Lévy-driven Continuous-time ARMA Processes

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Summary. Gaussian ARMA processes with continuous time parameter, otherwise known as stationary continuous-time Gaussian processes with rational spectral density, have been of interest for many years. (See for example the papers of Doob (1944), Bartlett (1946), Phillips (1959), Durbin (1961), Dzhaparidze (1970,1971), Pham-Din-Tuan (1977) and the monograph of Arató (1982).) In the last twenty years there has been a resurgence of interest in continuous-time processes, partly as a result of the very successful application of stochastic differential equation models to problems in finance, exemplified by the derivation of the Black-Scholes option-pricing formula and its generalizations (Hull and White (1987)). Numerous examples of econometric applications of continuous-time models are contained in the book of Bergstrom (1990). Continuous-time models have also been utilized very successfully for the modelling of irregularly-spaced data (Jones (1981, 1985), Jones and Ackerson (1990)). Like their discrete-time counterparts, continuous-time ARMA processes constitute a very convenient parametric family of stationary processes exhibiting a wide range of autocorrelation functions which can be used to model the empirical autocorrelations observed in financial time series analysis. In financial applications it has been observed that jumps play an important role in the realistic modelling of asset prices and derived series such as volatility. This has led to an upsurge of interest in Lévy processes and their applications to financial modelling. In this article we discuss second-order Lévy-driven continuous-time ARMA models, their properties and some of their financial applications, in particular to the modelling of stochastic volatility in the class of models introduced by Barndorff-Nielsen and Shephard (2001) and to the construction of a class of continuous-time GARCH models which generalize the COGARCH(1,1) process of Klüppelberg, Lindner and Maller (2004) and which exhibit properties analogous to those of the discrete-time GARCH($p.q$) process.
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1. Introduction

In financial econometrics, many discrete-time models (stochastic volatility, ARCH, GARCH and generalizations of these) are used to model the returns at regular intervals on stocks, currency investments and other assets. For example a GARCH process $(\xi_n)_{n\in\mathbb{N}}$ is frequently used to represent the increments, $\ln P_n - \ln P_{n-1}$, of the logarithms of the asset price P_n at times $1, 2, 3, \ldots$. These models capture many of the so-called *stylized features* of such data, e.g. tail heaviness, volatility clustering and dependence without correlation.

Various attempts have been made to capture the stylized features of financial time series using continuous-time models. The interest in continuous-time models stems from their use in modelling irregularly spaced data, their use in financial applications such as option-pricing and the current wide-spread availability of high-frequency data. In continuous-time it is natural to model the logarithm of the asset price itself, i.e. $G(t) = \ln P(t)$, rather than its increments as in discrete time.

One approach is via the stochastic volatility model of Barndorff-Nielsen and Shephard (2001) (see also Barndorff-Nielsen et al. (2002)), in which the volatility process V and the log asset price G satisfy the equations (apart from a deterministic rescaling of time),

$$dV(t) = -\lambda V(t)dt + dL(t), \tag{1.1}$$

$$dG(t) = (\gamma + \beta V(t))dt + \sqrt{V(t)}dW(t) + \rho dL(t), \tag{1.2}$$

where $\lambda > 0$, $L = (L(t))_{t\in\mathbb{R}_+}$ is a non-decreasing Lévy process and $W = (W(t))_{t\in\mathbb{R}_+}$ is standard Brownian motion independent of L. The volatility process V is taken to be a stationary solution of (1.1), in other words a *stationary Lévy-driven Ornstein-Uhlenbeck process* or a *continuous-time autoregression of order 1.* The background driving Lévy process L introduces the possibility of jumps in both the volatility and the log asset processes, a feature which is in accordance with empirical observations. It also allows for a rich class of marginal distributions, with possibly heavy tails. The autocorrelation function of the process V is $\rho(h) = \exp(-\lambda|h|)$. For modelling purposes this is quite restrictive, although the class of possible autocorrelations can be extended to a larger class of monotone functions if V is replaced by a superposition of such processes as in Barndorff-Nielsen (2001). However, as we shall see, a much wider class of not necessarily monotone autocorrelation functions for the volatility can be obtained by replacing the process V in (1.1) and (1.2) by a Lévy-driven continuous-time autoregressive moving average (CARMA) process as defined in Section 2. This class of processes constitutes a very flexible parametric family of stationary processes with a vast array of possible marginal distributions and autocorrelation functions.

Their role in continuous-time modelling is analogous to that of autoregressive moving average processes in discrete time. They belong to the more general class of Lévy-driven moving average process considered by Fasen (2004).

A continuous-time analogue of the GARCH(1,1) process, denoted COGARCH(1,1), has recently been constructed and studied by Klüppelberg et al. (2004). Their construction is based on an explicit representation of the discrete-time GARCH(1,1) volatility process which they use in order to obtain a continuous-time analogue. Since no such representation exists for higher-order discrete-time GARCH processes, a different approach is needed to construct higher-order models in continuous time. The Lévy-driven CARMA process plays a key role in this construction.

The present paper deals with *second-order* Lévy-driven continuous-time ARMA (denoted CARMA) processes, since for most financial applications processes with finite second moments are generally considered adequate. (Analogous processes without the second-order assumption are considered in Brockwell (2001).) In Section 2 we review the definition and properties, deriving the kernel and autocovariance functions, specifying the joint characteristic functions and discussing the issue of causality. Under the assumption of distinct autoregressive roots, some particularly tractable representations of the kernel, the autocovariance function and the process itself are derived. The question of recovering the driving process from a realization of the process on a (continuous) interval $[0, T]$ is also considered.

Section 3 considers connections between continuous-time and discrete-time ARMA processes.

In Section 4 we indicate the applications of CARMA processes to the modelling of stochastic volatility in the Barndorff-Nielsen-Shephard stochastic volatility model and in Section 5 their role in the construction of COGARCH models of order higher than (1,1).

Section 6 deals briefly with the well-established methods of inference for Gaussian CARMA processes and the far less developed question of inference for more general Lévy-driven processes.

Before proceeding further we need a few essential facts regarding Lévy processes. (For a detailed account of the pertinent properties of Lévy processes see Protter (2004) and for further properties see the books of Applebaum (2004), Bertoin (1996) and Sato (1999).) Suppose we are given a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le\infty}, P)$, where $\mathcal{F}_0$ contains all the P-null sets of $\mathcal{F}$ and $(\mathcal{F}_t)$ is right-continuous.

Definition 1 (Lévy Process). An adapted process $\{L(t), t \geq 0\}$ is said to be a Lévy process if

(i) $L(0) = 0$ a.s.,
(ii) $L(t) - L(s)$ is independent of $\mathcal{F}_s,\ 0 \le s < t < \infty$,
(iii) $L(t) - L(s)$ has the same distribution as $L(t - s)$ and
(iv) $L(t)$ is continuous in probability.

