Assessing the VaR of a portfolio using D-vine copula based multivariate GARCH models

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Abstract

We perform Bayesian joint estimation of a multivariate GARCH model where the dependence structure of the innovations across the univariate time series is given by a D-vine copula. Vine copulas are a flexible concept to extend bivariate copulas to the multivariate case. It is based on the idea that a multivariate copula can be constructed from (conditional) bivariate copulas. In particular it is possible to allow for symmetric dependence between some pairs of margins by using e.g. bivariate Student t or Gaussian copulas and asymmetric dependence between other pairs using e.g. bivariate Clayton or Gumbel copulas. A further advantage of D-vine copulas is that the resulting correlation matrix is always positive definite without imposing restrictions on the parameters. In contrast to likelihood based estimation methods a Bayesian approach always allows to construct valid interval estimates for any quantity which is a function of the model parameters. This provides the possibility to assess the uncertainty about Value at Risk (VaR) predictions. In a simulation study and two real data examples with up to 5 dimensions we compare the proposed model to a benchmark multivariate GARCH model with dependence structure of the innovations governed by a multivariate Student t copula. The proposed model shows a clearly better fit according to the DIC. The choice between the two models also affects

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the VaR predictions. We further study the error introduced by the widely used two step estimation approach in the VaR prediction. This shows that the two step estimation approach leads to an underestimation of the uncertainty of VaR predictions for simulated and real data.
Keywords: Multivariate GARCH model, D-vine copula, Bayesian inference, joint estimation, two step estimation, Value at Risk.

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1. Introduction

Since the proposal of (G)ARCH models by Engle (1982) and Bollerslev (1986) to account for variance heterogeneity in financial time series a number of multivariate extensions of GARCH models have been introduced. See Bauwens et al. (2006) for an overview. In particular the CCC GARCH model of Bollerslev (1990) and later the DCC GARCH of Tse and Tsui (2002) and Engle (2002) are based on multivariate Gaussian distributions, where care has to be taken to result in positive definite covariance matrices. Another way of extending univariate GARCH models is to use copulas for modeling the residual dependence between assets (e.g. Patton, 2006a,b; Jondeau and Rockinger, 2006; Ausin and Lopes, 2009; Min and Czado, 2010a). This has the advantage of allowing for non Gaussian dependencies. In many applications only the two dimensional case is considered.

In two dimensions many parametric copula families exist. Before the consideration of a pair copula construction (PCC), the class of multivariate copulas were sparse, consisting majorly out of elliptical copulas (see e.g. Frahm et al., 2003) and Archimedian copulas (see e.g. Nelsen, 1999) governed by a single parameter. PCC based vine copulas go back to Joe (1996) and Bedford and Cooke (2002). The major advantage over standard multivariate copulas are their flexibility. They are constructed using only bivariate copulas called pair copulas. They model the (conditional) distribution of certain pairs of variables, which make up a regular vine (R-vine) copula.

Many different factorizations are possible and Bedford and Cooke (2002) organized them using a sequence of trees. In the first tree the pairs of variables whose distribution enter the model directly are identified, while the second tree gives the indices for the distribution of pairs of variables conditioned on a single variable. The next tree indicates the indices for the distribution of pairs of variables conditional on two variables. The last tree identifies the distribution of a pair of variables conditioned on the remaining variables. Popular subclasses are C-vines and D-vines. See Kurowicka and Cooke (2006) for more details on this construction principle and Czado (2010) and Kurowicka and Joe (2011) for overviews, current developments and applications. In particular they nest the multivariate Gauss and t-copula. No restriction on the choice of the pair copula family is needed and they always result in positive definite correlation matrices. Different tail behavior for pairs of variables can be modeled, including asymmetric and heavy tail dependence. The flexibility of vine copula based GARCH models over alternative multivariate copulas has been demonstrated empirically in Fischer et al. (2009). To estimate the parameters of a specified vine copula model in I dimensions based on an i.i.d sample Aas et al. (2009) developed a sequential approach for the $I*(I+1)/2$ pair copula parameters of a D-vine copula. This requires only bivariate optimization if each pair copula in the D-vine is parametrized by a single parameter. Consistency and asymptotic normality of the sequential estimates are shown by Haff (2011). These sequential estimates can be used as a starting values for the joint maximum likelihood (ML) estimation of all copula parameters. Similar estimation methods are developed for C-vine copulas Czado et al. (2010) and Dißmann (2010) for R-vine copulas.

Returning to copula based GARCH models there are several ways to facilitate parameter estimation. Joint ML estimation for copula based GARCH models has been only used in two dimensions using maximization by parts in Liu and Luger (2009). Most commonly a two step estimation approach is applied, which avoids highdimensional optimization to determine the joint MLE of marginal and copula parameters. In this approach, univariate GARCH models are fitted to each margin separate and fitted innovations given by standardized residuals are determined in a first step. In a second step these innovations are either transformed parametrically (Shih and Louis, 1995; Joe and Xu, 1996) or nonparametrically (Oakes, 1994; Shih and Louis, 1995; Genest et al., 1995) to pseudo copula data.

Non parametric transformations of the innovations for each margin are popular, since one wants to guard against misspecifications of the marginal models. Chen and Fan (2006) develop the asymptotic theory under this misspecification and use their theory to develop tests for comparing different copula models. In addition Kim et al. (2007) quantify the effects of the marginal misspecification. They report large effects in 2 dimensional cases where the margins are severely misspecified.

In two step estimation approaches the copula parameters are then estimated based on the pseudo copula data. If ML estimation is used at each step of the two step estimation approaches the resulting estimates are consistent and asymptotically normal but not asymptotically efficient (Chen and Fan, 2006; Joe, 2005). Since the resulting asymptotic covariance matrix is often analytically intractable, it is estimated using numerical estimation of the Hessian matrix or bootstrapping (Joe, 2005; Chen and Fan, 2006; Min and Czado, 2010a). This might result for high dimensions in non negative definite Hessian matrix estimates or in extremely high computational effort when bootstrapping is used.

In contrast to likelihood based estimation methods a Bayesian approach always allows to construct valid interval estimates for parameters and any quantity which is a function of the parameters. Ardia (2008) applied Bayesian Markov Chain Monte Carlo (MCMC) methods to univariate GARCH models, while Ausin and Lopes (2009) develop joint Bayesian inference methods for two dimensional copula based GARCH models with time varying effects. Min and Czado (2010a) use a two step estimation approach, where GARCH margins are fitted separately and transformed nonparametrically to pseudo copula data. MCMC algorithms are then developed to estimate parameters in a D-vine copula with bivariate t-copulas as building blocks. The approach was illustrated in a five dimensional application. Subsequently Bayesian model selection methods were developed to simplify D-vine copula models by Min and Czado (2010b); Smith et al. (2010) again utilizing a two step estimation approach.

This opens the question about the tractability of joint Bayesian inference in copula based models in higher dimensions. Joint Bayesian inference in a D-vine copula with regression margins and Gaussian building blocks were developed by Lanzendörfer (2009), while a D-vine copula with AR(1) margins and t-copula building blocks were studied by Czado et al. (2011). We note the simple structure of the marginal models allowed.

The following tasks are tackled in the paper

- Since copula based multivariate GARCH models have been widely applied to financial data we want to demonstrate that a joint Bayesian estimation approach is feasible in higher dimensional copula based GARCH models. For this we choose GARCH(1,1) margins with t innovations coupled with a D-vine copula where the building blocks can be chosen individually from a catalog of 4 parametric bivariate copula families.

- For financial applications accurate forecasting of the Value at risk (VaR) is essential. Therefore we investigate the influence of the choice of the multivariate copula as well as the estimation approach (joint or two step) on these forecasts.

- The Bayesian approach allows to assess uncertainty in these forecasts. We want to quantify these when different copula models and/or different estimation approaches are used.

The paper is organized as follows: In Section 2 we will review D-vine copulas. The model definition and the estimation in a Bayesian setup using Markov Chain Monte Carlo (MCMC) are given in Section 3 and 4. In Section 5 we conduct a simulation study to assess the influence of misspecification of the copula model with regard to DIC (Spiegelhalter et al., 2002) and VaR. We further study the error introduced by the two step estimation approach in the estimation of the model parameters and one step ahead VaR values. Section 6 and 7 contain two real data applications involving US$ exchange rate data and mixed stock and bond index data, respectively. The article concludes with a discussion and an outlook in Section 8.

2. D-vine distributions and copulas

To explain the construction of D-vine copulas, we start with the fundamental theorem by Sklar (1959). He shows that a multivariate distribution function (cdf) $F_{1:I}(x_1, \ldots, x_I) := F_{(X_1,\ldots,X_I)'}(x_1, \ldots, x_I)$ of a random vector $\boldsymbol{X} = (X_1, \ldots, X_I)'$ with continuous margins $F_1(x_1), \ldots,$ $F_I(x_I)$ and corresponding quantile functions $F_1^{-1}(x_1), \ldots,$ $F_I^{-1}(x_I)$ can be written as

$$F_{1:I}(x_1, \ldots, x_I) \quad = \quad C_{1:I}(F_1(x_1), \ldots, F_I(x_I)), \tag{1}$$

where $C_{1:I}(u_1, \ldots, u_I) := F_{1:I}(F_1^{-1}(u_1), \ldots, F_I^{-1}(u_I))$ is called a copula function. The copula function can be identified as the cdf of $\boldsymbol{U} = (U_1, \ldots, U_I)$, where $U_i = F_i(X_i)\ \forall i = 1, \ldots, I$. From (1) we see that we can choose the dependency structure as captured by the copula function independently from the marginal distributions to construct multivariate distributions. Applying the chain rule for differentiation to (1) the joint density (pdf) of $\boldsymbol{X}$ can be identified as

$$f_{1:I}(x_1, \ldots, x_I) \quad = \quad c_{1:I}(F_1(x_1), \ldots, F_I(x_I)) f_1(x_1) \cdots f_I(x_I),$$

where $c_{1:I}$ is the corresponding pdf of $C_{1:I}$ and $f_i(\cdot)$ are marginal densities corresponding to X_i for $i = 1, \ldots, I$. In particular for $I = 2$ we can express the conditional pdf of X_1 given X_2 as

$$f_{1|2}(x_1|x_2) \quad = \quad \frac{f_{1:2}(x_1, x_2)}{f_2(x_2)} = c_{1:2}(F_1(x_1), F_2(x_2))f_1(x_1). \tag{2}$$

To ease the notation we denote by $f_{j|\boldsymbol{v}}$ and $F_{j|\boldsymbol{v}}$ the conditional pdf and cdf of X_j given $\boldsymbol{X_v} := (X_{j_1}, \ldots, X_{j_n})$ for a set of indices $\boldsymbol{v} := \{j_1, \ldots, j_n\}$, respectively. Additionally index sets of the form $\boldsymbol{v} = \{i, i+1, \ldots, j\}$ are denoted by $i : j$. A simple construction of a D-vine distribution was given by Czado (2010) based on a recursive factorization of the joint pdf and application of (1) to the bivariate conditional distributions. For convenience of the reader we recall the major steps of this development. First decompose $f_{1:I}$ as

$$f_{1:I}(x_1, \ldots, x_I) \quad = \quad [\prod_{j=2}^{I} f_{j|1:(j-1)}(x_j|\boldsymbol{x}_{1:(j-1)})]f_1(x_1). \tag{3}$$

