eqf21/008
Heavy Tails in Insurance

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Abstract

Large insurance losses happen infrequently, but they do happen. In this paper we present the standard distribution models used in fire, wind–storm or flood insurance and mention some insurance applications.

Keywords: compound Poisson process, Cramér-Lundberg model, integrated tail distribution, Pollaczek-Khinchine formula, regular variation, risk model, ruin probability, subexponential distribution.

In the class of heavy–tailed distribution functions, *subexponential* distribution functions are a special class which have just the right level of generality for risk measurement in insurance and finance models. The name arises from one of their properties, that their right tail decreases more slowly than any exponential tail. This implies that large values can occur in a sample with non–negligible probability, which proposes the subexponential distribution functions as natural candidates for situations, where extremely large values occur in a sample compared to the mean size of the data. Such a pattern is often seen in insurance data, for instance in fire, wind–storm or flood insurance (collectively known as catastrophe insurance), but also in data from finance and communications engineering, see for example [1, 6, 7]. Subexponential insurance claims can account for large fluctuations in the risk process of a company. Textbook accounts are in [2, 4, 6–8].

We present two defining properties of subexponential distribution functions. Let $(X_k)_{k\in\mathbb{N}}$ be i. i. d. positive random variables with distribution function F such

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that $F(x) < 1$ for all $x > 0$. Denote by $\overline{F}(x) = 1 - F(x)$ for $x \geq 0$, the tail of F, and for $n \in \mathbb{N}$,

$$\overline{F^{n*}}(x) = 1 - F^{n*}(x) = \mathbb{P}(X_1 + \cdots + X_n > x), \quad x \geq 0,$$

the tail of the n–fold convolution of F. We then say that F (or X) is subexponential (written $F \in \mathcal{S}$) if one of the following equivalent conditions holds:

(a) $\displaystyle\lim_{x \to \infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$ for some (all) $n \geq 2$,

(b) $\displaystyle\lim_{x \to \infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(\max(X_1, \ldots, X_n) > x)} = 1$ for some (all) $n \geq 2$.

The heavy-tailedness of $F \in \mathcal{S}$ is demonstrated by the implications

$$F \in \mathcal{S} \quad \Longrightarrow \quad \lim_{x \to \infty} \frac{\overline{F}(x - y)}{\overline{F}(x)} = 1 \quad \forall\, y \in \mathbb{R} \tag{1}$$

$$\Longrightarrow \quad \overline{F}(x)/\mathrm{e}^{-\varepsilon x} \xrightarrow{x \to \infty} \infty \quad \forall\, \varepsilon > 0. \tag{2}$$

A famous subclass of $\mathcal{S}$ is the class of distribution functions with regularly varying tails; see [3]. For a positive measurable function f we write $f \in \mathcal{R}(\alpha)$ for $\alpha \in \mathbb{R}$ (f is *regularly varying with index α*) if

$$\lim_{x \to \infty} \frac{f(tx)}{f(x)} = t^\alpha \quad \forall\, t > 0. \tag{3}$$

Let $\overline{F} \in \mathcal{R}(-\alpha)$ for $\alpha \geq 0$, then it has the representation

$$\overline{F}(x) = x^{-\alpha} \ell(x), \quad x > 0,$$

for some $\ell \in \mathcal{R}(0)$.

For regularly varying distribution tails we can check (a) for $n = 2$ by splitting the convolution integral and use partial integration to obtain

$$\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2 \int_0^{x/2} \frac{\overline{F}(x - y)}{\overline{F}(x)}\, dF(y) + \frac{(\overline{F}(x/2))^2}{\overline{F}(x)}, \quad x > 0.$$

Immediately, by (3), the last term tends to 0. The integrand satisfies $\overline{F}(x - y)/\overline{F}(x) \leq \overline{F}(x/2)/\overline{F}(x)$ for $0 \leq y \leq x/2$; hence, Lebesgue dominated convergence applies and, since F satisfies (1), the integral on the right hand side tends to 1 as $x \to \infty$. Examples of distribution functions with regularly varying tail include the Pareto, Burr, transformed beta (also called generalized F), log-gamma and stable distribution functions (see Table 1.2.6 in [4]).

