_{B_\delta}(x^2+y^2)\,\nu_{0t}^{(jk)}(dx,dy)\\ &\le \frac{1}{2}\int_{\{|x|\le\delta\}} x^2\,\nu_0^{(k)}(dx) + \frac{1}{2}\int_{\{|y|\le\delta\}} y^2\,\nu_0^{(j)}(dy) < \varepsilon \end{aligned}\tag{6.15}$$

for any positive ε and any $j,k=1,\dots,d$, if only δ is small enough. Then (6.15) yields, for every $j,k=1,\dots,d$,

$$\int_{B_\delta} |\sin x \sin y| \, \nu_{0t_n}^{(jk)}(dx,dy) \le \int_{B_\delta} |xy| \, \nu_{0t_n}^{(jk)}(dx,dy) < \varepsilon$$

for sufficiently small $\delta > 0$ and any n. Since every ν_{jk} is concentrated on the axes of $\mathbb{R}^2$, (6.14) implies that $\lim_{n\to\infty} \int_{B_\delta^c} \sin x \sin y \, \nu_{0t_n}^{(jk)}(dx,dy) = 0$. Thus

$$\lim_{n\to\infty} \int_{\mathbb{R}^2} \sin x \sin y \, \nu_{0t_n}^{(jk)}(dx,dy) = 0 \tag{6.16}$$

for every $j,k=1,\dots,d$.

From (6.10), (6.12) and (6.16) we infer that $\sigma_{jk}(t_n) \to 0$ as $n \to \infty$ for all $j,k = 1,\dots,d$. Since (t_n) is a subsequence of an arbitrary sequence $\tau_n \to \infty$, it follows that $\sigma_{jk}(t) \to 0$ as $t \to \infty$ and thus $\Sigma(t) \to 0$ as $t \to \infty$, i.e. (M1) holds.

To prove (M2), observe that, for any $n \in \mathbb{N}$,

$$\begin{aligned} \nu_{0t_n}\big(\underbrace{|x|^2 \cdot |y|^2}_{=\sum_{j,k=1}^d \left(x^{(k)}y^{(j)}\right)^2} > \delta^2 \big) &\le \sum_{j,k=1}^d \nu_{0t_n}^{(jk)} \left(\left(x^{(k)}y^{(j)}\right)^2 > \frac{\delta^2}{d^2} \right) \\ &\le \sum_{j,k=1}^d \nu_{0t_n}^{(jk)} \left(\left|x^{(k)}y^{(j)}\right| \ge \frac{\delta}{d} \right). \end{aligned}$$

Since

$$\limsup_{n\to\infty} \ \nu_{0t_n}^{(jk)} \left(\left|x^{(k)}y^{(j)}\right| \ge \frac{\delta}{d} \right) \overset{(6.14)}{\le} \nu_{jk} \left(\left|x^{(k)}y^{(j)}\right| \ge \frac{\delta}{d} \right) = 0$$

for every $j,k = 1,\dots,d$, we deduce $\lim_{n\to\infty} \nu_{0t_n}(|x| \cdot |y| > \delta) = 0$ for every $\delta > 0$. Again, since (t_n) is a subsequence of any arbitrary sequence $\tau_n \to \infty$, it follows that $\lim_{t\to\infty} \nu_{0t}(|x| \cdot |y| > \delta) = 0$ for every $\delta > 0$ and thus also $\lim_{t\to\infty} \nu_{0t}(\|x\| \cdot \|y\| > \delta) = 0$ for any $\delta > 0$ and any norm $\|\cdot\|$, i.e. (M2) is shown.

As already mentioned at the beginning of the proof, we can now conclude with the upcoming Lemma 6.6. □

In order to establish Lemma 6.6 we need the following multivariate generalization of [38, Lemma 1].