Every Lévy process has a unique modification which is càdlàg (right continuous with left limits) and which is also a Lévy process. We shall therefore assume that our Lévy process has these properties. The characteristic function of $L(t)$, $\phi_t(\theta) := E(\exp(i\theta L(t)))$, has the Lévy-Khinchin representation,

$$\phi_t(\theta) = \exp(t\xi(\theta)),\ \theta \in \mathbb{R}, \tag{1.3}$$

where

$$\xi(\theta) = i\theta m - \frac{1}{2}\theta^2 s^2 + \int_{\mathbb{R}_0} (e^{i\theta x} - 1 - ix\theta I_{\{|x|<1\}})\nu(dx), \tag{1.4}$$

for some $m \in \mathbb{R}$, $s \geq 0$, and measure ν on the Borel subsets of $\mathbb{R}_0 = \mathbb{R}\backslash\{0\}$. ν is known as the *Lévy measure* of the process L and satisfies the condition $\int_{\mathbb{R}_0} \min(1, |u|^2)\nu(du) < \infty$. If ν is the zero measure then $\{L(t)\}$ is Brownian motion with $E(L(t)) = mt$ and $\mathrm{Var}(L(t)) = s^2 t$. If $m = s^2 = 0$ and $\nu(\mathbb{R}_0) < \infty$, then $L(t) = at + P(t)$, where $\{P(t)\}$ is a compound Poisson process with jump-rate $\nu(\mathbb{R}_0)$, jump-size distribution $\nu/\nu(\mathbb{R}_0)$, and $a = -\int_{\mathbb{R}_0} \frac{u}{1+u^2}\nu(du)$. A wealth of distributions for $L(t)$ is attainable by suitable choice of the measure ν. See for example Barndorff-Nielsen and Shephard (2001). For the second-order Lévy processes (with which we are concerned in this paper), $E(L(1))^2 < \infty$. To avoid problems of parameter identifiability we shall assume throughout that L is scaled so that $\mathrm{Var}(L(1)) = 1$. Then $\mathrm{Var}(L(t)) = t$ for all $t \geq 0$ and there exists a real constant μ such that $EL(t) = \mu t$ for all $t \geq 0$. We shall then refer to the process L as a *standardized second-order Lévy process*, written henceforth as SSLP.

2. Second-order Lévy-driven CARMA Processes

A second-order Lévy-driven continuous-time ARMA(p, q) process, where p and q are non-negative integers such that $q < p$, is defined (see Brockwell (2001)) via the state-space representation of the formal equation,

$$a(D)Y(t) = \sigma b(D)DL(t),\ t \geq 0, \tag{2.1}$$

where σ is a strictly positive scale parameter, D denotes differentiation with respect to t, $\{L(t)\}$ is an SSLP,

$$a(z) := z^p + a_1 z^{p-1} + \cdots + a_p,$$

$$b(z) := b_0 + b_1 z + \cdots + b_{p-1} z^{p-1},$$

and the coefficients b_j satisfy $b_q = 1$ and $b_j = 0$ for $q < j < p$. The behaviour of the process is determined by the process L and the coefficients $\{a_j, 1 \leq j \leq p;\ b_j, 0 \leq j < q;\ \sigma\}$. In view of the scale parameter, σ, on the right-hand side of (2.1), there is clearly no loss of generality in assuming that $\mathrm{Var}(L(1)) = 1$, i.e. that L is an SSLP as defined at the end of Section 1. To avoid trivial and easily eliminated complications we shall assume that $a(z)$ and $b(z)$ have no common factors. The state-space representation consists of the *observation* and *state* equations,

$$Y(t) = \sigma \mathbf{b}' \mathbf{X}(t), \tag{2.2}$$

and

$$d\mathbf{X}(t) - A\mathbf{X}(t)dt = \mathbf{e}\, dL(t), \tag{2.3}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_p & -a_{p-1} & -a_{p-2} & \cdots & -a_1 \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} b_0 \\ b_1 \\ \vdots \\ b_{p-2} \\ b_{p-1} \end{bmatrix}.$$

(If $p = 1$, A is defined to be $-a_1$.) In the special case when $\{L(t)\}$ is standard Brownian motion, (2.3) is an Ito equation with solution $\{\mathbf{X}(t), t \geq 0\}$ satisfying

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-u)}\mathbf{e}\, dL(u), \tag{2.4}$$

where the integral is defined as the L^2 limit of approximating Riemann-Stieltjes sums S_n corresponding to the partition of the interval $[0, t]$ by the points $\{k/2^n, k \in \mathbf{Z}, 0 \leq k < 2^n t\}$ and $\{t\}$. If L is any second-order Lévy process the integral is defined in the same way. The continuous differentiability of the integrand in (2.4) implies that the sequence $\{S_n\}$ converges geometrically in L^2 and hence almost surely to the same limit. In fact the integral in (2.4) is a special case (with deterministic and continuously differentiable integrand) of integration with respect to a semimartingale as discussed in the book of Protter (2004). From (2.4) we can also write

$$\mathbf{X}(t) = e^{A(t-s)}\mathbf{X}(s) + \int_s^t e^{A(t-u)}\mathbf{e}\, dL(u), \quad \text{for all } t > s \geq 0, \tag{2.5}$$

which clearly shows (by the independence of increments of $\{L(t)\}$) that $\{\mathbf{X}(t)\}$ is Markov. The following propositions give necessary and sufficient conditions for stationarity of $\{\mathbf{X}(t)\}$.

Proposition 1 If $\mathbf{X}(0)$ is independent of $\{L(t), t \geq 0\}$ and $E(L(1)^2) < \infty$, then $\{\mathbf{X}(t)\}$ is weakly stationary if and only if the eigenvalues of the matrix A all have strictly negative real parts and $\mathbf{X}(0)$ has the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\, dL(u)$, i.e. $-A^{-1}\mathbf{e}\mu$ and $\int_0^\infty e^{Ay}\mathbf{e}\,\mathbf{e}' e^{A'y} dy$ respectively.

Proof. The eigenvalues of A must have negative real parts for the sum of the covariance matrices of the terms on the right of (2.4) to be bounded in t. If this condition is satisfied then $\{\mathbf{X}(t)\}$ converges in distribution as $t \to \infty$ to a random variable with the distribution of $\int_0^\infty e^{Au}\mathbf{e}\, dL(u)$. Hence, for weak stationarity, $\mathbf{X}(0)$ must have the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\, dL(u)$. Conversely if the eigenvalues of A all have negative real parts and if $\mathbf{X}(0)$ has the mean and covariance matrix of $\int_0^\infty e^{Au}\mathbf{e}\, dL(u)$, then a simple calculation using (2.4) shows that $\{\mathbf{X}(t)\}$ is weakly stationary.

Proposition 2. If $\mathbf{X}(0)$ is independent of $\{L(t), t \geq 0\}$ and $E(L(1)^2) < \infty$, then $\{\mathbf{X}(t)\}$ is strictly stationary if and only if the eigenvalues of the matrix A all have strictly negative real parts and $\mathbf{X}(0)$ has the distribution of $\int_0^\infty e^{Au}\mathbf{e}\, dL(u)$.

Proof. Necessity follows from Proposition 1. If the conditions are satisfied then strict stationarity follows from the fact that $\{\mathbf{X}(t)\}$ is a Markov process whose initial distribution is the same as its limit distribution.

Remark 1. It is convenient to extend the state process $\{\mathbf{X}(t), t \geq 0\}$ to a process with index set $(-\infty, \infty)$. To this end we introduce a second Lévy process $\{M(t), 0 \leq t < \infty\}$, independent of L and with the same distribution, and then define the following extension of L:

$$L^*(t) = L(t) I_{[0,\infty)}(t) - M(-t-) I_{(-\infty,0]}(t), \quad -\infty < t < \infty.$$

Then, provided the eigenvalues of A all have negative real parts, the process $\{\mathbf{X}(t)\}$ defined by

$$\mathbf{X}(t) = \int_{-\infty}^{t} e^{A(t-u)}\mathbf{e}\, dL^*(u), \tag{2.6}$$

is a strictly stationary process satisfying (2.5) (with L replaced by L^*) for all $t > s$ and $s \in (-\infty, \infty)$. Henceforth we shall refer to L^* as the background SSLP and denote it for simplicity by L rather than L^*.