Also the lognormal, the two Benktander families and the heavy-tailed Weibull (shape parameter less than 1) belong to $\mathcal{S}$; see again Table 1.2.6 in [4]. However, a direct proof of this is more difficult than for the regularly varying case. Subexponentiality is typically established via an integral test for the hazard rate known as *Pitman's criterion*, see [2] pp. 256–257..

The two main examples in insurance risk theory which uses the tail properties of subexponential distributions are aggregated claims tails and ruin probabilities. The *aggregated claims* denoted by A (eqf21/013) are defined as the sum of all claims to the insurance company in a given period. The usual model is

$$A = X_1 + \ldots + X_N$$

where N is the number of claims and $X_1, X_2, \ldots$ are the *claim sizes* in the given time interval ($N, (X_k)_{k \in \mathbb{N}}$ are assumed to be independent and $(X_k)_{k \in \mathbb{N}}$ is an i.i.d. sequence). The tail of A and the associated quantiles are important for assessing the probability of big losses and for Value-at-Risk (eqf15/004) calculations. The main result in the heavy-tailed case states that if the X_k are subexponential, then ($a(u) \sim b(u)$ as $u \to \infty$ means that $\lim_{u \to \infty} a(u)/b(u) = 1$)

$$\mathbb{P}(A > x) \ \sim \ \mathbb{E}(N)\mathbb{P}(X_1 > x), \quad \text{as } x \to \infty, \tag{4}$$

provided in addition $\mathbb{E}(z^N) < \infty$ for some $z > 1$ (cf. [2], Chapter IX, Lemma 2.2).

The classical insurance risk model is the *Cramér-Lundberg model* (cf. [2,4,6,8] and ef21/001), where the *claim times* occur at the jump times of a Poisson(λ) process $(N(t))_{t \geq 0}$ (eqf02/006) and the claims $(X_k)_{k \in \mathbb{N}}$ are again an i.i.d. sequence with finite mean. The *risk process* is for *initial reserve* $u \geq 0$ and *premium rate* $c > 0$ defined as

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k, \quad t \geq 0. \tag{5}$$

Then the *ruin probability* $\psi(u)$ *in infinite time* is the probability that R ever falls below 0, i.e. $\psi(u) = \mathbb{P}(\inf_{t \geq 0} R(t) < 0)$. Define the *integrated tail distribution function* F_I by

$$F_I(x) = \frac{1}{\mathbb{E}(X_1)} \int_0^x \overline{F}(y) dy, \quad x \geq 0. \tag{6}$$

If F_I is subexponential (which is satisfied for all subexponential distributions F mentioned above under the condition that they have a finite mean) and $\rho = \lambda \mathbb{E}(X_1)/c < 1$, then

$$\psi(u) \ \sim \ \frac{\rho}{1 - \rho} \int_u^\infty \overline{F_I}(y) \, dy, \quad \text{as } u \to \infty, \tag{7}$$

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a result that is often associated with the names of (in alphabetical order) Borovkov, Cohen, Embrechts, Pakes, Veraverbeke and von Bahr.

That F_I plays a role may be understood from the fact that F_I is the distribution function of the first undershoot of $R(t) - u$ below 0 under the condition that $R(t)$ falls below u in finite time. The number M of times, where $R(t)$ achieves a new local minimum in finite time plus 1, is geometrically distributed with parameter $(1 - \rho)$, i.e. $\mathbb{P}(M = n) = (1 - \rho)\rho^n$, $n \in \mathbb{N}_0$. This leads easily to the *Beekman-Bowers-Pollaczek-Khinchine formula*

$$\psi(u) = (1 - \rho) \sum_{n=1}^{\infty} \rho^n \overline{F_I^{n*}}(u), \quad u \geq 0\,.$$

from which (7) is an easy consequence.

For further aspects of ruin theory with heavy tails and a comprehensive set of recent references, see [5].

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