Lemma 6.5.
Assume that $\lim_{t\to\infty} \nu_{0t}(\|x\| \cdot \|y\| > \delta) = 0$ *for every* $\delta > 0$. *Then one also has*

(M3) $\lim_{t\to\infty} \int_{\{0<\|x\|^2+\|y\|^2\le 1\}} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) = 0.$

Proof. Fix $\varepsilon > 0$ and define for any $\delta \in (0,1)$ the sets

$$B_\delta := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d : \|x\|^2 + \|y\|^2 \le \delta^2\right\}$$

and $R_\delta := B_1 \backslash B_\delta$. Then, for every $\delta \in (0,1)$,

$$\begin{aligned} &\int_{\{0<\|x\|^2+\|y\|^2\le 1\}} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) \\ &\qquad = \int_{B_\delta} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) + \int_{R_\delta} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) =: I_1 + I_2. \end{aligned}$$

Taking advantage of stationarity of $(X_t)_{t\in\mathbb{R}}$, we obtain

$$\begin{aligned}|I_1| &\leq \frac{1}{2}\int_{B_\delta} \|x\|^2 + \|y\|^2 \, \nu_{0t}(d(x,y)) \\ &\leq \frac{1}{2}\left(\int_{\{\|x\|\leq\delta\}} \|x\|^2 \, \nu_{0t}(d(x,y)) + \int_{\{\|y\|\leq\delta\}} \|y\|^2 \, \nu_{0t}(d(x,y))\right) \\ &= \int_{\{\|x\|\leq\delta\}} \|x\|^2 \, \nu_0(dx) \leq \frac{\varepsilon}{2}\end{aligned}$$

for every δ sufficiently small.

We fix such a δ and set $l := \min\{\delta/2, \varepsilon/8q\}$ with $q := \nu_0\left(\left\{\|x\|^2 > \delta^2/2\right\}\right) < \infty$ and $C := R_\delta \cap \{\|x\| \cdot \|y\| > l\}$. Then

$$\begin{aligned}|I_2| &= \int_C \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) + \int_{R_\delta\backslash C} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) \leq \frac{1}{2}\nu_{0t}(C) + \frac{\varepsilon}{8q}\nu_{0t}(R_\delta\backslash C) \\ &\leq \frac{1}{2}\nu_{0t}(C) + \frac{\varepsilon}{8q}\left[\nu_{0t}\left(\|x\|^2 > \frac{\delta^2}{2}\right) + \nu_{0t}\left(\|y\|^2 > \frac{\delta^2}{2}\right)\right] \leq \frac{1}{2}\nu_{0t}\left(\|x\| \cdot \|y\| > l\right) + \frac{\varepsilon}{4}.\end{aligned}$$

Since $\nu_{0t}\left(\|x\| \cdot \|y\| > l\right) \leq \varepsilon/2$ if only t is large enough, we obtain

$$\int_{\left\{0<\|x\|^2+\|y\|^2\leq 1\right\}} \|x\| \cdot \|y\| \, \nu_{0t}(d(x,y)) \leq \varepsilon$$

for sufficiently large t. Letting $\varepsilon \searrow 0$, we obtain the desired result. □

The next lemma shows that conditions (M1) and (M2) together imply mixing and thus concludes the proof of Theorem 6.4.

Lemma 6.6.
Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary infinitely divisible process. Assume that

(M1) the covariance matrix function $\Sigma(t)$ of its Gaussian part tends to 0 as $t \to \infty$ and

(M2) $\lim_{t\to\infty} \nu_{0t}\left(\|x\| \cdot \|y\| > \delta\right) = 0$ for every $\delta > 0$.

Then the process $(X_t)_{t\in\mathbb{R}}$ is mixing.

Proof. Note first that, due to Lemma 6.5, also condition (M3) holds. We follow the proof of Theorem 6 in [42]. We shall denote $X_\tau = (X'_{s_1}, \ldots, X'_{s_m})'$ for any $\tau = (s_1, \ldots, s_m)' \in \mathbb{R}^m$. Then (cf. [42]) it is sufficient for $(X_t)_{t\in\mathbb{R}}$ to be mixing that for all $\tau = (s_1, \ldots, s_m)'$, $\mu = (u_1, \ldots, u_m)' \in \mathbb{R}^m$ and $z_1, z_2 \in \mathbb{R}^{md}$,

$$\lim_{t\to\infty} \mathbb{E}\left[e^{i\langle z_1, X_\tau\rangle + i\langle z_2, X_{\mu+t}\rangle}\right] = \mathbb{E}\left[e^{i\langle z_1, X_\tau\rangle}\right] \cdot \mathbb{E}\left[e^{i\langle z_2, X_\mu\rangle}\right] \tag{6.17}$$

where $\mu + t = (u_1 + t, \ldots, u_m + t)'$.

