Modelling and Measuring Multivariate Operational Risk with Lévy Copulas

Klaus Böcker *   Claudia Klüppelberg †

Abstract

Simultaneous modelling of operational risks occurring in different event type/business line cells poses a serious challenge for operational risk quantification. Here we invoke the new concept of Lévy copulas to model the dependence structure of operational loss events. We explain the consequences of this dependence concept for frequencies and severities of operational risk in detail. For important examples of the Lévy copula and heavy-tailed GPD tail severities we derive first order approximations for multivariate operational VAR.

1 Introduction

A required feature of any advanced measurement approach (AMA) of Basel II [3] for measuring operational risk is that it allows for explicit correlations between different operational risk events. More precisely, banks should allocate losses to one of eight business lines and to one of seven loss event types. The core problem here is multivariate modelling encompassing all different event type/business line cells, and thus the question how their dependence structure affects a bank's total operational risk.

The prototypical loss distribution approach (LDA) assumes that, for each cell $i = 1, \ldots, d$, the cumulated operational loss $S_i(t)$ up to time t is described by an aggregate loss process

$$S_i(t) = \sum_{k=1}^{N_i(t)} X_k^i, \quad t \geq 0, \tag{1.1}$$

*Risk Integration, Reporting & Policies - Risk Analytics and Methods - UniCredit Group, Munich Branch, email: klaus.boecker@hvb.de

†Center for Mathematical Sciences, Munich University of Technology, D-85747 Garching bei München, Germany, email: cklu@ma.tum.de, www.ma.tum.de/stat/

where for each i the sequence $(X_k^i)_{k\in\mathbb{N}}$ are independent and identically distributed (iid) positive random variables describing the magnitude of each loss event (loss severity), and $(N_i(t))_{t\geq 0}$ counts the number of losses in the time interval $[0,t]$ (called frequency process), independent of $(X_k^i)_{k\in\mathbb{N}}$.

The bank's total operational risk is then given by the stochastic process

$$S^+(t) := S_1(t) + S_2(t) + \cdots + S_d(t)\,, \quad t \geq 0\,. \tag{1.2}$$

The present literature suggests to model dependence between different operational risk cells by means of different concepts, which basically split into models for frequency dependence on the one hand and for severity dependence on the other hand. Some important techniques are

□ modelling dependence between the number of losses $N_1(t), \ldots, N_d(t)$ occurring within $t_1 = 1$ year via correlation or copulas, see e.g. Aue & Kalkbrenner [1], Bee [2], or Frachot, Roncalli & Salomon [13],

□ introducing coincident loss events by a common-shock model, see e.g. Lindskog & McNeil [14], or Powojowski, Reynolds & Tuenter [19],

□ modelling dependence between the severities of those losses that occur at the same points in time, see e.g. Chavez-Demoulin, Embrechts & Nešlehová [10],

□ for $t_1 = 1$ year, modelling dependence between the distribution functions of the aggregate marginal processes S_i for $i = 1, \ldots, d$ by means of distributional copulas.

For the practical implementation of LDA models, the first of the approaches above is probably most popular in the banking industry. Its main advantage is that correlation estimates between the yearly number of loss events within each operational risk cell can be quite easily calculated from empirical loss data. However, it has been reported by e.g. Aue & Kalkbrenner [1] or Bee [2] that the impact of a specific copula or the level of loss-number correlation (sometimes referred to as frequency correlation) has only little impact on the economic capital for operational risk. We will give some mathematical reasoning to support this observation later in this paper.

Here we suggest a different model, which is based on the new concept of Lévy copulas (see e.g. Cont & Tankov [11] and references therein, also Klüppelberg & Resnick [15] for a related concept). In contrast to the approaches above, dependence in frequency and severity between different cells is modelled at the same time using one and the same concept, namely the Lévy copula. This yields a model with comparably few parameters, which particularly in the light of spare data at hand may be a viable alternative to other, more complex models.

Our model has the same advantage as distributional copulas: the dependence structure between different cells can be separated from the univariate problem, i.e. the marginal loss processes S_i for $i = 1, \ldots, d$. Consequently, with a rather transparent dependence model, we are able to model the possibility of coincident losses occurring in different cells.

Since high-level OpVAR estimation is based on large severities, models can be based on exceedances over high thresholds and, consequently, generalised Pareto distributions (GPD) are natural models to consider. We derive closed form OpVAR asymptotics within multivariate models, where certain cells are dominated by such heavy-tailed GPD models. In case of equivalent heavy-tailed GPD distributions in different cells, we find OpVAR asymptotics for certain measures of association, namely complete dependence on the one hand and independence on the other.

Conclusion. Although complete dependence and independence are very extreme dependence structures, they are important as they allow us to study possible bandwidths for the aggregated, bank-wide OpVAR. Identifying heavy-tailed severities of similar order, we can treat rather realistic scenarios with little mathematical effort.

2 A Multivariate Generalization of the Standard LDA

We now want to motivate our approach for modelling multivariate operational risk and give some theoretical arguments for the use of Lévy copulas in this context. Needless to say, multivariate OpVAR is still in its infancy and so far the question regarding the right model cannot be answered only by statistical analysis because reliable data are often still not available. There exists, however, a model-theoretic rationale for our approach, which we want to briefly explain.

In accordance with the findings of a recent survey of the Basel Committee on Banking Supervision [4] about AMA practices at financial services firms, we assume within each cell i the following model. The loss frequency process N_i in (1.1) follows a homogeneous Poisson process with rate $\lambda_i > 0$, in particular, for every fixed $t > 0$,

$$P(N_i(t) = n)) = e^{-\lambda_i t} \frac{(\lambda_i t)^n}{n!}, \quad n \in \mathbb{N}_0 .$$

All loss severities within cell i are independent and have the same severity distribution function $F_i(x) = P(X^i \leq x)$ for $x \in [0, \infty)$. Then the aggregate loss (1.1) constitutes a compound Poisson process and, hence, is a Lévy process (the only Lévy process with piecewise constant sample paths). This kind of model is often referred to as (univariate) standard LDA model.

As a matter of fact, the definition of 56 different cells based on seven loss event types and eight business lines as suggested by the Basel Committee [3] is quite arbitrary. Actually, many banks are using a less dimensional cell matrix, which basically means e.g. that they apply a standard LDA to a *union* of some of the Basel II cells. Such a procedure, however, will only be consistent with the overall framework of a compound Poisson model, if we require that every additive conjunction of different cells again constitutes a univariate compound Poisson process, in particular, with common severity distribution $F_{i+j}(\cdot)$ and frequency λ_{i+j}, i.e. for such $i \neq j$,

$$S_i(\cdot) + S_j(\cdot) := S_{i+j}(\cdot) \quad \in \text{ compound Poisson processes}\,. \tag{2.1}$$

Or to put it another way, a natural requirement of a multivariate LDA model should be that it does not directly depend on the structure of the event type/business line matrix and thus on the business organization. As a direct consequence of the standard LDA model, rigorously applied to several operational risk cells we obtain an "invariance principle" any mathematical OpRisk model has to satisfy.

As we show in section 4 below, (2.1) holds true, whenever the vector of all marginals $(S_1(t), \ldots, S_d(t))_{t\geq 0}$ constitutes a d-dimensional compound Poisson process. Therefore, the problem is how the different one-dimensional compound Poisson processes $S_i(\cdot) = \sum_{k=1}^{N_i(\cdot)} X_k^i$ can be combined to form a d-dimensional compound Poisson process $S(t) = (S_1(t), \ldots, S_d(t))_{t\geq 0}$ with, in general, dependent components. If we are only interested in one fixed time point, say $t_1 = 1$ year, we can consider $(S_1(t_1), \ldots, S_d(t_1))$ simply as a vector of static random variables. Now, it is well-known that the dependence structure of a multidimensional random vector can be disentangled from its marginals by introducing a distributional copula. More precisely, Sklar's now famous theorem states that any multivariate distribution with continuous marginals can be transformed into a distribution with uniform marginals. Therefore, choosing an appropriate distributional copula C at t_1 we could write $S(t_1) = (S_1(t_1), \ldots, S_d(t_1)) = C(S_1(t_1), \ldots, S_d(t_1))$. However, switching on time-dependence again in the marginals, the process $(S(t))_{t\geq 0}$, will in general not be a multivariate compound Poisson process and thus, contradictory to our requirement (2.1), the multivariate model may not be invariant under a re-design of the cell matrix.

