A fractional credit model with long range dependent default rate

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Abstract

Motivated by empirical evidence of long range dependence in macroeconomic variables like interest rates we propose a fractional Brownian motion driven model to describe the dynamics of the short and the default rate in a bond market. Aiming at results analogous to those for affine models we start with a bivariate fractional Vasicek model for short and default rate, which allows for fairly explicit calculations. We calculate the prices of corresponding defaultable zero-coupon bonds by invoking Wick calculus. Applying a Girsanov theorem we derive today's prices of European calls and compare our results to the classical Brownian model.

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1 Introduction

The financial crisis showed that the mostly used Gaussian or, more general, Markov models may not be sufficient to catch the market structure for credit derivatives. One reason for this may be the fact that short rates and/or default rates, which are driven by macroeconomic variables like domestic gross products, supply and demand rates or volatilities, exhibit long range dependence, a statistical property which cannot be realistically modeled by Markov structures. Empirical evidence has been reported over the years and we refer to Henry and Zaffaroni [24] for details and further references. In particular, Backus and Zin [1] provide in their Section 4 evidence for long memory in the short rate process.

In this paper we start a thorough investigation considering bond and credit markets driven by fractional Brownian motions (fBms) with Hurst index $H > \frac{1}{2}$. We aim at results analogous to those obtained in recent years for affine models; see e.g. Duffie [9] and Duffie, Filipovic and Schachermayer [10]. This idea has been present in Biagini, Fuschini and Klüppelberg [2], where the focus was, however, on credit contagion.

In the present paper, we focus on structural results in a fBm driven market. We are facing two mathematical problems. Firstly, the non-Markovity implies that all past information will enter into prices. Secondly, our models are in general not semimartingales, so that we cannot use Itô calculus. However, we can apply pathwise or L^2-integration theory, also to obtain solutions to fBm driven stochastic differential equations (SDEs); cf. Buchmann and Klüppelberg [6] based on previous work by Zähle [37].

In this paper we mainly focus on the pricing of defaultable derivatives depending on the short rate and the default rate. Moreover, we concentrate in this paper on the Vasicek model (with possibly time dependent coefficient functions). We are aware of the fact that, as a Gaussian process, the short rate as well as the default rate can also take negative values. However, it is always possible to shift, and perhaps also scale, the model such that the probability of a path becoming negative is arbitrarily small. We leave more general models as, for instance, suggested in Buchmann and Klüppelberg [6] or the fractional Lévy driven versions of Fink and Klüppelberg [17] and Fink [16] for future research.

Apart from the driving fBm, the models we consider are well-known as two factor models in the literature and we refer to Filipovic [15], Schönbucher [34] and Bielecki and Rutkowski [4] for background reading on credit risk modeling. In Fink, Klüppelberg and Zähle [18] d factor models ($d \in \mathbb{N}$) are considered, however, the factors are assumed to be independent. In the present paper we derive a two factor Vasicek model from the fractional Heath-Jarrow-Morton (HJM) approach of Ohashi [30] based on previous work of Guasoni, Rásonyi and Schachermayer [21, 22]: As in the classical Brownian HJM setting of Heath, Jarrow and Morton [23] the whole forward curve is described under a measure $\mathcal{P}$. Since the dynamics are now modeled by a fBm, arbitrage opportunities may occur. However in a more realistic setting with proportional transaction costs, Ohashi showed that arbitrage strategies cannot be constructed anymore.

The existence of an average risk neutral measure $\mathcal{Q}$ can be proven and we can formally calculate prices of defaultable bonds or more general contingent claims under this measure as suggested in Sottinen and Valkeila [35]. On the other hand it is of course always possible to directly define prices via conditional expectations leading in general to an arbitrage-free model. In both situations, conditional distributions have to be considered and a Girsanov theorem and Fourier methods can be applied to find closed formulas for option prices.

Our paper is organized as follows. Section 2 will recall necessary preliminaries about fBm, prediction and the Wick product. In Section 3 we will derive a two factor fBm Vasicek model from the HJM approach of Ohashi [30]. In Section 4 we consider pricing of defaultable zero-coupon bonds as a special case of contingent claims. Section 5 is dedicated to option pricing, calculating today's price of a call on a zero-coupon bond directly. Afterwards Fourier methods are applied for more general options. Finally, we compare our results to the classical Brownian model.

2 Fractional Brownian motion: integrals, prediction and the Wick product

There are many examples, which consider the short and default rate as functions of state vectors of Markov processes; see e.g. Duffie, Filipovic and Schachermayer [10] or Schönbucher [34], Chapter 7. Processes driven by Brownian motion (Bm) are the most prominent ones. We will focus on the case where r and λ are given by Vasicek models, with possibly time-dependent coefficients, driven by fBms with Hurst indices strictly greater than $\frac{1}{2}$. This choice is motivated by the fact that macroeconomic variables like demand and supply, interest rates, or other economic activity measures often exhibit long range dependence, cf. Henry and Zaffaroni [24] for an overview.

We will always assume a given complete probability space $(\Omega, \mathcal{F}, \mathcal{P})$. For a family of random variables $(X(i))_{i\in I}$, I some index set, let $\sigma\overline{\{X(i), i \in I\}}$ denote the completion of the generated σ-algebra. The spaces of integrable and square integrable real functions are denoted by $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, respectively. On a compact interval $[0,T]$, the corresponding function spaces are denoted by $L^1([0,T])$ and $L^2([0,T])$. Furthermore $\|\cdot\|_2$ is the L^2-norm and $\mathbb{R}_+$ ($\mathbb{R}_-$) are the positive (negative) real half lines.

Recall that a fBm is a zero mean Gaussian process starting in 0 with stationary increments satisfying $(B^H(ct))_{t\geq 0} \stackrel{d}{=} c^H(B^H(t))_{t\geq 0}$ for every $c > 0$. The parameter $H \in (0,1)$ is the Hurst index and $\stackrel{d}{=}$ means equality of finite dimensional distributions. We also assume that B^H is standard; i.e. that $E[B^H(1)^2] = 1$. For general background on fBm we refer to Biagini et al. [3] or Samorodnitsky and Taqqu [33]. For the present paper we shall heavily draw from Pipiras and Taqqu [32].

It is appropriate in our context to use fractional calculus, which suggests to replace H by the fractional parameter $\kappa = H - \frac{1}{2} \in (-\frac{1}{2}, \frac{1}{2})$. In our long range dependence case, we shall work with $\kappa \in (0, \frac{1}{2})$, which implies that increments are positively correlated. We also recall that $\kappa = 0$ refers to standard Brownian motion and we shall write $B^0 = B$. In the sequel we shall work with two-sided processes, where a two-sided Brownian motion is defined as $B(t) = \mathbf{1}_{t\geq 0}B_1(t)+\mathbf{1}_{t<0}B_2(-t)$ for $t \in \mathbb{R}$, where B_1 and B_2 are independent standard Bm's. We also introduce a bivariate fBm $(B^\kappa, \bar{B}^{\bar{\kappa}}) = (B^\kappa(t), \bar{B}^{\bar{\kappa}}(t))_{t\in\mathbb{R}}$ with $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$. The dependence structure between the fBm's will be modeled as in Elliot and van der Hoek [14] by assuming that both processes arise through an integral representation driven by the same two-sided Bm $B = (B(t))_{t\in\mathbb{R}}$, which holds in $L^2(\Omega)$ and is stated in equation (3.7) of Pipiras and Taqqu [32]:

$$\begin{aligned} B^\kappa(t) &= \Gamma(\kappa+1)c_\kappa \int_{-\infty}^{\infty} \mathcal{I}_-^\kappa \mathbf{1}_{(0,t)}(s)dB(s), \quad c_\kappa := \frac{\sqrt{\Gamma(2\kappa+2)\sin((\kappa+1/2)\pi)}}{\Gamma(\kappa+1)}, \\ \bar{B}^{\bar{\kappa}}(t) &= \Gamma(\bar{\kappa}+1)c_{\bar{\kappa}} \int_{-\infty}^{\infty} \mathcal{I}_-^{\bar{\kappa}} \mathbf{1}_{(0,t)}(s)dB(s), \quad c_{\bar{\kappa}} := \frac{\sqrt{\Gamma(2\bar{\kappa}+2)\sin((\bar{\kappa}+1/2)\pi)}}{\Gamma(\bar{\kappa}+1)}, \end{aligned} \tag{2.1}$$

for $t \in \mathbb{R}$, with gamma function Γ and the classical Riemann-Liouville fractional integral defined for $\alpha > 0$ by

$$(\mathcal{I}_-^\alpha f)(x) = \frac{1}{\Gamma(\alpha)} \int_x^\infty f(t)(t-x)^{\alpha-1}dt$$

if the integrals exist for almost all $x \in \mathbb{R}$. We shall also need the fractional derivatives

$$(\mathcal{D}_-^\alpha f)(x) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dx} \int_x^\infty f(t)(t-x)^{-\alpha}dt$$

for $\alpha > -1$. The question of the existence of the fractional derivative $\mathcal{D}_-^\alpha f$ is more sophisticated and we refer to Zähle [37] for details. However, we will only take fractional derivatives of fractional integrals, where the orders fit together, therefore, existence will be always ensured.

Remark 2.1. (i) The two fBms arising from the same Bm have the economical interpretation that short rate and default rate are driven by the same market noise and macroeconomic factors. However, the influence of this noise may be different and depends on the long range dependence parameters as well as on the coefficient functions of the Langevin equations (as will be seen in section 4).

(ii) Of course, it is always possible to add several independent factors driven by independent Bms. Using such a technique different dynamics for short and default rate can be constructed.

From now on we will understand integration with respect to fBm in the $L^2(\Omega)$-sense of Pipiras and Taqqu [32] over whole the real line. Integrals on compacts shall be defined by restriction of the integrand function. However under certain additional assumptions also pathwise integrals (cf. Young [36]) will appear. When both types of integrals, which are defined as limits of sequences of step function integrals, exists, those sequences converge also in probability and therefore the resulting limits/integrals are the same in distribution.

The following proposition is a consequence of Theorem 3.2 and (3.13) of Pipiras and Taqqu [32].

Proposition 2.2. *Let $(B_t)_{t\in\mathbb{R}}$ be the two-sided Bm of (2.1) and $\kappa \in (0, \frac{1}{2})$. For every $f \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ the following integrals are equal in the $L^2(\Omega)$-sense:*

$$\int_{\mathbb{R}} f(s)dB^{\kappa}(s) \quad = \quad c_\kappa \Gamma(\kappa+1) \int_{\mathbb{R}} \mathcal{I}^{\kappa}_{-}(f)(s)dB(s)$$

A analogous result holds true for $\bar{B}^{\bar{\kappa}}$.

For $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\kappa \in (0, \frac{1}{2})$ the following inner product is finite:

$$\langle f, g\rangle_\kappa := \kappa(2\kappa+1) \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{2\kappa-1}dudv.$$

We shall denote the induced norm by $\|\cdot\|_\kappa$. Define further for $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and $\kappa, \bar{\kappa} \in (0, \frac{1}{2})$

$$\langle f, g\rangle_{\kappa,\bar{\kappa}} := \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)g(v)|u-v|^{\kappa+\bar{\kappa}-1}dudv$$

and denote the induced norm by $\|\cdot\|_{\kappa,\bar{\kappa}}$.

