Extremes of supOU processes

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Abstract

Barndorff-Nielsen and Shephard [3] investigate supOU processes as volatility models. Empirical volatility has tails heavier than normal, long memory in the sense that the empirical autocorrelation function decreases slower than exponential, and exhibits volatility clusters on high levels. We investigate supOU processes with respect to these stylized facts. The class of supOU processes is vast and can be distinguished by its underlying driving Lévy process. Within the classes of convolution equivalent distributions we shall show that extremal clusters and long range dependence only occur for supOU processes, whose underlying driving Lévy process has regularly varying increments. The results on the extremal behavior of supOU processes correspond to the results of classical Lévy-driven OU processes.

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1 Introduction

We investigate the extremal behavior of stationary supOU processes *(superposition of Ornstein-Uhlenbeck processes)* of the form

$$V_t = \int_{\mathbb{R}_+ \times \mathbb{R}} \mathrm{e}^{-r(t-s)} \, \mathbf{1}_{[0,\infty)}(t-s) \, d\Lambda(r, \lambda s) \quad \text{for } t \geq 0, \tag{1.1}$$

where $\lambda > 0$ and Λ is an *infinitely divisible independently scattered random measure* (i. d. i. s. r. m.). Such models coincide under weak regularity conditions with models introduced under the same acronym by Barndorff-Nielsen [1] aiming at volatility modelling. They allow for non-trivial extensions of OU (*Ornstein-Uhlenbeck*) type processes of the form

$$V_t = \int_{-\infty}^{t} \mathrm{e}^{-\lambda(t-s)} dL_{\lambda s} \quad \text{for } t \geq 0, \tag{1.2}$$

where $\lambda > 0$ and L is a Lévy process. The time-change by λ yields marginal distributions independent of λ. To guarantee that the volatility process V is positive, the Lévy process L is chosen as subordinator. The resulting price process has martingale term $dS_t = \sqrt{V_t}\, dB_t$, where B_t is a Brownian motion, independent of the volatility driving Lévy process. This model has been analyzed by Barndorff-Nielsen and Shephard [3].

An alternative continuous-time model has been suggested by Klüppelberg, Lindner and Maller [14]. In the COGARCH$(1,1)$ model, which is a continuous-time version of the GARCH$(1,1)$ process, the price process has martingale term $dS_t = \sqrt{V_t}\, dL_t$, where L is some arbitrary Lévy process and the volatility is given as solution of the SDE

$$dV_{t+} = (b - aV_t)\, dt + cV_t \, d[L, L]_t^{(d)} \tag{1.3}$$

for parameters $a, b > 0$ and $c \geq 0$, where $([L, L]_t^{(d)})_{t \geq 0}$ is the discrete part of the quadratic variation process of L.

Interestingly, although the two types of models seem at first sight to be quite different, they share many properties; see Klüppelberg, Lindner and Maller [15]. The models differ, however, in their extreme behavior. Whereas the large fluctuations in terms of the tail behavior of the volatility in the Barndorff-Nielsen and Shephard model (1.2) is inherited from the tail behavior of the increments of the Lévy process, the COGARCH model (1.3) exhibits under weak regularity conditions always Pareto-like tails. It has also been shown in Fasen, Klüppelberg and Lindner [12] that both models can only model volatility clusters, if they have Pareto-like tails;

i. e. the COGARCH model always does (under weak regularity conditions), and the OU-type model does, if the Lévy process has Pareto-like increments.

Besides volatility clustering, another issue in volatility modelling is the fact that many financial time series exhibit zero autocorrelation in the data, but a long range dependence effect in the volatility. Despite the ongoing debate for the origins of this effect, the modelling issue cannot just be ignored. Unfortunately, the autocovariance functions of both volatility models, the OU-type model and the COGARCH$(1, 1)$ decrease exponentially fast.

