Pair-copula constructions for modeling exchange rate dependence

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Abstract

In order to capture the dependency among exchange rates we construct semiparametric multivariate copula models with ARMA-GARCH margins. As multivariate copula models we utilize pair-copula constructions (PCC) such as regular and canonical vines. As building blocks of the PCC's we use bivariate t-copulas for different tail dependence between pairs of exchange rates. Alternatively we also consider a non Gaussian directed acyclic graph (DAG) model which can be imbedded as a special PCC. We apply these models to Euro exchange rates. A nonnested model comparison technique is developed to compare DAG, regular and canonical vine based models. This provides a modeling framework for constructing high dimensional joint models and allows extensions to asymmetric marginal models and time varying dependence models.

Keywords: multivariate copula, GARCH-ARMA margins, exchange rates, pair-copula construction, vines, directed acyclic graphs

1 Introduction

Copulas (see for example the books by Joe (1997), Nelsen (1999) and Cherubini, Luciano, and Vecchiato (2004)) are important tools to model dependence. While there are many bivariate copulas available for modelling bivariate dependence, the catalogue of multivariate copulas is less rich. It includes the multivariate Gauss, t-copula and more general elliptical copulas (see Frahm, Junker, and Szimayer (2003)) as well as Archimedian copulas (see for example Joe (1997) and McNeil and Nestlehova (2007)).

Recently Aas et al. (2007) have discovered the power of pair-copula constructions (PCC) for constructing highly flexible multivariate copulas. These constructions are based on early work by Joe (1996). Bedford and Cooke (2001) and Bedford and Cooke (2002) systemized these constructions and the book by Kurowicka and Cooke (2006) contains an overview on the subject. Aas et al. (2007) were the first to provide statistical inference procedures for these multivariate copulas. They used a maximum likelihood (ML) approach, while Min and Czado (2008) follow a Bayesian approach. The Bayesian approach using Markov chain Monte Carlo (MCMC) methods naturally gives credible intervals for the parameters and data sets of all sample sizes. In contrast to the Bayesian approach only asymptotic confidence intervals can be constructed using the asymptotic normality of the estimators. However the Fisher information is difficult to obtain and one has to resort as is done in this paper to utilizing a numerical approximation of the Hessian matrix.

PCC's consist of a cascade of arbitrary bivariate copulas modelling the dependence and conditional dependence between pairs of variables. In a first application to a four dimensional financial returns data set Aas et al. (2007) showed that this modelling approach is superior to a multivariate t-copula approach, where there is only a single parameter to model tail dependence between pairs. In contrast PCC's allow for different parameters for the tail dependence of each pair. A grouped t-copula (see Demarta and McNeil (2005)) can provide some flexibility in terms of modeling tail dependence, however the approach of Aas et al. (2007) is still more flexible. Archimedian copulas have also been extended to allow for more modeling flexibilities (see McNeil (2008) and Hofer (2007)). These nested Archimedian copulas however require parameter restrictions. In recent papers by Berg and Aas (2007) and Fischer et al. (2007) PCC's compare very well to a variety of other multivariate copulas. PCC's have also been utilized by Chollete et al. (2008) in a regime switching setup.

This paper considers semi-parametric dynamic models based on PCC's constructed in a similar fashion as Chen and Fan (2006a) with an application to foreign exchange rates. Estimation is based on two step estimation procedure, where first the marginal dynamic models are estimated and corresponding standardized residuals are computed. These are transformed non parametrically (see for example Andreou and Ghysels (2003), Chen and Fan (2006a) or Chen and Fan (2006b)) to the n-dimensional unit cube. Alternatively one can use a parametric transformation (see Engle and Sheppard (2001), Patton (2006) or Rockinger and Jondeau (2006)). In the second step the dependence between the transformed foreign exchange rates are modeled using a PCC copula. Here exploratory data analysis (EDA) helped to choose two different PCC's. Additionally the EDA also gave rise to a stochastic model on a directed acyclic graph where conditional distributions were modeled using PCC's. As pair-copulas in the PCC we chose bivariate t-copulas. Finally we compare the three model specifications.

2 Pair-copula constructions for high dimensional copulas

The starting point for constructing multivariate distribution is the well known recursive decomposition of a multivariate density into products of conditional densities. For this let $(X_1, ..., X_d)$ be a set of variables with joint distribution F and density f, respectively. Consider the decomposition

$$f(x_1, ..., x_d) = f(x_d|x_1, \cdots, x_{d-1}) f(x_1, \cdots, x_{d-1}) = \cdots = \prod_{t=2}^{d} f(x_t|x_1, \cdots, x_{t-1}) \times f(x_1). \quad (2.1)$$

Here $F(\cdot|\cdot)$ and $f(\cdot|\cdot)$ denote conditional cdf's and densities, respectively. Using Sklar's theorem for conditional bivariate densities we can reexpress $f(x_t|x_1,\cdots,x_{t-1})$ as

$$\begin{aligned} f(x_t|x_1,\cdots,x_{t-1}) &= \frac{f(x_{t-1},x_t|x_1,\cdots,x_{t-2})}{f(x_{t-1}|x_1,\cdots,x_{t-2})} \\ &= c_{t-1,t|1,\cdots,t-2} \times f(x_t|x_1,\cdots,x_{t-2}), \end{aligned} \tag{2.2}$$

where we use for arbitrary distinct indices $i,j,i_1,\cdots,i_k$ with $i<j$ and $i_1<\cdots<i_k$ the following abbreviation for a bivariate conditional copula density evaluated at conditional cdf's:

$$c_{i,j|i_1,\cdots,i_k} := c_{i,j|i_1,\cdots,i_k}(F(x_i|x_{i_1},\cdots,x_{i_k}),(F(x_j|x_{i_1},\cdots,x_{i_k})).$$

Using (2.2) we can write the following recursion for (2.1)

$$\begin{aligned} f(x_1,...,x_d) &= f(x_1)\times\prod_{t=2}^{d}\prod_{k=1}^{t-1} c_{t-k,t|1,\cdots,t-k-1}\times f(x_t) \\ &= \prod_{r=1}^{d} f(x_r)\times\prod_{t=2}^{d}\prod_{k=1}^{t-1} c_{t-k,t|1,\cdots,t-k-1} \\ &= \prod_{r=1}^{d} f(x_r)\times\prod_{j=1}^{d-1}\prod_{i=1}^{d-j} c_{j,j+i|1,\cdots,j-1}\ (j=t-k, j+i=t). \end{aligned} \tag{2.3}$$

Note that for $i=1,\cdots d-1$ we need to define $c_{1,i+1|10}$, which contains an impossible conditioning set. In this case we set $c_{1,i+1|10} := c_{1,i+1}$.

Bedford and Cooke (2001) and Bedford and Cooke (2002) noticed that they can represent this pair-copula decomposition (2.3) in a sequence of nested trees with undirected edges, which they call a vine. Edges in the trees denote the indices used for the conditional copula densities. Following Kurowicka and Cooke (2006) we recall for the convenience of the reader the definition of a regular vine. A regular vine on d variables consists of connected trees $T_1,\cdots T_{n-1}$ with nodes N_i and edges E_i for $i=1,\cdots,d-1$, which satisfy the following

1. T_1 has nodes $N_1=\{1,\cdot,d\}$ and edges E_1.
2. For $i=2,\cdots,d-1$ the tree T_i has nodes $N_i=E_{i-1}$.
3. Two edges in tree T_i are joined in tree T_{i+1} if they share a common node in tree T_i

The edges in tree T_i will be denoted by $jk|D$ where $j<k$ and D is the conditioning set. Note that in contrast to Kurowicka and Cooke (2006) we order the conditioned set $\{j,k\}$ to make the order of the arguments in the bivariate copulas unique. If D is the empty set, we denote the edge by jk. The notation of an edge e in T_i will depend on the two edges in T_{i-1}, which have a common node in T_{i-1}. Denote these edges by $a=j(a),k(a)|D(a)$ and $b=j(b),k(b)|D(b)$ with $V(a):=\{j(a),k(a),D(a)\}$ and $V(b):=\{j(b),k(b),D(b)\}$, respectively. The nodes a and b in tree T_i are therefore joined by edge $e=j(e),k(e)|D(e)$, where

$$\begin{aligned} j(e) &:= \min\{i : i\in V(a)\cup V(b)\setminus D(e)\} \text{ and } k(e):=\max\{i : i\in V(a)\cup V(b)\setminus D(e)\} \\ D(e) &:= V(a)\cap V(b). \end{aligned}$$

In Kurowicka and Cooke (2006) it is proven that these quantities are uniquely defined for regular vine trees.