Hence, the question is how dependence between different risk cells can be established by also conserving the compound Poisson property of the multivariate process over time. The answer leads us to the so-called Lévy measure, a key quantity in the theory of Lévy processes and thus for compound Poisson processes. A Lévy measure controls the jump behaviour of a Lévy process and has a very intuitive interpretation, in particular in the context of operational risk. The Lévy measure of a single operational risk cell measures the expected number of losses per unit time with a loss amount in a pre-specified interval. For our compound Poisson model, the Lévy measure Π_i of the cell process S_i is completely determined by the frequency parameter $\lambda_i > 0$ and the distribution function of the cell's severity, namely $\Pi_i([0, x)) := \lambda_i P(X^i \leq x) = \lambda_i F_i(x)$ for $x \in [0, \infty)$. Since here we are mainly interested in large operational losses, it is convenient to introduce the concept of a tail integral. A one-dimensional tail integral is simply the expected number of losses per unit time that are above a given threshold x:

$$\overline{\Pi}_i(x) := \Pi_i([x,\infty)) = \lambda_i P(X^i > x) = \lambda_i \overline{F}_i(x)\,, \quad x \in [0,\infty)\,. \tag{2.2}$$

In the dynamic framework of a multivariate Lévy process the multivariate Lévy measure controls the joint jump behaviour (per unit time) of all univariate components and contains all information of dependence between the components. Now, similarly to the fact that a multivariate distribution can be built from marginal distributions via a distributional copula, a multivariate tail integral

$$\overline{\Pi}(x_1,\ldots,x_d) = \Pi([x_1,\infty)\times\cdots\times[x_d,\infty))\,, \quad x \in [0,\infty]^d\,, \tag{2.3}$$

can be constructed from the marginal tail integrals (2.2) by means of a Lévy copula. This is the content of Sklar's theorem for Lévy processes with positive jumps, which basically says that every multivariate tail integral $\overline{\Pi}$ can be decomposed into its marginal tail integrals and a Lévy copula $\widehat{C}$ according to

$$\overline{\Pi}(x_1,\ldots,x_d) = \widehat{C}(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d))\,, \qquad x \in [0,\infty]^d\,. \tag{2.4}$$

For a precise formulation of this theorem we refer to Cont & Tankov [11], Theorem 5.6. Now we can define the following prototypical LDA model that we rely on in the rest of the paper. Since the multivariate tail integral (2.4) in turn defines a multivariate compound Poisson process $(S_1,\ldots,S_d)$ (cf. Cont & Tankov [11], Theorem 3.1) we arrive at the following.

Conclusion. The multivariate compound Poisson model based on Lévy copulas is the most natural and straight-forward extension of the well-known univariate standard LDA model to several dependent operational risk cells satisfying (2.1).

Definition 2.1. [Multivariate compound Poisson model]
□ *The vector of aggregate loss processes* $(S_1,\ldots,S_d)$ *is a d-dimensional compound Poisson process, which implies that for all* $i = 1,\ldots,d$ *the component* S_i *is a one-dimensional compound Poisson process with intensity* $\lambda_i > 0$ *and severity distribution* F_i.
□ *The dependence between different cells is modelled by a Lévy copula. For* $i = 1,\ldots,d$ *let* $\overline{\Pi}_i(\cdot) = \lambda_i \overline{F}_i(\cdot)$ *be the marginal tail integrals, where we assume that* $F(x) \in (0,1)$ *for all* $x \in (0,\infty)$, *and* $\widehat{C} : [0,\infty)^d \to [0,\infty)$ *be a Lévy copula. Then*

$$\overline{\Pi}(x_1,\ldots,x_d) = \widehat{C}(\overline{\Pi}_1(x_1),\ldots,\overline{\Pi}_d(x_d))$$

defines the tail integral of the d-dimensional compound Poisson process $S = (S_1,\ldots,S_d)$.

3 Dependent Operational Risk: a Bivariate Example

A bivariate model is particularly useful to illustrate how dependence modelling via Lévy copulas works. Therefore, we now focus on two operational risk cells (index $i = 1, 2$) with frequency parameters λ_i and severity distributions F_i so that the marginal tail integrals are given by $\overline{\Pi}_i(\cdot) = \lambda_i \overline{F}_i(\cdot)$ as explained in (2.2).

Before we consider the so-called Clayton Lévy copula in greater detail, we briefly mention how in general parametric Lévy copulas can be constructed. The following Proposition shows how Lévy copulas can be derived from distributional copulas and, therefore, ensures that there exists a wide variety of potentially useful Lévy copulas (see Cont & Tankov [11], Proposition 5.5).

Proposition 3.1. *Let C be a two-dimensional distributional copula and $f : [0,1] \to [0,\infty]$ an increasing convex function. Then*

$$\widehat{C}(u,v) = f(C(f^{-1}(u), f^{-1}(v))) \,, \quad u,v \in [0,\infty) \,,$$

defines a two-dimensional positive Lévy copula.

Example 3.2. [Clayton Lévy copula]
Henceforth, the dependence structure between two operational risk cells shall be modelled by a Clayton Lévy copula, which is similar to the well-known Clayton copula for distribution functions and parameterized by $\vartheta > 0$ (see Cont & Tankov [11], Example 5.5):

$$\widehat{C}_\vartheta(u,v) = (u^{-\vartheta} + v^{-\vartheta})^{-1/\vartheta} \,, \quad u,v \geq 0 \,.$$

□

We use this copula for mainly two reasons:

- This copula covers the whole range of positive dependence: For $\vartheta \to 0$ we obtain independence of the marginal processes given by $\widehat{C}_\perp(u,v) = u 1_{v=\infty} + v 1_{u=\infty}$, and losses in different cells never occur at the same time. For $\vartheta \to \infty$ we get the complete positive dependence Lévy copula given by $\widehat{C}_\|(u,v) = \min(u,v)$, and losses always occur at the same points in time. By varying ϑ, the cell dependence changes smoothly between these two extremes. However, it should be stressed that $\widehat{C}_\|$ only leads to completely dependent processes, if the marginal tail integrals are continuous. This is relevant for dependence of compound Poisson processes, which create by definition a discontinuity of the tail integral in 0, and this has to be discussed in detail below.

- The Clayton copula has a quite simple parametrization (only one parameter ϑ) and, as we will see later, it allows for precise analytical calculations regarding total aggregated OpVAR. Therefore, we consider the Clayton Lévy copula as particularly useful to investigate different dependence scenarios.

The question what dependence between operational risk cells actually means is not trivial, and with this regard we already mentioned some common modelling techniques in the introduction. Some of them are quite flexible and sophisticated, however, also very complex and often difficult to parameterize. In contrast, Lévy copulas and tail integrals together lead to a quite natural interpretation of dependence in the context of the multivariate compound Poisson model of Definition 2.1. To see this, we start with the following decomposition of the marginal tail integral $\overline{\Pi}_1$ for $x_1 \geq 0$,

$$\overline{\Pi}_1(x_1) \;=\; \Pi([x_1,\infty)\times[0,\infty))\,, \qquad x_1 \geq 0\,,$$

which basically measures the number of jumps larger than x_1 in the first component, regardless of the jumps in the second component (i.e. whether jumps with arbitrary size occur or not). This together with (2.3) and (2.4) leads to

$$\begin{aligned}\overline{\Pi}_1(x_1) &= \Pi([x_1,\infty)\times[0,\infty))\\ &= \Pi([x_1,\infty)\times\{0\}) + \lim_{x_2\downarrow 0}\Pi([x_1,\infty)\times[x_2,\infty))\\ &= \Pi([x_1,\infty)\times\{0\}) + \lim_{x_2\downarrow 0}\overline{\Pi}(x_1,x_2)\\ &= \Pi([x_1,\infty)\times\{0\}) + \lim_{x_2\downarrow 0}\widehat{C}(\overline{\Pi}_1(x_1),\overline{\Pi}_2(x_2))\\ &= \Pi([x_1,\infty)\times\{0\}) + \widehat{C}(\overline{\Pi}_1(x_1),\lambda_2)\\ &=: \overline{\Pi}_{\perp 1}(x_1) \;+\; \overline{\Pi}_{\| 1}(x_1)\;, \qquad x_1 \geq 0\,, \end{aligned} \tag{3.1}$$

where $\overline{\Pi}_{\perp 1}(\cdot)$ describes losses that occur in the first cell only without any simultaneous loss in the second cell. In contrast, $\overline{\Pi}_{\| 1}(\cdot)$ describes the expected number of losses per unit time above x_1 in the first cell that coincide with losses of arbitrary size in the second cell (occurring with frequency λ_2). Similarly we may write

$$\overline{\Pi}_2(x_2) \;=:\; \overline{\Pi}_{\perp 2}(x_2) \;+\; \overline{\Pi}_{\| 2}(x_2)\;, \qquad x_2 \geq 0\,. \tag{3.2}$$

Connected with these decompositions of the marginal tail integrals, we obtain the following split of the cells' aggregate loss processes (the time parameter t is dropped for simplicity):

$$\begin{aligned} S_1 &= S_{\perp 1} + S_{\| 1} = \sum_{k=1}^{N_{\perp 1}} X^1_{\perp k} + \sum_{l=1}^{N_{\|}} X^1_{\| l}\,,\\ S_2 &= S_{\perp 2} + S_{\| 2} = \sum_{m=1}^{N_{\perp 2}} X^2_{\perp m} + \sum_{l=1}^{N_{\|}} X^2_{\| l}\,, \end{aligned} \tag{3.3}$$

where $S_{\| 1}$ and $S_{\| 2}$ describe the aggregate losses of cell 1 and 2, respectively, that are generated by "common shocks", and $S_{\perp 1}$ and $S_{\perp 2}$ are independent loss processes. Note that apart from $S_{\|1}$ and $S_{\|2}$, all compound Poisson processes on the right-hand side of (3.3) are mutually independent.

So far all these considerations are regardless of a specific Lévy copula. However, it is clear that the relative “weights” of $S_{\|1}$ and $S_{\|2}$ compared to $S_{\perp 1}$ and $S_{\perp 2}$ directly reflect the dependence structure and so all their parameters can be written in terms of the Lévy copula. In the following we disentangle the dependence introduced by a Lévy copula and describe precisely, what it results for loss times and loss severities.

