First Order Approximations to Operational Risk – Dependence and Consequences

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Abstract

We investigate the problem of modelling and measuring multidimensional operational risk. Based on the very popular univariate loss distribution approach, we suggest an "invariance principle" which should be satisfied by any multidimensional operational risk model, and which is naturally fulfilled by our modelling technique based on the new concept of Pareto Lévy copulas. Our approach allows for a fully dynamic modelling of operational risk at any future point in time. We exploit the fact that operational loss data are typically heavy-tailed, and, therefore, we intensively discuss the concept of multivariate regular variation, which is considered as a very useful tool for various multivariate heavy-tailed phenomena. Moreover, for important examples of the Pareto Lévy copulas and appropriate severity distributions we derive first order approximations for multivariate operational Value-at-Risk.

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1. Introduction:

Three years ago, in Böcker and Klüppelberg (2005), we argued that operational risk could be a "long-term killer", and at this time maybe the most spectacular example for a bank failure caused by operational risk losses was Barings Bank after the rogue trader Nick Leeson had been hiding loss-making positions in financial derivatives. In the meanwhile other examples achieved doubtful fame, most recently Société Générale's loss of EUR 4.9 billion due to trader fraud and Bear Stearns near-death since it was not able to price its mortgage portfolios. Such examples clearly show the increased importance of a sound and reliable operational risk management, which consists of risk identification, monitoring and reporting, risk mitigation, risk controlling, and last but not least risk quantification. Needless to say, such catastrophic losses as mentioned above would have never been prevented just my measuring an operational Value-at-Risk (OpVaR). Often risk mitigation is primarily a matter of highly effective management and control processes. In the case of Société Générale, for instance, the question is how Jerome Kerviel was able to hide his massive speculative positions to the Dow Jones Eurostoxx 50 just by offsetting them with fictitious trades into the banking system.

Having said this, let us briefly − not only for a motivation to read this article − consider the question regarding the relevance of operational risk modelling. Maybe the simplest answer would be a reference to the regulatory requirements. Indeed, with the new framework of Basel II, the quantification of operational risk has become a conditio sine qua non for every financial institution. In this respect, the main intention for the so-called advanced measurement approaches (AMA) is to calculate a capital charge as a buffer against potential operational risk losses.

Another reason for building models, besides of making predictions, is that models can help us to gain a deeper understanding of a subject matter. This is one of our intentions in writing this article. We present a relatively simple model with only a few parameters, which allows us to gain interesting insight into the general behaviour of multivariate operational risk. Furthermore, our approach is appealing from a purely model-theoretic point of view because, as we will show in more detail below, it essentially is a straightforward generalization of the very popular loss distribution approach (LDA) to any dimension. The key feature of the one-dimensional, standard LDA model is splitting up the total loss amount over a certain time period into a frequency component, i.e. the number of losses, and a severity component, i.e. the individual loss amounts. In doing so, we assume the total aggregate operational loss of a bank up to a time horizon $t \geq 0$ to be represented by a compound Poisson process $\left(X_t^+\right)_{t\geq 0}$, which can be represented as

$$X_t^+ = \sum_{i=1}^{N_t^+} \Delta X_i^+, \quad t \geq 0. \qquad (1.1)$$

Let us denote the distribution function of X_t^+ by $G_t^+(\cdot) = P\left(X_t^+ \leq \cdot\right)$. As risk measure we use total OpVaR up to time t at confidence level κ, which is defined as the quantile

$$\mathrm{OpVaR}_t^+(\kappa) = G_t^{+\leftarrow}(\kappa) = \inf\left\{z \in \mathbb{R} : G_t^+(z) \geq \kappa\right\}, \quad \kappa \in (0,1), \qquad (1.2)$$

for κ near 1, e.g. 0.999 for regulatory purposes or, in the context of a bank's internal economic capital, even higher such as 0.9995.

Now, it is undoubted among experts and statistically justified by Moscadelli (2004) that operational loss data are heavy-tailed, and therefore we concentrate on Pareto-like severity distributions. In general, a severity distribution function F is said to be regularly varying with index $-\alpha$ for $\alpha > 0 \left(F \in \mathbb{R}_{-\alpha}\right)$, if

$$\lim_{t\to\infty} \frac{\overline{F}(xt)}{\overline{F}(t)} = x^{-\alpha}, \quad x > 0.$$

For such heavy tailed losses − and actually also for the more general class with subexponential distribution functions − it is now well-known and a consequence of Theorem 1.3.9 of Embrechts et. al (1997) (see Böcker and Klüppelberg (2005), Böcker and Sprittulla (2006)) that OpVaR at high confidence levels can be approximated by

$$\mathrm{OpVaR}_{\mathrm{t}}^{+}(\kappa) := G^{+\leftarrow}(\kappa) \sim F^{+\leftarrow}\left(1-\frac{1-\kappa}{\lambda^{+}}\right), \quad \kappa \uparrow 1 \qquad (1.3)$$

with $\lambda^{+} = E\left[N_1^{+}\right]$.

Figure 1 about here

Figure 1: Quality of approximation for a compound Poisson model with Pareto loss distribution.

Usually, however, total operational risk is not modelled by (1.1) directly, instead, operational risk is classified in different loss types and business lines. For instance, Basel II distinguishes 7 loss types and 8 business lines, yielding a matrix of 56 operational risk cells. Then, for each cell $i = 1,\ldots,d$ operational loss is modelled by a compound Poisson process model $(X_t^i)_{t\geq 0}$, and the bank's total aggregate operational loss is given as the sum

$$X_{\mathrm{t}}^{+} = X_t^1 + X_t^2 + \cdots + X_t^d, \qquad t \geq 0 .$$

The core problem of multivariate operational risk modelling here is, how to account for the dependence structure between the marginal processes. Several proposals have been made, see e.g. Frachot et al. (2004), Powojowski et al. (2002), or Chavez-Demoulin et al. (2005) just to mention a few. In general, however, all these approaches lead to a total aggregate loss process $\left(X_t^+\right)_{t\geq 0}$, which is not compound Poisson anymore and, thus, does not fit into the framework of (1.1). More generally, it is reasonable to demand that $\left(X_t^+\right)_{t\geq 0}$ does not depend on the design of the cell matrix, i.e whether the bank is using 56 or 20 cells within its operational risk model should in principal (i.e. abstracting from statistical estimation and data issues) not affect the bank's total OpVaR. In other words, a natural requirement of a multivariate operational risk model is that it is invariant under a re-design of the cell matrix and, thus, also under possible business re-organizations. Hence, we demand that every model should be closed with respect to the compound Poisson property, i.e. every additive composition of different cells must again constitute a univariate compound Poisson process with severity distribution function $F_{i+j}(\cdot)$ and frequency parameter λ_{i+j} for $i \neq j$:

$$X_t^i + X_t^j := X_t^{i+j} \quad \in \text{ compound Poisson processes}. \qquad (1.4)$$

The invariance principle formulated in (1.4) holds true, whenever the vector of all cell processes $\left(X_t^1,...,X_t^d\right)_{t\geq 0}$ constitutes a d-dimensional compound Poisson process. The dependence structure between the marginal processes is then described by a so-called Lévy copula, or, as we will do in this article, by means of a Pareto Lévy copula.

2. From Pareto copulas to Pareto Lévy copulas

Marginal transformations have been utilised in various fields. Certainly the most prominent in the financial area is the victory march of the copula, invoking marginal transformations resulting in a multivariate distribution function with uniform marginals. Whereas it is certainly convenient to automize certain procedures such as the normalisation to uniform marginals, this transformation is not always the best possible choice.