As a first result we provide an extension of Proposition 2.2:

Lemma 2.3. *For $0 \le t \le T$ let $c \in L^2[t,T]$. Let B^κ and $\bar{B}^{\bar{\kappa}}$ be fBm's as in (2.1). Assume further that $\kappa \le \bar{\kappa}$. Then the equality of following integrals holds in the $L^2(\Omega)$-sense:*

$$\int_t^T c(v)d\bar{B}^{\bar{\kappa}}(v) = \frac{c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)}{c_\kappa\Gamma(\kappa+1)} \int_{\mathbb{R}} \mathcal{I}^{\bar{\kappa}-\kappa}_{-}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v). \tag{2.2}$$

Proof. Set $a_\kappa := c_\kappa\Gamma(\kappa+1)$ and $a_{\bar{\kappa}} := c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)$. Then using repeatedly Proposition 2.2 we get

$$\begin{aligned}
\int_t^T c(v)d\bar{B}^{\bar{\kappa}}(v) &= \int_{\mathbb{R}} \mathbf{1}_{(t,T)}(v)c(v)d\bar{B}^{\bar{\kappa}}(v) \\
&= a_{\bar{\kappa}} \int_{\mathbb{R}} \mathcal{I}^{\bar{\kappa}}_{-}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB(v) \\
&= \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{D}^{\kappa}_{-}\mathcal{I}^{\bar{\kappa}}_{-}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v) \\
&= \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{D}^{\kappa}_{-}\mathcal{I}^{\kappa}_{-}\mathcal{I}^{\bar{\kappa}-\kappa}_{-}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v) \\
&= \frac{a_{\bar{\kappa}}}{a_\kappa} \int_{\mathbb{R}} \mathcal{I}^{\bar{\kappa}-\kappa}_{-}(\mathbf{1}_{(t,T)}(\cdot)c(\cdot))(v)dB^\kappa(v).
\end{aligned}$$

□

Derivatives pricing essentially means prediction, given information of the past. To approach this, we have to work on the compact interval $[0,T]$ for some $T > 0$ and define the fractional integral with finite time horizon for $\alpha > 0$,

$$(I^\alpha_{T-}f)(s) = \frac{1}{\Gamma(\alpha)} \int_s^T f(r)(r-s)^{\alpha-1}dr, \quad 0 \le s \le T. \tag{2.3}$$

For $f \in L^2(\mathbb{R})$ this integral always exists. We shall also need the fractional derivative with finite time horizon for $0 < \alpha < 1$

$$(D^{\alpha}_{T-}g)(u) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{du} \int_u^T g(s)(s-u)^{-\alpha} ds, \quad 0 < u < T. \tag{2.4}$$

As usual, we shall often write $I^{-\alpha}_{T-} = D^{\alpha}_{T-}$.

Recall that $\overline{\mathrm{sp}}_{[0,T]}(B^\kappa)$ is the closure in $L^2(\Omega)$ of all possible linear combinations of the increments of fBm on $[0, T]$.

In a Markov model, pricing formulas rely on the Markov property, which can certainly not be applied in models involving long range dependence processes.

Suppose we want to calculate the prediction

$$X^\kappa(t,T) := E[B^\kappa(T) \mid B^\kappa(s), s \in [0,t]], \quad 0 < t < T.$$

If $X^\kappa(t,T) \in \overline{\mathrm{sp}}_{[0,t]}(B^\kappa)$, then we hope that there exists some function $f \in L^2[0,T]$ such that $X^\kappa(t,T) = \int_0^t f(u) dB^\kappa(u)$. This is indeed true, and the formula has been derived by Gripenberg and Norros [20]. Pipiras and Taqqu [31] (Theorem 7.1) and Duncan [11] considered a similar situation for a fBm defined via a Molchan-Golosov kernel (cf. Molchan and Golosov [28] and Kleptsyna, LeBreton and Roubaud [26]). The following lemma is a slightly extended version of [20] and its proof is straightforward:

Proposition 2.4. *For $0 \le t < T$ and $\kappa \in (0, \frac{1}{2})$ let $c \in L^2[t,T]$. Then*

$$E\left[\int_t^T c(r) dB^\kappa(r) \Big| B^\kappa(r), r \in [0,t]\right] = \int_0^t \Psi^\kappa_c(t,T,u) dB^\kappa(u), \tag{2.5}$$

where

$$\begin{aligned} \Psi^\kappa_c(t,T,u) &= u^{-\kappa}(I^{-\kappa}_{t-}(I^{\kappa}_{T-}(\cdot)^\kappa c(\cdot)\mathbf{1}_{(t,T)}(\cdot)))(u) \\ &= \frac{\sin(\pi\kappa)}{\pi} u^{-\kappa}(t-u)^{-\kappa} \int_t^T \frac{z^\kappa (z-t)^\kappa}{z-u} c(z) dz. \end{aligned} \tag{2.6}$$

Moreover, $\Psi^\kappa_c(t,T,\cdot) \in L^2[0,t]$ for all $0 \le t \le T$.
Finally, the integral in (2.5) *is normally distributed with mean 0 and variance $\|\Psi^\kappa_c(t,T,\cdot)\mathbf{1}_{(0,t)}(\cdot)\|^2_\kappa$.*

Predicting exponentials of fBm driven integrals is more challenging and has been considered in Duncan [11]. However, Proposition 2 of that paper is not correct. An erratum is Duncan and Fink [12] and further considerations can be found in Fink, Klüppelberg and Zähle [18] and Fink [16].

From now on we will always assume that $\kappa \in (0, \frac{1}{2})$. We recall some basic properties of the Wick product for fBm and refer to Biagini et al. [3], Section 3, Elliot and van der Hoek [14] or Duncan, Hu and Pasik-Duncan [13] for details and background.

There are various ways to introduce the Wick product and we will follow mainly Section 3.1 of Biagini et al. [3]. Let $\kappa \in (0, \frac{1}{2})$. First we consider for $g : \mathbb{R} \to \mathbb{R}$ with $\|g\|_\kappa < \infty$ exponentials of the form

$$\varepsilon(g) := \exp\left\{\int_{\mathbb{R}} g(s) dB^\kappa(s) - \frac{1}{2}\|g\|_\kappa\right\} \tag{2.7}$$

like in (3.7) of [3]. The set $\mathcal{E}$ of linear combinations of these exponentials is dense in $L^p(\Omega)$ for all $p \geq 1$.

Definition 2.5. *For $g, h : \mathbb{R} \to \mathbb{R}$ with $\|g\|_\kappa, \|h\|_\kappa < \infty$ the* Wick product *of the exponentials of g, h is defined as*

$$\varepsilon(g) \diamond \varepsilon(h) := \varepsilon(g + h). \tag{2.8}$$

By bilinearity the Wick product is defined on the whole of $\mathcal{E}$. A classical density argument (see Theorem 3.1 of [3]) extends this definition now to L^p for all $p \geq 1$. The two main properties of the Wick product we need in this paper are summarized in the next proposition.

Proposition 2.6. *Let $c : \mathbb{R} \to \mathbb{R}$ with $\|c\|_\kappa < \infty$.*
(1) Define the Wick exponential by $\exp^\diamond(\cdot) := \sum_{i=0}^\infty ((\cdot)^{\diamond i}/i!)$. Then

$$\exp^\diamond \left\{ \int_\mathbb{R} c(s) dB^\kappa(s) \right\} = \exp \left\{ \int_\mathbb{R} c(s) dB^\kappa(s) - \frac{1}{2}\|c\|_\kappa \right\} = \varepsilon(c). \tag{2.9}$$

(2) Set $\mathcal{G}_t := \overline{\sigma\{B^\kappa(s), s \in (-\infty, t]\}}$ or $\mathcal{G}_t := \overline{\sigma\{B^\kappa(s), s \in [a, t]\}}$ for $-\infty < a < t < \infty$. Then

$$E\left[\exp^\diamond \left\{ \int_\mathbb{R} c(s) dB^\kappa(s) \right\} \middle| \mathcal{G}_t \right] = \exp^\diamond \left\{ E\left[\int_\mathbb{R} c(s) dB^\kappa(s) \middle| \mathcal{G}_t \right] \right\}$$

Proof. Part (1) is given by (3.25) of Biagini et al. [3] and part (2) is a consequence of (17) of Duncan [11] and the uniform convergence of the exponential Wick series. □

Proposition 2.7. *Let $(B^\kappa, \bar{B}^{\bar{\kappa}})$ be as in (2.1) and set*

$$\mathcal{G}_t^{B^\kappa} := \overline{\sigma\{B^\kappa(s),\ s \in [0, t]\}} \tag{2.10}$$

for $t > 0$ and $\mathcal{G}_t^B := \mathcal{G}_t^{B^\kappa} \vee \mathcal{G}_t^{\bar{B}^{\bar{\kappa}}}$. Then

$$\mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^B] = \mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B^\kappa}] \tag{2.11}$$

almost surely, for $0 \leq s < t$ and any bounded measurable function ψ.

Proof. Let $(W^\kappa, \bar{W}^{\bar{\kappa}}) = (W^\kappa(t), \bar{W}^{\bar{\kappa}}(t))_{t \in [0,T]}$ be a bivariate fBm defined on a compact interval via a Molchan-Golosov kernel (for details cf. Molchan and Golosov [28], Kleptsyna, LeBreton and Roubaud [26] and Norros, Valkeila and Virtamo [29])

$$W^\kappa(t) = d_k \int_0^t R_k(t, s) dW(s),$$
$$\bar{W}^{\bar{\kappa}}(t) = d_{\bar{\kappa}} \int_0^t R_{\bar{\kappa}}(t, s) dW(s),$$

$t \in [0, T]$, where d_k, $d_{\bar{\kappa}}$ are suitable constants and $W = (W(t))_{t \in [0,T]}$ is the same standard Brownian motion. Then in distribution

$$(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t))_{t \in [0,T]} \stackrel{d}{=} (W^\kappa(t), \bar{W}^{\bar{\kappa}}(t))_{t \in [0,T]} \tag{2.12}$$

since they are Gaussian processes with the same first and second order structure. Let

$$\mathcal{G}_t^{W^\kappa} := \overline{\sigma\{W^\kappa(s),\ s \in [a,t]\}}.$$

By (2.12) it follows that in distribution

$$\mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B^\kappa}] \stackrel{d}{=} \mathbb{E}[\psi(W^\kappa(t), \bar{W}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{W^\kappa}] \tag{2.13}$$

for all bounded measurable functions $\psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. $\square$

By Jost [25] we have

$$(\mathcal{G}_t^{W^\kappa})_{t\in[0,T]} = (\mathcal{G}_t^{W})_{t\in[0,T]} = (\mathcal{G}_t^{\bar{W}^{\bar{\kappa}}})_{t\in[0,T]}. \tag{2.14}$$

Consequently for all $0 < s < t$ and $\mathcal{G}_t^W := \mathcal{G}_t^{W^\kappa} \vee \mathcal{G}_t^{\bar{W}^{\bar{\kappa}}}$

$$\mathbb{E}[\psi(W^\kappa(t), \bar{W}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{W}] = \mathbb{E}[\psi(W^\kappa(t), \bar{W}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{W^\kappa}]$$

for all bounded measurable functions $\psi : (\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2)) \to (\mathbb{R}, \mathcal{B}(\mathbb{R}))$. Hence by (2.13) and (2.14) it follows that in distribution

$$\mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B}] \stackrel{d}{=} \mathbb{E}[\psi(W^\kappa(t), \bar{W}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{W^\kappa}] \tag{2.15}$$

for $0 < s < t$. With the same argument as in (2.13), we also have that in distribution

$$\mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B^\kappa}] \stackrel{d}{=} \mathbb{E}[\psi(W^\kappa(t), \bar{W}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{W^\kappa}] \tag{2.16}$$

for all bounded measurable functions ψ. Hence by (2.15) and (2.16) we have that in distribution

$$\mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B}] \stackrel{d}{=} \mathbb{E}[\psi(B^\kappa(t), \bar{B}^{\bar{\kappa}}(t)) \,|\, \mathcal{G}_s^{B^\kappa}] \tag{2.17}$$

for $0 < s < t$ and any bounded measurable function ψ. To conclude that (2.17) holds almost surely, we need Lemma 2.9.

Remark 2.8. *Note that the equality*

$$\mathcal{G}_t^B := \mathcal{G}_t^{B^\kappa} \vee \mathcal{G}_t^{\bar{B}^{\bar{\kappa}}} = \mathcal{G}_t^{B^\kappa} = \mathcal{G}_t^{\bar{B}^{\bar{\kappa}}}$$

for $t > 0$ has already been proved in Jost [25] and Pipiras and Taqqu [32] for $a = -\infty$.

Lemma 2.9. *Let X be a random variable in $L^1(\Omega, \mathcal{F}, \mathbb{P})$, where $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space. Let $\mathcal{E}$ be a sub-σ-algebra of $\mathcal{F}$. If*

$$\mathbb{E}[X \,|\, \mathcal{E}] \stackrel{d}{=} X \tag{2.18}$$

in distribution, then

$$\mathbb{E}[X \,|\, \mathcal{E}] = X \tag{2.19}$$

almost surely.