The vine representation of the PCC (2.3) for $d = 5$ is given in Figure 1. These PCC's are called canonical vines and can be characterized by requiring that each tree T_j has a unique node of degree $n - j$. The node in T_1 with maximal degree is called the root. In Figure 1 node 1 in T_1 is the root. Note that tree T_1 uniquely defines all subsequent trees in canonical vines. These multivariate distributions are especially useful, if one suspects that there exist a variable which influences all the others variables. In modeling foreign exchange rates the US dollar exchange rates might represent such a variable.

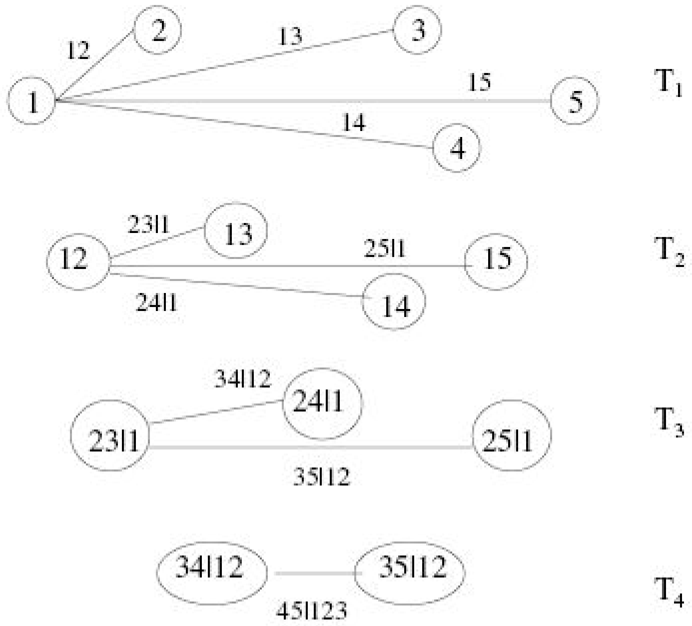

Figure 1: Five dimensional canonical vine representation using four nested trees

Kurowicka and Cooke (2006) showed in Theorem 4.2 that the joint density corresponding to a regular vine can be expressed as

$$f(x_1, ..., x_d) = \prod_{r=1}^{d} f(x_r) \times \prod_{i=1}^{d-1} \prod_{e \in E_i} c_{j(e),k(e)|D(e)}(F(x_{j(e)}|\boldsymbol{x}_{D(e)}), F(x_{k(e)}|\boldsymbol{x}_{D(e)})), \tag{2.4}$$

where $\boldsymbol{x}_D$ denotes the subvector of $\boldsymbol{x}$ indicated by the indices contained in D.

Using (2.4) it is straight forward to construct the likelihood for an i.i.d sample from a regular vine once conditional cdf's can be evaluated. Joe (1996) showed that for $v \in D$ and $D_{-v} := D \backslash v$

$$F(x_j|\boldsymbol{x}_D) = \frac{\partial C_{x_j,x_v|D_{-v}}(F(x_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}}))}{\partial F(x_v|\boldsymbol{x}_{D_{-v}})}. \tag{2.5}$$

For the special case where $D = \{v\}$ it follows that

$$F(x_j|x_v) = \frac{\partial C_{x_j,x_v}(F(x_j), F(x_v))}{\partial F(x_v)}.$$

In the case of uniform margins (i.e $F(x) = x$) this simplifies further for the parameterized copula cdf $C_{jv}(x_j, x_v) = C_{jv|\boldsymbol{\theta}_{jv}}(x_j, x_v|\boldsymbol{\theta}_{jv})$ to

$$h(x_j|x_v, \boldsymbol{\theta}_{jv}) := \frac{\partial C_{j,v|\boldsymbol{\theta}_{jv}}(x_j, x_v|\boldsymbol{\theta}_{jv})}{\partial x_v}. \tag{2.6}$$

We can use (2.6) to express conditional cdf's where D contains more than one element. In particular we have for $v \in D$

$$
\begin{aligned}
F(x_j|\boldsymbol{x}_D) &= \int_{-\infty}^{x_j} c_{jv|D_{-v}}(F(u_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}}))\, f(u_j|\boldsymbol{x}_{D_{-v}})\, \mathrm{d}u_j \\
&= \int_{-\infty}^{x_j} \frac{\partial^2 C_{jv|D_{-v}}(F(u_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}}))}{\partial F(u_j|\boldsymbol{x}_{D_{-v}})\, \partial F(x_v|\boldsymbol{x}_{D_{-v}})} \frac{\partial F(u_j|\boldsymbol{x}_{D_{-v}})}{\partial u_j}\, \mathrm{d}u_j \\
&= \frac{1}{\partial F(x_v|\boldsymbol{x}_{D_{-v}})} \int_{-\infty}^{x_j} \underbrace{\frac{\partial^2 C_{jv|D_{-v}}(F(u_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}}))}{\partial F(u_j|\boldsymbol{x}_{D_{-v}})} \frac{\partial F(u_j|\boldsymbol{x}_{D_{-v}})}{\partial u_j}}_{\frac{\partial}{\partial u_j} C_{jv|D_{-v}}(F(u_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}}))} \mathrm{d}u_j \\
&= \frac{\partial}{\partial F(x_v|\boldsymbol{x}_{D_{-v}})} C_{jv|D_{-v}}(F(x_j|\boldsymbol{x}_{D_{-v}}), F(x_v|\boldsymbol{x}_{D_{-v}})) \\
&= \frac{\partial}{\partial \eta} C_{jv|D_{-v}}(F(x_j|\boldsymbol{x}_{D_{-v}}), \eta)|_{\eta = F(x_v|\boldsymbol{x}_{D_{-v}})} \\
&= h(F(x_j|\boldsymbol{x}_{D_{-v}})|F(x_v|\boldsymbol{x}_{D_{-v}}), \boldsymbol{\theta}_{jv|D_{-v}}).
\end{aligned}
$$

This shows that the conditional cdf's with conditioning set D can be build up recursively using the h-function from conditional cdf's with lower dimensional conditioning set. For canonical vines these recursions can be explicitly derived and Aas et al. (2007) give exact algorithmic expression for calculating the likelihood in this case.

Since we would like to allow for tail dependence we will use the bivariate t-copula as a building block for the PCC's, therefore we need the corresponding h function. For the bivariate t-copula with parameters ρ and ν the h-function is given by

$$
h(x_j|x_v, \rho, \nu) = t_{\nu+1} \left\{ \frac{t_\nu^{-1}(x_j) - \rho\, t_\nu^{-1}(x_v)}{\sqrt{\frac{\left(\nu + (t_\nu^{-1}(x_v))^2\right)(1-\rho^2)}{\nu+1}}} \right\} \tag{2.7}
$$

and its inverse by

$$
h^{-1}(u_j|x_v, \rho, \nu) = t_\nu \left\{ t_{\nu+1}^{-1}(u_j) \sqrt{\frac{\left(\nu + (t_\nu^{-1}(x_v))^2\right)(1-\rho^2)}{\nu+1}} + \rho\, t_\nu^{-1}(x_v) \right\}, \tag{2.8}
$$

where $t_\nu^{-1}(\cdot)$ is the quantile function of the univariate standard t distribution with ν degrees of freedom zero mean and variance $\frac{\nu}{\nu-2}$ for $\nu > 2$. Note that ρ is the correlation between $t_\nu^{-1}(U)$ and $t_\nu^{-1}(V)$ and not between U and V. Here the random vector $(U, V)'$ is assumed to have a bivariate t-copula distribution with association parameter ρ and and degree of freedom parameter ν.