Simultaneous loss times. We begin with the frequency of simultaneous losses, which may in principle be arbitrarily small and, therefore, are given by

$$\lim_{x_1,x_2\downarrow 0} \overline{\Pi}(x_1,x_2) = \widehat{C}(\lambda_1,\lambda_2) = \lim_{x\downarrow 0} \overline{\Pi}_{\|2}(x) = \lim_{x\downarrow 0} \overline{\Pi}_{\|1}(x) =: \lambda_{\|} \,.$$

On one hand, in the case of independence, losses never occur at the same points in time; on the other hand, for complete positive dependence we have $\widehat{C}_{\|}(u,v) = \min(u,v)$. Obviously,

$$0 \le \lambda_{\|} \le \min(\lambda_1,\lambda_2)\,, \tag{3.4}$$

In particular, maximum dependence is reached if all losses in the cell with the smaller number of expected losses coincide with losses of the other cell.

A widespread concept for modelling dependence in operational risk is that of the frequency correlation between two aggregate loss processes. In the compound Poisson process approach (recall Sklar's theorem for Lévy copulas), the correlation between the number of losses $N_1(t)$ and $N_2(t)$ up to time t associated with S_1 and S_2, respectively, is simply given by

$$\rho(N_1(t),N_2(t)) \;=\; \frac{\operatorname{cov}(N_1(t),N_2(t))}{\sqrt{\operatorname{var}(N_1(t))\operatorname{var}(N_2(t))}} = \frac{\lambda_{\|}}{\sqrt{\lambda_1\,\lambda_2}}\,. \tag{3.5}$$

Obviously, for $\lambda_1 > \lambda_2$ the maximum possible frequency correlation is $\rho_{\max} = \sqrt{\lambda_2/\lambda_1}$. So, for two cells with $\lambda_1 \gg \lambda_2$ this frequency correlation is restricted to relatively low values.

Independent loss times. We now turn to the frequencies of the independent loss processes $S_{\perp 1}$ and $S_{\perp 2}$. Using (3.1) and (3.2) we can write their tail integrals for $x_1, x_2 \ge 0$ as

$$\begin{aligned} \overline{\Pi}_{\perp 1}(x_1) &= \overline{\Pi}_1(x_1) - \overline{\Pi}_{\|1}(x_1) = \overline{\Pi}_1(x_1) - \widehat{C}(\overline{\Pi}_1(x_1),\lambda_2)\,, \\ \overline{\Pi}_{\perp 2}(x_2) &= \overline{\Pi}_2(x_2) - \overline{\Pi}_{\|2}(x_2) = \overline{\Pi}_2(x_2) - \widehat{C}(\lambda_1,\overline{\Pi}_2(x_2))\,, \end{aligned} \tag{3.6}$$

so that

$$\lambda_{\perp 1} = \lim_{x\downarrow 0} \overline{\Pi}_{\perp 1}(x) = \lambda_1 - \lambda_{\|}\,, \qquad \lambda_{\perp 2} = \lim_{x\downarrow 0} \overline{\Pi}_{\perp 2}(x) = \lambda_2 - \lambda_{\|}\,. \tag{3.7}$$

Example 3.3. [Continuation of Example 3.2]
Recall the Clayton Lévy copula

$$\widehat{C}_\vartheta(u,v) = (u^{-\vartheta} + v^{-\vartheta})^{-1/\vartheta}, \quad u, v \geq 0$$

for $\vartheta \in (0, \infty)$. In this case we calculate the frequency of simultaneous jumps as

$$\lambda_\| = (\lambda_1^{-\vartheta} + \lambda_2^{-\vartheta})^{-1/\vartheta}, \tag{3.8}$$

and the frequency correlation is given by

$$\rho(N_1(t), N_2(t)) = \frac{\lambda_\|}{\sqrt{\lambda_1 \lambda_2}} = \frac{(\lambda_1^{-\vartheta} + \lambda_2^{-\vartheta})^{-1/\vartheta}}{\sqrt{\lambda_1 \lambda_2}}.$$

We show that, although the Clayton Lévy copula tends to $\widehat{C}_\infty(u,v) = \min(u,v)$ (i.e. the complete dependence copula) as $\vartheta \to \infty$, the processes S_1 and S_2 are not completely dependent. Take two cells with $\lambda_1 = 1000$ and $\lambda_2 = 10$, then in Figure 3.4 both $\lambda_\|$ and $\lambda_{\perp 1}$ are plotted as a function of the Lévy Clayton copula parameter ϑ. One can see that even for $\vartheta \to \infty$ there are non-simultaneous losses occurring in only the first cell with intensity $\lambda_{\perp 1} = \lambda_1 - \lambda_2 = 990$. Furthermore, the maximal possible correlation in this model is $\rho_{\max} = 10$ %. □

Conclusion. Since dependence of the loss frequency processes only influence the number of expected losses, it follows that frequency correlation for every model has only a very restricted impact on OpVAR.

Simultaneous loss severities and their distributional copula. Also the severity distributions of $X_\|^1$ and $X_\|^2$ as well as their dependence structure are determined by the Lévy copula. To see this, define the joint survival function as

$$\overline{F}_\|(x_1, x_2) := P(X_\|^1 > x_1, X_\|^2 > x_2) = \frac{1}{\lambda_\|} \widehat{C}(\overline{\Pi}_1(x_1), \overline{\Pi}_2(x_2)) \tag{3.9}$$

with marginals

$$\overline{F}_{\|1}(x_1) = \lim_{x_2 \downarrow 0} \overline{F}_\|(x_1, x_2) = \frac{1}{\lambda_\|} \widehat{C}(\overline{\Pi}_1(x_1), \lambda_2) \tag{3.10}$$

$$\overline{F}_{\|2}(x_2) = \lim_{x_1 \downarrow 0} \overline{F}_\|(x_1, x_2) = \frac{1}{\lambda_\|} \widehat{C}(\lambda_1, \overline{\Pi}_2(x_2)). \tag{3.11}$$

To explicitly extract the dependence structure between the severities of simultaneous losses $X_\|^1$ and $X_\|^2$ we use the concept of a distributional survival copula. In general, if $\overline{F}(x_1, x_2)$ is a joint survival function with continuous marginals $\overline{F}_i(x_i)$, there exists a unique survival copula $\widehat{S}$ such that $\overline{F}(x_1, x_2) = \widehat{S}(\overline{F}_1(x_1), \overline{F}_2(x_2))$ giving together with the rhs of (3.9) the relation between Lévy copula and survival copula.

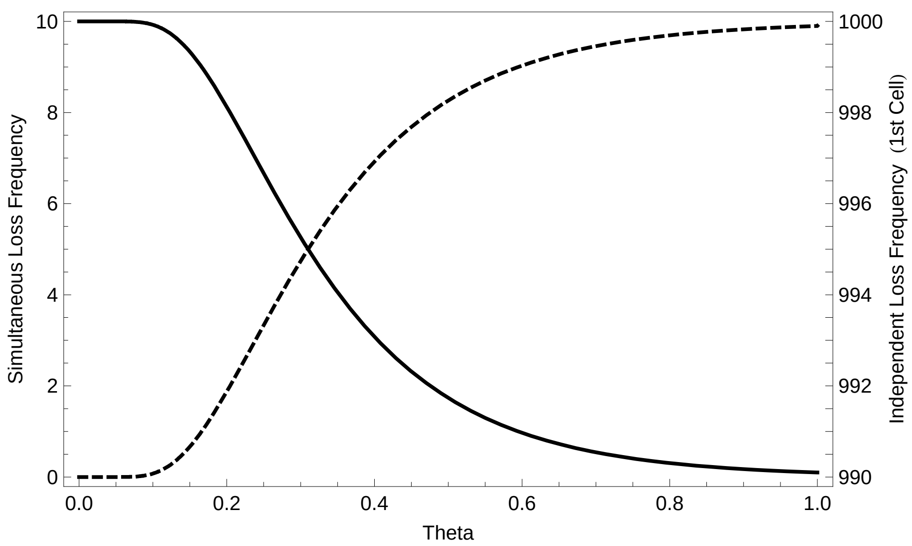

Figure 3.4. Example how the cells' loss frequencies are controlled by the Clayton Lévy copula for $\lambda_1 = 1000$ and $\lambda_2 = 10$. Left axis: frequency $\lambda_\|$ of the simultaneous loss processes $S_{\|1}$ and $S_{\|2}$ as a function of the Lévy Clayton copula parameter ϑ (dashed line). Right axis: frequency $\lambda_{\perp 1}$ of the independent loss process $S_{\perp 1}$ of the first cell as a function of the Lévy Clayton copula parameter ϑ (solid line).