Proof. Suppose (2.19) does not hold. Then there exists a convex function, such that the Jensen's inequality is strict, i.e.

$$\mathbb{E}[f(\mathbb{E}[X|\mathcal{E}])] < \mathbb{E}[\mathbb{E}[f(X)|\mathcal{E}]] = \mathbb{E}[f(X)].$$

But since (2.18) holds, we have

$$\mathbb{E}[f(\mathbb{E}[X|\mathcal{E}])] = \mathbb{E}[f(X)]$$

for all convex functions. This is a contradiction and it follows that (2.19) is verified. $\square$

For the remainder of this section define

$$\mathcal{G}_t \quad := \quad \sigma\overline{\{(B_s^\kappa, \bar{B}_s^{\bar{\kappa}}), s \in [0,t]\}}, \quad t \geq 0.$$

Next we need an analog of the erratum Duncan and Fink [12], where now in the exponential there is the sum of two integrals and the dependence between B^κ and $\bar{B}^{\bar{\kappa}}$ matters. We shall proceed as follows. First we transform both integrals with respect to B^κ and $\bar{B}^\kappa$, respectively, into one integral with respect to B^κ and invoke afterwards Proposition 2.4.

Proposition 2.10. *For $0 \leq t < T$ let $c, \bar{c} \in L^2[t,T]$. Further let B^κ and $\bar{B}^{\bar{\kappa}}$ be fBm's as in (2.1).*
Then

$$\begin{aligned} & E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\middle|\mathcal{G}_t\right] \\ = \quad & e^{W(t,T)-V(t,T)}\exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}, \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} W(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)c(\cdot)\right\|_\kappa^2 + 2\left\langle\mathbf{1}_{(t,T)}(\cdot)c(\cdot), \mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot)\right\rangle_{\kappa,\bar{\kappa}} + \left\|\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot)\right\|_{\bar{\kappa}}^2\right), \\ V(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|_\kappa^2 + 2\left\langle\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot), \mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\rangle_{\kappa,\bar{\kappa}}\right. \\ &\quad \left. + \left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)\right\|_{\bar{\kappa}}^2\right). \end{aligned}$$

and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (2.6) and belong to $L^2[0,t]$ for all $0 \leq t \leq T$.

Proof. To predict the exponential we transform it into a Wick exponential using Lemma 2.3 and then Proposition 2.6 as follows. W.l.o.g. assume that $\kappa \leq \bar{\kappa}$. Define $a_\kappa = c_\kappa\Gamma(\kappa+1)$ and $a_{\bar{\kappa}} = c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)$. Then by Lemma 2.3 and Proposition 2.6

$$\begin{aligned} & \exp\left\{\int_t^T c(v)dB^\kappa(v) + \int_t^T \bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\} \\ = \quad & \exp\left\{\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\} \\ = \quad & e^{W(t,T)}\exp^\diamond\left\{\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(v)c(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\} \end{aligned}$$

and, as preliminary version,

$$\begin{aligned} W(t,T) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(u)c(u)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(u)\right) \\ &\quad\times\left(\mathbf{1}_{(t,T)}(v)c(v)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(v)\right)|u-v|^{2\kappa-1}dudv \\ &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)c(\cdot)\right\|_\kappa^2+2\frac{a_{\bar{\kappa}}}{a_\kappa}\left\langle\mathbf{1}_{(t,T)}(\cdot)c(\cdot),\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(\cdot)\right\rangle_\kappa\right. \\ &\quad\left.+\left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2\left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(\cdot)\right\|_\kappa^2\right). \end{aligned}$$

Next we take the conditional expectation of the exponential integral, which is nothing else than an L^2 projection. Therefore, Proposition 2.6 and Proposition 2.7 apply, giving

$$\begin{aligned} &E\left[\exp\left\{\int_t^T c(v)dB^\kappa(v)+\int_t^T\bar{c}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\middle|\mathcal{G}_t\right] \\ &= e^{W(t,T)}E\left[\exp^\diamond\left\{\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(v)c(v)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\right\}\middle|\mathcal{G}_t\right] \\ &= e^{W(t,T)}\exp^\diamond\left\{E\left[\int_{\mathbb{R}}\left(\mathbf{1}_{(t,T)}(v)c(v)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(t,T)}(\cdot)\bar{c}(\cdot))(v)\right)dB^\kappa(v)\middle|\mathcal{G}_t\right]\right\}. \end{aligned}$$

Now transform the integral in the conditional expectation back and apply Proposition 2.4 and Proposition 2.7. Transforming the Wick exponential in a classical exponential yields the term

$$\begin{aligned} V(t,T) &= \frac{1}{2}\int_{\mathbb{R}}\int_{\mathbb{R}}\left(\mathbf{1}_{(0,t)}(u)\Psi_c^\kappa(t,T,u)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(u)\right) \\ &\quad\times\left(\mathbf{1}_{(0,t)}(v)\Psi_c^\kappa(t,T,v)+\frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(v)\right)|u-v|^{2\kappa-1}dudv \\ &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot)\right\|_\kappa^2+2\frac{a_{\bar{\kappa}}}{a_\kappa}\left\langle\mathbf{1}_{(0,t)}(\cdot)\Psi_c^\kappa(t,T,\cdot),\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(\cdot)\right\rangle_\kappa\right. \\ &\quad\left.+\left(\frac{a_{\bar{\kappa}}}{a_\kappa}\right)^2\left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,t)}(\cdot)\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot))(\cdot)\right\|_\kappa^2\right). \end{aligned}$$

Finally, we transform the indefinite integral $\mathcal{I}_-^{\bar{\kappa}-\kappa}$ within the conditional expectation back using Lemma 2.3. Combining these two steps yields (2.20). The final versions of $V(t,T)$ and $W(t,T)$ can be calculated by Lemma A.1 of the Appendix. $\square$

3 The market model

In this section we will derive a Vasicek model from the HJM approach of Ohashi [30]. As in the classical setting of Heath et al. [23] we model the whole term structure under a measure $\mathcal{P}$ and show that, under the realistic assumption of proportional transaction costs with proportionality factor $k>0$, arbitrage can be ruled out.

Ohashi's [30] work on a fractional HJM bond model with proportional transaction costs is based on an extension of the full support property of the logarithmic price processes in the set of continuous functions. This basic idea and its relevance to the absence of arbitrage was fully investigated by Guasoni, Rásonyi and Schachermayer [21]. Its remains to observe that these properties are only sufficient for the market to be arbitrage-free. In Guasoni, Rásonyi and Schachermayer [22] a fundamental theorem with necessary and sufficient conditions for risk-neutral asset pricing under proportional transaction costs has been derived.

The final time horizon of the market shall be $T^\star > 0$. We assume a given probability space $(\Omega, \mathcal{F}, \mathcal{P})$ endowed with a filtration $(\mathcal{F}_t)_{0\le t\le T^\star}$, representing the complete market information and satisfying the usual conditions of completeness and right continuity. Assume further the existence of the *forward rate process* $f = (f(t,T))_{0\le t\le T\le T^\star}$ on $(\Omega, \mathcal{F}, \mathcal{P})$ such that for each $0 \le T \le T^\star$ the process $(f(t,T))_{0\le t\le T}$ is adapted to $(\mathcal{F}_t)_{0\le t\le T}$. The stochastic process $r = (r(t))_{0\le t\le T^\star} := (f(t,t))_{0\le t\le T^\star}$ models the *short rate* and is assumed to be progressively measurable.

3.1 Tradable bonds and numeraire

We specify the dynamics of f by the linear stochastic differential equation

$$f(t,T) = f(0,T) + \int_0^t \alpha(s,T)ds + \int_0^t \sigma(s,T)dB^\kappa(s), \quad 0 \le t \le T \le T^\star, \tag{3.1}$$

where we will from now on impose the following assumptions:

(A1) The function $f : [0,T^\star] \to \mathbb{R}$, $T \mapsto f(0,T)$ is continuously differentiable.

(A2) The functions $\alpha, \sigma : [0,T^\star]^2 \to \mathbb{R}$ are continuous and bounded with $\sigma > 0$ on $[0,T^\star]^2$.

Remark 3.1. Under the conditions (A1) and (A2) the technical integrability assumptions (2.6)–(2.9) of Ohashi [30] are satisfied.

We assume the existence of the following tradable bonds:

$$P(t,T) = \exp\Big\{ - \int_t^T f(t,s)ds\Big\}, \quad 0 \le t \le T \le T^\star.$$

Definition 3.2. *Define for* $0 \le T \le T^\star$ *the* relative bond price *of a* T*-maturity bond by*

$$Z_t(T) := P(t,T)/S_0(t), \quad 0 \le t \le T \le T^\star,$$

with $(S_0(t))_{0\le t\le T^\star} := \big(\exp\big\{\int_0^t r(s)ds\big\}\big)_{0\le T\le T^\star}$ *for the* short rate $(r(t))_{0\le t\le T^\star} := (f(t,t))_{0\le T\le T^\star}$.

We will take the process $(S_0(t))_{0\le t\le T^\star}$ as the *numeraire*.

3.2 Trading strategies and the wealth process

We will now describe which trading strategies are admissible. To this end, we define the wealth process of a trading strategy by an integral over the whole relative bond price surface given by $Z = (Z_t(T))_{0\le t\le T\le T^\star}$ from Definition 3.2. For fixed $0 \le t \le T^\star$ denote $Z_t := (Z_t(T))_{t\le T\le T^\star}$.

First we have to specify the idea following Section 3 of Ohashi [30]: Let $\mathbb{B}([0,T^\star])$ denote the Borel sets of $[0,T^\star]$. Define admissible trading strategies in our market by the following procedure: Let $\mathcal{M}_{T^\star}$ be the space of all finite signed measures on $\mathbb{B}([0,T^\star])$ endowed with the total variation norm defined by

$$\|m\|_{TV} := \sup\{m(A)|A \in \mathbb{B}([0,T^\star])\} + |\inf\{m(A)|A \in \mathbb{B}([0,T^\star])\}|$$

for $m \in \mathcal{M}_{T^\star}$. Define further the total variation measure by

$$|m|(E) := \sup\{m(A)|A \in \mathbb{B}([0,T^\star]), A \subset E\} + |\inf\{m(A)|A \in \mathbb{B}([0,T^\star]), A \subset E\}|, \quad E \in \mathbb{B}([0,T^\star]).$$

Let φ be a measure-valued elementary process of the form

$$\varphi_t(\omega, A) = \sum_{i=0}^{N-1} \mathbf{1}_{F_i \times (t_i, t_{i+1}]}(\omega, t) m_i(A), \quad 0 \le t \le T^\star, \tag{3.2}$$

for $\omega \in \Omega$, $A \in \mathcal{F}$, $m_i \in \mathcal{M}_{T^\star}$, $0 = t_0 < \cdots < t_i < \cdots < t_N \le T^\star$ and $F_i \in \mathcal{F}_{t_i}$ for $i \in \{0, \dots, N-1\}$. The process φ represents an elementary trading strategy. Denote by $\mathcal{S}$ the set of all elementary processes of the form (3.2) endowed with the norm:

$$\|\varphi\|_{\mathcal{S}} := E\Big[\sup_{0\le t \le T^\star} \|\varphi_t\|^2_{TV}\Big].$$

To define the wealth process we must specify integration with respect to φ and Z: For $\varphi \in \mathcal{S}$ define the random variable

$$\Big(\int_0^t \varphi_s dZ_s\Big)(\omega) := \sum_{i=0}^{N-1} \mathbf{1}_{F_i}(\omega)([Z_{t_{i+1}\wedge t} - Z_{t_i \wedge t}] \bullet m_i)(\omega), \quad \omega \in \Omega,$$

where we set

$$(X \bullet m)(\omega) := \int_0^{T^\star} X(s)(\omega) m(ds), \quad \omega \in \Omega,$$

for a stochastic process $X = (X(s))_{0 \le s \le T^\star}$ and $m \in \mathcal{M}_{T^\star}$ such that the integral exists. From now on we will omit ω in the notation.