If one uses bivariate Gaussian copulas as building blocks for the PCC model, then we arrive at a multivariate Gauss copula. This follows from the facts that partial and conditional correlations are equal for elliptical distributions (see Baba and Sibuya (2005)) and that conditional distributions of normals are normal with a covariance independent of the conditioning value.

Bedford and Cooke (2002) provided a one-to-one relationship between unconditional and partial correlations for Gaussian distributions.

For multivariate t-distributions we also have that conditional distributions are again t distributed (see for example Chapter 5 of DeGroot (1970)), however the conditional covariance matrix depends on the conditioning value. This dependency however quickly disappears as the degree of freedom parameter increases and only involves a scaling factor. Further the conditional correlations of a multivariate t-distribution are independent of the conditioning value and are equal to the conditional correlations of a multivariate Gauss distribution. However since copula's are scaling invariant, it follows that a copula based on the PCC approach using bivariate conditional t-copulas $c_{j(e),k(e)|D(e)}$ with parameters $(\rho_{j(e),k(e)|D(e)}, \nu_{j(e),k(e)|D(e)})$ and satisfying $\nu_{j(e),k(e)|D(e)} = \nu + |D(e)| \forall e$ is the multivariate t-copula with common degree of freedom ν and association matrix R. Here $|\cdot|$ denotes the cardinality of a set and R is determined by the one-to-one-relationship between unconditional and partial correlations for the corresponding Gaussian distribution, i.e. $\nu_{j(e),k(e)|D(e)} = \infty$. Therefore the Gauss and multivariate t-copula are both nested with the class of copulas constructed using the PCC approach.

3 Semiparametric PCC copula-based multivariate dynamic models with ARMA-GARCH margins

In this paper we follow a two step estimation approach as for example taken by Chen and Fan (2006a). First we assume for each margin a dynamic model, which we estimate separately. Then we form standardized residuals and use the empirical distribution function to transform the standardized innovations to approximately i.i.d uniform variables for each margin. This allows for uncertainty in the distribution of the standardized residuals. Across margins these unit interval variables are dependent and we model their dependence structure using the PCC based vine copulas discussed in Section 2. In a second step the copula parameters are estimated. To be more precise consider a multivariate time series $\boldsymbol{X}_t = (X_{1t}, \cdots, X_{dt})'$ for $t = 1, \cdots, n$. Each marginal time series $\{X_{k1}, \cdots, X_{kn}\}$ follows a ARMA(P,Q)-GARCH(p,q) model, i.e. for each $k = 1, \cdots, d$ and $t = 1, \cdots, n$

$$\begin{aligned} X_{kt} - \mu_k &= \sum_{i=1}^{P} \psi_{ki}(X_{k,t-i} - \mu_k) + \epsilon_{kt} - \sum_{j=1}^{Q} \theta_{kj}\epsilon_{k,t-j} \\ \epsilon_{kt} &= \sigma_{kt}^2 \eta_{kt} \\ \sigma_{kt}^2 &= w_k + \sum_{i=1}^{q} \alpha_{ki}\epsilon_{k,t-i}^2 + \sum_{j=1}^{p} \beta_{kj}\sigma_{k,t-j}^2, \end{aligned} \tag{3.1}$$

where $\{\eta_{kt}, k = 1, \cdots, d; t = 1, \cdots, n\}$ are i.i.d. with $E(\eta_{kt}) = 0$ and $Var(\eta_{kt}) = 1$. Further η_{kt} are independent of $\{X_{ks} \forall s \leq t\}$. Copula-GARCH models were also considered by Rockinger and Jondeau (2006). Using quasi MLE we obtain parameter estimates for the marginal ARMA(P,Q)-GARCH(p,q) models and let $\{Z_{kt}, k = 1, \cdots, d; t = 1, \cdots, n\}$ the corresponding standardized residuals. Transform these standardized residuals nonparametrically to

$$U_{kt} = \hat{F}_n^{(k)}(Z_{kt}) \quad \forall k, t \text{ where } \hat{F}_n^{(k)}(x) := \frac{1}{n+1} \sum_{t=1}^{n} \chi_{(-\infty,x]}(Z_{tk}), \quad x \in \mathbb{R} \tag{3.2}$$

and $\chi_{(a,b]}(y)$ denotes the indicator function of y to the interval $(a, b]$. For the dependence model we assume now that the U_{kt} variables is an i.i.d sample of size n from a vine PCC copula on d

variables with joint density specified in (2.4). Note that one can use arbitrary bivariate copulas as building blocks for the construction. In the application we use bivariate t copulas to allow for different tail behavior of pairs of variables. Each bivariate copula term in (2.4) has its own parameters.

Parameter estimation for the copula parameters is facilitated using ML. For higher dimensional problems the number of parameters to be estimated can be substantial. Therefore we determine as in Aas et al. (2007) sensible starting values for the optimization required to determine the MLE's. In a first step consider all pairs of variables, which are identified in the first tree of the vine. Estimate the parameters corresponding to these pairs using any method you prefer. For example a bivariate t-copula pair the correlation parameter ρ is estimated using Kendall's τ and in second step the df parameter ν is maximized using the estimated ρ. For the copula parameters identified in the second tree, one first has to transform the data with the h function needed for the appropriate conditional cdf using estimated parameters to determine realizations needed in the second tree. For example we want to estimate the parameters of copula $c_{13|2}$. For this transform $\{U_{1,t}, U_{2,t}, U_{3,t}, t = 1, \cdots, n\}$ to $U_{1|2,t} := h(U_{1,t}|U_{2,t}, \hat{\boldsymbol{\theta}}_{12})$ and $U_{3|2,t} := h(U_{3,t}|U_{2,t}, \hat{\boldsymbol{\theta}}_{23})$, where $\hat{\boldsymbol{\theta}}_{12}$ and $\hat{\boldsymbol{\theta}}_{23}$ are the estimated parameters in the first tree. Now estimate $\boldsymbol{\theta}_{13|2}$ based on $\{U_{1|2,t}, U_{3|2,t}; t = 1, \cdots, n\}$. Continue sequentially with this procedure until all copula parameters of all trees are estimated. Note for trees T_i with $i \geq 2$ recursive applications of h functions are needed to transform to the appropriate conditional cdf.

Under the usual regularity conditions the MLE of the copula parameters based on (2.4) using the transformed data defined in (3.2) and assuming uniform margins are asymptotically unbiased and normally distributed with variance-covariance matrix given by the inverse of the Fisher information. However how to determine the Fisher information for PCC models efficiently is an open research question. In a first approach we use a numerical evaluation of the Hessian matrix to approximate the Fisher information for the foreign exchange data.

4 Statistical models for directed acyclic graphs

Stochastic models on directed acyclic graphs (DAG) have been used to describe dependencies between variables. For an introduction see the book by Edwards (2000) and for more advanced material see Cox and Wermuth (1996) and Lauritzen (1996). To fix ideas let $(X_1, ..., X_n)$ be a set of variables with joint distribution F and density f. As for PCC's we decompose the joint density into a product of conditional densities given by

$$f(x_1, ..., x_n) = f(x_n|x_1, \cdots, x_{n-1}) f(x_1, \cdots, x_{n-1}) = \cdots = \prod_{i=1}^{n} f(x_i|x_1, \cdots, x_{i-1}), \tag{4.1}$$

Now we assume that for some i the conditional density $f(x_i|x_1, \cdots, x_{i-1})$ does not depend on all predecessors $x_1, \cdots x_{i-1}$ but only on some. We denote the index subset of predecessors which influence the conditional density by $pa(i)$. Therefore we can rewrite (4.1) as

$$f(x_1, ..., x_n) = \prod_{i=1}^{n} f(x_i|\boldsymbol{x}_{pa(i)}), \tag{4.2}$$

where $f(x_i|\boldsymbol{x}_{pa(i)}) = f(x_i)$ for $pa(i) = \emptyset$. This recursive dependence structure can be represented as a DAG by drawing an directed arrow from each vertex in $pa(i)$ to i. It is common for continuous variables to assume a joint Gaussian distribution for F. In the foreign exchange example we assume t-distributions for certain conditional densities. This allows more generally to build up DAG's with non-Gaussian joint distribution.