So, it was pointed out e.g. by Klüppelberg and Resnick (2008) that, when asymptotic limit distributions and heavy tail behaviour of data is to be investigated, a transformation to standardised Pareto distributed marginals is much more natural than the transformation to uniform marginals. The analog technique, however applied to the Lévy measure, will prove to be useful also for our purpose, namely the examination of multivariate operational risk. Before we do this in some detail, let us briefly recap some of the arguments given in Klüppelberg and Resnick (2008).

Let $\boldsymbol{X} = \left(X^1, ..., X^d\right)$ be a random vector in $\mathbb{R}^d$ with distribution function F and one-dimensional marginal distribution functions $F_i(\cdot) = P\left(X^i \le \cdot\right)$ and assume throughout that they are continuous. Define for $d \in \mathbb{N}$

$$(2.1)$$

with $\overline{F}_i(\cdot) = 1 - F_i(\cdot)$. Note that $\mathcal{P}^i$ is standard Pareto distributed; i.e., for $i = 1, ..., d$ holds

$$P\left(\mathcal{P}^i > x\right) = x^{-1}, \qquad x \ge 1.$$

Definition 2.1. Suppose $\boldsymbol{X}$ has d.f. F with continuous marginals. Define $\mathcal{P}$ as in (2.1). Then we call the distribution C of $\mathcal{P}$ a Pareto copula.

Analogously to the standard distributional copula, the Pareto copula can be used to describe the dependence structure between different random variables.

Here, we do not use distributional copulas directly to model the dependence structure between the cells' aggregate loss processes $(X_t^i)_{t \ge 0}$. One reason is that, as described in the introduction, we are looking for a natural extension of the single cell LDA model, i.e. we require that also $(X_t^+)_{t\geq 0}$ is compound Poisson. This can be achieved by exploiting the fact that a compound Poisson process is a specific Lévy process, which allows us to invoke some Lévy structure analysis to derive OpVaR results. Our approach is similar to Böcker and Klüppelberg (2006, 2008), where we used standard Lévy copulas to derive analytic approximations for OpVaR. In this article, however, we use a transformation similar to (2.1), which leads us to the concept of *Pareto Lévy copulas*.

For a Lévy process the jump behaviour is governed by the so-called Lévy measure Π, which has a very intuitive interpretation, in particular in the context of operational risk. The Lévy measure of a single operational risk cell measures the expected number of losses per unit time with a loss amount in a pre-specified interval. Moreover, for our compound Poisson model, the Lévy measure Π_i of the cell process X^i is completely determined by the frequency parameter $\lambda_i > 0$ and the distribution function of the cell's severity, namely $\Pi_i([0,x]) := \lambda_i P(\Delta X^i \leq x) = \lambda_i F_i(x)$ for $x \in [0,\infty)$. Since here we are mainly interested in large operational losses, it is convenient to introduce the concept of a *tail measure*, sometimes also referred to as *tail integral. A* one-dimensional tail measure is simply the expected number of losses per unit time that are above a given threshold, which is in the case of a compound Poisson model given by:

$$\overline{\Pi}_i(x) := \Pi_i([x,\infty)) = \lambda_i P(\Delta X^i > x) = \lambda_i \overline{F}_i(x), \qquad x \in [0,\infty) \tag{2.2}$$

In particular, there is only a finite number of jumps per unit time, i.e. $\lim_{x\downarrow 0} \overline{\Pi}_i(x) = \lambda_i$.

Analogously, for a multivariate Lévy process the multivariate Lévy measure controls the jump behaviour (per unit time) of all univariate components and contains all information of dependence between the components. Hence, in this framework, dependence modelling between different operational risk cells is reduced to choosing appropriate multivariate Lévy measures. Since jumps are created by positive loss severities, the Lévy measure Π is concentrated on the punctured positive cone in $\mathbb{R}^d$ (the value $\mathbf{0}$ is taken out since Lévy measures can have a singularity in $\mathbf{0}$)

$$\mathbb{E} := [\mathbf{0}, \infty] \setminus \{\mathbf{0}\}.$$

Now, similarly to the fact that a multivariate distribution can be built from marginal distributions via a distributional (Pareto) copula, a multivariate tail measure (see also Böcker and Klüppelberg (2006), Definition 2.1)

$$\overline{\Pi}(\boldsymbol{x}) = \overline{\Pi}(x_1, ..., x_d) = \Pi([x_1, \infty) \times \ldots \times [x_d, \infty)), \qquad \boldsymbol{x} \in \mathbb{E}, \tag{2.3}$$

can be constructed from the marginal tail measures (2.2) by means of a Pareto Lévy copula. The marginal tail measures are found from (2.3) as expected by

$$\overline{\Pi}_i(x) = \overline{\Pi}(0, ..., x_i, ..., 0), \qquad x \in [0, \infty).$$

Definition 2.2. Let $(\boldsymbol{X}_t)_{t \geq 0}$ be a Lévy process with Lévy measure Γ that has standard 1-stable one-dimensional marginals. Then we call Γ a Pareto Lévy measure and the associated tail measure

$$\overline{\Gamma}(\boldsymbol{x}) = \Gamma([x_1, \infty) \times \ldots \times [x_d, \infty)) =: \hat{C}(x_1, ..., x_d), \qquad \boldsymbol{x} \in \mathbb{E},$$

is referred to as Pareto Lévy copula $\widehat{C}$.

We now can transform the marginal Lévy measures of a Lévy process analogously to (2.1), yielding standard 1-stable marginal Lévy processes with Lévy measures $\overline{\Gamma}_i(x) = x^{-1}$ for $x > 0$. Note that the transformed 1-stable Lévy processes are NOT compound Poisson anymore (even though they may have been before the transformation), instead they are of infinite variation and have an infinite number of small jumps per unit time expressed by $\lim_{x \downarrow 0} \overline{\Gamma}_{\mathrm{i}}(x) = \infty$. For definitions and references of stable Lévy processes see Cont and Tankov (2004).

Lemma 2.3. Let $(\boldsymbol{X}_t)_{t \geq 0}$ be a spectrally positive Lévy process (i.e. a Lévy process admitting only positive jumps) with Lévy measure Π on $\mathbb{E}$ and continuous marginal tail measures $\overline{\Pi}_1, \ldots, \overline{\Pi}_d$. Then

$$\overline{\Pi}(\boldsymbol{x}) = \Pi([x_1, \infty] \times \cdots \times [x_d, \infty]) = \hat{C}\left(\frac{1}{\overline{\Pi}_1(x_1)}, \ldots, \frac{1}{\overline{\Pi}_d(x_d)}\right), \qquad \boldsymbol{x} \in \mathbb{E},$$

and $\hat{C}$ is a Pareto Lévy copula.

Proof. Note that for all $\boldsymbol{x} \in \mathbb{E}$,

$$\hat{C}(x_1, \ldots, x_d) = \overline{\Pi}\left(\left(\frac{1}{\overline{\Pi}_1}\right)^{\leftarrow}(x_1), \ldots, \left(\frac{1}{\overline{\Pi}_d}\right)^{\leftarrow}(x_d)\right),$$

this implies for the one-dimensional marginal tail measures

$$\hat{C}(0, \ldots, x, \ldots, 0) = \overline{\Pi}_i \circ \left(\frac{1}{\overline{\Pi}_i}\right)^{\leftarrow}(x) = \frac{1}{x}, \qquad x \in [0, \infty) \qquad \square$$

The following is Sklar's Theorem for spectrally positive Lévy processes in the context of Lévy Pareto copulas. The proof is similar to the one of Theorem 5.6 of Cont and Tankov (2004).