The family of $\mathbb{R}^{2md}$-valued infinitely divisible random vectors $\{(X_\tau, X_{\mu+t})\}_{t\in\mathbb{R}}$ is tight. Therefore let $B_r := \left\{x \in \mathbb{R}^{2md} : |x|^2 \leq r^2\right\}$. Then

$$\begin{aligned}\mathbb{P}\left((X_\tau, X_{\mu+t}) \notin B_{\sqrt{2mr}}\right) &\leq \sum_{j=1}^m \mathbb{P}\left(\left|X_{s_j}\right| > r\right) + \mathbb{P}\left(\left|X_{u_j+t}\right| > r\right) \\ &= 2m \cdot \mathbb{P}\left(|X_0| > r\right) \to 0\end{aligned}$$

as $r \to \infty$. Hence the family is relatively compact w.r.t. the weak topology (i.e. the topology generated by weak convergence).

Let $(\gamma^1, \Sigma^1, \nu^1)$ and $(\gamma^2, \Sigma^2, \nu^2)$ be the characteristic triplets of $\mathscr{L}(X_\tau)$ and $\mathscr{L}(X_\mu)$, respectively. Consider an arbitrary sequence $\eta_n \in \mathbb{R}$, $\eta_n \to \infty$ and an accumulation point F of the associated sequence $\{\mathscr{L}(X_\tau, X_{\mu+\eta_n})\}_{n\in\mathbb{N}}$ as $n \to \infty$, i.e. there is a subsequence $(t_n)_{n\in\mathbb{N}}$ of $(\eta_n)_{n\in\mathbb{N}}$ s.t.

$$\mathscr{L}(X_\tau, X_{\mu+t_n}) \xrightarrow{w} F \quad \text{as } n \to \infty$$

where the accumulation point F is obviously (cf. [54, Lemma 7.8]) an infinitely divisible distribution on $\mathbb{R}^{2md}$ with some generating triplet (γ, Σ, ν). We denote by $(\gamma_n, \Sigma_n, \nu_n)$ the characteristic triplet of $\mathscr{L}(X_\tau, X_{\mu+t_n})$ for any $n \in \mathbb{N}$ and by $\Phi_n(z_1, z_2)$ its characteristic function at the point $(z_1, z_2) \in \mathbb{R}^{md} \times \mathbb{R}^{md}$. The logarithm of Φ_n can be written (cf. proof of Theorem 6.4) as

$$\begin{aligned}
\log \Phi_n(z_1, z_2) &= i\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \begin{pmatrix} \gamma^1 \\ \gamma^2 \end{pmatrix} \rangle - \frac{1}{2}\langle \begin{pmatrix} z_1 \\ z_2 \end{pmatrix}, \Sigma_n \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rangle \\
&+ \int_{\{|x|<\delta,\, |y|<\delta\}} e^{i\langle z_1, x\rangle + i\langle z_2, y\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2, y\rangle \mathbb{1}_{[0,1]}(\|y\|)\, \nu_n(d(x,y)) \\
&+ \int_{\{|x|\geq\delta \text{ or } |y|\geq\delta\}} e^{i\langle z_1, x\rangle + i\langle z_2, y\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2, y\rangle \mathbb{1}_{[0,1]}(\|y\|)\, \nu_n(d(x,y)) \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned}$$

We shall prove that $\log \Phi_n(z_1, z_2) \to \log \Phi_1(z_1) + \log \Phi_2(z_2)$ as $n \to \infty$ for all $z_1, z_2 \in \mathbb{R}^{md}$ where Φ_1 and Φ_2 are the characteristic functions of X_τ and X_μ, respectively.

Obviously $I_1 = i\langle z_1, \gamma^1\rangle + i\langle z_2, \gamma^2\rangle$ and due to the assumption (M1) the second term I_2 converges to $-1/2\langle z_1, \Sigma^1 z_1\rangle - 1/2\langle z_2, \Sigma^2 z_2\rangle$ as $n \to \infty$.