Remark 2. It is easy to check that the eigenvalues of the matrix A, which we shall denote by $\lambda_1, \ldots, \lambda_p$, are the same as the zeroes of the autoregressive polynomial $a(z)$. The corresponding right eigenvectors are

$$[1\ \lambda_j\ \lambda_j^2\ \ldots\ \lambda_j^{p-1}]',\ j = 1, \ldots, p,$$

We are now in a position to define the CARMA process $\{Y(t), -\infty < t < \infty\}$ under the condition that

$$\mathcal{R}e(\lambda_j) < 0,\ j = 1, \ldots, p. \tag{2.7}$$

Definition 2 (Causal CARMA Process). If the zeroes $\lambda_1, \ldots, \lambda_p$ of the autoregressive polynomial $a(z)$ satisfy (2.7), then the CARMA(p,q) process driven by the SSLP $\{L(t), -\infty < t < \infty\}$ with coefficients $\{a_j, 1 \leq j \leq p;\ b_j, 0 \leq j < q;\ \sigma\}$ is the strictly stationary process,

$$Y(t) = \sigma \mathbf{b}'\mathbf{X}(t),$$

where

$$\mathbf{X}(t) = \int_{-\infty}^{t} e^{A(t-u)}\mathbf{e}\, dL(u),$$

i.e.

$$Y(t) = \sigma \int_{-\infty}^{t} \mathbf{b}' e^{A(t-u)}\mathbf{e}\, dL(u). \tag{2.8}$$

Remark 3 (Causality and Non-causality). Under Condition (2.7) we see from (2.8) that $\{Y(t)\}$ is a *causal* function of $\{L(t)\}$, since it has the form

$$Y(t) = \int_{-\infty}^{\infty} g(t-u)\, dL(u), \tag{2.9}$$

where

$$g(t) = \begin{cases} \sigma \mathbf{b}' e^{At}\mathbf{e} & \text{if } t > 0, \\ 0 & \text{otherwise.} \end{cases} \tag{2.10}$$

The function g is referred to as the *kernel* of the CARMA process $\{Y(t)\}$. Under the condition (2.7), the function g defined by (2.10) can be written as

$$g(t) = \frac{\sigma}{2\pi} \int_{-\infty}^{\infty} e^{it\lambda} \frac{b(i\lambda)}{a(i\lambda)}\, d\lambda. \tag{2.11}$$

(To establish (2.11) when the eigenvalues $\lambda_1, \ldots, \lambda_p$ are distinct, we use the explicit expressions for the eigenvectors of A to replace e^{At} in (2.10) by its spectral representation. The same expression is obtained when the right side of (2.11) is evaluated by contour integration. When there are multiple eigenvalues, the result is obtained by separating the eigenvalues slightly and taking the limit as the repeated eigenvalues converge to their common value.) It is of interest to observe that the representation (2.9) and (2.11) of $\{Y(t)\}$ defines a strictly stationary process even under conditions less restrictive than (2.7), namely

$$\mathcal{R}e(\lambda_j) \neq 0,\ j = 1, \ldots, p.$$

Thus (2.9) and (2.11) provide a more general definition of CARMA process than Definition 2 above. However if any of the zeroes of $a(z)$ has real part greater than 0, the representation (2.9) of $\{Y(t)\}$ in terms of $\{L(t)\}$ will no longer be causal as is the case when (2.7) is satisfied. This distinction between causal and non-causal CARMA processes is analogous to the classification of discrete-time ARMA processes as causal or otherwise, depending on whether or not the zeroes of the autoregressive polynomial lie outside the unit circle (see e.g. Brockwell and Davis (1991)). From now on **we shall restrict attention to causal CARMA processes**, i.e. we shall assume that (2.7) holds, so that the general expression (2.11) for the kernel g can also be written in the form (2.10). However both forms of the kernel will prove to be useful.

Remark 4 (Second-order Properties). From the representation (2.8) of a causal CARMA process driven by the SSLP L with $EL(1) = \mu$, we immediately find that

$$EY(t) = -\sigma \mathbf{b}' A^{-1} \mathbf{e} \mu$$

and

$$\gamma(h) := \operatorname{cov}(Y(t+h), Y(t)) = \sigma^2 \mathbf{b}' \, e^{A|h|} \Sigma \, \mathbf{b}, \tag{2.12}$$

where

$$\Sigma = \int_0^\infty e^{Ay} \mathbf{e} \, \mathbf{e}' e^{A'y} dy.$$

From the representation (2.9) of $Y(t)$ we see that γ can also be expressed as

$$\gamma(h) = \operatorname{cov}(Y(t+h), Y(t)) = \int_{-\infty}^{\infty} \tilde{g}(h-u) g(u) du,$$

where $\tilde{g}(x) = g(-x)$ and g is defined in (2.11). Using the convolution theorem for Fourier transforms, we find that

$$\int_{-\infty}^{\infty} e^{-i\omega h} \gamma(h) dh = \sigma^2 \left| \frac{b(i\omega)}{a(i\omega)} \right|^2,$$

showing that the spectral density of the process Y is

$$f(\omega) = \frac{\sigma^2}{2\pi} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 \tag{2.13}$$

and the autocovariance function is

$$\gamma(h) = \frac{\sigma^2}{2\pi} \int_{-\infty}^{\infty} e^{i\omega h} \left| \frac{b(i\omega)}{a(i\omega)} \right|^2 \, d\omega. \tag{2.14}$$

The spectral density (2.13) is clearly a rational function of the frequency ω. The family of Gaussian CARMA processes is in fact identical to the class of stationary Gaussian processes with rational spectral density.

Remark 5 (Distinct Autoregressive Zeroes, the Canonical State Representation and Simulation of Y). When the zeroes $\lambda_1, \ldots, \lambda_p$ of $a(z)$ are distinct and satisfy the causality condition (2.7), the expression for the kernel g takes an especially simple form. Expanding the integrand in (2.11) in partial fractions and integrating each term gives the simple expression,

$$g(h) = \sigma \sum_{r=1}^{p} \frac{b(\lambda_r)}{a'(\lambda_r)} e^{\lambda_r h} I_{[0,\infty)}(h). \tag{2.15}$$

Applying the same argument to (2.14) gives a corresponding expression for the autocovariance function, i.e.

$$\gamma(h) = \mathrm{cov}(Y_{t+h}, Y_t) \;=\; \sigma^2 \sum_{j=1}^{p} \frac{b(\lambda_j)b(-\lambda_j)}{a'(\lambda_j)a(-\lambda_j)} e^{\lambda_j |h|}. \tag{2.16}$$

When the autoregressive roots are distinct we obtain a very useful representation of the CARMA(p, q) process Y from (2.15). Defining

$$\alpha_r = \sigma \frac{b(\lambda_r)}{a'(\lambda_r)}, \quad r = 1, \ldots, p, \tag{2.17}$$

we can write

$$Y(t) = \sum_{r=1}^{p} Y_r(t), \tag{2.18}$$

where

$$Y_r(t) = \int_{-\infty}^{t} \alpha_r e^{\lambda_r (t-u)} dL(u). \tag{2.19}$$

This expression shows that the component processes Y_r satisfy the simple equations,

$$Y_r(t) = Y_r(s) e^{\lambda_r (t-s)} + \int_s^t \alpha_r e^{\lambda_r (t-u)} dL(u), \quad t \geq s, \quad r = 1, \ldots, p. \tag{2.20}$$

Taking $s = 0$ and using Lemma 2.1 of Eberlein and Raible (1999), we find that

$$Y_r(t) = Y_r(0) e^{\lambda_r t} + \alpha_r L(t) + \int_0^t \alpha_r \lambda_r e^{\lambda_r (t-u)} L(u) du, \quad t \geq 0, \tag{2.21}$$

where the last integral is a Riemann integral and the equality holds for all finite $t \geq 0$ with probability 1. Defining

$$\mathbf{Y}(t) := [Y_1(t), \ldots, Y_p(t)]', \tag{2.22}$$

we obtain from (2.6), (2.15) and (2.19),

$$\mathbf{Y}(t) = \sigma B R^{-1} \mathbf{X}(t), \tag{2.23}$$

where $B = \mathrm{diag}\,[b(\lambda_i)]_{i=1}^{p}$ and $R = [\lambda_j^{i-1}]_{i,j=1}^{p}$. The initial values $Y_r(0)$ in (2.21) can therefore be obtained from those of the components of the state vector $\mathbf{X}(0)$. The process $\mathbf{Y}$ provides us with an alternative *canonical* state representation of $Y(t),\ t \geq 0$, namely

$$Y(t) = [1, \ldots, 1]\mathbf{Y}(t) \tag{2.24}$$

where $\mathbf{Y}$ is the solution of

$$d\mathbf{Y}(t) = \operatorname{diag}[\lambda_i]_{i=1}^{p} \mathbf{Y} dt + \sigma B R^{-1} \mathbf{e}\ dL. \tag{2.25}$$

with $\mathbf{Y}(0) = \sigma B R^{-1} \mathbf{X}(0)$.