5 Application: Foreign exchange rates

We consider seven time series of daily log-returns of foreign exchange rates of the British pound (GBP), US dollar (USD), Malaysian ringgit (MYR), Swiss franc (CHF), Japanese yen (JPY), Danisch crona (DKK) and Swedish krona (SEK) between May 13, 1985 and June 22, 2004 against the Euro. Before the introduction of the Euro, the exchange rate to the German mark was used and translated into Euro's with a rate of 1.95583 Euro for 1 German mark.

5.1 Marginal models for each foreign exchange rate

For each exchange rate series we first determined the appropriate ARMA(P,Q)-GARCH(p,q) model specified in (3.1). We used Ljung-Box tests to test for the independence of the estimated standardized residuals. This shows that a $ARMA(1,1) - GARCH(1,1)$ is sufficient to remove the time dependence in each of the marginal foreign exchange time series. However normal-QQ plots (see Figure 2) of the standardized residuals show that these are fat tailed, therefore normal margins are not appropriate. This indicates that the empirical transform (3.2) is more appropriate to achieve approximate uniform margins than a parametric normal transformation. The marginal heavy tailness might also be an indicator that a Gaussian copula on the transformed data is not appropriate. This will be verified later.

5.2 Dependence models for transformed foreign exchange rates

In the second step we remove the effect of the marginals by transforming the standardized residuals for each exchange rate series to approximately i.i.d uniform random variables by applying the probability integral transform based on the empirical distribution function (see transformation (3.2)). Figure 3 gives scatter plots of all pairs of transformed exchange change rates. For example we see that the the Malaysian ringgit - Euro exchange rate is highly dependent on the US dollar - Euro exchange rate.

To further explore the dependency structure we assume a bivariate t-copula for each pair of exchange rates. In particular we estimate Kendall's tau τ for each pair using the transformed data and estimate ρ by the relationship $\rho = \sin(\frac{\pi\tau}{2})$, which is valid for elliptical distributions (see Lindskog, McNeil, and Schmock (2003)). Finally we estimate the corresponding df parameter by maximizing the bivariate t-copula density with ρ fixed to its estimated value.

The results are given in Table 1. We see that the df parameters vary considerably between pairs indicating that a multivariate t-copula with a common df parameter is not sufficient. Therefore we consider different vine specifications. The first one is based on using the six pairs with smallest df values (bolded in Table 1) as edges in the first tree of the vine. The corresponding regular vine using its tree representation is given in Figure 4. Here we use the following abbreviations W_1 for GBP, W_2 for USD, W_3 for MYR, W_4 for CHF, W_5 for JPY, W_6 for DKK and W_7 for SEK. We denote this model by R-vine.

As second alternative we consider a canonical vine where the USD-Euro exchange rate is the root knot in the top tree. Recall that the identification of the canonical vine is completely determined by the first tree, which is given in Figure 5. Here the order is determined from the

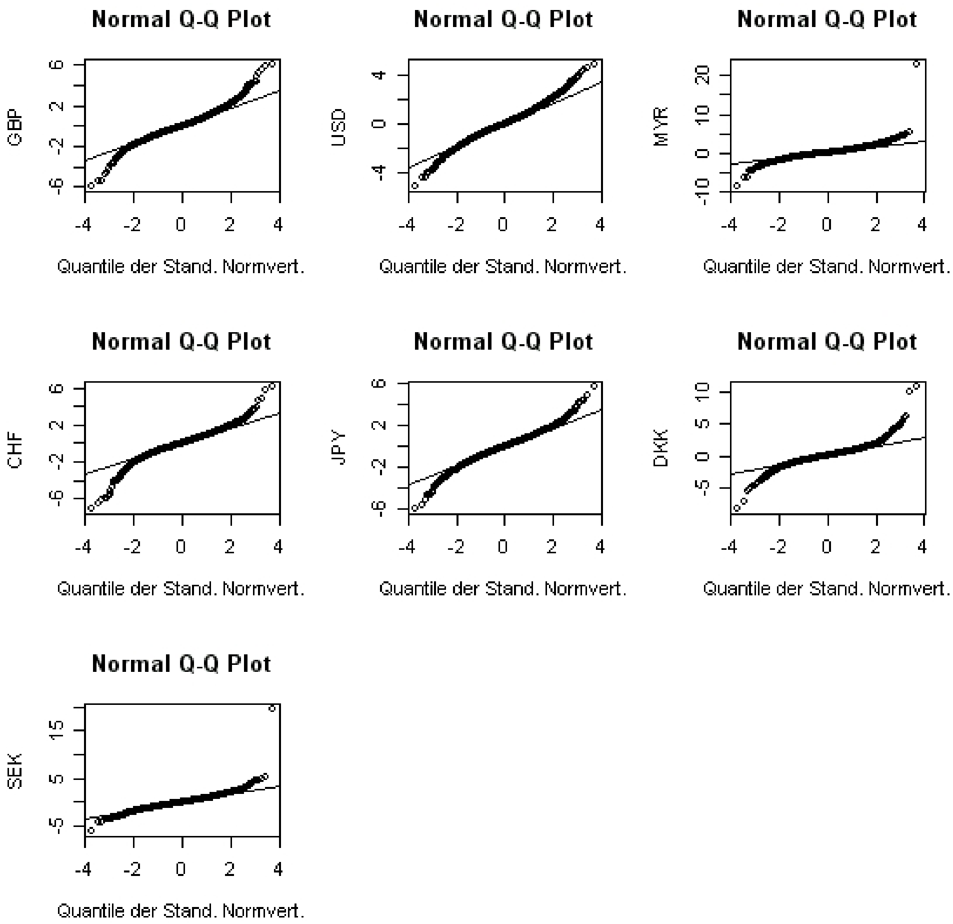

Figure 2: Normal-QQ-plots of standardized residuals for each exchange rate

smallest to the largest estimated pairwise df with the USD Euro exchange rate. For both models we use the procedure given before to determine starting values for maximizing the likelihood numerically. The starting values and the final MLE's with estimated standard errors for both models are given in Tables 2 and 3. They show that the starting values are quite close to the final ML values and that one is also quite close to the maximum log likelihood.

As already mentioned we estimate standard errors by using a numerical approximation of the Hessian matrix. They show that for the R-vine the MYR and SEK Euro exchange rates are independent given the USD and CHF Euro exchange rates, since $\rho_{\mathbf{W}_3,\mathbf{W}_7|\mathbf{W}_2,\mathbf{W}_4}$ is nonsignificant at the 1 % level and $\hat{\nu}_{\mathbf{W}_3,\mathbf{W}_7|\mathbf{W}_2,\mathbf{W}_4} > 30$ indicating a Gaussian copula for this conditional pair copula. Similar the MYR and DEK Euro exchange rates are independent given the USD, CHF and SEK Euro exchange rate. In the C-vine we can conclude that the USD and SEK Euro exchange rate are independent given the CHF Euro exchange rate. In both vine models we see that $\rho_{\mathbf{W}_2,\mathbf{W}_4}$ is nonsignificant but $\hat{\nu}_{\mathbf{W}_2,\mathbf{W}_4} < 6$ indicating that the USD and CHF Euro exchanges are uncorrelated but tail dependent.

Finally we consider a third model specification using DAG's. For this specification we consider again Table 1. We use the three highest correlations with the USD-Euro exchange rate and rank the remaining European countries according to their economic power. Figure 6 gives the corresponding DAG.