Theorem 2.4 (Sklar's Theorem for Pareto Lévy copulas)

Let $\overline{\Pi}$ be the tail measure of a d-dimensional spectrally positive Lévy process with marginal tail measures $\overline{\Pi}_1,...,\overline{\Pi}_d$. Then there exists a Pareto Lévy copula $\hat{C}:\mathbb{E}\to[0,\infty]$ such that for all $x_1,...,x_d \in \mathbb{E}$

$$\overline{\Pi}(x_1,...,x_d) = \hat{C}\left(\frac{1}{\overline{\Pi}_1(x_1)},...,\frac{1}{\overline{\Pi}_d(x_d)}\right). \qquad (2.4)$$

If the marginal tail measures are continuous on $[0,\infty]$, then $\hat{C}$ is unique. Otherwise, it is unique on $\operatorname{Ran}\left(\frac{1}{\overline{\Pi}_1}\right)\times\cdots\times\operatorname{Ran}\left(\frac{1}{\overline{\Pi}_d}\right)$. Conversely, if $\hat{C}$ is a Pareto Lévy-copula and $\overline{\Pi}_1,...,\overline{\Pi}_d$ are marginal tail measures, then $\overline{\Pi}$ defined in (2.4) is a joint tail measure with marginals $\overline{\Pi}_1,...,\overline{\Pi}_d$.

So-called Lévy copulas, as introduced in Cont and Tankov (2004) and also used in Böcker and Klüppelberg (2006, 2008), have one-dimensional marginal Lebesgue measures. As a consequence thereof, they do not have an interpretation as the Lévy measure of a one-dimensional Lévy process, because a Lévy measure is, for instance, finite on $[1,\infty)$.

From the construction above it is also clear that, if $\tilde{C}(x_1,\ldots,x_d)$ is a Lévy copula, then the associated Pareto Lévy copula $\hat{C}$ can be constructed by $\hat{C}(x_1,\ldots,x_d)=\tilde{C}(1/x_1,\ldots,1/x_d)$. Hence, the following examples follow immediately from those given in Böcker and Klüppelberg (2006):

Example 2.5. [Independence Pareto Lévy-copula].

Let $\boldsymbol{X}_t=\left(X_t^1,\ldots,X_t^d\right)$, $t\geq 0$, be a spectrally positive Lévy process with marginal tail measures $\overline{\Pi}_1,\ldots,\overline{\Pi}_d$. The components of $(\boldsymbol{X}_t)_{t\geq 0}$ are independent if and only if

$$\Pi(A) = \sum_{i=1}^{d} \Pi_i(A_i) \qquad A \in \mathfrak{B}(\mathbb{E}),$$

where $A_i = \{x \in \mathbb{R} : (0,...,0,x,0,...,0) \in A\}$, x stands at the i-th component, and $\mathfrak{B}(\mathbb{E})$ denotes the Borel sets of $\mathbb{E}$. This implies for the tail measure of $(\boldsymbol{X}_t)_{t\geq 0}$

$$\overline{\Pi}(x_1,\dots,x_d) = \overline{\Pi}_1(x_1)\, I_{\{x_2=\dots=x_d=0\}} + \cdots + \overline{\Pi}_d(x_d)\, I_{\{x_1=\dots=x_{d-1}=0\}},$$

giving a Pareto Lévy copula of

$$\hat{C}_{\perp}(\boldsymbol{x}) = x_1^{-1} I_{\{x_2=\dots=x_d=0\}} + \cdots + x_d^{-1} I_{\{x_1=\dots=x_{d-1}=0\}}.$$

The resulting Lévy process with Pareto Lévy copula $\hat{C}_{\perp}$ is a standard 1-stable process with independent components.

Example 2.6. [Complete (positive) dependence Pareto Lévy copula].

Let $\boldsymbol{X}_t = (X_t^1,...,X_t^d)$, $t \geq 0$, be a spectrally positive Lévy process with Lévy measure Π, which is concentrated on an increasing subset of $\mathbb{E}$. Then

$$\overline{\Pi}(\boldsymbol{x}) = \min\left(\overline{\Pi}^1(x_1),...,\overline{\Pi}^d(x_d)\right).$$

The corresponding Lévy copula is given by

$$\hat{C}_{\|}(\boldsymbol{x}) = \min\left(x_1^{-1},...,x_d^{-1}\right).$$

Example 2.7. [Archimedian Pareto Lévy copula].

Let $\phi : [0,\infty] \to [0,\infty]$ be strictly decreasing with $\phi(0) = \infty$ and $\phi(\infty) = 0$. Assume that $\phi^{\leftarrow}$ has derivatives up to order d with $(-1)^k \dfrac{d^k \phi^{\leftarrow}(t)}{dt^k} > 0$ for $k = 1,...,d$. Then the following is a Pareto Lévy copula

$$\hat{C}(\boldsymbol{x}) = \phi^{\leftarrow}\left(\phi\left(x_1^{-1}\right) + ... + \phi\left(x_d^{-1}\right)\right).$$

Example 2.8. [Clayton Pareto Lévy copula].

Take $\phi(t) = t^{-\theta}$ for $\theta > 0$. Then the Archimedean Pareto Lévy copula

$$\hat{C}_\theta(\boldsymbol{x}) = \left(x_1^\theta + ... + x_d^\theta\right)^{-1/\theta}$$

is called Clayton Pareto Lévy copula. Note that $\lim_{\theta\to\infty} \hat{C}_\theta(\boldsymbol{x}) = \hat{C}_{\parallel}(\boldsymbol{x})$ and $\lim_{\theta\to 0} \hat{C}_\theta(\boldsymbol{x}) = \hat{C}_{\perp}(\boldsymbol{x})$; i.e., this model covers the whole range of dependence.

3. Understanding the dependence structure

Recall our multivariate operational risk model, in which total aggregate loss is modelled by a compound Poisson process with representation (1.1), where $\left(N_t^+\right)_{t\ge 0}$ is the Poisson process counting the total number of losses and $\Delta X_1^+, ..., \Delta X_{N_t^+}^+$ denote all severities in the time interval $(0,t]$. In this model losses can occur either in one of the component processes or result from multiple simultaneous losses in different components. In the latter situation, ΔX_i^+ is the sum of all losses, which happen at the same time.

Based on a decomposition of the marginal Lévy measures, one can show (see e.g. Böcker and Klüppelberg (2008), Section 3) that the component processes can be decomposed into compound Poisson processes of single jumps and joint jumps. For $d = 2$ the cell's loss processes have the representation (the time parameter t is dropped for simplicity):

$$X^1 = X_\perp^1 + X_\parallel^1 = \sum_{k=1}^{N_\perp^1} \Delta X_{\perp k}^1 + \sum_{l=1}^{N_\parallel} \Delta X_{\parallel l}^1 \qquad (3.1)$$

$$X^2 = X_\perp^2 + X_\parallel^1 = \sum_{m=1}^{N_\perp^2} \Delta X_{\perp m}^2 + \sum_{l=1}^{N_\parallel} \Delta X_{\parallel l}^2$$

where $X_{\|1}$ and $X_{\|2}$ describe the aggregate losses of cell 1 and 2, respectively, that are generated by "common shocks", and $X_{\perp}^{1}$ and $X_{\perp}^{2}$ are independent loss processes. Note that apart from $X_{\|1}$ and $X_{\|2}$ all compound Poisson processes on the right-hand side of (3.1) are mutually independent.

