Ornstein-Uhlenbeck Processes and Extensions

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Summary. This paper surveys a class of Generalised Ornstein-Uhlenbeck (GOU) processes associated with Lévy processes, which has been recently much analysed in view of its applications in the financial modelling area, among others. We motivate the Lévy GOU by reviewing the framework already well understood for the "ordinary" (Gaussian) Ornstein-Uhlenbeck process, driven by Brownian motion; thus, defining it in terms of a stochastic differential equation (SDE), as the solution of this SDE, or as a time changed Brownian motion. Each of these approaches has an analogue for the GOU. Only the second approach, where the process is defined in terms of a stochastic integral, has been at all closely studied, and we take this as our definition of the GOU (see Eq. (12) below).

The stationarity of the GOU, thus defined, is related to the convergence of a class of "Lévy integrals", which we also briefly review. The statistical properties of processes related to or derived from the GOU are also currently of great interest, and we mention some of the research in this area. In practise, we can only observe a discrete sample over a finite time interval, and we devote some attention to the associated issues, touching briefly on such topics as an autoregressive representation connected with a discretely sampled GOU, discrete-time perpetuities, self-decomposability, self-similarity, and the Lamperti transform.

Some new statistical methodology, derived from a discrete approximation procedure, is applied to a set of financial data, to illustrate the possibilities.

1 Introduction

The Ornstein-Uhlenbeck (throughout: OU) process was proposed by Uhlenbeck and Ornstein (1930) in a physical modelling context, as an alternative to Brownian Motion, where some kind of mean reverting tendency is called for in order to adequately describe the situation being modelled. Since the original paper appeared, the model has been used in a wide variety of applications areas. In Finance, it is best known in connection with the Vasicek (1977) interest rate model. References to this (huge) literature are readily available via library and web searches, and we will not attempt to review it all here. However, to set the scene we will briefly discuss the standard (Gaussian) OU process, driven by Brownian Motion, and concentrate thereafter on some extensions that have recently attracted attention, especially in the financial modelling literature.

2 OU Process driven by Brownian Motion

The (one-dimensional) Gaussian OU process $X = (X_t)_{t\geq 0}$ can be defined as the solution to the stochastic differential equation (SDE)

$$\mathrm{d}X_t = \gamma(m - X_t)\mathrm{d}t + \sigma \mathrm{d}B_t,\ t > 0, \tag{1}$$

where γ, m, and $\sigma \geq 0$ are real constants, and B_t is a standard Brownian Motion (SBM) on $\mathbb{R}$. X_0, the initial value of X, is a given random variable (possibly, a constant), taken to be independent of $B = (B_t)_{t\geq 0}$. The parameter m can be formally eliminated from (1) by considering $X_t^{(m)} := X_t - m$ rather than X, but we will keep it explicit in view of some later applications.

Alternatively, we could define X in terms of a stochastic integral:

$$X_t = m\left(1 - e^{-\gamma t}\right) + \sigma e^{-\gamma t}\int_0^t e^{\gamma s}\mathrm{d}B_s + X_0 e^{-\gamma t},\ t \geq 0. \tag{2}$$

It is easily verified that X as defined by (2) satisfies (1) for any γ, m, σ, and choice of X_0; it is the *unique, strong Markov solution* to (1), cf. Protter (2005, p. 298). The stochastic integral in (2) is well defined and satisfies the properties outlined in Protter (2005), for example. In particular, $M_\cdot := \int_0^\cdot e^{\gamma s}\mathrm{d}B_s$ is a zero-mean martingale with respect to the natural filtration of B, whose quadratic variation is $[M,M]_\cdot = \int_0^\cdot e^{2\gamma s}\mathrm{d}s$. So $M_t = W_{[M,M]_t}$, $t \geq 0$, where W is an SBM (Protter 2005, p. 88). This leads to a third representation for X as a time changed Brownian motion:

$$X_t = m\left(1 - e^{-\gamma t}\right) + \sigma e^{-\gamma t} W_{(e^{2\gamma t}-1)/2\gamma} + X_0 e^{-\gamma t},\ t \geq 0. \tag{3}$$

Basic properties of X are easily derived from (1)–(3). In particular, conditional on X_0, and assuming X_0 has finite variance, X_t is Gaussian with expectation and covariance functions given by

$$EX_t = m\left(1 - e^{-\gamma t}\right) + e^{-\gamma t}EX_0,\ t \geq 0, \tag{4}$$

and

$$\mathrm{Cov}(X_u, X_t) = \frac{\sigma^2}{2\gamma}e^{-\gamma u}\left(e^{\gamma t} - e^{-\gamma t}\right) + e^{-\gamma(u+t)}\mathrm{Var}X_0,\ u \geq t \geq 0. \tag{5}$$

When, and only when, $\gamma > 0$, the limit $\lim_{t\to\infty}\int_0^t e^{-\gamma s}\mathrm{d}B_s$ exists almost surely (a.s.) as a finite random variable, which we can denote as $\int_0^\infty e^{-\gamma s}\mathrm{d}B_s$.

Using time reversal (for a fixed $t > 0$, $(B_s)_{0\le s\le t}$ has the same distribution as $(B_{t-s})_{0\le s\le t}$) we see from (2) that, for each $t \ge 0$, X_t has the same distribution as

$$\widetilde{X}_t := m\left(1 - e^{-\gamma t}\right) + \sigma \int_0^t e^{-\gamma s}\mathrm{d}B_s + X_0 e^{-\gamma t}, \tag{6}$$

so $\lim_{t\to\infty} \widetilde{X}_t$ exists a.s., is finite, and equals $\widetilde{X}_\infty := m + \sigma \int_0^\infty e^{-\gamma s}\mathrm{d}B_s$, when $\gamma > 0$. If X_t is "started with" initial value X_0, having the distribution of $\widetilde{X}_\infty$, and independent of $(X_t)_{t>0}$, then it is strictly stationary in the sense that the random vectors $(X_{t_1}, X_{t_2}, \ldots, X_{t_k})$ and $(X_{t_1+h}, X_{t_2+h}, \ldots, X_{t_k+h})$ have the same distribution, for any $k = 1, 2, \ldots$, $h > 0$, and $0 < t_1 < t_2 < \ldots < t_k < \infty$. In this case we can extend B_t to $(-\infty, 0)$, note that $\widetilde{X}_\infty :\stackrel{D}{=} m + \sigma \int_{-\infty}^0 e^{\gamma s}\mathrm{d}B_s$, independent of $(X_t)_{t\ge 0}$, and take $X_0 := m + \sigma \int_{-\infty}^0 e^{\gamma s}\mathrm{d}B_s$. From (2), then, we can write

$$X_t = m + \sigma e^{-\gamma t} \int_{-\infty}^t e^{\gamma s}\mathrm{d}B_s,\ t \ge 0. \tag{7}$$

Since B_t has stationary independent increments, from (7) we see that X is a stationary, Markovian, Gaussian process, which is continuous in probability. Conversely, any such process is a (stationary) version of a Gaussian OU process.