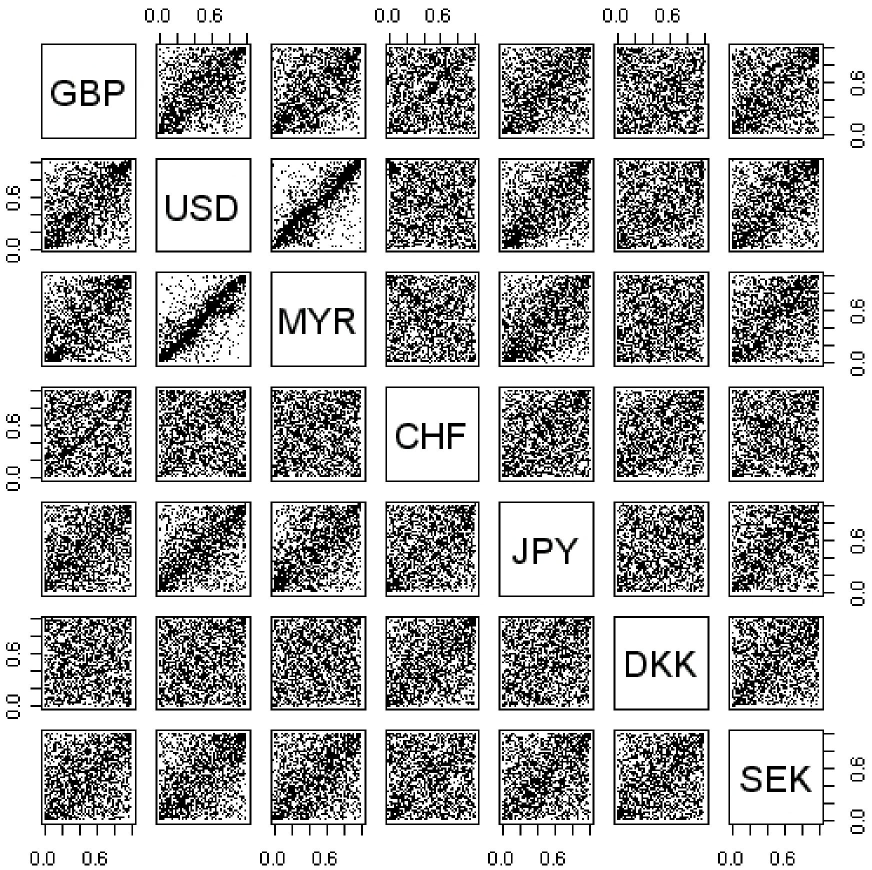

Figure 3: Scatter plots of the unit interval transformed standardized residuals for each exchange rate

	USD	**MYR**	**CHF**	**JPY**	**DKK**	**SEK**
GBP	4.80	6.18	**4.79**	5.87	5.81	6.16
USD		**2.00**	**4.40**	**3.59**	8.27	6.19
MYR			5.33	5.19	11.46	8.95
CHF				7.24	6.12	**5.05**
JPY					10.70	5.27
DKK						**5.19**
	USD	**MYR**	**CHF**	**JPY**	**DKK**	**SEK**
GBP	.48	.41	**.08**	.28	.01	.28
USD		**.81**	**-.01**	**.46**	.08	.42
MYR			.00	.45	.06	.33
CHF				.17	.25	**.09**
JPY					.14	.25
DKK						**.26**

Table 1: Estimated df (top) and estimated correlations (bottom) in a bivariate t-copula model for pairs of the transformed exchange rates

