Multivariate Models for Operational Risk

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Abstract

In Böcker and Klüppelberg (2005) we presented a simple approximation of Op-VaR of a single operational risk cell. The present paper derives approximations of similar quality and simplicity for the multivariate problem. Our approach is based on modelling of the dependence structure of different cells via the new concept of a Lévy copula.

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1 Introduction

The Basel II accord [4], which should be fully implemented by year-end 2007, imposes new methods of calculating regulatory capital that apply to the banking industry. Besides credit risk, the new accord focuses on operational risk, defined as the risk of losses resulting from inadequate or failed internal processes, people and systems, or from external events. Choosing the advanced measurement approach (AMA), banks can use their own internal modelling technique based on bank-internal and external empirical data.

A required feature of AMA is to allow for explicit correlations between different operational risk events. More precisely, according to Basel II banks should allocate losses to one of eight business lines and to one of seven loss event types. Therefore, the core problem here is the multivariate modelling encompassing all different risk type/business

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line cells. For this purpose, we consider a d-dimensional compound Poisson process $S = (S_1(t), S_2(t), \dots, S_d(t))_{t\geq 0}$ with cadlag (right continuous with left limits) sample paths. Each component has the representation

$$S_i(t) = \sum_{k=1}^{N_i(t)} X_k^i \,, \quad t \geq 0 \,,$$

where $N_i = (N_i(t))_{t\geq 0}$ is a Poisson process with rate $\lambda_i > 0$ (loss frequency) and $(X_k^i)_{k\in\mathbb{N}}$ is an iid sequence of positive random variables (loss severities), independent of the Poisson process N_i. The bank's total operational risk is then given by the stochastic process

$$S^+(t) := S_1(t) + S_2(t) + \cdots + S_d(t) \,, \quad t \geq 0 \,.$$

Note that S^+ is again a compound Poisson process; cf. Proposition 3.2.

A fundamental question is how the dependence structure between different cells affects the bank's total operational risk.

The present literature suggests to model dependence by introducing correlation between the Poisson processes (see e.g. Aue & Kalkbrenner [1], Bee [2], Frachot, Roncalli & Salomon [13], or Powojowski, Reynolds and Tuenter [21]), or by using a distributional copula on the random time points where operational loss occurs, or on the number of operational risk events (see Chavez-Demoulin, Embrechts and Nešlehová [10]). In all these approaches, each cell's severities are assumed to be independent and identically distributed (iid) as well as independent of the frequency process. A possible dependence between severities has to be modelled separately, yielding in the end to a rather complicated model. Given the fact that statistical fitting of a high-parameter model seems out of reach by the sparsity of the data, a simpler model is called for.

Our approach has the advantage of modelling dependence in frequency and severity at the same time yielding a model with comparably few parameters. Consequently, with a rather transparent dependence model, we are able to model coincident losses occurring in different cells. From a mathematical point of view, in contrast to the models proposed in Chavez-Demoulin et al. [10], we stay within the class of multivariate Lévy processes, a class of stochastic processes, which has been well studied also in the context of derivatives pricing; see e.g. Cont and Tankov [11].

Since operational risk is only concerned with losses, we restrict ourselves to Lévy processes admitting only positive jumps in every component, hereafter called *spectrally positive Lévy processes*. As a consequence of their independent and stationary increments, Lévy processes can be represented by the *Lévy-Khintchine formula*, which for a d-dimensional spectrally positive Lévy processes S without drift and Gaussian component simplifies to

$$E(e^{i\,(z,S_t)}) = \exp\left\{t \int_{\mathbb{R}_+^d} (e^{i(z,x)} - 1)\,\Pi(dx)\right\}, \quad z \in \mathbb{R}^d\,,$$

where Π is a measure on $\mathbb{R}_+^d = [0,\infty)^d$, called the *Lévy measure* of S and $(x,y) := \sum_{i=1}^d x_i y_i$ for $x, y \in \mathbb{R}^d$ denotes the inner product.

Whereas the dependence structure in a Gaussian model is well-understood, dependence in the Lévy measure Π is much less obvious. Nevertheless, as Π is independent of t, it suggests itself for modelling the dependence structure between the components of S. Such an approach has been suggested and investigated in Cont and Tankov [11], Kallsen and Tankov [15] and Barndorff-Nielsen and Lindner [3], and essentially models dependence between the jumps of different Lévy processes by means of so-called *Lévy copulas*.

In this paper we invoke Lévy copulas to model the dependence between different operational risk cells. This allows us to gain deep insight into the multivariate behaviour of operational risk defined as a high quantile of a loss distribution and referred to as *operational VaR* (OpVaR). In certain cases, we obtain closed-form approximations for OpVaR and, in this respect, this paper can be regarded as a multivariate extension of Böcker and Klüppelberg [6], where univariate OpVaR has been investigated.

Our paper is organised as follows. After stating the problem and reviewing the state of the art of operational risk modelling in the introduction, we present in Section 2 the necessary concepts and recall the results for the single cell model. In Section 3.1 we formulate the multivariate model and give the basic results, which we shall exploit later for the different dependence concepts. The total operational risk process is compound Poisson and we give the parameters explicitly, which results in the asymptotic form for total OpVaR. Before doing this we present in Section 3.2 asymptotic results for the OpVaR, when the losses of one cell dominate all the others. In Sections 3.3 and 3.4 we examine the cases of completely dependent and independent cells, respectively, and derive asymptotic closed-form expressions for the corresponding bank's total OpVaR. In doing so, we show that for very heavy-tailed data completely dependent OpVaR, which is asymptotically simply the sum of the single cell VaRs, is even smaller than for independent OpVaR. As a more general multivariate model we investigate in Section 3.5 the compound Poisson model with regularly varying Lévy measure. This covers the case of single cell processes, whose loss distributions are of the same order and have rather arbitrary dependece structure. This dependence structure, manifests in the so-called spectral measure, which carries the same information of dependence as the Lévy copula.

2 Preliminaries

2.1 Lévy Processes, Tail Integrals, and Lévy Copulas

Distributional copulas are multivariate distribution functions with uniform marginals. They are used for dependence modelling within the context of Sklar's theorem, which states that any multivariate distribution with continuous marginals can be transformed into a multivariate distribution with uniform marginals. This concept exploits the fact that distribution functions have values only in $[0, 1]$. In contrast, Lévy measures are in general unbounded on $\mathbb{R}^d$ and may have a non-integrable singularity at 0, which causes problems for the copula idea. Within the class of spectrally positive compound Poisson models, the Lévy measure of the cell process S_i is given by $\Pi_i([0, x)) = \lambda_i P(X^i \le x)$ for $x \in [0, \infty)$. It follows that the Lévy measure is a finite measure with total mass $\Pi_i([0, \infty)) = \lambda_i$ and, therefore, is in general not a probability measure. Since we are interested in extreme operational losses, we prefer (as is usual in the context of general Lévy process theory) to define a copula for the tail integral. Although we shall mainly work with compound Poisson processes, we formulate definitions and some results and examples for the slightly more general case of spectrally positive Lévy processes.

Definition 2.1. [Tail integral] *Let X be a spectrally positive Lévy process in $\mathbb{R}^d$ with Lévy measure Π. Its* tail integral *is the function $\overline{\Pi} : [0, \infty]^d \to [0, \infty]$ satisfying for $x = (x_1, \dots, x_d)$,*

(1) $\overline{\Pi}(x) = \Pi([x_1, \infty) \times \cdots \times [x_d, \infty))\,, \quad x \in [0, \infty)^d\,,$
where $\overline{\Pi}(0) = \lim_{x_1 \downarrow 0, \dots, x_d \downarrow 0} \Pi([x_1, \infty) \times \cdots \times [x_d, \infty))$
(this limit is finite if and only if X is compound Poisson);

(2) $\overline{\Pi}$ is equal to 0, if one of its arguments is ∞;

(3) $\overline{\Pi}(0, \dots, x_i, 0, \dots, 0) = \overline{\Pi}_i(x_i)$ for $(x_1, \dots, x_d) \in \mathbb{R}_+^d$, where $\overline{\Pi}_i(x_i) = \Pi_i([x_i, \infty))$ is the tail integral of component i.

Definition 2.2. [Lévy copula] *A d-dimensional* Lévy copula *of a spectrally positive Lévy process is a measure defining function $\widehat{C} : [0, \infty]^d \to [0, \infty]$ with marginals, which are the identity functions on $[0, \infty]$.*

The following is Sklar's theorem for spectrally positive Lévy processes.

Theorem 2.3. [Cont and Tankov [11], Theorem 5.6] *Let $\overline{\Pi}$ denote the tail integral of a d-dimensional spectrally positive Lévy process, whose components have Lévy measures $\Pi_1, \dots, \Pi_d$. Then there exists a Lévy copula $\widehat{C} : [0, \infty]^d \to [0, \infty]$ such that for* *all* $x_1,\ldots,x_d\in[0,\infty]$

$$\overline{\Pi}(x_1,\ldots,x_d)=\widehat{C}(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d)). \tag{2.1}$$

If the marginal tail integrals $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are continuous, then this Lévy copula is unique. Otherwise, it is unique on* $\mathrm{Ran}\overline{\Pi}_1\times\cdots\times\mathrm{Ran}\overline{\Pi}_d$.
Conversely, if $\widehat{C}$ *is a Lévy copula and* $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are marginal tail integrals of spectrally positive Lévy processes, then* (2.1) *defines the tail integral of a d-dimensional spectrally positive Lévy process and* $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$ *are tail integrals of its components.*

The following two important Lévy copulas model extreme dependence structures.

Example 2.4. [Complete (positive) dependence]
Let $S(t)=(S_1(t),\ldots,S_d(t))$, $t\ge 0$, be a spectrally positive Lévy process with marginal tail integrals $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$. Since all jumps are positive, the marginal processes can never be negatively dependent. Complete dependence corresponds to a Lévy copula

$$\widehat{C}_{\|}(x)=\min(x_1,\ldots,x_d)\,,$$

implying for the tail integral of S

$$\overline{\Pi}(x_1,\ldots,x_d)=\min(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d))$$

with all mass concentrated on $\{x\in[0,\infty)^d:\overline{\Pi}_1(x_1)=\cdots=\overline{\Pi}_d(x_d)\}$. □

Example 2.5. [Independence]
Let $S(t)=(S_1(t),\ldots,S_d(t))$, $t\ge 0$, be a spectrally positive Lévy process with marginal tail integrals $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$. The marginal processes are independent if and only if the Lévy measure Π of S can be decomposed into

$$\Pi(A)=\Pi_1(A_1)+\cdots+\Pi_d(A_d),\qquad A\in[0,\infty)^d \tag{2.2}$$

with $A_1=\{x_1\in[0,\infty):(x_1,0,\ldots,0)\in A\},\ldots,A_d=\{x_d\in[0,\infty):(0,\ldots,x_d)\in A\}$. Obviously, the support of Π are the coordinate axes. Equation (2.2) implies for the tail integral of S

$$\overline{\Pi}(x_1,\ldots,x_d)=\overline{\Pi}_1(x_1)\,1_{x_2=\cdots=x_d=0}+\cdots+\overline{\Pi}_d(x_d)\,1_{x_1=\cdots=x_{d-1}=0}\,.$$

It follows that the independence copula for spectrally positive Lévy processes is given by

$$\widehat{C}_{\perp}(x)=x_1\,1_{x_2=\cdots=x_d=\infty}+\cdots+x_d\,1_{x_1=\cdots=x_{d-1}=\infty}\,.$$

□

2.2 Subexponentiality and Regular Variation

As in Böcker and Klüppelberg [6], we work within the class of subexponential distributions to model high severity losses. For more details on subexponential distributions and related classes see Embrechts et al. [12], Appendix A3.