The "mean reversion" of X to the constant level m when $\gamma > 0$ can be inferred from (1); if X has diffused above m at some time, then the coefficient of the "dt" drift term is negative, so X will tend to move downwards immediately after, with the reverse holding if X is below m at some time.

Definitions (1) and (2) still make sense when $\gamma \le 0$. When $\gamma = 0$, X reduces to a zero mean Brownian motion (note that the parameter m is unidentified when $\gamma = 0$) and when $\gamma \le 0$ of course X is not stationary, in fact $|\int_0^t e^{-\gamma s}\mathrm{d}B_s|$ tends to infinity in probability as $t \to \infty$, so this is an "explosive" case.

3 Generalised OU Processes

There are many ways of generalising the Gaussian OU process, but we will concentrate here on a class of generalisations which has particular application in financial modelling, and has been recently studied intensely from this point of view. There are certainly applications of this class in other areas too.

The idea is to replace the dt and dB_t differentials in (1) with the differentials of other semimartingales, or, alternatively, replace the exponential function and Brownian motion in (2) or (3) with other processes. These are quite sweeping generalisations, and to keep the analysis manageable we restrict ourselves to a Lévy generalisation. This is already a profound one, and, apart from greatly increasing applicability, introduces many interesting and important analytical considerations, not least to do with the intricacies of the stochastic calculus. We passed over this aspect in Section 2 because the integrals involve only the *continuous* semimartingale B_t, and are relatively easy to handle. A general Lévy process has a *jump* component which requires special attention in the analysis. But the jumps introduce a modelling feature we wish to incorporate since they prove useful in some financial modelling situations, see, e.g., Geman, Madan and Yor (2000). Another aspect that becomes more interesting (and more difficult!) for jump processes is the statistical analysis; we discuss this below.

Before proceeding, we need to recall some properties of Lévy processes.

Background on Bivariate Lévy Processes

We refer to Bertoin (1996) and Sato (1999) for basic results and representations concerning Lévy processes (see also Protter 2005, Ch. I, Sect. 4). Univariate Lévy processes are also considered in Brockwell (2007). For the Generalised OU Process (GOU) we need some specialised material on bivariate Lévy processes, which we briefly review now.

The setup is as follows. Defined on $(\Omega, \mathcal{F}, P)$, a complete probability space, a bivariate Lévy process $(\xi_t, \eta_t)_{t\geq 0}$ is a stochastic process in $\mathbb{R}^2$, with càdlàg paths and stationary independent increments, which is continuous in probability. We take $(\xi_0, \eta_0) = (0,0)$ and associate with (ξ, η) its natural filtration $(\mathcal{F}_t)_{t\geq 0}$, the smallest right-continuous filtration for which $(\xi_t, \eta_t)_{t\geq 0}$ is adapted, completed to contain all P-null sets.

Especially important is the *Lévy exponent*, $\psi(\theta)$, which is defined in terms of the characteristic function of (ξ_t, η_t) via

$$Ee^{\mathrm{i}\langle(\xi_t,\eta_t),\theta\rangle} =: e^{t\psi(\theta)},$$

where $\langle\cdot,\cdot\rangle$ denotes inner product in $\mathbb{R}^2$. For a bivariate Lévy process the exponent is given by the Lévy-Khintchine representation:

$$\psi(\theta) = \mathrm{i}\langle A,\theta\rangle - \tfrac{1}{2}\langle\theta, \Sigma\theta\rangle + \iint_{|(x,y)|\leq 1} \left(e^{\mathrm{i}\langle(x,y),\theta\rangle} - 1 - \mathrm{i}\langle(x,y),\theta\rangle\right) \Pi_{\xi,\eta}(\mathrm{d}x,\mathrm{d}y) + \iint_{|(x,y)|>1} \left(e^{\mathrm{i}\langle(x,y),\theta\rangle} - 1\right) \Pi_{\xi,\eta}(\mathrm{d}x,\mathrm{d}y), \quad \text{for } \theta \in \mathbb{R}^2. \tag{8}$$

Here $|\cdot|$ is Euclidian distance in $\mathbb{R}^2$, $A = (A_1, A_2)$ is a nonstochastic 2–vector, $\Sigma = (\sigma_{rs})$ is a nonstochastic 2×2 non-negative definite matrix, and the Lévy measure, $\Pi_{\xi,\eta}$, is a measure on the Borel subsets of $\mathbb{R}^2 \setminus \{0\}$, with $\int(|(x,y)|^2 \wedge 1)\Pi_{\xi,\eta}(\mathrm{d}x,\mathrm{d}y) < \infty$. Though its value at $(0,0)$ is not relevant, for definiteness, we can take $\Pi_{\xi,\eta}\{(0,0)\} = 0$. In the literature, Lévy processes such that the Lévy measure of any neighbourhood in $\mathbb{R}^2 \setminus \{0\}$ whose closure contains 0 is infinite, are often described as having "infinite activity". Such processes have infinitely many jumps in every nonempty time interval, a.s. The remaining Lévy processes, that is, Lévy processes with "finite activity", are compound Poisson processes (possibly with a drift).

The component processes ξ_t and η_t are Lévy processes in their own right, having canonical triplets $(A_\xi, \sigma_{11}, \Pi_\xi)$ and $(A_\eta, \sigma_{22}, \Pi_\eta)$, say, where the Lévy measures are given by

$$\Pi_\xi\{\Lambda\} := \int_{\mathbb{R}} \Pi_{\xi,\eta}\{\Lambda, dy\} \quad \text{and} \quad \Pi_\eta\{\Lambda\} := \int_{\mathbb{R}} \Pi_{\xi,\eta}\{dx, \Lambda\}, \tag{9}$$

for Λ a Borel subset of $\mathbb{R} \setminus \{0\}$, and the centering constants are given by

$$A_\xi := A_1 + \int_{|x|\le 1} x \int_{|y|\ge\sqrt{1-x^2}} \Pi_{\xi,\eta}\{\mathrm{d}x, \mathrm{d}y\},$$

and similarly for A_η.