If we compare the left-hand and right-hand representation we can identify the parameters of the processes on the right-hand side. The parameters on the left-hand side are $\lambda_{1,}\lambda_2 > 0$ for the frequencies of the Poisson processes, which count the number of losses, and the severity distribution functions F_1, F_2; for details we refer to Böcker and Klüppelberg (2008).

Figure 2 about here

Figure 2: Decomposition of the domain of the tail measure $\overline{\Pi}^{+}(z)$ for $z = 6$ into a simultaneous loss part $\overline{\Pi}_{\|}(z)$ (grey) and independent parts $\overline{\Pi}_{\perp 1}(z)$ (solid black line) and $\overline{\Pi}_{\perp 2}(z)$ (dashed black line).

The frequency of simultaneous losses can be calculated from the bivariate tail measure $\overline{\Pi}(x_1, x_2)$ as the limit of arbitrarily small, simultaneous losses; i.e.

$$\lim_{x_1, x_2 \downarrow 0} \overline{\Pi}(x_1, x_2) = \hat{C}\left(\lambda_1^{-1}, \lambda_2^{-1}\right) = \lim_{x \downarrow 0} \overline{\Pi}_{\|2}(x) = \lim_{x \downarrow 0} \overline{\Pi}_{\|1}(x) = \lambda_{\|} \in \left[0, \min\left(\lambda_1, \lambda_2\right)\right].$$

Consequently, the frequency of independent losses must be

$$\lambda_{\perp 1} = \lim_{x \downarrow 0} \overline{\Pi}_{\perp 1}(x) = \lambda_1 \text{-} \lambda_{\|} \qquad \text{and} \quad \lambda_{\perp 2} = \lambda_2 - \lambda_{\|} .$$

By comparison of the Lévy measures we obtain further for the distribution functions of the independent losses

$$\overline{F}_{\perp 1}(x_1) = \frac{\lambda_1}{\lambda_{1\perp}} \overline{F}_1(x_1) - \frac{1}{\lambda_{1\perp}} \hat{C}\left(\left(\lambda_1 \overline{F}_1(x_1) \right)^{-1}, \lambda_2^{-1} \right)$$

$$\overline{F}_{\perp 2}(x_2) = \frac{\lambda_2}{\lambda_{2\perp}} \overline{F}_2(x_2) - \frac{1}{\lambda_{2\perp}} \hat{C}\left(\lambda_1^{-1}, \left(\lambda_2 \overline{F}_2(x_2) \right)^{-1} \right).$$

And, finally, the joint distribution functions of coincident losses and their marginals are given by

$$\overline{F}_{\|}(x_1, x_2) = P\left(X_{\|}^1 > x_1, X_{\|}^2 > x_2 \right) = \frac{1}{\lambda_{\|}} \hat{C}\left(\left(\lambda_1 \overline{F}_1(x_1) \right)^{-1}, \left(\lambda_2 \overline{F}_2(x_2) \right)^{-1} \right) \qquad (3.2)$$

$$\overline{F}_{\|1}(x_1) = \lim_{x_2 \downarrow 0} \overline{F}_{\|}(x_1, x_2) = \frac{1}{\lambda_{\|}} \hat{C}\left(\left(\lambda_1 \overline{F}_1(x_1) \right)^{-1}, \lambda_2^{-1} \right)$$

$$\overline{F}_{\|2}(x_2) = \lim_{x_1 \downarrow 0} \overline{F}_{\|}(x_1, x_2) = \frac{1}{\lambda_{\|}} \hat{C}\left(\lambda_1^{-1}, \left(\lambda_2 \overline{F}_2(x_2) \right)^{-1} \right).$$

Summarising our results, we can say that the Pareto Lévy copula approach is equivalent to a split of the cells' compound Poisson processes into completely dependent and independent parts. All parameters of these subprocesses can be derived from the Pareto Lévy copula, which we have shown here for the distribution functions and frequencies of the dependent and independent parts. Moreover, we would like to mention that the simultaneous loss distributions $F_{\|1}(\cdot)$ and $F_{\|2}(\cdot)$ are not independent, instead they are linked by a distributional copula, which can be derived from (3.2), see again Böcker and Klüppelberg (2008) for more details.

4. Approximating total OpVaR

In our model, OpVaR encompassing all risk cells is given by (1.2), which can asymptotically be approximated by (1.2). Needless to say, the parameters $\overline{F}^{+}$ and λ^{+}, which describe the bank's total OpVaR at aggregated level, depend on the dependence structure between different risk cells and thus on the Pareto Lévy copula. We now investigate various dependence scenarios, for which first order approximations like (1.3) are available. Our results yield valuable insight into the nature of multivariate operational risk.

One dominating cell severity

Although the first scenario is rather simple by assuming that high-severity losses mainly occur in one single risk cell, it is yet relevant in many practical situations. Note that the assumptions in the following result are very weak, no process structure is needed here.

Theorem 4.1 (Böcker and Klüppelberg (2006), Theorem 3.4, Corollary 3.5)

For fixed $t>0$ let X_t^i for $i=1,\dots,d$ have compound Poisson distributions. Assume that $\overline{F}_1 \in \mathcal{R}_{-\alpha}$ for $\alpha>0$. Let $\rho>\alpha$ and suppose that $E\left[\left(\Delta X^i\right)^{\rho}\right]<\infty$ for $i=2,\dots,d$. Then, regardless of the dependence structure between $X_t^1,\dots,X_t^d$,

$$P\left(X_t^+ > x\right) \sim EN_t^1 P\left(\Delta X^1 > x\right), \quad x\to\infty,$$

$$\mathrm{VaR}_t^+(\kappa) \sim F_1^{\leftarrow}\left(1-\frac{1-\kappa}{EN_t^1}\right) = \mathrm{OpVaR}_t^1(\kappa), \quad \kappa\uparrow 1.$$

This is a quite important result. It means that total operational risk measured at high confidence levels is dominated by the stand-alone OpVaR of that cell, where losses have

Pareto tails that are heavier than losses of other cells. Note that the assumptions of this theorem are also satisfied if the loss severity distribution is a mixture distribution, in which only the tail is explicitly assumed to be Pareto-like, whereas the body is modelled by any arbitrary distribution class. We have elaborated this in more detail in Böcker and Klüppelberg (2008), Section 5.

Multivariate compound Poisson model with completely dependent cells

Complete dependence for Lévy processes means that all cell processes jump together, i.e. losses always occur at the same time, necessarily implying that all frequencies must be equal, i.e. $\lambda := \lambda_1 = \cdots = \lambda_d$. It also implies that the mass of the multivariate Lévy measure Π is concentrated on

$$\left\{(x_1,\ldots,x_d) \in \mathbb{E} : \overline{\Pi}_1(x_1) = \cdots = \overline{\Pi}_d(x_d)\right\} = \left\{(x_1,\ldots,x_d) \in \mathbb{E} : F_1(x_1) = \cdots = F_d(x_d)\right\}$$

Let F_i be strictly increasing and continuous such that $F_i^{-1}(q)$ exists for all $q \in [0,1)$. Then

$$\begin{aligned}\Pi^+(z) \quad &= \Pi(\{(x_1,\ldots,x_d) \in \mathbb{E} : \sum_{i=1}^{d} x_i \geq z\}) \\ &= \Pi_1(\{x_1 \in (0,\infty) : x_1 + \sum_{i=2}^{d} F_i^{-1}\left(F_1(x_1)\right) \geq z\}), \qquad z > 0.\end{aligned}$$

This representation yields the following result.