Barndorff-Nielsen [1] suggests as a remedy the generalization of V to a supOU process. In this paper we want to investigate the extremal behavior of model (1.1) with respect to volatility clustering. As empirical findings indicate and economic reasoning supports, financial data can be modelled by a normal mixture model with tails ranging from exponential to Pareto. Consequently, it is indeed interesting to identify models with such tail behavior, long range dependence effect and volatility clusters in the extremes.

Our paper is organized as follows. We start in Section 2 with an introduction into supOU processes as given in (1.1) including necessary and sufficient conditions for the existence of a stationary version of (1.1). Moreover, we compare our definition with Barndorff-Nielsen's [1] slightly different definition and show that they coincide. In the context of extreme value theory we prefer working with representation (1.1) as it allows us to apply results for mixed MA processes as derived in Fasen [10, 11]. As we shall show in Section 2.2 supOU processes can model a wide range of correlation functions from exponential to polynomial decrease. Poisson shot noise processes as introduced in Section 2.3 present the basic structure for studying the extremal behavior. In Section 2.4 we present the class of convolution equivalent distributions, which will serve as models for the Lévy increments of supOU processes.

The extremal behavior of a supOU process, whose underlying driving Lévy process is in the class of convolution equivalent distributions, is classified by the tail behavior of the random variable $L_1 = \Lambda(\mathbb{R}_+ \times [0, 1])$, so that we have to distinguish between different regimes for L_1. In Section 3 we investigate the link between the tail behavior of the Lévy increments in the class of convolution equivalent distributions, represented by L_1, the stationary distribution V_0 of the supOU process, and $\sup_{0\leq t\leq 1} V_t$. In Section 4 we study the extremal behavior of V via marked point processes, which characterize the distributions of the locations of extremes on high levels. Moreover, we derive the distribution of cluster sizes of high level extremes and the normalizing constants of running maxima. Our findings are summarized in Section 5.

As not to disturb the flow of arguments we postpone classical definitions and concepts to an Appendix.

Throughout the paper we shall use the following notation. We abbreviate distribution function by d. f. and random variable by r. v. For any d. f. F we denote its tail $\overline{F} = 1 - F$ and $F * G$ for the convolution of F with the d. f. G. For two r. v. s X and Y with d. f. s F and G we write $X \stackrel{d}{=} Y$ if $F = G$, and by $\stackrel{n\to\infty}{\Longrightarrow}$ we denote weak convergence for $n \to \infty$. For two functions f and g we write $f(x) \sim g(x)$ as $x \to \infty$, if $\lim_{x\to\infty} f(x)/g(x) = 1$. We also denote $\mathbb{R}_+ = (0, \infty)$. For $x \in \mathbb{R}$, we define $x^+ = \max\{x, 0\}$.

2 The model

Let $\mathcal{T}$ be a σ-ring on $\mathbb{R}_+ \times \mathbb{R}$ (i. e. countable unions of sets in $\mathcal{T}$ belong to $\mathcal{T}$ and if $A, B \in \mathcal{T}$ with $A \subset B$ then $B\backslash A \in \mathcal{T}$) and let $\Lambda = \{\Lambda(A) : A \in \mathcal{T}\}$ be an i. d. i. s. r. m., which means by definition that all finite dimensional distributions are infinitely divisible and for all disjoint sets $(A_n)_{n\in\mathbb{N}}$ in $\mathcal{T}$ we have that $(\Lambda(A_n))_{n\in\mathbb{N}}$ is an independent sequence and $\Lambda\left(\bigcup_{n=1}^{\infty} A_n\right) = \sum_{n=1}^{\infty} \Lambda(A_n)$ almost surely (a. s.). We work with i. d. i. s. r. m. s, whose characteristic function can be written in the form

$$\mathbb{E}\exp(iu\Lambda(A)) = \exp(\psi(u)\Pi(A)) \quad \text{for } u \in \mathbb{R}, \tag{2.1}$$