Notice that the canonical representation of the process Y reduces the problem of simulating CARMA(p,q) processes with distinct autoregressive roots to the much simpler problem of simulating the (possibly complex-valued) component CAR(1) processes (2.19) and adding them together.

Example 1 (The CAR(1) Process). The CAR(1) (or stationary Ornstein-Uhlenbeck) process satisfies (2.1) with $b(z) = 1$ and $a(z) = z - \lambda$ where $\lambda < 0$. From (2.15) and (2.16) we immediately find that $g(h) = e^{\lambda h} I_{[0,\infty)}(h)$ and $\gamma(h) = \sigma^2 e^{\lambda|h|}/(2|\lambda|)$. In this case the 1×1 matrices B and R are both equal to 1 so the (1-dimensional) state vectors $\mathbf{X}$ and $\mathbf{Y}$ are identical and the state-space representation given by (2.2) and (2.3) is already in canonical form. Equations (2.18) and (2.19) reduce to

$$Y(t) = Y_1(t)$$

and

$$Y_1(t) = \sigma \int_{-\infty}^{t} e^{\lambda(t-u)} dL(u)$$

respectively (since $\lambda_1 = \lambda$ and $\alpha_1 = \sigma$).

Example 2 (The CARMA(2,1) Process). In this case $b(z) = b_0 + z$, $a(z) = (z - \lambda_1)(z - \lambda_2)$ and the real parts of λ_1 and λ_2 are both negative. Assuming that $\lambda_1 \neq \lambda_2$, we find from (2.15) that

$$g(h) = (\alpha_1 e^{\lambda_1 h} + \alpha_2 e^{\lambda_2 h}) I_{[0,\infty)}(h)$$

where $\alpha_r = \sigma(b_0 + \lambda_r)/(\lambda_r - \lambda_{3-r}),\ r = 1, 2$. An analogous expression for $\gamma(h)$ can be found from (2.16). From (2.23) the canonical state vector is

$$\mathbf{Y}(t) = \begin{bmatrix} Y_1(t) \\ Y_2(t) \end{bmatrix} = \frac{\sigma}{\lambda_1 - \lambda_2} \begin{bmatrix} \lambda_2(b_0 + \lambda_1) & -(b_0 + \lambda_1) \\ -\lambda_1(b_0 + \lambda_2) & b_0 + \lambda_2 \end{bmatrix} \mathbf{X}(t)$$

and the canonical representation of Y is, from (2.18) and (2.19),

$$Y(t) = Y_1(t) + Y_2(t)$$

where

$$Y_r(t) = \int_{-\infty}^{t} \alpha_r e^{\lambda_r(t-u)} dL(u), \ r = 1, 2,$$

and $\alpha_r = \sigma(b_0 + \lambda_r)/(\lambda_r - \lambda_{3-r}), \ r = 1, 2.$

Remark 6 (The Joint Distributions). Since the study of Lévy-driven CARMA processes is largely motivated by the need to model processes with non-Gaussian joint distributions, it is important to go beyond a second-order characterization of these processes. From Proposition 2 we already know that the marginal distribution of $Y(t)$ is that of $\int_0^\infty g(u)dL(u)$, where g is given by (2.11) or, if the autoregressive roots are distinct and the causality conditions (2.7) are satisfied, by (2.15). Using the expression (1.3) for the characteristic function of $L(t)$, we find that the cumulant generating function of $Y(t)$ is

$$\log E(\exp(i\theta Y(t))) = \int_0^\infty \xi(\theta g(u))du, \tag{2.26}$$

showing that the distribution of $Y(t)$, like that of $L(t)$, is infinitely divisible. In the special case of the CAR(1) process the distribution of $Y(t)$ is also self-decomposable (see Barndorff-Nielsen and Shephard (2001), Theorem 2.1, and the accompanying references). More generally it can be shown (see Brockwell (2001)) that the cumulant generating function of $Y(t_1), Y(t_2), \ldots, Y(t_n)$, $(t_1 < t_2 < \cdots < t_n)$ is

$$\text{(2.27)}\ \log E[\exp(i\theta_1 Y(t_1) + \cdots + i\theta_n Y(t_n))] =$$

$$\int_0^\infty \xi\left(\sum_{i=1}^n \theta_i g(t_i + u)\right) du + \int_0^{t_1} \xi\left(\sum_{i=1}^n \theta_i g(t_i - u)\right) du +$$

$$\int_{t_1}^{t_2} \xi\left(\sum_{i=2}^n \theta_i g(t_i - u)\right) du + \cdots + \int_{t_{n-1}}^{t_n} \xi\left(\theta_n g(t_n - u)\right) du.$$

If $\{L(t)\}$ is a compound Poisson process with finite jump-rate λ and bilateral exponential jump-size distribution with probability density $f(x) = \frac{1}{2}\beta e^{-\beta|x|}$, then the corresponding CAR(1) process (see Example 1) has marginal cumulant generating function,

$$\kappa(\theta) = \int_0^\infty \xi(\theta e^{-cu})du,$$

where $\xi(\theta) = \lambda\theta^2/(\beta^2 + \theta^2)$. Straightforward evaluation of the integral gives

$$\kappa(\theta) = -\frac{\lambda}{2c} \ln\left(1 + \frac{\theta^2}{\beta^2}\right),$$

showing that $Y(t)$ has a symmetrized gamma distribution, or more specifically that $Y(t)$ is distributed as the difference between two independent gamma distributed random variables with exponent $\lambda/(2c)$ and scale parameter β. In particular, if $\lambda = 2c$, the marginal distribution is bilateral exponential. For more examples see Barndorff-Nielsen and Shephard (2001).