Definition 2.6. [Subexponential distributions] *Let* $(X_k)_{k\in\mathbb{N}}$ *be iid random variables with distribution function* F*. Then* F *(or sometimes* $\overline{F}$*) is said to be a* subexponential distribution function ($F \in \mathcal{S}$) *if*

$$\lim_{x\to\infty} \frac{P(X_1 + \cdots + X_n > x)}{P(\max(X_1, \ldots, X_n) > x)} = 1 \quad \textit{for some (all) } n \geq 2.$$

The interpretation of subexponential distributions is therefore that their iid sum is likely to be very large because of one of the terms being very large. The attribute *subexponential* refers to the fact that the tail of a subexponential distribution decays slower than any exponential tail, i.e. the class $\mathcal{S}$ consists of heavy-tailed distributions and is therefore appropriate to describe typical operational loss data. Important subexponential distributions are Pareto, lognormal and Weibull (with shape parameter less than 1).

As a useful semiparametric class of subexponential distributions, we introduce distributions, whose far out right tails behave like a power function. We present the definition for arbitrary functions, since we shall need this property not only for distribution tails, but also for quantile functions as e.g. in Proposition 2.14.

Definition 2.7. [Regularly varying functions] *Let* f *be a positive measureable function. If for some* $\alpha \in \mathbb{R}$

$$\lim_{x\to\infty} \frac{f(xt)}{f(x)} = t^{-\alpha}\,, \quad t > 0\,, \tag{2.3}$$

then f *is called* regularly varying with index $-\alpha$.

Here we consider loss variables X whose distribution tails are regularly varying.

Definition 2.8. [Regularly varying distribution tails] *Let* X *be a positive random variable with distribution tail* $\overline{F}(x) := 1 - F(x) = P(X > x)$ *for* $x > 0$*. If for* $\overline{F}$ *relation* (2.3) *holds for some* $\alpha \geq 0$*, then* X *is called* regularly varying with index $-\alpha$ *and denoted by* $\overline{F} \in \mathcal{R}_{-\alpha}$*. The quantity* α *is also called the* tail index *of* F*.*
Finally we define $\mathcal{R} := \cup_{\alpha\geq 0}\mathcal{R}_{-\alpha}$*.*

Remark 2.9. (a) As already mentioned, $\mathcal{R} \subset \mathcal{S}$.
(b) Regularly varying distribution functions have representation $\overline{F}(x) = x^{-\alpha}L(x)$ for $x \geq 0$, where L is a *slowly varying function* ($L \in \mathcal{R}_0$) satisfying $\lim_{x\to\infty} L(xt)/L(x) = 1$

for all $t > 0$. Typical examples are functions, which converge to a positive constant or are logarithmic as e.g. $L(\cdot) = \ln(\cdot)$.
(c) The classes $\mathcal{S}$ and $\mathcal{R}_{-\alpha}$, $\alpha \geq 0$, are closed with respect to *tail-equivalence*, which for two distribution functions (or also tail integrals) is defined as $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = c$ for $c \in (0, \infty)$.
(d) We introduce the notation $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$, meaning that the quotient of right-hand and left-hand side tends to 1; i.e. $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = 1$.
(e) In Definition 2.8 we have used a functional approach to regular variation. Alternatively, regular variation can be reformulated in terms of vague convergence of the underlying probability measures, and this turns out to be very useful when we consider in Section 3.5 below multivariate regular variation; see e.g. Resnick [23], Chapter 3.6. This measure theoretical approach will be used in Section 3.4 to define multivariate regularly varying Lévy measures. □

Distributions in $\mathcal{S}$ but not in $\mathcal{R}$ include the heavy-tailed Weibull distribution and the lognormal distribution. Their tail decreases faster than tails in $\mathcal{R}$, but less fast than an exponential tail. The following definition will be useful.

Definition 2.10. [Rapidly varying distribution tails] *Let X be a positive random variable with distribution tail $\overline{F}(x) := 1 - F(x) = P(X > x)$ for $x > 0$. If*

$$\lim_{x\to\infty} \frac{\overline{F}(xt)}{\overline{F}(x)} = \begin{cases} 0\,, & \text{if } t > 1\,, \\ \infty & \text{if } 0 < t < 1\,. \end{cases}$$

then $\overline{F}$ is called rapidly varying, *denoted by $\overline{F} \in \mathcal{R}_\infty$.*

2.3 Recalling the Single Cell Model

Now we are in the position to introduce an LDA model based on subexponential severities. We begin with the univariate case. Later, when we consider multivariate models, each of its d operational risk processes will follow the univariate model defined below.

Definition 2.11. [Subexponential compound Poisson (SCP) model]

(1) The severity process.
The severities $(X_k)_{k\in\mathbb{N}}$ are positive iid random variables with distribution function $F \in \mathcal{S}$ describing the magnitude of each loss event.

(2) The frequency process.
The number $N(t)$ of loss events in the time interval $[0, t]$ for $t \geq 0$ is random, where $(N(t))_{t\geq 0}$ is a homogenous Poisson process with intensity $\lambda > 0$. In particular,

$$P(N(t) = n) = p_t(n) = e^{-\lambda t} \frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0\,.$$

(3) *The severity process and the frequency process are assumed to be independent.*

(4) The aggregate loss process.
The aggregate loss $S(t)$ in $[0,t]$ constitutes a process

$$S(t) = \sum_{k=1}^{N(t)} X_k\,, \quad t \geq 0\,.$$

Of main importance in the context of operational risk is the *aggregate loss distribution function*, given by

$$G_t(x) = P(S(t) \leq x) = \sum_{n=0}^{\infty} p_t(n)\, F^{n*}(x), \quad x \geq 0, \quad t \geq 0\,, \tag{2.4}$$

with

$$p_t(n) = P(N_t = n) = e^{-\lambda t}\,\frac{(\lambda t)^n}{n!}, \quad n \in \mathbb{N}_0\,,$$

and $F(\cdot) = P(X_k \leq \cdot)$ is the distribution function of X_k, and $F^{n*}(\cdot) = P(\sum_{k=1}^{n} X_k \leq \cdot)$ is the n-fold convolution of F with $F^{1*} = F$ and $F^{0*} = I_{[0,\infty)}$.

Now, OpVaR is just a quantile of G_t. The following defines the OpVaR of a single cell process, the so-called *stand alone* VaR.

Definition 2.12. [Operational VaR (OpVaR)] *Suppose G_t is a loss distribution function according to eq. (2.4). Then,* operational VaR *up to time t at confidence level κ,* $\mathrm{VaR}_t(\kappa)$, *is defined as its κ-quantile*

$$\mathrm{VaR}_t(\kappa) = G_t^{\leftarrow}(\kappa)\,, \quad \kappa \in (0,1)\,,$$

where $G_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R}: \; G_t(x) \geq \kappa\}$, $0 < \kappa < 1$, *is the (left continuous) generalized inverse of G_t. If G_t is strictly increasing and continuous, we may write* $\mathrm{VaR}_t(\kappa) = G_t^{-1}(\kappa)$.

In general, $G_t(\kappa)$—and thus also OpVaR—cannot be analytically calculated so that one depends on techniques like Panjer recursion, Monte Carlo simulation, and fast Fourier transform (FFT), see e.g. Klugman et al. [17]. Recently, based on the asymptotic identity $\overline{G}_t(x) \sim \lambda\, t \overline{F}(x)$ as $x \to \infty$ for subexponential distributions, Böcker and Klüppelberg [6] have shown that for a wide class of LDA models closed-form approximations for OpVaR at high confidence levels are available. For a more natural definition in the context of high quantiles we express $\mathrm{VaR}_t(\kappa)$ in terms of the tail $\overline{F}(\cdot)$ instead of $F(\cdot)$. This can easily be achieved by noting that $1/\overline{F}$ is increasing, hence,

$$F^{\leftarrow}(\kappa) \;=\; \inf\{x \in \mathbb{R}: \; F(x) \geq \kappa\} \;=: \left(\frac{1}{\overline{F}}\right)^{\leftarrow}\left(\frac{1}{1-\kappa}\right), \quad 0 < \kappa < 1\,. \tag{2.5}$$

In [6] we have shown that

$$G_t^{\leftarrow}(\kappa) = F^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda t}(1+o(1))\right), \quad \kappa \uparrow 1, \tag{2.6}$$

or, equivalently using (2.5),

$$\left(\frac{1}{\overline{G}_t}\right)^{\leftarrow}\left(\frac{1}{1-\kappa}\right) = \left(\frac{1}{\overline{F}}\right)^{\leftarrow}\left(\frac{\lambda t}{1-\kappa}(1+o(1))\right), \quad \kappa \uparrow 1.$$

In the present paper we shall restrict ourselves to situations, where the right-hand side of (2.6) is asymptotically equivalent to $F^{\leftarrow}\big(1 - \frac{1-\kappa}{\lambda t}\big)$ as $\kappa \uparrow 1$. That this is not always the case for $F \in \mathcal{S}$ shows the following example.

Example 2.13. Consider $(1/\overline{F})^{\leftarrow}(y) = \exp(y + y^{1-\varepsilon})$ for some $0 < \varepsilon < 1$ with $y = 1/(1-\kappa)$, i.e. $\kappa \uparrow 1$ equivalent to $y \to \infty$. Then $(1/\overline{F})^{\leftarrow}(y) = \exp(y(1+o(1))$, but $(1/\overline{F})^{\leftarrow}(y)/e^y = \exp(y^{1-\varepsilon}) \to \infty$ as $y \to \infty$. This situation typically occurs, when $\overline{F} \in \mathcal{R}_0$, i.e. for extremely heavy-tailed models. □

The reason is given by the following equivalences, which we will often use throughout this paper. We present a short proof, which can be ignored by those readers interested mainly in the OpVar application.

Proposition 2.14. *(1)* [Regular variation] *Let $\alpha > 0$. Then*
(i) $\overline{F} \in \mathcal{R}_{-\alpha} \quad \Leftrightarrow \quad (1/\overline{F})^{\leftarrow} \in \mathcal{R}_{1/\alpha}$,
(ii) $\overline{F}(x) = x^{-\alpha}L(x)$ *for* $x \geq 0 \quad \Leftrightarrow \quad (1/\overline{F})^{\leftarrow}(z) = z^{1/\alpha}\widetilde{L}(z)$ *for* $z \geq 0$,
where L and $\widetilde{L}$ are slowly varying functions,
(iii) $\overline{F}(x) \sim \overline{G}(x)$ *as* $x \to \infty \quad \Leftrightarrow \quad (1/\overline{F})^{\leftarrow}(z) \sim (1/\overline{G})^{\leftarrow}(z)$ *as* $z \to \infty$.