A (càdlàg) Lévy process has countably many jumps at most, a.s. We set $(\xi_{s-}, \eta_{s-}) := \lim_{u\uparrow s}(\xi_u, \eta_u)$ for $s > 0$, and denote the jump process by

$$\Delta(\xi,\, \eta)_t := (\Delta\xi_t,\, \Delta\eta_t) = (\xi_t - \xi_{t-},\, \eta_t - \eta_{t-}), \quad t \ge 0$$

(with $(\xi_{0-}, \eta_{0-}) = 0$). If Λ is a Borel subset of $\mathbb{R}^2 \setminus \{0\}$, then the expected number of jumps of (ξ, η) of (vector) magnitude in Λ occuring during any unit time interval equals $\Pi\{\Lambda\}$, i.e., for any $t > 0$,

$$\Pi\{\Lambda\} = E \sum_{t<s\le t+1} 1_{\{(\Delta\xi_s, \Delta\eta_s)\in\Lambda\}}. \tag{10}$$

Corresponding exactly to the decomposition in (8) is the Lévy-Ito representation of the process as a shift vector plus Brownian plus "small jump" plus "large jump" components:

$$(\xi_t, \eta_t) = (A_1, A_2)t + (B_{\xi,t}, B_{\eta,t}) + (\xi_t^{(1)}, \eta_t^{(1)}) + (\xi_t^{(2)}, \eta_t^{(2)}). \tag{11}$$

Here $(B_{\xi,t}, B_{\eta,t})_{t\ge 0}$ is a Brownian motion on $\mathbb{R}^2$ with mean $(0, 0)$ and covariance matrix $t\Sigma$, $(\xi_t^{(1)}, \eta_t^{(1)})_{t\ge0}$ is a discontinuous (pure jump) process with jumps of magnitude not exceeding 1, which may be of bounded variation on compact time intervals (that is, $\sum_{0<s\le t} |\Delta(\xi, \eta)_s| < \infty$ a.s. for all $t > 0$), or of unbounded variation, and $(\xi_t^{(2)}, \eta_t^{(2)})_{t\ge0}$ is a pure jump process with jumps of magnitude always exceeding 1; thus it is a compound Poisson process. The truncation point "1" is arbitrary and can be replaced by any other positive number at the expense only of redefining the shift vector (A_1, A_2). The representation (11) is a great aid to intuition as well as being indispensible in many analyses.

The couple $(\xi_t^{(1)}, \eta_t^{(1)})$ (also a bivariate Lévy process, as is $(\xi_t^{(2)}, \eta_t^{(2)})$) have finite moments of all orders, and by adjusting the centering vector A if necessary we can take $E\xi_1^{(1)} = E\eta_1^{(1)} = 0$. Moments of $\xi_t^{(2)}$ and $\eta_t^{(2)}$ are not necessarily finite; conditions for this to be so, in terms of the canonical measures, are in Sato (1999, p.159, and p.163, ff.). In general, one or more of the components on the righthand side of (11) may not be present, i.e., degenerates to 0. The bivariate Lévy then correspondingly degenerates to a simpler form.

Lévy OU Processes

As a starting point for the generalisation we could use (1), (2), or (3). In our general setting these three definitions do not produce the same process. Each is interesting in its own right, but what is presently known in the literature as the *Generalised OU process* proceeds from (2), and we will adhere to this usage. Thus, we take a bivariate Lévy process (ξ, η) and write

$$X_t = m\left(1 - e^{-\xi_t}\right) + e^{-\xi_t}\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s + X_0 e^{-\xi_t},\ t \geq 0, \tag{12}$$

where X_0 is independent of $(\xi_t, \eta_t)_{t\geq 0}$, and assumed $\mathcal{F}_0$-measurable. Considerations of the stochastic calculus require us to be precise in specifying the filtration with respect to which the integral in (12) is defined, and we take it to be the natural filtration $(\mathcal{F}_t)_{t\geq 0}$. The Lévy processes ξ and η are semimartingales, so the stochastic integral in (12) is well defined without further conditions; in particular, no moment conditions on ξ or η are needed.

One motivation for studying (12) is that special cases of it occupy central positions in certain models of financial time series; the Lévy driven OU processes of Barndorff-Nielsen and Shephard (2001a, 2001b, 2003) and the COGARCH process of Klüppelberg, Lindner and Maller (2004) are recent examples.

The GOU as defined in (12) seems to have been first considered by Carmona, Petit and Yor (1997); it is also implicit in the paper of de Haan and Karandikar (1989), where it occurs as a natural continuous time generalisation of a random recurrence equation. It has been studied in some detail by Lindner and Maller (2005), with emphasis on carrying over some of the properties enjoyed by the Gaussian OU. Other applications are in option pricing (Yor (1992, 2001)), insurance and perpetuities (Harrison (1977), Dufresne (1990), Paulsen and Hove (1999)), and risk theory (Klüppelberg and Kostadinova (2006)). Many of these place further restrictions on ξ and η; for example, ξ may be independent of η, or one or another or both of ξ or η may be a Brownian motion or compound Poisson process, etc. To begin with, we make no assumptions on ξ or η (not even independence), and investigate some general properties of X_t.

Thus, it is the case that X_t is a time homogeneous Markov process (Carmona et al. 1997, Lemma 5.1), and it is elementary that $(X_t)_{t\geq 0}$ is strictly stationary if and only if X_t converges in distribution to X_0, as $t \to \infty$. To study when this occurs, stationarity is related to the convergence of a certain stochastic integral in Lindner and Maller (2005). But which integral? Let us note that in general there is no counterpart of the equality (in distribution) of (2) and (6). That is, in general, $e^{-\xi_t}\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s$ does not have the same distribution (even for a fixed $t > 0$) as $\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s$, as might at first be thought via a time-reversal argument. The correct relationship is given in Proposition 2.3 of Lindner and Maller (2005):

$$e^{-\xi_t}\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s \stackrel{D}{=} \int_0^t e^{-\xi_{s-}}\mathrm{d}L_s, \text{ for each } t>0, \tag{13}$$

where L_t is a Lévy process constructed from ξ and η as follows:

$$L_t := \eta_t + \sum_{0<s\le t}(e^{-\Delta\xi_s}-1)\Delta\eta_s - t\,\mathrm{Cov}(B_{\xi,1},B_{\eta,1}), \quad t\ge 0. \tag{14}$$

Here "Cov" denotes the covariance of the Brownian components of ξ and η. In general $L_t \neq \eta_t$, but when ξ and η are independent, for example, they have no jumps in common, a.s., and the covariance term is 0, so (14) gives $L_t \equiv \eta_t$, and the integral on the righthand side of (13) then equals $\int_0^t e^{-\xi_{s-}}\mathrm{d}\eta_s$.