Theorem 4.2 (Böcker and Klüppelberg (2006), Theorem 3.6)

Consider a multivariate compound Poisson process $\boldsymbol{X}_t = \left(X_t^1,\ldots,X_t^d\right), t \geq 0$, with completely dependent cell processes and strictly increasing and continuous severity distributions F_i. Then $\left(X_t^+\right)_{t\geq 0}$ is compound Poisson process with parameters

$$\lambda^+ = \lambda \;\text{ and }\; \overline{F}^+(z) = \overline{F}_1\left(H^{-1}(z)\right), \qquad z > 0,$$

where $H(z) := z + \sum_{i=2}^{d} F_i^{-1}(F_1(z))$. If $F^+ \in \mathcal{R}_{-\alpha}$ for $\alpha \in (0,\infty)$, then

$$\mathrm{OpVaR}_t^+(\kappa) \sim \sum_{i=1}^{d} \mathrm{OpVaR}_t^i(\kappa), \quad \kappa \uparrow 1,$$

Where $\mathrm{OpVaR}_t^i(\cdot)$ denotes the stand alone OpVaR of cell i.

Corollary 4.3. Assume that the conditions of Theorem 4.2 hold and that $F_1 \in \mathcal{R}_{-\alpha}$ for $\alpha \in (0,\infty)$ and

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = c_i \in [0,\infty).$$

Assume further that $c_i \neq 0$ for $i = 1,\ldots,b \leq d$ and $c_i = 0$ for $i = b+1,\ldots,d$. Then

$$\mathrm{OpVaR}_t^+(\kappa) \sim \sum_{i=1}^{b} c_i^{1/\alpha} \mathrm{OpVaR}_t^1(\kappa), \quad \kappa \uparrow 1$$

Note how the result of Theorem 4.2 resembles the proposals of the Basel Committee on Banking Supervision (2006), in which a bank's total capital charge for operational risk measured as OpVaR at confidence level of 99.9 % is usually the sum of the risk capital charges attributed to the different risk type/business line cells. Hence, following our model, regulators implicitly assume that losses in different categories always occur simultaneously.

Multivariate compound Poisson model with independent cells

The other extreme case we want to consider is independence between different cells. For a general Lévy process independence means that not two cell processes ever jump together. Consequently, the mass of the Lévy measure is concentrated on the axes, cf. Example 2.5, so that we have

$$\overline{\Pi}^{+}(z) = \overline{\Pi}_1(z_1) + \cdots + \overline{\Pi}_d(z_d) \qquad z \geq 0.$$

Theorem 4.4. Assume that $\boldsymbol{X}_t = \left(X_t^1, \ldots, X_t^d\right), t \geq 0$, has independent cell processes. Then $\left(X_t^{+}\right)_{t\geq 0}$ is a compound Poisson process with parameters

$$\lambda^{+} = \lambda_1 + \cdots + \lambda_d \quad \text{and} \quad \overline{F}^{+}(z) = \frac{1}{\lambda^{+}}\left[\lambda_1 \overline{F}_1(z) + \cdots + \lambda_d \overline{F}_d(z)\right], \quad z \geq 0.$$

If $F_1 \in \mathcal{R}_{-\alpha}$ for $\alpha \in (0,\infty)$ and for all $i = 2,\ldots,d$,

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = c_i \in [0,\infty) ,$$

then

$$\text{OpVaR}_t^{+}(\kappa) \sim F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{(\lambda_1 + c_2\lambda_2 + \cdots + c_d\lambda_d)\,t}\right), \qquad \kappa \uparrow 1. \qquad (4.1)$$

If we compare (4.1) to the formula for the single-cell OpVaR (1.3), we see that multivariate OpVaR in the case of independent cells can be expressed by the stand-alone OpVaR of the first cell with adjusted frequency parameter $\lambda := \lambda_1 + c_2\lambda_2 + \cdots + c_d\lambda_d$. Actually, we will see in the next section that this is possible for much more general dependence structures, namely those belonging to the class of multivariate regular variation.

Multivariate regularly varying Lévy measure

Multivariate regular variation is an appropriate mathematical tool for discussing heavy tail phenomena as they occur for instance in operational risk. We begin with regular variation of random vectors or, equivalently, of multivariate distribution functions.

The idea is to have regular variation not only in some (or all) marginals, but along every ray starting in $\mathbf{0}$ and going through the positive cone to infinity. Clearly, this limits the possible dependence structures between the marginals, however, such models are still flexible enough to be broadly applied to various fields such as telecommunication, insurance, and last but not least VaR analysis in the banking industry. Furthermore, many of the dependence models implying multivariate regular variation can still be solved and analysed analytically.

Let us consider a positive random vector $\boldsymbol{X}$ with distribution function F that is − as our Lévy measure Π − concentrated on $\mathbb{E}$. Moreover, we introduce for $\boldsymbol{x} \in \mathbb{E}$ the following sets (for any Borel set $A \subset \mathbb{E}$ its complement in $\mathbb{E}$ is denoted by A^c):

$$[\mathbf{0}, \boldsymbol{x}]^c = \mathbb{E} \setminus [\mathbf{0}, \boldsymbol{x}] = \{\boldsymbol{y} \in \mathbb{E} : \max_{1 \le i \le d} \frac{y_i}{x_i} > 1\}.$$

Assume there exists a Radon measure ν on $\mathbb{E}$ (i.e. a Borel measure that is finite on compact sets) such that

$$\lim_{t \to \infty} \frac{1 - F(t\boldsymbol{x})}{1 - F(t\mathbf{1})} = \lim_{t \to \infty} \frac{P\left(t^{-1}\boldsymbol{X} \in [\mathbf{0}, \boldsymbol{x}]^c\right)}{P\left(t^{-1}\boldsymbol{X} \in [\mathbf{0}, \mathbf{1}]^c\right)} = \nu\left([\mathbf{0}, \boldsymbol{x}]^c\right) \tag{4.2}$$

holds for all $\boldsymbol{x} \in \mathbb{E}$, which are continuity points of the function $\nu\left([\mathbf{0}, \cdot]^c\right)$. One can show that the above definition (4.2) implies that ν has a homogeneity property; i.e. there exists some $\alpha > 0$ such that

$$\nu\left([\mathbf{0}, s\boldsymbol{x}]^c\right) = s^{-\alpha}\nu\left([\mathbf{0}, \boldsymbol{x}]^c\right), \quad s > 0, \tag{4.3}$$

and we say that F is multivariate regularly varying with index $-\alpha$ $\left(F \in \mathcal{R}_{-\alpha}\right)$. Condition (4.2) also says that $F(t\mathbf{1})$ as a function of t is in $\mathcal{R}_{-\alpha}$. Define now $b(t)$ to satisfy $\overline{F}\left(b(t)\mathbf{1}\right) \sim t^{-1}$ as $t \to \infty$. Then, replacing t by $b(t)$ in (4.2) yields

$$\lim_{t\to\infty} tP\left(\frac{\boldsymbol{X}}{b(t)} \in [\mathbf{0}, \boldsymbol{x}]^c\right) = \nu\left([\mathbf{0}, \boldsymbol{x}]^c\right). \tag{4.4}$$

In (4.4) the random variable $\boldsymbol{X}$ is normalised by the function $b(\cdot)$. As explained in Resnick (2007), Section 6.5.6, normalisation of all components by the same function $b(\cdot)$ implies that the marginal tails of $\boldsymbol{X}$ satisfy for $i, j \in \{1, \ldots, d\}$

$$\lim_{x\to\infty} \frac{\bar{F}_i(x)}{\bar{F}_j(x)} = \frac{c_i}{c_j},$$

where $c_i, c_j \in [0, \infty)$. Assuming $c_1 > 0$ we set w.l.o.g. $c_1 = 1$. Then we can also choose $b(t)$ such that for $t \to \infty$

$$\bar{F}_1(b(t)) \sim t^{-1} \Leftrightarrow \quad b(t) \sim \left(\frac{1}{\bar{F}_1}\right)^{\leftarrow}(t) \tag{4.5}$$

and, by substituting in (4.4), we obtain a limit on the left-hand side of (4.4) with the same scaling structure as before.