Applying (3) to the conditional distribution of X_1 and X_j given $\boldsymbol{X}_{2:(j-1)}$ we can express each factor in (3) as

$$\begin{aligned} f_{j|1:(j-1)}(x_j|\boldsymbol{x}_{1:(j-1)}) \quad = \quad & c_{1j|2:(j-1)}(F_{1|2:(j-1)}(x_1|\boldsymbol{x}_{2:(j-1)}), F_{j|2:(j-1)}(x_j|\boldsymbol{x}_{2:(j-1)})) \\ & f_{j|2:(j-1)}(x_j|x_{2:(j-1)}). \end{aligned} \tag{4}$$

To achieve tractability of the model we make the assumption that the copula corresponding to the bivariate distribution of X_1 and X_j given $\boldsymbol{X}_{2:(j-1)} = \boldsymbol{x}_{2:(j-1)}$ does not depend on the conditioning value $\boldsymbol{x}_{2:(j-1)}$. Therefore we denote the corresponding copula pdf by $c_{1j|2:(j-1)}$. This restriction is however not severe (see Haff et al., 2010). Using (4) recursively we get

$$\begin{aligned} f_{j|1:(j-1)}(x_j|\boldsymbol{x}_{1:(j-1)}) \quad = \quad & [\prod_{k=1}^{j-2} c_{kj|(k+1):(j-1)}(F_{k|(k+1):(j-1)}(x_k|\boldsymbol{x}_{(k+1):(j-1)}), \\ & F_{j|(k+1):(j-1)}(x_j|\boldsymbol{x}_{(k+1):(j-1)}))]c_{j-1,j}(F_{j-1}(x_{j-1}), F_j(x_j))f_j(x_j) \end{aligned} \tag{5}$$

for the factors in (3). A similar tractability assumption is made. Inserting (5) into (3) we rewrite (3) as

$$\begin{aligned} f_{1:I}(x_{1:I}) &= [\prod_{j=2}^{I}\prod_{k=1}^{j-1} c_{kj|(k+1):(j-1)}(F_{k|(k+1):(j-1)}(x_k|\boldsymbol{x}_{(k+1):(j-1)}), F_{j|(k+1):(j-1)}(x_j|\boldsymbol{x}_{(k+1):(j-1)}))] \\ & [\prod_{i=1}^{I} f_i(x_i)]. \end{aligned} \tag{6}$$

Here we made the convention that a conditional pdf (cdf) with empty conditioning set corresponds to an unconditional pdf (cdf). The conditional cdf's $F_{j|\boldsymbol{v}}$ occurring in (6) for $\boldsymbol{v} = \{k, j_1, \ldots, j_n\}$ and $k < j$ can be obtained recursively (see Joe (1996) for details) as

$$F_{j|\boldsymbol{v}}(x_j|\boldsymbol{x}_{\boldsymbol{v}}) = \frac{\partial C_{kj|\boldsymbol{v}_{-k}}(F_{k|\boldsymbol{v}_{-k}}(x_k|\boldsymbol{x}_{\boldsymbol{v}_{-k}}), F_{j|\boldsymbol{v}_{-k}}(x_j|\boldsymbol{x}_{\boldsymbol{v}_{-k}}))}{\partial F_{k|\boldsymbol{v}_{-k}}(x_k|\boldsymbol{x}_{\boldsymbol{v}_{-k}})}, \tag{7}$$

where $\boldsymbol{v}_{-k} = \{j_1, \ldots, j_n\}$. We call the right hand side of (6) the pdf of a D-vine distribution with marginal densities f_i and pairwise copula densities $c_{kj|(k+1):(j-1)}$ for $j = 2, \ldots, I$ and $k = 1, \ldots, j-1$. These copula densities can be chosen individually by the modeler. A D-vine copula is a D-vine distribution where the marginal densities are uniform densities on $[0, 1]$.

3. D-vine copula based multivariate GARCH models

We construct for multivariate time series data $\boldsymbol{Y}_t = (Y_{1t}, \ldots, Y_{It})'$ for $t = 1, \ldots, T$ a copula based GARCH model. In particular we assume for the ith margin a GARCH(1,1) model given by

$$\begin{aligned} Y_{it}|h_{it} &:= \sqrt{h_{it}}\epsilon_{it}, \quad \epsilon_{it} \sim G_{\boldsymbol{\varphi}_i} \quad \text{independent for} \quad t = 1, \ldots, T & (8) \\ h_{it} &:= \omega_i + \alpha_i Y_{i,t-1}^2 + \beta_i h_{i,t-1}, \quad t = 1, \ldots, T & (9) \\ Y_{i0} &:= 0, \quad h_{i0} := 0, & (10) \end{aligned}$$

where $\omega_i > 0$, $\alpha_i \geq 0$, $\beta_i \geq 0$ and $\boldsymbol{\varphi}_i$ are unknown marginal parameters. Here $G_{\boldsymbol{\varphi}_i}$ denotes the corresponding cdf of the innovation distribution with mean zero and unit variance. Later we need the quantile and density function, which we denote by $G_{\boldsymbol{\varphi}_i}^{-1}$ and $g_{\boldsymbol{\varphi}_i}$, respectively. In our examples we use an appropriately scaled Student t distribution, i.e. $\epsilon_{it} := \frac{\nu_i^{(G)}-2}{\nu_i^{(G)}}\tilde{\epsilon}_{it}$ where $\tilde{\epsilon}_{it}$ is standard univariate t distributed with $\nu_i^{(G)}$ degrees of freedom. The quantity h_{it} can be interpreted as conditional variance of Y_{it} given h_{it}.

The innovation vectors $\boldsymbol{\epsilon}_t := (\epsilon_{1t}, \ldots, \epsilon_{It})'$, $t = 1, \ldots, T$ are assumed to be i.i.d for $t = 1, \ldots, T$. The dependence structure among the components of $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \ldots, \epsilon_{It})'$ for fixed t is assumed to be given by an I dimensional D-vine distribution. The pdf of the innovation vector $\boldsymbol{\epsilon}_t$ is therefore given by

$$\begin{aligned} f_{\boldsymbol{\epsilon}_t}(\epsilon_{1t}, \ldots, \epsilon_{It}) &= [\prod_{j=2}^{I} \prod_{k=1}^{j-1} c_{kj|(k+1):(j-1)}(F_{k|(k+1):(j-1)}(\epsilon_{kt}|\epsilon_{k+1,t}, \ldots, \epsilon_{j-1,t}), \\ & \quad F_{j|(k+1):(j-1)}(\epsilon_{jt}|\epsilon_{k+1,t}, \ldots, \epsilon_{j-1,t}))][\prod_{i=1}^{I} g_{\boldsymbol{\varphi}_i}(\epsilon_{it})]. \end{aligned} \tag{11}$$

Here $F_{l|(k+1):(j-1)}(\cdot)$ denotes the conditional cdf of $\epsilon_{l,t}$ given $\epsilon_{k+1,t}, \ldots, \epsilon_{j-1,t}$. The unconditional cdfs that are passed to the copulas are the marginal cdfs of the innovations $F_i(\epsilon_{it}) = G_{\boldsymbol{\varphi}_i}(\epsilon_{it})$ and the conditional cdfs are defined according to the recursion (7).

The likelihood of the observations $\boldsymbol{Y}_t$ given the conditional variances $\boldsymbol{h}_t = (h_{1t}, \ldots, h_{It})'$ for $t = 1, \ldots, T$ can be obtained from (11) by a change of variables using the transformation $\boldsymbol{y}_t := (y_{1t}, \ldots, y_{It})' = \boldsymbol{a}(\epsilon_{1t}, \ldots, \epsilon_{It}) := (\epsilon_{1t}\sqrt{h_{1t}}, \ldots, \epsilon_{It}\sqrt{h_{1t}})'$. Since the determinant of the Jacobian matrix from $\boldsymbol{a}^{-1}(y_{1t}, \ldots, y_{It}) = (\frac{y_{1t}}{\sqrt{h_{1t}}}, \ldots, \frac{y_{It}}{\sqrt{h_{It}}})$ is given by $\prod_{i=1}^{I} \frac{1}{\sqrt{h_{it}}}$ we get

$$\begin{aligned} f_{\boldsymbol{Y}_t|\boldsymbol{h}_t}(\boldsymbol{y}_t|\boldsymbol{h}_t) &= \left[\prod_{j=2}^{I} \prod_{k=1}^{j-1} c_{kj|(k+1):(j-1)}\left(F_{k|(k+1):(j-1)}\left(\frac{y_{kt}}{\sqrt{h_{kt}}}\middle|\frac{y_{k+1,t}}{\sqrt{h_{k+1,t}}}, \ldots, \frac{y_{j-1,t}}{\sqrt{h_{j-1,t}}}\right),\right.\right. \\ & \quad \left.\left. F_{j|(k+1):(j-1)}\left(\frac{y_{jt}}{\sqrt{h_{jt}}}\middle|\frac{y_{k+1,t}}{\sqrt{h_{k+1,t}}}, \ldots, \frac{y_{j-1,t}}{\sqrt{h_{j-1,t}}}\right)\right)\right]\left[\prod_{i=1}^{I} \frac{1}{\sqrt{h_{it}}} g_{\boldsymbol{\varphi}_i}\left(\frac{y_{it}}{\sqrt{h_{it}}}\right)\right] \end{aligned} \tag{12}$$

The variance parameters of the i-th marginal time series are denoted by $\boldsymbol{\eta}_i := (\omega_i, \alpha_i, \beta_i)'$ and jointly by $\boldsymbol{\eta} := (\boldsymbol{\eta}_1', \ldots, \boldsymbol{\eta}_I')'$. The marginal parameters of the GARCH innovations are collected in $\boldsymbol{\varphi} := (\boldsymbol{\varphi}_1', \ldots, \boldsymbol{\varphi}_I')'$. The vector of pair copula parameters is denoted by $\boldsymbol{\theta} := \{\boldsymbol{\theta}_{kj|(k+1):(j-1)}', j = 2, \ldots, I, k = 1, \ldots, j-1\}$.

We will call the model defined in this section a **G-DV** model. To distinguish between Gaussian and Student t innovations we use the notation **NG** and **tG**, respectively. The types of the pair copulas in one D-vine copula can either be the same or we can have different types of pair copulas. In the first case we write **NDV**, **tDV**, **CDV** and **GDV** for a D-vine copula with all Gaussian, Student t, Clayton and Gumbel pair copulas, respectively. In the second case, i.e. a mixed D-vine copula, we write **mDV**. For the independent I dimensional copula and the multivariate Student t copula with a common degree of freedom (df) we will write **I** and **tM**, respectively. So tG-tDV is a G-DV model with Student t innovations for each margin and a D-vine copula consisting of all Student t pair copulas, while tG-tM is a multivariate GARCH model with Student t innovations of the margins and a multivariate Student t copula for the dependence structure of the innovations. We will also compare joint estimation with two step estimation, where for the latter we will use the prefix **2s**, i.e. 2s-tG-tDV denotes a tG-tDV model that is estimated using a two step estimation approach.