Example 3.5. [Continuation of Examples 3.2 and 3.3]
For the Clayton copula a straight-forward calculation using (3.9)–(3.11) shows that the survival copula $\widehat{S}_\vartheta$ for the tail severity distributions $\overline{F}_{\|1}(\cdot)$ and $\overline{F}_{\|2}(\cdot)$ is the well-known distributional Clayton copula; i.e. for $\vartheta > 0$,

$$\widehat{S}_\vartheta(u,v) = (u^{-\vartheta} + v^{-\vartheta} - 1)^{-1/\vartheta}, \qquad 0 \le u, v \le 1\,.$$

Consequently, the distribution functions $F_{\|1}$ and $F_{\|2}$ (and thus the simultaneous losses $X_\|^1$ and $X_\|^2$) are linked by a copula C_ϑ that is related to $\widehat{S}_\vartheta$ via

$$\begin{aligned} C_\vartheta(u,v) &= \widehat{S}_\vartheta(1-u, 1-v) + u + v - 1 \\ &= ((1-u)^{-\vartheta} + (1-v)^{-\vartheta} - 1)^{-1/\vartheta} + u + v - 1,\ 0 \le u, v \le 1\,. \end{aligned} \tag{3.12}$$

Specifically, for $\vartheta \to \infty$ we obtain the complete dependence distributional copula $C_\|(u,v) = \min(u,v)$, implying comonotonicity of the simultaneous losses $X_\|^1$ and $X_\|^2$. □

Independent loss severities and their distributions. We obtain from (3.6) for the severity distributions of non-simultaneous losses

$$\begin{aligned}\overline{F}_{\perp 1}(x_1) &= \frac{\lambda_1}{\lambda_1^{\perp}}\overline{F}_1(x_1) - \frac{1}{\lambda_1^{\perp}}\widehat{C}(\lambda_1\overline{F}_1(x_1), \lambda_2)\,,\\ \overline{F}_{\perp 2}(x_1) &= \frac{\lambda_2}{\lambda_2^{\perp}}\overline{F}_2(x_2) - \frac{1}{\lambda_2^{\perp}}\widehat{C}(\lambda_1, \lambda_2\overline{F}_2(x_2))\,.\end{aligned}$$

Let us summarize the interpretation of multivariate operational risk as it is suggested by our model.

- Dependence between different cells is solely due to the occurrence of simultaneous loss events in different cells.
- There are two types of losses: independent ones, which happen in one single cell only and dependent ones, which happen simultaneously. The severity distributions of dependent losses are themselves coupled by a distributional copula, which can be derived from the Lévy copula (3.9), e.g. (3.12) in the case of a Clayton Lévy copula. In particular, it follows that in general $F_{\|1}$ and $F_{\|2}$ are different from F_1 and F_2, respectively. Also $F_{\perp 1}$ and $F_{\perp 2}$ are different from F_1 and F_2 as well as from $F_{\|1}$ and $F_{\|2}$, respectively.
- Independence of different cells means that their losses never happen at the same time, whereas complete dependence is equivalent to losses that always occur together.

This pattern is depicted in Figures 5.6-5.8 for the Clayton Lévy copula, where sample paths and occurrence times of the bivariate compound Poisson model are simulated for different parameters $\vartheta = 0.3, 1$ and 7 of the Clayton Lévy copula. For the purpose of a clearer illustration of the dependence structure, both cells are assumed to have identical frequencies of $\lambda_1 = \lambda_2 = 10$ and Pareto distributed severities with tail parameters $\alpha_1 = 1.2$ and $\alpha_2 = 2$, and scale parameters $\theta_1 = \theta_2 = 1$. According to (3.8), the percentage average number of common losses related to the different ϑ used are 10%, 50%, and 90%. The simulation is based on Algorithm 6.15 of Cont & Tankov [11], which can be used for arbitrary severity distributions as well.

Conclusion. The Lévy copula influences the distributions of the simultaneous losses and the non-simultaneous ones in different cells.

4 Analytical Approximations for Operational VAR

4.1 Preliminaries

In this section we turn to the quantification of total operational loss encompassing all operational risk cells and, therefore, we focus on the total aggregate loss process S^+ defined in (1.2). Our goal is to provide some general insight to multivariate operational

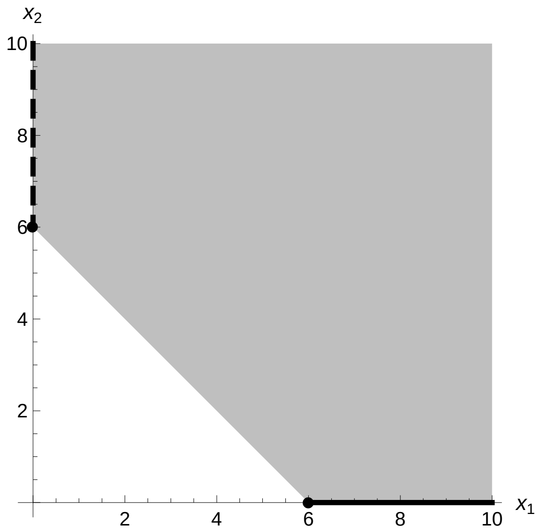

Figure 4.1. Decomposition of the domain of the tail integral $\overline{\Pi}^+(z)$ for $z = 6$ into a simultaneous loss part $\overline{\Pi}_{\|}^+(z)$ (grey area) and independent parts $\overline{\Pi}_{\perp 1}(z)$ (solid black line) and $\overline{\Pi}_{\perp 2}(z)$ (dashed black line).

risk and to find out, how different dependence structures (modelled by Lévy copulas) affect OpVAR. The tail integral associated with S^+ is given by

$$\overline{\Pi}^+(z) = \Pi(\{(x_1, \dots, x_d) \in [0,\infty)^d : \sum_{i=1}^{d} x_i \geq z\}), \quad z \geq 0, \tag{4.1}$$

measuring the expected number of aggregate losses per unit time leading to a total loss larger than z. For $d = 2$ we can write

$$\overline{\Pi}^+(z) \;=\; \overline{\Pi}_{\perp 1}(z) + \overline{\Pi}_{\perp 2}(z) + \overline{\Pi}_{\|}^+(z), \qquad z \geq 0, \tag{4.2}$$

where $\overline{\Pi}_{\perp 1}(\cdot)$ and $\overline{\Pi}_{\perp 2}(\cdot)$ are the independent parts defined in (3.1)-(3.2) and

$$\overline{\Pi}_{\|}^+(z) \;=\; \Pi(\{(x_1, x_2) \in (0,\infty)^2 : x_1 + x_2 \geq z\}), \qquad z \geq 0,$$

describes the dependent part due to simultaneous loss events. This is depicted in Figure 4.1, where the support of $\overline{\Pi}_{\|}^+(\cdot)$ is shaded in orange, and the support of $\overline{\Pi}_{\|1}(\cdot)$ and $\overline{\Pi}_{\|2}(\cdot)$ are solid black and dashed black lines, respectively.

Our basic model is a multivariate compound Poisson process, which implies that the total aggregate loss process S^+ is again compound Poisson with frequency parameter and severity distribution

$$\lambda^+ = \lim_{z \downarrow 0} \overline{\Pi}^+(z) \quad \text{and} \quad F^+(z) = 1 - \overline{F}^+(z) = 1 - \frac{\overline{\Pi}^+(z)}{\lambda^+}, \quad z \geq 0. \tag{4.3}$$

This result has been exploited already in Böcker & Klüppelberg [7] and Bregman & Klüppelberg [9] and it will prove useful to determine a bank's total operational risk consisting of several cells.

Conclusion. Though operational risk is usually modeled by separating several event type/business line cells, a bank's total OpVAR can be thought of as effectively being a compound Poisson process with risk inter-arrival times being exponentially distributed with finite mean λ^+ and loss severities, which are independent and identically distributed with distribution function $F^+(\cdot)$.

Stand-alone OpVAR revisited. Stand-alone OpVAR at confidence levels $\kappa \in (0,1)$ and time horizon t is the κ-quantile of the aggregate loss distribution, i.e.

$$\mathrm{VAR}_t(\kappa) = G_t^{\leftarrow}(\kappa) = \inf\{x \in \mathbb{R} : \; P(S(t) \leq x) \geq \kappa\} .$$

In general, even stand-alone OpVAR cannot be calculated analytically. In Böcker & Klüppelberg [6], however, it was shown that stand-alone OpVAR at a high confidence level can be approximated by a closed-form expression, if the loss severity is subexponential, i.e. heavy-tailed. As heavy-tailedness of severity distributions is not debated and indeed statistically justified by Moscadelli [17], we consider in the sequel this approximation, which can be written as

$$\mathrm{VAR}_t(\kappa) \sim F^{\leftarrow}\left(1 - \frac{1-\kappa}{EN(t)}\right), \quad \kappa \uparrow 1 , \tag{4.4}$$

where "$\sim$" means that the ratio of left and right hand side converges to 1. Moreover, $EN(t)$ is the cell's expected number of losses in the time interval $[0,t]$. Important examples for subexponential distributions are lognormal, Weibull, and Pareto. Equation (4.4) can be interpreted that only one single very big loss event instead of the accumulation of several small events determines overall OpVAR and is therefore often called "single-loss approximation" of OpVAR, see Böcker & Sprittulla [8].

A first glance at multivariate OpVAR. Let us extend the idea of an asymptotic OpVAR approximation to the multivariate model. In doing so, we exploit the fact that S^+ is a one-dimensional compound Poisson process with parameters as in (4.3). In particular, if F^+ is subexponential, we can apply (4.4) to estimate total OpVAR. In combination with the Conclusion drawn from the dependence of the loss frequency processes before this leads immediately to the already mentioned very important observation regarding multivariate operational risk:

Conclusion. Total OpVAR is asymptotically only impacted by the expected number of total loss events, $EN^+(t) = EN_1(t) + \cdots + EN_d(t) = \lambda^+$ for $t \geq 0$. It follows that frequency correlation for every model has only a very restricted impact on OpVAR and does not deserve much attention.