(2) [Rapid variation] *If $\overline{F}, \overline{G} \in \mathcal{R}_\infty$ such that $\overline{F}(x) \sim \overline{G}(x)$ as $x \to \infty$, then $(1/\overline{F})^{\leftarrow}(z) \sim (1/\overline{G})^{\leftarrow}(z)$ as $z \to \infty$.*

Proof. (1) Proposition 1.5.15 of Bingham, Goldie and Teugels [5] ensures that regular variation of $1/\overline{F}$ is equivalent to regular variation of its (generalised) inverse and provides the representation. Proposition 0.8(vi) of Resnick [22] gives the asymptotic equivalence.
(2) Theorem 2.4.7(ii) of [5] applied to $1/\overline{F}$ ensures that $(1/\overline{F})^{\leftarrow} \in \mathcal{R}_0$. Furthermore, tail equivalence of F and G implies that $(1/\overline{F})^{\leftarrow}(z) = (1/\overline{G})^{\leftarrow}(z(1+o(1))) = (1/\overline{G})^{\leftarrow}(z)(1+o(1))$ as $z \to \infty$, where we have used that the convergence in Definition 2.8 is locally uniformly. □

Theorem 2.15. [Analytical OpVaR for the SCP model] *Consider the SCP model.*

(i) *If* $\overline{F} \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$, *then* $\mathrm{VaR}_t(\kappa)$ *is asymptotically given by*

$$\mathrm{VaR}_t(\kappa) = \left(\frac{1}{\overline{G}_t}\right)^{\leftarrow} \left(\frac{1}{1-\kappa}\right) \sim F^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda t}\right), \quad \kappa \uparrow 1\,. \tag{2.7}$$

(ii) *The severity distribution tail belongs to* $\mathcal{R}_{-\alpha}$ *for* $\alpha > 0$, *i.e.* $\overline{F}(x) = x^{-\alpha} L(x)$ *for* $x \geq 0$ *and some slowly varying function* L *if and only if*

$$\mathrm{VaR}_t(\kappa) \sim \left(\frac{\lambda t}{1-\kappa}\right)^{1/\alpha} \widetilde{L}\left(\frac{1}{1-\kappa}\right), \quad \kappa \uparrow 1\,, \tag{2.8}$$

where $\widetilde{L}\left(\frac{1}{1-\cdot}\right) \in \mathcal{R}_0$.

Proof. (i) is a consequene of Böcker and Klüppelberg [6] in combination with Proposition 2.14.
(ii) By Definition 2.12, $\mathrm{VaR}_t(\kappa) = G^{\leftarrow}(\kappa)$. In our SCP model we have $\overline{G}_t(x) \sim \lambda t \overline{F}(x)$ as $x \to \infty$. From Proposition 2.14 it follows that

$$\left(\frac{1}{\overline{G}_t}\right)^{\leftarrow} \left(\frac{1}{1-\kappa}\right) \sim \left(\frac{1}{\overline{F}}\right)^{\leftarrow} \left(\frac{\lambda t}{1-\kappa}\right) = \left(\frac{\lambda t}{1-\kappa}\right)^{1/\alpha} \widetilde{L}\left(\frac{\lambda t}{1-\kappa}\right), \quad \kappa \uparrow 1\,,$$

and the result follows. □

We refrain from giving more information on the relationship between L and $\widetilde{L}$ (which can be found in [5]) as it is rather involved and plays no role in our paper. When such a model is fitted statistically, then L and $\widetilde{L}$ are usually replaced by constants; see Embrechts et al. [12], Chapter 6. In that case $L \equiv \theta^\alpha$ results in $\widetilde{L} \equiv \theta$ as in the following example. To indicate that the equivalence of Theorem 2.15(ii) does not extend to subexponential distribution tails in $\mathcal{R}_\infty$ we refer to Example 3.11.

We can now formulate the analytical VaR theorem for subexponential severity tails. A precise result can be obtained for Pareto distributed severities. Pareto's law is the prototypical parametric example for a heavy tailed distribution and suitable for operational risk modelling, see e.g. Moscadelli [20].

Example 2.16. [Poisson-Pareto LDA]
The *Poisson-Pareto LDA* is an SCP model, where the severities are Pareto distributed with

$$\overline{F}(x) = \left(1 + \frac{x}{\theta}\right)^{-\alpha}, \quad x > 0,$$

with parameters $\alpha, \theta > 0$. Here, OpVaR can be calculated explicitly and satisfies

$$\mathrm{VaR}_t(\kappa) \sim \theta \left[\left(\frac{\lambda t}{1-\kappa}\right)^{1/\alpha} - 1\right] \sim \theta \left(\frac{\lambda t}{1-\kappa}\right)^{1/\alpha}, \quad \kappa \uparrow 1\,. \tag{2.9}$$

□

3 Multivariate Loss Distribution Models

3.1 The Lévy Copula Model

The SCP model of the previous section can be used for estimating OpVaR of a single cell, sometimes referred to as the cell's *stand alone* OpVaR. Then, a first approximation to the bank's total OpVaR is obtained by summing up all different stand alone VaR numbers. Indeed, the Basel committee requires banks to sum up all their different operational risk estimates unless sound and robust correlation estimates are available; cf. [4], paragraph 669(d). Moreover, this "simple-sum VaR" is often interpreted as an upper bound for total OpVaR, with the implicit understanding that every other (realistic) cell dependence model necessarily reduces overall operational risk.

However, as is well recognised (cf. Table 5.3 in [7]), simple-sum VaR may even underestimate total OpVaR when severity data is heavy-tailed, which in practice it is, see e.g. Moscadelli [20]. Therefore, to obtain a more accurate and reliable result, one needs more general models for multivariate operational risk.

Various models have been suggested. Most of them are variations of the following scheme. Fix a time horizon $t > 0$, and model the accumulated losses of each operational risk cell $i = 1, \ldots, d$ by a compound Poisson random variable $S_i(t)$. Then, in general, both the dependence of loss sizes in different cells as well as the dependence between the frequency variables $N_i(t)$ is modelled by appropriate copulas, where for the latter one has to take the discreteness of these variables into account. Considering this model as a dynamic model in time, it does not constitute a multivariate compound Poisson model but leads outside the well-studied class of Lévy processes. This can be easily seen as follows: since a Poisson process jumps with probability 0 at any fixed time $s > 0$, we have for any jump time s of $N_j(\cdot)$ that $P(\Delta N_i(s) = 1) = 0$ for $i \neq j$, hence any two of such processes almost surely never jump at the same time. However, as described in Section 2.1, dependence in multivariate compound Poisson processes—as in every multivariate Lévy process—means dependence in the jump measure, i.e. the possibility of joint jumps. Finally, from a statistical point of view such a model requires a large number of parameters, which, given the sparsity of data in combination with the task of estimating high quantiles, will be almost impossible to fit.

We formulate a multivariate compound Poisson model and apply Sklar's theorem for Lévy copulas. Invoking a Lévy copula allows for a low number of parameters and introduces a transparent dependence structure in the model; we present a detailed example in Section 3 of [7].

Definition 3.1. [Multivariate SCP model] *The* multivariate SCP model *consists of:*

(1) Cell processes.
All operational risk cells, indexed by $i = 1, \ldots, d$, *are described by an SCP model with aggregate loss process* S_i, *subexponential severity distribution function* F_i *and Poisson intensity* $\lambda_i > 0$, *respectively.*

(2) Dependence structure.
The dependence between different cells is modelled by a Lévy copula. More precisely, let $\overline{\Pi}_i : [0,\infty) \to [0,\infty)$ *be the tail integral associated with* S_i, *i.e.* $\overline{\Pi}_i(\cdot) = \lambda_i \overline{F}_i(\cdot)$ *for* $i = 1, \ldots, d$, *and let* $\widehat{C} : [0,\infty)^d \to [0,\infty)$ *be a Lévy copula. Then*

$$\overline{\Pi}(x_1, \ldots, x_d) = \widehat{C}(\overline{\Pi}_1(x_1), \ldots, \overline{\Pi}_d(x_d))$$

defines the tail integral of the d*-dimensional compound Poisson process* $S = (S_1, \ldots, S_d)$.

(3) Total aggregate loss process.
The bank's total aggregate loss process *is defined as*

$$S^+(t) = S_1(t) + S_2(t) + \cdots + S_d(t)\,, \qquad t \geq 0$$

with tail integral

$$\overline{\Pi}^+(z) = \Pi(\{(x_1, \ldots, x_d) \in [0,\infty)^d : \sum_{i=1}^{d} x_i \geq z\})\,, \quad z \geq 0\,. \tag{3.1}$$

The following result states an important property of the multivariate SCP model.

Proposition 3.2. *Consider the multivariate SCP model of Definition 3.1. Its total aggregate loss process* S^+ *is compound Poisson with frequency parameter*

$$\lambda^+ = \lim_{z \downarrow 0} \overline{\Pi}^+(z)$$

and severity distribution

$$F^+(z) = 1 - \overline{F}^+(z) = 1 - \frac{\overline{\Pi}^+(z)}{\lambda^+}\,, \quad z \geq 0\,.$$

Proof. Projections of Lévy processes are Lévy processes. For every compound Poisson process with intensity $\lambda > 0$ and only positive jumps with distribution function F, the tail integral of the Lévy measure is given by $\overline{\Pi}(x) = \lambda \overline{F}(x)$, $x > 0$. Consequently, $\lambda = \overline{\Pi}(0)$ and $\overline{F}(x) = \overline{\Pi}(x)/\lambda$. We apply this relation to the Lévy process S^+ and obtain the total mass λ^+ of S^+, which ensures that S^+ is compound Poisson with the parameters as stated. □

Note that S^+ does not necessarily define a one dimensional SCP model because F^+ need not to be subexponential, even if all components are. This has been investigated for sums of independent random variables in great detail; see e.g. the review paper Goldie and Klüppelberg [14], Section 5. For dependent random variables we present in Examples 3.10 and 3.11 two situations, where $F^+ \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$. In that case we can apply (2.7) to estimate total OpVaR, which shall now be defined precisely.

Definition 3.3. [Total OpVaR] *Consider the multivariate SCP model of Definition 3.1. Then,* total OpVaR *up to time t at confidence level κ is the κ-quantile of the total aggregate loss distribution $G_t^+(\cdot) = P(S^+(t) \le \cdot)$:*

$$\mathrm{VaR}_t^+(\kappa) = G_t^{+\leftarrow}(\kappa)\,, \quad \kappa \in (0,1)\,,$$

with $G_t^{+\leftarrow}(\kappa) = \inf\{z \in \mathbb{R} : \ G_t^+(z) \ge \kappa\}$ for $0 < \kappa < 1$.

