Integrated insurance risk models with exponential Lévy investment

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Abstract

We consider an insurance risk model for the cashflow of an insurance company, which invests its reser.v.e into a portfolio consisting of risky and riskless assets. The price of the risky asset is modeled by an exponential Lévy process. We derive the integrated risk process and the corresponding discounted net loss process. We calculate certain quantities as characteristic functions and moments. We also show under weak conditions stationarity of the discounted net loss process and derive the left and right tail behaviour of the model. Our results show that the model carries a high risk, which may originate either from large insurance claims or from the risky investment.

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1 Introduction

Throughout this paper let $(\Omega,\, \mathcal{F},\, (\mathcal{F}_t)_{t\geq 0},\, P)$ be a filtered complete probability space on which all stochastic quantities are defined. The filtration $(\mathcal{F}_t)_{t\geq 0}$ is right continuous and all stochastic processes to be defined in this paper are adapted. We define first the *insurance risk process* as in the Cramér-Lundberg model by

$$U(t) = u + ct - S(t)\,, \quad t \geq 0\,,$$

where $u > 0$ is the *initial risk reser.v.e*, $c > 0$ is the constant *premium rate* and the *total claim amount process* is defined as compound Poisson process $S(t) = \sum_{j=1}^{N(t)} Y_j$, $t \geq 0$. The claim sizes $(Y_j)_{j\in\mathbb{N}}$ are independent and identically distributed (iid) random variables (r.v.'s) with common distribution function F supported on the whole of $\mathbb{R}^+ = (0, \infty)$ and finite mean μ. The claims arrive at random time points $0 < T_1 < T_2 < \cdots$ and the *claim arrival process* $N(t) = \mathrm{card}\{k \geq 1 : T_k \leq t\}$ for $t > 0$ with $N(0) = 0$ is a homogeneous Poisson process with intensity $\lambda > 0$. Finally, $(N(t))_{t\geq 0}$ and $(Y_j)_{j\in\mathbb{N}}$ are independent processes.

This classical model is extended by allowing for investment of the risk reser.v.e. We consider an insurer who invests its reser.v.e into a Black-Scholes type market consisting of a *bond* and some *stock*, modeled by an exponential Lévy process. Their respective price processes follow the equations

$$X_0(t) = e^{\delta t} \quad \text{and} \quad X_1(t) = e^{L(t)}\,, \quad t \geq 0\,. \tag{1.1}$$

The constant $\delta > 0$ is the *riskless interest rate*. The process $(L(t))_{t\geq 0}$ is a Lévy process with *characteristic exponent* Ψ, i.e. $E[\exp(isL(t))] = \exp(t\Psi(s))$, $s \in \mathbb{R}$, $t \geq 0$, where Ψ has Lévy-Khintchine representation

$$\Psi(s) = is\gamma - \frac{\sigma^2}{2}s^2 + \int_{\mathbb{R}} \left(e^{isx} - 1 - isx1_{\{|x|\leq 1\}}\right)\, \nu(dx)\,, \quad s \in \mathbb{R}, \tag{1.2}$$

with $\gamma \in \mathbb{R}$, $\sigma \geq 0$ and *Lévy measure* ν satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(x^2 \wedge 1)\nu(dx) < \infty$. The *characteristic triplet* (γ, σ^2, ν) determines the Lévy process. For general Lévy process theory we refer to the monographs by Cont and Tankov [2] or Sato [24].

For allocation of the reser.v.e among the riskless and the risky asset we use the so-called *constant mix strategy;* i.e. the initial proportions which are invested into bond and stock remain constant over a predetermined planning horizon; see e.g. Emmer, Klüppelberg and Korn [5], Section 2. Such a strategy is dynamic in the sense that it requires at every instance of time a rebalancing of the portfolio depending on the corresponding price changes. We denote by $\theta \in [0, 1]$ the fraction of the reser.v.e invested into the risky asset; we call θ the *investment strategy*.

To derive the investment process we follow the calculations in Emmer, Klüppelberg and Korn [5] and Emmer and Klüppelberg [4]. We state first the corresponding SDEs for the price processes, where we use Itô's formula:

$$\begin{aligned} dX_0(t) &= \delta X_0\,(t)\,dt\,, \quad t>0\,, \quad X_0(0)=1\,, \\ dX_1(t) &= X_1(t-)\,d\widehat{L}(t) \\ &= X_1(t-)\left(dL(t)+\frac{\sigma^2}{2}\,dt+e^{\Delta L(t)}-1-\Delta L(t)\right)\,,\ \ t>0\,,\ \ X_1(0)=1\,, \end{aligned}$$

where $\Delta L(t,\omega) = L(t,\omega) - L(t-,\omega)$ for each $\omega \in \Omega$ denotes the jump of L at time $t > 0$. The process $\widehat{L}$ is such that $e^{L(t)} = \mathcal{E}(\widehat{L}(t))$, $t \geq 0$, where $\mathcal{E}$ denotes the stochastic exponential of a process (see, e.g. Protter [23], Section 2.8, or Cont and Tankov [2], Section 8.4.2).

Definition 1.1. *For $\theta \in [0,1]$ we define the* investment process *as the solution to the SDE*

$$dX_\theta(t) \;=\; X_\theta(t-)\left((1-\theta)\delta dt+\theta d\widehat{L}(t)\right)\,, \quad t>0\,, \quad X_\theta(0)=1. \tag{1.3}$$

This approach is based on self-financing portfolios and hence classical in financial portfolio optimization; see Korn [14], Section 2.1. The following is a consequence of Itô's Lemma.

Lemma 1.2. *The SDE* (1.3) *has the solution*

$$X_\theta(t) \;=\; \mathcal{E}(\widehat{L}_\theta(t)) = e^{L_\theta(t)}\,, \quad t\geq 0\,, \tag{1.4}$$

where $\widehat{L}_\theta(t) = (1-\theta)\delta t + \theta\widehat{L}(t)$ and L_θ is such that $\mathcal{E}(\widehat{L}_\theta(t)) = e^{L_\theta(t)}$.

Lemma 1.3. [Emmer and Klüppelberg [4], Lemma 2.5]
The process $(L_\theta(t))_{t\geq 0}$ is a Lévy process with characteristic exponent Ψ_θ, and the characteristic triplet $(\gamma_\theta, \sigma^2_\theta, \nu_\theta)$ is given by

$$\begin{aligned} \gamma_\theta &= \gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \\ &\quad + \int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\leq 1\}} - \theta x 1_{\{|x|\leq 1\}})\nu(dx)\,, \\ \sigma^2_\theta &= \theta^2\sigma^2\,, \\ \nu_\theta(A) &= \nu\left(\{x\in\mathbb{R} : \log(1+\theta(e^x-1))\in A\}\right) \textit{ for any Borel set } A\subset\mathbb{R}\,. \end{aligned}$$

$\square$

Remark 1.4. (i) Besides the characteristic exponents Ψ and Ψ_θ we shall also need the *Laplace exponents* given by

$$\begin{aligned} \varphi(s) &= \Psi(is) = \log E\left[e^{-sL(1)}\right], & (1.5)\\ \varphi_\theta(s) &= \Psi_\theta(is) = \log E\left[e^{-sL_\theta(1)}\right], & (1.6) \end{aligned}$$

provided they exist. If $\varphi(s) < \infty$, then $E\left[e^{-sL(t)}\right] = e^{t\varphi(s)} < \infty$ for all $t \geq 0$, see Sato [24], Theorem 25.17. As we show in Lemma A.1(c), $E\left[e^{sL_\theta(1)}\right] < \infty$ for all $\theta \in [0,1]$ provided $E\left[e^{sL(1)}\right] < \infty$.

(ii) A jump of size ΔL of L leads to a jump of size $e^{\Delta L} - 1$ of $\widehat{L}$ and to a jump of size $\Delta L_\theta = \log(1 + \theta(e^{\Delta L} - 1)) > \log(1-\theta)$ of L_θ. In other words, ν_θ is the image measure of ν under the transformation $x \mapsto \log(1 + \theta(e^x - 1))$. This explains the requirement $\theta \leq 1$.

(iii) If L is a process of finite variation, then L_θ is as well. Indeed,

$$\begin{aligned} \int_{|x|\leq 1} |x|\nu_\theta(dx) &= \int_{|\log(1+\theta(e^x-1))|\leq 1} |\log(1+\theta(e^x-1))|\nu(dx) \\ &\leq \int_{-\infty}^{-1} |\log(1+\theta(e^x-1))|\nu(dx) + \int_{-1}^{p} |\log(1+\theta(e^x-1))|\nu(dx)\,, \end{aligned}$$

where $p = \log(1 + \theta^{-1}(e-1)) > 0$. Then $\int_{-\infty}^{-1} |\log(1+\theta(e^x-1))|\nu(dx) \leq |\log(1-\theta)| \int_{-\infty}^{-1} \nu(dx) < \infty$ and, because of the finite variation of L, also $\int_{-1}^{p} |\log(1+\theta(e^x - 1))|\nu(dx) \leq \int_{-1}^{p} |x|\nu(dx) < \infty$ holds. $\square$

The goal of this paper is to study the integrated risk process, which allows for risk assessment of the insurance and investment risk at the same time. This process is defined in Section 2. We assume throughout this paper that investment process and total claim amount process are independent, which allows for a very explicit analysis of the integrated risk process.

In Section 3 the stationary version of the integrated risk process, the discounted net loss process (DNLP), is defined and investigated. The model fits into the framework of *generalized Ornstein-Uhlenbeck processes*, which have recently attracted much attention. Due to the special structure of our model we derive more specific results than in the more general case treated in Lindner and Maller [16]. We start with stationarity conditions and compare the process to its natural embedded discrete skeleton process; i.e. the process sampled at the claim arrival times. Our most important results in this section concern the tail behaviour of the stationary distribution. We show in particular that the stationary distribution of the continuous time process and the discrete time process coincide. We analyse two different regimes, which both lead to Pareto tails of the stationary distribution. The reasons, however, are different. If the claims have finite moments of sufficiently high order, under weak regularity conditions, both tails of the stationary DNLP are mainly determined by the Laplace exponent φ_θ in (1.6), i.e. heavy tails are mainly caused by properties of the investment process. In contrast to that, if the investment process has finite moments of sufficiently high order, but the claims have regularly varying tail, then the right tail of the stationary DNLP inherits the tail of the claim size distribution. In this case the left tail is of lower order than the right tail. We also discuss the influence of the investment strategy θ on the tail behaviour. To obtain these results, we use the theory of stochastic recurrence equations, see Goldie [7], Grey [9] and Konstantinides and Mikosch [13]. Technicalities are summarized together with most of the proofs in an Appendix.

The results of this paper are exploited in Kostadinova [15], where the optimal investment strategy θ is determined subject to a risk constraint of the insurer.

Throughout this paper we use the following notations. For $a \in \mathbb{N}$ we set $a^+ = \max(0, a)$ and $a^- = \max(0, -a)$; we also define $\log^+ a = \max(0, \log a)$ for $a > 0$. Furthermore, we write $\int_a^b := \int_{(a,b]}$ for $a < b$ in $\mathbb{R}$. We also denote $\lceil x \rceil = \min\{n \in \mathbb{N} : x \le n\}$ for $x > 0$ and recall that for $x > y$ we estimate $x - y - 1 < \lceil x \rceil - \lceil y \rceil < x - y + 1$.

2 The integrated risk process

We start by defining the integrated risk process as the total risk reser.v.e, i.e. the result of the insurance business and the net gains of the investment.

Definition 2.1. *With the quantities as introduced in Section 1 we define the* integrated risk process (IRP) *as the solution to the SDE*

$$dU_\theta(t) = c\,dt - dS(t) + U_\theta(t-)\left((1-\theta)\delta\,dt + \theta d\widehat{L}(t)\right), \quad t > 0, \quad U_\theta(0) = u\,. \tag{2.1}$$

Lemma 2.2. *The SDE (2.1) has the solution*

$$U_\theta(t) = e^{L_\theta(t)}\left(u + \int_0^t e^{-L_\theta(v)}\,(cdv - dS(v))\right), \quad t \ge 0\,. \tag{2.2}$$

Proof. Define

$$Z(t) = \int_0^t e^{-L_\theta(v-)}(cdv - dS(v)) = \int_0^t e^{-L_\theta(v)}\,(cdv - dS(v))\,, \quad t \ge 0\,. \tag{2.3}$$

Equality holds as the independent processes L_θ and S have no common jumps almost surely (a.s.) (see Cont and Tankov [2], Proposition 5.3). The integration by parts formula gives

$$d(X_\theta(t)Z(t)) = X_\theta(t-)dZ(t) + Z(t-)dX_\theta(t) + d[X_\theta, Z]_t\,, \quad t > 0\,,$$

where $[X_\theta, Z]$ denotes the quadratic covariation process of X_θ and Y. Using again that the processes L_θ and S have no common jumps a.s., an application of Theorems 28 and 29, Chapter II of Protter [23] yields $[X_\theta, Z]_t \equiv 0$. Thus, as $X_\theta(t-)dZ(t) = cdt - dS(t)$, we have

$$\begin{aligned} d\left(X_\theta(t)Z(t)\right) &= X_\theta(t-)dZ(t) + dX_\theta(t)Z(t-) \\ &= cdt - dS(t) + dX_\theta(t)\int_0^{t-} e^{-L_\theta(v-)}(cdv - dS(v))\,, \quad t > 0\,. \end{aligned}$$

Finally, from the last equality and from (1.3), (2.2) and (2.3) we get for $t > 0$

$$\begin{aligned} dU_\theta(t) &= udX_\theta(t) + cdt - dS(t) + dX_\theta(t)\int_0^{t-} e^{-L_\theta(v-)}(cdv - dS(v)) \\ &= cdt - dS(t) + X_\theta(t-)\left(u + \int_0^{t-} e^{-L_\theta(v)}(cdv - dS(v))\right)\frac{dX_\theta(t)}{X_\theta(t-)} \\ &= cdt - dS(t) + U_\theta(t-)\left((1-\theta)\delta\, dt + \theta d\widehat{L}(t)\right)\,. \end{aligned}$$