It has been observed that total OpVAR is presumably affected by business volume at time. Actually, this belief is a basic assumption both for the Basic Indicator Approach and the Standardized Approach of Basel II [3], where capital charges for operational risk are scaled by gross income. This idea can be included in the above multivariate compound Poisson model by adapting the frequency. For each $i = 1, \ldots, d$, we leave the severity models X^i untouched, as well as the independence of the severities of the (no longer homogeneous) Poisson process N_i. However, instead of a constant intensity, we model a time-dependent frequency depending on business volume: in each single cell we replace $EN_i(t) = \lambda_i t$ by $EN_i(t) = \int_0^t \lambda_i(s)\, ds$ for $t > 0$. This is then plugged into formula (4.4) for the stand-alone OpVAR of cell i and consequently into formula (4.3) $EN^+(t) = \lambda^+ t$ for total OpVAR.

5 Results for the heavy-tailed GPD model

In general, analytic results for multivariate OpVAR (similar to (4.4) in the univariate case) do not exist for arbitrary Lévy copulas and severity distributions, but can be obtained by simulation analysis as shown in Figures 5.6-5.8. However, focusing our attention on special cases of the dependence structure and the severity distributions, useful and simply to apply closed-form results can be attained.

We now consider the very typical situation that operational losses are heavy-tailed. Instead of using the general concept of regular variation as in Böcker & Klüppelberg [7], we invoke here the so-called POT method (an achronym for "peaks over threshold"), which is a classical technique of extreme value theory. The POT method is based on the fact that (under weak regularity conditions, see Embrechts, Klüppelberg & Mikosch [12], Section 3.4 for details) loss data above a high threshold u follow a generalized Pareto distribution (GPD). The body of the severity distribution is estimated by the empirical distribution function, i.e. for losses with moderate size below the threshold u any arbitrary distribution is possible. This situation is depicted in Figure 5.1, which schematically shows the probability density function of such a mixed severity distribution. The appropriateness of heavy-tailed GPD models in the context of operational risk has been justified very convincingly e.g. in Moscadelli [17]; an example for its practical implementation can be found in Nguyen & Ottmann [18]. Finally, closed-form results for stand-alone OpVAR as well as expected shortfall using GPD tail severities are provided in Böcker [5].

In such a model, the heavy-tailed severity distribution above a high threshold $u > 0$ (i.e. high severity loss) is parameterized for $\xi, \beta, > 0$ by

$$\overline{F}(x) = w \left(1 + \xi \, \frac{x - u}{\beta}\right)^{-1/\xi}, \quad x > u > 0\,, \tag{5.1}$$

where $w = w(u) \in (0,1)$ describes the relative number of losses above u, sometimes

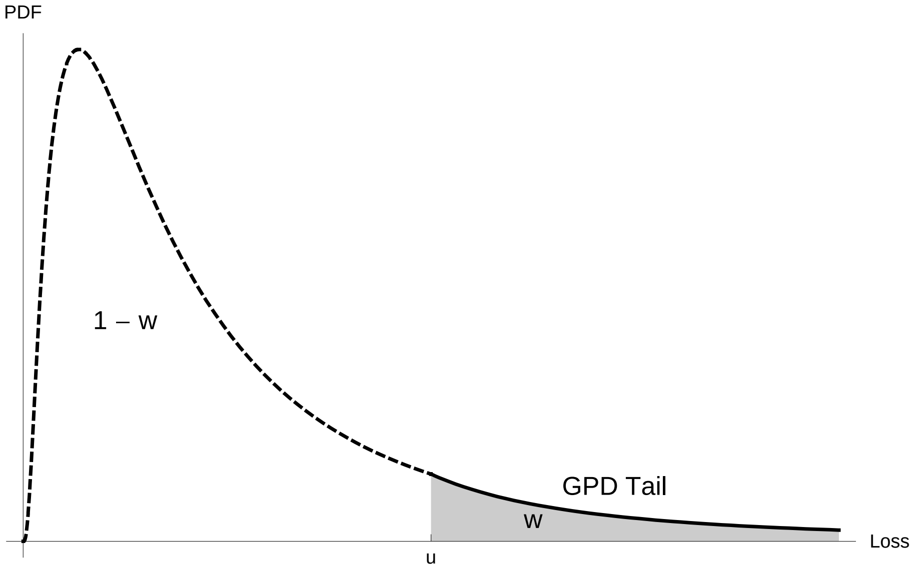

Figure 5.1. Probability density function of a model where high severity losses above a threshold $u > 0$ follow a GPD (solid line). The distribution body (dotted line) is differently modelled by e.g. a lognormal or Weibull distribution. The tail weight w corresponds to the shaded area under the curve.

referred to as the tail weight of F, and ξ is called the tail parameter. Note that (as $\xi > 0$ is required) the GPD is a Pareto distribution including besides the shape parameter ξ also a location and scale parameter. This proves in particular useful for statistical analyses. Moreover, in contrast to the general concept of regular variation (cf. [6, 15]), it makes the slowly varying function precise, namely a constant: $\overline{F}(x) \sim w(\xi/\beta)^{-1/\xi} x^{-1/\xi}$ as $x \to \infty$. As a consequence of (4.4), analaytic stand-alone OpVAR is then for fixed $t > 0$ given explicitly by (see again Böcker [5])

$$\mathrm{VAR}_t(\kappa) \sim u + \frac{\beta}{\xi}\left[\left(\frac{w\,\lambda\,t}{1-\kappa}\right)^{\xi} - 1\right], \quad \kappa \uparrow 1 \tag{5.2}$$

where $w\,\lambda$ denotes the expected number of losses per unit time above a threshold u. Note that the threshold u enters the OpVAR approximation in two different ways. First, explicitly by the linear term on the right-hand side of (5.2). However, in practical calculations, for typical parameterizations of LDA models OpVAR at high confidence level κ will be approximated sufficiently well by neglecting u and only evaluating the second term on the right-hand side of (5.2). Second, and even more important, the threshold u implicitly enters OpVAR by influencing w and thus the effective loss frequency used in formula (5.2). Hence our result again shows how important a careful, sound and proper calibration of an operational risk models is in order not to fall into the *model and calibration risk's trap.*

Our first analytical result for total OpVAR deals with the case of one cell severity dominating all the others and holds for arbitrary Lévy copulas. We have, shown in [7], Theorem 3.4 and Corollary 3.5, that for multivariate compound Poisson models for fixed $t > 0$, whenever one Pareto-like severity tail dominates the others, the stand-alone OpVAR of this most dangerous severity distribution as in (4.4) dominates total OpVAR. We restate this important result here in the case of the GPD model.

Theorem 5.2. *Consider the compound Poisson model of Definition 2.1 with arbitrary Lévy copula and assume that large losses above a high threshold $u > 0$ in the first cell have a GPD tail with tail weight w and parameters $\xi, \beta > 0$, given by (5.1). Assume that $\lambda > 0$ denotes the frequency in the first cell and its severity distribution F_1 as above is tail-dominant to all other cell severities (which apart from that can have arbitrary distribution functions), i.e. $\overline{F}_i(x)/\overline{F}_1(x) \to 0$ as $x \to \infty$ for all $i = 2, \ldots, d$. Then S^+ is a compound Poisson process with*

$$P(S^+(t) > z) \sim \lambda\, t\, \overline{F}_1(z) = \lambda\, t\, w \left(1 + \xi \frac{z-u}{\beta}\right)^{-1/\xi}, \quad z \to \infty\,.$$

Furthermore, total OpVAR is asymptotically given by

$$\mathrm{VAR}_t^+(\kappa) \sim \mathrm{VAR}_t^1(\kappa) \sim u + \frac{\beta}{\xi}\left[\left(\frac{w\,\lambda\,t}{1-\kappa}\right)^{\xi} - 1\right] \sim \frac{\beta}{\xi}\left(\frac{w\,\lambda\,t}{1-\kappa}\right)^{\xi}, \qquad \kappa \uparrow 1\,,$$

i.e. total OpVAR *is asymptotically dominated by the stand-alone* OpVAR *of the first cell.*

The assumptions of Theorem 5.2 may in many cases be quite realistic, however, it is of course possible that two ore more cells' severity distributions are tail equivalent. To present an approximation of OpVAR in such a situation, we consider a GPD-Poisson model where the severity distributions F_i of the first (say) $b \le d$ cells are tail equivalent, both with tail parameter $\xi > 0$, and dominant to all other cells, i.e.

$$\lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = \frac{w_i}{w_1}\left(\frac{\beta_i}{\beta_1}\right)^{1/\xi}, \; i = 1, \ldots, b, \qquad \lim_{x\to\infty} \frac{\overline{F}_i(x)}{\overline{F}_1(x)} = 0\,, \; i = b+1, \ldots, d\,, \quad (5.3)$$

and thus $0 \le \xi_i < \xi$, $i = b+1, \ldots, d$. The case $\xi_i = 0$ corresponds to a tail lighter than any Pareto tail. Naturally, $b = 1$ is a special case of Theorem 5.2. Even in this quite simple setup, for arbitrary Lévy copula an analytic approximation for total OpVAR cannot be given. However, in the important cases of complete positive dependence and independence, closed-form approximations can be found.