As to I_4, we have (cf. (6.14))

$$\begin{aligned}
I_4 &\overset{n\to\infty}{\to} \int_{\{|x|\geq\delta \text{ or } |y|\geq\delta\}} e^{i\langle z_1, x\rangle + i\langle z_2, y\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|) - i\langle z_2, y\rangle \mathbb{1}_{[0,1]}(\|y\|)\, \nu(d(x,y)) \\
&= \int_{\{|x|\geq\delta\}} e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\, \nu^1(dx) \\
&\qquad + \int_{\{|y|\geq\delta\}} e^{i\langle z_2, y\rangle} - 1 - i\langle z_2, y\rangle \mathbb{1}_{[0,1]}(\|y\|)\, \nu^2(dy)
\end{aligned}$$

since, letting $x = \left(x^{(1)'}, \ldots, x^{(m)'}\right)' \in \left(\mathbb{R}^d\right)^m$ and $y = \left(y^{(1)'}, \ldots, y^{(m)'}\right)' \in \left(\mathbb{R}^d\right)^m$,

$$\begin{aligned}
\nu\left(|x| \cdot |y| > \delta\right) &\leq \liminf_{n\to\infty}\ \nu_n\left(|x| \cdot |y| > \delta\right) \\
&\leq \liminf_{n\to\infty} \sum_{j,k=1}^{m} \nu_{0,u_k - s_j + t_n}\left(\left|x^{(j)}\right| \cdot \left|y^{(k)}\right| > \frac{\delta}{m}\right) \\
&\overset{(M2)}{=} 0
\end{aligned}$$

for any $\delta > 0$ which shows in particular that $\nu(|x| \cdot |y| > 0) = 0$.

Analogously to x and y we denote the $\mathbb{R}^d$-components of z_1 and z_2 by $z_1^{(j)}$ and $z_2^{(j)}$, respectively. Concerning I_3, a simple Taylor expansion yields for any $\delta > 0$ small enough

$$I_3 = -\frac{1}{2}\Bigg[\int_{\{|x|<\delta,\,|y|<\delta\}} \left(\sum_{j=1}^{m}\langle z_1^{(j)}, x^{(j)}\rangle\right)^2 + \left(\sum_{j=1}^{m}\langle z_2^{(j)}, y^{(j)}\rangle\right)^2 \nu_n(d(x,y))$$

$$+ 2\int_{\{|x|<\delta,\,|y|<\delta\}} \left(\sum_{j,k=1}^{m}\langle z_1^{(j)}, x^{(j)}\rangle\langle z_2^{(k)}, y^{(k)}\rangle\right) \nu_n(d(x,y))\Bigg] + R$$

with

$$6\,|R| \leq \int_{\{|x|<\delta,\,|y|<\delta\}} |\langle z_1, x\rangle + \langle z_2, y\rangle|^3 + o\left(\left(|x|^2+|y|^2\right)^{3/2}\right) \nu_n(d(x,y))$$

$$\leq \left|\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right|^3 \cdot \sqrt{2} \cdot \int_{\{|x|<\delta,\,|y|<\delta\}} \left|\begin{pmatrix} x \\ y \end{pmatrix}\right|^3 \nu_n(d(x,y))$$

$$\leq \left|\begin{pmatrix} z_1 \\ z_2 \end{pmatrix}\right|^3 \cdot 2\delta \cdot \left(\int_{\{0<|x|<\delta\}} |x|^2\,\nu^1(dx) + \int_{\{0<|y|<\delta\}} |y|^2\,\nu^2(dy)\right)$$

and thus $6\,|R| < \varepsilon$ for any positive ε if only δ is small enough. Moreover, for again sufficiently small $\delta > 0$ and every $j, k = 1, \ldots, m$,

$$\int_{\{|x|<\delta,\,|y|<\delta\}} \left|\langle z_1^{(j)}, x^{(j)}\rangle\langle z_2^{(k)}, y^{(k)}\rangle\right| \nu_n(d(x,y))$$