□

Example 2.3. [Geometric Brownian motion with jumps]
Assume that the log returns of the risky asset are modeled by

$$L(t) = \xi t + \sigma W(t) + C(t)\,, \quad t \geq 0\,,$$

where $\xi \in \mathbb{R}$, $\sigma > 0$, $(W(t))_{t\geq 0}$ is a standard Brownian motion and $C(t) = \sum_{j=1}^{M(t)} Z_j$, $t \geq 0$, is a compound Poisson process given by a homogeneous Poisson process $(M(T))_{t\geq 0}$ with intensity η and jump sizes represented by the generic r.v. Z. The Laplace exponent of L is given by

$$\varphi(s) = -\xi s + \sigma^2\frac{s^2}{2} + \eta(Ee^{-sZ} - 1)\,.$$

Note that L has drift $\gamma = E[L(1)] = \xi + \eta EZ$. By Lemma 1.3

$$L_\theta(t) = \xi_\theta t + \sigma_\theta W(t) + C_\theta(t)\,, \quad t \geq 0\,,$$

where C_θ is a compound Poisson process with the same jump intensity η as C and jump size $\log(1 + \theta(e^Z - 1))$. Moreover, $\xi_\theta = \xi\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)$ and $\sigma_\theta^2 = \theta^2\sigma^2$. The Laplace exponent of L_θ is given by

$$\varphi_\theta(s) = -\xi_\theta s + \sigma_\theta^2\frac{s^2}{2} + \int_{-\infty}^{\infty}(e^{-sx} - 1)\nu_\theta(dx) = -\xi_\theta s + \sigma_\theta^2\frac{s^2}{2} + \eta(E(1 + \theta(e^Z - 1))^{-s} - 1)\,,$$

and L_θ has drift

$$\gamma_\theta = \gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) + \eta(E[\log(1 + \theta(e^Z - 1))] - EZ)\,.$$

In case of the classical geometric Brownian motion model with drift; i.e. if $C(t) \equiv 0$, we have $\xi = \gamma$ and

$$\gamma_\theta = \theta\gamma + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta) \quad \text{and} \quad \sigma_\theta^2 = \theta^2\sigma^2\,. \tag{2.4}$$

As $L(t) = \gamma t + \sigma W(t)$, $t \geq 0$, we have $\widehat{L}(t) = (\gamma + \sigma^2/2)t + \sigma W(t)$, $t \geq 0$. The SDE (2.1) for U_θ reduces in this case to

$$dU_\theta(t) = cdt - dS(t) + U_\theta(t-)\Big(\big((1-\theta)\delta + \theta(\gamma + \frac{\sigma^2}{2})\big)dt + \theta\sigma dW(t)\Big)\,, t > 0\,, \quad U_\theta(0) = u\,. \tag{2.5}$$

Geometric Brownian motion as investment process in an integrated risk management context has been investigated by various authors. Paulsen [20, 21] considers the case, when $\theta = 1$, and Frolova Kabanov and Pergamenchchikov [6], when $\delta = 0$; these models are mathematically equivalent to ours.

Hipp and Plum [10, 11] analyse a model, when the insurance company invests into risky assets, not necessarily financed from the risk reser.v.e. In contrast to that, in our model the trading strategy θ is constant and $\theta \in [0,1]$, i.e. short selling is not allowed and the portfolio is self-financing.

More general models with exponential Lévy investment are considered in Paulsen [22], see also Tang and Tsitsiashvili [25]. The main focus in all these papers is on ruin estimation.

□

Example 2.4. [VG Lévy process as risky investment process]
The variance gamma process (VG), suggested by Madan and Seneta [17], is a normal mixture model, i.e. obtained by time change of an independent Brownian motion. The time changing process is a gamma Lévy process C, where $C(1) \stackrel{d}{=} \Gamma(\eta,\, r)$, i.e. the density is given by $f_\Gamma(x) = r^\eta x^{\eta-1} e^{-rx}/\Gamma(\eta)$, $x > 0$, for parameters $r, \eta > 0$. The characteristic triplet of C is $(0,\, 0,\, \nu_\Gamma)$ where $\nu_\Gamma(dx) = 1_{\{x>0\}}\eta x^{-1} e^{-rx}\, dx$. A non-symmetric VG model is given by

$$L(t) = \xi t + W(C(t))\,, \quad t \geq 0\,,$$

where $\xi > 0$ and W is Brownian motion with drift $a < 0$ and variance b^2. This makes it possible to model the usually obser.v.ed positive drift in combination with downwards jumps of the price process. Mean and variance of $L(1)$ are given by $\gamma = E[L(1)] = \xi + a\eta/r$ and $\mathrm{var}(L(1)) = b^2\eta/r + a^2\eta/r^2$. For the Laplace exponent of L we have

$$\varphi(s) = -\xi s - \eta \log\left(1 - \frac{1}{r}\big(b^2\frac{s^2}{2} - sa\big)\right)\,, \quad s \in \mathbb{R}\,. \tag{2.6}$$

The Lévy measure of L is given by

$$\nu(dx) = \frac{r^2}{\eta|x|} \exp\Big(\frac{a}{b^2}x - \frac{\sqrt{a^2+2b^2r^2/\eta}}{b^2}|x|\Big)\,dx\,, \quad x \in \mathbb{R}\,. \tag{2.7}$$

If $\theta < 1$, the characteristic triplet of L_θ calculated by Lemma 1.3 shows that L_θ is no longer a VG Lévy process. However, as L is of finite variation, also L_θ is and its Laplace exponent is given by

$$\varphi_\theta(s) = -\xi_\theta s + \int_{\mathbb{R}} (e^{-sx}-1)\nu_\theta(dx) = -\xi_\theta s + \int_{x>\log(1-\theta)} \big((1+\theta(e^x-1))^{-s}-1\big)\nu(dx)\,,$$

where $\xi_\theta = \theta\xi + (1-\theta)\delta$ and ν is as in (2.7). We refer to Cont and Tankov [2], Section 4, for more details. □

3 The discounted net loss process

We aim at measuring the risk of an insurance business at the beginning of a planning period. Hence, following long tradition in insurance, we work with the discounted integrated risk process. From a mathematical point of view we want to work with a stationary process aiming at a reasonable statistical risk assessment. Taking all this into account we introduce the discounted net loss process.

3.1 Definition, characteristic function and moments

Definition 3.1. *With the quantities introduced in Section 1 we define the* discounted net loss process (DNLP) *by*

$$V_\theta(t) = u - e^{-L_\theta(t)}U_\theta(t) = \int_0^t e^{-L_\theta(v)}\,(dS(v) - c\,dv)\,, \quad t \geq 0\,. \tag{3.1}$$

First we calculate the chf and the moment functions of V_θ.

Lemma 3.2. *For $t \geq 0$ denote by $\widehat{v}_{\theta,t}(s) = E\exp(isV_\theta(t))$ and $\widehat{f}(s) = Ee^{isY}$, $s \in \mathbb{R}$. Then*

$$\widehat{v}_{\theta,t}(s) \;=\; E\left[\exp\left(\int_0^t \Big(\lambda(\widehat{f}(se^{-L_\theta(v)})-1) - icse^{-L_\theta(v)}\Big)\,dv\right)\right].$$

Moreover, the following moment representations hold.
(a) Assume that $\varphi(1) < \infty$ and $EY = \mu < \infty$. Then for $t \geq 0$,

$$EV_\theta(t) = (\lambda\mu - c)\int_0^t Ee^{-L_\theta(v)}\,dv = \frac{c-\lambda\mu}{\varphi_\theta(1)}(1-e^{t\varphi_\theta(1)})\,.$$

(b) Assume that $\varphi(2) < \infty$ and $EY^2 = \mu_2 < \infty$. Then for $t \geq 0$,

$$\mathrm{var}(V_\theta(t)) \;=\; \lambda\mu_2 \int_0^t Ee^{-2L_\theta(v)}\,dv + (c-\lambda\mu)^2 \int_0^t \int_0^t \mathrm{cov}(e^{-L_\theta(v)}, e^{-L_\theta(v')})\,dv'\,dv\,.$$

(c) Assume that $\varphi(2) < \infty$ and $EY^2 = \mu_2 < \infty$. Then for $0 \leq y \leq t$,

$$\mathrm{cov}(V_\theta(y),\, V_\theta(t)) = \mathrm{var}(V_\theta(y)) + (\lambda\mu - c)^2 \int_0^{t-y} Ee^{-L_\theta(v)}dv \int_0^y \mathrm{cov}(e^{-L_\theta(v)},\, e^{-L_\theta(y)})\,dv\,.$$

Proof. We apply Lemma 15.1 in Cont and Tankov [2]: for every $t > 0$ and left continuous function $g:\ [0,\, t] \to \mathbb{R}$ and a Lévy process Z with characteristic exponent Ψ_Z,

$$E\left[\exp\left(i\int_0^t g(v)\,dZ(v)\right)\right] = \exp\left(\int_0^t \Psi_Z(g(v))\,dv\right)\,. \tag{3.2}$$

Setting $Z(t) = S(t) - ct$, $t \geq 0$, we obtain $\Psi_Z(s) = \lambda(\widehat{f}(s) - 1) - ics$, $s \in \mathbb{R}$. Conditioning on the sample path of L up to time t, and using the notation $E_L\,[E\,[\cdot]] = E\,[E\,[\cdot \mid L(v),\, v \in (0,\, t]]]$ for $t \geq 0$, we have by independence of L and S for $s \in \mathbb{R}$,

$$\begin{aligned}
\widehat{v}_{\theta,\,t}(s) &= E_L\left[E\left[\exp\left(is\int_0^t e^{-L_\theta(v)}dZ(v)\right)\right]\right]\\
&= E\left[\exp\left(\int_0^t \Psi_Z(se^{-L_\theta(v)})dv\right)\right]\\
&= E\left[\exp\left(\int_0^t \left(\lambda(\widehat{f}(se^{-L_\theta(v)}) - 1) - icse^{-L_\theta(v)}\right)dv\right)\right].
\end{aligned}$$

The moments of the process V_θ (if they exist) can be obtained by taking derivatives of the chf in 0. For the autocovariance function we also need $E\left[V_\theta(y)e^{-L_\theta(y)}\right]$ for $y \geq 0$. We apply (3.2) again and obtain

$$\begin{aligned}
&E\left[\exp\left(isV_\theta(y)e^{-L_\theta(y)}\right)\right]\\
&\quad= E\left[\exp\left(\int_0^y \left(\lambda\left(\widehat{f}(se^{-L_\theta(v)-L_\theta(y)}) - 1\right) - isce^{-L_\theta(v)-L_\theta(y)}\right)\,dv\right)\right]\,.
\end{aligned}$$

Taking the first derivative of this chf in 0 we obtain

$$E\left[V_\theta(y)e^{-L_\theta(y)}\right] \;=\; (\lambda\mu - c)E\left[\int_0^y e^{-L_\theta(v)-L_\theta(y)}\,dv\right]\,.$$

For the autocovariance function we calculate for $0 \leq y < t$

$$\begin{aligned}
\mathrm{cov}(V_\theta(t),\, V_\theta(y)) &= E\,[V_\theta(t)V_\theta(y)] - E\,[V_\theta(t)]\,E\,[V_\theta(y)]\\
&= E\,[V_\theta(y)E\,[V_\theta(t) \mid \mathcal{F}_y]] - E\,[E\,[V_\theta(t) \mid \mathcal{F}_y]]\,E\,[V_\theta(y)]\,.
\end{aligned}$$

We calculate the conditional expectation

$$
\begin{aligned}
E[V_\theta(t)\,|\,\mathcal{F}_y] &= E\left[V_\theta(y) + e^{-L_\theta(y)} \int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v)\,|\,\mathcal{F}_y\right] \\
&= V_\theta(y) + e^{-L_\theta(y)} E\left[\int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v)\right],
\end{aligned}
$$

where the last equality holds by the independent increments of L. By the stationarity increments property of L and Z we obtain

$$
\int_y^t e^{-(L_\theta(v)-L_\theta(y))}\, dZ(v) \stackrel{d}{=} \int_0^{t-y} e^{-L_\theta(v)}\, dZ(v) \stackrel{d}{=} V_\theta(t-y)\,,
$$

where the r.v. $V_\theta(t-y)$ is independent of $\mathcal{F}_y$. Hence we can write

$$
\begin{aligned}
&\operatorname{cov}(V_\theta(t),\, V_\theta(y)) \\
&= \operatorname{var}(V_\theta(y)) + E\left[V_\theta(t-y)\right]\left(E\left[V_\theta(y)e^{-L_\theta(y)}\right] - E\left[V_\theta(y)\right]E\left[e^{-L_\theta(y)}\right]\right) \\
&= \operatorname{var}(V_\theta(y)) + (\lambda\mu - c)E\left[V_\theta(t-y)\right] \\
&\quad \times \int_0^y \left(E\left[X_\theta^{-1}(v)X_\theta^{-1}(y)\right] - E\left[X_\theta^{-1}(v)\right]E\left[X_\theta^{-1}(y)\right]\right) dv\,,
\end{aligned}
$$

which implies (c). □

Remark 3.3. Note that for $\varphi_\theta(1) < 0$ we have $\lim_{t\to\infty} EV_\theta(t) = (\lambda\mu - c)/|\varphi_\theta(1)|$. Under the net profit condition $c > \lambda\mu$ the right hand side is negative. This can be interpreted that in this situation the mean profit is positive. □

3.2 Stationarity of the discounted net loss process

We are interested in possible stationarity of the discounted net loss process V_θ. The following example is well-known in the case of $c = 0$. For a pure bond strategy, i.e. when $\theta = 0$, the DNLP converges to a r.v. with finite left endpoint. In particular, when the insurance claims are exponentially distributed, the discounted net loss process converges to a gamma distribution.

Example 3.4. [Pure bond strategy]
For $\theta = 0$ we have $L_\theta(t) = \delta t$. Then for $s \in \mathbb{R}$ we get

$$
\begin{aligned}
E\left[e^{isV_0(t)}\right] &= \exp\left(\lambda \int_0^t \left(E\exp\left(ise^{-\delta v}Y\right) - 1\right) dv\right) \exp\left(-isc\int_0^t e^{-\delta v}\, dv\right) \\
&= \exp\left(\frac{\lambda}{\delta}\int_{e^{-\delta t}}^1 \left(E\exp\left(ise^{-\delta v}Y\right) - 1\right)\frac{1}{y}\, dv\right) \exp\left(isc\,\frac{e^{-\delta t}-1}{\delta}\right) \\
&\to \exp\left(\frac{\lambda}{\delta}\int_0^1 \left(Ee^{isyY} - 1\right)\frac{1}{y}\, dy\right) e^{-isc/\delta}, \quad t \to \infty\,.
\end{aligned}
$$

Hence, $V_0(t)$ converges in distribution and we denote by V_0^∞ the limit r.v. From the limit result above follows that V_0^∞ can be decomposed to $V_0^\infty = V_{0,+}^\infty - c/\delta$ where the random variable $V_{0,+}^\infty$ is a.s. positive. From this follows that the stationary r.v. V_0^∞ has no left tail and its left endpoint is $-c/\delta$.