Theorem 5.3. *Consider a compound Poisson model with cell processes $S_1, \ldots, S_d$ with GPD severity tails (and arbitrary distribution body) satisfying (5.3). Let $\mathrm{VAR}_t^i(\cdot)$ be the stand-alone* OpVAR *of cell i as in (5.2).*

(1) *If all cells are completely dependent with the same frequency* λ *for all cells, then* S^+ *is a compound Poisson process with parameters*

$$\lambda^+ = \lambda \qquad \text{and} \qquad \overline{F}^+(z) \sim w^+ \left(1 + \xi \frac{z - u_1}{\beta_1}\right)^{-1/\xi} \sim \left(\sum_{i=1}^b \frac{\beta_i}{\xi}\, w_i^\xi\right)^{1/\xi} z^{-1/\xi}, \quad z \to \infty,$$

where $w^+ = (\sum_{i=1}^b \frac{\beta_i}{\beta_1} w_i^\xi)^{1/\xi}$. *Total* OpVAR *is asymptotically given by*

$$\begin{aligned} \mathrm{VAR}^+_{\|t}(\kappa) &\sim u_1 + \frac{\beta_1}{\xi}\left[\left(\frac{w^+\lambda t}{1-\kappa}\right)^\xi - 1\right] \\ &\sim \sum_{i=1}^b \frac{\beta_i}{\xi}\left(\frac{w_i \lambda\, t}{1-\kappa}\right)^\xi \sim \sum_{i=1}^b \mathrm{VAR}^i_t(\kappa), \qquad \kappa \uparrow 1. \end{aligned} \tag{5.4}$$

(2) *If all cells are independent, then* S^+ *is a compound Poisson process with parameters*

$$\begin{aligned} \lambda^+ &= \lambda_1 + \cdots + \lambda_d \quad \text{and} \\ \overline{F}^+(z) &\sim w^+ \left(1 + \xi \frac{z - u_1}{\beta_1}\right)^{-1/\xi} \sim \frac{1}{\lambda^+}\sum_{i=1}^b w_i\, \lambda_i \left(\frac{\beta_i}{\xi}\right)^{1/\xi} z^{-1/\xi}, \quad z \to \infty, \end{aligned} \tag{5.5}$$

where $w^+ = \frac{1}{\lambda^+}\sum_{i=1}^b (\frac{\beta_i}{\beta_1})^{1/\xi}\lambda_i w_i$. *Total* OpVAR *is asymptotically given by*

$$\begin{aligned} \mathrm{VAR}^+_{\perp t}(\kappa) &\sim u_1 + \frac{\beta_1}{\xi}\left[\left(\frac{w^+\lambda t}{1-\kappa}\right)^\xi - 1\right] \\ &\sim \left(\sum_{i=1}^b \left(\frac{\beta_i}{\xi}\right)^{1/\xi} \frac{w_i \lambda_i\, t}{1-\kappa}\right)^\xi \sim \left(\sum_{i=1}^b \left(\mathrm{VAR}^i_t(\kappa)\right)^{1/\xi}\right)^\xi, \qquad \kappa \uparrow 1. \end{aligned} \tag{5.6}$$

Proof. The proof follows along the lines of the proof of Theorem 3.6 in [7].
(1) Complete dependents means all cell processes jump simultaneously, hence $\lambda^+ = \lambda$. For the heavy-tailed GPD model the function $H(x_1) := x_1 + \sum_{i=2}^d F_i^{-1}(F_1(x_1))$ specifies for some constant $c > 0$ and for $x_1 > \min_{i=1,\dots,d} u_i > 0$ to

$$H(x_1) = c + \sum_{i=1}^b \frac{\beta_i}{\beta_1}\left(\frac{w_1}{w_i}\right)^{-\xi} x_1 + \sum_{i=b+1}^d \frac{\beta_i}{\xi_i}\left(1 + \xi\frac{x_1 - u_1}{\beta_1}\right)^{\frac{\xi_i}{\xi}} \left(\frac{w_1}{w_i}\right)^{-\xi_i},$$

and thus

$$H(x_1) \sim C x_1 \quad \text{with} \quad C := \sum_{i=1}^b \frac{\beta_i}{\beta_1}\left(\frac{w_i}{w_1}\right)^\xi, \quad x_1 \to \infty.$$

Hence, $H^{-1}(z) \sim z/C$ as $z \to \infty$, which implies by the fact that $\overline{\Pi}^+(z) = \overline{\Pi}^1(H^{-1}(z))$ to

$$\overline{\Pi}^+(z) \sim \lambda \overline{F}_1(z/C) \sim \lambda C^{1/\xi}\, \overline{F}_1(z), \quad z \to \infty.$$

Together with (5.1) this leads to

$$\overline{F}^{+}(z) \sim \left(\sum_{i=1}^{b} \frac{\beta_i}{\beta_1} \left(\frac{w_i}{w_1} \right)^{\xi} \right)^{1/\xi} w_1 \left(1 + \xi \, \frac{z - u_1}{\beta_1} \right)^{-1/\xi} \sim \left(\sum_{i=1}^{b} \frac{\beta_i}{\xi} \, w_i^{\xi} \right)^{1/\xi} z^{-1/\xi} \,, \quad z \to \infty \,,$$

from which together with (5.2) we finally arrive at (5.4).
(2) In this situation we have

$$\overline{\Pi}^{+}(z) \;=\; \overline{\Pi}_1(z) + \cdots + \overline{\Pi}_d(z) \,, \quad z \geq 0 \,.$$

This implies immediately

$$\lambda^{+} = \lambda_1 + \cdots + \lambda_d \quad \text{and} \quad \overline{F}^{+}(z) \sim \frac{1}{\lambda^{+}} [\lambda_1 \overline{F}_1(\cdot) + \cdots + \lambda_d \overline{F}_d(\cdot)] \,.$$

With (5.3) we thus conclude that

$$\lim_{z \to \infty} \frac{\overline{F}^{+}(z)}{\overline{F}_1(z)} = \frac{1}{\lambda^{+}} \sum_{i=1}^{b} \lambda_i \, \frac{w_i}{w_1} \left(\frac{\beta_i}{\beta_1} \right)^{1/\xi} \,,$$

from which (5.5) and finally (5.6) follows. □

Conclusion. In both situations of extreme dependence $\overline{F}^{+}(z) \sim \dfrac{w^{+}}{w_1} \overline{F}_1(z)$ as $z \to \infty$. The dependence influences the constant w^{+} only.

We would like to emphasize that also for more general dependence structures this pattern remains. When dependence is modelled by a multivariate regular variation model, this has been shown in Klüppelberg and Resnick [15]; see also [7], Theorem 3.18.

On one hand, Theorem 5.3, which holds for arbitrary heavy-tailed severity distributions as long as (5.1) holds approximately for large severities, states that in the case of complete dependence, total asymptotic OpVAR is simply the sum of the dominating cell's asymptotic stand-alone OpVARs. On the other hand, for independent cells' severities, total OpVAR can be expressed in terms of a generalized mean M_p by

$$M_p(a_1, \ldots, a_n) := \left(\frac{1}{n} \sum_{k=1}^{n} a_k^p \right)^{1/p} \,, \quad a_k \geq 0, \quad p \neq 0 \,,$$

and (5.6) can be written for $b \leq d$ as

$$\mathrm{VAR}^{+}_{\perp t}(\kappa) \;\sim\; b^{\xi} \, M_{1/\xi}(\mathrm{VAR}^1_t(\kappa), \ldots, \mathrm{VAR}^b_t(\kappa)) \,, \qquad \kappa \uparrow 1 \,.$$

Formally, the complete dependent case (5.4) can also be expressed by M_p, namely

$$\mathrm{VAR}^{+}_{\| t}(\kappa) \;\sim\; b \, M_1(\mathrm{VAR}^1_t(\kappa), \ldots, \mathrm{VAR}^b_t(\kappa)) \,, \qquad \kappa \uparrow 1 \,.$$

A fundamental difference between both extreme dependence models is that, due to the dynamical dependence concept of Lévy copulas, the completely dependent model implies identical frequency λ for all cells, whereas the independent model allows for different cell frequencies. However, if high-severity losses mainly occur in one, say the first cell, both models yield the same asymptotic total OpVAR, namely the stand-alone VAR of the first cell; see Theorem 5.2.

Recall that simple-sum OpVAR (5.4) is often suggested as an upper bound for total OpVAR. This is also the basis for the new proposals of Basel II, where the standard procedure for calculating capital charges for operational risk is just the simple-sum OpVAR. Hence, our calculation has shown that regulators implicitly assume complete dependence between different cells as worst case scenario, meaning that losses within different business lines or risk categories always happen at the same instants of time. Moreover, they assume completely dependent loss severities.