For $\varphi \in \mathcal{S}$ and proportional transactions costs with proportionality factor $k > 0$ define the wealth process $V^k(\varphi)$ via

$$V_t^k(\varphi) := \sum_{i=0}^{N-1} \mathbf{1}_{F_i}[Z_{t_{i+1}\wedge t} - Z_{t_i\wedge t}] \bullet m_i - k \sum_{i=0}^{N-1} \mathbf{1}_{F_i} Z_{t_i \wedge t} \bullet |\varphi_{t_{i+1}\wedge t} - \varphi_{t_{i1}\wedge t}| - k Z_t \bullet |\varphi_t|, \quad 0 \le t \le T^\star.$$

Let $\bar{\mathcal{S}}$ be the completion of $\mathcal{S}$ with respect to the norm $\|\cdot\|_{\mathcal{S}}$. Then equation (3.4) of Ohashi [30] shows that $\int_0^\cdot \varphi_s dZ_s$ and $V_\cdot^k(\varphi)$ can be defined for all $\varphi \in \bar{\mathcal{S}}$, as the next theorem states. The proof can be found in Ohashi [30], p. 1559-1560.

Theorem 3.3. *Let the dynamics of f be given via* (3.1) *such that (A1) and (A2) hold. Assume proportional transactions costs with proportionality factor $k > 0$. For $\varphi \in \bar{\mathcal{S}}$ let further the random variable*

$$\sup_\pi \sum_{t_i \in \pi} \|\varphi_{t_{i+1}} - \varphi_{t_i}\|_{TV} \tag{3.3}$$

be in $L^2(\Omega)$, where the supremum is taken over all partitions π of $[0,T^\star]$. Then for each sequence φ^n of elementary processes with $\lim_{n\to\infty}\varphi^n=\varphi$ in $\bar{\mathcal{S}}$ we have that

$$\lim_{n\to\infty} E\Big[\sup_{0\le t\le T^\star}|V_t^k(\varphi^n)-V_t^k(\varphi)|\Big]=0,\quad k>0,$$

where

$$V_t^k(\varphi):=\int_0^t\varphi_s dZ_s-k\int_0^t Z_s d|\varphi_s|-kZ_t\bullet|\varphi_t|,\quad 0\le t\le T^\star.$$

Now we can define admissible trading strategies in our bond market.

Definition 3.4. *(a) For proportional transactions costs with proportionality factor $k>0$ a trading strategy $\varphi\in\bar{\mathcal{S}}$ is called* admissible*, if it is adapted, the random variable* (3.3) *is in $L^2(\Omega)$ and if there exists $M>0$ such that $V_t^k(\varphi)\ge -M$ a.s. for every $0\le t\le T^\star$.*

(b) An admissible trading strategy $\varphi\in\bar{\mathcal{S}}$ is called an arbitrage opportunity with proportional transactions costs with proportionality factor $k>0$ *if $V_{T^\star}^k(\varphi)\ge 0$ a.s. and $P(V_{T^\star}^k(\varphi)>0)>0$.*

*(c) The market is called k-*arbitrage free *with proportional transactions costs with proportionality factor $k>0$, if for every admissible trading strategy $\varphi\in\bar{\mathcal{S}}$, $V_{T^\star}^k(\varphi)\ge 0$ a.s. implies $V_{T^\star}^k(\varphi)=0$ a.s.*

3.3 No-arbitrage and average risk neutral measure

Set $\phi_\kappa(x):=\kappa(2\kappa+1)|x|^{2\kappa-1}$ for $x\in\mathbb{R}$ and define

$$\begin{aligned}\widetilde{\alpha}(s,T) &:= \sigma(s,T)\int_0^s\int_0^{T-\theta}\sigma(\theta,\theta+x)\phi_\kappa(s-\theta)dxd\theta\\ &+ \int_0^{T-s}\sigma(s,s+x)dx\int_0^s\sigma(\theta,T)\phi_\kappa(s-\theta)d\theta,\quad 0\le s\le T\le T^\star.\end{aligned}\tag{3.4}$$

Furthermore we impose from now on the following assumptions:

(A3) The function $(t,T)\mapsto\int_0^{T-t}\sigma(t,t+s)ds$ is λ-Hölder continuous on $0\le t\le T\le T^\star$ for all $1/2<\lambda<1$.

(A4) There exists an integrable function $\gamma:(-\infty,T^\star]\to\mathbb{R}$ such that $\sigma(t,T)\gamma(t)=\widetilde{\alpha}(t,T)-\alpha(t,T)$ for $0\le t\le T\le T^\star$. Furthermore there is $\vartheta:(-\infty,T^\star]\to\mathbb{R}$ square-integrable satisfying

$$\int_{-\infty}^t\gamma(s)ds=\Gamma(\kappa+1)c_\kappa\int_{-\infty}^t[\mathcal{I}_-^\kappa\mathbf{1}_{(0,t)}(s)]\vartheta(s)ds,\quad 0\le t\le T^\star.\tag{3.5}$$

Now we can state the main theorem of this section.

Theorem 3.5. *Let the dynamics of f be given via* (3.1) *such that (A1) - (A4) hold and assume proportional transactions costs with proportionality factor $k > 0$. Then there exists a probability measure $\mathcal{Q} \sim \mathcal{P}$ such that for all $0 \le t \le T \le T^\star$*

$$E_{\mathcal{Q}}[Z_t(T)] = P(0,T) \tag{3.6}$$

holds. The market is k-arbitrage free.

Proof. Since $\sigma(\cdot,\cdot) > 0$ is continuous and bounded by assumption (A2), it is also square-integrable on $[0, T^\star]^2$. Therefore, the conditions of Lemma 2.3 and Theorem 3.1 of Ohashi [30] are met. It follows that the market is k-arbitrage free with proportional transactions costs with proportionality factor $k > 0$ and a measure $\mathcal{Q}$, satisfying equation (3.6), exists. □

Motivated by equation (3.6) and using Remark 3.8 of Ohashi [30], we can price contingent claims under the measure $\mathcal{Q}$ in a formal way similar to Sottinen and Valkeila [35]: The price of an integrable, $\mathcal{F}_T$-measurable contingent claim X with maturity $T \in [0, T^\star]$ at time t is given by

$$E_{\mathcal{Q}}\Big[X \exp\Big\{ - \int_t^T r(s)ds\Big\}\Big|\mathcal{F}_t\Big], \quad 0 \le t \le T \le T^\star. \tag{3.7}$$

We want to point out that Theorem 3.5 says nothing about the uniqueness of the pricing measure $\mathcal{Q}$ and in fact there might be more than one. However, firstly, the stated property (3.6) motivates us to use such a measure as a natural choice. And secondly, as explained before Corollary 3.1 in Ohashi [30], there is a canonical choice for the measure change under which (3.6) holds. This leads to $\mathcal{Q}$ as in Theorem 3.5.

3.4 Dynamics of the short rate under $\mathcal{Q}$

We need to be aware of the dynamics of the forward rate process under the measure $\mathcal{Q}$. Recall the definition in equation (3.4), then by Theorem 3.1 Ohashi [30] we have that the model (3.1) under $\mathcal{Q}$ has the following form

$$f(t,T) = f(0,T) + \int_0^t \widetilde{\alpha}(s,T)ds + \int_0^t \sigma(s,T)d\tilde{B}^\kappa(s), \quad 0 \le t \le T \le T^\star, \tag{3.8}$$

where $\tilde{B}^\kappa = (\tilde{B}^\kappa(t))_{t\in[0,T^\star]}$ is a $\mathcal{Q}$-fBm.

Equation (3.7) shows that the payoff of a contingent claim must be discounted by the short rate r before taking the conditional expectation. In this paper we are interested in models for which r is given by a fractional Vasicek model. To derive the Vasicek dynamics we have to impose a separability assumption on the volatility coefficient. This is similar to the situation in the classical Brownian HJM model of Heath et al. [23]. For details we refer to Section 5.3 of Brigo and Mercurio [5] and, in particular, Proposition 2.1 of Carverhill [7].

Assumption 3.6. *In addition to Assumptions (A1) - (A4) the volatility coefficient $\sigma(\cdot,\cdot)$ factorizes: $\sigma(t,T) = \xi(t)\nu(T)$, $0 \le t \le T \le T^\star$, where $\xi(\cdot)$ and $\nu(\cdot)$ are strictly positive and $\nu(\cdot)$ is differentiable. Further $\xi(\cdot)$ is of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$.*

Now we calculate the short rate for $0 \leq t \leq T^\star$ under $\mathcal{Q}$

$$\begin{aligned} r(t) &= f(t,t) = f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds + \int_0^t \sigma(s,t)d\tilde{B}^\kappa(s) \\ &= f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds + \nu(t)\int_0^t \xi(s)d\tilde{B}^\kappa(s). \end{aligned}$$

Furthermore,

$$\begin{aligned} &\int_0^t \widetilde{\alpha}(s,t)ds \\ =\ & \int_0^t \Big[\sigma(s,t)\int_0^s\int_0^{t-\theta} \sigma(\theta,\theta+x)\phi_\kappa(s-\theta)dxd\theta + \int_0^{t-s}\sigma(s,s+x)dx\int_0^s \sigma(\theta,t)\phi_\kappa(s-\theta)d\theta\Big]ds \\ =\ & \nu(t)\int_0^t \Big[\int_0^s\int_\theta^t \xi(s)\xi(\theta)\nu(x)\phi_\kappa(s-\theta)dxd\theta + \int_s^t \xi(s)\nu(x)dx\int_0^s \xi(\theta)\phi_\kappa(s-\theta)d\theta\Big]ds \\ =\ & \nu(t)\Big[\int_0^t\int_0^s \xi(s)\xi(\theta)\epsilon(\theta,t)\phi_\kappa(s-\theta)d\theta ds + \int_0^t\int_0^s \xi(s)\xi(\theta)\epsilon(s,t)\phi_\kappa(s-\theta)d\theta ds\Big], \end{aligned}$$

where $\epsilon(s,t) := \int_s^t \nu(x)dx$.

The function $t \mapsto f(0,t)$ is by assumption differentiable. Further we have the following lemma.

Lemma 3.7. *The function* $[0,T^\star] \to \mathbb{R}, t \mapsto \int_0^t \widetilde{\alpha}(s,t)ds$ *is differentiable.*

Proof. Since ν is by Assumption (A1) differentiable, we just need to show that the lemma is true for

$$t \mapsto \int_0^t\int_0^s (\xi(s)\xi(\theta)\epsilon(\theta,t)\phi_\kappa(s-\theta)d\theta ds \quad \textit{and} \tag{3.9}$$

$$t \mapsto \int_0^t\int_0^s (\xi(s)\xi(\theta)\epsilon(s,t)\phi_\kappa(s-\theta)d\theta ds. \tag{3.10}$$

We start by showing that the integrand function of (3.9) is differentiable in t. This follows by the classical rule for differentiation under the integral sign since for all $0 \leq t \leq T^\star$

$$\Big|\frac{\partial}{\partial t}\xi(s)\xi(\theta)\epsilon(\theta,t)\phi_\kappa(s-\theta)\Big| = \xi(s)\xi(\theta)\phi_\kappa(s-\theta)\nu(t) \leq C\xi(s)\xi(\theta)\phi_\kappa(s-\theta)$$

for some constant $C > 0$ since ν is differentiable. Applying the Leipniz rule a second time shows that (3.9) is differentiable. Similar arguments work for (3.10). ☐

By Lemma 3.7 we conclude that $t \mapsto A(t) := f(0,t) + \int_0^t \widetilde{\alpha}(s,t)ds$ is differentiable. Under $\mathcal{Q}$ we have that for $0 \leq t \leq T^\star$

$$r(t) = A(t) + \nu(t)\int_0^t \xi(s)d\tilde{B}^\kappa(s).$$

Therefore we have for $0 \leq s \leq t \leq T^\star$ by using a pathwise product rule and density formula (like in Theorem A.4 of Fink and Klüppelberg [17])

$$\begin{aligned} r(t) - r(s) &= A(t) - A(s) + \nu(t)\int_0^t \xi(u)d\tilde{B}^\kappa(u) - \nu(s)\int_0^s \xi(u)d\tilde{B}^\kappa(u) \\ &= \int_s^t A'(u)du + \int_s^t \Big(\int_0^u \xi(v)d\tilde{B}^\kappa(v)\Big)d\nu(u) + \int_s^t \nu(u)d\Big(\int_0^u \xi(v)d\tilde{B}^\kappa(v)\Big) \\ &= \int_s^t A'(u)du + \int_s^t \nu'(u)\Big(\int_0^u \xi(v)d\tilde{B}^\kappa(v)\Big)du + \int_s^t \nu(u)\xi(u)d\tilde{B}^\kappa(u) \\ &= \int_s^t \Big[A'(u) + \nu'(u)\frac{r(u)-A(u)}{\nu(u)}\Big]du + \int_s^t \nu(u)\xi(u)d\tilde{B}^\kappa(u) \\ &= \int_s^t \Big[k(u) - a(u)r(u)\Big]du + \int_s^t \sigma(u)d\tilde{B}^\kappa(u), \end{aligned}$$

where $k(t) = A'(t) - \frac{\nu'(t)}{\nu(t)}A(t)$, $a(t) = -\frac{\nu'(t)}{\nu(t)}$ and $\sigma(t) = \sigma(t,t) = \xi(t)\nu(t)$. Therefore, the short rate is described by a Vasicek dynamic under $\mathcal{Q}$.