Remark 7 (Recovering the driving noise process). For statistical modelling, one needs to know or to postulate an appropriate family of models for the driving Lévy process L. It would be useful therefore to recover the realized driving process, for given or estimated values of $\{a_j, 1 \leq j \leq p; b_j, 0 \leq j < q; \sigma\}$, from a realization of Y on some finite interval $[0,T]$. This requires knowledge of the initial state vector $\mathbf{X}(0)$ in general, but if this is available (as for example when a CARMA$(p,0)$ process is observed continuously on $[0,T]$), or if we are willing to assume a plausible value for $\mathbf{X}(0)$, then an argument due to Pham-Din-Tuan (1977) can be used to recover $\{L(t), 0 \leq t \leq T\}$. We shall assume in this Remark that the polynomial b (as well as the polynomial a) has all its zeroes in the left half-plane. This assumption is analogous to that of invertibility in discrete time. Since the covariance structure of our Lévy-driven process is exactly the same (except for slight notational changes) as that of Pham-Din-Tuan's Gaussian CARMA process and since his result holds for Gaussian CARMA processes with arbitrary mean (obtained by adding a constant to the zero-mean process) his L^2-based spectral argument can be applied directly to the Lévy-driven CARMA process to give, for $t \geq 0$,

$$
\begin{aligned}
L(t) = \sigma^{-1} &\left[Y^{(p-q-1)}(t) - Y^{(p-q-1)}(0)\right] \\
&- \int_0^t \left[\sum_{j=1}^{q} b_{q-j} X^{(p-j)}(s) - \sum_{j=1}^{p} a_j X^{(p-j)}(s)\right] ds,
\end{aligned}
\tag{2.28}
$$

where $Y^{(p-q-1)}$ denotes the derivative of order $p-q-1$ of the CARMA process Y and $X^{(0)}, \ldots, X^{(p-1)}$ are the components of the state process $\mathbf{X}$ (the component $X^{(j)}$ being the j^{th} derivative of $X^{(0)}$). $\mathbf{X}(t)$ can be expressed in terms of Y and $\mathbf{X}(0)$ by noting that (2.2) characterizes $X^{(0)}$as a CARMA$(q,0)$ process driven by the process $\{\sigma^{-1} \int_0^t Y(s)ds\}$. Making use of this observation, introducing the $q \times 1$ state vector $\mathbf{X}_q(t) := [X^{(0)}(t), \ldots, X^{(q-1)}(t)]'$ and proceeding exactly as we did in solving the CARMA equations in Section 2, we find that, for $q \geq 1$,

$$
\mathbf{X}_q(t) = \mathbf{X}_q(0)e^{Bt} + \sigma^{-1} \int_0^t e^{B(t-u)} \mathbf{e}_q Y(u)du, \tag{2.29}
$$

where

$$
B = \begin{bmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -b_0 & -b_1 & -b_2 & \cdots & -b_{q-1} \end{bmatrix} \quad \text{and} \quad \mathbf{e}_q = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},
$$

with $B := -b_0$ if $q = 1$, while for $q = 0$,

$$X^{(0)}(t) = \sigma^{-1}Y(t). \tag{2.30}$$

The remaining derivatives of $X^{(0)}$ up to order $p-1$ can be determined from (2.29) or (2.30), completing the determination of the state vector $\mathbf{X}(t)$. Having recovered $\mathbf{X}(t)$, the SSLP is found from (2.28).

To illustrate the use of (2.28) and (2.29) or (2.30), we consider the CAR(1) process of Example 1. In this case $a(z) = z - \lambda$, $b(z) = 1$ and the (one-dimensional) state vector is, from (2.30), $X(t) = \sigma^{-1}Y(t)$. Substituting into (2.28) gives

$$L(t) = \sigma^{-1}\left[Y(t) - Y(0) - \lambda\int_0^t Y(s)ds\right]. \tag{2.31}$$

It is easy to check directly that if Y is a Lévy-driven CARMA(1,0) process with parameters $a_1(= -\lambda)$ and σ and if L is the Lévy process defined by (2.31), then

$$Y(t) = Y(0)e^{\lambda t} + \sigma\int_0^t e^{\lambda(t-u)}dL(u), \tag{2.32}$$

since the last integral can be rewritten, by Lemma 2.1 of Eberlein and Raible (1999), as $\sigma L(t) + \sigma\int_0^t \lambda e^{\lambda(t-u)}L(u)du$. Making this replacement and substituting from (2.31) for L, we see that the right-hand side of (2.32) reduces to $Y(t)$.

In the case when the autoregressive roots are distinct, we can use the transformation (2.23) to recover the canonical state process $\mathbf{Y}$ defined by (2.19) and (2.22) from $\mathbf{X}$. Then applying the argument of Pham-Din-Tuan to the component processes Y_r we obtain p (equivalent) representations of $L(t)$, namely

$$L(t) = \alpha_r^{-1}\left[Y_r(t) - Y_r(0) - \lambda_r\int_0^t Y_r(s)ds\right],\ r = 1,\ldots,p. \tag{2.33}$$

Although Pham-Din-Tuan's result was derived with real-valued processes in mind, it is easy to check directly, as in the CARMA(1,0) case, that if Y is a Lévy-driven CARMA(p,q) process with parameters $\{a_j, 1 \le j \le p;\ b_j, 0 \le j < q;\ \sigma\}$ and L is the Lévy process satisfying the equations (2.33) with possibly complex-valued Y_r and λ_r, then

$$Y_r(t) = Y_r(0)e^{\lambda_r t} + \int_0^t \alpha_r e^{\lambda_r(t-u)}dL(u),\ t \ge 0,\ r = 1,\ldots,p,$$

and these equations imply, with (2.23), that the state process $\mathbf{X}$ satisfies

$$\mathbf{X}(t) = e^{At}\mathbf{X}(0) + \int_0^t e^{A(t-u)}\mathbf{e}dL(u),\ t \ge 0,$$

showing that $Y = \sigma\mathbf{b}'\mathbf{X}$ is indeed the CARMA(p,q) process with parameters $\{a_j, 1 \le j \le p;\ b_j, 0 \le j < q;\ \sigma\}$ driven by L. Thus we have arrived at p very simple (equivalent) representations of the driving SSLP, any of which can be computed from the realization of Y, the value of $\mathbf{X}(0)$ and the parameters of the CARMA process. Of course for calculations it is simplest to choose a value of r in (2.34) for which λ_r is real (if such an r exists).

3. Connections with Discrete-time ARMA Processes

The discrete-time ARMA(p,q) process $\{Y_n\}$ with autoregressive coefficients $\phi_1,\ldots,\phi_p$, moving average coefficients $\theta_1,\ldots,\theta_q$, and white noise variance σ_d^2, is defined to be a (weakly) stationary solution of the p^{th} order linear difference equations,

$$\phi(B)Y_n = \theta(B)Z_n,\ n = 0, \pm1, \pm2, \ldots, \tag{3.1}$$

where B is the backward shift operator ($BY_n = Y_{n-1}$ and $BZ_n = Z_{n-1}$ for all n), $\{Z_n\}$ is a sequence of uncorrelated random variables with mean zero and variance σ_d^2 (abbreviated to $\{Z_n\} \sim \text{WN}(0,\sigma_d^2)$) and

$$\phi(z) := 1 - \phi_1 z - \cdots - \phi_p z^p,$$

$$\theta(z) := 1 + \theta_1 z + \cdots + \theta_q z^q,$$

with $\theta_q \neq 0$ and $\phi_p \neq 0$. We define $\phi(z) := 1$ if $p = 0$ and $\theta(z) := 1$ if $q = 0$. We shall assume that the polynomials $\phi(z)$ and $\theta(z)$ have no common zeroes and that $\phi(z) = 1 - \phi_1 z - \cdots - \phi_p z^p$ is non-zero for all complex z such that $|z| \leq 1$. This last condition guarantees the existence of a unique stationary solution of (3.1) which is also causal, i.e. is expressible in the form $Y_n = \sum_{j=0}^{\infty} \psi_j Z_{n-j}$ for some absolutely summable sequence $\{\psi_j\}$. It is evident from this representation that the mean of the ARMA process defined by (3.1) is zero. The process $\{Y_n\}$ is said to be an ARMA(p,q) process with mean μ if $\{Y_n - \mu\}$ is an ARMA(p,q) process. A more restrictive definition of ARMA process imposes the further requirement that the random variables Z_n be independent and identically distributed, in which case we write $\{Z_n\} \sim \text{IID}(0,\sigma_d^2)$. The process $\{Y_n\}$ is then strictly (as well as weakly) stationary and we shall refer to $\{Y_n\}$ as a *strict ARMA process*. If we impose the further constraint that each Z_n is Gaussian, then we write $\{Z_n\} \sim \text{IIDN}(0,\sigma_d^2)$ and refer to $\{Y_n\}$ as a *Gaussian ARMA process*.