The Gaussian D-vine copula is equivalent to a multivariate Gaussian copula. The conditional correlations correspond to partial correlations and there exist a one-to-one correspondence between partial and unconditional correlations (Bedford and Cooke, 2002; Kurowicka and Cooke, 2006). Also a multivariate Student t-copula with a common df parameter ν can be represented as a special D-vine copula. The association parameters in a D-vine copula with all pair t-copulas correspond to partial correlations. If the df parameters of each pair copula satisfy

$$\nu_{kj|(k+1):(j-1)} = \nu + (j-1-k), \quad j = 2, \ldots, I, \quad k = 1, \ldots, j-1, \tag{13}$$

then the D-vine copula with all pair t-copulas is a multivariate t-copula with df ν.

Defining $D_t := diag(\sqrt{h_{1t}}, \ldots, \sqrt{h_{It}})'$ and $R := \mathrm{Cor}(\boldsymbol{\epsilon}_t)$ to be the correlation of the innovations $\boldsymbol{\epsilon}_t = (\epsilon_{1t}, \ldots, \epsilon_{It})'$ induced by the D-vine copula, we have $\mathrm{Cov}(\boldsymbol{Y}_t|\boldsymbol{Y}_{1:t-1}) = D_t R D_t$. This implies that the conditional covariance $\mathrm{Cov}(\boldsymbol{Y}_t|\boldsymbol{Y}_{1:t-1})$ is time varying, despite a time constant correlation R. This shows that in the case of Gaussian pair copulas the G-DV model can be regarded as a CCC type model discussed in Bollerslev (1990). Note that for any choice of partial correlations of a G-tDV or G-NDV model the resulting correlation matrix R is positive definite. This shows that G-DV models can be regarded as non Gaussian extensions to CCC models. An algorithm to sample from a G-DV model is given in Appendix A.

3.1. Selection of the D-vine copula structure

The structure of a D-vine copula is fixed up to the order in which the variables enter the D-vine copula specification (6). An order and its inverse order lead to the same D-vine copula. Therefore there are $I!/2$ possible D-vine structures. Since the number of possible D-vine copula structures is increasing rapidly with the dimension I it is not possible in reasonable time to estimate all possible D-vine copula models and compare them using a model fit criterion like DIC. Aas et al. (2009) suggest to find an order of the variables X_i, $i = 1, \dots, I$ in which the pairs (X_j, X_{j+1}) for $j = 1, \dots, I-1$ have the strongest dependence among all other orderings of the variables. In some application a natural order of the components exist, such if one wants to jointly model measurements taken at different times of the day. The problem of searching for an optimal D-vine structure is still an open problem. However this is not the focus here.

4. Bayesian Inference using MCMC

4.1. Prior specifications

Priors are chosen to enforce parameter restrictions. For positive marginal parameters we choose a normal $N(0, \sigma^2)$ distribution truncated to $[0, \infty)$. We denote this distribution by $N_{[0,\infty)}(0, \sigma^2)$. Its mean and variance is given by $2\phi(0)\sigma$ and $(2\phi(0)\sigma)^2$, respectively. Here $\phi(\cdot)$ denotes the standard normal pdf. For the df parameter of GARCH(1,1) models with Student t innovations, we want to enforce certain finite moments. Therefore we choose the translated exponential distribution $\text{Exp}(\delta, \lambda)$ with pdf

$$f(x; \delta, \lambda) = \lambda \exp(-\lambda(x - \delta))\mathbb{1}_{(\delta,\infty)}(x). \tag{14}$$

For a finite mean of a Student t innovation we just set $\delta = 2$. This choice also puts higher prior probabilities to smaller df values, thus inducing high prior probabilities to heavier tail innovations. Each prior for a marginal parameter is assumed to be independent.

For the copula parameters we impose independent priors to the parameters of each pair copula. Since we have no prior information about the dependence we enforce a uniform prior on $[-1, 1]$ for the induced Kendalls τ parameter in case of a Student or Gaussian copula and on $[0, 1]$ in case of a Clayton or Gumbel copula. The prior densities for the association parameter of each pair copula are obtained by a change of variables. This covers the cases of Gaussian, Clayton and Gumbel pair copulas. For a Student t pair copula we choose again a translated exponential distribution for the df parameter. Finally we assume prior independence between the marginal and copula parameters. The prior choices are summarized in Table 1.

Marginal GARCH Priors	Copula Priors		
	Bivariate Copula	Kendalls τ	Prior
$\omega_i \sim N_{[0,\infty)}(0,\sigma^2_{\omega_i})$	Gaussian	$\tau = 2\arcsin(\theta)/\pi$	$f(\theta) = \frac{\pi}{2}\frac{1}{\sqrt{1-\theta^2}}\mathbb{1}_{[-1,1]}(\theta)$
$\alpha_i \sim N_{[0,\infty)}(0,\sigma^2_{\alpha_i})$	Clayton	$\tau = 1 - 2/\theta$	$f(\theta) = \frac{2}{\theta^2}\mathbb{1}_{[0,1)}(\theta)$
$\beta_i \sim N_{[0,\infty)}(0,\sigma^2_{\beta_i})$	Gumbel	$\tau = 1 - 1/\theta$	$f(\theta) = \frac{1}{\theta^2}\mathbb{1}_{[1,\infty)}(\theta)$
$\nu_i^{(G)} \sim \text{Exp}(2,\lambda_i^{(G)})$	Student	$\tau = 2\arcsin(\theta)/\pi$	$f(\theta) = \frac{\pi}{2}\frac{1}{\sqrt{1-\theta^2}}\mathbb{1}_{[-1,1]}(\theta)$ $\nu^{(C)} \sim \text{Exp}(\delta^{(C)},\lambda^{(C)})$

Table 1: Prior choice for the marginal and copula parameters and relationship between the association parameter of the copula and Kendall's τ for different pair copula types. To keep notation simple we omit the indices for the copula parameters. We denote the parameters of a bivariate Student t copula by $\boldsymbol{\theta} = (\theta, \nu^{(C)})'$.

4.2. Posterior distribution

The posterior distribution of the marginal parameters $\boldsymbol{\eta}$ and $\boldsymbol{\varphi}$, the copula parameters $\boldsymbol{\theta}$ and the auxiliary variables $\boldsymbol{h} = (\boldsymbol{h}'_1,\ldots,\boldsymbol{h}'_T)'$ given the data $\boldsymbol{Y} = (\boldsymbol{Y}'_1,\ldots,\boldsymbol{Y}'_T)'$ can be derived as:

$$
\begin{aligned}
f(\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta},\boldsymbol{h}|\boldsymbol{Y}) &\propto f(\boldsymbol{Y},\boldsymbol{h},\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta}) \\
&= [\prod_{t=1}^{T} f(\boldsymbol{Y}_t,\boldsymbol{h}_t|\boldsymbol{Y}_1,\ldots,\boldsymbol{Y}_{t-1},\boldsymbol{h}_1,\ldots,\boldsymbol{h}_{t-1},\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta})]f(\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta}) \\
&= [\prod_{t=1}^{T} f(\boldsymbol{Y}_t,\boldsymbol{h}_t|\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta})]f(\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta}) \\
&= [\prod_{t=1}^{T} f(\boldsymbol{Y}_t|\boldsymbol{h}_t,\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta})f(\boldsymbol{h}_t|\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta})]f(\boldsymbol{\eta},\boldsymbol{\varphi},\boldsymbol{\theta}) \\
&= [\prod_{t=1}^{T} f(\boldsymbol{Y}_t|\boldsymbol{h}_t,\boldsymbol{\varphi},\boldsymbol{\theta})f(\boldsymbol{h}_t|\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta})]f(\boldsymbol{\eta})f(\boldsymbol{\varphi})f(\boldsymbol{\theta}). \qquad (15)
\end{aligned}
$$

The conditional densities $f(\boldsymbol{Y}_t|\boldsymbol{h}_t,\boldsymbol{\varphi},\boldsymbol{\theta})$ for $t = 1,\ldots,T$ are given in (12). The factors $f(\boldsymbol{h}_t|\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta})$ for $t = 1,\ldots,T$ are deterministic and given by $f(\boldsymbol{h}_t|\boldsymbol{Y}_{t-1},\boldsymbol{h}_{t-1},\boldsymbol{\eta}) = \prod_{i=1}^{I}\mathbb{1}_{\omega_i+\alpha_i y^2_{i,t-1}+\beta_i h_{i,t-1}}(h_{it})$, where the indicator function $\mathbb{1}_A(x)$ equals one if $x \in A$ and zero otherwise. Note that $Y_{i0} = 0$ and $h_{i0} = 0$ for $i = 1,\ldots,I$ according to the model definition in (10). The prior densities $f(\boldsymbol{\eta})$, $f(\boldsymbol{\varphi})$ and $f(\boldsymbol{\theta})$ are given in Section 4.1.

A sample from the posterior distribution of the parameters is obtained by MCMC methods. For details see Appendix B. Appendix B also contains the description of the Bayesian two step estimation approach used in Section 5-7. Further it contains the description of the MCMC algorithm used for the multivariate Student copula as a special case of the D-vine copula.

4.3. *Prediction of the Value at Risk (VaR)*

Model comparison will be based amongst others on the VaR. More specifically, we will compare different models based on the posterior predictive distribution of the VaR for a portfolio of the margins. A portfolio is a weighted average of the margins. The VaR is a lower quantile of the distribution of a portfolio. The absolute value of the $(1-\alpha)*100\%$ VaR from the predictive distribution of a portfolio gives the loss that is not exceeded with probability α. An algorithm to sample from the posterior predictive distribution of the VaR can be found in Appendix C.

5. Simulation study

We study the performance of Bayesian joint and two step estimation of tG-tDV and tG-mDV models including model misspecifications effects. As performance criteria we use the deviance information criteria (DIC) of Spiegelhalter et al. (2002) and evaluate the performance of the one step ahead posterior predictive pdf of a 99% VaR for an equally weighted portfolio. Finally we quantify the effects of using a two step instead of a joint approach with regard to posterior median estimates and the length of 99% posterior credible intervals.

We choose 8 parameter scenarios for tG-tDV or tG-mDV models in 4 dimensions requiring 6 pair copulas as given in Table 2. To keep notation simple we use alternative indices defined by $(1,2,3,4,5,6)' := (12,23,34,13|2,24|3,14|23)'$. The 8 scenarios include different marginal

	Marginal Parameters				**Copula Parameters**		
Scenario	ω_i	α_i	β_i	$\nu_i^{(G)}$	Copula Types	$(\theta_1,\ldots,\theta_6)$	$(\nu_1^{(C)},\ldots,\nu_6^{(C)})$
1	0.01	0.07	0.92	6	(t,t,t,t,t,t)	0.8	3
2	0.01	0.07	0.92	6	(t,t,t,t,t,t)	0.3	3
3	0.01	0.07	0.92	6	(t,t,t,t,t,t)	0.3	(3,10,5,3,3,3)
4	0.01	0.07	0.92	6	(C,C,C,t,t,C)	(1,1,1,0.5,0.5,1)	(-,-,-,5,5,-)
5	0.05	0.25	0.7	6	(t,t,t,t,t,t)	0.8	3
6	0.05	0.25	0.7	6	(t,t,t,t,t,t)	0.3	3
7	0.05	0.25	0.7	6	(t,t,t,t,t,t)	0.3	(3,10,5,3,3,3)
8	0.05	0.25	0.7	6	(C,C,C,t,t,C)	(1,1,1,0.5,0.5,1)	(-,-,-,5,5,-)

Table 2: Parameter specification of the simulated data. A single value for a vector means that all elements of the vector have this value. Copula type: t (Student t); C (Clayton).