But even in the general case, (13) can be used to investigate the large time behaviour of X_t, because necessary and sufficient conditions for the convergence (a.s., or, in distribution) of Lévy integrals of the form $\int_0^\infty e^{-\xi_{t-}}\mathrm{d}L_t$ have been worked out by Erickson and Maller (2005), phrased in terms of quite simple functionals of the canonical triplet of (L,η), which is easily obtained from the canonical triplet of (ξ,η) via (14). Except for a degenerate case, necessary and sufficient is that $\lim_{t\to\infty}\xi_t=\infty$ a.s., together with a kind of log-moment condition involving only the *marginal* measures of ξ and η. The divergence criterion $\lim_{t\to\infty}\xi_t=\infty$ a.s. is also easily expressed in terms of the canonical measure of ξ_t. The stationarity criterion, given in Theorem 2.1 of Lindner and Maller (2005), is that $(X_t)_{t\ge 0}$ is strictly stationary, for an appropriate choice of X_0, if and only if the integral $\int_0^\infty e^{-\xi_{t-}}\mathrm{d}L_t$ converges (a.s., or, equivalently, in distribution), or else X_t is indistinguishable from a constant process.

From these results we see that a study of the GOU process can be reduced in part to a study of the exponential Lévy integral $\int_0^\infty e^{-\xi_{t-}}\mathrm{d}L_t$, and this program is continued in Erickson and Maller (2007) (conditions for convergence of stochastic integrals), Bertoin, Lindner and Maller (2007) and Kondo, Maejima and Sato (2006) (continuity properties of the integral), and Maller, Müller and Szimayer (2007) (discrete approximation and statistical properties).

We took as starting point in this section a generalisation of (2), via (12). (12) has direct relevance to stochastic volatility and other models in finance, among other possible applications. On the other hand, modelling by SDEs such as (1) (the Langevin equation) can arise directly from a physical situation; e.g., the interpretation of (1) as describing the motion of a particle under a restraining force proportional to its velocity. The counterpart of (1) for the GOU is the SDE

$$\mathrm{d}X_t = (X_{t-}-m)\mathrm{d}U_t + \mathrm{d}L_t, \ t\ge 0, \tag{15}$$

where (U,L) is a bivariate Lévy process. Suppose this holds for a U whose Lévy measure attributes no mass to $(-\infty,-1]$, and define a Lévy process ξ by

$\xi_t = -\log \mathcal{E}(U)_t$, where $\mathcal{E}(U)$ denotes the *stochastic exponential* of U, namely, the solution to the SDE $\mathrm{d}\mathcal{E}(U)_t = \mathcal{E}(U)_{t-}\mathrm{d}U_t$ with $\mathcal{E}(U)_0 = 1$; see Protter (2005, p.85). Then define a Lévy process η_t by

$$\eta_t := L_t - \sum_{0<s\le t} (1 - e^{-\Delta\xi_s})\Delta L_s + t\,\mathrm{Cov}(B_{\xi,1}, B_{L,1}), \quad t \ge 0. \tag{16}$$

With these definitions, (12) is the unique (up to indistinguishability) solution to (15). To verify this, use integration by parts in (12) together with Eq. (2.10) of Lindner and Maller (2005). The fact that the Lévy measure of U attributes no mass to $(-\infty, -1]$ ensures that $\mathcal{E}(U)$ is positive. Conversely, if, for a given bivariate Lévy process (ξ, η), L satisfies (14), and U satisfies $\xi_t = -\log \mathcal{E}(U)_t$, then X_t as defined in (12) satisfies (15), and, further, the Lévy measure of U attributes no mass to $(-\infty, -1]$. See Protter (2005, p.322) and Yoeurp (1979) for further discussion.

A third approach to generalising an OU is to consider more general time changes. Monroe (1978), generalising Lévy's result for continuous local martingales, showed that *any* semimartingale can be obtained as a time changed Brownian motion. Thus we can write $\int_0^t e^{\xi_{s-}}\mathrm{d}\eta_s = W_{T_t}$, for an SBM W and an increasing semimartingale $(T_t)_{t\ge 0}$, leading to another kind of generalisation of (3). The properties of such a class are also unexplored, so far as we know. Other versions of time changed Brownian motions have been used in many situations; see, e.g., Anh, Heyde and Leonenko (2002), for a financial application.

Self-Decomposability, Self-Similarity, Class L, Lamperti Transform

Consider the case when $m = 0$ and $\xi_t = \gamma t$, $\gamma > 0$, is a pure drift in (12):

$$X_t = e^{-\gamma t}\int_0^t e^{\gamma s}\mathrm{d}\eta_s + X_0 e^{-\gamma t},\ t \ge 0. \tag{17}$$

Say that (the distribution of) a random variable X is *semi-self-decomposable* if X has the same distribution as $aX + Y^{(a)}$, for a constant $0 < a < 1$, for some random variable $Y^{(a)}$, independent of X, possibly depending on a. If an equality in distribution $X \stackrel{D}{=} aX + Y^{(a)}$ can be achieved for *all* $a \in (0, 1)$, X is said to be *self-decomposable*. See Sato (1999, Section 15). This property can also be described as saying that the distribution of X is *of Class L*; this is a subclass of the infinitely divisible distributions which can be obtained as the limit laws of normed, centered, sums of independent (but not necessarily identically distributed) random variables. See Feller (1971, p. 588). Class L contains but is not confined to the stable laws, which are the limit laws of normed, centered, sums of i.i.d. random variables.

A potential limiting value of X_t in (17) as $t \to \infty$ is the random variable $X_\infty := \int_0^\infty e^{-\gamma t}\mathrm{d}\eta_t$, if finite, and then X_t is stationary if $X_0 \stackrel{D}{=} X_\infty$. Wolfe (1982) showed that a random variable X is self-decomposable if and only if it has the representation

$$X \stackrel{D}{=} \int_0^\infty e^{-\gamma t} \mathrm{d}\eta_t,$$

for some Lévy process η with $E \log^+ |\eta| < \infty$ (and then X is a.s. finite), and, further, that the canonical triplets of X_t (the Lévy process with the distribution of X when $t = 1$) and η_t are then connected in a simple way. He made crucial use of the formula

$$Ee^{\mathrm{i}\theta \int_a^b f(s)\mathrm{d}\eta_s} = Ee^{\int_a^b \Psi_\eta(-\theta f(s))\mathrm{d}s}, \ 0 \le a < b < \infty, \theta \in \mathbb{R}, \tag{18}$$

where f is a bounded continuous function in $\mathbb{R}$ and $\Psi_\eta(\theta) := -\log\left(Ee^{\mathrm{i}\theta\eta_1}\right)$ (e.g., Bichteler (2002, Lemma 4.6.4, p. 256)).