Our goal in this paper is to investigate multivariate SCP models and find useful approximations in a variety of dependence structures.

3.2 Losses Dominant in One Cell

Before we discuss Lévy copula dependence structures we formulate a very general result for the situation, where the losses in one cell are regularly varying and dominate all others. Indeed the situation of the model is such that it covers arbitrary dependence structures, including also the practitioner's models described above.

Assume for fixed $t > 0$ for each cell model a compound Poisson random variable. Dependence is introduced by an arbitrary correlation or copula for $(N_1(\cdot), \dots, N_d(\cdot))$ and an arbitrary copula between the severity distributions $F_1(\cdot) = P(X^1 \le \cdot), \dots, F_d(\cdot) = P(X^d \le \cdot)$. Recall that the resulting model $(S_1(t), \dots, S_d(t))_{t\ge0}$ does NOT constitute a multivariate compound Poisson process and so is not captured by the multivariate SCP model of Definition 3.1. We want to calculate an approximation for the tail $P(S_1(t) + S_2(t) > x)$ for large x and total OpVaR for high levels κ. We formulate the result in arbitrary dimension.

Theorem 3.4. *For fixed $t > 0$ let $S_i(t)$ for $i = 1, \dots, d$ have compound Poisson distributions. Assume that $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ for $\alpha > 0$. Let $\rho > \alpha$ and suppose that $E[(X^i)^\rho] < \infty$ for $i = 2, \dots, d$. Then regardless of the dependence structure between $(S_1(t), \dots, S_d(t))$,*

$$\begin{aligned} P(S_1(t) + \cdots + S_d(t) > x) &\sim EN_1(t)\, P(X^1 > x)\,, \quad x \to \infty\,, \\ \mathrm{VaR}_t^+(\kappa) &\sim F_1^{\leftarrow}\Big(1 - \frac{1-\kappa}{EN_1(t)}\Big) = \mathrm{VaR}_t^1(\kappa)\,, \quad \kappa \uparrow 1\,. \end{aligned} \tag{3.2}$$

Proof. Consider $d = 2$. Note first that

$$\frac{P(S_1(t)+S_2(t)>x)}{P(X^1>x)} \tag{3.3}$$

$$= \sum_{k,m=1}^{\infty} P(N_1(t)=k, N_2(t)=m) \frac{P\Big(\sum_{i=1}^{k} X_i^1 + \sum_{j=1}^{m} X_j^2 > x\Big)}{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)} \frac{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)}{P\Big(X^1 > x\Big)}.$$

We have to find conditions such that we can interchange the limit for $x \to \infty$ and the infinite sum. This means that we need uniform estimates for the two ratios on the right-hand side for $x \to \infty$. We start with an estimate for the second ratio: Lemma 1.3.5 of Embrechts et al. [12] applies giving for arbitrary $\varepsilon > 0$ and all $x > 0$ a finite positive constant $K(\varepsilon)$ so that

$$\frac{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)}{P(X^1>x)} \le K(\varepsilon)(1+\varepsilon)^k .$$

For the first ratio we proceed as in the proof of Lemma 2 of Klüppelberg, Lindner and Maller [18]. For arbitrary $0 < \delta < 1$ we have

$$\frac{P\Big(\sum_{i=1}^{k} X_i^1 + \sum_{j=1}^{m} X_j^2 > x\Big)}{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)} \le \frac{P\Big(\sum_{i=1}^{k} X_i^1 > x(1-\delta)\Big)}{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)} + \frac{P\Big(\sum_{j=1}^{m} X_j^2 > x\delta\Big)}{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)} \tag{3.4}$$

Regular variation of the distribution of X^1 implies regular variation of the distribution of $\sum_{i=1}^k X_i^1$ with the same index $-\alpha$. We write for the first term

$$\frac{P\Big(\sum_{i=1}^{k} X_i^1 > x(1-\delta)\Big)}{P\Big(X^1 > x(1-\delta)\Big)} \frac{P\Big(X^1 > x(1-\delta)\Big)}{P\Big(X^1 > x\Big)} \frac{P\Big(X^1 > x\Big)}{P\Big(\sum_{i=1}^{k} X_i^1 > x\Big)}.$$

For the first ratio we use the same estimate as above and obtain for all $x > 0$ the upper bound $K'(\varepsilon)(1+\varepsilon)^k$. For the second ratio, using the so-called Potter bounds (cf. Theorem 1.5.6 (iii) of Bingham, Goldie and Teugels [5]), for every chosen constants $a > 0, A > 1$, we obtain an upper bound $A(1-\delta)^{-(\alpha+a)}$ uniformly for all $x \ge x_0 \ge 0$. The third ratio is less or equal to 1 for all k and x.

As the denominator of the second term of the rhs of (3.4) is regularly varying, it can be bounded below by $x^{-(\alpha+\rho')}$ for some $0 < \rho' < \rho - \alpha$. By Markov's inequality, we obtain for the numerator

$$P\Big(\sum_{j=1}^{m} X_j^2 > x\delta\Big) \le (x\delta)^{-\rho} E\Big[\big(\sum_{j=1}^{m} X_j^2\big)^{\rho}\Big].$$

The so-called c_ρ-inequality (see e.g. Loéve [19], p. 157) applies giving

$$E\big[(\sum_{j=1}^{m} X_j^2)^\rho\big] \le m\, c_\rho E(X_j^2)^\rho$$

for $c_\rho = 1$ or $c_\rho = 2^{\rho-1}$, according as $\rho \le 1$ or $\rho > 1$. We combine these estimates and obtain in (3.3) for $x \ge x_0 > 0$,

$$\begin{aligned}
&\frac{P(S_1(t)+S_2(t)>x)}{P(X^1>x)} \\
&\le \sum_{k,m=1}^{\infty} P(N_1(t)=k, N_2(t)=m) \qquad (3.5) \\
&\quad \Big(K'(\varepsilon)(1+\varepsilon)^k A(1-\delta)^{-(\alpha+a)} + x^{\alpha+\rho'}(x\delta)^{-\rho}\, m\, c_\rho\, E[(X_j^2)^\rho]\Big)\, K(\varepsilon)(1+\varepsilon)^k\,.
\end{aligned}$$

Now note that $x^{\alpha+\rho'-\rho}$ tends to 0 as $x \to \infty$. Furthermore, we have

$$\begin{aligned}
\sum_{k,m=0}^{\infty} P(N_1(t)=k, N_2(t)=m) &= 1\,, \\
\sum_{k,m=1}^{\infty} P(N_1(t)=k, N_2(t)=m)\, k &= \sum_{k=1}^{\infty} P(N_1(t)=k)\, k = EN_k(t) < \infty\,.
\end{aligned}$$

Consequently, the rhs of 3.5 converges. By Pratt's Lemma (see e.g. Resnick [22], Ex. 5.4.2.4), we can interchange limit and infinite sum on the rhs of (3.3) and obtain

$$\lim_{x\to\infty} \frac{P(S_1(t)+S_2(t)>x)}{P(X^1>x)} = \sum_{k=1}^{\infty} P(N_1(t)=k)\, k = EN_1(t)\,.$$

The result for $d > 2$ follows by induction.
Approximation (3.2) holds by Theorem 2.15(1). □

Within the context of multivariate compound Poisson models, the proof of this result simplifies. Moreover, since a possible singularity of the tail integral in 0 is of no consequence, it even holds for all spectrally positive Lévy processes. We formulate this as follows.

Proposition 3.5. *Consider a multivariate spectrally positive Lévy process and suppose that* $\overline{\Pi}_1 \in \mathcal{R}_{-\alpha}$. *Furthermore, assume that for all* $i = 2, \ldots, d$ *the integrability condition*

$$\int_{x\ge1} x^\rho\, \Pi_i(dx) < \infty \qquad (3.6)$$

for some $\rho > \alpha$ *is satisfied. Then*

$$\lim_{z\to\infty} \frac{\overline{\Pi}^+(z)}{\overline{\Pi}_1(z)} = 1\,. \qquad (3.7)$$

Moreover,

$$\mathrm{VaR}_t^+(\kappa) \sim \mathrm{VaR}_t^1(\kappa)\,, \qquad \kappa \uparrow 1\,, \tag{3.8}$$

i.e. total OpVaR is asymptotically dominated by the stand alone OpVaR of the first cell.

Proof. We first show that (3.7) holds. From equation (3.6) it follows that for $i = 2, \ldots, d$

$$\lim_{z\to\infty} z^\rho\, \overline{\Pi}_i(z) = 0\,. \tag{3.9}$$

Since $\alpha < \rho$, we obtain from regular variation for some slowly varying function L, invoking (3.9),

$$\lim_{z\to\infty} \frac{\overline{\Pi}_i(z)}{\overline{\Pi}_1(z)} = \lim_{z\to\infty} \frac{z^\rho \overline{\Pi}_i(z)}{z^{\rho-\alpha} L(z)} = 0\,, \quad i = 2, \ldots, d\,,$$

because the numerator tends to 0 and the denominator to ∞. (Recall that $z^\varepsilon L(z) \to \infty$ as $z \to \infty$ for all $\varepsilon > 0$ and $L \in \mathcal{R}_0$.)

We proceed by induction. For $d = 2$ we have by the decomposition as in (3.4)

$$\overline{\Pi}_2^+(z) := \overline{\Pi}^+(z) \le \overline{\Pi}_1(z(1-\varepsilon)) + \overline{\Pi}_2(z\,\varepsilon), \quad z > 0, \quad 0 < \varepsilon < 1.$$

It then follows that

$$\limsup_{z\to\infty} \frac{\overline{\Pi}_2^+(z)}{\overline{\Pi}_1(z)} \le \lim_{z\to\infty} \frac{\overline{\Pi}_1(z(1-\varepsilon))}{\overline{\Pi}_1(z)} + \lim_{z\to\infty} \frac{\overline{\Pi}_2(z\,\varepsilon)}{\overline{\Pi}_1(z\,\varepsilon)} \frac{\overline{\Pi}_1(z\,\varepsilon)}{\overline{\Pi}_1(z)} = (1-\varepsilon)^{-\alpha}\,. \tag{3.10}$$

Similarly, $\overline{\Pi}_2^+(z) \ge \overline{\Pi}_1((1+\varepsilon)z)$ for every $\varepsilon > 0$. Therefore,

$$\liminf_{z\to\infty} \frac{\overline{\Pi}_2^+(z)}{\overline{\Pi}_1(z)} \ge \lim_{z\to\infty} \frac{\overline{\Pi}_1((1+\varepsilon)z)}{\overline{\Pi}_1(z)} = (1+\varepsilon)^{-\alpha}\,. \tag{3.11}$$

Assertion (3.7) follows for $\overline{\Pi}_2^+$ from (3.10) and (3.11). This implies that $\overline{\Pi}_2^+ \in \mathcal{R}_\alpha$. Now replace $\overline{\Pi}_1$ by $\overline{\Pi}_2^+$ and $\overline{\Pi}_2^+$ by $\overline{\Pi}_3^+$ and proceed as above to obtain (3.7) for general dimension d. Finally, Theorem 2.15(1) applies giving (3.8). $\square$

This result is mostly applied in terms of the following corollary, which formulates a direct condition for (3.10) and (3.11) to hold.