To formulate analogous definitions for Lévy measures note first that we can rewrite (4.2) by means of the distribution of $\boldsymbol{X}$ as:

$$\lim_{t\to\infty} \frac{P_{\boldsymbol{X}}\left(t[\mathbf{0}, \boldsymbol{x}]^c\right)}{P_{\boldsymbol{X}}\left(t[\mathbf{0}, \mathbf{1}]^c\right)} = \nu\left([\mathbf{0}, \boldsymbol{x}]^c\right).$$

and, similarly, (4.4) as

$$\lim_{t\to\infty} tP_{\boldsymbol{X}}\left(b(t)[\mathbf{0}, \boldsymbol{x}]^c\right) = \lim_{t\to\infty} tP_{\boldsymbol{X}}\left(\left[\mathbf{0}, b(t)\boldsymbol{x}\right]^c\right) = \nu\left([\mathbf{0}, \boldsymbol{x}]^c\right) \tag{4.6}$$

Then, the analogue expression to (4.2) for a Lévy measure Π is simply

$$\lim_{t\to\infty}\frac{\Pi\left(t[\mathbf{0},\boldsymbol{x}]^{c}\right)}{\Pi\left(t[\mathbf{0},\mathbf{1}]^{c}\right)}=\lim_{t\to\infty}\frac{\Pi\left(\{\boldsymbol{y}\in\mathbb{E}:y_1>tx_1\text{ or }\cdots\text{ or }y_d>tx_d\}\right)}{\Pi\left(\{\boldsymbol{y}\in\mathbb{E}:y_1>t\text{ or }\cdots\text{ or }y_d>t\}\right)}=\nu\left([\mathbf{0},\boldsymbol{x}]^{c}\right) \tag{4.7}$$

for all $\boldsymbol{x}\in\mathbb{E}$, which are continuity points of the function $\nu\left([\mathbf{0},\cdot]^{c}\right)$. Summarising what we have so far yields the following definition of multivariate regular variation for Lévy measures, now formulated in analogy to (4.4) or (4.6), respectively:

Definition 4.5. [Multivariate regular variation for spectrally positive Lévy processes]
Let Π be a Lévy measure of a spectrally positive Lévy process on $\mathbb{E}$. Assume that there exists a function $b:(0,\infty)\to(0,\infty)$ satisfying $b(t)\to\infty$ as $t\to\infty$ and a Radon measure ν on $\mathbb{E}$ such that

$$\lim_{t\to\infty} t\Pi\left(\left[\mathbf{0},b(t)\boldsymbol{x}\right]^{c}\right)=\nu\left([\mathbf{0},\boldsymbol{x}]^{c}\right) \tag{4.8}$$

holds for all $\boldsymbol{x}\in\mathbb{E}$ which are continuity points of the function $\nu\left([\mathbf{0},\cdot]^{c}\right)$. Then ν satisfies the homogeneity property

$$\nu\left([\mathbf{0},s\boldsymbol{x}]^{c}\right)=s^{-\alpha}\nu\left([\mathbf{0},\boldsymbol{x}]^{c}\right),\quad s>0$$

for some $\alpha>0$ and the Lévy measure Π is called multivariate regularly varying with index $-\alpha$ $\left(\Pi\in\mathcal{R}_{-\alpha}\right)$.

As before in the case of a regularly varying random vector $\boldsymbol{X}$, we assume that in (4.8) we can choose one single scaling function $b(\cdot)$, which applies to all marginal tail measures. In analogy to (4.5) we can therefore set

$$\bar{\Pi}_1\left(b(t)\right)\sim t^{-1}\quad\Leftrightarrow\quad b(t)\sim\left(\frac{1}{\bar{\Pi}_1}\right)^{\leftarrow}(t),\qquad t\to\infty\,. \tag{4.9}$$

As explained above, normalisation of all components by the same function $b(\cdot)$ implies that the marginal tail measures satisfy for $i, j \in \{1,\ldots,d\}$

$$\lim_{x\to\infty}\frac{\overline{\Pi}_i(x)}{\overline{\Pi}_j(x)}=\frac{c_i}{c_j}, \qquad c_i, c_j \in [0,\infty) \ . \tag{4.10}$$

As we have already said, multivariate regular variation is just a special way of describing dependence between multivariate heavy-tailed measures. Therefore it is natural to ask, under which conditions a given Pareto Lévy copula is in line with multivariate regular variation. Starting with a multivariate tail measure $\overline{\Pi}$, we know from Lemma 2.3 that we can derive its Pareto Lévy copula $\hat{C}$ by normalising the marginal Lévy measures to standard 1-stable marginal Lévy processes, i.e.

$$\overline{\Pi}_i(x) \to \ \overline{\Gamma}_i(x)=\overline{\Pi}_i \circ \Big(\frac{1}{\overline{\Pi}_i}\Big)^{\leftarrow}(x)=\frac{1}{x}, \quad x\in[0,\infty) \ . \tag{4.11}$$

We now consider the question under which conditions the resulting multivariate tail measure $\overline{\Gamma}$ of a standardised 1-stable Lévy process corresponds to a multivariate regularly varying Lévy measure Γ (automatically with index -1).

Example 4.6. [Pareto Lévy copula and multivariate regular variation].

Let $\boldsymbol{X}_t=(X_t^1, X_t^2), t\geq 0$, be a spectrally positive Lévy process with Lévy measure Π on $[0,\infty)^d$. Transforming the marginal Lévy measure by $(1/\overline{\Pi}_i)^{\leftarrow}(x)$, we obtain the Pareto Lévy copula $\hat{C}(x_1,x_2)=\Gamma([x_1,\infty)\times[x_2,\infty))$ of $(\boldsymbol{X}_t)_{t\geq 0}$.

From (4.11) we know that, if Γ is regularly varying, then with index -1, and, therefore, we set $b(t)=t$. Obviously, for $b(t)=t$ we are in the standard case and all marginals are standard Pareto distributed with $\alpha=1$. Then, because in general we have

$$\Pi\left(\left[\mathbf{0},(x_1,x_2)^\top\right]^c\right)=\bar{\Pi}_1(x_1)+\bar{\Pi}_2(x_2)-\bar{\Pi}(x_1,x_2), \qquad (x_1,x_2)\in\mathbb{E}, \qquad (4.12)$$

we immediately get for the left-hand side of (4.8) for the transformed Lévy measure Γ

$$t\Gamma\left(\left[\mathbf{0},(tx_1,tx_2)^\top\right]^c\right)=\frac{1}{x_1}+\frac{1}{x_2}-t\hat{C}(tx_1,tx_2).$$

For bivariate regular variation with index -1, the right-hand side above must converge for $t\to\infty$ to a Radon measure ν on $\mathbb{E}$, more precisely,

$$\frac{1}{x_1}+\frac{1}{x_2}-t\hat{C}(tx_1,tx_2)\to\nu\left(\left[\mathbf{0},(x_1,x_2)^\top\right]^c\right)$$

with

$$\nu\left(\left[\mathbf{0},(sx_1,sx_2)^\top\right]^c\right)=s^{-1}\nu\left(\left[\mathbf{0},(x_1,x_2)^\top\right]^c\right), \quad s>0 .$$

This is clearly the case if the Lévy Pareto copula $\hat{C}$ is homogenous of order -1 and, more generally, if there is a Radon measure μ such that

$$\lim_{t\to\infty} t\hat{C}(tx_1,tx_2)=\mu\left(\left[\mathbf{0},(x_1,x_2)^\top\right]^c\right), \quad (x_1,x_2)\in\mathbb{E}.$$

A more general result is the following, which links multivariate regular variation to the dependence concept of a Pareto Lévy copula.