An H*–self-similar* process $(X_t)_{t\ge0}$ is such that $(X_{at})_{t\ge0}$ has the same distribution as $(a^H X_t)_{t\ge0}$, for some constant $H > 0$, and each $a > 0$. Sato (1991) showed that a random variable X_1 is self-decomposable if and only if for each $H > 0$ its distribution is the distribution at time 1 of an H–self-similar process. An H–self-similar *Lévy* process must have $H \ge 1/2$; and then X_t is an α–stable process with index $\alpha = 1/H \in (0, 2]$.

The *Lamperti Transform* of an H–self-similar process $(X_t)_{t\ge0}$ is the (stationary) process $Y_t := e^{-tH} X_{e^t}$, $t \ge 0$. Lamperti (1962, 1972) showed, conversely, that any stationary process Y can be represented in this form. Thus, in summary, we have a correspondence between a stationary process Y_t, an H–self-similar process X_t, a self-decomposable random variable X_1, the class L, and the integral $\int_0^\infty e^{-\gamma t}\mathrm{d}\eta_t$. Jeanblanc, Pitman and Yor (2002) give an elegant linking approach to these.

The integral $\int_0^\infty e^{-\xi_t}\mathrm{d}t$ (assumed convergent) is self-decomposable when ξ is spectrally negative, but *not* in general; (in fact, it is not even infinitely divisible in general). These results are due to Samorodnitsky (reported in Klüppelberg et al. (2004)). Thus, *a fortiori*, the integral $\int_0^\infty e^{-\xi_t}\mathrm{d}\eta_t$ is not in general self-decomposable. See also Kondo et al. (2006) for further results.

4 Discretisations

Autoregressive Representation, and Perpetuities

Given a Lévy process L_t with $E \log^+ |L_1| < \infty$ and constants $h > 0$ and $\gamma > 0$, let $(Q_n)_{n=1,2,\ldots}$ be i.i.d. with the distribution of $e^{-\gamma h}\int_0^h e^{\gamma s}\mathrm{d}L_s$. Then (Wolfe 1982) the discrete time process (time series) defined recursively by

$$Z_n = e^{-\gamma h} Z_{n-1} + Q_n, \ n = 1, 2, \ldots, \quad \text{with } Z_0 = 0, \tag{19}$$

converges in distribution as $n \to \infty$ to a random variable with the distribution of the (a.s. finite) integral $\int_0^\infty e^{-\gamma t}\mathrm{d}L_t$. Thus the stationary distribution of an OU process driven by Lévy motion can be obtained from the behaviour at large times of an autoregressive time series. Conversely, Wolfe showed that if $(Q_n)_{n=1,2,\ldots}$ are given i.i.d. random variables with $E \log^+ |Q_n| < \infty$, and Z_n

are defined by the recursion in (19) with $\gamma > 0$ and $h = 1$, then there is a Lévy process L_t with $E \log^+ |L_1| < \infty$ such that the process X_t as defined in (17) satisfies $X_n = Z_n$, $n = 1, 2, \ldots$, if and only if $Q_1 \stackrel{D}{=} e^{-\gamma} \int_0^1 e^{\gamma s} \mathrm{d}L_s$. He further gave necessary and sufficient conditions for the latter property to hold; namely, a random variable Q has the same distribution as $e^{-\gamma} \int_0^1 e^{\gamma s} \mathrm{d}L_s$, for a given $\gamma > 0$ and Lévy process L_t with $E \log^+ |L_1| < \infty$, if and only if $\prod_{j=0}^{\infty} E(e^{\mathrm{i}\rho^j \theta Q})$ is the characteristic function of a distribution in class L, where $\rho = e^{-\gamma}$. See also Sato (1999, Section 17).

(19) is a special case of a discrete time "perpetuity". More generally, we may replace the coefficient $e^{-\gamma h}$ in (19) by a random sequence, M_n, say, such that $(Q_n, M_n)_{n=1,2,\ldots}$ is an i.i.d. sequence of 2-vectors. Then Z_n is a kind of analogue of the Lévy integral $\int_0^t e^{-\xi_{s-}} d\eta_s$; see, e.g., Lindner and Maller (2005) for a discussion. Random sequences related to perpetuities have received much attention in the literature as models for a great variety of phenomena, including but not restricted to the actuarial area. We refer to Vervaat (1979), Goldie and Maller (2000), Nyrhinen (1999, 2001).

Statistical Issues: Estimation and Hypothesis Testing

There are methods of estimation of parameters in continuous time models based on hypothetical continuous time observation of a process over a finite interval, and the testing of hypotheses about them, for example, in a likelihood framework (Liptser and Shiryaev (1978), Basawa and Prakasa Rao (1980), Heyde (1997), Kutoyants (2004)), which provide much insight. But in practise we can only observe in discrete time, and have to think how to approximate the parameters in the original continuous time model from a finite (discrete) sample. Furthermore, observation in practise can only be carried out over a finite time interval, whereas frequently in Statistics we may wish to employ a large sample theory, or, in the case of a time series, let the observation time grow large, to derive benchmark distributions for parameter estimates and test statistics which are free from finite sample effects.

Consequently, in approximating a continuous by a discrete time process, we can proceed in one or both of two ways. One is to form a series of approximations on a finite time interval $[0, T]$, which is subdivided into finer and finer grids, so that in the limit the discrete approximations converge, hopefully, to the original continuous, time process (in some mode); alternatively, we can sample at discrete points in a finite time interval $[0, T]$ and let $T \to \infty$ to get asymptotic distributions; or, thirdly, we can attempt to combine both methods in some way.