When the claims are exponentially distributed with density $f(y) = e^{-y/\mu}/\mu$, $y > 0$, and chf $\widehat{f}(s) = Ee^{isY} = (1 - is\mu)^{-1}$, $s \in \mathbb{R}$; then we get for $s \in \mathbb{R}$

$$\lim_{t\to\infty} E\left[e^{isV_0(t)}\right] = \exp\left(\frac{\lambda}{\delta}\int_0^1 \left(\frac{1}{1-is\mu y} - 1\right)\frac{1}{y}\,dy\right) e^{-isc/\delta} = e^{-isc/\delta}\,(1-is\mu)^{-\lambda/\delta}\,.$$

We recognise $(1-is\mu)^{-\lambda/\delta}$ as the chf of a gamma distributed r.v. $X \stackrel{d}{=} \Gamma(\frac{\lambda}{\delta}, \frac{1}{\mu})$ with density $f_X(x) = \mu^{-\lambda/\delta} x^{-1+(\lambda/\delta)} \exp(-x/\mu)/\Gamma(\lambda/\delta)$, $x > 0$. Consequently, we have shown that

$$V_0(t) \quad \stackrel{d}{\to} \quad V_0^\infty \stackrel{d}{=} \Gamma(\frac{\lambda}{\delta}, \frac{1}{\mu}) - \frac{c}{\delta}\,, \quad t \to \infty\,.$$

For $c = 0$ this is a well known result (see e.g. the introduction in Nilsen and Paulsen [18] and references therein). □

We now turn to the discounted net loss process for $\theta > 0$. As this process is an exponential functional of a Lévy process and fits in the framework of generalized OU processes, and L_θ and S are independent processes, the NASCs of Proposition 2.4 of Lindner and Maller [16] apply to our situation. Whenever $L(t) \to \infty$ a.s. and the tail $\overline{F}(x) = 1 - F(x)$, $x > 0$, of the claim size distribution decreases to 0 not too slowly, then there exists a finite r.v. $V_\theta^{\infty,c}$ such that

$$V_\theta(t) \quad \stackrel{\text{a.s.}}{\to} \quad V_\theta^{\infty,c}\,, \quad t \to \infty\,. \tag{3.3}$$

Unfortunately, for very few examples the stationary distribution is known. The following examples can be found in Carmona, Petit and Yor [1]. We present them in terms of our insurance application.

Example 3.5. [Geometric Brownian motion as risky investment process and small claims; continuation of Example 2.3]
Let the risky asset be modeled by a geometric Brownian motion. Then, according to Example 2.3, the resulting investment process is also geometric Brownian motion with parameters γ_θ and σ_θ given in (2.4). When the claims of the portfolio are sufficiently small, it is possible to approximate the total claim amount process by Brownian motion. We consider this situation and take $(S(t)-ct)_{t\geq 0}$ as Brownian motion with drift $\lambda\mu - c < 0$ and variance $\lambda\mu$. Then $V_\theta^{\infty,c}$ follows a Pearson type IV distribution with density

$$f(x) = const.\,(1+x^2)^{-(\gamma_\theta/\sigma_\theta^2)+1/2} \exp\left(-\frac{2}{\sigma_\theta}\frac{c-\lambda}{\sqrt{\lambda\mu}}\arctan x\right)\,, \quad x \in \mathbb{R}\,.$$ □

Example 3.6. [Geometric Brownian motion as risky investment process, exponential claims and no premiums; continuation of Example 2.3]
Let the risky asset be modeled by a geometric Brownian motion and γ_θ and σ_θ be given by (2.4). Assume that $c = 0$ and that the insurance claims are exponentially distributed with mean μ. Then it is shown in Nilsen and Paulsen [18] that

$$V_\theta^{\infty,c} \stackrel{d}{=} \frac{X}{Z}\,,$$

where $X \sim \Gamma(b, \frac{1}{\mu})$ with density $f_X(x) = \mu^{-b} x^{b-1} e^{-x/\mu}/\Gamma(b)$, $x > 0$, and is independent of Z, which is beta distributed with density

$$f_Z(x) = \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b+1)} x^{a-1}(1-x)^{b-1}\,, \quad 0 < x < 1\,,$$

where

$$a = \frac{2\gamma_\theta}{\sigma_\theta^2} \quad \text{and} \quad b = \frac{\gamma_\theta}{\sigma_\theta^2}\left(\sqrt{1 + \frac{2\lambda\sigma_\theta^2}{\gamma_\theta^2}} - 1\right).$$

Straightforward calculations show that the density of $1/Z$

$$f_{1/Z}(x) \sim \frac{\Gamma(a+b+1)}{\Gamma(a)\Gamma(b+1)} x^{-a-1}\,, \quad x \to \infty\,.$$

Hence the corresponding distribution tail

$$\overline{F}_{1/Z}(x) \sim \frac{\Gamma(a+b+1)}{a\,\Gamma(a)\Gamma(b+1)} x^{-a}\,, \quad x \to \infty\,.$$

On the other hand, the r.v. X has light right tail and is independent of Z. Consequently, by Breiman's classical result

$$P(V_\theta^{\infty,c} > x) \sim const.\, x^{-a}\,, \quad x \to \infty\,,$$

with $a = 2\gamma_\theta/\sigma_\theta^2$. This will be confirmed by our result in Theorem 4.4; see also Example 4.5 below. □

For more general models the theory of discrete and continuous time perpetuities can provide at least the tail behaviour of such models. The advantage of our model lies in the fact that it has a natural discrete time skeleton, which allows us to apply standard methods from the theory of random recurrence equations.

Let $T_j = \sum_{k=1}^{j} E_k$, $j \in \mathbb{N}$, be the claim arrival times, where $(E_k)_{k\in\mathbb{N}}$ is a sequence of iid exponentially distributed r.v.'s with parameter λ. We denote by E a generic r.v. of $(E_k)_{k\in\mathbb{N}}$. Recall that Y is a generic claim size. This allows us to introduce a natural discretization of the process V_θ given by $(V_\theta(T_k))_{k\in\mathbb{N}_0}$, which will prove useful. We denote by

$$(A_\theta,\, B_\theta) = \left(Y e^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}\,dv\,,\, e^{-L_\theta(E)}\right). \tag{3.4}$$

Proposition 3.7. *Set $T_0 = 0$ and note that $N(T_k) = k$. For $k \in \mathbb{N}$ define*

$$\begin{aligned} A_{\theta,k} &= \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v)-L(T_{k-1}))}(dS(v) - cdv)\,, \\ B_{\theta,k} &= e^{-(L_\theta(T_k)-L_\theta(T_{k-1}))}\,. \end{aligned}$$

(a) Then $((A_{\theta,k}, B_{\theta,k}))_{k\in\mathbb{N}}$ is a sequence of iid bivariate r.v.'s with the same distribution as the vector in (3.4)*.*

(b) Define $(V_{\theta,k})_{k\in\mathbb{N}}$ by the following backward stochastic recurrence equation

$$V_{\theta,0} = 0 \quad \textit{and} \quad V_{\theta,k} = \sum_{m=1}^{k} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j}\,, \quad k \in \mathbb{N}\,. \tag{3.5}$$

Then $V_\theta(T_k) = V_{\theta,k}$ for all $k \in \mathbb{N}$.

Proof. (a) is an immediate consequence of the stationary and independent increments of Lévy processes.
(b) For $k \in \mathbb{N}$ we have

$$\begin{aligned} V_\theta(T_k) &= \int_0^{T_k} e^{-L_\theta(v)}\,(dS(v) - cdv) \\ &= \int_0^{T_{k-1}} e^{-L_\theta(v)}\,(dS(v) - cdv) + \int_{T_{k-1}}^{T_k} e^{-L_\theta(v)}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + e^{-L_\theta(T_{k-1})} \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v)-L_\theta(T_{k-1}))}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + \prod_{j=1}^{k-1} e^{-(L_\theta(T_j)-L_\theta(T_{j-1}))} \int_{T_{k-1}}^{T_k} e^{-(L_\theta(v)-L_\theta(T_{k-1}))}\,(dS(v) - cdv) \\ &= V_\theta(T_{k-1}) + A_{\theta,k} \prod_{j=1}^{k-1} B_{\theta,j}\,. \end{aligned}$$

We have used that for every Lévy process the stationary and independent increments property also holds for the random time inter.v.als defined by $(T_j)_{j\in\mathbb{N}}$. Equation (3.5) follows then by iteration. □

For the sequence $(V_{\theta,k})_{k\in\mathbb{N}_0}$ Goldie and Maller [8] derive also for this discrete time process NASCs for stationarity; see their Theorem 2.1. In our insurance context it is, however, more natural to work with moment conditions, which are slightly weaker. They are stated and discussed in Corollary 4.1 of [8], where also precise references to earlier work can be found. In Lemma A.3 we show that these conditions are satisfied in our model under weak conditions. We obtain the following result.

Theorem 3.8. *Assume that Y is supported on the whole of $\mathbb{R}^+$ and $EY = \mu < \infty$, $E[L(1)] > 0$ and $\varphi_\theta(1) < \lambda$. Let $(V_{\theta,k})_{k\in\mathbb{N}_0}$ be defined as in* (3.5).
(a) Then

$$V_{\theta,k} \quad \stackrel{\text{a.s.}}{\to} \quad V_\theta^\infty = \sum_{m=1}^{\infty} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j}\,, \quad k \to \infty\,, \tag{3.6}$$

where the series on the rhs converges absolutely with probability 1.
Moreover, V_θ^∞ satisfies the identity in law

$$V_\theta^\infty \stackrel{d}{=} A_\theta + B_\theta V_\theta^\infty\,, \tag{3.7}$$

where V_θ^∞ and $(A_\theta,\, B_\theta)$ are independent.
(b) Let the discounted net loss process $(V_\theta(t))_{t\ge 0}$ be defined by equation (3.1)*. Then $V_\theta(t)$ converges a.s. if and only if $V_{\theta,k}$ does and*

$$V_\theta^\infty = V_\theta^{\infty,c} \quad a.s. \tag{3.8}$$

Proof. (a) Stationarity of the discrete time sequence $(V_{\theta,k})_{k\in\mathbb{N}}$ is usually proved via the corresponding *backward stochastic recurrence equation*. In order to prove (3.7) we introduce r.v.'s $\widetilde{V}_{\theta,k}$ for $k \in \mathbb{N}$ invoking the same iid sequence $((A_{\theta,k}, B_{\theta,k}))_{k\in\mathbb{N}}$ as above.

$$\widetilde{V}_{\theta,0} = 0 \quad \text{and} \quad \widetilde{V}_{\theta,k} \quad = \quad A_{\theta,k} + \widetilde{V}_{\theta,k-1} B_{\theta,k} = \sum_{m=1}^{k} A_{\theta,m} \prod_{j=m+1}^{k} B_{\theta,j}\,, \quad k \in \mathbb{N}\,.$$

We obser.v.e that for every $k \in \mathbb{N}$

$$((A_{\theta,j}, B_{\theta,j}))_{1\le j\le k} \stackrel{d}{=} ((A_{\theta,k-j+1}, B_{\theta,k-j+1}))_{1\le j\le k}\,,$$

implying that

$$\sum_{m=1}^{k} A_{\theta,m} \prod_{j=1}^{m-1} B_{\theta,j} \stackrel{d}{=} \sum_{m=1}^{k} A_{\theta,m} \prod_{j=m+1}^{k} B_{\theta,j}\,,$$

hence $V_{\theta,k} \stackrel{d}{=} \widetilde{V}_{\theta,k}$ for all $k \in \mathbb{N}$. The result goes back to Kesten [12] (see his Theorem 5); Proposition 8.4.3 in Embrechts, Klüppelberg and Mikosch [3] also states the result with proof. The conditions in that proposition hold due to Lemma A.3 in Appendix 4.2.

(b) Consider the continuous time process V_θ.

$$\begin{aligned} V_\theta(t) \quad &\stackrel{a.s.}{=} \quad \int_0^{T_{N(t)}} e^{-L_\theta(v)}(dS(v) - cdv) + \int_{T_{N(t)}}^{t} e^{-L_\theta(v)}(dS(v) - cdv) \\ &= \quad V_{\theta,N(t)} + e^{-L_\theta(T_{N(t)})} \int_{T_{N(t)}}^{t} e^{-(L_\theta(v) - L_\theta(T_{N(t)}))}\,(dS(v) - cdv)\,, \quad t \ge 0 =,\,, \end{aligned}$$

where in the last line the integral is independent of the first summand. Since $N(t) \stackrel{\text{a.s.}}{\to} \infty$ as $t \to \infty$, we know from part (a) that $V_{\theta,N(t)} \stackrel{\text{a.s.}}{\to} V_\theta^\infty$ as $t \to \infty$. Moreover, as $EL(1) > 0$, we have by Lemma A.1(b) that $E[L_\theta(1)] > 0$ and hence $e^{-L_\theta(T_{N(t)})} \stackrel{\text{a.s.}}{\to} 0$ as $t \to \infty$. As $t - T_{N(t)} \stackrel{d}{=} E$, the last integral is a finite random variable. This implies (3.8). $\square$

Remark 3.9. The condition $\varphi_\theta(1) < \lambda$ is needed to show Lemma A.3(a) which, together with Lemma A.3(b), ensures a.s. convergence of $V_{\theta,k}$ as $k \to \infty$ to a finite r.v. When $\varphi_\theta(1) \le 0$, the limit variable has finite mean, whereas for $\varphi_\theta(1) \in (0, \lambda)$ it has infinite mean; see also Lemma 3.2 and Remark 3.3.

4 Tail behaviour of the discounted net loss process

From now on in most of our results we exclude the pure bond strategy and assume that $\theta \in (0, 1]$. Moreover, we assume that the conditions of Theorem 3.8 hold. Then the stationary random variable V_θ^∞ exists and satisfies the fixed point equation (3.7). As we are interested in distributional properties of V_θ^∞ we can work with the continuous time process or with the discrete skeleton process as they both lead to the same a.s. limit.

Our next goal is the tail behaviour of V_θ^∞. To this end we start with some preliminary results on Laplace transforms. Note that the condition $\delta < \log E[e^{L(1)}] = \varphi(-1)$ is quite natural. Indeed, it guarantees that the expected value of the risky stock investment is larger than the riskless bond investment.