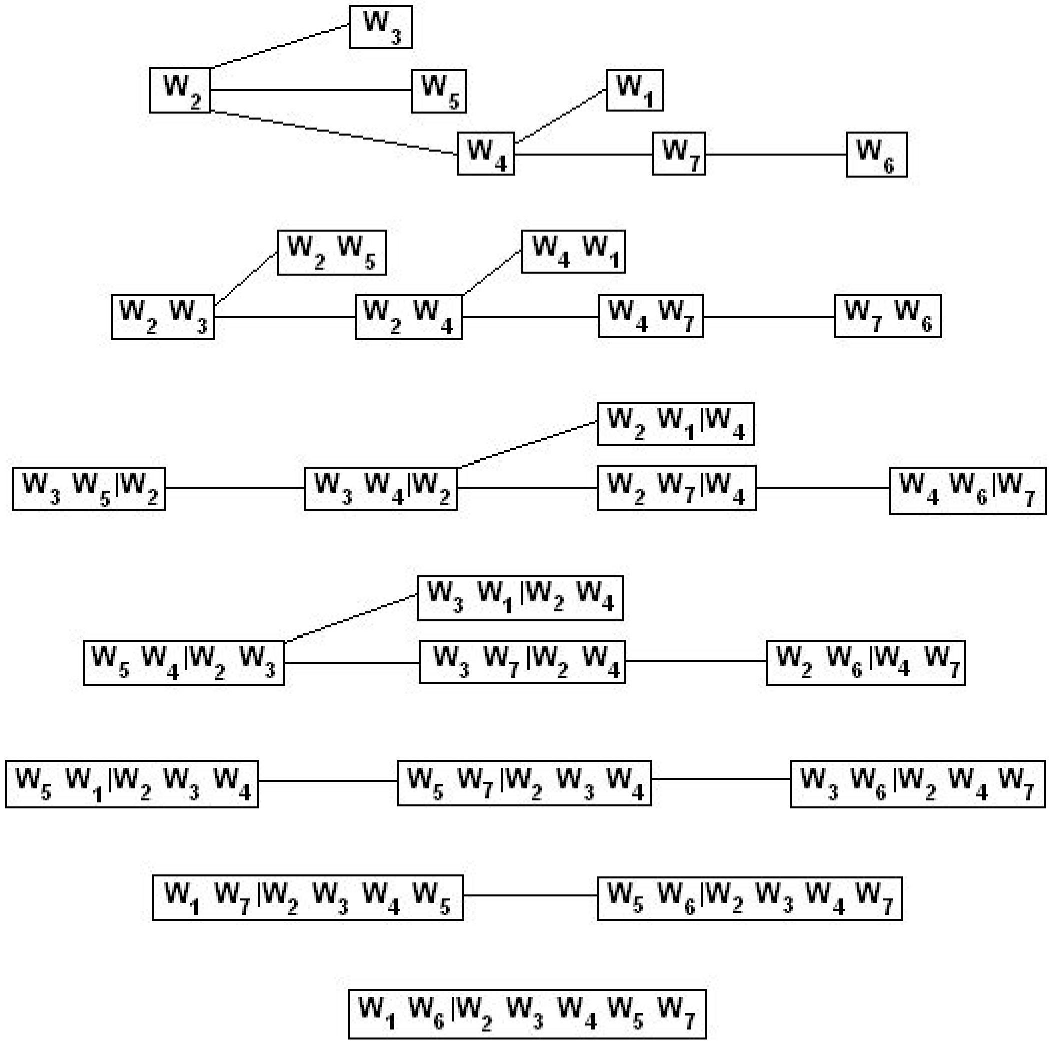

Figure 4: Tree structure for R-vine model for the exchange rate data

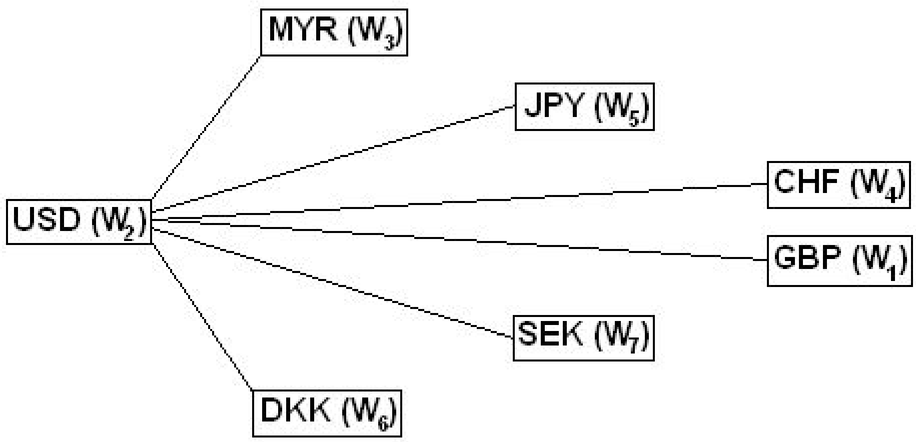

Figure 5: First tree of the canonical vine for the Euro exchange data

Parameter	start	final	std. error
$\rho_{\mathbf{W}_2,\mathbf{W}_3}$	0.85	0.86	0.005
$\rho_{\mathbf{W}_2,\mathbf{W}_5}$	0.49	0.50	0.012
$\rho_{\mathbf{W}_2,\mathbf{W}_4}$	-0.01	-0.01	0.016
$\rho_{\mathbf{W}_4,\mathbf{W}_1}$	0.09	0.09	0.016
$\rho_{\mathbf{W}_4,\mathbf{W}_7}$	0.09	0.11	0.015
$\rho_{\mathbf{W}_7,\mathbf{W}_6}$	0.28	0.28	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_5\|\mathbf{W}_2}$	0.15	0.14	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_4\|\mathbf{W}_2}$	0.03	0.03	0.014
$\rho_{\mathbf{W}_2,\mathbf{W}_1\|\mathbf{W}_4}$	0.50	0.50	0.011
$\rho_{\mathbf{W}_2,\mathbf{W}_7\|\mathbf{W}_4}$	0.45	0.44	0.012
$\rho_{\mathbf{W}_4,\mathbf{W}_6\|\mathbf{W}_7}$	0.24	0.24	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_4\|\mathbf{W}_2,\mathbf{W}_3}$	0.21	0.21	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_4}$	0.06	0.05	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_4}$	-0.05	-0.03	0.014
$\rho_{\mathbf{W}_2,\mathbf{W}_6\|\mathbf{W}_4,\mathbf{W}_7}$	-0.02	-0.03	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4}$	0.05	0.05	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4}$	0.07	0.06	0.015
$\rho_{\mathbf{W}_3,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_4,\mathbf{W}_7}$	-0.02	-0.01	0.014
$\rho_{\mathbf{W}_1,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	0.11	0.10	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_7}$	0.07	0.05	0.014
$\rho_{\mathbf{W}_1,\mathbf{W}_6\|\mathbf{W}_{V\setminus\{1,6\}}}$	-0.09	-0.10	0.015
$\nu_{\mathbf{W}_2,\mathbf{W}_3}$	2.00	2.00	0.103
$\nu_{\mathbf{W}_2,\mathbf{W}_5}$	3.59	3.64	0.237
$\nu_{\mathbf{W}_2,\mathbf{W}_4}$	4.40	5.79	0.519
$\nu_{\mathbf{W}_4,\mathbf{W}_1}$	4.79	5.81	0.566
$\nu_{\mathbf{W}_4,\mathbf{W}_7}$	5.05	6.56	0.709
$\nu_{\mathbf{W}_7,\mathbf{W}_6}$	5.19	5.63	0.566
$\nu_{\mathbf{W}_3,\mathbf{W}_5\|\mathbf{W}_2}$	19.92	20.31	3.825
$\nu_{\mathbf{W}_3,\mathbf{W}_4\|\mathbf{W}_2}$	24.28	24.39	6.398
$\nu_{\mathbf{W}_2,\mathbf{W}_1\|\mathbf{W}_4}$	5.46	4.94	0.419
$\nu_{\mathbf{W}_2,\mathbf{W}_7\|\mathbf{W}_4}$	8.13	7.24	0.905
$\nu_{\mathbf{W}_4,\mathbf{W}_6\|\mathbf{W}_7}$	8.71	8.34	1.296
$\nu_{\mathbf{W}_5,\mathbf{W}_4\|\mathbf{W}_2,\mathbf{W}_3}$	19.62	19.71	5.173
$\nu_{\mathbf{W}_3,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_4}$	32.23	32.30	9.812
$\nu_{\mathbf{W}_3,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_4}$	300.00	300.00	438.419
$\nu_{\mathbf{W}_2,\mathbf{W}_6\|\mathbf{W}_4,\mathbf{W}_7}$	18.68	18.66	5.804
$\nu_{\mathbf{W}_5,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4}$	20.14	20.23	5.088
$\nu_{\mathbf{W}_5,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4}$	14.01	14.24	2.720
$\nu_{\mathbf{W}_3,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_4,\mathbf{W}_7}$	300.00	300.00	270.102
$\nu_{\mathbf{W}_1,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	15.77	15.89	3.455
$\nu_{\mathbf{W}_5,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_7}$	300.00	300.00	617.258
$\nu_{\mathbf{W}_1,\mathbf{W}_6\|\mathbf{W}_{V\setminus\{1,6\}}}$	13.25	13.39	2.703
Log-Likelihood	6471.22	6487.56	

Table 2: Estimated start and final ML parameters with estimated standard errors of R-vine

parameter	start	final	std. error
$\rho_{\mathbf{W}_2,\mathbf{W}_3}$	0.85	0.86	0.005
$\rho_{\mathbf{W}_2,\mathbf{W}_5}$	0.49	0.50	0.012
$\rho_{\mathbf{W}_2,\mathbf{W}_4}$	-0.01	-0.01	0.016
$\rho_{\mathbf{W}_2,\mathbf{W}_1}$	0.51	0.50	0.011
$\rho_{\mathbf{W}_2,\mathbf{W}_7}$	0.45	0.44	0.012
$\rho_{\mathbf{W}_2,\mathbf{W}_6}$	0.09	0.09	0.015
$\rho_{\mathbf{W}_3,\mathbf{W}_5\|\mathbf{W}_2}$	0.15	0.14	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_4\|\mathbf{W}_2}$	0.03	0.03	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_1\|\mathbf{W}_2}$	0.06	0.05	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_7\|\mathbf{W}_2}$	-0.04	-0.02	0.014
$\rho_{\mathbf{W}_3,\mathbf{W}_6\|\mathbf{W}_2}$	-0.02	-0.01	0.014
$\rho_{\mathbf{W}_5,\mathbf{W}_4\|\mathbf{W}_2,\mathbf{W}_3}$	0.21	0.20	0.014
$\rho_{\mathbf{W}_5,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3}$	0.08	0.07	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3}$	0.09	0.08	0.015
$\rho_{\mathbf{W}_5,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3}$	0.13	0.12	0.014
$\rho_{\mathbf{W}_4,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	0.10	0.10	0.015
$\rho_{\mathbf{W}_4,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	0.11	0.11	0.015
$\rho_{\mathbf{W}_4,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	0.25	0.25	0.014
$\rho_{\mathbf{W}_1,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	0.11	0.10	0.015
$\rho_{\mathbf{W}_1,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	-0.06	-0.07	0.015
$\rho_{\mathbf{W}_6,\mathbf{W}_7\|\mathbf{W}_{V\setminus\{6,7\}}}$	0.27	0.26	0.017
$\nu_{\mathbf{W}_2,\mathbf{W}_3}$	2.00	2.00	0.103
$\nu_{\mathbf{W}_2,\mathbf{W}_5}$	3.59	3.61	0.231
$\nu_{\mathbf{W}_2,\mathbf{W}_4}$	4.40	4.51	0.342
$\nu_{\mathbf{W}_2,\mathbf{W}_1}$	4.80	5.59	0.710
$\nu_{\mathbf{W}_2,\mathbf{W}_7}$	6.19	5.41	0.463
$\nu_{\mathbf{W}_2,\mathbf{W}_6}$	8.27	8.55	1.234
$\nu_{\mathbf{W}_3,\mathbf{W}_5\|\mathbf{W}_2}$	19.92	20.74	4.020
$\nu_{\mathbf{W}_3,\mathbf{W}_4\|\mathbf{W}_2}$	24.28	24.82	5.288
$\nu_{\mathbf{W}_3,\mathbf{W}_1\|\mathbf{W}_2}$	25.49	25.80	6.668
$\nu_{\mathbf{W}_3,\mathbf{W}_7\|\mathbf{W}_2}$	77.26	77.31	38.992
$\nu_{\mathbf{W}_3,\mathbf{W}_6\|\mathbf{W}_2}$	55.42	55.54	23.671
$\nu_{\mathbf{W}_5,\mathbf{W}_4\|\mathbf{W}_2,\mathbf{W}_3}$	19.62	19.66	5.512
$\nu_{\mathbf{W}_5,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3}$	14.28	14.88	2.870
$\nu_{\mathbf{W}_5,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3}$	12.20	12.67	2.330
$\nu_{\mathbf{W}_5,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3}$	38.91	38.92	19.611
$\nu_{\mathbf{W}_4,\mathbf{W}_1\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	9.77	9.70	1.553
$\nu_{\mathbf{W}_4,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	9.83	10.19	1.709
$\nu_{\mathbf{W}_4,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_5}$	9.67	8.53	1.381
$\nu_{\mathbf{W}_1,\mathbf{W}_7\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	16.57	16.84	3.874
$\nu_{\mathbf{W}_1,\mathbf{W}_6\|\mathbf{W}_2,\mathbf{W}_3,\mathbf{W}_4,\mathbf{W}_5}$	11.72	12.19	2.428
$\nu_{\mathbf{W}_6,\mathbf{W}_7\|\mathbf{W}_{V\setminus\{6,7\}}}$	8.84	9.42	2.123
Log-Likelihood	6465.81	6475.54	