Corollary 3.6. *Consider a multivariate spectrally positive Lévy process and suppose that $\overline{\Pi}_1 \in \mathcal{R}_{-\alpha}$. Furthermore, assume that for all $i = 2, \ldots, d$*

$$\lim_{z\to\infty} \frac{\overline{\Pi}_i(z)}{\overline{\Pi}_1(z)} = 0\,.$$

Then (3.7) *and* (3.8) *hold.*

Hence, for arbitrary dependence structures, when the severity of *one* cell has regularly varying tail dominating those of all other cells, total OpVaR is tail-equivalent to the OpVaR of the dominating cell. This implies that the bank's total loss at high confidence levels is likely to be due to one big loss occurring in one cell rather than an accumulation of losses of different cells regardless of the dependence structure.

From our equivalence results of Proposition 2.14 and Theorem 2.15 this is not a general property of the completely dependent SCP model. We shall see in Example 3.11 below that the following does NOT hold in general for $x \to \infty$ (equivalently $\kappa \uparrow 1$):

$$\overline{F}_i(x) = o(\overline{F}_1(x)) \quad \Longrightarrow \quad \mathrm{VaR}_t^i(\kappa) = o(\mathrm{VaR}_t^1(\kappa)) \,, \quad i = 2, \ldots, d \,.$$

We now study two very basic multivariate SCP models in more detail, namely the completely dependent and the independent one. Despite their extreme dependence structure, both models provide interesting and valuable insight into multivariate operational risk.

3.3 Multivariate SCP Model with Completely Dependent Cells

Consider a multivariate SCP model and assume that its cell processes $S_i,\ i = 1, \ldots, d$, are completely positively dependent. In the context of Lévy processes this means that they always jump together, implying that also the expected number of jumps per unit time of all cells, i.e. the intensities λ_i, must be equal,

$$\lambda := \lambda_1 = \cdots = \lambda_d \,. \tag{3.12}$$

The severity distributions F_i, however, can be different. Indeed, from Example 2.4 we infer that in the case of complete dependence, all Lévy mass is concentrated on

$$\{(x_1, \ldots, x_d) \in [0, \infty)^d : \overline{\Pi}_1(x_1) = \cdots = \overline{\Pi}_d(x_d)\} \,,$$

or, equivalently,

$$\{(x_1, \ldots, x_d) \in [0, \infty)^d : F_1(x_1) = \cdots = F_d(x_d)\} \,. \tag{3.13}$$

Until further notice, we assume for simplicity that all severity distributions F_i are strictly increasing and continuous so that $F_i^{-1}(q)$ exists for all $q \in [0, 1)$. Together with (3.13), we can express the tail integral of S^+ in terms of the marginal $\overline{\Pi}_1$.

$$\begin{aligned} \overline{\Pi}^+(z) &= \Pi(\{(x_1, \ldots, x_d) \in [0, \infty)^d : \sum_{i=1}^d x_i \geq z\}) \\ &= \Pi_1(\{x_1 \in [0, \infty) : x_1 + \sum_{i=2}^d F_i^{-1}(F_1(x_1)) \geq z\}) \,, \quad z \geq 0 \,. \end{aligned}$$

Set $H(x_1) := x_1 + \sum_{i=2}^d F_i^{-1}(F_1(x_1))$ for $x_1 \in [0,\infty)$ and note that it is strictly increasing and therefore invertible. Hence,

$$\overline{\Pi}^+(z) \;=\; \Pi_1(\{x_1 \in [0,\infty) : x_1 \geq H^{-1}(z)\}) \;=\; \overline{\Pi}_1\left(H^{-1}(z)\right), \quad z \geq 0\,. \qquad (3.14)$$

Now we can derive an asymptotic expression for total OpVaR.

Theorem 3.7. [OpVaR for the completely dependent SCP model] *Consider a multivariate SCP model with completely dependent cell processes $S_1, \ldots, S_d$ and strictly increasing and continuous severity distributions F_i. Then, S^+ is compound Poisson with parameters*

$$\lambda^+ = \lambda \qquad \text{and} \qquad \overline{F}^+(\cdot) = \overline{F}_1\left(H^{-1}(\cdot)\right). \qquad (3.15)$$

If furthermore $\overline{F}^+ \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$, total OpVaR is asymptotically given by

$$\mathrm{VaR}_t^+(\kappa) \sim \sum_{i=1}^d \mathrm{VaR}_t^i(\kappa)\,, \quad \kappa \uparrow 1\,, \qquad (3.16)$$

where $\mathrm{VaR}_t^i(\kappa)$ denotes the stand alone OpVaR of cell i.

Proof. Expression (3.15) immediately follows from (3.12) and (3.14),

$$\lambda^+ = \lim_{z\to 0} \overline{\Pi}^+(z) = \lim_{z\to 0} \lambda\,\overline{F}_1\left(H^{-1}(z)\right) = \lambda\,\overline{F}_1\left(\lim_{z\to 0} H^{-1}(z)\right) = \lambda\,.$$

If $\overline{F}^+ \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$, we may use (2.7) and the definition of H to obtain

$$\begin{aligned} \mathrm{VaR}_t^+(\kappa) \;&\sim\; H\left[F_1^{-1}\left(1 - \frac{1-\kappa}{\lambda\, t}\right)\right] \;=\; F_1^{-1}\left(1 - \frac{1-\kappa}{\lambda\, t}\right) + \cdots + F_d^{-1}\left(1 - \frac{1-\kappa}{\lambda\, t}\right) \\ &\sim\; \mathrm{VaR}_t^1(\kappa) + \cdots + \mathrm{VaR}_t^d(\kappa), \qquad \kappa \uparrow 1\,. \end{aligned}$$

□

Theorem 3.7 states that for the completely dependent SCP model, total asymptotic OpVaR is simply the sum of the asymptotic stand alone cell OpVaRs. Recall that this is similar to the new proposals of Basel II, where the standard procedure for calculating capital charges for operational risk is just the simple-sum VaR. Or stated another way, regulators implicitly assume complete dependence between different cells, meaning that losses within different business lines or risk categories always happen at the same instants of time. This is often considered as the worst case scenario, which, however, in the heavy-tailed case can be grossly misleading.

The following example describes another regime for completely depenendent cells.

Example 3.8. [Identical severity distributions]
Assume that all cells have identical severity distributions, i.e. $F := F_1 = \ldots = F_d$. In this case we have $H(x_1) = d\, x_1$ for $x_1 \geq 0$ and, therefore,

$$\overline{\Pi}^{+}(z) = \lambda \overline{F}\left(\frac{z}{d}\right), \quad z \geq 0 .$$

If furthermore $\overline{F} \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$, it follows that $\overline{F}^{+}(\cdot) = \overline{F}(\cdot\,/d)$ is, and we obtain

$$\mathrm{VaR}_t^{+}(\kappa) \sim d\, \overline{F}\left(1 - \frac{1-\kappa}{\lambda\, t}\right), \quad \kappa \uparrow 1 .$$

□

We can derive very precise asymptotics in the case of dominating regularly varying severities.

Proposition 3.9. *Assume that the conditions of Theorem 3.7 hold. Assume further that $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ with $\alpha > 0$ and that for all $i = 2, \ldots, d$ there exist $c_i \in [0, \infty)$ such that*

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = c_i . \tag{3.17}$$

Assume that $c_i \neq 0$ for $2 \leq i \leq b \leq d$ and $c_i = 0$ for $i \leq b+1 \leq d$. For $\overline{F}_1(x) = x^{-\alpha} L(x)$, $x \geq 0$, let $\widetilde{L}$ be the function as in Theorem 2.15(ii). Then

$$\mathrm{VaR}_t^{+}(\kappa) \;\sim\; \sum_{i=1}^{b} c_i^{1/\alpha}\, \mathrm{VaR}_t^{1}(\kappa) \;\sim\; \sum_{i=1}^{b} c_i^{1/\alpha} \left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha} \widetilde{L}\left(\frac{1}{1-\kappa}\right), \quad \kappa \uparrow 1 .$$

Proof. From Theorem 2.15(ii) we know that

$$\mathrm{VaR}_t^{1}(\kappa) \sim \left(\frac{\lambda\, t}{1-\kappa}\right)^{1/\alpha} \widetilde{L}\left(\frac{1}{1-\kappa}\right), \quad \kappa \uparrow 1 ,$$

where $\widetilde{L}\left(\frac{1}{1-\cdot}\right) \in \mathcal{R}_0$. Note: If all $c_i = 0$ holds for $i = 2, \ldots, d$ then Corollary 3.6 applies. So assume that $c_i \neq 0$ for $2 \leq i \leq b$. From (3.17) and Resnick [22], Proposition 0.8(vi), we get $F_i^{\leftarrow}(1 - \frac{1}{z}) \sim c_i^{1/\alpha} F_1^{\leftarrow}(1 - \frac{1}{z})$ as $z \to \infty$ for $i = 1, \ldots, d$. This yields for $x_1 \to \infty$

$$\begin{aligned} H(x_1) &= x_1 + \sum_{i=2}^{d} F_i^{-1}(1 - \overline{F}_1(x_1)) \\ &= x_1 + \sum_{i=2}^{d} c_i^{1/\alpha} F_1^{-1}\left(1 - \overline{F}_1(x_1)\right)(1 + o_i(1)) \\ &= x_1 \sum_{i=1}^{b} c_i^{1/\alpha}\,(1 + o(1)) , \end{aligned}$$

where we have $c_1 = 1$. Defining $C := \sum_{i=1}^b c_i^{1/\alpha}$, then $H(x_1) \sim Cx_1$ as $x_1 \to \infty$, and hence $H^{-1}(z) \sim z/C$ as $z \to \infty$, which implies by (3.14) and regular variation of $\overline{F}_1$

$$\overline{\Pi}^+(z) \;=\; \overline{\Pi}_1(H^{-1}(z)) \;\sim\; \lambda\,\overline{F}_1(z/C) \;\sim\; \lambda\, C^\alpha\, \overline{F}_1(z)\,, \quad z \to \infty\,.$$

Obviously, $\overline{F}^+(z) = C^\alpha\,\overline{F}_1(z) \in \mathcal{R}_{-\alpha}$ and Theorem 3.7 applies. By (2.8) together with the fact that all summands from index $b+1$ on are of lower order, (3.16) reduces to

$$\begin{aligned}
\mathrm{VaR}_t^+(\kappa) \;&\sim\; F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda\,t}\right) + \cdots + F_b^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda\,t}\right)\\
&\sim\; F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{\lambda\,t\,C^\alpha}\right)\\
&\sim\; \sum_{i=1}^b c_i^{1/\alpha}\left(\frac{\lambda\,t}{1-\kappa}\right)^{1/\alpha}\widetilde{L}\left(\frac{1}{1-\kappa}\right), \quad \kappa \uparrow 1\,.
\end{aligned}$$

□

An important example of Proposition 3.9 is the Pareto case.