Lemma 4.1. *Let $\theta \in (0, 1]$ and assume that $0 < E[L(1)] < \infty$, and either $\sigma > 0$ or $\nu((-\infty, 0)) > 0$. Define $\mathcal{V}_\infty = \{v \ge 0 : \varphi_1(v) < \infty\}$ and assume that $v_1^* = \sup \mathcal{V}_\infty \notin \mathcal{V}_\infty$.*

(a) Then there exists a unique positive $\kappa = \kappa(\theta) > 0$ such that $\varphi_\theta(\kappa) = 0$. Moreover, $\varphi_\theta'(\kappa) > 0$ and

$$\kappa(\theta) \begin{cases} > 1 & \text{if } \varphi_\theta(1) < 0\,, \\ = 1 & \text{if } \varphi_\theta(1) = 0\,, \\ < 1 & \text{if } \varphi_\theta(1) \in (0, \lambda)\,. \end{cases} \tag{4.9}$$

(b) Let $\delta < \varphi(-1)$. For fixed $s > 0$ the function $\varphi_\theta(s)$ is strictly convex in θ.

(c) Let $\delta < \varphi(-1)$. Then the function $\kappa(\theta)$ as defined in (a) is decreasing in θ.

Proof. (a) First note that $\varphi_\theta(0) = 0$ for all $\theta \in (0, 1]$. Moreover, $\varphi_1'(0) = -E[L] < 0$ and $\lim_{v \to v_1^*} \varphi_1(v) = \infty$. Hence the existence of $\kappa(1)$ follows from convexity.
Now assume that $\theta \in (0, 1)$. For $\theta < 1$ we set $p = \log((1 + \theta^{-1}(e - 1)) > 0$, $q = -\infty$, if $\theta^{-1}(1 - e^{-1}) \ge 1$, and $q = \log(1 - \theta^{-1}(e^{-1} - 1)) < 0$, if $\theta^{-1}(1 - e^{-1}) < 1$. Then for $s \in \mathbb{R}^+$

we get

$$\begin{aligned}\int_{|x|\geq 1} e^{-sx}\nu_\theta(dx) &= \int_{|\log(1+\theta(e^x-1))|\geq 1} (1+\theta(e^x-1))^{-s}\nu(dx) \\ &= \int_{-\infty}^{q} (1+\theta(e^x-1))^{-s}\nu(dx) + \int_{p}^{\infty} (1+\theta(e^x-1))^{-s}\nu(dx) \\ &\leq (1-\theta)^{-s}\int_{-\infty}^{q}\nu(dx) + e^{-s}\int_{p}^{\infty}\nu(dx) \;<\; \infty\,.\end{aligned}$$

By Proposition 3.14 in Cont and Tankov [2] $\varphi_\theta(s) < \infty$.
On the other hand, for $0 < \theta \leq 1$ we have

$$Ee^{-vL_\theta(1)} \;\geq\; P(L_\theta(1) < 0)E[e^{-vL_\theta(1)} \,|\, L_\theta(1) < 0)]\,.$$

Note that $\lim_{v\to\infty} E[e^{-vL_\theta(1)} \,|\, L_\theta(1) < 0] = \infty$. Since $P(L_\theta(1) < 0) > 0$ holds by Lemma A.2, we have

$$\lim_{v\to\infty} Ee^{-vL_\theta(1)} = \lim_{v\to\infty} e^{\varphi_\theta(v)} = \infty\,. \tag{4.10}$$

Then the existence of $\kappa(\theta)$ for $\theta \in (0, 1)$ follows from the above together with the convexity of φ_θ and the fact that $\varphi'_\theta(0) = -E[L_\theta(1)] \in (-\infty, 0)$ (see Lemma A.1 (a,b)).

(b) We consider the function $\varphi_\theta(s) =: \varphi(\theta, s)$ for $\theta \in [0, 1]$ and $s > 0$ as a function in two variables. Then

$$\begin{aligned}\varphi(\theta, s) &:= -\left(\delta + \theta(\gamma + \frac{\sigma^2}{2} - \delta)\right)s + \frac{\sigma^2}{2}s(s+1)\theta^2 \\ &\quad + \int_{\mathbb{R}} \left((1+\theta(e^x-1))^{-s} - 1 + s\theta x 1_{\{|x|\leq 1\}}\right)\nu(dx)\,,\end{aligned}$$

and we investigate $\varphi(\theta, s)$ as a function of θ. First notice that

$$\begin{aligned}\frac{\partial}{\partial\theta}\varphi(\theta, s) &= -\left(\gamma + \frac{\sigma^2}{2} - \delta\right)s + \sigma^2 s(s+1)\theta \\ &\quad -s\int_{\mathbb{R}}\left(\frac{(e^x-1)}{(1+\theta(e^x-1))^{s+1}} - x1_{\{|x|\leq 1\}}\right)\nu(dx)\,.\end{aligned}$$

As $\delta < \varphi(-1)$, we have $\frac{\partial}{\partial\theta}\varphi(0, s) = -(\varphi(-1) - \delta)s < 0$. Secondly,

$$\frac{\partial^2}{\partial\theta^2}\varphi(\theta, s) = s(s+1)\left(\sigma^2 + \int_{\mathbb{R}}\frac{(e^x-1)^2}{(1+\theta(e^x-1))^{s+2}}\nu(dx)\right) > 0\,, \tag{4.11}$$

i.e. the function $\varphi(\theta, s)$ is strictly convex in θ.

(c) From (b) follows that for each fixed $s > 0$ there exists some $\theta_*(s) > 0$, in which $\varphi(\theta, s)$ attains its minimum. Consider $0 \leq \theta_1 < \theta_2 \leq 1$. We shall show that $\kappa(\theta_1) > \kappa(\theta_2) > 0$.

To this end fix $s = \kappa(\theta_2)$ and consider its corresponding value $\theta_*(s) = \operatorname{argmin}_\theta \varphi(\theta, s)$. Assume first that $\theta_*(\kappa(\theta_2)) \geq \theta_2$. Then $\varphi(\theta,\, \kappa(\theta_2))$ is decreasing in $[0,\, \theta_2)$ and hence

$$0 = \varphi(\theta_2,\, \kappa(\theta_2)) \leq \varphi(0,\, \kappa(\theta_2)) = -\delta\kappa(\theta_2) < 0\,,$$

which is a contradiction. Hence $\theta_*(\kappa(\theta_2)) < \theta_2$. So for $\theta \in [0,\, \theta_*(\kappa(\theta_2))\,)$, $\varphi(\theta,\, \kappa(\theta_2))$ is decreasing function of θ and for $\theta \in (\theta_*(\kappa(\theta_2)),\, 1]$, $\varphi(\theta,\, \kappa(\theta_2))$ is an increasing function of θ. Next consider $\theta_1 < \theta_*(\kappa(\theta_2))$. This implies

$$\varphi(\theta_1,\, \kappa(\theta_2)) < \varphi(0,\, \kappa(\theta_2)) < 0 = \varphi(\theta_1,\, \kappa(\theta_1))\,,$$

which – by convexity of $\varphi(\theta,\, s)$ in s (for fixed θ) – implies $\kappa(\theta_1) > \kappa(\theta_2)$. Assume now that $\theta_*(\kappa(\theta_2)) \leq \theta_1 < \theta_2$, where $\varphi(\theta,\, s)$ is increasing in θ. Then

$$\varphi(\theta_1,\, \kappa(\theta_2)) < \varphi(\theta_2,\, \kappa(\theta_2)) = 0 = \varphi(\theta_1,\, \kappa(\theta_1))\,,$$

hence, again by convexity of $\varphi(\theta,\, s)$ in s, we obtain $\kappa(\theta_1) > \kappa(\theta_2)$. This completes the proof. □

4.1 Claims with finite moment of order κ

Theorem 4.1 of Goldie [7] guarantees under natural conditions, which hold by Lemma A.4 that V_θ^∞ has a heavy left **or** right tail. In the context of risk management, however, only the right tail is of prime interest. Invoking the theory of large deviations as suggested in the context of ruin theory by Nyrhinen [19] gives us a method to decide about right and left tails separately. The next lemma concerns properties of

$$l_\theta(s) = \log E[e^{-sL_\theta(E)}] = \log E[B_\theta^s] = \log \frac{\lambda}{\lambda - \varphi_\theta(s)}\,. \tag{4.12}$$

First note that $l_\theta(s) < \infty$ on $\mathcal{S}_\theta = \{v \geq 0\, :\, \varphi_\theta(v) < \lambda\}$ and $\sup \mathcal{S}_\theta \notin \mathcal{S}_\theta$.

Lemma 4.2. *Let the conditions of Lemma 4.1 hold and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Then the following hold.*
(a) l_θ is strictly convex and continuously differentiable on the interior of $\mathcal{S}_\theta$ and $l_\theta(\kappa) = 0$.
(b) There exists $\beta = \beta(\theta) > 0$ such that $l_\theta(\kappa + \beta) < \infty$.
(c) $l'_\theta(\kappa) \in (0,\, \infty)$ and $P(B_\theta > 1) > 0$.

Proof. (a) follows as in Lemma 4.1 and by definition of κ.
For (b) we note that from $\varphi_\theta(0) = \varphi_\theta(\kappa) = 0$ and strict convexity follows that $\varphi_\theta(s) < 0$ for $s \in (0,\, \kappa)$. As $\lambda > 0$, there exist $b \in (0, \lambda)$ and $\beta > 0$, such that $\varphi_\theta(\kappa + \beta) = b < \lambda$. Hence, for this $\beta > 0$ we have $l_\theta(\kappa + \beta) < \infty$.

The first part of (c) follows from $l'_\theta(\kappa) = \lambda^{-1}\varphi'_\theta(\kappa) \in (0,\infty)$ as $\varphi'_\theta(\kappa) \in (0,\infty)$.
For the second part of (c) we use continuity in probability of Lévy processes. Since $P(L_\theta(1) < 0) > 0$, there exist $\eta, \epsilon > 0$, such that for $t \in (1-\epsilon, 1+\epsilon)$ we have $P(L_\theta(t) < 0) \geq \eta$. Then

$$P(B_\theta > 1) = P(L_\theta(E) < 0) \geq \lambda \int_{1-\epsilon}^{1+\epsilon} P(L_\theta(z) < 0) e^{-\lambda z} dz \geq \eta\lambda \int_{1-\epsilon}^{1+\epsilon} e^{-\lambda z} dz > 0\,.$$

□

Next we present a large deviations result, which is proved in Appendix B. The idea is taken from Nyrhinen [19] and his proof is adapted to our situation.

Lemma 4.3. *Let the conditions of Lemma 4.1 hold and $\kappa(\theta) \in (0,\infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Assume also that Y is supported on the whole of $\mathbb{R}^+$ and that $EY < \infty$. Then*

$$\liminf_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \geq -\kappa(\theta) \quad \text{and} \quad \liminf_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} \geq -\kappa(\theta)\,. \tag{4.13}$$

In the following result we show that V_θ^∞ has heavy left **and** right tail. It is an application of Theorem 4.7 of Goldie [7] in combination with Lemma 4.3.

Theorem 4.4. *Assume that the conditions of Theorem 3.8 and Lemma 4.1 hold. Let $\kappa = \kappa(\theta) \in (0,\infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Assume also that Y is supported on the whole of $\mathbb{R}^+$. Let β be as in Lemma 4.2(b) and assume that*

$$EY^{\kappa+\beta} < \infty\,. \tag{4.14}$$

Then there exist constants $C_\pm$ such that for $x \to \infty$

$$P(V_\theta^\infty > x) = C_+\, x^{-\kappa} + O(x^{-(\kappa+\beta/2)}) \quad \text{and} \quad P(V_\theta^\infty < -x) = C_-\, x^{-\kappa} + O(x^{-(\kappa+\beta/2)})\,. \tag{4.15}$$

Moreover,

$$C_\pm = C_\pm(\theta) = \frac{1}{\kappa\, m} E\left[\left((A_\theta + B_\theta V_\theta^\infty)^\pm\right)^\kappa - \left((B_\theta V_\theta^\infty)^\pm\right)^\kappa\right] > 0\,, \tag{4.16}$$

where

$$m = m(\theta) = \frac{1}{\lambda}\varphi'_\theta(\kappa(\theta)) \in (0,\infty)\,. \tag{4.17}$$

Proof. Lemma 4.2 guarantees that $m = \varphi'_\theta(\kappa)/\lambda = l'_\theta(\kappa) \in (0,\infty)$. The rate result (4.15) holds by Theorem 4.7 in Goldie [7], where it is possible to choose the parameter $\beta = \beta(\theta) > 0$ in Theorem 3.2 of Goldie [7] small enough such that the contour integral in formula (3.8) vanishes. Note also that we can always choose the same β for the left and right tail. For more details see also the discussions after Theorems 3.1 and 3.2 in Goldie [7]. The conditions of this theorem are satisfied: (3.3) and (3.4) hold by the choice of β and Lemma 4.2(b).

Furthermore, $E[|A_\theta|^{\kappa+\beta}] < \infty$ by (4.14) and Lemma A.4(c). Define the probability law $\eta(dx) = e^{\kappa x} P(\log B_\theta \in dx)$. This is spread out as $\log B_\theta = -L_\theta(E)$ is. The corresponding first moment is positive as $\varphi_\theta'(\kappa) > 0$ by Lemma 4.1, and the second moment is finite, since the moment generating function exists in a neighbourhood of 0. Finally, $\widetilde{\eta}(\beta) = \varphi_\theta(\kappa + \beta) < \infty$ by Lemma 4.2(b).

To prove that $C_+(\theta) > 0$ we apply Lemma 4.3. Assume that $C_+(\theta) = 0$. Then from (4.15) follows that there exists some constant $M \in (0, \infty)$ and some x_0 such that

$$P(V_\theta^\infty > x) \le M x^{-(\kappa+\beta/2)}, \quad x > x_0,$$

which implies, taking logarithms,

$$\frac{\log P(V_\theta^\infty > x)}{\log x} \le \frac{\log M}{\log x} - \kappa - \frac{\beta}{2}.$$

Now letting x tend to ∞ and making use of (4.13) we get the following inequality chain

$$-\kappa \le \liminf_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \le \lim_{x\to\infty} \frac{\log P(V_\theta^\infty > x)}{\log x} \le -\kappa - \frac{\beta}{2},$$

which is a contradiction to $\beta > 0$. Hence $C_+ > 0$.