Remark 3.8. This also matches the situation for $n = 1$ of the interest rate market in Fink, Klüppelberg and Zähle [18], Section 4, where the authors considered a fractional Vasicek model directly under $\mathcal{Q}$.

4 Pricing a defaultable zero-coupon bond

Using the results of Section 3 we see that the model (3.1) has basically the same structure under the pricing measure $\mathcal{Q}$, which can be seen by (3.8). Motivated by this, we will directly model under $\mathcal{Q}$ from now on. Therefore, for the rest of the paper, let $(B^\kappa, \bar{B}^{\bar{\kappa}})$ (as defined in Section 2) be a bivariate fBm under $\mathcal{Q}$.

On the one hand we could use equation (3.7) to calculate prices of defaultable bonds and contingent claims. On the other hand we can also always directly define bond and derivative prices via (3.7) without the approach of Section 3, i.e. through conditional expectations leading directly to an arbitrage-free setting since these quantities are per definition martingales. Independent of the chosen approach, the following calculations are valid.

Now we will consider defaultable bonds as specific contingent claims. Let H be the *default indicator process* given by

$$H(t) = \mathbf{1}_{\{\tau \leq t\}}, \quad 0 \leq t \leq T^\star,$$

where τ is an $(\mathcal{F}_t)_{0\leq t\leq T^\star}$-stopping time, representing the default time of some firm or financial instrument. We denote by $(\mathcal{H}_t)_{0\leq t\leq T^\star}$ the filtration generated by H. We assume further that there exists a subfiltration $(\mathcal{G}_t)_{0\leq t\leq T^\star}$ of $(\mathcal{F}_t)_{0\leq t\leq T^\star}$ such that

$$\mathcal{F}_t := \mathcal{G}_t \vee \mathcal{H}_t, \quad 0 \leq t \leq T^\star,$$

Assumption 4.1. *Remaining in the framework of most reduced-form credit risk models in the literature we assume that there is a $(\mathcal{G}_t)_{0\leq t\leq T^\star}$-progressive stochastic process $\lambda = (\lambda_t)_{0\leq t\leq T^\star}$*

modeling the intensity of H with the following properties (see also Corollary 5.1.5 of Bielecki and Rutkowski [4]): λ is positive, $\int_0^t \lambda(s)ds < \infty$ a.s. for all $0 \leq t \leq T^\star$, and it satisfies

$$\mathcal{Q}(\tau > t \mid \mathcal{G}_t) = E[1 - H(t) \mid \mathcal{G}_t] = \exp\left\{-\int_0^t \lambda(s)ds\right\}. \tag{4.1}$$

Moreover, defining $\mathcal{G}_\infty := \bigvee_{0 \leq t \leq T^\star} \mathcal{G}_t$, for all bounded $\mathcal{G}_\infty$-measurable random variables η, we have

$$E[\eta \mid \mathcal{F}_t] = E[\eta \mid \mathcal{G}_t], \quad 0 \leq t \leq T^\star. \tag{4.2}$$

We call λ the default rate.

Now we have to specify the joint dynamics of r and λ. Recall the bivariate fBm from Section 2. The dependence between B^κ and $\bar{B}^{\bar{\kappa}}$ is then given by the covariance function (see (2.17) of Elliot and van der Hoek [14]) for $0 \leq s, t \leq T^\star$ as

$$\mathrm{Cov}(B^\kappa(t), \bar{B}^{\bar{\kappa}}(s)) = \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)}[|t|^{\kappa+\bar{\kappa}+1} + |s|^{\kappa+\bar{\kappa}+1} + |t-s|^{\kappa+\bar{\kappa}+1}].$$

Using (2.1) we can show the following proposition by approximation with step functions.

Proposition 4.2. *Let $f, g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then*

$$\begin{aligned} & E\left[\int_{\mathbb{R}} f(s)dB^\kappa(s) \int_{\mathbb{R}} g(s)d\bar{B}^{\bar{\kappa}}(s)\right] \\ = \ & \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)}(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1)\int_{\mathbb{R}}\int_{\mathbb{R}} f(u)g(v)|u-v|^{\kappa+\bar{\kappa}-1}dudv. \end{aligned}$$

Now we model the short rate r and the default rate λ as pathwise solutions to Langevin equations

$$dr(t) = (k(t) - a(t)r(t))dt + \sigma(t)dB^\kappa(t), \quad r(0) = r_0 \in \mathbb{R}, \tag{4.3}$$

$$d\lambda(t) = (\bar{k}(t) - \bar{a}(t)\lambda(t))dt + \bar{\sigma}(t)d\bar{B}^{\bar{\kappa}}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R}, \tag{4.4}$$

where $k(\cdot), \bar{k}(\cdot), a(\cdot), \bar{a}(\cdot)$ are continuous and locally integrable on $[0, T^\star]$. Further we assume that $\sigma(\cdot), \bar{\sigma}(\cdot) > 0$ are continuous and that $\sigma(\cdot), 1/\sigma(\cdot)$ are of bounded p-variation for some $0 < p < 1/(\frac{1}{2} - \kappa)$ and that $\bar{\sigma}(\cdot), 1/\bar{\sigma}(\cdot)$ are of bounded $\bar{p}$-variation for some $0 < \bar{p} < 1/(\frac{1}{2} - \bar{\kappa})$ on $[0, T^\star]$.

Although both fBms are driven by the same Bm, its influence can vary through different coefficient functions of the Langevin equations.

Note that it is also possible to model different dynamics in r and λ by adding several independent factors driven by independent Brownian motions as explained in Remark 2.1.

Lemma 4.3. *Under the above conditions the pathwise solutions of the SDEs* (4.3) *and* (4.4) *are given for $0 \leq t \leq T \leq T^\star$ by*

$$r(T) = r(t)e^{-\int_t^T a(u)du} + \int_t^T e^{-\int_s^T a(u)du}k(s)ds + \int_t^T e^{-\int_s^T a(u)du}\sigma(s)dB^\kappa(s), \tag{4.5}$$

$$\lambda(T) = \lambda(t)e^{-\int_t^T \bar{a}(u)du} + \int_t^T e^{-\int_s^T \bar{a}(u)du}\bar{k}(s)ds + \int_t^T e^{-\int_s^T \bar{a}(u)du}\bar{\sigma}(s)d\bar{B}^{\bar{\kappa}}(s), \tag{4.6}$$

where the fBm integrals can be considered in the $L^2(\Omega)$- or pathwise sense, cf. Young [36].

Now the non-observable fBms can be replaced by observable processes given by solutions to (4.3) and (4.4).

Proposition 4.4. *Under the above conditions we have for* $0 \le t \le T \le T^\star$

$$dB^\kappa(t) = \left(-\frac{k(t)}{\sigma(t)} + \frac{a(t)}{\sigma(t)} r(t)\right) dt + \frac{1}{\sigma(t)} dr(t) \quad \text{and} \quad d\bar{B}^{\bar{\kappa}}(t) = \left(-\frac{\bar{k}(t)}{\bar{\sigma}(t)} + \frac{\bar{a}(t)}{\bar{\sigma}(t)} \lambda(t)\right) dt + \frac{1}{\bar{\sigma}(t)} d\lambda(t).$$

Proof. By (4.3) and (4.4) we have for $0 \le t \le T \le T^\star$

$$\int_t^T e^{-\int_s^T a(v)dv} \sigma(s) dB^\kappa(s) = r(T) - r(t) e^{-\int_t^T a(v)dv} - \int_t^T e^{-\int_s^T a(v)dv} k(s) ds$$

and, applying a density formula (which can be applied by Theorem A.4 of Fink and Klüppelberg [17]) we get for $0 \le t \le T \le T^\star$

$$\begin{aligned} B^\kappa(T) - B^\kappa(t) &= \int_t^T \frac{e^{\int_u^T a(v)dv}}{\sigma(u)} d\left(-\int_u^T e^{-\int_s^T a(v)dv} \sigma(s) dB^\kappa(s)\right) \\ &= \int_t^T \frac{e^{\int_u^T a(v)dv}}{\sigma(u)} d\left(\int_u^T e^{-\int_s^T a(v)dv} k(s) ds + r(u) e^{-\int_u^T a(v)dv} - r(T)\right) \\ &= -\int_t^T \frac{k(u)}{\sigma(u)} du + \int_t^T \frac{a(u)}{\sigma(u)} r(u) du + \int_t^T \frac{1}{\sigma(u)} dr(u). \end{aligned}$$

The second equation can be obtained similarly. $\square$

Corollary 4.5. *Let* $0 \le t \le s \le T^\star$*. Then the sum* $r(s) + \lambda(s)$ *is normally distributed with mean zero and variance given by*

$$\begin{aligned} & \kappa(2\kappa+1) \int_t^s \int_t^s e^{-\int_u^s a(w)dw - \int_v^s a(w)dw} \sigma(u)\sigma(v) |u-v|^{2\kappa-1} du dv \\ + \quad & 2\rho(\kappa+\bar{\kappa})(\kappa+\bar{\kappa}+1) \int_t^s \int_t^s e^{-\int_u^s a(w)dw - \int_v^s \bar{a}(w)dw} \sigma(u)\bar{\sigma}(v) |u-v|^{\kappa+\bar{\kappa}-1} du dv \qquad (4.7) \\ + \quad & \bar{\kappa}(2\bar{\kappa}+1) \int_t^s \int_t^s e^{-\int_u^s \bar{a}(w)dw - \int_v^s \bar{a}(w)dw} \bar{\sigma}(u)\bar{\sigma}(v) |u-v|^{2\bar{\kappa}-1} du dv, \end{aligned}$$

where the covariance of the two integrals is given in line (4.7)*. Here*

$$\rho = \frac{c_\kappa c_{\bar{\kappa}} \Gamma(\kappa+1)\Gamma(\bar{\kappa}+1)}{2\sin(\pi(\kappa+\bar{\kappa}+1)/2)\Gamma(\kappa+\bar{\kappa}+2)} \ge 0.$$

Remark 4.6. Corollary 4.5 implies that short rate and default rate are positively correlated, which makes sense economically. A high default rate indicates a higher probability of default before maturity. An investor will, therefore, request a compensation by a higher interest rate before taking this risk.

The information filtration given by the short rate and the default rate process is

$$\mathcal{G}_t \quad = \quad \sigma\overline{\{(r_s, \lambda_s), s \in [0,t]\}} = \sigma\overline{\{(B_s^\kappa, \bar{B}_s^{\bar{\kappa}}), s \in [0,t]\}} \quad 0 \le t \le T^\star.$$

Let $0 \le t \le T \le T^\star$ and X be $\mathcal{F}_T$-measurable. Since the contingent claim $\mathbf{1}_{\{\tau>T\}}$ is also $\mathcal{F}_T$-measurable we can apply the pricing formula (3.7) to get for the price of the defaultable $\mathcal{F}_T$-measurable contingent claim $\mathbf{1}_{\{\tau>T\}}X$ at time t

$$E\Big[\mathbf{1}_{\{\tau>T\}}Xe^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big]. \tag{4.8}$$

Considering (4.8) and Lemma 13.2 of Filipovic [15] the price of a defaultable $\mathcal{F}_T$-measurable contingent claim $\mathbf{1}_{\{\tau>T\}}X$ is for $0 \le t \le T \le T^\star$ given by

$$\overline{B}(t,T) \;=\; E\Big[\mathbf{1}_{\{\tau>T\}}Xe^{-\int_t^T r(s)ds}\Big|\mathcal{F}_t\Big] = \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_t^T (r(s)+\lambda(s))ds}X\Big|\mathcal{G}_t\Big]. \tag{4.9}$$

Setting $X=1$ we get the situation of a defaultable zero-coupon bond. The following is our main result of this section, manifesting a similar structure for the price as in the affine Markovian case.