Theorem 4.7 (Böcker and Klüppelberg (2006), Theorem 3.16)

Let $\bar{\Pi}$ be a multivariate tail measure of a spectrally positive Lévy process on $\mathbb{E}$.Assume that the marginal tail measures $\bar{\Pi}_i$ are regularly varying with index $-\alpha$ for some $\alpha > 0$. Then the following holds.

(1) The Lévy measure Π is multivariate regularly varying with index $-\alpha$ if and only if the standardised Lévy measure Γ is regularly varying with index -1.

(2) If the Pareto Lévy copula $\hat{C}$ is homogeneous of order -1 and $0 < \alpha < 2$, then $\hat{C}$ is the Lévy measure of a multivariate α-stable process.

Example 4.8 [Visualisation of the Clayton Pareto Lévy copula].

Recall the Clayton Lévy copula $\hat{C}(x_1, x_2) = \left(x_1^\theta + x_2^\theta\right)^{-1/\theta}$ for $x_1, x_2 > 0$. From Definition 2.2 we know that

$$\hat{C}(x_1, x_2) = \Gamma\left([x_1, \infty) \times [x_2, \infty)\right), \quad (x_1, x_2) \in \mathbb{E},$$

and the corresponding marginal processes have been standardised to be 1-stable Lévy processes. Since $\hat{C}$ is homogeneous of order -1, we know from Theorem 4.7 that the bivariate Lévy process is a 1-stable process. We can also conclude that, if the marginal Lévy tail measures $\bar{\Pi}_1$ and $\bar{\Pi}_2$ before standardising the marginals were regularly varying with some index $-\alpha$, then the Lévy measure Π was bivariate regularly varying tail with index $-\alpha$.

Note also that

$$\Gamma\left([\mathbf{0}, \boldsymbol{x}]^c\right) = \bar{\Gamma}_1(x_1) + \bar{\Gamma}_2(x_2) - \hat{C}(x_1, x_2) = \frac{1}{x_1} + \frac{1}{x_2} - \left(\left(\frac{1}{x_1}\right)^{-\theta} + \left(\frac{1}{x_2}\right)^{-\theta}\right)^{-1/\theta}.$$

The homogeneity can be used as follows to allow for some visualisation of the dependence. We transform to polar coordinates by setting $x_1 = r\cos\varphi$ and $x_2 = r\sin\varphi$ for $r = |x| = \sqrt{x_2^1 + x_2^2}$ and $\varphi \in [0, \pi/2]$. From the homogeneity property it follows

$$\Gamma\left([\mathbf{0}, \boldsymbol{x}]^c\right) = r^{-1}\Gamma\left(\left[\mathbf{0}, (\cos\varphi, \sin\varphi)^\top\right]^c\right) =: \Gamma(r,\varphi).$$

This is depicted in Figure 3 where $\Gamma(r,\varphi)$ is plotted for $r=1$ as a function of φ, and thus the Clayton dependence structure is plotted as a measure on the quatercircle.

Figure 3a and 3b about here

Figure 3: Plot of the Pareto Lévy copula in polar coordinates $\hat{C}(r,\varphi) = \overline{\Gamma}(r,\varphi)$ as a function of the angle $\varphi \in (0, \pi/2)$ for r=1 and different values for the dependence parameter.

Left Plot: $\theta = 1{,}8$ (dotted line), $\theta = 0{,}7$ (dashed line), $\theta = 0{,}3$ (solid line).

Right Plot: $\theta = 2{,}5$ (solid line), $\theta = 5$ (dashed line), $\theta = 10$ (dotted line), and $\theta = \infty$ (complete positive dependence, long-dashed line).

It is worth mentioning that all we have said so far about multivariate regular variation of Lévy measures holds true for general spectrally positive Lévy processes. We now turn back again to the problem of calculating total OpVaR and consider a multivariate compound Poisson process, whose Lévy measure Π is multivariate regularly varying according to (4.8). In particular, this implies tail equivalence of the marginal Lévy measures, and we can write (4.10) with some $\tilde{c}_i \in (0,\infty)$ as

$$\tilde{c}_i := \lim_{x\to\infty} \frac{\overline{\Pi}_i(x)}{\overline{\Pi}_1(x)} = \frac{\lambda_i \overline{F}_i(x)}{\lambda_1 \overline{F}_1(x)} =: \frac{\lambda_i}{\lambda_1} c_i \qquad (4.13)$$

i.e. $\lim_{x\to\infty} \overline{F}_i(x)/\overline{F}_1(x) = c_i$. We avoid, situations, where for some i we have $\tilde{c}_i = 0$, corresponding to cases in which for $x \to \infty$ the tail measure $\overline{\Pi}_i(x)$ decays faster than $x^{-\alpha}$, i.e. in (4.13) we only consider the heaviest tail measures, all of tail index $-\alpha$. This makes sense, because we know from our discussion at the beginning of this section that only the heaviest-tailed risk cells contribute to total OpVaR, see Theorem 4.1.

When calculating total aggregated OpVaR, we are interested in the sum of these tail equivalent, regularly varying marginals, i.e. we have to calculate the tail measure

$$\overline{\Pi}^+(z) = \Pi\left(\left\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x_i > z\right\}\right), \quad z > 0.$$

Analogously to Resnick (2007), Proposition 7.3, p. 227, the tail measure $\overline{\Pi}^+$ is also regularly varying with index $-\alpha$, more precisely we have

$$\lim_{t\to\infty} t\overline{\Pi}^+(b(t)z) = \nu\left\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x_i > z\right\} = z^{-\alpha}\,\nu\left\{\boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x_i > 1\right\} =: z^{-\alpha}\,\nu^+(1,\infty] \,. \qquad (4.14)$$

Now, let us choose the scaling function $b(t)$ so that $\overline{\Pi}_1(b(t)) \sim t^{-1}$. Then we have

$$\lim_{z\to\infty} \frac{\overline{\Pi}^+(z)}{\overline{\Pi}_1(z)} = \lim_{t\to\infty} \frac{t\overline{\Pi}^+(b(t))}{t\overline{\Pi}_1(b(t))} = \nu^+(1,\infty] \,.$$

This implies the following result for aggregated OpVaR.

Theorem 4.9 (Böcker and Klüppelberg (2006), Theorem 3.18).