To prove that $C_- > 0$ note that $P(V_\theta^\infty < -x) = P(-V_\theta^\infty > x)$. Moreover, $-V_\theta^\infty$ is the almost sure limit of the random recurrence equation

$$-V_{\theta,0} = 0 \quad \text{and} \quad -V_{\theta,n} = \sum_{m=1}^{n} (-A_{\theta,m}) \prod_{j=1}^{m-1} B_{\theta,j}, \quad n \in \mathbb{N},$$

with $(A_{\theta,k},\, B_{\theta,k})$ as defined in (3.5). Hence, Lemma 4.3 applies. □

Theorem 4.4 says that V_θ^∞ has left **and** right Pareto-like tails. By Lemma 4.1(c) the Pareto index $\kappa = \kappa(\theta)$ is decreasing in θ. This can be interpreted that the more we invest into the risky asset the heavier the tail of the stationary DNLP becomes. More risky investment increases the risk.

Example 4.5. [Dangerous investment]
In this example we demonstrate that investment into risky stock can be dangerous, although the insurance claims are moderate. Assume for simplicity that the claims have moments of all order. Let the conditions on the investment process in Theorem 3.8 be satisfied so that there exists an a.s. limit V_θ^∞ of the DNLP. Theorem 4.4 gives

$$P(V_\theta^\infty > x) \sim C_+(\theta) x^{-\kappa}, \quad x \to \infty, \tag{4.18}$$

where $\kappa = \kappa(\theta)$ is determined by the investment process only. Intuitively, in this case the extremes of the investment process dominate the extremes of the resulting integrated risk process.

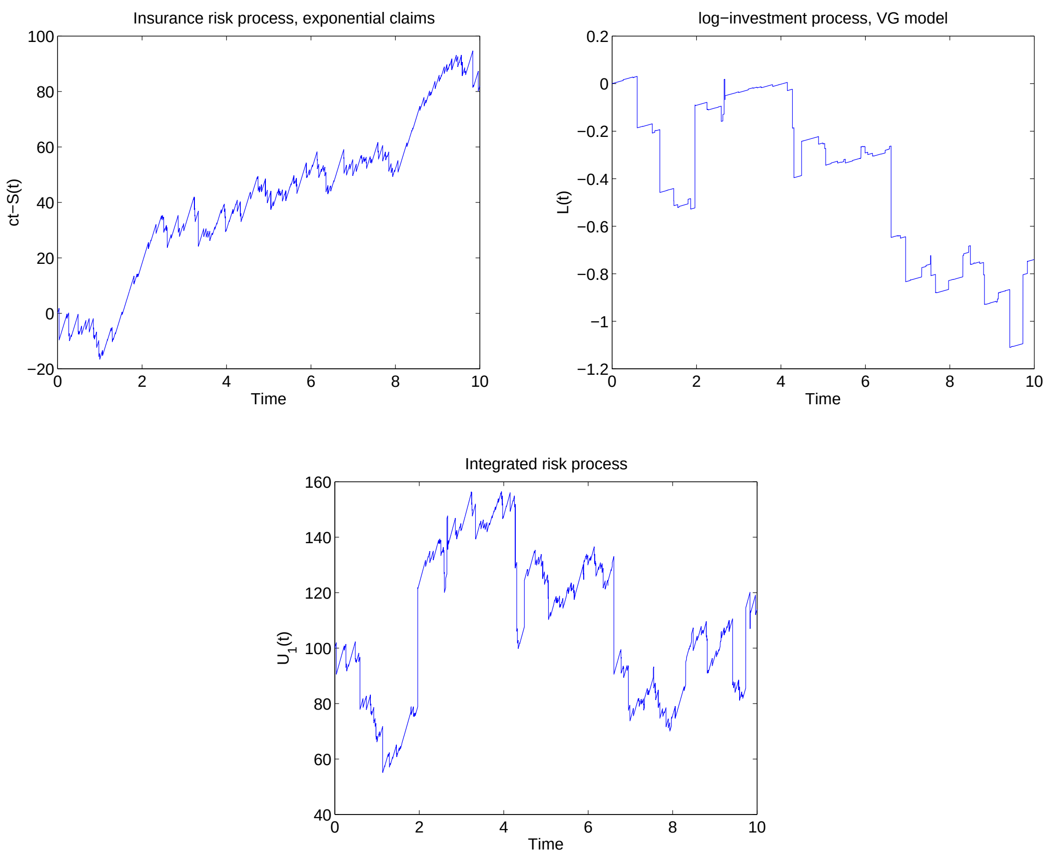

Figure 1: *Upper left plot:* sample path of the insurance tisk process with premium rate $c = 50$, intensity of the Poisson claim arrival process $\lambda = 20$ and exponentially distributed claims with mean $\mu = 2$, i.e. $\overline{F}(x) = e^{-x/2}$, $x > 0$. *Upper right plot:* sample path of the log-investment process for investment strategy $\theta = 1$ (pure stock investment) and the VG process $L(t) = qt + W_{a,b}(S_\Gamma(t))$, $t > 0$, with parameters $q = 0.05$. W is Brownian motion with drift $a = -0.01$, variance $b^2 = 0.04 - a^2$ and $\mathrm{var}(\mathrm{S}_\Gamma) = 1$. *Lower plot:* sample path of the resulting IRP with initial capital $u = 100$. The time horizon is $T = 10$. It is clearly seen that the jumps of the IRP are dominated by the jumps of the investment process.

The parameter κ can only be calculated explicitly, if the price process of the risky asset is geometric Brownian motion; see Example 2.3. Then the investment process is again geometric Brownian motion given by

$$X_\theta(t) = \exp(\gamma_\theta t + \sigma_\theta W(t)) , \quad t \geq 0 ,$$

with γ_θ and σ_θ as in (2.4). The value κ is the unique positive solution to

$$\varphi_\theta(s) = -\gamma_\theta s + \frac{\sigma_\theta^2}{2} s^2 = 0$$

given by

$$\kappa = \kappa(\theta) = \frac{2\gamma_\theta}{\sigma_\theta^2} = \frac{2}{\sigma^2\theta^2}\left(\gamma\theta + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)\right) .$$

In the case of Brownian motion with jumps with distribution Z (Example 2.3), κ is given as the unique positive solution to

$$\varphi_\theta(s) = -\xi_\theta s + \sigma_\theta^2 \frac{s^2}{2} + \eta(E[(1+\theta(e^Z - 1))^{-s}] - 1) = 0\,,$$

where ξ_θ and σ_θ are given in Example 2.3. Even in this simple case $\kappa(\theta)$ can only be found by numerical methods. The problem becomes even more difficult for a VG Lévy process (Example 2.4) or any other process with infinite jump activity.

In Figure 2 we have plotted the value $\kappa(\theta)$ as a function of the investment strategy θ for three different models for the risky asset. Recall that by Lemma 4.1(c) $\kappa(\theta)$ is decreasing in θ for all Lévy models. This means that in all models more investment into the risky asset leads to a heavier tail of V_θ^∞; i.e. more risky investment yields a higher risk.

We compare a Brownian motion model with two different VG models. The parameters are chosen such that mean and variance of the log returns of the risky asset are the same in all models. As we can see in Figure 2, jumps in the model yield a smaller κ, corresponding to a heavier tail of V_θ^∞. Higher intensity of large negative jumps yields also a smaller κ. □

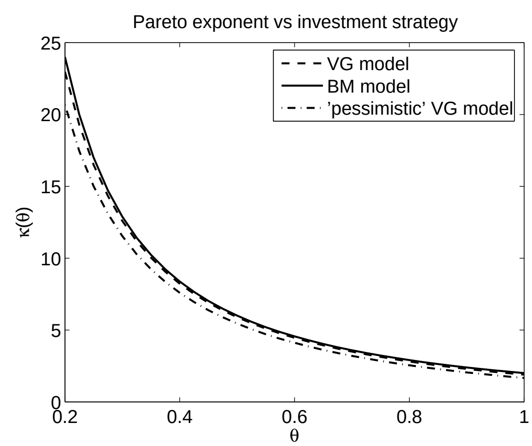

Figure 2: The Pareto exponent $\kappa(\theta)$ in (4.18) as a function of the investment strategy θ. We compare the following investment models: Brownian motion model with drift 0.04 and volatility 0.2, a VG model with parameters as in Figure 1 and a more pessimistic VG model of the form $L(t) = qt + W_{a,b}(S_\Gamma(t))$ where $q = 0.14$, $a = -0.1$ and $b^2 = 0.04 - a^2$ (more large negative jumps, which are compensated by a larger deterministic drift q).

4.2 Regularly varying claims

In this section we consider claim size distributions satisfying $\overline{F}(x) = x^{-\alpha}\ell(x)$, $x > 0$, where $\lim_{x\to\infty} \ell(xt)/\ell(x) = 1$ for all $t > 0$; i.e. $\overline{F}$ is regularly varying with index $\alpha > 1$ and we require throughout that $\alpha < \kappa$. Here $\kappa = \kappa(\theta) \in (1, \infty)$ is the unique value satisfying $\varphi_\theta(\kappa) = 0$ for some fixed $\theta \in (0, 1]$ as defined in Lemma 4.1. In this case $EY^\kappa = \infty$, hence this is a different situation than in Section 4.1. In the next proposition we shall see that in this case the tail of the stationary r.v. V_θ^∞ is determined by the tail behaviour of Y.

Theorem 4.6. *Let V_θ^∞ be the stationary solution to the stochastic recurrence equation (3.5). Let $\kappa = \kappa(\theta) \in (1, \infty)$ be the unique value satisfying $\varphi_\theta(v) = 0$. Assume that the claim size Y has distribution with regularly varying tail for some $\alpha \in (1, \kappa(\theta))$. Then the following assertions hold.*

(a) Right tail. *V_θ^∞ has also regularly varying tail with index α, more precisely,*

$$P(V_\theta^\infty > x) \sim \frac{\lambda}{|\varphi_\theta(\alpha)|} P(Y > x)\,, \quad x \to \infty\,. \tag{4.19}$$

(b) Left tail. *Assume that $\sigma > 0$ or $\nu(-\infty, 0) > 0$. In the case when L is of finite variation, assume that either the drift is non-zero, or that for no $r > 0$ the support of the Lévy measure ν_θ is concentrated on $r\mathbb{Z}$. Then*

$$\limsup_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} = -\kappa\,.$$

In particular,

$$\lim_{x\to\infty} \frac{P(V_\theta^\infty < -x)}{P(V_\theta^\infty > x)} = 0\,.$$

Proof. (a) Recall that $\varphi_\theta(0) = \varphi_\theta(\kappa)$ and φ_θ is strictly convex in s; i.e. $\varphi_\theta(s) < 0$ for all $0 < s < \kappa$. As $\alpha < \kappa$ we have $\varphi_\theta(\alpha) < 0$. Hence

$$E[B_\theta^\alpha] = \frac{\lambda}{\lambda - \varphi_\theta(\alpha)} < 1\,,$$

and there exists some $\beta > 0$ such that $E[B_\theta^{\alpha+\beta}] < \infty$. Then, if we can show that A_θ is regularly varying with index α, it follow directly from Proposition 2.4 in Konstantinides and Mikosch [13] that

$$P(V_\theta^\infty > x) \sim \frac{1}{(1 - E[B_\theta^\alpha])} P(A_\theta > x)\,, \quad x \to \infty\,.$$

As there exists some $\beta > 0$, such that $E[B_\theta^{\alpha+\beta}] < \infty$, from Breiman's classical result, see Lemma 2.2 in [13], follows that

$$P(Ye^{-L_\theta(E)} > x) = P(YB_\theta > x) \sim E[B_\theta^\alpha] P(Y > x)\,, \quad x \to \infty\,. \tag{4.20}$$

Define $\xi = Ye^{-L_\theta(E)}$ and $\eta = c\int_0^E e^{-L_\theta(v)}\,dv$, then both r.v.'s ξ and η are a.s. positive. On the one hand we estimate

$$P(\xi - \eta > x) \le P(\xi > x)\,. \tag{4.21}$$

On the other hand, for every $\epsilon > 0$ we calculate

$$\begin{aligned} P(\xi-\eta > x) + P(\eta > \epsilon x) &\ge P(\xi - \eta > x\,,\, \eta \le \epsilon x) + P(\eta > \epsilon x) \\ &\ge P(\xi > (1+\epsilon)x,\, \eta \le \epsilon x) + P(\eta > \epsilon x,\, \xi > (1+\epsilon)x) \\ &= P(\xi > (1+\epsilon)x)\,. \end{aligned}$$

This implies

$$P(\xi - \eta > x) \ge P(\xi > (1+\epsilon)x) - P(\eta > \epsilon x)\,. \tag{4.22}$$

As $1 < \alpha < \kappa$, by (A.2) we know that $E[\eta^\alpha] < \infty$. As a consequence, for every $\epsilon > 0$ follows $\lim_{x\to\infty} x^\alpha P(\eta > \epsilon x) = 0$. This together with (4.20) and inequalities (4.21) and (4.22) implies the following estimates for the tail of A_θ as $x \to \infty$:

$$E[B_\theta^\alpha]\frac{P(Y > (1+\epsilon)x)}{P(Y > x)} \sim \frac{P(\xi > (1+\epsilon)x)}{P(Y > x)} \le \frac{P(A_\theta > x)}{P(Y > x)} \le \frac{P(\xi > x)}{P(Y > x)} \to E[B_\theta^\alpha]\,.$$

Letting $x \to \infty$ on the left hand side, and then $\epsilon \to 0$ gives (4.19).

(b) First notice that

$$P(V_\theta^\infty < -x) \le P\left(c\int_0^\infty \exp(-L_\theta(v))dv > x\right)\,. \tag{4.23}$$

The r.v. $V_\theta^{\infty,-} = c\int_0^\infty \exp(-L_\theta(v))dv$ satisfies the fixed point equation

$$V_\theta^{\infty,-} \stackrel{d}{=} A_\theta^- + B_\theta V_\theta^{\infty,-}\,,$$

for $A_\theta^- = c\int_0^E \exp(-L_\theta(v))dv > 0$ and $B_\theta = \exp(-L_\theta(E)) > 0$ a.s.. By (A.2), setting $g = \kappa > 1$, $E[|A_\theta^-|^\kappa] < \infty$. Therefore we may apply Theorem 4.1 and Lemma 2.2 of Goldie [7] and we get for some constant $C > 0$

$$P\Big(c\int_0^\infty \exp(-L_\theta(v))dv > x\Big) \sim C\,x^{-\kappa} \quad x \to \infty\,.$$

Inequality (4.23) ensures that

$$\limsup_{x\to\infty} \frac{\log P(V_\theta^\infty < -x)}{\log x} \le -\kappa\,.$$

From this follows that for every $\varepsilon > 0$ there exists some $x_0 = x_0(\varepsilon)$ such that $P(V_\theta^\infty < -x) \le x^{-\kappa+\varepsilon}$ holds for all $x \ge x_0$; on the other hand, due to (4.19), also $P(V_\theta^\infty > x) \ge x^{-\alpha+\varepsilon/2}$ for all $x \ge x_0$. Since $\alpha < \kappa$, for ε small enough, for all $x > x_0$ we get

$$\frac{P(V_\theta^\infty < -x)}{P(V_\theta^\infty > x)} \le x^{-(\kappa-\alpha-\varepsilon/2)} \to 0\,, \quad x \to \infty\,. \qquad \square$$

Theorem 4.6(a) gives a Pareto-like right tail. In the context of risk management this is the important tail as it describes the likelihood of large losses.