Example 3.10. [Pareto distributed severities]
Consider a multivariate SCP model with completely dependent cells and Pareto distributed severities as in Example 2.16. Then we obtain for the c_i

$$\lim_{x\to\infty}\frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \left(\frac{\theta_i}{\theta_1}\right)^\alpha, \quad i = 1,\ldots,b, \quad \text{and} \quad \lim_{x\to\infty}\frac{\overline{F}_i(x)}{\overline{F}_1(x)} = 0\,, \quad i = b+1,\ldots,d\,,$$

for some $1 \le b \le d$. This, together with Proposition 3.9 leads to

$$\overline{F}^+(z) \sim \left(\sum_{i=1}^b \frac{\theta_i}{\theta_1}\right)^\alpha \left(1 + \frac{z}{\theta_1}\right)^{-\alpha} \sim \left(\sum_{i=1}^b \theta_i\right)^\alpha z^{-\alpha}, \quad z \to \infty\,.$$

Finally, from (2.9) and (3.16) we obtain total OpVaR as

$$\mathrm{VaR}_t^+(\kappa) \sim \sum_{i=1}^b \mathrm{VaR}_t^i(\kappa) \sim \sum_{i=1}^b \theta_i\left(\frac{\lambda\,t}{1-\kappa}\right)^{1/\alpha}, \qquad \kappa \uparrow 1\,.$$

□

We conclude this session with an example showing that Corollary 3.6 does not hold for every general dominating tail.

Example 3.11. [Weibull severities]
Consider a bivariate SCP model with completely dependent cells and assume that the cells' severities are Weibull distributed according to

$$\overline{F}_1(x) = \exp(-\sqrt{x/2}) \qquad \text{and} \qquad \overline{F}_2(x) = \exp(-\sqrt{x})\,, \quad x > 0\,. \tag{3.18}$$

Note that $\overline{F}_{1,2} \in \mathcal{S} \cap \mathcal{R}_\infty$. Equation (3.18) immediately implies that $\overline{F}_2(x) = o(\overline{F}_1(x))$. We find that $H(x_1) = \frac{3}{2} x_1$ implying that $\overline{F}^+ \in \mathcal{S} \cap \mathcal{R}_\infty$, since

$$\overline{F}^+(z) = \exp(-\sqrt{z/3})\,, \quad z > 0\,. \tag{3.19}$$

It is remarkable that in this example the total severity (3.19) is heavier tailed than the stand alone severities (3.18), i.e. $F_{1,2}(x) = o(F^+(x))$ as $x \to \infty$. However, from

$$\mathrm{VaR}_t^1(\kappa) \sim 2\left[\ln\left(\frac{1-\kappa}{\lambda\, t}\right)\right]^2 \quad \text{and} \quad \mathrm{VaR}_t^2(\kappa) \sim \left[\ln\left(\frac{1-\kappa}{\lambda\, t}\right)\right]^2, \qquad \kappa \uparrow 1\,,$$

we find that the stand alone VaRs are of the same order of magnitude:

$$\lim_{\kappa \uparrow 1} \frac{\mathrm{VaR}_t^2(\kappa)}{\mathrm{VaR}_t^1(\kappa)} = \frac{1}{2}\,.$$

Nevertheless, equation (3.16) of Theorem 3.7 still holds,

$$\mathrm{VaR}_t^+(\kappa) \;\sim\; 3\left[\ln\left(\frac{1-\kappa}{\lambda\, t}\right)\right]^2 = \mathrm{VaR}_t^1(\kappa) + \mathrm{VaR}_t^2(\kappa)\,, \qquad \kappa \uparrow 1\,.$$

□

3.4 Multivariate SCP Model with Independent Cells

Let us now turn to a multivariate SCP model where the cell processes S_i, $i = 1, \dots, d$, are independent and so any two of the component sample paths almost surely do not ever jump together. Therefore, we may write the tail integral of S^+ as

$$\overline{\Pi}^+(z) = \Pi([z,\infty) \times \{0\} \times \cdots \times \{0\}) + \cdots + \Pi(\{0\} \times \cdots \times \{0\} \times [z,\infty))\,, \quad z \ge 0\,.$$

Recall from Example 2.5 that in the case of independence all mass of the Lévy measure Π is concentrated on the axes. Hence,

$$\begin{aligned}
\Pi([z,\infty) \times \{0\} \times \cdots \times \{0\}) &= \Pi([z,\infty) \times [0,\infty) \times \cdots \times [0,\infty)),\\
\Pi(\{0\} \times [z,\infty) \times \cdots \times \{0\}) &= \Pi([0,\infty) \times [z,\infty) \times \cdots \times [0,\infty)),\\
&\vdots\\
\Pi(\{0\} \times \{0\} \times \cdots \times [z,\infty)) &= \Pi([0,\infty) \times [0,\infty) \times \cdots \times [z,\infty)),
\end{aligned}$$

and we obtain

$$\begin{aligned}
\overline{\Pi}^+(z) &= \Pi([z,\infty) \times [0,\infty) \times \cdots \times [0,\infty)) + \cdots + \Pi([0,\infty) \times \cdots \times [0,\infty) \times [z,\infty))\\
&= \overline{\Pi}_1(z) + \cdots + \overline{\Pi}_d(z)\,.
\end{aligned} \tag{3.20}$$

Now we are in the position to derive an asymptotic expression for total OpVaR in the case of independent cells.

Theorem 3.12. [OpVaR for the independent SCP model] *Consider a multivariate SCP model with independent cell processes $S_1, \dots, S_d$. Then S^+ defines a one-dimensional SCP model with parameters*

$$\lambda^+ = \lambda_1 + \cdots + \lambda_d \quad \textit{and} \quad \overline{F}^+(z) = \frac{1}{\lambda^+}\left[\lambda_1 \overline{F}_1(z) + \cdots + \lambda_d \overline{F}_d(z)\right], \quad z \geq 0. \quad (3.21)$$

If $\overline{F}_1 \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$ and for all $i = 2, \dots, d$ there exist $c_i \in [0, \infty)$ such that

$$\lim_{x \to \infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = c_i\,, \quad (3.22)$$

then, setting $C_\lambda = \lambda_1 + c_2\lambda_2 + \cdots + c_d\lambda_d$, total OpVaR can be approximated by

$$\mathrm{VaR}_t^+(\kappa) \sim F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{C_\lambda\, t}\right), \qquad \kappa \uparrow 1\,. \quad (3.23)$$

Proof. From Proposition 3.2 we know that S^+ is a compound Poisson process with parameters λ^+ (here following from (3.20)) and F^+ as in (3.21) from which we conclude

$$\lim_{z \to \infty} \frac{\overline{F}^+(z)}{\overline{F}_1(z)} = \frac{1}{\lambda^+}[\lambda_1 + c_2\,\lambda_2 + \cdots + c_d\,\lambda_d] = \frac{C_\lambda}{\lambda^+} \in (0, \infty)\,,$$

i.e.

$$\overline{F}^+(z) \sim \frac{C_\lambda}{\lambda^+}\overline{F}_1(z)\,, \quad z \to \infty\,. \quad (3.24)$$

In particular, $\overline{F}^+ \in \mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$ and S^+ defines a one-dimensional SCP model. From (2.7) and (3.24) total OpVaR follows as

$$\mathrm{VaR}_t^+(\kappa) \sim F^{+\leftarrow}\left(1 - \frac{1-\kappa}{\lambda^+\, t}\right) \sim F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{C_\lambda\, t}\right), \qquad \kappa \uparrow 1\,. \qquad \square$$

Example 3.13. [Multivariate SCP model with independent cells]
(1) Assume that $c_i = 0$ for all $i \geq 2$; i.e. $\overline{F}_i(x) = o(\overline{F}_1(x)),\ i = 2, \dots, d$. We then have $C_\lambda = \lambda_1$ and it follows from (3.23) that independent total OpVaR asymptotically equals the stand alone OpVaR of the first cell. In contrast to the completely dependent case (confer Proposition 3.9 and Example 3.11), this holds for the class $\mathcal{S} \cap (\mathcal{R} \cup \mathcal{R}_\infty)$ and not only for $F_1 \in \mathcal{R}$.

(2) Consider a multivariate SCP model with independent cells and Pareto distributed severities so that the constants c_i of Theorem 3.12 are given by

$$\lim_{x \to \infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \left(\frac{\theta_i}{\theta_1}\right)^\alpha, \quad i = 1, \dots, b, \quad \text{and} \quad \lim_{x \to \infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = 0\,, \quad i = b+1, \dots, d\,,$$

for some $b \geq 1$. Then

$$C_\lambda = \sum_{i=1}^{b} \left(\frac{\theta_i}{\theta_1}\right)^{\alpha} \lambda_i$$

and the distribution tail $\overline{F}^{+}$ satisfies

$$\overline{F}^{+}(z) \;=\; \frac{1}{\lambda^+} \sum_{i=1}^{b} \lambda_i \left(1 + \frac{z}{\theta_i}\right)^{-\alpha} \sim \frac{1}{\lambda^+} \sum_{i=1}^{b} \lambda_i \, \theta_i^{\alpha} \, z^{-\alpha} \,, \qquad z \to \infty \,.$$

It follows that

$$\mathrm{VaR}_t^{+}(\kappa) \sim \left(\frac{t \sum_{i=1}^{b} \lambda_i \, \theta_i^{\alpha}}{1-\kappa}\right)^{1/\alpha} = \left(\sum_{i=1}^{b} \left(\mathrm{VaR}_t^{i}(\kappa)\right)^{\alpha}\right)^{1/\alpha} \,, \qquad \kappa \uparrow 1 \,,$$

where $\mathrm{VaR}_t^{i}(\kappa)$ denotes the stand alone OpVaR of cell i according to (2.9). For identical cell frequencies $\lambda := \lambda_1 = \cdots = \lambda_b$ this further simplifies to

$$\mathrm{VaR}_t^{+}(\kappa) \sim \left(\frac{\lambda \, t}{1-\kappa}\right)^{1/\alpha} \left(\sum_{i=1}^{b} \theta_i^{\alpha}\right)^{1/\alpha} \,, \quad \kappa \uparrow 1 \,.$$

□

Example 3.14. [Continuation of Example 3.11]
Consider a bivariate SCP model with independent cells and Weibull distributed severities according to (3.18). Accoding to Theorem 3.12 we have $C_\lambda = \lambda_1$ and independent total OpVaR is asymptotically given by

$$\mathrm{VaR}_t^{+}(\kappa) \;\sim\; \mathrm{VaR}_t^{1}(\kappa) \;\sim\; 2 \left[\ln\left(\frac{1-\kappa}{\lambda \, t}\right)\right]^2 \,, \qquad \kappa \uparrow 1 \,.$$

□

3.5 Multivariate SCP Models of Regular Variation

As the notion of regular variation has proved useful for one-dimensional cell severity distributions, it seems natural to exploit the corresponding concept for the multivariate model. The following definition extends regular variation to the Lévy measure, which is our natural situation; the books Resnick [22, 23] are useful sources for insight into multivariate regular variation.