This viewpoint is in the heavy-tailed case grossly misleading. To see this, assume the same frequency λ in all cells, also for the independent model, and denote by $\mathrm{VAR}^+_{\|}(\kappa)$ and $\mathrm{VAR}^+_{\perp}(\kappa)$ completely dependent and independent total OpVAR, respectively. Then, from (5.4) and (5.6), as a consequence of convexity ($0 < \xi < 1$) and concavity ($\xi > 1$) of the function $x \mapsto x^{1/\xi}$, we obtain

$$\frac{\mathrm{VAR}^+_{\perp}(\kappa)}{\mathrm{VAR}^+_{\|}(\kappa)} \sim \frac{\left(\sum_{i=1}^b w_i\, \beta_i^{1/\xi}\right)^\xi}{\sum_{i=1}^b w_i^\xi\, \beta_i} \quad \begin{cases} < 1\,, & 0 < \xi < 1\,, \\ = 1\,, & \xi = 1\,, \\ > 1\,, & \xi > 1\,. \end{cases} \tag{5.7}$$

This result says that for heavy-tailed severity data with GPD tail given by (5.1) subadditivity of OpVAR is violated because the sum of stand-alone OpVARs is smaller than independent total OpVAR. This is a direct consequence of the Pareto-like tail, which we assumed for the loss severity distribution and is well-known in the financial literature; cf. Rootzén & Klüppelberg [20]. Nevertheless, to give an example for operational risk, consider two cells with constant stand-alone OpVAR of EUR 100 million, each calculated from a GPD model with fixed parameters $\beta_1 = \beta_2 = 1$, $w_1 = w_2 = 1$, and common tail parameter $\xi = \xi_1 = \xi_2$. Table 5.4 compares, for a realistic range of ξ-values (cf. Moscadelli [17])), total OpVAR both for completely dependent and independent data. Obviously, for $\xi > 1$, total OpVAR increases superlinearly, when taking on two independent risks, for example by opening two new subsidiaries in different parts of the world.

Even if we assume $0 < \xi < 1$ for all operational risk cells and thus $\mathrm{VAR}^+_{\|}(\kappa) > \mathrm{VAR}^+_{\perp}(\kappa)$, we obtain an interesting result concerning the relative "diversification benefit" in operational risk defined as $(\mathrm{VAR}^+_{\|} - \mathrm{VAR}^+_{\perp})/\mathrm{VAR}^+_{\|}$. Often diversification is understood to be directly linked to the notion of correlation – and particularly in the context of

$1/\xi$	$\text{VAR}_{\parallel}^{+}$	$\text{VAR}_{\perp}^{+}$
1.2		178.2
1.1		187.8
1.0	200.0	200.0
0.9		216.0
0.8		237.8
0.7		269.2

Table 5.4. *Comparison of total* OpVAR *for two operational risk cells (each with stand-alone* VAR *of EUR 100 million) in the case of complete dependence* ($\parallel$) *and independence* ($\perp$) *for different values of the tail parameter* ξ *in the relevant area* (*cf.* (5.7)).

operational risk – to the loss-number correlation $\rho(N_1(t), N_2(t))$. For heavy-tailed data, however, it is well-known that correlation (even if it exists) is a misleading concept to describe diversification within a portfolio. Consider e.g. Figure 5.5, where the relative diversification benefit for two operational risk cells with $\theta_1 = \theta_2$ is plotted as a function of the tail parameter ξ. Obviously, relative diversification is very sensitive with regards to the value of ξ. In contrast to that, the loss-number correlation ρ of both models is constant, and it follows from (3.5) that $\rho_{\parallel} = 1$ and $\rho_{\perp} = 0$. Over and above, we know that dependence models, with regards to the frequency correlation have asymptotically undistinguishable OpVARs. In particular in the case of GPD heavy-tailed severities, total OpVAR is given by (5.4).

Conclusion. Instead of trying to estimate precise frequency correlations between different cells, all effort should be directed into a more accurate modelling of the loss severity distribution.

A final word of warning. It is beyond all dispute that operational risk is very material in most financial institutions. However, risk severities can be extreme by their very nature, recall for instance prominent examples such as Barings Bank (loss \$1.3 billion) or Sumitomo Corp. (loss \$2.6 billion). Moreover, our analysis shows that multivariate high-confidence OpVAR is very sensible to the parametrization of the severity distribution, an issue, which has already been pointed out by Mignola & Ugoccioni [16] for the univariate case. Altogether, this confirms the view that capital charges are not always the best way to deal with operational risk, and that risk measurement has always to be complemented by sound risk management and control processes.

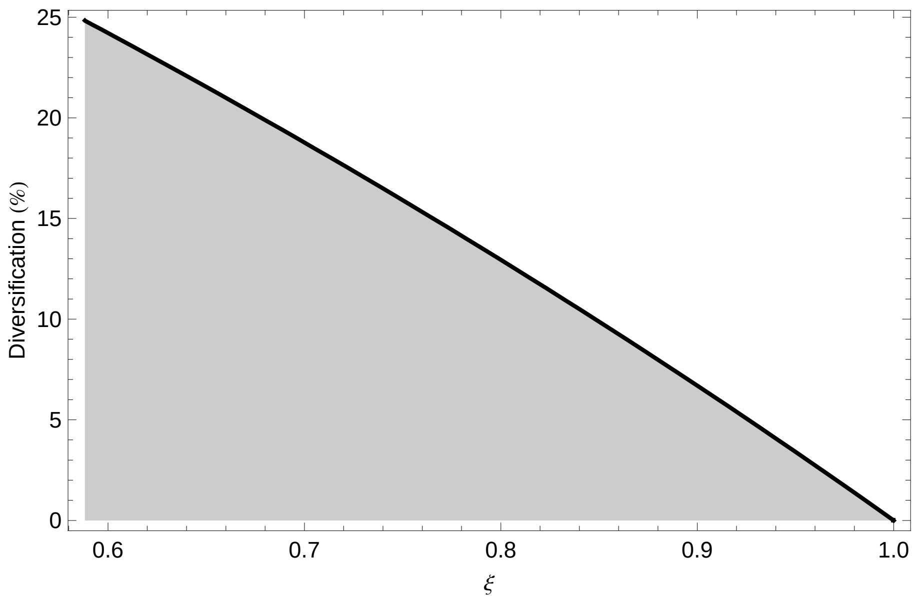

Figure 5.5. Plot of the relative diversification benefit $\frac{\mathrm{VAR}_{\parallel}^{+} - \mathrm{VAR}_{\perp}^{+}}{\mathrm{VAR}_{\parallel}^{+}} = 1 - 2^{\xi - 1}$ as given by (5.7) for two operational risk cells as a function of the tail parameter ξ.

About the Authors/Disclaimer

Klaus Böcker is senior risk controller at UniCredit Group Italiano, Munich Branch. Claudia Klüppelberg holds the chair of Mathematical Statistics at the Center for Mathematical Sciences of the Munich University of Technology. The opinions expressed in this article are those of the authors and do not reflect the views of UniCredit Group.

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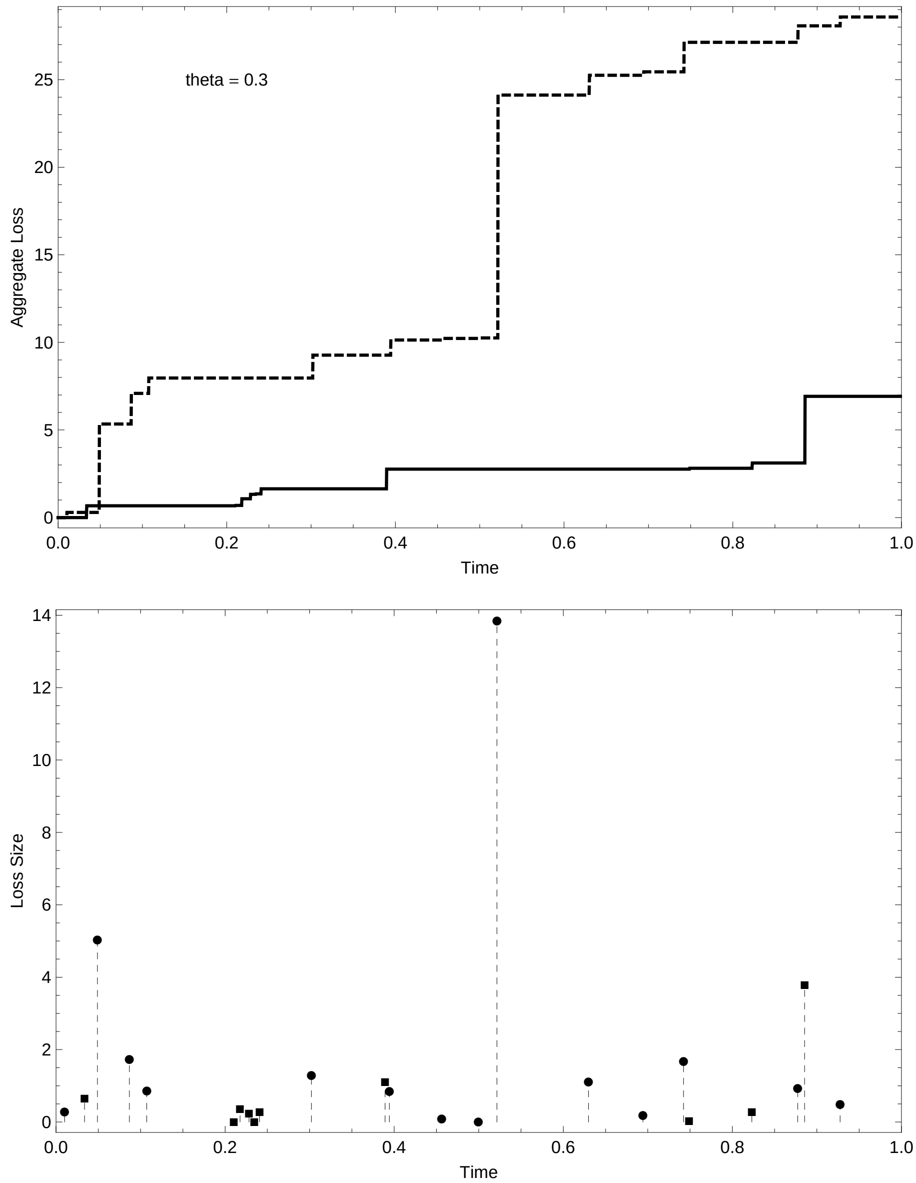

Figure 5.6. Simulation of the bivariate compound Poisson model as of Definition 2.1 with Clayton Lévy copula with parameter $\vartheta = 0.3$ (light dependence). Top panel: sample paths of the aggregate loss processes. Bottom panel: severity and occurrence times of losses. The univariate compound Poisson processes have frequencies of $\lambda_1 = \lambda_2 = 10$ and Pareto distributed severities with parameters $\alpha_1 = 1/\xi = 1.2$ and $\alpha_1 = 1/\xi = 2$.