$$\leq \left|z_1^{(j)}\right| \cdot \left|z_2^{(k)}\right| \cdot \int_{\{|x|<\delta,\,|y|<\delta\}} \left|x^{(j)}\right| \cdot \left|y^{(k)}\right| \nu_n(d(x,y))$$

$$\leq \left|z_1^{(j)}\right| \cdot \left|z_2^{(k)}\right| \cdot \int_{\left\{0<\|x^{(j)}\|^2+\|y^{(k)}\|^2\leq 1\right\}} \left|x^{(j)}\right| \cdot \left|y^{(k)}\right| \nu_{0,u_k-s_j+t_n}\left(d\left(x^{(j)}, y^{(k)}\right)\right)$$

$$\overset{n\to\infty}{\longrightarrow} 0$$

by virtue of (M3). Finally

$$\left|\frac{1}{2}\int_{\{|x|<\delta,\,|y|<\delta\}} \langle z_1, x\rangle^2\,\nu_n(d(x,y)) + \int_{\{0<|x|<\delta\}} e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\,\nu^1(dx)\right|$$

$$\leq J_1 + J_2$$

with

$$J_1 = \left|\frac{1}{2}\int_{\{0<|x|<\delta,\,|y|<\delta\}} \langle z_1, x\rangle^2\,\nu_n(d(x,y)) - \frac{1}{2}\int_{\{0<|x|<\delta\}} \langle z_1, x\rangle^2\,\nu_n(d(x,y))\right|$$

$$\leq \int_{\{0<|x|<\delta,\,|y|\geq\delta\}} \langle z_1, x\rangle^2\,\nu_n(d(x,y)) \leq |z_1|^2 \cdot \int_{\{0<|x|<\delta\}} |x|^2\,\nu^1(dx)$$

and

$$J_2 = \left|\frac{1}{2}\int_{\{0<|x|<\delta\}} \langle z_1, x\rangle^2\,\nu_n(d(x,y)) + \int_{\{0<|x|<\delta\}} e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\,\nu^1(dx)\right|$$

$$= \left|\int_{\{0<|x|<\delta\}} \frac{1}{2}\langle z_1, x\rangle^2 + e^{i\langle z_1, x\rangle} - 1 - i\langle z_1, x\rangle \mathbb{1}_{[0,1]}(\|x\|)\,\nu^1(dx)\right|$$

$$\leq |z_1|^3\,\delta \cdot \int_{\{0<|x|<\delta\}} |x|^2\,\nu^1(dx).$$

An analogous result is obviously true for the second addend of the first term of I_3.

Putting all this together we obtain

$$\lim_{n\to\infty} \log \Phi_n(z_1, z_2) = \log \Phi_1(z_1) + \log \Phi_2(z_2) \quad \text{for all } z_1, z_2 \in \mathbb{R}^{md}$$

and thus the desired result in (6.17) which completes the proof. □

From the foregoing results we can derive the following corollary:

Corollary 6.7. *Let $(X_t)_{t\in\mathbb{R}}$ be an $\mathbb{R}^d$-valued strictly stationary infinitely divisible process. Then, with the previous notation, $(X_t)_{t\in\mathbb{R}}$ is mixing iff*

$$\lim_{t\to\infty} \left\{ \|\Sigma(t)\| + \int_{\mathbb{R}^{2d}} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) \right\} = 0. \tag{6.18}$$

Proof. Obviously (6.18) implies (M1) and (M2) and thus, due to Lemma 6.6, (6.18) implies mixing.

Conversely, if $(X_t)_{t\in\mathbb{R}}$ is mixing, then by investigating again Theorem 6.4 and Lemma 6.6 (and their proofs) we know that (M1) holds. Moreover (cf. (6.14))

$$\nu_{0t}^{(jk)}\Big|_{B_\delta^c} \xrightarrow{w} \nu_{jk}|_{B_\delta^c} \quad \text{as } t \to \infty \tag{6.19}$$

for every $\delta > 0$ s.t. $\nu_{jk}(\partial B_\delta) = 0$ and any $j, k = 1, \ldots, d$. From the proof of Theorem 6.4 we further know that the Lévy measures ν_{jk} are concentrated on the axes of $\mathbb{R}^2$. Now choose $\delta > 0$ s.t. (6.15) and (6.19) hold, then we have

$$\limsup_{t\to\infty} \int_{\mathbb{R}^2} (1 \wedge |xy|)\, \nu_{0t}^{(jk)}(dx, dy) \le \varepsilon + \limsup_{t\to\infty} \int_{B_\delta^c} (1 \wedge |xy|)\, \nu_{0t}^{(jk)}(dx, dy) = \varepsilon.$$