GARCH(1,1) specifications as well as different tail and (a)symmetry behaviors of the D-vine copula. The marginal GARCH(1,1) specifications are weakly stationary with finite second moments. For the GARCH specification $\omega_i = 0.01$, $\alpha_i = 0.07$ and $\beta_i = 0.92$ the 4th moment exist and is equal to 5.91, while the other specification does not have a finite 4th moment. This specification also shows a higher persistence $\alpha_i + \beta_i$ compared to the other one.

From each scenario we simulated a data set with $T = 1000$ and estimated 4 models using a joint Bayesian approach: (1) The correct **tG-tDV** (Scenario 1, 2, 3, 5, 6, 7) or **tG-mDV** (Scenario 4, 6) model; (2) A **tG-NDV** model assuming a Gaussian D-Vine copula which is equivalent to a multivariate Gaussian copula; (3) A **tG-I** model assuming independence between the marginal univariate time series which is equivalent to estimate univariate GARCH models for each marginal time series; (4) A **tG-tM** model, assuming a multivariate Student t copula. The tG-tM model is estimated using fully Bayesian joint estimation in contrast to e.g. Fischer et al. (2009) who used two step ML estimation. Additionally, we estimate the tG-tDV or tG-mDV model using two step estimation **2s-tG-tDV** or **2s-tG-mDV** described in Appendix B. This is a fully Bayesian two step estimation in contrast to Min and Czado (2010a) who use a mixed ML-Bayesian two step estimation.

The priors are, where possible, chosen that the prior mean is equal to the true parameter. For Scenario 1 we therefore choose $\omega_i \sim N_{(0,\infty)}(0, 0.005/\phi(0))$, $\alpha_i \sim N_{(0,\infty)}(0, 0.035/\phi(0))$, $\beta_i \sim N_{(0,\infty)}(0, 0.46/\phi(0))$, $\nu_i^{(G)} \sim Exp(2, 1/4)$, $i = 1, \dots 4$ and $\tau_j \sim U(-1, 1)$, $\nu_j^{(C)} \sim Exp(2, 1/3)$, $j = 1, \dots, 6$. The priors for other scenarios are chosen accordingly. For the multivariate Student t copula we choose the prior mean of the single df parameter to be the true df parameter of the copula c_{12}.

For each model we obtain 4000 MCMC samples, running the MCMC algorithm for 60000 iterations after a pre-run of 10000 to determine appropriate proposal variances and a burn in of 5000, taking every 15th sample. In case of the 2s-tG-tDV or 2s-tG-mDV estimation approach, this is done for both steps.

First we compare the four models and the two step estimation approach based on the Bayesian model fit criterion DIC (Spiegelhalter et al., 2002) shown in Table 3 for each of the 8 scenarios. As expected, the true tG-tDV or tG-mDV model shows the best fit for all 8 scenarios. The tG-NDV model and the tG-I model are far away from the best model in terms of DIC. The tG-tM model shows a relatively good fit for the Scenarios 1, 2, 5 and 6. This is not surprising since these scenarios are relatively close to a multivariate Student t copula and the tG-tM models require 5 fewer parameters compared to the tG-tDV models. For Scenarios 3 and 7 with different df of the Student t pair copulas in the D-vine copula the fit of the tG-tM model

	Joint estimation				two step estimation
Scenario	tG-I	tG-NDV	tG-tM	tG-t/mDV	2s-tG-t/mDV
1	8475	2562	1941	**1920**	2322
2	8590	8125	7557	**7546**	7582
3	8605	8134	7745	**7687**	7719
4	8580	6863	6775	**6408**	6473
5	8198	2668	1668	**1647**	2067
6	8406	8172	7363	**7354**	7392
7	8411	8154	7542	**7485**	7522
8	8325	6602	6510	**6149**	6218

Table 3: Simulated data: DIC Table.

is notable worse compared to the best fitting tG-tDV model. For Scenarios 4 and 8 with different pair copula types in the D-vine copula, including asymmetric pairwise dependence, the fit of the tG-tM model is clearly worse than for the true tG-mDV model.

The two step estimation approach 2s-tG-tDV shows a significant loss of fit in all 8 scenarios compared to the joint estimation approach tG-tDV which is especially high in Scenario 1 and 5. These are the scenarios with the highest correlations in the D-vine copula.

We now have a look at the small sample properties of the two step estimation approach. Table 4 and 5 show the quotients of the posterior median estimates and the 99% credibility interval lengths for the parameters from the two step estimation approach compared to the joint estimation approach, respectively. A quotient of > 1 means that the median or credibility interval length of the two step estimation approach is larger compared to the joint estimation approach. We see that for the marginal parameters the length of the credibility interval is systematically overestimated by the two step estimation approach. For the copula parameters the length of the credibility interval is systematically underestimated by the two step estimation approach compared to the joint estimation approach. Further there are considerable differences between the posterior median estimates of the two approaches. The posterior medians for the association parameters are slightly but systematically underestimated.

Last we consider the effect that model misspecification and the error introduced by the two step estimation approach for the correct tG-tDV/tG-mDV model have on the one step ahead VaR predictions. For each scenario Figure 1 shows the estimated one step ahead posterior predictive pdf of the 99% VaR for an equally weighted portfolio of the four different estimated models and the two step estimation approach. A point estimate for the VaR can be obtained e.g. by the

	Scenario							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Marginal GARCH Parameters								
ω_1	0.95	0.66	0.66	1	**1.47**	**1.34**	**1.37**	**1.7**
α_1	0.82	0.77	0.76	0.8	0.98	1	0.99	**1.05**
β_1	**1.01**	**1.03**	**1.03**	**1.02**	0.92	0.94	0.93	0.89
$\nu_1^{(G)}$	0.97	0.86	0.85	0.89	0.96	0.87	0.86	0.9
ω_2	**1.23**	0.8	0.73	**1.16**	0.89	0.69	0.66	0.97
α_2	0.98	**1.07**	**1.08**	**1.09**	0.86	0.85	0.83	0.91
β_2	0.99	1	1	0.99	**1.03**	**1.15**	**1.17**	**1.05**
$\nu_2^{(G)}$	**1.17**	1	**1.03**	0.94	**1.17**	0.97	**1.01**	0.91
ω_3	0.94	0.75	0.8	0.91	**1.42**	0.97	0.97	**1.04**
α_3	0.75	0.81	0.88	0.81	0.9	0.9	0.94	0.82
β_3	**1.01**	**1.02**	**1.01**	**1.01**	0.93	**1.03**	**1.01**	**1.05**
$\nu_3^{(G)}$	0.99	0.95	**1.01**	**1.01**	0.97	0.95	1	1
ω_4	0.9	**1.06**	**1.16**	**1.25**	**1.27**	0.86	0.9	**1.09**
α_4	0.84	**1.06**	**1.01**	**1.11**	0.89	0.93	0.92	**1.08**
β_4	**1.01**	0.99	0.99	0.99	0.95	**1.04**	**1.04**	0.95
$\nu_4^{(G)}$	**1.1**	0.97	0.92	0.96	**1.11**	0.96	0.92	0.95
Copula Parameters								
θ_1	0.99	0.97	0.97	**1.01**	0.98	0.95	0.97	0.99
θ_2	0.99	0.95	0.97	0.97	0.99	0.96	0.97	0.97
θ_3	0.99	0.94	0.96	0.95	1	0.95	0.98	0.96
θ_4	1	0.99	1	0.99	**1.01**	0.97	0.98	0.99
θ_5	1	0.98	0.97	1	0.99	0.97	1	1
θ_6	1	0.99	0.97	**1.01**	0.98	0.96	0.98	0.98
$\nu_1^{(C)}$	1	0.99	**1.02**	-	**1.01**	**1.01**	1	-
$\nu_2^{(C)}$	**1.02**	**1.02**	0.94	-	**1.06**	**1.02**	0.96	-
$\nu_3^{(C)}$	**1.01**	**1.02**	0.97	-	**1.09**	**1.01**	0.93	-
$\nu_4^{(C)}$	0.99	0.99	0.98	0.99	1	0.99	**1.01**	**1.01**
$\nu_5^{(C)}$	**1.01**	**1.05**	**1.02**	**1.03**	**1.14**	**1.05**	0.97	1
$\nu_6^{(C)}$	**1.05**	**1.04**	**1.04**	-	**1.08**	**1.03**	0.98	-

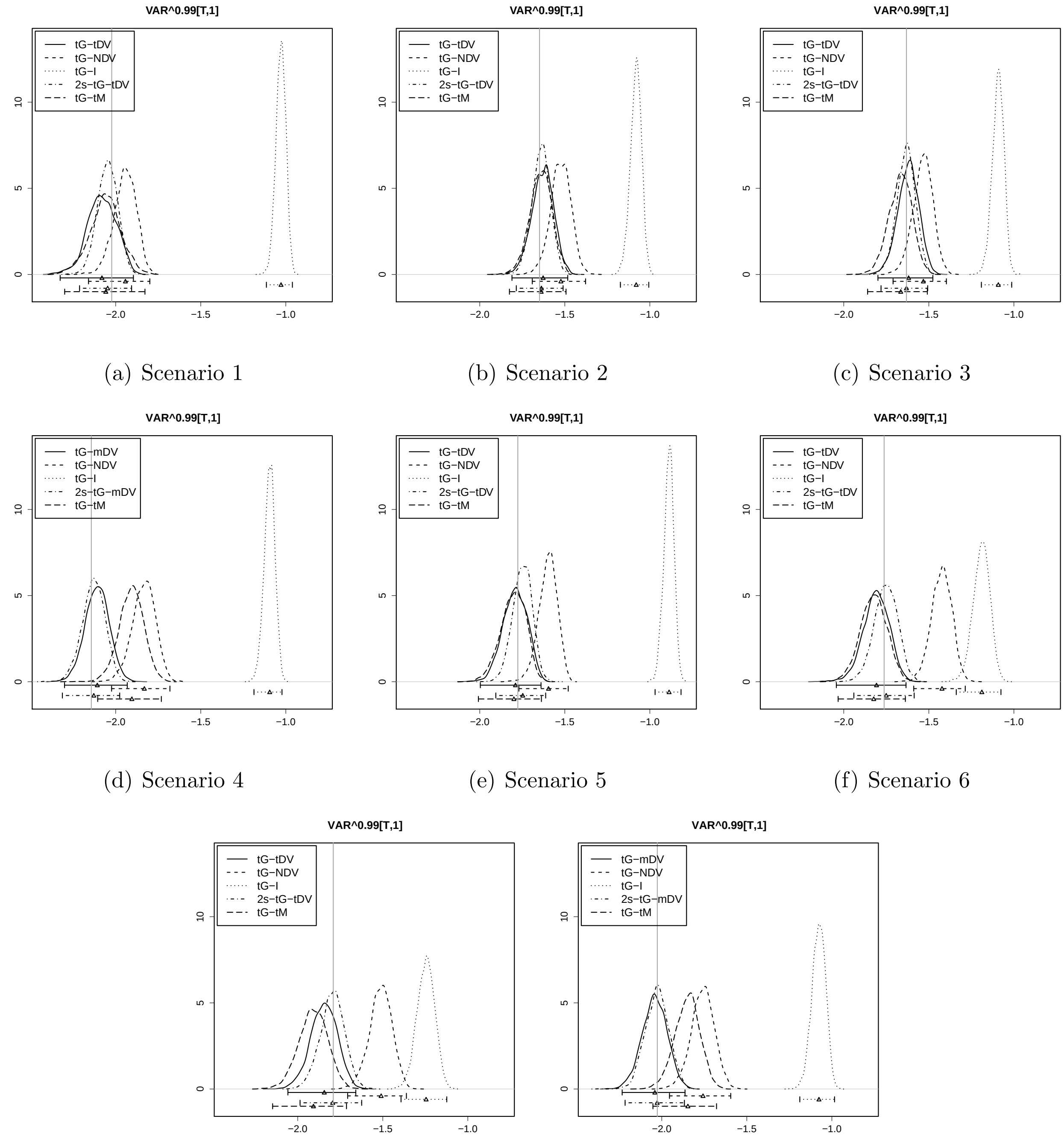
Table 4: The quotient of the posterior median estimates for the parameters from the two step estimation approach compared to the joint estimation approach for the eight simulation scenarios. A quotient of > 1 means that the median length of the two step estimation approach is larger compared to the joint estimation approach. (Bold: "value" > 1)