As one might expect, there are many structural similarities between ARMA and CARMA processes. In the case when the polynomial $\phi(z)$ has distinct zeroes and $q < p$, there is an analogue of (2.16) for the autocovariance function of the ARMA process, namely

$$\gamma_d(h) = -\sigma_d^2 \sum_{j=1}^{p} \frac{\lambda_j^{|h|+1}\theta(\lambda_j)\theta(\lambda_j^{-1})}{\phi(\lambda_j)\phi'(\lambda_j^{-1})}, \quad h = 0, \pm1, \pm2, \ldots. \tag{3.2}$$

There is also a corresponding *canonical representation* analogous to that in Remark 5 of Section 2. It takes the form (cf. (2.18) and (2.19)),

$$Y_n = \sum_{r=1}^{p} Y_{r,n}, \tag{3.3}$$

and

$$Y_{r,n} = \sum_{k=-\infty}^{n} \beta_r \xi_r^{n-k} Z_k, \quad r = 1, \ldots, p \tag{3.4}$$

where ξ_r^{-1}, $r = 1, \ldots, p$, are the (distinct) zeroes of $\phi(z)$, and

$$\beta_r = -\xi_r \frac{\theta(\xi_r^{-1})}{\phi'(\xi_r^{-1})}, \quad r = 1, \ldots, p. \tag{3.5}$$

From (3.4) we also obtain the relations (cf. (2.20)),

$$Y_{r,n} = \xi_r Y_{r,n-1} + \beta_r Z_n, \quad n = 0, \pm 1, \ldots; \; r = 1, \ldots, p. \tag{3.6}$$

Remark 8. When $q < p$ and the autoregressive roots are distinct, the equations (2.19) and (3.6) show that both the CARMA and ARMA processes can be represented as a sum of autoregressive processes of order 1. Note however that in both cases the component processes are not independent and are in general complex valued.

Example 3 (The AR(1) Process). The defining equation (3.1) with $\phi(z) = 1 - \xi z$ and $\theta(z) = 1$ is clearly already in canonical form and, since $\beta_1 = 1$, equations (3.3) and (3.4) take the form

$$Y_n = Y_{1,n}$$

where

$$Y_{1,n} = \sum_{k=-\infty}^{n} \xi^{n-k} Z_k. \tag{3.7}$$

Example 4 (The ARMA(2,1) Process). In this case $\phi(z) = (1 - \xi_1 z)(1 - \xi_2 z)$, where we assume that $|\xi_1| < 1$, $|\xi_2| < 1$ and $\xi_1 \neq \xi_2$. The moving average polynomial is $\theta(z) = 1 + \theta_1 z$ and the white noise variance is σ_d^2. From (3.5) we find that

$$\beta_r = \frac{\xi_r + \theta_1}{\xi_r - \xi_{3-r}}, \quad r = 1, 2. \tag{3.8}$$

The canonical representation of the ARMA(2,1) process is thus

$$Y_n = Y_{1,n} + Y_{2,n},$$

where

$$Y_{r,n} = \beta_r \sum_{k=-\infty}^{n} \xi_r^{n-k} Z_k, \quad r = 1, 2, \tag{3.9}$$

with β_r, $r = 1, 2$, as defined in (3.7).

If Y is a Gaussian CARMA process defined as in Section 2 with standard Brownian motion as the driving process, then it is well-known (see e.g. Doob (1944), Phillips (1959), Brockwell (1995)) that the sampled process $(Y(n\delta))_{n\in\mathbf{Z}}$ with fixed $\delta > 0$ is a (strict) Gaussian ARMA(r, s) process with $0 \le s < r \le p$ and spectral density

(3.10
$$f_\delta(\omega) = \sum_{k=-\infty}^{\infty} \delta^{-1} f_Y(\delta^{-1}(\omega + 2k\pi)), \quad -\pi \le \omega \le \pi,$$

where $f_Y(\omega)$, $-\infty < \omega < \infty$, is the spectral density of the original CARMA process.

If L is non-Gaussian, the sampled process will have the same spectral density and autocovariance function as the process obtained by sampling a Gaussian CARMA process with the same parameters, driven by Brownian motion with the same mean and variance as L. Consequently from a second-order point of view the two sampled processes will be the same. However, except in the case of the CAR(1) process, the sampled process will not generally be a strict ARMA process.

If Y is the CAR(1) process in Example 1, the sampled process is the strict AR(1) process satisfying

$$Y(n\delta) = e^{\lambda\delta} Y((n-1)\delta) + Z_n, \quad n = 0, \pm 1, \ldots, \tag{3.11}$$

where

$$Z_n = \sigma \int_{(n-1)\delta}^{n\delta} e^{\lambda(n\delta - u)} dL(u). \tag{3.12}$$

The noise sequence $\{Z(n)\}$ is i.i.d. and $Z(n)$ has the infinitely divisible distribution with log characteristic function $\int_0^\delta \xi(\sigma\theta e^{\lambda u})du$, where $\xi(\theta)$ is the log characteristic function of $L(1)$ as in (1.3). For the CARMA(p, q) process with $p > 1$ the situation is more complicated. If the autoregressive roots $\lambda_1, \ldots, \lambda_p$, are all distinct, then from (2.18) and (2.19) the sampled process $\{Y(n\delta)\}$ is the sum of the strict AR(1) component processes $\{Y_r(n\delta)\}$, $r = 1, \ldots, p$, satisfying

$$Y_r(n\delta) = e^{\lambda_r\delta} Y_r((n-1)\delta) + Z_r(n), \quad n = 0, \pm 1, \ldots,$$

where

$$Z_r(n) = \alpha_r \int_{(n-1)\delta}^{n\delta} e^{\lambda_r(n\delta - u)} dL(u),$$

and α_r is given by (2.17).

The following question is important if we estimate parameters of a CARMA process by fitting a discrete-time ARMA(p,q) process with $q<p$ to regularly spaced data and then attempt to find the parameters of a CARMA process whose values at the observation times have the same distribution as the values of the fitted ARMA process at those times. The critical question here is whether or not such a CARMA process exists.

If a given Gaussian ARMA(p,q) process with $q<p$ is distributed as the observations at integer times of *some* Gaussian CARMA process it is said to be *embeddable*. Embeddability depends on the polynomials $\phi(z)$ and $\theta(z)$. Many, but not all, Gaussian ARMA processes are embeddable. For example the ARMA(1,0) process (3.1) with $\phi(z)=1-\phi_1 z$ and white-noise variance σ_d^2 can be embedded, if $0<\phi<1$, in the Gaussian CAR(1) process defined by (2.1) with $a(z)=z-\log(\phi_1)$, $b(z)=1$ and $\sigma^2=-2\log(\phi_1)\sigma_d^2/(1-\phi_1^2)$ and, if $-1<\phi_1<0$, it can be embedded in a CARMA(2,1) process (see Chan and Tong (1987)). However Gaussian ARMA processes for which $\theta(z)=0$ has a root on the unit circle are not embeddable in *any* CARMA process (see Brockwell and Brockwell (1999)). The class of non-embeddable Gaussian ARMA processes also includes ARMA(2,1) processes with autocovariance functions of the form $\gamma(h)=C_1\xi_1^{|h|}+C_2\xi_2^{|h|}$, where ξ_1 and ξ_2 are distinct values in $(0,1)$ and $C_1\log(\xi_1)+C_2\log(\xi_2)>0$. Such ARMA processes exist since there are infinitely many values of C_1 and C_2 satisfying the latter condition for which γ is a non-negative-definite function on the integers.