Letting $\varepsilon \searrow 0$ we deduce $\lim_{t\to\infty} \int_{\mathbb{R}^2} (1 \wedge |xy|)\, \nu_{0t}^{(jk)}(dx, dy) = 0$ for any $j, k = 1, \ldots, d$. Finally

$$\begin{aligned} \int_{\mathbb{R}^{2d}} \left(1 \wedge \sum_{k=1}^d |x_k| \cdot \sum_{j=1}^d |y_j|\right) \nu_{0t}(d(x,y)) &\le \sum_{j,k=1}^d \int_{\mathbb{R}^{2d}} (1 \wedge |x_k y_j|)\, \nu_{0t}(d(x,y)) \\ &= \sum_{j,k=1}^d \int_{\mathbb{R}^2} (1 \wedge |xy|)\, \nu_{0t}^{(jk)}(dx, dy) \overset{t\to\infty}{\longrightarrow} 0. \end{aligned}$$

This clearly implies $\lim_{t\to\infty} \int_{\mathbb{R}^{2d}} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) = 0$ as well and hence (6.18) is shown. □

In view of an application to MCARMA processes in the upcoming section, we now consider a stationary infinitely divisible process of the form

$$\int_{\mathbb{R}} f_t(s)\, L(ds), \quad t \in \mathbb{R},$$

where $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process in $\mathbb{R}^d$ and $f_t(s) : \mathbb{R} \to M_{k\times d}(\mathbb{R})$, $(s,t) \mapsto f_t(s)$, is measurable (cf. end of Section 6.1 for the univariate mixing results in such a case).

Theorem 6.8. *Let $(X_t)_{t\in\mathbb{R}} \overset{\mathscr{D}}{=} \left(\int_{\mathbb{R}} f_t(s)\, L(ds)\right)_{t\in\mathbb{R}}$ be a strictly stationary infinitely divisible process where $L = (L_t)_{t\in\mathbb{R}}$ is a Lévy process in $\mathbb{R}^d$ with generating triplet (γ, Σ, ν) and $f_t(s) : \mathbb{R} \to M_{k\times d}(\mathbb{R})$, $(s,t) \mapsto f_t(s)$, is measurable. Then $(X_t)_{t\in\mathbb{R}}$ is mixing iff*

$$\lim_{t\to\infty} \left\{ \left\| \int_{\mathbb{R}} f_0(s)\Sigma f_t(s)' ds \right\| + \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f_0(s)x\| \cdot \|f_t(s)x\|)\, \nu(dx)\, ds \right\} = 0.$$

Proof. Since we can write

$$\begin{pmatrix} X_0 \\ X_t \end{pmatrix} = \int_{\mathbb{R}} \begin{pmatrix} f_0(s) \\ f_t(s) \end{pmatrix} L(ds), \quad t \in \mathbb{R},$$

we immediately obtain the covariance matrix function of the Gaussian part of $(X_t)_{t\in\mathbb{R}}$ (cf. (2.8)) by

$$\Sigma(t) = \int_{\mathbb{R}} f_0(s)\Sigma f_t(s)' ds, \quad t \in \mathbb{R}.$$

The Lévy measure ν_{0t} of $\mathscr{L}(X_0, X_t)$ is given (cf. (2.9)) by

$$\nu_{0t}(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A(f_0(s)x, f_t(s)x)\, \nu(dx)\, ds, \quad A \in \mathscr{B}(\mathbb{R}^{2k}_*).$$