To simplify notation we denote by $\mathbb{E} := [0,\infty]^d \setminus \{0\}$, where 0 is the zero vector in $\mathbb{R}^d$. Then we introduce for $x \in \mathbb{E}$ the complement

$$[0,x]^c := \mathbb{E} \setminus [0,x] = \{y \in \mathbb{E} : \max_{1 \leq i \leq d} \frac{y_i}{x_i} > 1\} \,.$$

We also recall that a Radon measure is a measure, which is finite on all compacts. Finally, henceforth all operations and order relations of vectors are taken componentwise.

As already mentioned in Remark 2.9 (e), multivariate regular variation is best formulated in terms of vague convergence of measures. For spectrally positive Lévy processes we work in the space of non-negative Radon measures on $\mathbb{E}$. From Lemma 6.1 of Resnick [23], p. 174, however, it suffices to consider regions $[0, x]^c$ for $x \in \mathbb{E}$ which determine the convergence, and this is how we formulate our results.

Definition 3.15. [Multivariate regular variation]
(a) *Let Π be a Lévy measure of a spectrally positive Lévy process in $\mathbb{R}_+^d$. Assume that there exists a function $b : (0, \infty) \to (0, \infty)$ satisfying $b(t) \to \infty$ as $t \to \infty$ and a Radon measure ν on $\mathbb{E}$, called the limit measure, such that*

$$\lim_{t\to\infty} t\,\Pi([0, b(t)\,x]^c) = \nu([0, x]^c)\,, \quad x \in \mathbb{E}\,. \tag{3.25}$$

Then we call $\overline{\Pi}$ multivariate regularly varying.

(b) *The measure ν has a scaling property: there exists some $\alpha > 0$ such that for every $s > 0$*

$$\nu([0, sx]^c) = s^{-\alpha}\nu([0, x]^c)\,, \quad x \in \mathbb{E}\,, \tag{3.26}$$

i.e. $\nu([0, \cdot]^c)$ is homogeneous *of order $-\alpha$, and $\overline{\Pi}$ is called* multivariate regularly varying with index $-\alpha$.

Remark 3.16. (a) In (3.25), the scaling of all components of the tail integral by the same function $b(\cdot)$ implies

$$\lim_{x\to\infty} \frac{\overline{\Pi}_i(x)}{\overline{\Pi}_j(x)} = c_{ij} \in [0, \infty]\,,$$

for $1 \le i, j \le d$. We now focus on the case that all $\overline{\Pi}_i$ are tail-equivalent, i.e. $c_{ij} > 0$ for some i, j. In particular, we then have marginal regular variation $\overline{\Pi}_i \in \mathcal{R}_{-\alpha}$ with the same tail index, and thus for all $i = 1, \ldots, d$

$$\begin{aligned} \lim_{t\to\infty} t\,\overline{\Pi}_i(b(t)\,x) &= \nu([0, \infty] \times \cdots \times (x, \infty] \times [0, \infty] \times \cdots \times [0, \infty]) \\ &= \nu_i(x, \infty] = c_i\,x^{-\alpha}\,, \qquad x > 0\,, \end{aligned} \tag{3.27}$$

for some $c_i > 0$.
(b) Specifically, if $\overline{\Pi}_1$ is *standard regularly varying* (i.e. with index $\alpha = 1$ and slowly varying function $L \equiv 1$), we can take $b(t) = t$.
(c) There exists also a broader definition of multivariate regular variation which allows for different α_i in each marginal; see Theorem 6.5 of Resnick [23], p. 204. However, we have already dealt with the situation of dominant marginals and, hence, the above definition is the relevant one for us. □

From the point of view of dependence structure modeling, multivariate regular variation is basically a special form of multivariate dependence. Hence, a natural question in this context is how multivariate regular variation is linked to the dependence concept of a Lévy copula.

Theorem 3.17. [Lévy copulas and multivariate regular variation] *Let $\overline{\Pi}$ be a multivariate tail integral of a spectrally positive Lévy process in $\mathbb{R}_+^d$. Assume that the marginal tail integrals $\overline{\Pi}_i$ are regularly varying with index $-\alpha$. Then the following assertions hold.*
(1) If the Lévy copula $\widehat{C}$ is a homogeneous function of order 1, then $\overline{\Pi}$ is multivariate regularly varying with index $-\alpha$.
(2) The tail integral $\overline{\Pi}$ is multivariate regularly varying with index $-\alpha$ if and only if the Lévy copula $\widehat{C}$ is regularly varying with index 1; i.e.

$$\lim_{t\to\infty}\frac{\widehat{C}(t\,(x_1,\ldots,x_d))}{\widehat{C}(t\,(1,\ldots,1))}=g(x_1,\ldots,x_d)\,,\quad (x_1,\ldots,x_d)\in[0,\infty)^d\,, \tag{3.28}$$

and $g(s\,x)=s\,g(x)$ for $x\in[0,\infty)^d$.

Proof. (1) For any Lévy copula $\widehat{C}$, we can write the Lévy measure $\Pi([0,x]^c)$ for $x\in\mathbb{E}$ as

$$\begin{aligned}
\Pi([0,x]^c) &= \Pi\{y\in\mathbb{E}: y_1>x_1 \text{ or } \cdots \text{ or } y_d>x_d\}\\
&= \sum_{i=1}^d \overline{\Pi}_i(x_i)-\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^d \widehat{C}(\overline{\Pi}_{i_1}(x_{i_1}),\overline{\Pi}_{i_2}(x_{i_2}))\\
&\quad + \sum_{\substack{i_1,i_2,i_3=1\\ i_1<i_2<i_3}}^d \widehat{C}(\overline{\Pi}_{i_1}(x_{i_1}),\overline{\Pi}_{i_2}(x_{i_2}),\overline{\Pi}_{i_3}(x_{i_3}))\\
&\quad +\cdots+(-1)^{d-1}\widehat{C}(\overline{\Pi}_{i_1}(x_{i_1}),\ldots,\overline{\Pi}_{i_d}(x_{i_d}))\,.
\end{aligned}$$

The homogeneity allows interchange of the factor t with $\widehat{C}$, which, together with marginal regular variation as formulated in (3.27), yields the limit as in (3.25):

$$\begin{aligned}
\lim_{t\to\infty} t\,\Pi([0,b(t)\,x]^c) &= \sum_{i=1}^d \nu_i(x_i,\infty]-\sum_{\substack{i_1,i_2=1\\ i_1<i_2}}^d \widehat{C}(\nu_{i_1}(x_{i_1},\infty],\nu_{i_2}(x_{i_2},\infty])\\
&\quad +\cdots+(-1)^{d-1}\,\widehat{C}(\nu_{i_1}(x_{i_1},\infty],\ldots,\nu_{i_d}(x_{i_d},\infty])\\
&= \nu\{y\in\mathbb{E}: y_1>x_1 \text{ or } \cdots \text{ or } y_d>x_d\}\\
&= \nu([0,x]^c)\,,\qquad x\in\mathbb{E}\,.
\end{aligned} \tag{3.29}$$

(2) This follows from the same calculation as in the proof of (1) by observing that asymptotic interchange of the factor t with $\widehat{C}$ is possible if and only if (3.28) holds. □

Remark 3.18. (a) For definition (3.28) of multivariate regular variation of arbitrary functions we refer to Bingham et al. [5], Appendix 1.4.
(b) The general concept of multivariate regular variation of measures with possibly different marginals requires different normalizing functions $b_1(\cdot), \dots, b_d(\cdot)$ in (3.25). In that case marginals are usually transformed to standard regular variation with $\alpha = 1$ and $L \equiv 1$. In this case the scaling property (3.26) in the limit measure ν always scales with $\alpha = 1$. This is equivalent to all marginal Lévy processes being one-stable. In this situation the multivariate measure ν defines a function $\psi(x_1, \dots, x_d)$, which models the dependence between the marginal Lévy measures, and which is termed a *Pareto Lévy copula* in Klüppelberg and Resnick [16] as well as in Böcker and Klüppelberg [8]. Furthermore, according to Corollary 3.2 of [16], the limit measure ν is the Lévy measure of a one-stable Lévy process in $\mathbb{R}_+^d$ if and only if all marginals are one-stable and the Pareto Lévy copula ψ is homogeneous of order -1 (cf. Theorem 3.17 for the classical Lévy copula). In the context of multivariate regular variation this approach seems to be more natural than the classical Lévy copula with Lebesgue marginals. $\square$

We now want to apply the results above to the problem of calculating total OpVaR. Assume that the tail integral $\overline{\Pi}$ is multivariate regularly varying according to (3.25), implying tail equivalence of the marginal severity distributions. We then have the following result:

Theorem 3.19. [OpVaR for the SCP model with multivariate regular variation] *Consider an SCP model with multivariate regularly varying cell processes $S_1, \dots, S_d$ with index $-\alpha$ and limit measure ν in* (3.25)*. Assume further that the severity distributions F_i for $i = 1, \dots, d$ are strictly increasing and continuous. Then, S^+ is compound Poisson with*

$$\overline{F}^+(z) \sim \nu^+(1, \infty] \frac{\lambda_1}{\lambda^+} \overline{F}_1(z) \in \mathcal{R}_{-\alpha}, \quad z \to \infty, \tag{3.30}$$

where $\nu^+(z, \infty] = \nu\{x \in \mathbb{E} : \sum_{i=1}^d x_i > z\}$ for $z > 0$. Furthermore, total OpVaR is asymptotically given by

$$\mathrm{VaR}_t(\kappa) \sim F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{t\,\lambda_1\,\nu^+(1,\infty]}\right), \quad \kappa \uparrow 1. \tag{3.31}$$

Proof. First recall that multivariate regular variation of $\overline{\Pi}$ implies regular variation of the marginal tail integrals, i.e. $\overline{\Pi}_i \in \mathcal{R}_{-\alpha}$ for all $i = 1, \dots, d$. In analogy to relation (3.8) of Klüppelberg and Resnick [16], we can calculate the limit measure ν^+ of the tail integral $\overline{\Pi}^+$ by

$$\begin{aligned} \lim_{t\to\infty} t\,\overline{\Pi}^+(b(t)\,z) &= \nu^+(z,\infty] \\ = \nu\{x \in \mathbb{E} : \sum_{i=1}^d x_i > z\} &= z^{-\alpha}\nu\{x \in \mathbb{E} : \sum_{i=1}^d x_i > 1\}, \end{aligned}$$

i.e. $\overline{\Pi}^+$ and thus $\overline{F}^+$ are regularly varying of index $-\alpha$. Now we can choose $b(t)$ so that $\lim_{t\to\infty} t\overline{\Pi}_1(b(t)) = 1$ and thus

$$\lim_{z\to\infty} \frac{\overline{\Pi}^+(z)}{\overline{\Pi}_1(z)} = \lim_{t\to\infty} \frac{t\overline{\Pi}^+(b(t))}{t\overline{\Pi}_1(b(t))} = \nu^+(1,\infty]\,.$$

Relation (3.30) follows immediately, and (3.31) by Theorem 2.15(1). □

There are certain situations, where the limit measure $\nu^+(1,\infty]$ and so also total OpVaR can be explicitly calculated. In the following example we present a result for $d = 2$.