Table 3: Estimated start and final ML parameters with estimated standard errors of C-vine

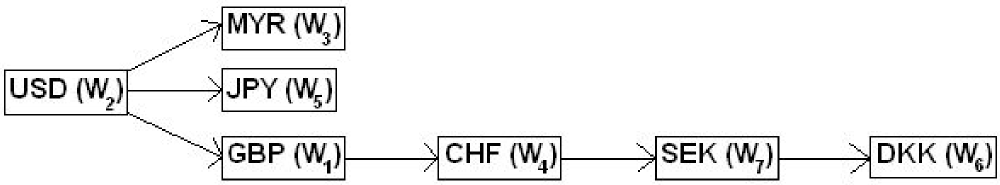

Figure 6: DAG model investigated for the exchange rate data

According to equation (4.2) the corresponding joint density for $(\mathbf{W}_1, ..., \mathbf{W}_7)$ is given by

$$\begin{aligned} f(\mathbf{w}_1, ..., \mathbf{w}_7) &= f(\mathbf{w}_2) \cdot f(\mathbf{w}_3|\mathbf{w}_2) \cdot f(\mathbf{w}_5|\mathbf{w}_2) \cdot f(\mathbf{w}_1|\mathbf{w}_2) \\ &\cdot f(\mathbf{w}_4|\mathbf{w}_1) \cdot f(\mathbf{w}_7|\mathbf{w}_4) \cdot f(\mathbf{w}_6|\mathbf{w}_7). \end{aligned}$$

Using Sklar's theorem we can rewrite the conditional densities using pair copula densities and obtain

$$\begin{aligned} f(\mathbf{w}_1, ..., \mathbf{w}_7) &= f(\mathbf{w}_1) \cdot f(\mathbf{w}_2) \cdot f(\mathbf{w}_3) \cdot f(\mathbf{w}_4) \cdot f(\mathbf{w}_5) \cdot f(\mathbf{w}_6) \cdot f(\mathbf{w}_7) \\ &\cdot c_{23}(F(\mathbf{w}_2), F(\mathbf{w}_3)) \cdot c_{25}(F(\mathbf{w}_2), F(\mathbf{w}_5)) \cdot c_{12}(F(\mathbf{w}_1), F(\mathbf{w}_2)) \\ &\cdot c_{14}(F(\mathbf{w}_1), F(\mathbf{w}_4)) \cdot c_{47}(F(\mathbf{w}_4), F(\mathbf{w}_7)) \cdot c_{67}(F(\mathbf{w}_7), F(\mathbf{w}_6)). \end{aligned} \quad (5.1)$$

Considering Figure 6 and the joint density (5.1) we see that we can imbed this DAG into a regular vine, where the top tree is given by Figure 6 with undirected edges. The corresponding pair copula terms in the trees below are set identically to 1, which corresponds to conditional independence. In particular the density specified in (5.1) assumes the following conditional independencies:

$\boldsymbol{W}_3 \perp \boldsymbol{W}_5|\boldsymbol{W}_2; \boldsymbol{W}_1 \perp \boldsymbol{W}_3|\boldsymbol{W}_2; \boldsymbol{W}_2 \perp \boldsymbol{W}_4|\boldsymbol{W}_1; \boldsymbol{W}_1 \perp \boldsymbol{W}_7|\boldsymbol{W}_4; \boldsymbol{W}_4 \perp \boldsymbol{W}_6|\boldsymbol{W}_7$

$\boldsymbol{W}_1 \perp \boldsymbol{W}_5|\boldsymbol{W}_2, \boldsymbol{W}_3; \boldsymbol{W}_3 \perp \boldsymbol{W}_4|\boldsymbol{W}_1, \boldsymbol{W}_2; \boldsymbol{W}_2 \perp \boldsymbol{W}_7|\boldsymbol{W}_1, \boldsymbol{W}_4; \boldsymbol{W}_1 \perp \boldsymbol{W}_6|\boldsymbol{W}_4, \boldsymbol{W}_7$

$\boldsymbol{W}_4 \perp \boldsymbol{W}_5|\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3; \boldsymbol{W}_3 \perp \boldsymbol{W}_7|\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_4; \boldsymbol{W}_2 \perp \boldsymbol{W}_6|\boldsymbol{W}_1, \boldsymbol{W}_4, \boldsymbol{W}_7$

$\boldsymbol{W}_5 \perp \boldsymbol{W}_7|\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3, \boldsymbol{W}_4; \boldsymbol{W}_3 \perp \boldsymbol{W}_6|\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_4, \boldsymbol{W}_7$

$\boldsymbol{W}_5 \perp \boldsymbol{W}_6|\boldsymbol{W}_1, \boldsymbol{W}_2, \boldsymbol{W}_3, \boldsymbol{W}_4, \boldsymbol{W}_4$

Note that not all stochastic models on directed acyclic graphs can be embedded in a single regular vine. For an example where this does not work see Chapter 5 of Kurowicka and Cooke (2006) and Hanea et al. (2006).

For parameter estimation we determine again starting values for the ML estimation of Model (5.1) and optimize the log likelihood. The results are given in Table 4. Here all parameters are highly significant and all df parameters are less than 6 indicating high tail dependence between all pairs occuring in the DAG specification (5.1). It remains to investigate whether this simple specification is sufficient enough to fit the data well. This question is studied in the next section.

Finally we investigate if the fit of the PCC copula models such as the R-vine or C-vine are superior to a standard multivariate Gaussian or t-copula with a common degree of freedom (df). The Gauss copula model gives a loglikelihood of 4908.31 while a multivariate t-copula with common df gives a value of 5903.37. Since the Gauss copula is nested within the multivariate t-copula with common df, we see using the likelihood ratio test (LRT) that the t-copula with common df is preferred over the Gauss copula. Since also the t-copula with common df is nested within the R-vine (loglikelihood = 6487.56) and C-vine (loglikelihood = 6475.54) model LRT's show that the t-copula with common df is inferior to both the R-vine and the C-vine model.