Example 4.7. [Dangerous claims]
In this example we demonstrate how large insurance claims may dominate the extremes in the integrated risk process. Let the claims have Pareto-like tail with exponent $\alpha > 1$, i.e. $P(Y > x) \sim C_Y x^{-\alpha}$, $x \to \infty$, for some constant $C_Y > 0$. Then the claims have finite moments up to order α, including a finite mean, but no moments of order larger than α. Let the conditions on the investment process in Theorem 3.8 be satisfied. Then there exists an a.s. limit V_θ^∞ of the DNLP. Theorem 4.6 applies: recall first from Lemma 4.1(c), if $\alpha < \kappa(1)$, then $\alpha < \kappa(\theta)$ for all $\theta \in [0, 1]$). In this case,

$$P(V_\theta^\infty > x) \sim C(\theta) x^{-\alpha}, \quad x \to \infty. \tag{4.24}$$

The investment process enters only into the constant $C(\theta) = \lambda\mu C_Y / |\varphi_\theta(\alpha)|$. Intuitively, in this case the large insurance claims dominate the extremes of the resulting IRP. This is illustrated in Figure 3.
The constant $C(\theta)$ can be calculated explicitly for models such that $\varphi_\theta(\alpha)$ can be calculated. In principle this holds for the geometric Brownian motion model, and also for special cases of the geometric Brownian motion with jumps (Example 2.3). For processes with infinite jump activity (Example 2.4), the constant $C(\theta)$ has to be computed numerically. In Figure 4 we have plotted the Pareto constant $C(\theta)$ as a function of the investment strategy θ for three different models for the risky asset. □

Appendix

A Conditions for stationarity and Pareto tail approximation

The first lemma concerns the connection between the expectation and the moment generating function of $L_\theta(1)$ and those of $L(1)$.

Lemma A.1. *Let $\theta \in [0, 1]$.*
(a) If $E[L(1)] < \infty$, then also $E[L_\theta(1)] < \infty$.
(b) If $E[L(1)] > 0$, then also $E[L_\theta(1)] > 0$.
(c) If $\varphi(s) = E\left[e^{-sL(1)}\right] < \infty$, then $\varphi_\theta(s) = E\left[e^{-sL_\theta(1)}\right] < \infty$.

Proof. (a) $E[L(1)] < \infty$ is equivalent to $\int_{\mathbb{R}} x 1_{\{|x|>1\}}\, \nu(dx) < \infty$. Note that this formulation is chosen as a particular way to control the large jumps of the process. The cut-off points -1

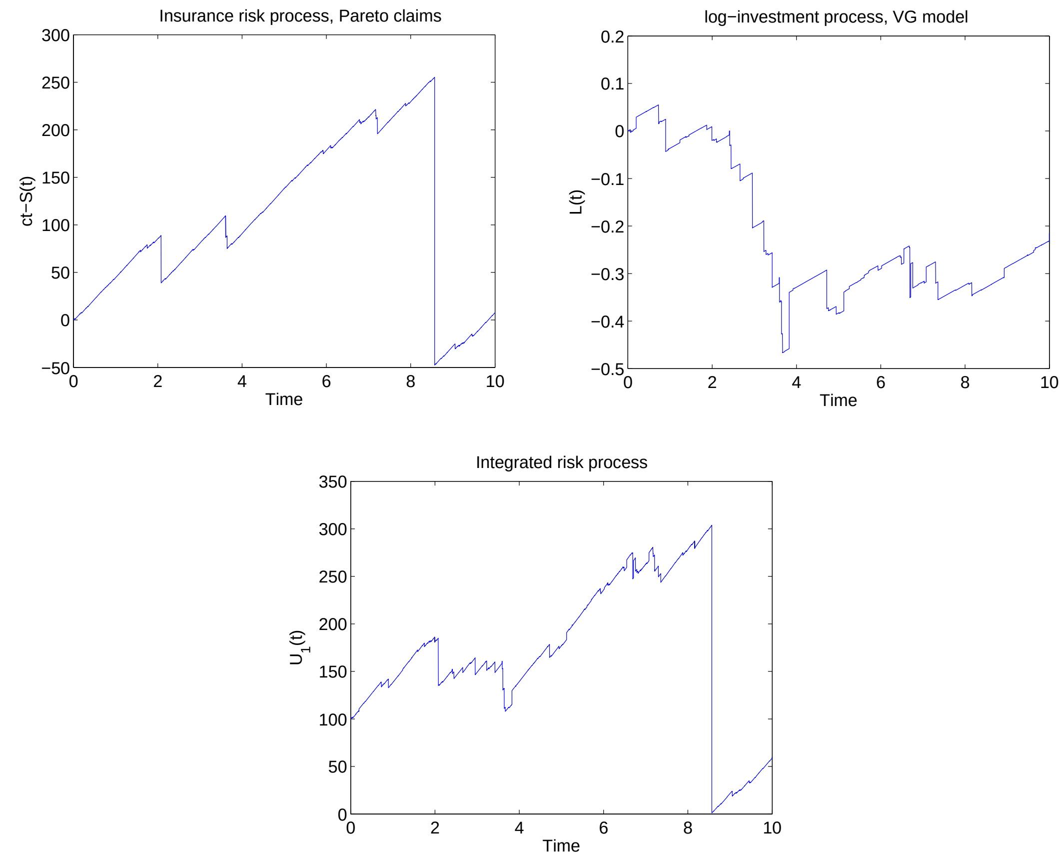

Figure 3: *Upper left plot:* sample path of the insurance process with premium rate $c = 50$, intensity of the Poisson claim arrival process $\lambda = 20$ and Pareto distributed claims with mean $\mu = 2$ and Pareto exponent $\alpha = 1.1$, i.e. $\overline{F}(x) = (\frac{0.2}{0.2+x})^{-1.1}$, $x > 0$. *Upper right plot:* sample path of the investment process for investment strategy $\theta = 1$ (pure stock investment) and log returns of the risky asset modeled by a VG process with parameters as in Figure 1. *Lower plot:* sample path of the resulting IRP with initial capital $u = 100$. The time horizon is $T = 10$. It is clearly seen that the large jumps of the IRP are dominated by the large insurance claims.

and 1 can be chosen arbitrarily and need not have the same modulus. By Remark 1.4(ii), the large jumps are of the form $\log(1+\theta(e^{\Delta L(1)}-1))$. Since $\log(1+\theta(e^x-1)) \geq \log(1-\theta)$, i.e. negative jumps are bounded below, we only need to control large positive jumps. Note that by l'Hospital's rule

$$\lim_{x\to\infty} \frac{\log(1+\theta(e^x-1))}{x} = 1\,.$$

This implies that for large enough $h > 0$

$$\int_h^\infty \log(1+\theta(e^x-1))\nu(dx) \leq \int_h^\infty (x+\varepsilon)\nu(dx) < \infty\,.$$

(b) First note that, whenever the expectations are finite, then $E[L(1)] = \gamma + \int_{\mathbb{R}} x 1_{\{|x|>1\}}\,\nu(dx)$ and $E[L_\theta(1)] = \gamma_\theta + \int_{\mathbb{R}} x 1_{\{|x|>1\}}\,\nu_\theta(dx)$ (see, e.g. Sato [24], E25.12, p. 163). Now assume

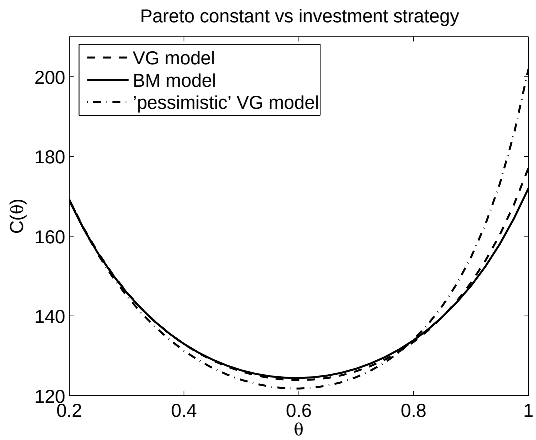

Figure 4: The Pareto constant $C(\theta)$ in (4.24) as a function of the investment strategy θ. We compare a Brownian motion and two VG models. The parameters of the models are as in Figure 2. Note that the more risky the investment model, the larger is the difference between the minimal and the maximal value of $C(\theta)$; i.e. between the minimal and the maximal value of the tail of V_θ^∞.

that $E[L(1)] > 0$. By Lemma 1.3, setting $(1-\theta)(\delta + \frac{\sigma^2}{2}\theta) =: a > 0$, we obtain

$$\begin{aligned} E[L_\theta(1)] &= \gamma\theta + a + \int_{\mathbb{R}} \log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|>1\}}\nu(dx) \\ &\quad + \int_{\mathbb{R}} (\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\le 1\}} - \theta x 1_{\{|x|\le 1\}})\nu(dx) \\ &= a + \theta E[L(1)] + \int_{\mathbb{R}} (\log(1+\theta(e^x-1)) - \theta x)\,\nu(dx) > 0\,, \end{aligned}$$

since the integrand is positive.

(c) Recall that

$$\varphi(s) = \Psi(is) = -\gamma s + \frac{\sigma^2}{2}s^2 + \int_{\mathbb{R}} (e^{-sx} - 1 + sx1_{\{|x|\le 1\}})\nu(dx)$$

and we assume that $\varphi(s) < \infty$, equivalently, the integral being finite. We consider the corresponding integral for $\varphi_\theta(s)$. We denote

$$\begin{aligned} h(\theta) &:= \int_{\mathbb{R}} (e^{-sx} - 1 + sx1_{\{|x|\le 1\}})\nu_\theta(dx) \\ &= \int_{\mathbb{R}} \left((1+\theta(e^x-1))^{-s} - 1 - s\log(1+\theta(e^x-1))1_{\{|\log(1+\theta(e^x-1))|\le 1\}}\right)\nu(dx)\,. \end{aligned}$$

Now the function $h(\theta)$ is continuous in $\theta \in [0,1]$. Moreover, $h(0) = 0$ and

$$h(1) = \int_{\mathbb{R}} \left(e^{sx} - 1 - sx1_{\{|x|\le 1\}}\right)\nu(dx) = \varphi(s) < \infty\,,$$

hence $h(\theta)$ is finite for all $\theta \in [0,1]$. $\square$

Lemma A.2. *If $\sigma > 0$ or $\nu((-\infty, 0)) > 0$, then $P(L_\theta(1) < 0) > 0$ for all $\theta \in (0,1]$.*

Proof. If $\sigma > 0$, then by Lemma 1.3 also $\sigma_\theta > 0$, and the Gaussian component guarantees the result. On the other hand, if $\nu((-\infty, 0)) > 0$, then Remark 1.4(ii) ensures also downwards jumps of L_θ, giving again the result. For more details we refer to Sato [24], Section 24. $\square$

Lemma A.3. *Assume that $EY = \mu < \infty$, $E[L(1)] > 0$ and $\varphi_\theta(1) < \lambda$. Then for the r.v.'s A_θ and B_θ defined in (3.4) we have*

(a) $E \log^+ |A_\theta| \leq \dfrac{\lambda\mu + c}{\lambda - \varphi_\theta(1)} < \infty\,;$

(b) $-\infty \leq E \log |B_\theta| = -\frac{1}{\lambda} E[L_\theta(1)] < 0\,.$

Proof. We first prove (b). From Lemma A.1(a) we know that $E[L_\theta(1)] > 0$. Moreover, if $E[L_\theta(1)] < \infty$, then $E[L_\theta(t)] = tE[L_\theta(1)]$ (see Sato [24], E25.12, Formula (25.7) at p. 163). Then we obtain

$$E[\log |B_\theta|] = -E[L_\theta(E)] = -\lambda \int_0^\infty E[L_\theta(z)] e^{-\lambda z}\, dz = -\frac{E[L_\theta(1)]}{\lambda}\,.$$

For the proof of (a) we use that for every r.v. $X > 0$ a.s., also $\log X < X$ a.s. and $\max(0, \log X) \leq X$; hence $E[\max(0, \log X)] \leq EX$. Then we estimate

$$\begin{aligned} E[\log^+ |A_\theta|] \;&=\; E\left[\log^+ \left| Y e^{-L_\theta(E)} - c \int_0^E e^{-L_\theta(v)}\, dv \right|\right] \\ \leq\; E\left[\left| Y e^{-L_\theta(E)} - c \int_0^E e^{-L_\theta(v)}\, dv \right|\right] \;&\leq\; \mu E\left[e^{-L_\theta(E)}\right] + cE\left[\int_0^E e^{-L_\theta(v)}\, dv\right]. \end{aligned}$$

Now for the first summand we calculate for $\varphi_\theta(1) < \lambda$,

$$E\left[e^{-L_\theta(E)}\right] = \lambda \int_0^\infty e^{\varphi_\theta(1) z} e^{-\lambda z}\, dz = \frac{\lambda}{\lambda - \varphi_\theta(1)} < \infty\,.$$

For the second summand we write

$$E\left[\int_0^E e^{-L_\theta(v)}\, dv\right] = \lambda \int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)} dv\right) e^{-\lambda z}\, dz\,.$$

If $\varphi_\theta(1) = 0$, then the last term is equal to $1/\lambda < \infty$. If $\varphi_\theta(1) \neq 0$, then $\varphi_\theta(1) < \lambda$,

$$\lambda \int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)} dv\right) e^{-\lambda z}\, dz = \frac{\lambda}{\varphi_\theta(1)} \int_0^\infty e^{-z(\lambda - \varphi_\theta(1))} dz - \frac{1}{\varphi_\theta(1)} = \frac{1}{\lambda - \varphi_\theta(1)} < \infty\,.$$

$\square$

Lemma A.4. *Let the conditions of Lemma 4.1 be satisfied and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Then*

(a) $E[B_\theta^\kappa] = 1$;

(b) $E\left[B_\theta^\kappa \log^+ B_\theta\right] < \infty$;

(c) If $E[Y^q] < \infty$ for some $q \geq 1$, then $E\left[|A_\theta|^{\min(q,\kappa+\beta)}\right] < \infty$ for $\beta > 0$ as in Lemma 4.2(b).