Discretely Sampled Process

Discrete sampling of an OU process on an equispaced grid over a finite time horizon $T > 0$ produces an autoregressive (AR) time series, as follows. Suppose X_t satisfies (1), and fix a compact interval $[0, T]$, $T > 0$. Then

$$X_{i,n} = X_{iT/n}, \; i = 0, 1, \ldots, n, \; \text{for } n = 1, 2, \ldots, \tag{20}$$

is the discretely sampled process. From (1) we can write

$$X_{i,n} = (1-\alpha_n)m + \alpha_n X_{i-1,n} + \sigma_n \varepsilon_{i,n},\ i = 1,2,\ldots,n, \tag{21}$$

where

$$\alpha_n = e^{-\gamma T/n}, \quad \sigma_n^2 = \sigma^2(1-e^{-2\gamma T/n})/(2\gamma), \tag{22}$$

and

$$\varepsilon_{i,n} := \frac{\sigma}{\sigma_n}\int_0^{T/n} e^{\gamma(s-T/n)} \mathrm{d}B_{s+(i-1)T/n}. \tag{23}$$

(21) is a system of autoregressions, where the $(\varepsilon_{i,n})_{i=1,2,\ldots,n}$ are i.i.d. standard normal random variables for each $n = 1,2,\ldots$.

Next, embed each $X_{i,n}$ into a continuous time process $X_n(t)$ by setting

$$X_n(t) = X_{i-1,n},\ \text{for}\ (i-1)T/n \le t < iT/n,\ i = 1,2,\ldots,n. \tag{24}$$

Then $X_n(t) \to X_t$, uniformly on $[0,T]$, in probability, as $n \to \infty$.

Szimayer and Maller (2004) carry out the above procedure, but with a Lévy process L_t, satisfying $EL_1 = 0$ and $EL_1^2 = 1$, replacing B_t in (1) and consequently in (23). The $\varepsilon_{i,n}$ in (23) remain i.i.d. $(0,1)$ random variables, though in general of course they are no longer normally distributed. Szimayer and Maller (2004) used a Quasi-Maximum Likelihood (QML) approach, whereby a likelihood for the observations is written down as if the $\varepsilon_{i,n}$ were normally distributed, and estimates and test statistics calculated from it, but then the normality assumption is discarded for the rest of the analysis. They test the hypothesis $H_0 : \gamma = 0$, of no mean reversion in the model (so X_t reduces to L_t, a pure Lévy process). This hypothesis test has the nonstandard feature that the long term equilibrium parameter m "disappears under the null"; it cannot be identified from (1) when $\gamma = 0$. Methods of Davies (1977, 1987) are available for handling this. Szimayer and Maller (2004) work out the asymptotic distribution (as the mesh size tends to 0, over the compact interval $[0,T]$) of the QML statistic for testing H_0, as a function of the underlying Lévy process L_t. That asymptotic distribution of course depends on T, and as $T \to \infty$, Szimayer and Maller (2004) show further that it tends to the distribution of a random variable related to the Dickey-Fuller unit root test in econometrics. This procedure is an example of estimating on a finite grid whose mesh size shrinks to 0, after which the observation window expands to infinity.

Approximating the COGARCH

The COGARCH is a continuous-time dynamic model suggested by Klüppelberg, Lindner and Maller (2004) to generalise the popular GARCH (Generalised Autoregressive Conditional Heteroscedasticity) model now commonly used in (discrete) time series analysis. The COGARCH is defined by

$$G_t = \int_0^t \sigma_{s-} \mathrm{d}L_s,\ t \ge 0, \tag{25}$$

where L_t is a "background driving Lévy process", and σ_t, the volatility process, satisfies

$$\sigma_t^2 = \beta e^{-X_t} \int_0^t e^{X_s}\, \mathrm{d}s + \sigma_0^2 e^{-X_t},\ t \geq 0, \tag{26}$$

for constants $\beta > 0$ and $\sigma_0^2 > 0$. (26) is a version of the GOU (12), with η_t replaced by a pure drift, and ξ_t replaced by X_t. The latter is just a notational change; the X_t in (26) is also a Lévy process, defined in terms of the original L_t by

$$X_t = \eta t - \sum_{0<s\leq t} \log(1 + \varphi(\Delta L_s)^2),\ t \geq 0, \tag{27}$$

for parameters $\eta > 0$ and $\varphi > 0$. Note that only one source of randomness, L_t, underlies both the process itself and the volatility process; this is an important feature of the discrete time GARCH models, preserved in the COGARCH.

Further analysis of the COGARCH is in Klüppelberg et al. (2004, 2006), where stationarity properties are related to the convergence of a Lévy integral. See also Lindner (2007). Statistical issues, especially, fitting the COGARCH to data, are in Haug et al. (2007), Müller (2007), and Maller, Müller and Szimayer (2007). The latter paper proposes a discretisation of the COGARCH in the same spirit as we discussed above for the Lévy driven OU model. Using a first-jump approximation of a Lévy process originally developed in Szimayer and Maller (2006) for an option pricing application, Maller et al. (2007) show that the COGARCH can be obtained as a limit of discrete time GARCH processes defined on the same probability space. This allows advantage to be taken of currently existing methods in time-series modeling and econometrics for this well-established process class.

The procedure is as follows. Take a sequence of integers $(N_n)_{n\geq 1}$ with $\lim_{n\to\infty} N_n = \infty$, and a finite interval $[0, T]$, $T > 0$, with a deterministic partitioning $0 = t_0(n) < t_1(n) < \ldots < t_{N_n}(n) = T$. Let $\Delta t_i(n) := t_i(n) - t_{i-1}(n)$ for $i = 1, 2, \ldots, N_n$, and assume $\Delta t_n := \max_{i=1,\ldots,N_n} \Delta t_i(n) \to 0$ as $n \to \infty$. Given the COGARCH parameters (β, η, φ), define the process

$$G_{i,n} = G_{i-1,n} + \sigma_{i-1,n} \sqrt{\Delta t_i(n)}\varepsilon_{i,n},\ \text{for } i = 1, 2, \ldots, N_n,\ \text{with } G_{0,n} = 0, \tag{28}$$

with an accompanying variance process:

$$\sigma_{i,n}^2 = \beta \Delta t_i(n) + \left(1 + \varphi \Delta t_i(n) \varepsilon_{i,n}^2\right) e^{-\eta \Delta t_i(n)} \sigma_{i-1,n}^2,\ i = 1, 2, \ldots, N_n. \tag{29}$$

Here, for each $n \geq 1$, $(\varepsilon_{i,n})_{i=1,\ldots,N_n}$ is a sequence of independent random variables with $\mathbb{E}\varepsilon_{1,n} = 0$ and $\mathbb{E}\varepsilon_{1,n}^2 = 1$ constructed pathwise from the driving Lévy process L_t in (25) and its characteristics; and $\sigma_{0,n}^2$ is a given random variable, independent of the $\varepsilon_{i,n}$. (28) and (29) define a kind of discrete time GARCH-type recursion with scaling by the time increments $\Delta t_i(n)$.