where Π is a measure on $\mathbb{R}_+ \times \mathbb{R}$, which is the product of a probability measure π on $\mathbb{R}_+$ and the Lebesgue measure on $\mathbb{R}$, and

$$\psi(u) = ium - \frac{1}{2}u^2\sigma^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{iux} - 1 - iu\kappa(x)\right) \nu(dx) \quad \text{for } u \in \mathbb{R}$$

with $\kappa(x) = \mathbf{1}_{[-1,1]}(x)$. The function ψ is the cumulant generating function of an infinitely divisible r. v. with *generating triplet* (m, σ^2, ν), where $m \in \mathbb{R}$, $\sigma^2 \geq 0$, and ν is a measure on $\mathbb{R}$, called *Lévy measure*, satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1 \wedge |x|^2)\,\nu(dx) < \infty$. The *generating quadruple* (m, σ^2, ν, π) determines completely the distribution of Λ.

The *underlying driving Lévy process*

$$L_t = \Lambda(\mathbb{R}_+ \times [0, t]) \quad \text{for } t \geq 0 \tag{2.2}$$

has generating triplet (m, σ^2, ν).

2.1 Existence and stationarity of the model

The following result guarantees existence, infinite divisibility and stationarity of the model and ensures the equivalence of (1.1) and the supOU model as defined in Barndorff-Nielsen [1]. For the comparison we recall first that integrals of the form $\int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{e}^{-r(t-s)}\, \mathbf{1}_{[0,\infty)}(t-s)\, d\Lambda(r,\lambda s)$ are defined for each fixed $t \geq 0$ as limit in probability of simple functions (cf. Rajput and Rosinski [16], Theorem 2.7). Hence, V_t is defined a. s. for each fixed t.

Proposition 2.1 *Let (m, σ^2, ν) be the generating triplet of an infinitely divisible distribution with*

$$\int_{|x|>1} \log(1+|x|)\, \nu(dx) < \infty\,. \tag{2.3}$$

Define $T : \mathbb{R}_+ \times \mathbb{R} \to \mathbb{R}_+ \times \mathbb{R}$ by $T(r,s) = (r, r^{-1}s)$. Then the following hold:

(a) Let $\widetilde{\pi}$ be a probability measure on $\mathbb{R}_+$ with $\lambda := \int_{\mathbb{R}_+} r\, \widetilde{\pi}(dr) < \infty$ and $\widetilde{\Lambda}$ be an i. d. i. s. r. m. with generating quadruple $(\widetilde{m}, \widetilde{\sigma}^2, \widetilde{\nu}, \widetilde{\pi})$. Then $\Lambda = \widetilde{\Lambda} \circ T^{-1}$ is an i. d. i. s. r. m. with generating quadruple $(\lambda\widetilde{m}, \lambda\widetilde{\sigma}^2, \lambda\widetilde{\nu}, \pi)$, where $\pi(dr) = \lambda^{-1} r \widetilde{\pi}(dr)$.

(b) Let π be a probability measure on $\mathbb{R}_+$ with $\lambda^{-1} := \int_{\mathbb{R}_+} r^{-1}\, \pi(dr) < \infty$ and Λ be an i. d. i. s. r. m. with generating quadruple (m, σ^2, ν, π). Then $\widetilde{\Lambda} = \Lambda \circ T$ is an i. d. i. s. r. m. with generating quadruple $(\lambda^{-1}m, \lambda^{-1}\sigma^2, \lambda^{-1}\nu, \widetilde{\pi})$, where $\widetilde{\pi}(dr) = \lambda r^{-1} \pi(dr)$.