posterior mode, median or mean. As expected, for the tG-tDV or tG-mDV model the true VaR lies inside the 99% credibility interval for all eight scenarios and the posterior median is quite close to the true value. For the tG-I model assuming independence we observe that the VaR is significantly overestimated, i.e. the absolute value of the VaR is underestimated, in all scenarios. In the case of a tG-NDV estimation model the VaR is significantly overestimated in five out of eight scenarios. These are all scenarios with a lower GARCH persistence (Scenarios 5-8)

	Scenario							
	1	**2**	**3**	**4**	**5**	**6**	**7**	**8**
Marginal GARCH Parameters								
ω_1	**1.69**	0.9	0.93	**1.66**	**3**	**1.79**	**1.87**	**2.46**
α_1	**1.75**	**1.16**	**1.12**	**1.42**	**2.18**	**1.46**	**1.49**	**2**
β_1	**1.98**	**1.05**	**1.03**	**1.45**	**3.02**	**1.66**	**1.67**	**2.04**
$\nu_1^{(G)}$	**1.5**	0.89	0.93	**1.24**	**1.6**	0.98	1	**1.35**
ω_2	**2.3**	0.97	0.77	**1.62**	**1.91**	**1.06**	0.97	**1.33**
α_2	**1.82**	**1.19**	**1.06**	**1.53**	**1.64**	**1.12**	**1.01**	**1.23**
β_2	**2.47**	**1.08**	0.93	**1.54**	**2.19**	**1.2**	**1.11**	**1.26**
$\nu_2^{(G)}$	**1.82**	**1.14**	**1.18**	**1.21**	**2.03**	**1.05**	**1.16**	**1.11**
ω_3	**1.51**	0.89	0.89	**1.17**	**2.52**	**1.23**	**1.21**	**1.68**
α_3	**1.49**	**1.08**	**1.1**	**1.2**	**1.98**	**1.22**	**1.09**	**1.27**
β_3	**1.69**	**1.05**	**1.09**	**1.2**	**2.57**	**1.3**	**1.16**	**1.54**
$\nu_3^{(G)}$	**1.51**	**1.1**	**1.12**	**1.4**	**1.59**	**1.07**	**1.12**	**1.36**
ω_4	**1.76**	**1.57**	**1.57**	**2.28**	**2.91**	**1.24**	**1.28**	**1.83**
α_4	**1.71**	**1.43**	**1.23**	**2.07**	**2.03**	**1.21**	**1.2**	**1.64**
β_4	**2.02**	**1.48**	**1.44**	**2.24**	**3.24**	**1.28**	**1.27**	**1.78**
$\nu_4^{(G)}$	**1.82**	**1.15**	**1.01**	**1.44**	**2.13**	**1.1**	1	**1.43**
Copula Parameters								
θ_1	0.87	0.99	1	0.89	0.95	0.97	0.97	0.87
θ_2	0.87	0.96	0.95	0.86	0.9	**1.01**	0.96	0.86
θ_3	0.87	0.96	0.93	0.81	0.8	0.93	0.93	0.85
θ_4	0.94	1	0.97	0.98	0.85	0.95	0.93	0.96
θ_5	0.92	0.95	0.93	0.93	0.99	0.99	**1.02**	0.97
θ_6	0.94	0.97	0.95	0.96	**1.06**	0.93	1	0.94
$\nu_1^{(C)}$	0.84	0.9	0.93	-	0.86	0.91	0.89	-
$\nu_2^{(C)}$	0.83	0.91	0.86	-	0.96	0.89	0.91	-
$\nu_3^{(C)}$	0.83	0.89	0.91	-	0.98	0.84	0.83	-
$\nu_4^{(C)}$	0.91	0.95	0.92	0.98	**1.01**	0.93	0.89	0.98
$\nu_5^{(C)}$	0.94	0.96	0.95	0.95	**1.11**	0.97	0.86	0.9
$\nu_6^{(C)}$	**1.1**	**1.03**	0.99	-	**1.14**	0.94	0.94	-

Table 5: The quotient of the 99% credibility interval length for the parameters from the two step estimation approach compared to the joint estimation approach for the eight simulation scenarios. A quotient of > 1 means that the credibility interval length of the two step estimation approach is larger compared to the joint estimation approach. (Bold: "value" > 1)

and with asymmetric pairwise dependencies (Scenario 4 and 8). The tG-tM model assuming a multivariate Student t copula is very close to the tG-tDV model and the true VaR for scenarios with common t pair copulas (Scenario 1, 2, 5 and 6). However for Scenarios 3 and 7 assuming different df of the Student t pair copulas we already observe differences. For Scenarios 4 and 8 assuming different pair copula types in the D-vine copula, including asymmetric pairwise dependence, there are considerable differences between the tG-tM model and the tG-mDV



(a) Scenario 1 (b) Scenario 2 (c) Scenario 3

(d) Scenario 4 (e) Scenario 5 (f) Scenario 6

(g) Scenario 7 (h) Scenario 8

Figure 1: Estimated one step ahead posterior predictive pdf of the 99% VaR from an equally weighted portfolio. Below: 99% credibility intervals (lines) and median (triangle). The true one step ahead prediction of the 99% VaR is indicated by a vertical line.

model. For Scenario 8 the true VaR is not even inside the 99% credibility interval of posterior predictive distribution from the VaR for the tG-tM model.

Now we examine the effect of parameter estimation error of the two step estimation approach has on the VaR estimation. In terms of the posterior mode estimate the two step estimation approach for the tG-tDV/tG-mDV model, i.e. 2s-tG-tDV/2s-tG-mDV, differs a bit from the joint estimation approach but not too much. However the two step estimation approach underestimates the uncertainty about the VaR. The length of the 99% credibility interval for the two step estimation approach is between 6% (Scenario 8) and 29% (Scenario 1) shorter than for the joint estimation approach in the 8 considered scenarios with an average of 14%.

6. Application to exchange rate data

We consider a five dimensional dataset consisting of daily US $ exchange rates of Japanese yen (JPY), British pound sterling (GBP), Australian dollar (AUD), Canadian dollar (CAD) and Brazilian real (BRL) observed from July 25, 2005 until July 17, 2009 ($I = 5$, $T = 1005$). We estimate the four models considered in Section 5; tG-mDV, tG-NDV, tG-I and tG-tM and the two step estimation approach for the tG-mDV model, i.e. 2s-tG-mDV to the data.
For the tG-mDV model we have to choose two things:

1. The order of margins, i.e. which margins are connected by an unconditional copula. This completely specifies the structure of the D-vine copula.
2. The copula families for all unconditioned and conditioned pairs.

The structure is chosen by the approach of Aas et al. (2009) which suggest to put the pairs with the highest correlations together. More precisely we examine all pairwise empirical Kendall's $\hat{\tau}$. We select the pair (i, j) with highest $\hat{\tau}_{ij}$. In the next step we select a variable $r \neq i, j$ with highest empirical Kendall's $\hat{\tau}$ value among $A_i \cup B_j$, where $A_i := \{\hat{\tau}_{si}, s \neq i, j\}$ and $B_j := \{\hat{\tau}_{sj}, s \neq i, j\}$. If the maximum is attained in A_i we choose the order $r - i - j$ otherwise $i - j - r$ in the D-vine copula. We proceed in this way until a complete order of the D-vine copula is established. The resulting order is JPY - GBP - AUD - CAD - BRL.
The copula families for each pair are chosen according to the highest AIC (Akaike, 1974). Brechmann et al. (2010) justify this way of proceeding through simulation. In particular for every unconditional bivariate copula in the D-vine copula we estimate all considered copula families (Gaussian, Student, Clayton and Gumbel) using ML and choose the copula with the lowest AIC. To choose the copula families of the pair copulas with one variable in the conditioning set, we calculate the copula data of these copulas based on the estimated unconditional copulas using (7) and choose the best fitting copula according to the AIC. The procedure is continued for the pair copulas with larger number of variables in the conditioning set until

copula families are specified for all bivariate copulas. The chosen pair copula families for this data set are given in the first column of Table 7.

The prior specifications are shown in Table D.10. Priors for the marginal GARCH parameters were chosen similar to ML estimates of GARCH(1,1) models fitted to several financial time series. Priors for the df parameters were chosen to approximately include the Gaussian case.

The DIC for the four models and the two step estimation approach is shown in Table 6. The tG-mDV model fits the data best. The fit of the tG-tM model is worse compared to the tG-mDV model. The tG-NDV model shows a significantly worse fit compared to the best fitting tG-mDV model, while the tG-I model is far away from the other models in terms of DIC. The two step estimation approach 2s-tG-mDV shows a significant loss in fit compared to the joint estimation approach tG-mDV according to the DIC.

tG-I	tG-NDV	tG-tM	tG-mDV	2s-tG-mDV
-35541	-36808	-36923	**-36938**	-36886

Table 6: DIC values for several models applied to the exchange rate data.

Figure 2 (top row) shows the estimated one step ahead posterior predictive distribution of the 90%, 99% and 99.9% VaR from an equally weighted portfolio for the 4 estimated models and the two step estimation approach. While the tG-I is far away from the best fitting models, we see that the 90% VaR for the other 3 models are relatively close, while in case of the 99% VaR and especially the 99.9% VaR we see differences.

The two step estimation approach 2s-tG-mDV clearly overestimates the VaR compared to the joint estimation approach tG-mDV. Further we observe that the uncertainty about the VaR is underestimated by the two step estimation approach compared to the joint estimation approach as in the simulation study in Section 5.