Theorem 4.7. *Let $0 \le t < T \le T^\star$. Set $D(t,T) := \int_t^T e^{-\int_t^s a(u)du}ds$, $\bar{D}(t,T) := \int_t^T e^{-\int_t^s \bar{a}(u)du}ds$ and assume that $D(\cdot,T)\sigma(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot) \in L^2[t,T]$. Then*

$$\overline{B}(t,T) \;=\; \mathbf{1}_{\{\tau>t\}}e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)}, \tag{4.10}$$

where

$$\begin{aligned} A(t,T) \;=\; & V(t,T) - W(t,T) + \int_t^T \big(D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v)\big)dv \\ & + \int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v). \end{aligned}$$

Here $V(t,T)$, $W(t,T)$ are given in Proposition 2.10 and $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot)$ are as in (2.6) with $c(\cdot) = D(\cdot,T)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,T)\bar{\sigma}(\cdot)$. Furthermore, $\log(\overline{B}(t,T))$ is normally distributed with

$$\begin{aligned} E[\log(\overline{B}(t,T))] \;=\; & -D(t,T)e^{-\int_0^t a(u)du}r(0) - \bar{D}(t,T)e^{-\int_0^t \bar{a}(u)du}\lambda(0) \\ & -D(t,T)\int_0^t e^{-\int_v^t a(u)du}k(v)dv - \bar{D}(t,T)\int_0^t e^{-\int_v^t \bar{a}(u)du}\bar{k}(v)dv \\ & -\int_t^T (D(v,T)k(v) + \bar{D}(v,T)\bar{k}(v))dv - V(t,T) + W(t,T)\,, \\ \mathrm{Var}(\log(\overline{B}(t,T)) \;=\; & \Big\|\Big(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a(u)du}\sigma(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\|_\kappa^2 \\ & +2\Big\langle\Big(\Psi_c^\kappa(t,T,\cdot) + D(t,T)e^{-\int_\cdot^t a(u)du}\sigma(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot), \\ & \Big(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}(u)du}\bar{\sigma}(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\rangle_{\kappa,\bar{\kappa}} \\ & +\Big\|\Big(\Psi_{\bar{c}}^{\bar{\kappa}}(t,T,\cdot) + \bar{D}(t,T)e^{-\int_\cdot^t \bar{a}(u)du}\bar{\sigma}(\cdot)\Big)\mathbf{1}_{[0,t]}(\cdot)\Big\|_{\bar{\kappa}}^2. \end{aligned}$$

Proof. The case $t = 0$ is trivial by (4.7), so let $t > 0$. We obtain from Lemma 4.3 and Fubini's Theorem (Theorem 1 of Krvavich and Mishura [27])

$$\begin{aligned}
\int_t^T (r(s)+\lambda(s))ds &= \int_t^T \Big[r(t)e^{-\int_t^s a(u)du} + \int_t^s e^{-\int_v^s a(u)du} k(v)dv + \int_t^s e^{-\int_v^s a(u)du}\sigma(v)dB^\kappa(v)\Big]ds \\
&\quad + \int_t^T \Big[\lambda(t)e^{-\int_t^s \bar{a}(u)du} + \int_t^s e^{-\int_v^s \bar{a}(u)du}\bar{k}(v)dv + \int_t^s e^{-\int_v^s \bar{a}(u)du}\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\Big]ds \\
&= D(t,T)r(t) + \int_t^T D(v,T)k(v)dv + \int_t^T D(v,T)\sigma(v)dB^\kappa(v) \\
&\quad + \bar{D}(t,T)\lambda(t) + \int_t^T \bar{D}(v,T)\bar{k}(v)dv + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v). \qquad (4.11)
\end{aligned}$$

By Proposition 2.10 we have

$$\begin{aligned}
&E\left[\exp\left\{\int_t^T D(v,T)\sigma(v)dB^\kappa(v) + \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Bigg|\,\mathcal{G}_t\right] \\
&= e^{W(t,T)-V(t,T)}\exp\left\{\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) + \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}.
\end{aligned}$$

Now we get for the price of the defaultable zero-coupon bond by

$$\begin{aligned}
\overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}E\Big[e^{-\int_0^T (r(s)+\lambda(s))ds}\Big|\mathcal{G}_t\Big] \\
&= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\
&\quad \times E\left[\exp\left\{-\int_t^T D(v,T)\sigma(v)dB^\kappa(v) - \int_t^T \bar{D}(v,T)\bar{\sigma}(v)d\bar{B}^{\bar{\kappa}}(v)\right\}\Bigg|\,\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}}e^{-D(t,T)r(t)-\int_t^T D(v,T)k(v)dv-\bar{D}(t,T)\lambda(t)-\int_t^T \bar{D}(v,T)\bar{k}(v)dv} \\
&\quad \times e^{W(t,T)-V(t,T)}\exp\left\{-\int_0^t \Psi_c^\kappa(t,T,v)dB^\kappa(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\} \\
&= \mathbf{1}_{\{\tau>t\}}e^{-A(t,T)-D(t,T)r(t)-\bar{D}(t,T)\lambda(t)}
\end{aligned}$$

with $A(t,T)$, c and $\bar{c}$ as given in the assertion. The formulas for the expectation and variance of $\log(\overline{B}(t,T))$ can be obtained by simple calculations. □

Remark 4.8. If we compare (4.10) with Proposition 7.2 of Schönbucher [34], we realize that in the case $t = 0$ the zero-coupon bond prices differ only by a deterministic factor. However, if we calculate the price at time $t > 0$, the whole paths of the fractional Brownian motions up to time t enter because of the dependent increments. Those integrals do not appear in a Markovian model.

By Proposition 4.4 we rewrite the bond price in terms of r and λ.

Corollary 4.9. *In the situation of Theorem 4.7 we have for* $0 \le t < T \le T^\star$

$$\begin{aligned}
\overline{B}(t,T) &= \mathbf{1}_{\{\tau>t\}}\exp\Big\{-\widetilde{A}(t,T) - D(t,T)r(t) - \bar{D}(t,T)\lambda(t)\Big\} \\
&\quad \times \exp\Big\{-\int_0^t (\Psi_c^\kappa(t,T,v)\frac{a(t)}{\sigma(t)}r(t) + \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{\bar{a}(t)}{\bar{\sigma}(v)}\lambda(t))dv\Big\} \\
&\quad \times \exp\Big\{-\int_0^t \Psi_c^\kappa(t,T,v)\frac{1}{\sigma(v)}dr(v) - \int_0^t \Psi_{\bar{c}}^{\bar{\kappa}}(t,T,v)\frac{1}{\bar{\sigma}(v)}d\lambda(v)\Big\}
\end{aligned}$$

where $\Psi_c^\kappa(t,T,\cdot)$, $\Psi_{\bar c}^{\bar\kappa}(t,T,\cdot)$ *are as in* (2.6) *with* $c(\cdot)=D(\cdot,T)\sigma(\cdot)$, $\bar c(\cdot)=\bar D(\cdot,T)\bar\sigma(\cdot)$ *and*

$$\widetilde{A}(t,T) \;=\; V(t,T)-W(t,T)+\int_t^T \big(D(v,T)k(v)+\bar D(v,T)\bar k(v)\big)dv$$

with $W(t,T)$ *and* $V(t,T)$ *as in Proposition 2.10.*

5 Option pricing

In this section we explain how derivatives prices can be calculated. First we aim for a European call price with a defaultable zero-coupon bond as underlying. Today's price can be found similar to the classical Brownian case and a closed formula is obtained. For more general options and times, we apply Fourier techniques.

In Theorem 5.2 below we will price a European call option invoking a change of numéraire. Therefore, we need a Girsanov theorem. For the elementary case, where the drift of a fBm is changed by a deterministic factor, the measure change has been derived in Norros, Valkeila and Virtamo [29], Theorem 4.1, using pathwise integration. In our case we need some result for the other direction. We need to know the distribution of a fBm after a given measure change. Theorem 3.3 of Duncan, Hu and Pasik-Duncan [13] considers a general situation, which we can use. Moreover, their result also covers the result of [29].

Proposition 5.1. *Let* $0\le t<T<S\le T^\star$. *Consider a European call at strike* $K>0$ *and maturity* T *based on a defaultable zero-coupon bond maturing at time* S *as underlying given by the contingent claim*

$$\mathbf{1}_{\{\tau>T\}}(\overline{B}(T,S)-K)_+.$$

At time t *the price* $\mathcal{V}(t,T,S)$ *is given by*

$$\begin{aligned}\mathcal{V}(t,T,S) &= \mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}(\overline{B}(T,S)-K)_+\Big|\,\mathcal{G}_t\right]\\ &= \mathbf{1}_{\{\tau>t\}}\overline{B}(t,T)E^T\left[(\overline{B}(T,S)-K)_+\big|\,\mathcal{G}_t\right],\end{aligned}\tag{5.1}$$

where E^T *is the expectation with respect to the* T*-forward measure defined by the Radon-Nikodym derivative*

$$\frac{d\mathcal{Q}^T}{d\mathcal{Q}} \;=\; \exp\left\{-\int_0^T(r(s)+\lambda(s))ds\right\}e^{-\overline{B}(0,T)}.\tag{5.2}$$

Proof. The first equality follows by (4.9). As in the classical Bm case we calculate the European call price by means of a T-forward measure (using the expressions defined in Theorem 4.7)

$$\begin{aligned}\frac{d\mathcal{Q}^T}{d\mathcal{Q}} &= \exp\left\{-\int_0^T(r(s)+\lambda(s))ds\right\}e^{-\overline{B}(0,T)}\\ &= \exp\left\{-\int_0^T D(v,T)\sigma(v)dB^\kappa(v)-\int_0^T\bar D(v,T)\bar\sigma(v)d\bar B^{\bar\kappa}(v)-W(0,T)\right\}\end{aligned}$$

Using Bayes' theorem for conditional expectations we obtain (5.1). □

Denote by N the standard normal distribution function.

Theorem 5.2. *Let $0 < T < S \le T^\star$. At time 0 the price $\mathcal{V}(0,T,S)$ of a European call at strike $K > 0$ and maturity T based on a defaultable zero-coupon bond maturing at time S as underlying is given by*

$$\begin{aligned}&\mathcal{V}(0,T,S) \;=\; \overline{B}(0,T)\\ &\times \left\{ e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)} N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S) \right) - KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} \right) \right\}\end{aligned}$$

with

$$\begin{aligned} A(0,T,S) \;&=\; V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar{D}(v,S)\bar{k}(v) \big) dv \\ &+D(T,S)\left(r(0)e^{-\int_0^T a(u)du} + \int_0^T e^{-\int_v^T a(u)du} k(v)dv \right) \\ &+\bar{D}(T,S)\left(\lambda(0)e^{-\int_0^T \bar{a}(u)du} + \int_0^T e^{-\int_v^T \bar{a}(u)du} \bar{k}(v)dv \right) \\ &- \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot) \right\rangle_\kappa - \left\langle \Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{(0,T)}(\cdot) \right\rangle_{\kappa,\bar{\kappa}} \\ &- \left\langle \bar{\Phi}(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot) \right\rangle_{\kappa,\bar{\kappa}} - \left\langle \bar{\Phi}(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar{\sigma}(\cdot)\mathbf{1}_{(0,T)}(\cdot) \right\rangle_{\bar{\kappa}} \end{aligned}$$

where $V(T,S), W(T,S)$ are as in Proposition 2.10. Furthermore,

$$\begin{aligned} \Sigma(0,T,S)^2 \;&=\; Var\left(-\int_0^T \Phi(v)dB^\kappa(v) - \int_0^T \bar{\Phi}(v)d\bar{B}^{\bar{\kappa}}(v) \right) \\ &= \left\| \mathbf{1}_{(0,T)}(\cdot)\Phi(\cdot) \right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(0,T)}(\cdot)\Phi(\cdot), \mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot) \right\rangle_{\kappa,\bar{\kappa}} + \left\| \mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot) \right\|_{\bar{\kappa}}^2 . \end{aligned} \tag{5.3}$$

Here we have set

$$\begin{aligned} \Phi(\cdot) \;&:=\; \Psi_c^\kappa(S,T,\cdot) + D(T,S)e^{-\int_\cdot^S a(u)du}\sigma(\cdot) \quad \text{and} \\ \bar{\Phi}(\cdot) \;&:=\; \Psi_{\bar{c}}^{\bar{\kappa}}(S,T,\cdot) + \bar{D}(T,S)e^{-\int_\cdot^S \bar{a}(u)du}\bar{\sigma}(\cdot), \end{aligned} \tag{5.4}$$

where $\Psi_c^\kappa(S,T,\cdot)$, $\Psi_{\bar{c}}^{\bar{\kappa}}(S,T,\cdot)$ are as in (2.6) with $c(\cdot) = D(\cdot,S)\sigma(\cdot)$, $\bar{c}(\cdot) = \bar{D}(\cdot,S)\bar{\sigma}(\cdot)$.