The problem of finding a CARMA process whose *autocovariance function* at integer lags matches that of a given non-Gaussian ARMA process is clearly equivalent to the problem of embedding a Gaussian ARMA process as described above.

However the determination of a Lévy-driven CARMA process (if there is one) whose sampled process has the same *joint distributions* as a given non-Gaussian ARMA process is more difficult. For example, from (3.11) and (3.12) we see that in order to embed a discrete-time AR(1) in a CAR(1) process, the driving noise sequence $\{Z_n\}$ of the AR(1) process must be i.i.d. with an infinitely divisible distribution, and the coefficient ϕ in the autoregressive polynomial $(1-\phi z)$ must be positive. Given such a process, with coefficient $\phi\in(0,1)$ and white-noise characteristic function $\exp(\psi(\theta))$, it is embeddable in a CAR(1) process (which must have autoregressive polynomial $a(z)=z-\lambda$, where $\lambda=\log(\phi)$) if and only if there exists a characteristic function $\exp(\rho(\theta))$ such that

$$\int_0^1 \rho(\theta e^{\lambda u})du=\psi(\theta),\ \text{for all } \theta\in\mathbb{R}, \tag{3.13}$$

and then $\exp(\rho(\theta)t)$ is the characteristic function of $\sigma L(t)$ for the CAR(1) process in which the AR(1) process can be embedded. It is easy to check that if $\psi(\theta)=-\sigma_d^2\theta^2/2$, i.e. if Z_n is normally distributed with mean zero and variance σ_d^2, then (3.13) is satisfied if $\rho(\theta)=-\lambda\sigma_d^2\theta^2/(1-e^{2\lambda})$, i.e. if

$\sigma L(1)$ is normally distributed with mean zero and variance $2\lambda\sigma_d^2/(1-e^{2\lambda})$. (More generally if Z_n is symmetric α-stable with $\psi(\theta) = -c|\theta|^\alpha$, $c > 0$, $\alpha \in (0,2]$, (3.13) is satisfied if $\rho(\theta) = -\alpha c\lambda|\theta|^\alpha/(1-e^{2\lambda})$, i.e. if $\sigma L(1)$ also has a symmetric α-stable distribution. If $\alpha \in (0,2)$ the processes do not have finite variance but the embedding is still valid.)

4. An Application to Stochastic Volatility Modelling

In the stochastic volatility model (1.1) and (1.2) of Barndorff-Nielsen and Shephard, the volatility process V is a CAR(1) (or stationary Ornstein-Uhlenbeck) process driven by a non-decreasing Lévy process L. With this model the authors were able to derive explicit expressions for quantities of fundamental interest, such as the integrated volatility. Since the process V can be written,

$$V(t) = \int_{-\infty}^{t} e^{-\lambda(t-u)} dL(u),$$

and since both the kernel, $g(u) = e^{-\lambda u} I_{(0,\infty)}(u)$, and the increments of the driving Lévy process are non-negative, the volatility is non-negative as required. A limitation of the use of the Ornstein-Uhlenbeck process however (and of linear combinations with non-negative coefficients of independent Ornstein-Uhlenbeck processes) is the constraint that the autocorrelations $\rho(h),\ h \geq 0$, are necessarily non-increasing in h.

Much of the analysis of Barndorff-Nielsen and Shephard can however be carried out after replacing the Ornstein-Uhlenbeck process by a CARMA process with non-negative kernel driven by a non-decreasing Lévy process. This has the advantage of allowing the representation of volatility processes with a larger range of autocorrelation functions than is possible in the Ornstein-Uhlenbeck framework. For example, the CARMA(3,2) process with

$$a(z) = (z+0.1)(z+0.5-i\pi/2)(z+0.5-i\pi/2) \quad \text{and} \quad b(z) = 2.792 + 5z + z^2$$

has non-negative kernel and autocovariance functions,

$$g(t) = 0.8762e^{-0.1t} + \left(0.1238\cos\frac{\pi t}{2} + 2.5780\sin\frac{\pi t}{2}\right)e^{-0.5t}, \quad t \geq 0,$$

and

$$\gamma(h) = 5.1161e^{-0.1h} + \left(4.3860\cos\frac{\pi h}{2} + 1.4066\sin\frac{\pi h}{2}\right)e^{-0.5h}, \quad h \geq 0,$$

respectively, both of which exhibit damped oscillatory behaviour.

There is of course a constraint imposed upon the allowable CARMA processes for stochastic volatility modelling by the requirement that the kernel g be non-negative. Conditions on the coefficients which guarantee non-negativity of the kernel have been considered by Brockwell and Davis (2001) and Todorov and Tauchen (2004) for the CARMA(2,1) process with real autoregressive roots and, more generally by Tsai and Chan (2004). In his analysis of the German Mark/US Dollar exchange rate series from 1986 through 1999, Todorov (2005) finds that a good fit to the autocorrelation function of the realized volatility is provided by a CARMA(2,1) model with two real autoregessive roots.

A class of long-memory Lévy-driven CARMA processes was introduced by Brockwell (2004) and Brockwell and Marquardt (2005) by replacing the kernel g in (2.9) by the kernel,

$$g_d(t) = \int_{-\infty}^{\infty} e^{it\lambda}(i\lambda)^{-d}\frac{b(i\lambda)}{a(i\lambda)}d\lambda,$$

with $0 < d < 0.5$. The resulting processes, which exhibit hyperbolic rather than geometric decay in their autocorrelation functions, must however be driven by Lévy processes with zero mean, and such Lévy processes cannot be non-decreasing. Long-memory Lévy-driven CARMA processes cannot therefore be used directly for the modelling of stochastic volatility. They can however be used for the modelling of mean-corrected log volatility in order to account for the frequently observed long memory in such series.

5. Continuous-time GARCH Processes

A continuous-time analog of the GARCH(1,1) process, denoted COGARCH(1,1), has recently been constructed and studied by Klüppelberg et al. (2004). Their construction uses an explicit representation of the discrete-time GARCH(1,1) process to obtain a continuous-time analog. Since no such representation exists for higher-order discrete-time GARCH processes, a different approach is needed to construct higher-order continuous-time analogs. For a detailed discussion of continuous-time GARCH processes see the article of Lindner (2007) in the present volume.

Let $(\varepsilon_n)_{n\in N_0}$ be an iid sequence of random variables. For any non-negative integers p and q, the discrete-time GARCH(p,q) process $(\xi_n)_{n\in\mathbb{N}_0}$ is defined by the equations,

$$\begin{aligned} \xi_n &= \sigma_n\varepsilon_n, \\ \sigma_n^2 &= \alpha_0 + \alpha_1\xi_{n-1}^2 + \ldots + \alpha_p\xi_{n-p}^2 + \beta_1\sigma_{n-1}^2 + \ldots + \beta_q\sigma_{n-q}^2, \end{aligned} \tag{5.1}$$

where $s := \max(p, q)$, the initial values $\sigma_0^2, \ldots, \sigma_{s-1}^2$ are assumed to be iid and independent of the iid sequence $(\varepsilon_n)_{n\geq s}$, and $\xi_n = G_{n+1} - G_n$ represents the increment at time n of the log asset price process $(G_n)_{n\in\mathbb{N}_0}$. In continuous-time it is more convenient to define the GARCH process as a model for $(G_t)_{t\geq 0}$ rather than for its increments as in discrete-time.