Proof. From (4.12) we know that $E[B_\theta^s] = e^{l_\theta(s)}$, hence (a) follows directly from the definition of $\kappa(\theta)$ as in Lemma 4.2.

Part (b) is a simple consequence of Lemma 4.2(b).

To prove (c) we consider two cases. Define $r := \kappa + \beta$. Assume first that $r \leq 1 \leq q$ and obser.v.e that then $\varphi_\theta(1) \geq 0$. As the function $f(x) = x^r$ is concave on $\mathbb{R}^+$, $|x+y|^r \leq |x|^r + |y|^r$ for every x, $y \in \mathbb{R}$. Hence we estimate

$$\begin{aligned} E[|A_\theta|^r] &= E\left[\left|Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}\,dv\right|^r\right] \\ &\leq E\left[\left(Ye^{-L_\theta(E)}\right)^r\right] + c^r E\left[\left(\int_0^E e^{-L_\theta(v)}\,dv\right)^r\right]. \end{aligned}$$

The first term on the rhs of the inequality is finite as $EY^r < \infty$ and $Ee^{-rL_\theta(E)} < \infty$ and both r.v.'s are independent. For the second term Jensen's inequality yields

$$\begin{aligned} E\left[\left(\int_0^E e^{-L_\theta(v)}\,dv\right)^r\right] &= \lambda\int_0^\infty E\left[\left(\int_0^z e^{-L_\theta(v)}\,dv\right)^r\right] e^{-\lambda z}\,dz \\ \leq \lambda\int_0^\infty \left(\int_0^z Ee^{-L_\theta(v)}\,dv\right)^r e^{-\lambda z}\,dz &= \lambda\int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)}\,dv\right)^r e^{-\lambda z}\,dz\,. \end{aligned}$$

If $\varphi_\theta(1) = 0$ then the last term is equal to $E[E^r] < \infty$ as E is exponentially distributed. If $\varphi_\theta(1) \neq 0$, then we have

$$\begin{aligned} \lambda\int_0^\infty \left(\int_0^z e^{v\varphi_\theta(1)}\,dv\right)^r e^{-\lambda z}\,dz &= \frac{\lambda}{\varphi_\theta(1)}\int_0^\infty \left(e^{z\varphi_\theta(1)} - 1\right)^r e^{-\lambda z}\,dz \\ &\leq \frac{\lambda}{\varphi_\theta(1)}\int_0^\infty e^{-z(\lambda - r\varphi_\theta(1))}\,dz < \infty\,, \end{aligned}$$

provided that $\varphi_\theta(1) < \lambda/g$, which is satisfied for $\varphi_\theta(1) < \lambda$ for $g \leq 1$.

Now assume that $g = \min(r, q) > 1$. Then the function $f(x) = x^g$ is convex giving

$$\begin{aligned} E[|A_\theta|^g] &= E\left[\left|Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}dv\right|^g\right] \\ &\leq 2^{g-1}\left(E\left[Y^g e^{-gL_\theta(E)}\right] + c^g E\left[\left(\int_0^E e^{-L_\theta(v)}dv\right)^g\right]\right). \end{aligned} \tag{A.1}$$

Now again from Jensen's inequality for the second expectation above we have

$$
\begin{aligned}
E\left[\left(\int_0^E e^{-L_\theta(v)}dv\right)^g\right] &= \lambda\int_0^\infty E\left[\left(\int_0^z e^{-L_\theta(v)}dv\right)^g\right]e^{-\lambda z}dz \\
\le \lambda\int_0^\infty z^{g-1}\int_0^z E[e^{-gL_\theta(v)}]dve^{-\lambda z}dz &= \lambda\int_0^\infty z^{g-1}\int_0^z e^{v\varphi_\theta(g)}]dve^{-\lambda z}dz < \infty \quad \text{(A.2)}
\end{aligned}
$$

as $Ee^{-gL_\theta(v)} \le 1$. For the first expectation in (A.1) we have $E\left[Y^g e^{-gL_\theta(E)}\right] = E[Y^g]E[B_\theta^g] < \infty$ by part (a), and the proof is completed. □

B Proof of Lemma 4.3

Lemma 4.3 is a large deviations result. We largely follow Nyrhinen [19] with adaptations to our situation. We introduce some notations first. Set

$$
m(\theta) = l'_\theta(\kappa(\theta))\,. \tag{B.1}
$$

For $d \in (0,\, 1/m(\theta))$, $\epsilon' > 0$ and $n \in \mathbb{N}$ define the subsets $D_n = D_n(d,\, \epsilon')$ and $E_n = E_n(d,\, \epsilon')$ of Ω by

$$
\begin{aligned}
D_n &= \left\{\omega \in \Omega : \sup_{0<\alpha\le 1/m(\theta)-d}\left|\frac{1}{n}\sum_{j=1}^{\lceil \alpha n\rceil}\log B_{\theta,j} - \alpha m(\theta)\right| \le \epsilon'\right\}, \\
E_n &= \left\{\omega \in \Omega : \sup_{j=1,\dots,\lceil(1/m(\theta)-d)n\rceil}|A_{\theta,j}| \le e^{\epsilon' n}\right\}.
\end{aligned} \tag{B.2}
$$

The following lemma is the key for the proof of Lemma 4.3.

Lemma B.1. *Let the conditions of Lemma 4.1 hold and $\kappa = \kappa(\theta) \in (0, \infty)$ be the unique value satisfying $\varphi_\theta(\kappa) = 0$. Let also $EY < \infty$. Then for every $d \in (0,\, 1/m(\theta))$ there exists some $\epsilon' > 0$ such that*

$$
\liminf_{n\to\infty}\frac{\log P(D_n(d,\, \epsilon') \cap E_n(d,\, \epsilon'))}{n} \ \ge\ -\kappa(\theta)\,. \tag{B.3}
$$

Proof. Recall that under the probability measure P the sequence $((A_{\theta,k},\, B_{\theta,k}))_{k\in\mathbb{N}}$ consists of iid random vectors all distributed like $(A_\theta,\, B_\theta)$ as defined in (3.4). Define a new probability measure Q by

$$
dQ(y_1,\, y_2) = y_2^{\kappa(\theta)}dP(y_1,\, y_2)\,,
$$

for $(y_1,\, y_2) \in \mathbb{R}^2$ and such that $((A_{\theta,k},\, B_{\theta,k}))_{k\in\mathbb{N}}$ is again a sequence of iid random vectors with respect to Q. We denote by E_Q the expectation under Q. Then, for $k \in \mathbb{N}$ and any measurable function $f : \mathbb{R}^{2k} \to \mathbb{R}$ we have

$$
\begin{aligned}
&E\left[f((A_{\theta,1},\, B_{\theta,1}), \ldots, (A_{\theta,k},\, B_{\theta,k}))\right] \\
&= \int_{\mathbb{R}^k} \int_{(0,\infty)^k} f((y_1^1,\, y_2^1), \ldots, (y_1^k,\, y_2^k))\, dP(y_1^1,\, y_2^1) \cdots dP(y_1^k,\, y_2^k) \\
&= \int_{\mathbb{R}^k} \int_{(0,\infty)^k} f((y_1^1,\, y_2^1), \ldots, (y_1^k,\, y_2^k))\, (y_2^1)^{-\kappa(\theta)} dQ(y_1^1,\, y_2^1) \, \cdots \, (y_2^k)^{-\kappa(\theta)} dQ(y_1^k,\, y_2^k) \\
&= E_Q\left[\Big(\prod_{j=1}^k B_{\theta,j} \Big)^{-\kappa(\theta)} f((A_{\theta,1},\, B_{\theta,1}), \ldots, (A_{\theta,k},\, B_{\theta,k})) \right]. \qquad \text{(B.4)}
\end{aligned}
$$

Take $\epsilon'' \in (0,\, \epsilon')$. Then using (B.4) and the definition of D_n we estimate

$$
\begin{aligned}
& P(D_n(d,\, \epsilon') \cap E_n(d,\, \epsilon')) \\
\geq \;& P(D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon'')) = E\left[1\{D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon'')\}\right] \\
= \;& E_Q\left[\exp\left(-\kappa(\theta)(-\log B_{\theta,1} - \cdots - \log B_{\theta,\lceil(1/m(\theta)-d)n\rceil})\right) 1\{D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon'')\}\right] \\
\geq \;& \exp\left(-\kappa(\theta) n \left(\epsilon'' + 1 - d m(\theta)\right)\right) Q(D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon'')) \,.
\end{aligned}
$$

As $0 < dm(\theta) < 1$ follows

$$
\frac{\log P(D_n(d,\, \epsilon') \cap E_n(d,\, \epsilon'))}{n} \;\; \geq \;\; -\kappa(\theta)(1+\epsilon'') + \frac{\log Q(D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon''))}{n} \,.
$$

Then, if we can show that

$$
\lim_{n\to\infty} Q(D_n(d,\, \epsilon'') \cap E_n(d,\, \epsilon'')) \;\; = \;\; 1\,, \qquad \text{(B.5)}
$$

we obtain (B.3) after letting $\epsilon'' \to 0$. For the proof of (B.5) it is sufficient to show that

$$
\lim_{n\to\infty} Q(D_n(d,\, \epsilon'')) = 1 \quad \text{and} \quad \lim_{n\to\infty} Q(E_n(d,\, \epsilon'')) = 1\,. \qquad \text{(B.6)}
$$

We start with the lhs of (B.6). Note that by Lemma 4.2(b) there exists some s in a neighborhood of 0 such that

$$
E_Q\left[B_\theta^s\right] = E\left[B_\theta^{s+\kappa(\theta)}\right] < \infty\,.
$$

This implies for such s

$$
\begin{aligned}
E_Q\left[B_\theta^s\right] \;&=\; \frac{d}{ds}\left(\log E_Q\left[B_\theta^s\right]\right)_{|s=0} = \frac{d}{ds}\left(\log E\left[B_\theta^{s+\kappa(\theta)}\right]\right)_{|s=0} \\
&=\; \frac{d}{ds}\left(\log E\left[B_\theta^s\right]\right)_{|s=\kappa(\theta)} = l_\theta'(\kappa(\theta)) = m(\theta)\,.
\end{aligned}
$$

From the above follows that for the sum $S_n = \log B_{\theta,1} + \cdots + \log B_{\theta,n}$ the SLLN holds under the measure Q, i.e.

$$
\frac{S_n}{n} \overset{Q-a.s.}{\to} m(\theta)\,, \quad n \to \infty\,.
$$

For $x > 0$ we have $xn/\lceil xn \rceil \to 1$ as $n \to \infty$, hence for $0 < \alpha \le 1/m(\theta) - d$ we obtain

$$\frac{S_{\lceil \alpha n \rceil}}{n} \overset{Q-a.s.}{\rightarrow} \alpha m(\theta)\,, \quad n \to \infty\,,$$

from which follows the lhs of (B.6), i.e.

$$Q\left(\left|S_{\lceil \alpha n \rceil}/n - \alpha m(\theta)\right| \le \epsilon''\,,\ 0 < \alpha \le 1/m(\theta) - d\right) \to Q(D_n(d,\,\epsilon'')) = 1\,, \quad n \to \infty\,.$$

To show the rhs of (B.6) first note that, if $EY < \infty$, then Lemma 4.2(c) implies that $E[|A_\theta|^{\min(1,\kappa)}] < \infty$. Second, by Hölder's inequality, for $p,\,q > 0$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$ we get for some $s > 0$

$$E_Q\left[|A_\theta|^s\right] \;=\; E\left[B_\theta^\kappa\,|A_\theta|^s\right] \le \left(E\left[B_\theta^{\kappa p}\right]\right)^{1/p}\left(E\left[|A_\theta|^{sq}\right]\right)^{1/q}\,.$$

We can choose $p > 1$ such that $\kappa p < \kappa + \beta$, where $\beta > 0$ is as in Lemma 4.2(b) and $s' > 0$ such that $s'q < \min(\kappa, 1)$. In other words, Hölder's inequality guarantees the existence of some $s' > 0$, such that

$$E_Q[|A_\theta|^{s'}] < \infty\,.$$

Then for this $s' > 0$ we estimate

$$E_Q\left[|A_\theta|^{s'}\right] \ge E_Q\left[|A_\theta|^{s'}\,1\{|A_\theta| \ge e^{\epsilon'' n}\}\right] \ge e^{s'\epsilon'' n} Q(|A_\theta| \ge e^{\epsilon'' n})\,. \tag{B.7}$$

Furthermore, for this $s' > 0$, using that $(A_{\theta,j})_{j\in\mathbb{N}}$ is a sequence of iid r.v.'s and (B.7), we have

$$\begin{aligned} Q(E_n(d,\,\epsilon'')) &= Q\left(|A_{\theta,j}| \le e^{\epsilon'' n}\,,\ j = 1,\ldots,\lceil (1/m(\theta) - d)n \rceil\right) \\ &= 1 - \lceil (1/m(\theta) - d)n \rceil\, Q(|A_\theta| > e^{\epsilon'' n}) \\ &\ge 1 - \lceil (1/m(\theta) - d)n \rceil\, e^{-s'\epsilon'' n} E_Q\left[|A_\theta|^{s'}\right]\,. \end{aligned}$$

Now, as $E_Q[|A_\theta|^{s'}] < \infty$, after letting in the last expression $n \to \infty$, we get the rhs of (B.6). This completes the proof. □

Proof of Lemma 4.3. By Lemma 4.2(c) we have that $P(B_\theta > 1) > 0$, from which follows the existence of some $b > 1$ satisfying $P(|B_\theta - b| < \epsilon) > 0$ for $\epsilon \in (0, b-1)$. Then

$$0 < P(|B_\theta - b| < \epsilon) \;=\; \lambda \int_0^\infty P\left(\log(b-\epsilon) < -L_\theta(z) < \log(b+\epsilon)\right) e^{-\lambda z}\, dz\,,$$

and, therefore, there exists some $t > 0$ such that $P(\log(b-\epsilon) < -L_\theta(t) < \log(b+\epsilon)) > 0$. From Theorem 24.3 in Sato we know that $L_\theta(t)$ has unbounded support for every $t > 0$, hence by continuity in probability,

$$P(\log(b-\epsilon) < -L_\theta(v) < \log(b+\epsilon) \text{ for } v \in (0,t]) > 0 \tag{B.8}$$