The discrete time processes are then embedded into continuous time by

$$G_n(t) := G_{i,n} \quad \text{and} \quad \sigma_n^2(t) := \sigma_{i,n}^2 \text{ when } t \in (t_{i-1}(n), t_i(n)],\ 0 \leq t \leq T, \tag{30}$$

with $G_n(0) = 0$ and $\sigma_n^2(0) = \sigma_{0,n}^2$. A key result of Maller et al. (2007) is that, as $n \to \infty$ (so $\Delta t(n) \to 0$), the Skorokhod distance between $(G_n(\cdot), \sigma_n(\cdot))$ and $(G(\cdot), \sigma(\cdot))$, over $[0, T]$, converges in probability to 0; thus, in particular, $(G_n(\cdot), \sigma_n(\cdot))$ converges in distribution to $(G(\cdot), \sigma(\cdot))$ in $\mathbb{D}[0, T] \times \mathbb{D}[0, T]$, where $\mathbb{D}[0, T]$ is the space of càdlàg stochastic process on $[0, T]$.

Maller et al. (2007) use this result to motivate an estimation procedure for the COGARCH parameters in terms of estimates of the parameters of the discrete GARCH approximating process. Via some simulations, this is shown to work somewhat better, in some selected situations, than the Haug et al. (2007) method, at least as judged by the mean square error of the estimates.

As an example application, we fitted the COGARCH model to a series of 33,480 log-prices of the Intel stock traded on the NYSE between February 1 and June 6, 2002, observed every minute from 09:36am to 04:00pm. The data is from the TAQ data base provided by the NYSE. We removed the overnight jumps and a linear trend from the data, then fitted a GARCH model by the QML method as described above, thus obtaining estimates $(\widehat{\beta}, \widehat{\varphi}, \widehat{\eta})$ of (β, φ, η). Then with G_t as the log stock price at time t, an estimate of the volatility process $(\sigma_t^2)_{t \geq 0}$ can be calculated recursively from

$$\widehat{\sigma}_n^2 = \widehat{\beta} + (1 - \widehat{\eta})\widehat{\sigma}_{n-1}^2 + \widehat{\varphi}(G_n - G_{n-1})^2, \ n = 1, 2, \ldots$$

(Haug et al. 2007). Figure 1 shows that the resulting volatility sequence (for the first 1,000 observations) compares reasonably well with the absolute log returns. Further discussion is in Maller et al. (2007).

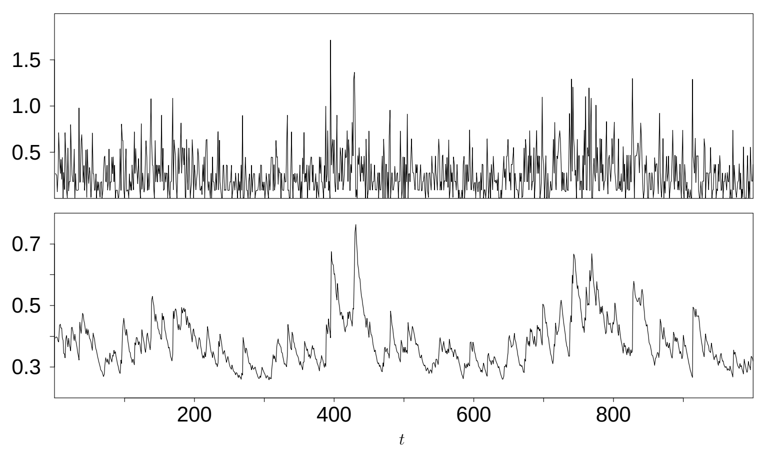

Fig. 1. Top: 1,000 minute-by-minute absolute log returns on Intel stock. Bottom: Corresponding estimated annualised volatilities for Intel data.

5 Conclusion

This survey cannot do justice in the space available to the many theoretical and practical studies, past and ongoing, related to the OU and GOU processes. We note in particular Brockwell, Erdenebaatar and Lindner (2006) (a COGARCH(p, q)); Masuda (2004) (a multidimensional OU process); Aalen and Gjessing (2004) (an interesting connection between the finance and survival analysis applications); Novikov (2004) (passage time problems); Kondo et al. (2006) (multidimensional exponential Lévy integrals); and the list goes on. Despite all this activity, much remains to be done, as we have suggested throughout the discussion, to add to our understanding of the stochastic processes themselves, and their statistical properties.

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References

1. Aalen OO, Gjessing HK (2004) Survival models based on the Ornstein-Uhlenbeck process. Lifetime Data Analysis 10:407–423
2. Anh VV, Heyde CC, Leonenko NN (2002) Dynamic models of long-memory processes driven by Lévy noise. J. Appl. Probab. 39:730–747
3. Barndorff-Nielsen OE, Shephard, N (2001a) Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics (with discussion). J. Roy. Statist. Soc. Ser. B 63:167–241
4. Barndorff-Nielsen OE, Shephard N (2001b) Modelling by Lévy processes for financial econometrics. In: Barndorff-Nielsen OE, Mikosch T, Resnick S (eds) Lévy processes, theory and applications. Birkhäuser, Boston
5. Barndorff-Nielsen OE, Shephard N (2003) Integrated OU processes and non-Gaussian OU-based stochastic volatility models. Scand. J. Statist. 30:277-295.
6. Basawa IV, Prakasa Rao BLS (1980) Statistical inference for stochastic processes. Academic Press, London New York
7. Bertoin J (1996) Lévy processes. Cambridge University Press, Cambridge
8. Bichteler K (2002) Stochastic integration with jumps. Cambridge University Press, Cambridge
9. Brockwell P (2007) Lévy-driven continuous time ARMA processes. In: Andersen TG, Davis RA, Kreiss J-P, Mikosch T (eds) Handbook of financial time series. Springer, Berlin Heidelberg New York
10. Brockwell P, Erdenebaatar C, Lindner A (2006) Continuous time GARCH processes. Annals of Appl. Prob. 16:790–826
11. Carmona P, Petit F, Yor M (1997) On the distribution and asymptotic results for exponential functionals of Lévy processes. In: Yor M (ed) Exponential functionals and principal values related to Brownian motion. Bibl. Rev. Mat. Iberoamericana, Madrid