Parameter	start	final	std. error
$\rho_{\mathbf{W}_2,\mathbf{W}_3}$	0.85	0.86	0.01
$\rho_{\mathbf{W}_2,\mathbf{W}_5}$	0.49	0.49	0.01
$\rho_{\mathbf{W}_2,\mathbf{W}_1}$	0.51	0.51	0.01
$\rho_{\mathbf{W}_1,\mathbf{W}_4}$	0.09	0.09	0.02
$\rho_{\mathbf{W}_4,\mathbf{W}_7}$	0.09	0.10	0.02
$\rho_{\mathbf{W}_7,\mathbf{W}_6}$	0.28	0.28	0.02
$\nu_{\mathbf{W}_2,\mathbf{W}_3}$	2.00	2.00	0.11
$\nu_{\mathbf{W}_2,\mathbf{W}_5}$	3.59	3.54	0.23
$\nu_{\mathbf{W}_2,\mathbf{W}_1}$	4.80	4.76	0.40
$\nu_{\mathbf{W}_1,\mathbf{W}_4}$	4.79	4.77	0.41
$\nu_{\mathbf{W}_4,\mathbf{W}_7}$	5.05	5.03	0.44
$\nu_{\mathbf{W}_7,\mathbf{W}_6}$	5.19	5.17	0.48
Log-Likelihood	5428.76	5432.41	

Table 4: Estimated start and final ML estimates with estimated standard errors for the DAG model of the exchange rate data

5.3 Model selection among the dependency models

We focus on model selection among the dependency models, i.e. the two vine specifications and the DAG model. One difficulty is that the dependency models are based on the transformed exchange rate data, i.e. in the following we neglect the estimation error for the marginals. Another difficulty is that all considered dependency models are nonnested. Therefore we follow the approach by Vuong (1989). Consider two non-nested parametric models $P_{\boldsymbol{\theta}} := \{F(\cdot|\boldsymbol{\theta}), \boldsymbol{\theta} \in \Theta \subset \mathbb{R}^p\}$ and $P_\gamma := \{F(\cdot|\boldsymbol{\gamma}), \boldsymbol{\gamma} \in \Gamma \subset \mathbb{R}^q\}$ with densities $f(\boldsymbol{X}|\boldsymbol{\theta})$ and $g(\boldsymbol{X}|\boldsymbol{\gamma})$, respectively. Determine the likelihood ratio statistics based on the sample $(\boldsymbol{X}_1, \cdots, \boldsymbol{X}_n)$ with parameter estimates $\hat{\boldsymbol{\theta}}_n$ and $\hat{\gamma}_n$

$$LR_n(\hat{\boldsymbol{\theta}}_n, \hat{\gamma}_n) := \sum_{t=1}^{n} \log \frac{f(\boldsymbol{X}_t|\hat{\boldsymbol{\theta}}_n)}{g(\boldsymbol{X}_t|\hat{\gamma}_n)}$$

and variance estimate

$$\hat{\omega}_n := \frac{1}{n} \sum_{t=1}^{n} \left[\log \frac{f(\boldsymbol{X}_t|\hat{\boldsymbol{\theta}}_n)}{g(\boldsymbol{X}_t|\hat{\gamma}_n)}\right]^2 - \left[\frac{1}{n} \log \frac{f(\boldsymbol{X}_t|\hat{\boldsymbol{\theta}}_n)}{g(\boldsymbol{X}_t|\hat{\gamma}_n)}\right]^2.$$

Vuong (1989) considered the following hypotheses

$$H_0 : E(\log \frac{f(\boldsymbol{X}|\boldsymbol{\theta})}{g(\boldsymbol{X}|\boldsymbol{\gamma})}) = 0$$

meaning $P_{\boldsymbol{\theta}}$ and $P_{\boldsymbol{\gamma}}$ are equivalent, against

$$H_f : E(\log \frac{f(\boldsymbol{X}|\boldsymbol{\theta})}{g(\boldsymbol{X}|\boldsymbol{\gamma})}) > 0$$

meaning $P_{\boldsymbol{\theta}}$ is better than $P_{\boldsymbol{\gamma}}$ or

$$H_g : E(\log \frac{f(\boldsymbol{X}|\boldsymbol{\theta})}{g(\boldsymbol{X}|\boldsymbol{\gamma})}) < 0$$

H_0 : equivalence of	R-vine and C-vine	C-vine and DAG	R-vine and DAG
test statistic	0.0227	1.005	3.3580
test decision	accept H_0	accept H_0	reject H_0 and R-vine better than DAG

Table 5: Test statistic and decision for pairwise comparison of the dependence models for the exchange rate data set

	R-vine	C-vine	DAG
D	0.0015	0.0017	1.32

Table 6: Sum of squared differences between simulated and empirical pairwise correlations for each dependency model of the exchange rate data

meaning $P_{\boldsymbol{\theta}}$ is worse than $P_{\boldsymbol{\gamma}}$. Here the expectation is to be taken under the true unknown model. He showed that an asymptotic α adjusted level test is given as follows

- Accept H_0 if $-q_{\alpha/2} + n^{-\frac{1}{2}} \frac{(p-q)}{\hat{\omega}_n} \leq n^{-\frac{1}{2}} \frac{LR_n(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\gamma}}_n)}{\hat{\omega}_n} \leq q_{\alpha/2} + n^{-\frac{1}{2}} \frac{(p-q)}{\hat{\omega}_n}$
- Accept H_f if $n^{-\frac{1}{2}} \frac{LR_n(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\gamma}}_n)}{\hat{\omega}_n} > q_{\alpha/2} + n^{-\frac{1}{2}} \frac{(p-q)}{\hat{\omega}_n}$
- Accept H_g if $-q_{\alpha/2} + n^{-\frac{1}{2}} \frac{(p-q)}{\hat{\omega}_n} < n^{-\frac{1}{2}} \frac{LR_n(\hat{\boldsymbol{\theta}}_n, \hat{\boldsymbol{\gamma}}_n)}{\hat{\omega}_n}$

Here q_α is the $1-\alpha$ quantile of the standard normal distribution. Considering three nonnested models we make three pairwise comparisons using the regular vine density (2.4) together with the tree structure given in Figure 4 (R-vine), the canonical vine density (2.3) (C-vine) and finally the density (5.1) for the DAG model. The corresponding results are given in Table 5. This clearly shows the preference for the R-vine specification over the DAG specification, while the two vine specification cannot be distinguished at $\alpha = .05$. The C-vine and DAG formulation also have to be considered equivalent at $\alpha = .05$, however the positive value of the test statistics points to a slightly better fit for the C-vine specification.

As a final model comparison we simulate 100 data sets from each dependency model using the corresponding ML parameter estimates. Based on these 100 simulations we determine the mean values of the simulated pairwise correlations $\hat{C}^i$ for dependency model i. As model comparison measure we use the sum of squared distance of all mean simulated to the empirical pairwise correlations $\widehat{\mathrm{Cor}}(\mathbf{W}))_{k,j}$, given by

$$D_i = \sum_{k=1}^{6} \sum_{j=k+1}^{7} \left[\widehat{\mathrm{Cor}}(\mathbf{W})_{k,j} - \hat{C}^i_{k,j} \right]^2 .$$

The results are given in Table 6 showing that the vine specifications are preferable over the DAG specification. There is a very slight preference for the R-vine specification.

6 Summary and discussion

This paper presents an analysis of Euro exchange rates using PCC copula based multivariate models with ARMA(P,Q)-GARCH(p,q) margins. Parameter estimation is facilitated in two steps, first using separate QMLE for each ARMA(P,Q)-GARCH(p,q) margin, then using the probability integral transformation to transform the estimated standardized innovations to observations with approximately uniform margins. The dependency among these uniform margin observations is modelled using three different PCC copula specifications including a first application of a non-Gaussian regular vine specification and a DAG model built on conditional t-distributions demonstrating the usefulness of PCC's.

Model selection among these non-nested models was done based on tests constructed using the approach taken by Vuong (1989) ignoring the separate estimation of the margins. One can adjust for the marginal estimation using the methods developed in Chen and Fan (2006a) since ARMA(P,Q)-GARCH(p,q) margins are allowed in their class of SCOMDY models. The adjustment however requires the efficient determination of derivatives with respect to all variables and parameters of the copula density. For the PCC specification given in (2.4) this is subject of current research.

The main contribution of this paper is to provide a modeling framework to construct high dimensional copulas, which can be extended to capture further data features such as marginal asymmetry and time varying dependency. For marginal asymmetries one can allow for nonsymmetric GARCH effects, as being considered in Dias and Embrechts (2007). Chollete et al. (2008) use for example the the skewed-t-GARCH of Hansen (1994). There exits many models for time varying dependency effects such as the BEKK model of Engle and Kroner (1995), the DCC model of Engle and Sheppard (2001) or the TVC model of Tse and Tsui (2002). In contrast to Patton (2006) and Dias and Embrechts (2007) this would allow to investigate more than two time series jointly. Again these extensions and model selection among these PCC copula-based models is a subject of ongoing research.

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