Then we have

$$
\begin{aligned}
q &= P(A_\theta > 0,\, |B_\theta - b| < \epsilon) \\
&= P(Ye^{-L_\theta(E)} - c\int_0^E e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon) \\
&\geq P(Y(b-\epsilon) - c\int_0^E e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon) \\
&= P\left(Y(b-\epsilon) - c\int_0^E e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon \mid Y > \frac{b+\epsilon}{b-\epsilon}\,c\,y\right) P\left(Y > \frac{b+\epsilon}{b-\epsilon}\,c\,y\right)\,,
\end{aligned}
$$

where $y > 0$ is arbitrary. As the support of Y is the whole of $\mathbb{R}^+$, the claims can come arbitrarily large, hence $q_1 := P(Y > \frac{b+\epsilon}{b-\epsilon}\,c\,y) > 0$. Note that we do not need to fix $y > 0$ at this step and therefore we can still apply (B.8) a few lines later to estimate q. We estimate

$$
\begin{aligned}
q &\geq q_1 P\left(y(b+\epsilon) - \int_0^E e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon\right) \\
&= q_1\lambda \int_0^\infty P\left(y(b+\epsilon) - \int_0^z e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(z)} - b| < \epsilon\right) e^{-\lambda z}\,dz \\
&\geq q_1\lambda \int_0^\infty P\left(y(b+\epsilon) - \int_0^z e^{-L_\theta(v)}dv > 0\,,\, |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz \\
&\geq q_1\lambda \int_0^\infty P\left(y - z > 0,\, |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz \\
&\geq q_1\lambda \int_0^y P\left(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\,dz\,.
\end{aligned}
$$

Therefore, $q \geq q_1\lambda P(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,y))P(E < y)$. The probability $P(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,y)) > 0$ is selected such that (B.8) is satisfied. Consequently, we may choose numbers $b > 1$ and $\epsilon \in (0,\, b-1)$, such that

$$
q = P(|B_\theta - b| < \epsilon,\, A_\theta > 0) > 0\,. \tag{B.9}
$$

To prove our result we take some $d \in (0,\, 1/m(\theta))$, where $m(\theta)$ is as in (B.1), and some small number $\epsilon' > 0$, which we shall fix later. Recall the sets $D_n = D_n(d,\, \epsilon')$ and $E_n = E_n(d,\, \epsilon')$ in (B.2). and set $m = 1 + \lceil \alpha n \rceil$ for $0 < \alpha < 1/m(\theta) - d$. Then for $\omega \in D_n$ we have (cf. (B.2))

$$
\log B_{\theta,1} + \cdots + \log B_{\theta,m-1} \leq (\epsilon' + \alpha m(\theta))n \leq \big(\epsilon' + \frac{m-1}{n} m(\theta)\big)n \leq (\epsilon' + 1 - dm(\theta))n\,.
$$

For $m \in \mathbb{N}$ set $\Pi_m = \prod_{j=1}^m B_{\theta,j}$ and $\Pi_0 = 1$. For sufficiently large $n \in \mathbb{N}$ and $\omega \in D_n \cap E_n$

we estimate, starting with the definition in (3.5),

$$\begin{aligned}
V_{\theta,\lceil(1/m(\theta)-d)n\rceil} &= \sum_{m=1}^{\lceil(1/m(\theta)-d)n\rceil} A_{\theta,m}\Pi_{m-1} \\
&\geq -e^{\epsilon' n} \sum_{m=1}^{\lceil(1/m(\theta)-d)n\rceil} \exp\left(\sum_{j=1}^{m-1} \log B_{\theta,j}\right) \\
&\geq -\lceil(1/m(\theta)-d)n\rceil\, e^{\epsilon' n} e^{n(\epsilon'+1-dm(\theta))} \\
&> -e^{(3\epsilon'+1-dm(\theta))n}. \qquad \text{(B.10)}
\end{aligned}$$

The last inequality holds as for all $\epsilon' > 0$ and sufficiently large $n \in \mathbb{N}$ we have $\lceil(1/m(\theta)-d)n\rceil < e^{\epsilon' n}$.

Let $d' \in (0,\, d)$. For $n \in \mathbb{N}$ introduce the following subset of Ω

$$F_n = \{\omega \in \Omega \,|\, A_{\theta,j} > 0,\, |B_{\theta,j} - b| < \epsilon,\, j = \lceil(1/m(\theta)-d)n\rceil + 1, \ldots, \lceil(1/m(\theta)-d')n\rceil\}\ .$$

As the index sets are disjoint, F_n is independent of $D_n \cap E_n$. From (B.9) we conclude

$$P(F_n) \geq q^{(d-d')n+1}\,. \qquad \text{(B.11)}$$

Further, for sufficiently large $n \in \mathbb{N}$ and $\omega \in F_n$ we consider the increment (recall that $b - \epsilon > 1$)

$$V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil} = \sum_{m=\lceil(1/m(\theta)-d)n\rceil+1}^{\lceil(1/m(\theta)-d')n\rceil} A_{\theta,m}\Pi_{m-1}\ >\ 0 \qquad \text{(B.12)}$$

Next we define for $n \in \mathbb{N}$ one more subset of Ω:

$$G_n = \left\{\omega \in \Omega \,|\, A_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} > 1\right\}\ ,$$

From $x_F = \infty$ follows that A_θ has infinite right endpoint; hence

$$P(G_n) = P(A_\theta > 1) = r > 0\,. \qquad \text{(B.13)}$$

Finally for sufficiently large $n \in \mathbb{N}$ we consider for $\omega \in D_n \cap E_n \cap F_n \cap G_n$

$$\begin{aligned}
V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} &= \left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} - V_{\theta,\lceil(1/m(\theta)-d')n\rceil}\right) \\
&\quad + \left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil}\right) + V_{\theta,\lceil(1/m(\theta)-d)n\rceil} \\
&> \Pi_{\lceil(1/m(\theta)-d')n\rceil} - e^{n(1-m(\theta)d+3\epsilon')}\,, \qquad \text{(B.14)}
\end{aligned}$$

where we have used that

(i) $V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} - V_{\theta,\lceil(1/m(\theta)-d')n\rceil} \geq \Pi_{\lceil(1/m(\theta)-d')n\rceil}$ for $\omega \in G_n$;

(ii) $V_{\theta,\lceil(1/m(\theta)-d')n\rceil} - V_{\theta,\lceil(1/m(\theta)-d)n\rceil} \geq 0$ for $\omega \in F_n$ from (B.12);

(iii) $V_{\theta,\lceil(1/m(\theta)-d)n\rceil} > -e^{n(1-m(\theta)d+3\epsilon')}$ for $\omega \in D_n \cap E_n$ from (B.10).

For $\omega \in D_n \cap F_n$ we estimate the product in (B.14) using the definitions of D_n and F_n

$$\begin{aligned}
\Pi_{\lceil(1/m(\theta)-d')n\rceil} &= \exp\left(\sum_{j=1}^{\lceil(1/m(\theta)-d)n\rceil} \log B_{\theta,j}\right) \times \exp\left(\sum_{\lceil(1/m(\theta)-d)n\rceil+1}^{\lceil(1/m(\theta)-d')n\rceil} \log B_{\theta,j}\right) \\
&\geq \exp((-\epsilon' + 1 - m(\theta)d)n)(b-\epsilon)^{(d-d')n-1}\,,
\end{aligned} \tag{B.15}$$

where $b - \epsilon > 1$. By fixing ϵ' such that $5\epsilon' = (d - d')\log(b - \epsilon)$ we obtain the following lower bound in (B.15)

$$\Pi_{\lceil(1/m(\theta)-d')n\rceil} \geq \frac{1}{b-\epsilon}\exp((4\epsilon' + 1 - m(\theta)d)n)\,.$$

Using this in (B.14) results in the following inequality

$$\begin{aligned}
V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} &\geq \exp((1-m(\theta)d)n)\left(\frac{1}{b-\epsilon}\exp(n4\epsilon') - \exp(n3\epsilon')\right) \\
&> \exp((1-m(\theta)d)n)\,,
\end{aligned} \tag{B.16}$$

where for the last inequality we have used that for sufficiently large $n \in \mathbb{N}$ holds

$$\exp(4\epsilon' n) > (b-\epsilon)(\exp(3\epsilon' n) - 1)\,.$$

We derived inequality (B.14) for sufficiently large $n \in \mathbb{N}$ for $\omega \in (D_n \cap E_n) \cap F_n \cap G_n$, where $D_n \cap E_n$, F_n and G_n are mutually independent. Hence, together with (B.11) and (B.13), taking logarithm and dividing by n, we obtain the following inequality

$$\begin{aligned}
&\frac{\log P\left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} > \exp\left((1-m(\theta)d)n\right)\right)}{n} \\
&\geq \frac{\log P(D_n \cap E_n)}{n} + \frac{\log P(G_n)}{n} + \frac{\log P(F_n)}{n} \\
&= \frac{\log P(D_n \cap E_n)}{n} + \frac{\log r}{n} + (d - d' + \frac{1}{n})\log q\,.
\end{aligned}$$

Now we let $n \to \infty$ and make use of (B.3) resulting into

$$\liminf_{n\to\infty} \frac{\log P\left(V_{\theta,\lceil(1/m(\theta)-d')n\rceil+1} > \exp\left((1-m(\theta)d)n\right)\right)}{n} \geq -\kappa(\theta) + (d-d')\log q\,.$$

Finally, letting $d' \to d$ and $d \to 0$ and substituting $n = \log x$, we obtain

$$\liminf_{x\to\infty} \frac{\log P(V_{\theta,\lceil \log x/m(\theta)\rceil+1} > x)}{\log x} \geq -\kappa(\theta)\,. \tag{B.17}$$

Denote now $k := k(x) = \lceil \log x/m(\theta)\rceil + 1$ and note that, due to the iid increments property of the Lévy processes,

$$\begin{aligned}
V_\theta^\infty &= \int_0^\infty e^{-L_\theta(v)}(dS(v) - cdv) \\
&= V_\theta(T_k) + \int_{T_k}^\infty e^{-L_\theta(v)}(dS(v) - cdv) \\
&= V_\theta(T_k) + e^{-L_\theta(T_k)} \int_{T_k}^\infty e^{-(L_\theta(v)-L_\theta(T_k))}(dS(v) - cdv) \\
&\stackrel{d}{=} V_\theta(T_k) + e^{-L_\theta(T_k)} \widetilde{V}_\theta^\infty\,,
\end{aligned}$$

where $\widetilde{V}_\theta^\infty$ is a copy of V_θ^∞, independent of $\mathcal{F}_{T_k}$. Furthermore, recalling from Proposition 3.7(b) that $V_{\theta,k} = V_\theta(T_k)$ we can write

$$\frac{\log P(V_\theta^\infty > x)}{\log x} \geq \frac{\log P(V_{\theta,k} > x)}{\log x} + \frac{\log P(\widetilde{V}_\theta^\infty \geq 0)}{\log x}\,.$$

Note that $P(\widetilde{V}_\theta^\infty \geq 0) > 0$ because of (3.6) and the fact that (because of full support of Y on $\mathbb{R}^+$) we have $P(A_\theta > 0) > 0$. Letting x tend to infinity and making use of (B.17) gives (4.13).

To prove the rhs of (4.13) it suffices to show (B.9) and (B.13) for the r.v. $-A_\theta$. Again we take $b > 1$ such that $P(|B_\theta - b| < \epsilon) > 0$ for all $\epsilon \in (0, b-1)$.

$$\begin{aligned}
q &= P(-A_\theta > 0,\, |B_\theta - b| < \epsilon) \;=\; P\left(c\int_0^E e^{-L_\theta(v)}dv - Ye^{-L_\theta(E)} > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon\right) \\
&\geq P\left(c\int_0^E e^{-L_\theta(v)}dv - Y(b+\epsilon) > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon\right) \\
&= P\left(c\int_0^E e^{-L_\theta(v)}dv - Y(b+\epsilon) > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon \,\Big|\, Y < \frac{b-\epsilon}{b+\epsilon}\,c\,y\right) P\left(Y < \frac{b-\epsilon}{b+\epsilon}\,c\,y\right)\,,
\end{aligned}$$

where $y > 0$ is arbitrary. As the support of Y is the whole of $\mathbb{R}^+$, the claims can come arbitrarily close to 0, hence $q_1 := P(Y < \frac{b-\epsilon}{b+\epsilon}\,c\,y) > 0$. Note that we do not need to fix $y > 0$ at this step and therefore we can still apply (B.8) a few lines later to estimate q.

Therefore

$$
\begin{aligned}
q &\geq q_1 P\left(\int_0^E e^{-L_\theta(v)} dv - y(b-\epsilon) > 0\,,\, |e^{-L_\theta(E)} - b| < \epsilon\right) \\
&= q_1 \lambda \int_0^\infty P\left(\int_0^z e^{-L_\theta(v)} dv - y(b-\epsilon) > 0\,,\, |e^{-L_\theta(z)} - b| < \epsilon\right) e^{-\lambda z}\, dz \\
&\geq q_1 \lambda \int_0^\infty P\left(\int_0^z e^{-L_\theta(v)} dv - y(b-\epsilon) > 0\,,\, |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right)\, e^{-\lambda z}\, dz \\
&\geq q_1 \lambda \int_0^\infty P\left(z - y > 0,\, |e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\, dz \\
&\geq q_1 \lambda \int_y^\infty P\left(|e^{-L_\theta(v)} - b| < \epsilon \text{ for } v \in (0,z]\right) e^{-\lambda z}\, dz \;>\; 0\,.
\end{aligned}
$$

The proof of the rhs of (4.13) follows by repetition of all steps of the proof of the lhs of (4.13), replacing A_θ and $V_{\theta,k}$ for $k \in \mathbb{N}$ by $-A_\theta$ and $-V_{\theta,k}$, $k \in \mathbb{N}$, respectively. To this end we still have to show that

$$
r \;=\; P(-A_\theta > 1) > 0 \tag{B.18}
$$

Indeed, from the infinite right end point of E follows

$$
\begin{aligned}
& P(c \int_0^E e^{-L_\theta(v)} dv - Y e^{-L_\theta(E)} > 1) \\
&\geq \int_0^\infty P(c \int_0^E e^{-L_\theta(v)} dv - y e^{-L_\theta(E)} > 0\,,\, \left|e^{-L_\theta(v)} - b\right| < \epsilon \text{ for } v \in (0,E))\, dF(y) \\
&\geq \int_0^\infty P(E > \frac{y(b+\epsilon)}{c(b-\epsilon)})\, dF(y) \;>\; 0\,.
\end{aligned}
$$

□

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