Figure 2 (bottom row) shows the estimated posterior predictive pdf of the covariance stationarity condition $CSC = \alpha + \beta - 1$ for the for marginal GARCH(1,1) models of the tG-mDV model. A GARCH(1,1) process is covariance stationary if $CSC < 0$. The copula has only influence on the dependency between the marginal GARCH(1,1) models but not on the marginal distribution. The properties of the margins with respect to stationarity and existence of moments are therefore retained in the multivariate model. We see that there is a high estimated posterior probability for the five time series to be stationary with a relatively high posterior estimate for

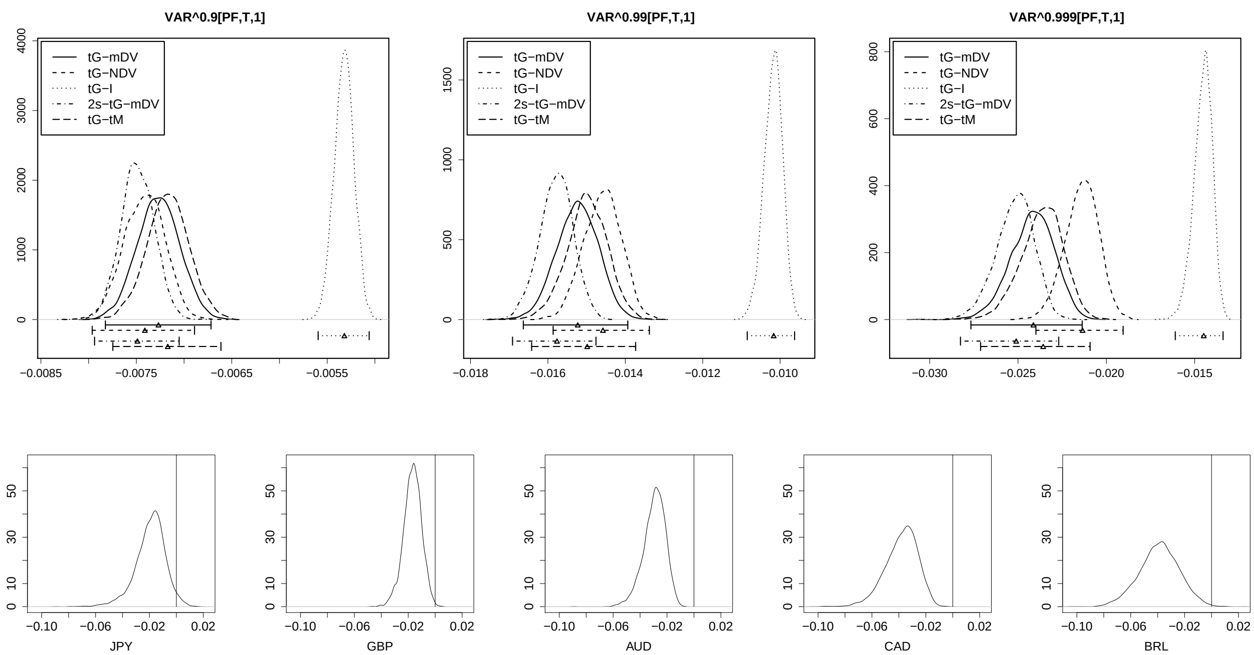


Figure 2: tG-tDV: Exchange rate data: Top row: Estimated one step ahead posterior predictive distribution of the 90% (left), 99%(middle) and 99.9%(right) VaR for an equally weighted portfolio. Below: 99% credibility intervals (lines) and median (triangle). Bottom row: Estimated posterior pdf of the covariance (weak) stationarity condition $CSC = \alpha + \beta - 1$ for each margin. (A GARCH(1,1) process is covariance stationary if $CSC < 0$).

the persistence $(\alpha_i + \beta_i)$.

Table 7 shows a summary of the posterior distribution from the parameters for the tG-mDV and the 2s-tG-mDV estimation approach. The median and CI ratios of the parameters estimates confirm the results from the simulation study in Section 5.

7. Application to a mixed portfolio of stock and bond indices

As a second application we consider a dataset of two stock and two bond indices: Treasury Yield 5 Years (^FVX), CBOE Interest Rate 10-Year T-No (^TNX), iShares Dow Jones US Utilities (IDU) and iShares Dow Jones US Healthcare (IYH). The data are given as the daily log returns from February 26, 2006 until February 25, 2010. Usually a portfolio would not only contain these four assets. However, for simplicity we concentrate on these four assets between which an asymmetric dependency can be observed as can be seen in the following.

The structure in the D-vine copula and the copula families are chosen in the same way as in Section 6. The resulting order is IDU - IYH - ^FVX - ^TNX. The chosen copula families are given in the first column of Table 9.

	tG-mDV			2s-tG-mDV			Quotients	
	2.5%	50%	97.5%	2.5%	50%	97.5%	median	CI length
Marginal GARCH parameters								
ω_{JPY}	5.9e-07	1.2e-06	2.6e-06	6.7e-07	1.4e-06	3.1e-06	**1.17**	**1.17**
α_{JPY}	0.03	0.06	0.10	0.04	0.07	0.12	**1.26**	**1.19**
β_{JPY}	0.87	0.93	0.95	0.85	0.91	0.94	0.98	**1.25**
$\nu^{(G)}_{JPY}$	3.80	4.93	6.76	4.00	5.30	7.39	**1.07**	**1.21**
ω_{GBP}	4.8e-07	8.0e-07	1.3e-06	4.7e-07	8.9e-07	1.6e-06	**1.11**	**1.33**
α_{GBP}	0.03	0.05	0.07	0.05	0.07	0.11	**1.40**	**1.61**
β_{GBP}	0.91	0.93	0.95	0.87	0.91	0.94	0.98	**1.58**
$\nu^{(G)}_{GBP}$	4.94	6.89	10.63	5.16	7.65	12.44	**1.11**	**1.34**
ω_{AUD}	1.2e-06	1.9e-06	3.1e-06	8.2e-07	1.6e-06	3.1e-06	0.88	**1.20**
α_{AUD}	0.03	0.05	0.08	0.07	0.11	0.16	**1.97**	**1.90**
β_{AUD}	0.88	0.92	0.94	0.82	0.88	0.92	0.96	**1.52**
$\nu^{(G)}_{AUD}$	5.49	7.51	10.63	4.83	6.88	10.34	0.92	**1.03**
ω_{CAD}	9.8e-07	1.7e-06	2.8e-06	4.3e-07	9.3e-07	1.8e-06	0.56	0.74
α_{CAD}	0.04	0.06	0.09	0.06	0.08	0.13	**1.41**	**1.40**
β_{CAD}	0.85	0.90	0.94	0.86	0.90	0.93	1.00	0.97
$\nu^{(G)}_{CAD}$	4.92	6.88	10.38	4.42	6.39	9.74	0.93	0.97
ω_{BRL}	2.4e-06	3.8e-06	6.0e-06	1.5e-06	2.9e-06	5.0e-06	0.75	1.00
α_{BRL}	0.11	0.15	0.20	0.16	0.22	0.29	**1.45**	**1.45**
β_{BRL}	0.76	0.81	0.85	0.72	0.78	0.83	0.96	**1.28**
$\nu^{(G)}_{BRL}$	7.41	11.73	20.92	5.95	9.11	16.19	0.78	0.79
Copula parameters								
(t) $\theta_{JPY,GBP}$	0.16	0.23	0.30	0.16	0.23	0.29	0.99	0.94
(t) $\theta_{GBP,AUD}$	0.54	0.59	0.63	0.54	0.58	0.62	0.99	0.89
(t) $\theta_{AUD,CAD}$	0.51	0.56	0.60	0.49	0.53	0.57	0.96	0.84
(t) $\theta_{CAD,BRL}$	0.35	0.41	0.46	0.33	0.38	0.44	0.94	0.94
(t) $\theta_{JPY,AUD\|GBP}$	-0.14	-0.07	-0.00	-0.11	-0.04	0.02	0.62	0.97
(N) $\theta_{GBP,CAD\|AUD}$	0.11	0.17	0.23	0.11	0.18	0.24	**1.03**	**1.03**
(t) $\theta_{AUD,BRL\|CAD}$	0.29	0.35	0.41	0.29	0.35	0.40	0.98	0.95
(N) $\theta_{JPY,CAD\|GBP,AUD}$	-0.18	-0.12	-0.06	-0.17	-0.11	-0.05	0.93	0.93
(t) $\theta_{GBP,BRL\|AUD,CAD}$	-0.11	-0.05	0.02	-0.10	-0.04	0.03	0.78	0.95
(N) $\theta_{JPY,BRL\|GBP,AUD,CAD}$	-0.24	-0.18	-0.12	-0.24	-0.18	-0.12	1.00	0.92
$\nu^{(C)}_{JPY,GBP}$	2.82	3.82	5.50	3.01	3.93	5.45	**1.03**	0.95
$\nu^{(C)}_{GBP,AUD}$	5.29	8.77	17.38	5.37	8.80	17.81	1.00	**1.07**
$\nu^{(C)}_{AUD,CAD}$	6.00	9.93	19.47	6.50	10.90	21.78	**1.10**	**1.05**
$\nu^{(C)}_{CAD,BRL}$	4.69	7.20	13.01	4.93	7.60	14.17	**1.05**	**1.13**
$\nu^{(C)}_{JPY,AUD\|GBP}$	4.82	7.59	14.08	5.16	8.07	15.46	**1.06**	**1.09**
$\nu^{(C)}_{AUD,BRL\|CAD}$	4.93	7.69	14.07	4.86	7.28	13.25	0.95	0.89
$\nu^{(C)}_{GBP,BRL\|AUD,CAD}$	6.86	11.82	23.75	7.06	12.25	24.28	**1.04**	**1.15**

Table 7: Exchange rate data: Estimated posterior quantiles for the parameters based on a tG-mDV and 2s-tG-mDV estimation approach, respectively, and the quotients of the posterior median estimates and the credibility interval lengths of the two step estimation approach compared to the joint estimation approach. A quotient of > 1 means that the median or credibility interval length of the two step estimation approach is larger compared to the joint estimation approach (Bold: "value" > 1). The copula type is given in front of the (first) copula parameter of each pair copula (Student: (t), Gaussian: (N)).

Additionally to the described tG-mDV model we again estimate a tG-tM , tG-NDV and tG-I model and the two step estimation approach for the tG-mDV model, i.e. 2s-tG-mDV. We use the same prior specifications as in Section 6. Table 8 presents the DIC for the four estimated models and the two step estimation approach. The tG-mDV model shows a clearly better fit to the data than the other three models including the model assuming a multivariate Student t copula, tG-tM. This also gives evidence that mixed symmetric and asymmetric pairwise dependence can be found in a portfolio of real data.

tG-I	tG-NDV	tG-tM	tG-mDV	2s-tG-mDV
-23142	-26129	-26219	**-26255**	-26141

Table 8: DIC values for several models applied to the mixed stock and bond index data.