Consider a multivariate compound Poisson model $\boldsymbol{X}_t = (X_t^1, \ldots, X_t^d), t \geq 0$, with multivariate regularly varying Lévy measure Π with index $-\alpha$ and limit measure ν in (4.8). Assume further that the severity distributions F_i for $i = 1, \ldots, d$ are strictly increasing and continuous. Then $\left(X_t^+\right)_{t \geq 0}$ is a compound Poisson process with parameters satisfying for $z \to \infty$

$$\lambda^+ \bar{F}^+(z) \sim \nu^+(1,\infty] \lambda_1 \bar{F}_1(z) \in \mathcal{R}_{-\alpha} , \tag{4.15}$$

where $\nu^+(1,\infty] = \nu\left\{ \boldsymbol{x} \in \mathbb{E} : \sum_{i=1}^{d} x_i > 1 \right\}$. Furthermore, total OpVaR is asymptotically given by

$$\mathrm{OpVaR}_t(\kappa) \sim F_1^{\leftarrow}\left(1 - \frac{1-\kappa}{t\lambda_1 \nu^+(1,\infty]} \right), \quad \kappa \uparrow 1 . \tag{4.16}$$

We notice that for the wide class of regularly varying distributions, total OpVaR can effectively be written in terms of the severity distribution of the first cell. Specifically, the right-hand side of (4.16) can be understood as the stand-alone, asymptotic OpVaR of the first cell with an adjusted frequency parameter, namely $\lambda_1 \nu^+(1,\infty]$. What remains is to find examples, where $\nu^+(1,\infty]$ can be calculated analytically or numerically to understand better the influence of certain dependence parameter.

Revisiting the case of independent operational risk cells

Before we present some explicit results for the Clayton Pareto Lévy copula below, let us consider again the particularly easy case with independent cells. Since then all mass is on the positive axes, we obtain

$$\nu^+(1,\infty] = \nu_1(1,\infty] + \cdots + \nu_d(1,\infty]. \tag{4.17}$$

From $\bar{\Pi}_1(\mathrm{b}(t)) \sim t^{-1}$ as $t \to \infty$ it follows for the tail measure of the first cell

$$\lim_{t\to\infty} t\bar{\Pi}_1(b(t)z) = z^{-\alpha} = \nu_1(z,\infty]. \tag{4.18}$$

For $i = 2, \ldots, d$ we obtain by using (4.13)

$$\lim_{t\to\infty} t\overline{\Pi}_i\left(b(t)\right) = \lim_{t\to\infty}\frac{\overline{\Pi}_i\left(b(t)z\right)}{\overline{\Pi}_1\left(b(t)\right)} = \lim_{u\to\infty}\frac{\overline{\Pi}_i(uz)}{\overline{\Pi}_i(u)}\frac{\overline{\Pi}_i(u)}{\overline{\Pi}_1(u)} = \tilde{c}_i\, z^{-\alpha} = \nu_i\left(z,\infty\right], \qquad (4.19)$$

and, therefore, altogether $\nu^+\left(1,\infty\right] = 1 + \sum_{i=2}^{d}\tilde{c}_i$. By (4.15) together with $\lambda_1\tilde{c}_i = \lambda_i c_i$ we finally recover the result of Theorem 4.4.

Two explicit results for the Clayton Lévy copula

Let us consider a bivariate example where the marginal Lévy measures are not independent, and thus the limit measure $\nu^+(z,\infty]$ is not just the sum of the marginal limit measures as in (4.17). Instead, $\nu^+\left(z,\infty\right]$ has to be calculated by taking also mass between the positive axes into account, which can be done by representing $\nu^+\left(z,\infty\right]$ as an integral over a density.

First, note that from (4.12) together with (4.18) and (4.19), it follows in the case of a Pareto Lévy copula that for all $(x_1,x_2)\in\mathbb{E}$

$$\nu\left(\left[\mathbf{0},(x_1,x_2)^\top\right]^c\right) = x_1^{-\alpha} + \tilde{c}_2\, x_2^{-\alpha} - \left(x_1^{\alpha\theta} + \tilde{c}_2^{-\theta} x_2^{\alpha\theta}\right)^{-1/\theta},$$

which after differentiating yields

$$\nu'\left(x_1,x_2\right) = \tilde{c}_2^{-\theta}\,\alpha^2\left(1+\theta\right)x_1^{-\alpha(1+\theta)-1}x_2^{\alpha\theta-1}\left(1+\tilde{c}^{-\theta}\left(x_2/x_1\right)^{\alpha\theta}\right)^{-1/\theta-2}.$$

Hence, we can calculate

$$\nu^+\left(1,\infty\right] = \nu\left(\left(1,\infty\right]\times\left[0,\infty\right]\right) + \int_0^1\int_{1-x_1}^{\infty}\nu'\left(x_1,x_2\right)dx_1dx_2$$

$$= 1 + \alpha\int_0^1\left(1+\tilde{c}_2^{-\theta}\left(x_1^{-1}-1\right)^{\alpha\theta}\right)^{-1/\theta-1}x_1^{-1-\alpha}dx_1$$

and, by substitution $u = x_1^{-1}-1$, we obtain

$$\nu^{+}\left(1,\infty\right] = 1+\alpha\int_0^\infty \left(1+\tilde{c}_2^{-\theta}u^{\alpha\theta}\right)^{-1/\theta-1}(1+u)^{\alpha-1}du$$

$$= 1+\tilde{c}_2^{1/\alpha}\int_0^\infty \left(1+s^{\theta}\right)^{-1/\theta-1}(s^{-1/\alpha}+\tilde{c}_2^{1/\alpha})^{\alpha-1}ds$$

$$= 1+\tilde{c}_2^{1/\alpha}E\left[\left(\tilde{c}_2^{1/\alpha}+Y_\theta^{-1/\alpha}\right)^{\alpha-1}\right] \qquad (4.20)$$

where Y_θ is a positive random variable with density $g\left(s\right)=\left(1+s^{\theta}\right)^{-1/\theta-1}$, independent of all parameters but the Pareto copula parameter θ.

Example 4.10. (a) For $\alpha=1$ we have $\nu^{+}\left(1,\infty\right]=1+\tilde{c}_2$, which implies that, regardless of the dependence parameter θ, total OpVaR for all $0<\theta<\infty$ is asymptotically equal to the independent OpVaR.

(b) If $\alpha\theta=1$, then

$$\nu^{+}\left(1,\infty\right]=\frac{\tilde{c}_2^{1+1/\alpha}-1}{\tilde{c}_2^{1/\alpha}-1},$$

with $\tilde{c}_2=(\lambda_2/\lambda_1)c_2$ as defined in (4.13).

Figure 4 about here

Figure 4: Illustration of the tail measure $\nu^{+}\left(1,\infty\right]$ as a function of α for different values of θ. We have chosen $\theta=0.3$ (light dependence, solid line), $\theta=1$ (medium dependence, dashed line), and $\theta=10$ (strong dependence, dotted-dashed line). Morover, the long-dashed line corresponds to the independent case.

Figure 4 illustrates the tail measure $\nu^+(1,\infty]$ as given in (4.20) for different values of θ and α. Note that according to (4.16), OpVaR increases with $\nu^+(1,\infty]$. Hence, Figure 4 shows that in the case of $\alpha > 1$, a higher dependence (i.e. a higher number of joint losses in the components) leads to a higher OpVaR, whereas for $\alpha < 1$, it is the other way around: a lower dependence (i.e. a lower number of joint losses in the components) leads to a higher OpVaR. This again shows, how things may go awry for extremely heavy-tailed distributions. Due to the non-convexity of the OpVaR for $\alpha < 1$ diversification is not only not present, but the opposite effect occurs. Finally, note that for $\theta \to \infty$, independence occurs and $\nu^+(1,\infty] = 1 + \tilde{c}_2$ is constant as indicated by the horizontal long-dashed line in Figure 4.

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