(c) For Λ and $\widetilde{\Lambda}$ as in (a) and (b) define for $t \geq 0$,

$$\begin{aligned} V_t &= \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{e}^{-r(t-s)}\, \mathbf{1}_{[0,\infty)}(t-s)\, d\Lambda(r,\lambda s), \\ X_t &= \int_{-\infty}^{\infty} \mathrm{e}^{-rt} \int_{-\infty}^{rt} \mathrm{e}^{s}\, d\widetilde{\Lambda}(r,\lambda s)\,. \end{aligned}$$

Then, $V_t = X_t$ a. s. for $t \geq 0$ and, hence, V is a version of X and vice versa. Furthermore, V (and hence X) has a stationary version. For $d \in \mathbb{N}$ let $-\infty = t_0 < t_1 < \ldots < t_d < \infty$ and $u_1, \ldots, u_d \in \mathbb{R}$. The finite dimensional distributions of the stationary process V have the cumulant generating function

$$\begin{aligned} &\log \mathbb{E} \exp(i(u_1 V_{t_1} + \ldots + u_d V_{t_d})) \\ &\quad = \sum_{m=1}^{d} \int_0^{\infty} \int_{t_{m-1}}^{t_m} \lambda \psi\Big(\sum_{j=m}^{d} u_j \mathrm{e}^{-r(t_j - s)} \Big)\, ds\, \pi(dr). \end{aligned} \tag{2.4}$$

The results *(a)* and *(b)* follow by simple calculations of the characteristic functions of the finite dimensional distributions of Λ and $\widetilde{\Lambda}$. Statement *(c)* is the consequence of the change of variables in *(a)* and *(b)*, respectively, and Barndorff-Nielsen [1], Theorem 3.1 (cf. Rajput and Rosinski [16], Proposition 2.6). Condition (2.3) and $\int_{\mathbb{R}_+} r^{-1}\, \pi(dr) < \infty$ are necessary and sufficient for the existence of a stationary version of V (and hence X).

Throughout this paper we shall assume that V is a measurable, separable and stationary version of the supOU process as given in (1.1) and that $\mathbb{P}(\sup_{0\leq t\leq 1}|V_t| < \infty) = 1$.

Remark 2.2 *(i)* By (2.4) the cumulant generating function of the stationary distribution is given by

$$\log \mathbb{E} \exp(iuV_0) = \int_0^\infty \int_{-\infty}^0 \lambda\psi\left(u e^{rs}\right) \, ds \, \pi(dr) = \int_{-\infty}^0 \psi\left(u e^{s}\right) \, ds \quad \text{for } u \in \mathbb{R}. \quad (2.5)$$

This is the cumulant generating function of a stationary OU-type process (1.2) driven by the underlying driving Lévy process L as given in (2.2). Then, V_0 has absolutely continuous Lévy measure ν_V with

$$\nu_V\,(dx) = x^{-1}\nu\,[x,\infty)\; dx \quad \text{for } x > 0, \quad (2.6)$$

and is *selfdecomposable* (Proposition A.5). Note that the stationary distribution of V_0 is independent of π.

(ii) Positivity of V, which is needed for volatility processes, can be guaranteed by choosing L as a *subordinator*; i. e. ν has only support on $\mathbb{R}_+$ with $\int_{(0,\infty)}(1\wedge x)\,\nu(dx) < \infty$, $\sigma^2 = 0$ and $m = \int_0^\infty \kappa(x)\,\nu(dx)$.

The following examples serve as motivation.

Example 2.3 *(a)* If π has only support in some $\lambda > 0$, i. e. $\pi(\{\lambda\}) = 1$, then (2.4) reduces to the cumulant generating function of the d-dimensional distribution of an OU-type process. Thus, (1.1) defines the usual OU-type process (1.2).