12. Davies RB (1977) Hypothesis testing when a nuisance parameter is present only under the alternative. Biometrika 64:247–254
13. Davies RB (1987) Hypothesis testing when a nuisance parameter is present only under the alternative. Biometrika 74:33–43
14. De Haan L, Karandikar RL (1989). Embedding a stochastic difference equation in a continuous-time process. Stoch. Proc. Appl. 32:225–235
15. Dufresne D (1990) The distribution of a perpetuity, with application to risk theory and pension funding. Scand. Actuar. J. 9:39–79
16. Erickson KB, Maller, RA (2004) Generalised Ornstein-Uhlenbeck processes and the convergence of Lévy integrals, Séminaire de Probabilités XXXVIII, Lecture Notes in Mathematics Vol. 1857, 70–94. Springer-Verlag Heidelberg.
17. Erickson KB, Maller, RA (2007) Finiteness of functions of integrals of Lévy processes, Proc. Lond. Math. Soc. 94:386–420.
18. Feller W (1971) An introduction to probability theory and its applications II. Wiley, New York
19. Geman H, Madan DB, Yor M (2000) Asset prices are Brownian motion: only in business time. In: Avellaneda M (ed) Quantitative analysis in financial markets, Vol 2. World Scientific Publishing Company, Singapore
20. Goldie CM, Maller RA (2000) Stability of perpetuities. Ann. Probab. 28:1195–1218
21. Harrison JM (1977) Ruin problems with compounding assets. Stoch. Proc. Appl. 5:67–79
22. Haug S, Klüppelberg C, Lindner A, Zapp M (2007) Method of moments estimation in the COGARCH(1,1) model. The Econ. J. 10:320–341
23. Heyde CC (1997) Quasi-likelihood and its application: a general approach to optimal parameter estimation. Springer, Berlin Heidelberg New York
24. Jeanblanc M, Pitman J, Yor M (2002) Self-similar processes with independent increments associated with Lévy and Bessel processes. Stoch. Proc. Appl. 100:223–231
25. Karatzas I, Shreve SE (1998) Methods of mathematical finance. Springer, Berlin Heidelberg New York
26. Klüppelberg C, Kostadinova R (2006) Integrated insurance risk models with exponential Lévy investment. Insurance: Math. and Econ., to appear.
27. Klüppelberg C, Lindner A, Maller RA (2004) A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour. J. Appl. Prob. 41:601–622.
28. Klüppelberg C, Lindner A, Maller RA (2006) Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In: Kabanov Yu, Liptser R., Stoyanov J (eds) From stochastic calculus to mathematical finance, Shiryaev Festschrift. Springer, Berlin Heidelberg New York
29. Kondo H, Maejima M, Sato K (2006) Some properties of exponential integrals of Lévy processes and examples. Elect. Comm. in Probab. 11:291–303. University (www.math.keio.ac.jp/local/maejima/)
30. Kutoyants YA (2004) Statistical inference for ergodic diffusion processes. Springer, Berlin Heidelberg New York
31. Lindner A (2007) Continuous time approximations to both GARCH and stochastic volatility models. In: Andersen TG, Davis RA, Kreiss J-P, Mikosch T (eds) Handbook of financial time series. Springer, Berlin Heidelberg New York
32. Lindner A, Maller RA (2005) Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. Stoch. Proc. Appl. 115:1701–1722

33. Liptser RS, Shiryaev AN (1978) Statistics of random processes II: applications. Springer, Berlin Heidelberg New York
34. Madan DB, Carr PP, Chang EC (1998) The Variance Gamma process and option pricing. European Finance Review 2:79–105
35. Maller RA, Müller G, Szimayer A (2006) The COGARCH as a limit of GARCH models. Preprint, Munich University of Technology (http://www-m4.ma.tum.de/Papers/)
36. Masuda H (2004) On multidimensional Ornstein-Uhlenbeck processes driven by a general Lévy process. Bernoulli 10:97–120
37. Monroe, I (1978) Processes that can be embedded in Brownian motion. Ann. Probab. 6:42–56
38. Müller G (2006) MCMC estimation of the COGARCH(1,1) model. Preprint, Munich University of Technology (http://www-m4.ma.tum.de/Papers/)
39. Novikov AA (2004) Martingales and first-exit times for the Ornstein-Uhlenbeck process with jumps. Theory Probab. Appl. 48:288–303
40. Nyrhinen H (1999) On the ruin probabilities in a general economic environment. Stoch. Proc. Appl. 83:319–330
41. Nyrhinen H (2001) Finite and infinite time ruin probabilities in a stochastic economic environment. Stoch. Proc. Appl. 92:265–285
42. Paulsen J, Hove A (1999) Markov chain Monte Carlo simulation of the distribution of some perpetuities. Adv. Appl. Prob. 31:112–134
43. Protter P (1990) Stochastic integration and differential equations: a new approach. Springer, Berlin Heidelberg New York
44. Sato K (1991) Self-similar processes with independent increments. Prob. Th. Rel. Fields 89:285–300
45. Sato K (1999) Lévy processes and infinitely divisible distributions. Cambridge University Press, Cambridge
46. Szimayer A, Maller RA (2004) Testing for mean-reversion in processes of Ornstein-Uhlenbeck type. Stat. Inf. for Stoch. Proc. 7:95-113
47. Szimayer A, Maller RA (2007) Discrete time, finite state space almost sure and $\mathcal{L}_1$ approximation schemes for Lévy processes. Stoch. Proc. Appl, to appear.
48. Uhlenbeck GE, Ornstein LS (1930) On the theory of the Brownian motion. Phys. Rev. 36:823–841.
49. Vasicek OA (1977) An equilibrium characterisation of the term structure. J. Fin. Econ. 5:177–188
50. Vervaat W (1979) On a stochastic difference equation and a representation of non-negative infinitely divisible random variables. Adv. Appl. Prob. 11:750–783
51. Wolfe SJ (1982) On a continuous analogue of the stochastic difference equation $X_n = \rho X_{n-1} + B_n$. Stoch. Proc. Appl. 12:301–312
52. Yoeurp Ch. (1979) Solution explicite de l'équation $Z_t = 1 + \int_0^t |Z_{s-}| dX_s$. Séminaire de Probabilités X, Lecture Notes in Mathematics, XXXVIII Vol. 511, 614–619. Springer-Verlag Heidelberg.
53. Yor M (1992) Sur certaines fonctionnelles du mouvement brownien réel. J. Appl. Probab. 29:202–208
54. Yor M (2001) Exponential functionals of Brownian motion and related processes. Springer, Berlin Heidelberg New York