Equation (5.1) shows that the volatility process $(V_n := \sigma_n^2)_{n\in\mathbb{N}_0}$ can be viewed as a "self-exciting" ARMA($q, p-1$) process driven by the noise sequence $(V_{n-1}\varepsilon_{n-1}^2)_{n\in\mathbb{N}}$. This observation suggests defining a continuous time GARCH model of order (p, q) for the log asset price process $(G_t)_{t\geq 0}$ by

$$dG_t = \sqrt{V_t}\, dL_t, \quad t > 0, \quad G_0 = 0,$$

where $(V_t)_{t\geq 0}$ is a left-continuous non-negative CARMA$(q, p-1)$ process driven by a suitable replacement for the discrete time driving noise sequence $(V_{n-1}\varepsilon_{n-1}^2)_{n\in\mathbb{N}}$. By choosing the driving process to be

$$R_t = \int_0^t V_s d[L,L]_s^{(d)}, \quad \text{i.e.} \quad dR_t = V_t\, d[L,L]_t^{(d)}.$$

where $[L,L]^{(d)}$ is the *discrete part of the quadratic covariation* of the Lévy process L, we obtain the COGARCH(p,q) process, which has properties analogous to those of the discrete-time GARCH process and which includes the COGARCH(1,1) process of Klüppelberg et al.(2004) as a special case. The precise definition is as follows.

Definition 3 (COGARCH(p,q) process). If p and q are integers such that $q \geq p \geq 1$, $\alpha_0 > 0$, $\alpha_1, \ldots, \alpha_p \in \mathbb{R}$, $\beta_1, \ldots, \beta_q \in \mathbb{R}$, $\alpha_p \neq 0$, $\beta_q \neq 0$, and $\alpha_{p+1} = \ldots = \alpha_q = 0$, we define the $(q \times q)$–matrix B and the vectors $\mathbf{a}$ and $\mathbf{e}$ by

$$B = \begin{bmatrix} 0 & 1 & 0 & \ldots & 0 \\ 0 & 0 & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ -\beta_q & -\beta_{q-1} & -\beta_{q-2} & \ldots & -\beta_1 \end{bmatrix}, \quad \mathbf{a} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \vdots \\ \alpha_{q-1} \\ \alpha_q \end{bmatrix}, \quad \mathbf{e} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix},$$

with $B := -\beta_1$ if $q = 1$. Then if $L = (L_t)_{t\geq 0}$ is a Lévy process with non-trivial Lévy measure, we define the (left-continuous) *volatility process* $V = (V_t)_{t\geq 0}$ with parameters B, $\mathbf{a}$, α_0 and driving Lévy process L by

$$V_t = \alpha_0 + \mathbf{a}'\mathbf{Y}_{t-}, \quad t > 0, \quad V_0 = \alpha_0 + \mathbf{a}'\mathbf{Y}_0,$$

where the *state process* $\mathbf{Y} = (\mathbf{Y}_t)_{t\geq 0}$ is the unique càdlàg solution of the stochastic differential equation

$$d\mathbf{Y}_t = B\mathbf{Y}_{t-}\, dt + \mathbf{e}(\alpha_0 + \mathbf{a}'\mathbf{Y}_{t-})\, d[L,L]_t^{(d)}, \quad t > 0,$$

with initial value $\mathbf{Y}_0$, independent of the driving Lévy process $(L_t)_{t\geq 0}$. If the process $(V_t)_{t\geq 0}$ is strictly stationary and non-negative almost surely, we say that $G = (G_t)_{t\geq 0}$, given by

$$dG_t = \sqrt{V_t}\, dL_t, \quad t > 0, \quad G_0 = 0,$$

is a *COGARCH(p,q) process* with parameters B, $\mathbf{a}$, α_0 and driving Lévy process L .

Conditions for the existence of a non-negative stationary solution of the equations for V and the properties of the resulting volatility and COGARCH(p,q)

processes, including conditions for the existence of moments of order k, are studied in the paper of Brockwell et al. (2006). In particular it is shown under mild conditions that the process of increments

$$G_t^{(r)} := G_{t+r} - G_t,$$

for any fixed $r > 0$, has the characteristic GARCH properties,

$$EG_t^{(r)} = 0, \quad \mathrm{cov}(G_{t+h}^{(r)}, G_t^{(r)}) = 0 \quad h \geq r,$$

while the squared increment process $G^{(r)2}$ has a non-zero autocovariance function, expressible in terms of the defining parameters of the process. The autocovariance function of the stationary volatility process, if it exists, is that of a CARMA process, just as the discrete-time GARCH volatility process has the autocovariance function of an ARMA process.

6. Inference for CARMA Processes.

Given observations of a CARMA(p,q) process at times $0 \leq t_1 < t_2 < \ldots < t_N$, there is an extensive literature on maximum *Gaussian* likelihood estimation of the parameters. This literature however does not address the question of identifying and estimating parameters for the driving process when it is not Gaussian. In the general case we can write, from (2.2) and (2.5),

$$Y(t_i) = \sigma \mathbf{b}'\mathbf{X}(t_i), \quad i = 1, \ldots, N, \tag{6.1}$$

where

$$\mathbf{X}(t_i) = e^{A(t_i - t_{i-1})}\mathbf{X}(t_{i-1}) + \int_{t_{i-1}}^{t_i} e^{A(t_i - u)}\mathbf{e}\, dL(u), \quad i = 2, \ldots, N, \tag{6.2}$$

and $\mathbf{X}(t_1)$ has the distribution of $\int_0^\infty e^{Au}\mathbf{e}dL(u)$. The *observation equations* (6.1) and *state equations* (6.2) are in precisely the form required for application of the discrete-time Kalman recursions (see e.g. Brockwell and Davis (1991)) in order to compute numerically the best one-step linear predictors of $Y_2, \ldots, Y_N$, and hence the Gaussian likelihood of the observations in terms of the coefficients $\{a_j, 1 \leq j \leq p;\ b_j, 0 \leq j < q;\ \sigma\}$. Jones (1981) used this representation, together with numerical maximization of the calculated Gaussian likelihood, to compute maximum Gaussian likelihood estimates of the parameters for time series with irregularly spaced data. A similar approach was used in a more general setting by Bergstrom (1985). If the observations are uniformly spaced an alternative approach due to Phillips (1959) is to fit a discrete-time ARMA model to the observations and then to determine a Gaussian CARMA process in which the discrete-time process can be embedded. (Recalling the results of Section 3 however, it may be the case that there is no CARMA process in which the fitted ARMA process can be embedded.)

For a CAR(p) process observed continuously on the time interval $[0,T]$, Hyndman (1993) derived continuous-time analogues of the discrete-time Yule-Walker equations for estimating the coefficients. For a Gaussian CARMA process observed continuously on $[0,T]$, the exact likelihood function was determined by Pham-Din-Tuan (1977) who also gave a computational algorithm for computing approximate maximum likelihood estimators of the parameters which are asymptotically normal and efficient. The determination of the exact likelihood, conditional on the initial state vector $\mathbf{X}(0)$, can also be carried out for non-linear Gaussian CAR(p) processes and maximum conditional likelihood estimators expressed in terms of stochastic integrals (see Brockwell et al. (2006), where this method of estimation is applied to threshold CAR processes observed at closely spaced times, using sums to approximate the stochastic integrals involved.)

For Lévy-driven CARMA processes, estimation procedures which take into account the generally non-Gaussian nature of L are less well-developed. One approach is to estimate the parameters $\{a_j, 1 \leq j \leq p;\ b_j, 0 \leq j < q;\ \sigma\}$ by maximizing the Gaussian likelihood of the observations using (6.1) and (6.2). If the process is observed continuously on $[0,T]$, these estimates and the results of Remark 2.8 can be used to recover, for any observed or assumed $\mathbf{X}(0)$, a realization of L on $[0,T]$. The increments of this realization can then be examined and a driving Lévy process chosen whose increments are compatible with the increments of the recovered realization of L. If the CARMA process is observed at closely-spaced discrete time points then a discretized version of this procedure can be used. This work is currently in progress.

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