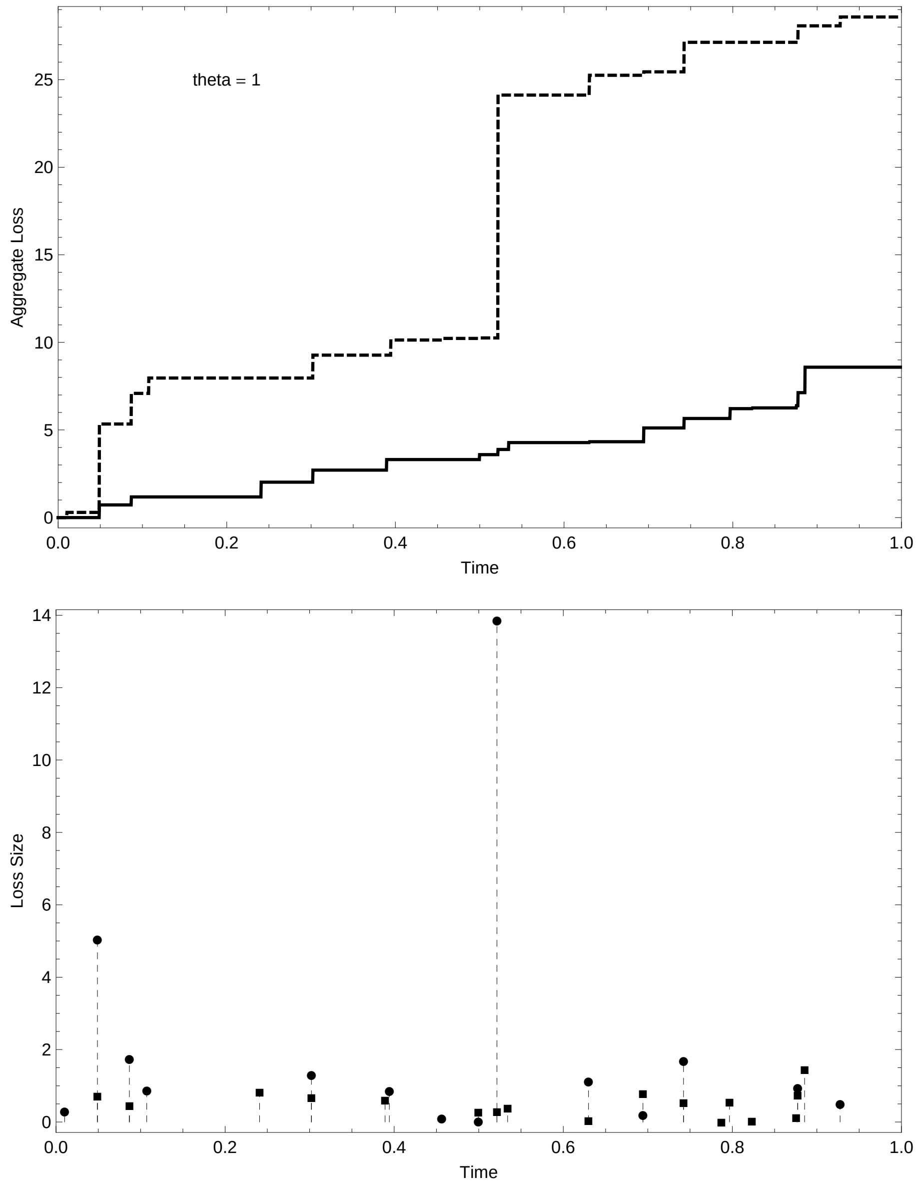

Figure 5.7. Simulation of the bivariate compound Poisson model as of Definition 2.1 with Clayton Lévy copula with parameter $\vartheta = 1$ (medium dependence). Top panel: sample paths of the aggregate loss processes. Bottom panel: severity and occurrence times of losses. The univariate compound Poisson processes have frequencies of $\lambda_1 = \lambda_2 = 10$ and Pareto distributed severities with parameters $\alpha_1 = 1/\xi = 1.2$ and $\alpha_1 = 1/\xi = 2$.

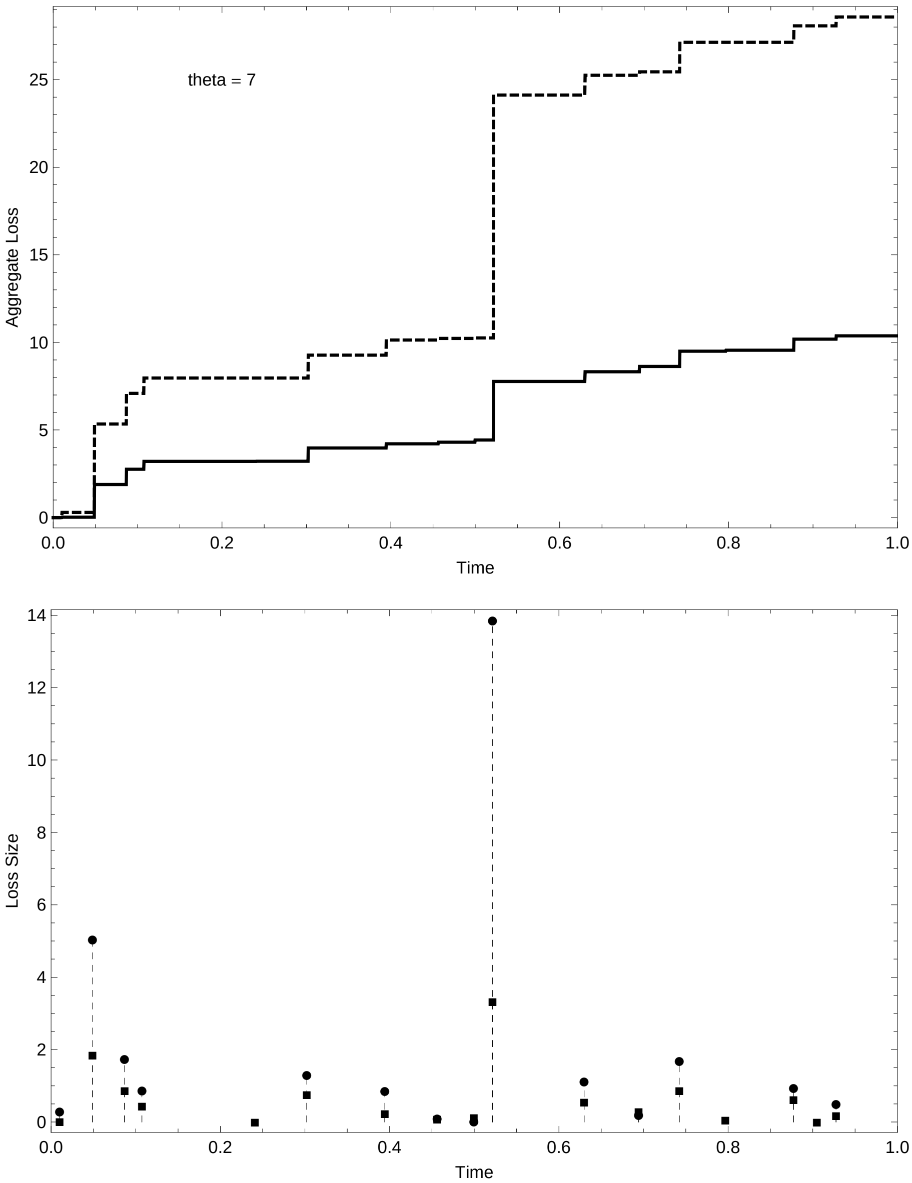

Figure 5.8. Simulation of the bivariate compound Poisson model as of Definition 2.1 with Clayton Lévy copula with parameter $\vartheta = 7$ (strong dependence). Top panel: sample paths of the aggregate loss processes. Bottom panel: severity and occurrence times of losses. The univariate compound Poisson processes have frequencies of $\lambda_1 = \lambda_2 = 10$ and Pareto distributed severities with parameters $\alpha_1 = 1/\xi = 1.2$ and $\alpha_1 = 1/\xi = 2$.

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