Thus

$$\int_{\mathbb{R}^{2k}} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f_0(s)x\| \cdot \|f_t(s)x\|)\, \nu(dx)\, ds$$

and Corollary 6.7 completes the proof. □

6.3 Application to MCARMA processes

Let us now consider a $S\alpha S$ or a regularly varying MCARMA(p,q) process $(Y_t)_{t\in\mathbb{R}}$ which can be represented as the moving average

$$Y_t = \int_{-\infty}^{\infty} f(t-s)\, L(ds), \quad t \in \mathbb{R}, \tag{6.20}$$

with kernel function

$$f(s) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{i\mu s} P(i\mu)^{-1} Q(i\mu)\, d\mu, \quad s \in \mathbb{R},$$

where P and Q are the autoregressive and moving average polynomial, respectively. The Lévy process L is accordingly supposed to be $S\alpha S$ or regularly varying.

Applying [1, Theorem 4.3.16] immediately yields strict stationarity of $S\alpha S$ and regularly varying MCARMA processes. The moving averages in (6.20) are in particular infinitely divisible processes. We get the following analogue to [41, Proposition 3.27]:

Proposition 6.9. *If the driving Lévy process has generating triplet (γ, Σ, ν), then the distribution of the MCARMA process Y_t in (6.20) is infinitely divisible for all $t \in \mathbb{R}$ and the generating triplet of the stationary distribution is $(\gamma_Y^\infty, \Sigma_Y^\infty, \nu_Y^\infty)$ where*

$$\gamma_Y^\infty = \int_{\mathbb{R}} f(s)\gamma\, ds + \int_{\mathbb{R}} \int_{\mathbb{R}^d} f(s)x \left(\mathbb{1}_{\{\|f(s)x\|\le 1\}} - \mathbb{1}_{\{\|x\|\le 1\}}\right) \nu(dx)\, ds,$$

$$\Sigma_Y^\infty = \int_{\mathbb{R}} f(s)\Sigma f(s)' ds$$

and

$$\nu_Y^\infty(A) = \int_{\mathbb{R}} \int_{\mathbb{R}^d} \mathbb{1}_A(f(s)x)\, \nu(dx)\, ds, \quad A \in \mathscr{B}(\mathbb{R}^d_*).$$

We use Theorem 6.8 in order to derive the mixing property (and thus also weak mixing and ergodicity) for the MCARMA processes in (6.20). Observe first (see (4.21)) that

$$\|f(t-s)\| \le C$$

for some $C > 0$ and any $t, s \in \mathbb{R}$. Then, for some $c > 0$,

$$\left\| \int_{\mathbb{R}} f(-s)\Sigma f(t-s)' ds \right\| \le c\, \|\Sigma\| \cdot \int_{\mathbb{R}} \|f(-s)\| \cdot \|f(t-s)\|\, ds \overset{t\to\infty}{\to} 0$$

due to the Dominated Convergence Theorem since the integrand converges pointwise to 0 as $t \to \infty$ and can be bounded by $C \cdot \|f(-s)\|$ which is an integrable majorant. The second condition in Theorem 6.8, namely

$$\int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f(-s)x\| \cdot \|f(t-s)x\|)\, \nu(dx)\, ds \overset{t\to\infty}{\to} 0, \tag{6.21}$$

is verified in the following lemma. Hence, Theorem 6.8 yields the mixing property for $S\alpha S$ and regularly varying MCARMA processes.

Lemma 6.10.
The $S\alpha S$ (or alternatively regularly varying) MCARMA process satisfies (6.21).

Proof. First note that, for any $\varepsilon > 0$,

$$\lambda^1 \left(\{s \in \mathbb{R} :\ \|f(-s)\| > \varepsilon,\ \|f(t-s)\| > \varepsilon\}\right) \overset{t\to\infty}{\to} 0. \tag{6.22}$$

Indeed, for any $\varepsilon > 0$ there is some $n_\varepsilon \in \mathbb{N}$ such that

$$\{s \in \mathbb{R} :\ \|f(-s)\| > \varepsilon\} \subseteq [-n_\varepsilon, n_\varepsilon]$$

since $f(-s) \to 0$ as $s \to \pm\infty$. Thus, for any $t > 2n_\varepsilon$,

$$\begin{aligned} &\lambda^1(\{s \in \mathbb{R} : \|f(-s)\| > \varepsilon,\ \|f(t-s)\| > \varepsilon\}) \\ &\qquad \le \lambda^1\left(\{s \in [-n_\varepsilon, n_\varepsilon] :\ \|f(t-s)\| > \varepsilon\}\right) = 0. \end{aligned}$$