Example 3.20. [Clayton Lévy copula]
The Clayton Lévy copula is for $\delta > 0$ defined as

$$\widehat{C}(u_1,\dots,u_d) = (u_1^{-\delta} + \cdots + u_d^{-\delta})^{-1/\delta}\,, \quad u_1,\dots,u_d \in (0,\infty)\,.$$

In Figure 3.21 we show sample paths of two dependent compound Poisson processes, where the dependence is modelled via a Clayton Lévy copula for different parameter values. With increasing dependence parameter δ we see more joint jumps.

Note that $\widehat{C}$ is homogenous of order 1. Hence, from Theorem 3.17, if $\overline{\Pi}_i \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ the Lévy measure is multivariate regularly varying with index $-\alpha$. To calculate $\nu^+(1,\infty]$, we follow Section 3.2 of [16]. According to Remark 3.16 (a) we can set

$$\lim_{x\to\infty} \frac{\overline{\Pi}_2(x)}{\overline{\Pi}_1(x)} = c \in (0,\infty)\,, \tag{3.32}$$

i.e. we assume that both tail integrals are tail-equivalent. Choosing $\overline{\Pi}_1(b(t)) \sim t^{-1}$ we have

$$\lim_{t\to\infty} t\overline{\Pi}_1(b(t)x_1) = \nu_1(x_1,\infty] = x_1^{-\alpha}$$

and

$$\lim_{t\to\infty} t\overline{\Pi}_2(b(t)x_2) = \lim_{t\to\infty} \frac{\overline{\Pi}_2(b(t)x_2)}{\overline{\Pi}_1(b(t))} = \lim_{u\to\infty} \frac{\overline{\Pi}_2(ux_2)}{\overline{\Pi}_2(u)}\frac{\overline{\Pi}_2(u)}{\overline{\Pi}_1(u)} = c\,x_2^{-\alpha}\,.$$

Then we obtain from (3.29) we obtain for $d = 2$

$$\nu([0,(x_1,x_2)]^c) = x_1^{-\alpha} + c\,x_2^{-\alpha} - \left[x_1^{\alpha\delta} + c^{-\delta}\,x_2^{\alpha\delta}\right]^{-1/\delta}\,, \quad x_1 > 0,\, x_2 > 0\,.$$

By differentiating we obtain the density ν' for $0 < \delta < \infty$ (the completely positive dependent case ($\delta \to \infty$) and the independent case ($\delta \to 0$) are not covered by the following calculation) as

$$\nu'(x_1,x_2) = c^{-\delta}\,\alpha^2(1+\delta)\,x_1^{-\alpha(1+\delta)-1} x_2^{\alpha\delta-1}\left(1 + c^{-\delta}\big(\frac{x_2}{x_1}\big)^{\alpha\delta}\right)^{-1/\delta-2}\,,\ x_1 > 0,\, x_2 > 0\,.$$

We then can write

$$
\begin{aligned}
\nu^+(1,\infty] &= \nu\{(x_1,x_2)\in\mathbb{E} : x_1+x_2>1\}\\
&= \nu((1,\infty]\times[0,\infty]) + \int_0^1\int_{1-x_1}^{\infty} \nu'(x_1,x_2)\,dx_2\,dx_1\\
&= \nu_1(1,\infty] + \int_0^1\int_{1-x_1}^{\infty} \nu'(x_1,x_2)\,dx_2\,dx_1\\
&= 1+\alpha\int_0^1 \left(1+c^{-\delta}\big(\frac{1}{x_1}-1\big)^{\alpha\delta}\right)^{-1/\delta-1} x_1^{-1-\alpha}dx_1\,,
\end{aligned}
$$

and substituting $v=\frac{1}{x_1}-1$ we obtain

$$
\begin{aligned}
\nu^+(1,\infty] &= 1+\alpha\int_0^{\infty}\left(1+c^{-\delta}v^{\alpha\delta}\right)^{-1/\delta-1}(1+v)^{\alpha-1}dv\\
&= 1+c^{1/\alpha}\int_0^{\infty}\left(1+s^{\delta}\right)^{-1/\delta-1}(c^{1/\alpha}+s^{-1/\alpha})^{\alpha-1}ds\,. \qquad (3.33)
\end{aligned}
$$

Since $g(y):=(1+y^{\delta})^{-1/\delta-1}, y>0$, is the density of a positive random variable Y_δ, we finally arrive at

$$
\begin{aligned}
\nu^+(1,\infty] &= 1+c^{1/\alpha}\,E[(c^{1/\alpha}+Y_\delta^{-1/\alpha})^{\alpha-1}]\\
&=: 1+c^{1/\alpha}\,C(\alpha,\delta)\,.
\end{aligned}
$$

Then, an analytical approximation for OpVaR follows together with expression (3.31),

$$
\mathrm{VaR}_t^+(\kappa)\sim F_1^{\leftarrow}\left(1-\frac{1-\kappa}{\lambda_1(1+c^{1/\alpha}\,C(\alpha,\delta))\,t}\right),\quad \kappa\uparrow 1\,. \qquad (3.34)
$$

Note that $\mathrm{VaR}_t^+(\kappa)$ increases with $C(\alpha,\delta,c,\lambda_1,\lambda_2)$. For $\alpha=1$ the constant $C(1,\delta)=1$ implies that total OpVaR for all Clayton parameters in the range $0<\delta<\infty$ is given by

$$
\mathrm{VaR}_t^+(\kappa)\sim F_1^{\leftarrow}\left(1-\frac{1-\kappa}{\lambda_1(1+c)\,t}\right)=F_1^{\leftarrow}\left(1-\frac{1-\kappa}{(\lambda_1+c_2\lambda_2)\,t}\right),\quad \kappa\uparrow 1\,,
$$

which is (independent of the dependence parameter δ) equal to the independent OpVaR of Theorem 3.12. Note also the relation $c=\lambda_2/\lambda_1 c_2$ between the different constants in (3.22) and (3.32). Furthermore, $\nu^+(1,\infty]$ is greater or less than 2, according as α is greater or less than 1, respectively. If $\alpha\,\delta=1$ we can solve the integral in (3.33) (similarly to Example 3.8 of Bregman and Klüppelberg [9]) and obtain

$$
\nu^+(1,\infty]=\frac{c^{1-1/\alpha}-1}{c^{1/\alpha}-1}\,.
$$

□

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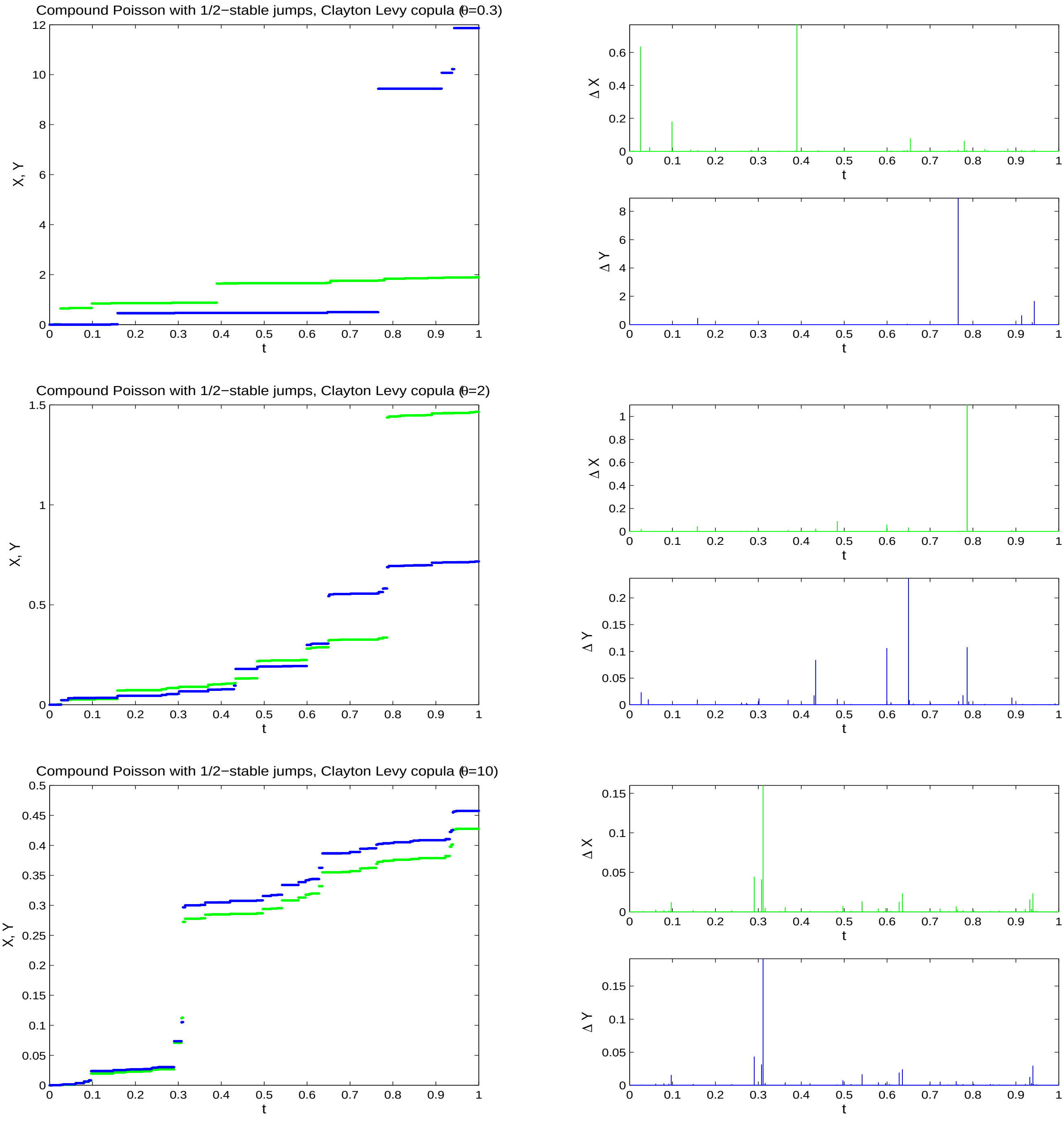

Figure 3.21. Two-dimensional LDA Clayton-1/2-stable model (the severity distribution belongs to $\mathcal{R}_{-1/2}$) for different dependence parameter values. *Left column:* compound processes, *right column:* frequencies and severities.
Upper row: $\delta = 0.3$ (low dependence), *middle row:* $\delta = 2$ (medium dependence), *lower row:* $\delta = 10$ (high dependence).