Proof. W.l.o.g. assume $\bar{\kappa} \ge \kappa$. Recall $\overline{B}(S,T)$ from Theorem 4.7. We replace $r(S)$ and $\lambda(S)$ as in the proof of Theorem 4.7 by the solutions to the SDEs given in (4.5). Then we collect those terms, which are deterministic and those, which are not. This yields the following definition of a function F on the paths of the fBm B^κ as

$$\begin{aligned}&F(B^\kappa) := \\ &\left(\exp\left\{ -\bar{A}(0,T,S) - \int_{\mathbb{R}} \left(\Phi(v)\mathbf{1}_{(0,T)}(v) + \frac{a_{\bar{\kappa}}}{a_\kappa}\mathcal{I}_-^{\bar{\kappa}-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar{\Phi}(\cdot))(v) \right) dB^\kappa(v) \right\} - K \right)_+ .\end{aligned}$$

with Φ and $\bar{\Phi}$ as in (5.4) and

$$\begin{aligned} \bar{A}(0,T,S) \;&=\; V(T,S) - W(T,S) + \int_T^S \big(D(v,S)k(v) + \bar{D}(v,S)\bar{k}(v) \big) dv \\ &+D(T,S)\left(r(0)e^{-\int_0^T a(u)du} + \int_0^T e^{-\int_v^T a(u)du} k(v)dv \right) \\ &+\bar{D}(T,S)\left(\lambda(0)e^{-\int_0^T \bar{a}(u)du} + \int_0^T e^{-\int_v^T \bar{a}(u)du} \bar{k}(v)dv \right) \end{aligned}$$

where $V(T,S), W(T,S)$ are as in Proposition 2.10. Starting with (5.1) from Proposition 5.1 we obtain

$$
\begin{aligned}
\mathcal{V}(0,T,S) &= \overline{B}(0,T)E^T[(\overline{B}(T,S)-K)_+] \\
&= \overline{B}(0,T)E^T\left[F(B^\kappa)\right] \\
&= \overline{B}(0,T)E\left[F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right].
\end{aligned}
$$

with

$$
\Upsilon(v) := -\left(D(v,T)\sigma(v)\mathbf{1}_{(0,T)}(v) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar{D}(\cdot,T)\bar\sigma(\cdot))(v)\right).
$$

For the last equality we applied Theorem 3.3 of Duncan, Hu and Pasik-Duncan [13] to calculate the expectation under the T-forward measure $\mathcal{Q}^T$. (In fact, we have to extend their result to Wick exponentials defined on the whole of $\mathbb{R}$ as in (2.7).) We further calculate

$$
\begin{aligned}
& F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dvds\right) \\
= & \left(\exp\left\{-\bar{A}(0,T,S) - \int_{\mathbb{R}}\left(\Phi(v)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(v)\right)dB^\kappa(v)\right.\right. \\
& \left.\left. - \left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_\kappa\right\} - K\right)_+ .
\end{aligned}
$$

With Lemma A.1 of the Appendix one can show that

$$
\begin{aligned}
& -\left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot) + \frac{a_{\bar\kappa}}{a_\kappa}\mathcal{I}_-^{\bar\kappa-\kappa}(\mathbf{1}_{(0,T)}(\cdot)\bar\Phi(\cdot))(\cdot), \Upsilon(\cdot)\right\rangle_\kappa \\
= & \left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_\kappa + \left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\kappa,\bar\kappa} \\
& + \left\langle\bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\kappa,\bar\kappa} + \left\langle\bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\bar\kappa}.
\end{aligned}
$$

Collecting all terms and transforming the integral back we finally arrive at

$$
\begin{aligned}
& F\left(B^\kappa + \kappa(2\kappa+1)\int_{-\infty}^{\cdot}\left(\int_{-\infty}^{\infty}\Upsilon(v)|v-s|^{2\kappa-1}dv\right)ds\right) \\
= & \left(\exp\left\{-A(0,T,S) - \int_0^T\Phi(v)dB^\kappa(v) - \int_0^T\bar\Phi(v)d\bar{B}^{\bar\kappa}(v)\right\} - K\right)_+,
\end{aligned}
$$

where

$$
\begin{aligned}
& A(0,T,S) \\
:= & \bar{A}(0,T,S) - \left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_\kappa - \left\langle\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\kappa,\bar\kappa} \\
& - \left\langle\bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), D(\cdot,T)\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\kappa,\bar\kappa} - \left\langle\bar\Phi(\cdot)\mathbf{1}_{(0,T)}(\cdot), \bar{D}(\cdot,T)\bar\sigma(\cdot)\mathbf{1}_{(0,T)}(\cdot)\right\rangle_{\bar\kappa}.
\end{aligned}
$$

Finally, we can calculate the expectation in the pricing formula. This works now exactly as in the case of the classical Black-Scholes setting, since the appearing integrals are Gaussian. This results in

$$
\begin{aligned}
\mathcal{V}(0,T,S) &= \overline{B}(0,T) \\
&\quad \times E\left[\left(\exp\left\{-A(0,T,S) - \int_0^T \Phi(v)dB^{\kappa}(v) - \int_0^T \bar{\Phi}(v)d\bar{B}^{\bar{\kappa}}(v)\right\} - K\right)_+\right] \\
&= \overline{B}(0,T) \\
&\quad \times e^{-A(0,T,S)}E\left[\left(\exp\left\{- \int_0^T \Phi(v)dB^{\kappa}(v) - \int_0^T \bar{\Phi}(v)d\bar{B}^{\bar{\kappa}}(v)\right\} - e^{A(0,T,S)}K\right)_+\right] \\
&= \overline{B}(0,T) \times \left\{e^{\frac{\Sigma(0,T,S)^2}{2} - A(0,T,S)} N\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)} + \Sigma(0,T,S)\right)\right. \\
&\quad \left. - KN\left(-\frac{A(0,T,S)+\log(K)}{\Sigma(0,T,S)}\right)\right\}
\end{aligned}
$$

where $\Sigma(0,T,S)^2$ is defined in (5.3). The expression for the variance can be deduced by calculating the characteristic functions analogously to the moment generating functions in Proposition 2.10, then apply Lemma A.1 of the Appendix to rewrite the appearing norms and scalar products. □

Remark 5.3. We want to compare the price (5.3) to the European call price in a classical Brownian Vasicek model. For simplicity we choose a model with constant coefficient functions. Given two dependent standard Brownian motions $B, \bar{B}$ with correlation $\rho > 0$, we model the short and hazard rate by the SDEs

$$
\begin{aligned}
dr(t) &= (k - ar(t))dt + \sigma dB(t), \quad r(0) = r_0 \in \mathbb{R}, \\
d\lambda(t) &= (\bar{k} - \bar{a}\lambda(t))dt + \bar{\sigma} d\bar{B}(t), \quad \lambda(0) = \lambda_0 \in \mathbb{R},
\end{aligned}
$$

where we will assume that $\sigma, \bar{\sigma} > 0$. We know by Proposition 5.3 of Schönbucher [34] that this model eventually boils down to a two-factor short rate model. Using for example Theorem 4.2.1 of Brigo and Mercurio [5], today's price of the defaultable zero-coupon bond is given by

$$
\overline{B}(0,T) = \exp\left\{-A(0,T) - \frac{k}{a}\left[T - \frac{e^{-aT}-1}{a}\right] - \frac{\bar{k}}{\bar{a}}\left[T - \frac{e^{-\bar{a}T}-1}{\bar{a}}\right] - \frac{1-e^{-aT}}{a}r_0 - \frac{1-e^{-\bar{a}T}}{\bar{a}\lambda_0}\right\}
$$

with

$$
\begin{aligned}
A(0,T) &= -\frac{1}{2}\left(\frac{\sigma^2}{a^2}\left[T + \frac{2}{a}e^{-aT} - \frac{1}{2a}e^{-2aT} - \frac{3}{2a}\right] + \frac{\bar{\sigma}^2}{\bar{a}^2}\left[T + \frac{2}{\bar{a}}e^{-\bar{a}T} - \frac{1}{2\bar{a}}e^{-2\bar{a}T} - \frac{3}{2\bar{a}}\right]\right. \\
&\quad \left. + 2\rho\frac{\sigma\bar{\sigma}}{a\bar{a}}\left[T + \frac{e^{-aT}-1}{a} + \frac{e^{-\bar{a}T}-1}{\bar{a}} - \frac{e^{-(a+\bar{a})T}-1}{a+\bar{a}}\right]\right).
\end{aligned}
$$

Let $0 \leq T \leq S \leq T^\star$. Applying Theorem 4.2.2 of Brigo and Mercurio [5] we get for the price $\mathcal{V}(0,T,S)$ of a call option with maturity T and strike K, written on a defaultable zero-coupon bond maturing at time S:

$$
\begin{aligned}
&\mathcal{V}(0,T,S) \\
&= \overline{B}(0,S)N\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} + \frac{1}{2}\Sigma(0,T,S)\right) - \overline{B}(0,T)KN\left(\frac{\log\left(\frac{\overline{B}(0,S)}{K\overline{B}(0,T)}\right)}{\Sigma(0,T,S)} - \frac{1}{2}\Sigma(0,T,S)\right),
\end{aligned}
$$

where

$$\begin{aligned}\Sigma^2(0,T,S) &= \frac{\sigma^2}{2a^3}\Big(1-e^{-a(S-T)}\Big)^2\Big(1-e^{-2a(T-t)}\Big)+\frac{\bar\sigma^2}{2\bar a^3}\Big(1-e^{-\bar a(S-T)}\Big)^2\Big(1-e^{-2\bar a(T-t)}\Big)\\ &= 2\rho\frac{\sigma\bar\sigma}{a\bar a(a+\bar a)}\Big(1-e^{-a(S-T)}\Big)\Big(1-e^{-\bar a(S-T)}\Big)^2\Big(1-e^{-(a+\bar a)(T-t)}\Big).\end{aligned}$$

Note now that the main structure of bond and call prices is the same in both models, especially today's bond prices differ only by a deterministic multiplicative factor; however, if we look further "into the future" the path of the fBm does matter, which results in a more complex option price.

We want to emphasize that we have in the situation of (5.3)

$$\overline{B}(0,T)e^{\frac{\Sigma(0,T,S)^2}{2}-A(0,T,S)}\neq\overline{B}(0,S)$$

and, therefore, do not get exactly the same structure as in the Brownian case.

Numerical evaluations of the formulas in the fractional case are significantly more complicated than in the classical Brownian model. Especially calculating the norms $\|\cdot\|_\kappa$ is challenging due to the singularity of the weight function $(x,y)\mapsto|x-y|^{2\kappa-1}$ on the diagonal. For some graphs depicting bond prices for different fractional parameters we refer to Section 4 of Fink et al. [18].

The following pricing method allows for more general payoff functions, but it is less explicit. Note that it also includes the European call price calculated explicitly in Theorem 5.2.