Now fix an arbitrary $\varepsilon > 0$. We set $B_r := \left\{(x,y) \in \mathbb{R}^d \times \mathbb{R}^d :\ \|x\|^2 + \|y\|^2 \le r^2\right\}$. Note that in order to establish (6.14) we did not use the assumption that the process is mixing. Hence there is some $R > 1$ and some $t_0 > 0$ s.t.

$$\sup_{t \ge t_0} \nu_{0t}(B_R^c) \le \varepsilon.$$

Thus, for any $t \ge t_0$,

$$\begin{aligned} \int_{\mathbb{R}} \int_{\mathbb{R}^d} (1 \wedge \|f(-s)x\| \cdot \|f(t-s)x\|)\, \nu(dx)\, ds &= \int_{\mathbb{R}^{2d}} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) \\ &\le \int_{B_R} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) + \varepsilon. \end{aligned}$$

Then, for any $\delta > 0$,

$$
\begin{aligned}
&\int_{B_R} (1 \wedge \|x\| \cdot \|y\|)\, \nu_{0t}(d(x,y)) \\
&\quad = \int_{\{\|f(-s)\|\le\delta\}\cup\{\|f(t-s)\|\le\delta\}} \int_{\{(f(-s)x, f(t-s)x)\in B_R\}} (1 \wedge \|f(-s)x\| \cdot \|f(t-s)x\|)\, \nu(dx)\, ds \\
&\qquad + \int_{\{\|f(-s)\|>\delta,\, \|f(t-s)\|>\delta\}} \int_{\{(f(-s)x, f(t-s)x)\in B_R\}} (1 \wedge \|f(-s)x\| \cdot \|f(t-s)x\|)\, \nu(dx)\, ds \\
&\quad =: I_1 + I_2.
\end{aligned}
$$

Since

$$
\min\{\|u\| \cdot \|v\|, 1\} \le R \cdot \min\{\|u\|, 1\} \cdot \min\{\|v\|, 1\},
$$

provided that $\max\{\|u\|, \|v\|\} \le R$, we obtain

$$
\begin{aligned}
I_1 &\le R \cdot \int_{\{\|f(-s)\|\le\delta\}\cup\{\|f(t-s)\|\le\delta\}} \int_{\mathbb{R}^d} (1 \wedge \|f(-s)x\|) \cdot (1 \wedge \|f(t-s)x\|)\, \nu(dx)\, ds \\
&\le R \cdot \left(\int_{\{\|f(-s)\|\le\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(t-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\qquad + R \cdot \left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{\{\|f(t-s)\|\le\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(t-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&= R \cdot \left(\int_{\{\|f(-s)\|\le\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\qquad\qquad \times \left\{ \left(\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu_t(dy) \right)^{1/2} + \left(\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu_0(dy) \right)^{1/2} \right\} \\
&= 2R \cdot \left(\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu_0(dy) \right)^{1/2} \cdot \left(\int_{\{\|f(-s)\|\le\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\le \varepsilon
\end{aligned}
$$

if only δ is sufficiently small. Now fix such a δ and consider I_2. Analogously, one obtains

$$
\begin{aligned}
I_2 &\le R \cdot \left(\int_{\{\|f(-s)\|>\delta,\, \|f(t-s)\|>\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{\mathbb{R}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(t-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2} \\
&= R \cdot \left(\int_{\mathbb{R}^d} \left(1 \wedge \|y\|^2\right) \nu_0(dy) \right)^{1/2} \\
&\qquad\qquad \times \left(\int_{\{\|f(-s)\|>\delta,\, \|f(t-s)\|>\delta\}} \int_{\mathbb{R}^d} \left(1 \wedge \|f(-s)x\|^2\right) \nu(dx)\, ds \right)^{1/2}
\end{aligned}
$$

where the right-hand side converges to 0 as $t \to \infty$ by virtue of (6.22) and thus (6.21) is shown. $\square$

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