Figure 3 (top row) shows the estimated one step ahead posterior predictive distribution of the 90%, 99% and 99.9% VaR for an equally weighted portfolio. Especially in case of 99.9% we see

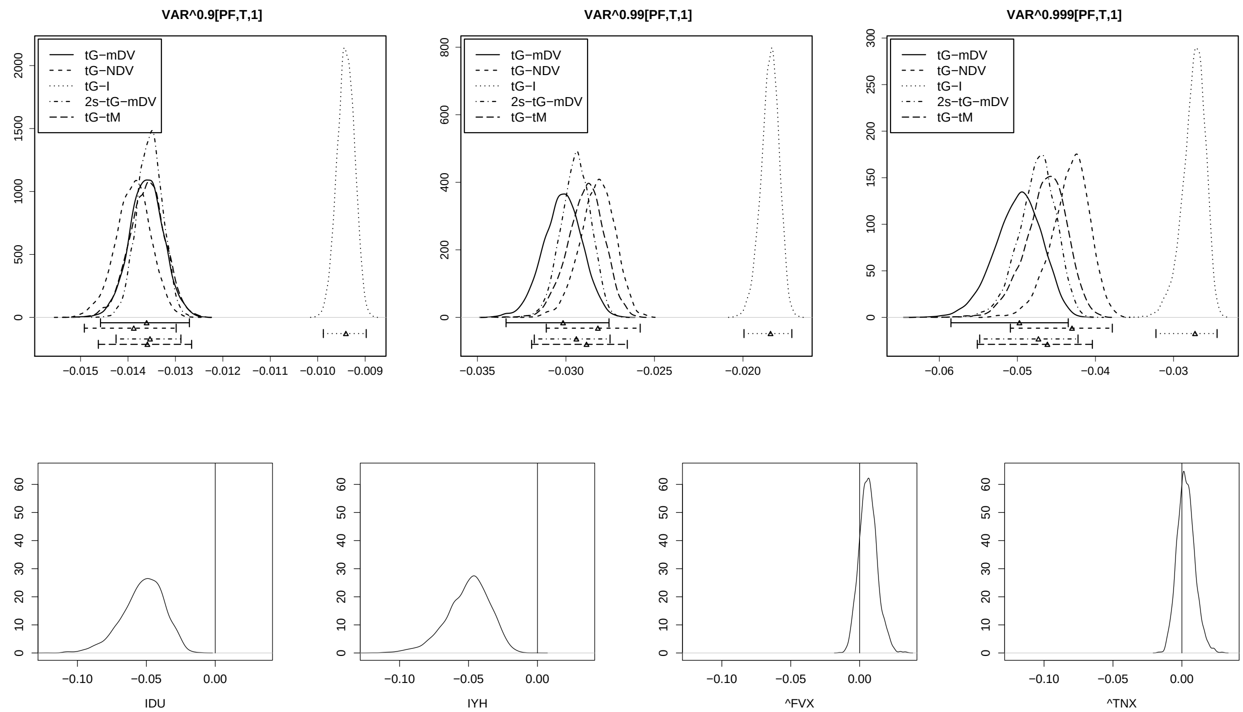


Figure 3: tG-tDV: Mixed stocks and bonds: Top row: Estimated one step ahead posterior predictive distribution of the 90% (left), 99%(middle) and 99.9%(right) VaR for an equally weighted portfolio. Below: 99% credibility intervals (lines) and median (triangle). Bottom row: Estimated posterior pdf of the covariance (weak) stationarity condition $CSC = \alpha + \beta - 1$ for each margin. (A GARCH(1,1) process is covariance stationary if $CSC < 0$).

clearly different estimates for the four models. Compared to the best fitting tG-mDV model the 99.9% VaR is overestimated by the other three models including the model assuming a multivariate Student t copula. Further we see that the order of the estimated posterior medians depends on the level of the VaR (90%, 99%, 99.9%). While compared to the tG-mDV the tG-NDV model underestimates the 90% VaR it overestimates the 99.9% VaR. Figure 3 (bottom row) shows the posterior distribution of the covariance stationarity condition $CSC = \alpha + \beta - 1$ for each margin of the tG-mDV model. While the stock indices look stationary the bond indices have a relatively large posterior probability of being non-stationary.

Table 9 shows a summary of the posterior distribution from the parameters for the tG-mDV model and the 2s-tG-mDV estimation approach. The median and credibility interval length ratios of the parameters estimates confirm the results from the simulation study in Section 5.

	tG-mDV			**2s-tG-mDV**			**Quotients**	
	2.5%	50%	97.5%	2.5%	50%	97.5%	median	CI length
Marginal GARCH parameters								
ω_{IDU}	4.0e-06	6.7e-06	1.1e-05	2.3e-06	4.3e-06	7.4e-06	0.64	0.73
α_{IDU}	0.07	0.10	0.15	0.09	0.13	0.19	**1.27**	**1.12**
β_{IDU}	0.78	0.84	0.89	0.79	0.84	0.89	1.00	0.93
$\nu^{(G)}_{IDU}$	6.89	10.67	19.26	6.28	9.88	18.86	0.93	**1.03**
ω_{IYH}	3.0e-06	5.1e-06	8.7e-06	1.7e-06	3.5e-06	6.6e-06	0.69	0.90
α_{IYH}	0.06	0.09	0.14	0.08	0.12	0.18	**1.27**	**1.31**
β_{IYH}	0.79	0.86	0.91	0.79	0.86	0.90	1.00	**1.02**
$\nu^{(G)}_{IYH}$	5.12	7.05	10.43	4.55	6.37	9.76	0.90	**1.01**
$\omega_{\wedge FVX}$	1.4e-06	2.6e-06	4.4e-06	1.6e-06	3.4e-06	6.5e-06	**1.34**	**1.78**
$\alpha_{\wedge FVX}$	0.06	0.09	0.12	0.06	0.10	0.15	**1.15**	**1.73**
$\beta_{\wedge FVX}$	0.90	0.92	0.94	0.86	0.90	0.94	0.98	**1.83**
$\nu^{(G)}_{\wedge FVX}$	4.64	5.81	7.50	4.89	6.89	10.18	**1.19**	**1.87**
$\omega_{\wedge TNX}$	1.1e-06	2.0e-06	3.5e-06	1.5e-06	3.1e-06	5.7e-06	**1.52**	**1.87**
$\alpha_{\wedge TNX}$	0.06	0.09	0.12	0.06	0.09	0.14	**1.08**	**1.46**
$\beta_{\wedge TNX}$	0.89	0.92	0.94	0.86	0.90	0.94	0.99	**1.70**
$\nu^{(G)}_{\wedge TNX}$	5.19	6.74	9.22	5.44	7.94	12.87	**1.18**	**2.00**
Copula parameters								
(t) $\theta_{IDU,IYH}$	0.61	0.65	0.69	0.60	0.64	0.67	0.98	0.92
(C) $\theta_{IYH,\wedge FVX}$	0.28	0.36	0.46	0.28	0.36	0.45	1.00	0.93
(t) $\theta_{\wedge FVX,\wedge TNX}$	0.95	0.95	0.96	0.94	0.95	0.96	1.00	0.87
(C) $\theta_{IDU,\wedge FVX\|IYH}$	0.01	0.06	0.14	0.01	0.06	0.13	0.95	0.95
(N) $\theta_{IYH,\wedge TNX\|\wedge FVX}$	-0.14	-0.08	-0.02	-0.13	-0.07	-0.01	0.92	0.95
(N) $\theta_{IDU,\wedge TNX\|IYH,\wedge FVX}$	-0.07	-0.01	0.05	-0.07	-0.01	0.05	0.39	0.92
$\nu^{(C)}_{IDU,IYH}$	5.04	8.13	15.69	4.91	7.85	15.35	0.97	0.93
$\nu^{(C)}_{\wedge FVX,\wedge TNX}$	3.07	4.32	6.48	3.30	4.57	6.69	**1.06**	**1.04**

Table 9: Mixed stock and bond index data: Estimated posterior quantiles for the parameters based on a tG-mDV and 2s-tG-mDV estimation approach, respectively, and the quotients of the posterior median estimates and the credibility interval lengths of the two step estimation approach compared to the joint estimation approach. A quotient of > 1 means that the median or credibility interval length of the two step estimation approach is larger compared to the joint estimation approach (Bold: "value" > 1). The copula type is given in front of the (first) copula parameter of each pair copula (Student: (t), Gaussian: (N), Clayton: (C)).

8. Conclusion and Outlook

We performed Bayesian joint estimation for a multivariate GARCH model where the univariate margins follow GARCH(1,1) models and the dependence between the innovations across the univariate time series is given by a D-vine copula. Vine copulas are flexible class of multivariate copulas using only (conditional) bivariate copulas. It allows for symmetric dependence between some pairs of margins and asymmetric dependence between other pairs. D-vine copulas have the advantage that the resulting correlation matrix is always positive definite. In contrast to likelihood based estimation methods a Bayesian approach always allows to construct valid interval estimates for any quantity which is a function of the model parameters. This provides the possibility to assess the uncertainty about VaR predictions. In a simulation study and two real data applications we compared the approach to other models including a multivariate GARCH model assuming a multivariate Student t copula for the dependence of the innovations. Model comparison was based on the DIC and one step ahead VaR. We further compared a Bayesian two step estimation approach with the Bayesian joint estimation approach to assess the small sample properties of the parameter estimates from the Bayesian two step estimation approach and their effect on the VaR predictions.

Our findings were the following:

- The D-vine copula can provide a clearly better fit according to the DIC compared to the multivariate Student t copula for the dependence of the GARCH innovations which has been shown in an application to a dataset of stock and bond indices. This is due to the fact that in a D-vine copula it is possible to allow for symmetric dependence for some (conditional) pairs of margins and asymmetric dependence for other pairs. This is not possible in a multivariate Student copula where all dependencies are symmetric. Is is also not possible for the multivariate extension of Clayton of Gumbel copulas, where all pairwise dependencies are asymmetric.

- The choice between a D-vine copula and a multivariate Student t copula for the dependence of the margins clearly affects the VaR prediction.

- The application to a dataset of stock and bond indices shows that both symmetric and asymmetric pairwise dependencies can be present within one portfolio.

- Compared to the Bayesian joint estimation approach the Bayesian two step estimation approach can show a considerable lack of fit according to the DIC.

- The Bayesian two step estimation approach further leads to an underestimation of the uncertainty of one step ahead VaR. This result is not directly transferable to the ML case. However comparable results can also be expected in the ML case. This would mean that the variance of a two step ML estimator for the VaR is smaller compared to the joint ML estimator which is efficient.

An interesting extension of the model would be to allow for time variation of the copula parameters in the D-vine copula. Here the time varying bivariate dependency copula model of

Almeida and Czado (2010) can be used as a model for the bivariate building blocks. An alternative would be to allow for an observation driven time variation of the copula parameters as considered in Ausin and Lopes (2009). A natural extension would also be the extension to other vines like C-vines or more general R-vines. Financial data often show asymmetric dependence that can not be captured by a Clayton or Gumbel copula. In this case it would be interesting to include BB1 or BB7 pair copulas that allow for individual tail dependence in the upper and lower tail. Since the considered bivariate asymmetric copulas are restricted to positive correlations one could include rotated versions to allow for negative correlations in these copula families. It is well known that beside asymmetric dependency one can find asymmetric conditional variance in ups and downs caused by the leverage effect (Black, 1976). It would be interesting to see how these two asymmetries work together, by considering margins that follow e.g. a GJR model (Glosten et al., 1993). A further extension would be to allow for ARMA-GARCH margins or higher orders of the GARCH margins.