Theorem 5.4. *Let $0\le t<T\le T^\star$. Denote by X an $\mathcal{F}_T$-measurable payoff of the form*

$$X=\mathbf{1}_{\{\tau>T\}}f\left(\int_0^T\phi(s)dB^\kappa(s)+\int_0^T\bar\phi(s)d\bar B^{\bar\kappa}(s)\right)$$

for some $f:\mathbb{R}\to\mathbb{R}$ and $\phi,\bar\phi\in L^2[0,T]$. Assume further that there exist $b>0$ and $z\in\mathbb{R}$ such that $f_+^{b,z}(\cdot):=e^{-b\cdot}f(\cdot)\mathbf{1}_{[z,\infty)}(\cdot)$ and $f_-^{b,z}(\cdot):=e^{b\cdot}f(\cdot)\mathbf{1}_{(-\infty,z)}(\cdot)$ are in $L^1(\mathbb{R})\cap L^2(\mathbb{R})$. Define for $\xi\in\mathbb{R}$ and $\star\in\{+,-\}$

$$\Phi^{\xi,\star}(\cdot):=D(\cdot,T)\sigma(\cdot)-(i\xi\star b)\phi(\cdot),\quad \bar\Phi^{\xi,\star}(\cdot):=\bar D(\cdot,T)\bar\sigma(\cdot)-(i\xi\star b)\bar\phi(\cdot).\tag{5.5}$$

Then the price of X at time t is given by

$$\begin{aligned}&\mathcal{V}(t,T)=\mathbf{1}_{\{\tau>t\}}E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}X\Big|\mathcal{G}_t\right] \qquad (5.6)\\ =\ &\mathbf{1}_{\{\tau>t\}}\exp\left\{-\int_t^T D(s,T)k(s)ds-\int_t^T\bar D(s,T)\bar k(s)ds-D(t,T)r(t)-\bar D(t,T)\lambda(t)\right\}\\ &\times\frac{1}{2\pi}\int_{\mathbb{R}}\left[\exp\left\{V^\xi(t,T)-W^\xi(t,T)-\int_0^t\Psi^\kappa_{c_\xi^+}(t,T,v)dB^\kappa(v)-\int_0^t\Psi^{\bar\kappa}_{\bar c_\xi^+}(t,T,v)d\bar B^{\bar\kappa}(v)\right\}\widehat{f_+^{b,z}}(\xi)\right.\\ &\left.+\exp\left\{V^\xi(t,T)-W^\xi(t,T)-\int_0^t\Psi^\kappa_{c_\xi^-}(t,T,v)dB^\kappa(v)-\int_0^t\Psi^{\bar\kappa}_{\bar c_\xi^-}(t,T,v)d\bar B^{\bar\kappa}(v)\right\}\widehat{f_-^{b,z}}(\xi)\right]d\xi\end{aligned}$$

where $c_\xi^\star(\cdot) = \Phi^{\xi,\star}(\cdot)$ *and* $\bar{c}_\xi^\star(\cdot) = \bar{\Phi}^{\xi,\star}(\cdot)$,

$$
\begin{aligned}
W^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(t,T)}(\cdot)\Phi^{\xi,\star}(\cdot)\right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(t,T)}(\cdot)\Phi^{\xi,\star}(\cdot), \mathbf{1}_{(t,T)}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\rangle_{\kappa,\bar{\kappa}}\right. \\
&\quad \left. + \left\|\mathbf{1}_{(t,T)}(\cdot)\bar{\Phi}^{\xi,\star}(\cdot)\right\|_{\bar{\kappa}}^2\right), \\
V^{\xi,\star}(t,T) &= \frac{1}{2}\left(\left\|\mathbf{1}_{(0,t)}(\cdot)\Psi^\kappa_{c_\xi^\star}(t,T,\cdot)\right\|_\kappa^2 + 2\left\langle \mathbf{1}_{(0,t)}(\cdot)\Psi^\kappa_{c_\xi^\star}(t,T,\cdot), \mathbf{1}_{(0,t)}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}_\xi^\star}(t,T,\cdot)\right\rangle_{\kappa,\bar{\kappa}}\right. \\
&\quad \left. + \left\|\mathbf{1}_{(0,t)}(\cdot)\Psi^{\bar{\kappa}}_{\bar{c}_\xi^\star}(t,T,\cdot)\right\|_{\bar{\kappa}}^2\right), \qquad (5.7)
\end{aligned}
$$

with $\widehat{f_+^{b,z}}$ *and* $\widehat{f_-^{b,z}}$ *the Fourier transforms of* $f_+^{b,z}$ *and* $f_-^{b,z}$ *respectively.*

Proof. Applying - as in the theorem before - Lemma 13.2 of Filipovic [15] we obtain (5.6). For some $a < 0$ and $z \in \mathbb{R}$ we have

$$
f(x) = e^{bx}[e^{-bx}f(x)\mathbf{1}_{[z,\infty)}(x)] + e^{-bx}[e^{bx}f(x)\mathbf{1}_{(-\infty,z)}(x)] =: e^{bx}f_+^{b,z}(x) + e^{-bx}f_-^{b,z}(x). \qquad (5.8)
$$

Denote by $\widehat{f_+^{b,z}}$ and $\widehat{f_-^{b,z}}$ the Fourier transforms of $f_+^{b,z}$ and $f_-^{b,z}$ respectively. Using classical Fourier analysis we obtain for $\xi, x \in \mathbb{R}$ and $\star \in \{+,-\}$

$$
\widehat{f_\star^{b,z}}(\xi) = \int_{\mathbb{R}} e^{-i\xi x} f_\star^{b,z}(x)dx, \quad f_\star^{b,z}(x) = \frac{1}{2\pi}\int_{\mathbb{R}} e^{i\xi x}\widehat{f_\star^{b,z}}(\xi)d\xi,
$$

where we used the fact that $f_+^{b,z}$ and $f_-^{b,z}$ are in $L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Set

$$
J(t,T) := \int_t^T \phi(s)dB^\kappa(s) + \int_t^T \bar{\phi}(s)d\bar{B}^{\bar{\kappa}}(s).
$$

We get by the definition and (5.8)

$$
\begin{aligned}
X &= f\left(J(0,T)\right) \\
&= e^{bJ(0,T)} f_+^{b,z}\left(J(0,T)\right) + e^{-bJ(0,T)} f_-^{b,z}\left(J(0,T)\right) \\
&= \frac{1}{2\pi}\int_{\mathbb{R}} \left(e^{(i\xi+b)J(0,T)}\widehat{f_+^{b,z}}(\xi) + e^{(i\xi-b)J(0,T)}\widehat{f_-^{b,z}}(\xi)\right)d\xi
\end{aligned}
$$

Since by normality $E[e^{bJ(0,T)}] < \infty$ for all $b \in \mathbb{R}$ we can interchange expectation and integration as follows using (4.11)

$$
\begin{aligned}
\mathcal{V}(t,T) &= \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}X\middle|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}} E\left[e^{-\int_t^T(r(s)+\lambda(s))ds}\frac{1}{2\pi}\int_{\mathbb{R}}\left[e^{(i\xi+b)J(0,T)}\widehat{f_+^{b,z}}(\xi) + e^{(i\xi-b)J(0,T)}\widehat{f_-^{b,z}}(\xi)\right]d\xi\middle|\mathcal{G}_t\right] \\
&= \mathbf{1}_{\{\tau>t\}} e^{C(t,T)+D(t,T)r(t)+\bar{D}(t,T)\lambda(t)} \\
&\quad \times\frac{1}{2\pi}\int_{\mathbb{R}}\left[E\left[e^{G(t,T)+(i\xi+b)J(0,T)}\middle|\mathcal{G}_t\right]\widehat{f_+^{b,z}}(\xi) + E\left[e^{G(t,T)+(i\xi-b)J(0,T)}\middle|\mathcal{G}_t\right]\widehat{f_-^{b,z}}(\xi)\right]d\xi
\end{aligned}
$$

with $C(t,T) := -\int_t^T D(v,T)k(v)dv - \int_t^T \bar{D}(v,T)\bar{k}(v)dv$ and $G(t,T) := -\int_t^T D(s,T)\sigma(s)dB^{\kappa}(s) - \int_t^T \bar{D}(s,T)\bar{\sigma}(s)d\bar{B}^{\bar{\kappa}}(s)$. The case $t=0$ is now again simple because we just need to calculate the expectations. Let further be $t>0$. Prediction works now the same way as in Proposition 2.10 and we obtain for $\star \in \{+,-\}$ with $\Phi^{\xi,\star}$ and $\bar{\Phi}^{\xi,\star}$ as in (5.5):

$$
\begin{aligned}
& E\left[e^{G(t,T)+(i\xi\star b)J(0,T)}\middle|\,\mathcal{G}_t\right] = e^{(i\xi\star b)J(0,t)}E\left[e^{G(t,T)+(i\xi\star b)J(t,T)}\middle|\,\mathcal{G}_t\right] \\
= \;& e^{(i\xi\star b)J(0,t)}E\left[e^{-\int_t^T (D(s,T)\sigma(s)-(i\xi\star b)\phi(s))dB^{\kappa}(s)-\int_t^T(\bar{D}(s,T)\bar{\sigma}(s)-(i\xi\star b)\bar{\phi}(s))d\bar{B}^{\bar{\kappa}}(s)}\middle|\,\mathcal{G}_t\right] \\
= \;& e^{(i\xi\star b)J(0,t)}E\left[e^{-\int_t^T \Phi^{\xi,\star}(s)dB^{\kappa}(s)-\int_t^T \bar{\Phi}^{\xi,\star}(s)d\bar{B}^{\bar{\kappa}}(s)}\middle|\,\mathcal{G}_t\right] \\
= \;& e^{(i\xi\star b)J(0,t)+V^{\xi,\star}(t,T)-W^{\xi,\star}(t,T)}\exp\left\{-\int_0^t \Psi^{\kappa}_{c^\star_\xi}(t,T,v)dB^{\kappa}(v) - \int_0^t \Psi^{\bar{\kappa}}_{\bar{c}^\star_\xi}(t,T,v)d\bar{B}^{\bar{\kappa}}(v)\right\}
\end{aligned}
$$

where $c^\star_\xi(\cdot) = \Phi^{\xi,\star}(\cdot)$ and $\bar{c}^\star_\xi(\cdot) = \bar{\Phi}^{\xi,\star}(\cdot)$ and $W^{\xi,\star}(t,T)$, $V^{\xi,\star}(t,T)$ are as in (5.7). □

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Appendix

Lemma A.1. *Given the situation of Section 2 we assume $\bar{\kappa} \geq \kappa$. Similar results hold true for $\bar{\kappa} < \kappa$. Let $f : \mathbb{R} \to \mathbb{R}$ with $||f(\cdot)||_{\kappa} < \infty$ and $g \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$. Then we have*

$$
\begin{aligned}
\left(\frac{a_{\bar{\kappa}}}{a_{\kappa}}\right)^2 \left\|\mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\|_{\kappa}^2 &= \|g(\cdot)\|_{\bar{\kappa}}^2, &\text{(A.1)}\\
\frac{a_{\bar{\kappa}}}{a_{\kappa}}\left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_{\kappa} &= \langle f(\cdot), g(\cdot)\rangle_{\kappa,\bar{\kappa}}, &\text{(A.2)}
\end{aligned}
$$

where $a_{\kappa} := c_{\kappa}\Gamma(\kappa+1)$ and $a_{\bar{\kappa}} := c_{\bar{\kappa}}\Gamma(\bar{\kappa}+1)$.

Proof. We know from Lemma 2.3 that in the $L^2(\Omega)$-sense

$$\int_{\mathbb{R}} g(v)d\bar{B}^{\bar{\kappa}}(v) = \frac{a_{\bar{\kappa}}}{a_{\kappa}}\int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v)dB^{\kappa}(v)$$

and, therefore, variances are equal. Equation (A.1) follows. Furthermore we have with Lemma 2.3 again in the $L^2(\Omega)$-sense

$$\int_{\mathbb{R}} f(v)dB^{\kappa}(v)\int_{\mathbb{R}} g(v)d\bar{B}^{\bar{\kappa}}(v) = \int_{\mathbb{R}} f(v)dB^{\kappa}(v)\frac{a_{\bar{\kappa}}}{a_{\kappa}}\int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v)dB^{\kappa}(v)$$

and, therefore, by Proposition 4.2

$$
\begin{aligned}
\langle f(\cdot), g(\cdot)\rangle_{\kappa,\bar{\kappa}} &= E\left[\int_{\mathbb{R}} f(v)dB^{\kappa}(v)\int_{\mathbb{R}} g(v)d\bar{B}^{\bar{\kappa}}(v)\right] \\
&= E\left[\int_{\mathbb{R}} f(v)dB^{\kappa}(v)\frac{a_{\bar{\kappa}}}{a_{\kappa}}\int_{\mathbb{R}} \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(v)dB^{\kappa}(v)\right] \\
&= \frac{a_{\bar{\kappa}}}{a_{\kappa}}\left\langle f(\cdot), \mathcal{I}_-^{\bar{\kappa}-\kappa}(g(\cdot))(\cdot)\right\rangle_{\kappa}
\end{aligned}
$$

□

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