An open question is still the choice of the best or at least a good fitting D-vine copula among the many choices. Min and Czado (2010b) and Min and Czado (2010a) used the computer intensive Bayesian model selection approaches of Green (1995) and Congdon (2006), respectively, to choose the pair copula families in a D-vine. These approaches could be extended to include the D-vine order. One could also transfer the concepts of truncation and simplification as considered in Brechmann et al. (2010) and Brechmann and Czado (2011) to the Bayesian framework to facilitate higher dimensional applications ($I > 20$).

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Appendix A. Simulation from the G-DV model

Algorithm 1 Simulation from the G-DV model

Obtain T samples from a I dimensional D-vine copula, i.e. $u_{1:I,1:T}$, using Algorithm 2 described in Aas et al. (2009)

for $i \leftarrow 1 : I$

  Start with $h_{i,0} = 0$, $y_{i,0} = 0$.

  for $t \leftarrow 1 : T$

    $h_{i,t} = \omega_i + \alpha_i y_{i,t-1}^2 + \beta_i h_{i,t-1}; \quad \epsilon_{i,t} = G_{\boldsymbol{\varphi}_i}^{-1}(u_{i,t}); \quad y_{i,t} = \sqrt{h_{i,t}}\epsilon_{i,t}$

  end for t

end for i

Appendix B. Update of the parameters within the MCMC algorithm

Appendix B.1. Update of the marginal GARCH parameters

Variance parameters: We first update the variance parameters $\boldsymbol{\eta}_i = (\omega_i, \alpha_i, \beta_i)'$ of each margin $i = 1, \ldots, I$ within the MCMC algorithm individually for 10000 iterations. We therefore use individual Metropolis-Hastings (MH) steps (Metropolis et al., 1953; Hastings, 1970) with univariate Gaussian random walk proposals truncated at 0,

$$\omega_i^*|\omega_i \sim N_{[0,\infty)}(\omega_i, \tilde{\sigma}_{\omega_i}^2), \quad \alpha_i^*|\alpha_i \sim N_{[0,\infty)}(\alpha_i, \tilde{\sigma}_{\alpha_i}^2), \quad \beta_i^*|\beta_i \sim N_{[0,\infty)}(\beta_i, \tilde{\sigma}_{\beta_i}^2),$$

where ω_i^*, α_i^* and β_i^* are the proposed values and ω_i, α_i and β_i are the current values of the parameters. The variances $\tilde{\sigma}_{\omega_i}^2$, $\tilde{\sigma}_{\alpha_i}^2$ and $\tilde{\sigma}_{\beta_i}^2$ are tuned to achieve acceptance rates between 20% and 80% as proposed in Besag et al. (1995). Since the samples of the variance parameters are highly correlated, we use the first 8000 samples after a burn in of 2000 to get an approximation of the covariance matrix from the posterior distribution for the variance parameters of each magin, $\widetilde{\Sigma}_i$, and proceed with a joint update for $\boldsymbol{\eta}_i = (\omega_i, \alpha_i, \beta_i)'$, $i = 1, \ldots, I$ using a MH step with a three dimensional Gaussian random walk proposal with $\widetilde{\Sigma}_i$ as proposal covariance matrix, which is again truncated at 0 for all three parameters,

$$\boldsymbol{\eta}_i^*|\boldsymbol{\eta}_i \sim N_{[0,\infty)^3}(\boldsymbol{\eta}_i, \widetilde{\Sigma}_i).$$

Here $\boldsymbol{\eta}_i^*$ is the proposed value and $\boldsymbol{\eta}_i$ the current value of the parameter vector in the MCMC algorithm and $N_{[0,\infty)^3}(\boldsymbol{\mu}, \Sigma)$ denotes the trivariate normal distribution truncated to $[0,\infty)^3$, with mean and covariance matrix parameter $\boldsymbol{\mu}$ and Σ, respectively. This leads to lower autocorrelation between the samples with acceptance rates ranging between 7% and 42% for the considered simulated and real data. A similar procedure is used in Ausin and Lopes (2009).

Update of the df parameters: The df parameters are updated using individual MH steps using truncated Gaussian random walk proposals, i.e.

$$\nu_i^{(G)*}|\nu_i^{(G)} \quad \sim \quad N_{[\delta_i^{(G)},\infty)}\left(\nu_i^{(G)}, \tilde{\sigma}^2_{\nu_i^{(G)}}\right),$$

$i = 1, \ldots, I$, where $\nu_i^{(G)*}$ is the proposed value and $\nu_i^{(G)}$ the current value of the parameter. Here $\tilde{\sigma}^2_{\nu_i^{(G)}}$ is the proposal variance, that is tuned to get acceptance rates between 20% and 80%.

Appendix B.2. Update of the copula parameters

The copula parameters $\theta_{kj|(k+1):(j-1)}$ and $\nu^{(C)}_{kj|(k+1):(j-1)}$, $j = 2, \ldots, I$, $k = 1, \ldots, j-1$, are updated individually within the MCMC algorithm, using MH steps with Gaussian random walk proposals, truncated to the domain of the parameters. The proposal variances are again tuned in order to get acceptance rates between 20% and 80%. To facilitate reading we will omit the indices in the following.

Update of the association parameters: In case of a Gaussian or Student pair copula, we use a truncated random walk proposal for the association parameter,

$$\theta^*|\theta \quad \sim \quad N_{[-1,1]}\left(\theta, \tilde{\sigma}^2_\theta\right),$$

where θ is the current state, θ^* is the proposed value, and $\tilde{\sigma}^2_\theta$ is the proposal variance. In the case of a Clayton or Gumbel pair copula, we use a Gaussian random walk proposal for Kendall's τ, which gives the following proposal distributions for θ,

$$\begin{aligned} f(\theta^*|\theta) \quad &:= \quad \phi_{[0,\infty]}\left(1 - \frac{2}{\theta^*}; 1 - \frac{2}{\theta}, \tilde{\sigma}^2_\theta\right)\frac{2}{\theta^2} \\ f(\theta^*|\theta) \quad &:= \quad \phi_{[1,\infty]}\left(\frac{1}{\theta^*}; \frac{1}{\theta}, \tilde{\sigma}^2_\theta\right)\frac{1}{\theta^2} \end{aligned}$$

in case of a Clayton or Gumbel pair copula term, respectively.

Update of the df parameters: The df parameters are updated using individual MH steps with truncated Gaussian random walk proposals, i.e.

$$\nu^{(C)*}|\nu^{(C)} \quad \sim \quad N_{[\delta^{(C)},\infty)}\left(\nu^{(C)}, \tilde{\sigma}^2_{\nu^{(C)}}\right), \tag{B.1}$$

where $\nu^{(C)*}$ is the proposed value, $\nu^{(C)}$ the current value and $\tilde{\sigma}^2_{\nu^{(C)}}$ the proposal variance.

The multivariate Student t copula is estimated as a special case of a D-vine copula. The prior and proposal distribution for the df parameter of the multivariate Student t copula is chosen to be the same as the prior and proposal distribution for a Student t pair copula in a D-vine copula (Section 4.1 and (B.1)). The likelihood of the multivariate Student t copula for the proposed and current value of the df parameter, that is needed in the acceptance probability of the MH step, can be evaluated as a special case of a D-vine copula with all Student t pair copulas. The df of the corresponding D-vine copula can be calculated from the proposed and current value of the df parameter for the multivariate Student t copula using (13). The other parameters of the tG-tM model are updated in the same way as the corresponding parameters in the tG-tDV model.

In case of a Bayesian two step estimation we first estimate the margins using a G-I model, i.e. a multivariate GARCH model assuming independence. Then we calculate the copula data using the posterior mode estimates of the parameters of the G-I model and the probability integral transform. In a second step we estimate a DV model for the copula data.

Appendix C. Prediction of the Value at Risk (VaR)

Algorithm 2 Generation of a sample of size R from the one step ahead posterior predictive distribution of the $(1-\alpha)*100\%$ VaR of an equally weighted portfolio based on an MCMC sample of size R from the posterior distribution of the parameters, $(\boldsymbol{\eta}^{(r)'}, \boldsymbol{\varphi}^{(r)'}, \boldsymbol{\theta}^{(r)'})'$, $r = 1, \ldots, R$, and the observed data $\boldsymbol{y}$.

for $r \leftarrow 1 : R$

  Start with $h_{i,0} = 0$, $y_{i,0} = 0$.

  for $t \leftarrow 1 : (T+1)$

    Calculate $h_{it}^{(r)} = \omega_i^{(r)} + \alpha_i^{(r)} y_{i,t-1}^2 + \beta_i^{(r)} h_{i,t-1}^{(r)}$

  end for t

for $s \leftarrow 1 : S$

Obtain a sample from an I dimensional D-vine copula with parameters $\boldsymbol{\theta}^{(r)}$, i.e. $u^{(r,s)}_{1:I,T+1}$, using Algorithm 2 described in Aas et al. (2009)

for $i \leftarrow 1 : I$

Calculate $\epsilon^{(r,s)}_{i,T+1} = G^{-1}_{\boldsymbol{\varphi}^{(r)}_i}(u^{(r,s)}_{i,T+1})$; $\quad y^{(r,s)}_{i,T+1} = \sqrt{h^{(r)}_{i,T+1}}\epsilon^{(r,s)}_{i,T+1}$

end for i

Calculate the portfolio value $\bar{y}^{(r,s)}_{T+1} := I^{-1}\sum_{i=1}^{I} y^{(r,s)}_{i,T+1}$

end for s

Calculate $VaR^{(1-\alpha)*100\%}(\bar{\boldsymbol{y}}^{(r)}_{T+1})$, i.e. the empirical α quantile of $\bar{\boldsymbol{y}}^{(r)}_{T+1} := (\bar{y}^{(r,1)}_{T+1}, \ldots, \bar{y}^{(r,S)}_{T+1})'$.

end for r

The true predictive VaR of the simulated data is obtained by using the true parameter values instead of the MCMC samples in Algorithm 2. We used $S = 100000$ to simulate from the one step ahead posterior predictive distribution of the VaR and $S = 10000000$ to get an approximation for the true predictive VaR. Sensitivity analyses showed that S is chosen large enough to get sufficient precision.

Appendix D. Prior specification for the real data applications

Prior distribution	Mean	0.5%	2.5%	50%	97.5%	99.5%
$\omega_i \sim N_{(0,\infty)}(0, 0.01/\phi(0))$	0.02	0.00016	0.00079	0.01691	0.05618	0.07036
$\alpha_i \sim N_{(0,\infty)}(0, 0.035/\phi(0))$	0.07	0.00055	0.00275	0.05917	0.19664	0.24627
$\beta_i \sim N_{(0,\infty)}(0, 0.46/\phi(0))$	0.92	0.00723	0.03613	0.77772	2.58445	3.23665
$\nu_i^{(G)} \sim Exp(2, 1/5)$	7	2.0	2.1	5.5	20.4	28.5
$\nu_j^{(C)} \sim Exp(2, 1/5)$	7	2.0	2.1	5.57	20.4	28.5

Table D.10: Prior specification with corresponding prior means and quantiles.

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