(b) Let π be a discrete probability measure with $\pi(\{\lambda_k\}) = p_k$ for $k \in \mathbb{N}$ and $\lambda_k > 0$. Then the assumption $\lambda^{-1} := \int_{\mathbb{R}_+} r^{-1}\,\pi(dr) < \infty$ is equivalent to $\sum_{k=1}^\infty p_k\lambda_k^{-1} < \infty$. By (2.4) the cumulant generating function of the d-dimensional distribution is given by

$$\log \mathbb{E} \exp(i(u_1V_{t_1} + \ldots + u_dV_{t_d})) = \sum_{k=1}^\infty \sum_{m=1}^d \int_{t_{m-1}}^{t_m} \lambda p_k \psi\left(\sum_{j=m}^d u_j e^{-\lambda_k(t_j-s)}\right) ds.$$

Consequently, V has the same distribution as the superposition of independent OU processes,

$$\sum_{k=1}^\infty \int_{-\infty}^t e^{-\lambda_k(t-s)}\, dL^{(k)}_{\lambda s} \quad \text{for } t \geq 0\,,$$

where $(L^{(k)})_{k\in\mathbb{N}}$ are independent Lévy processes with characteristic triplets $(p_k m, p_k\sigma^2, p_k\nu)$.

2.2 Dependence structure

Provided the underlying driving Lévy process has finite second moment the autocorrelation function ρ of the stationary supOU process (1.1) can be calculated taking derivatives with respect to u_1 and u_2 in (2.4) and taking the limit for $u_1, u_2 \to 0$. We obtain

$$\rho(h) = \lambda \int_0^\infty r^{-1} \mathrm{e}^{-hr} \, \pi(dr) \quad \text{for } h \geq 0. \tag{2.7}$$

For a discrete probability measure π as given in Example 2.3 we obtain

$$\rho(h) = \lambda \sum_{k=1}^{\infty} p_k \lambda_k^{-1} \mathrm{e}^{-h\lambda_k} \quad \text{for } h \geq 0. \tag{2.8}$$

Remark 2.4 On the one hand the correlation function (2.7) of a supOU process depends only on the probability measure π and is independent of the generating triplet (m, σ^2, ν) of the underlying driving Lévy process. On the other hand the stationary distribution V_0 depends only on (m, σ^2, ν) and is independent of π, represented by the cumulant generating function given in (2.5). Thus, supOU processes can model the stationary distribution and the correlation function independently. This opens the way to a simple statistical fitting of such models. More about supOU models and applications to financial data can be found in Barndorff-Nielsen and Shephard [2,3].

There are various notions of long range dependence, all having in common that the correlation function should decrease slower than exponential. We shall work with the following definition.

Definition 2.5 *A stationary process with correlation function ρ exhibits* long range dependence, *if there exists a $H \in (0, 1/2)$ and a a slowly varying function l (see Definition A.2), such that*

$$\rho(h) \sim l(h) h^{-2H} \quad \textit{for } h \to \infty.$$

We observe that long range dependence implies that $\int_0^\infty \rho(h) \, dh = \infty$.

The following result explains how long range dependence can be introduced into supOU models. Essentially, the measure π needs sufficient mass near 0. We write $\pi(r)$ for $\pi((0, r])$.

Proposition 2.6 *Let V be a stationary supOU process as in (1.1) and L be as in (2.2) with $\mathbb{E} L_1^2 = 1$. We denote by ρ the correlation function of V. Suppose l is slowly varying and $H > 0$. Then*

$$\widetilde{\pi}(r) \sim (2H)^{-1} l(r^{-1}) r^{2H} \quad \textit{for } r \to 0, \tag{2.9}$$

if and only if

$$\rho(h) \sim \Gamma(2H) l(h) h^{-2H} \quad \textit{for } h \to \infty. \tag{2.10}$$

If

$$\pi(r) \sim \lambda^{-1}(2H+1)^{-1} l(r^{-1}) r^{2H+1} \quad \textit{for } r \to 0, \tag{2.11}$$

then (2.9) and, hence, (2.10) follow. The converse, i. e. (2.10) *implies* (2.11) *holds, provided that* π *is absolutely continuous with density* π', *and* $r^{-1}\pi'(r)$ *is monotone on* $(0, r_0)$ *for some* $r_0 > 0$.

Proof. The equivalence of (2.9) and (2.10) is a consequence of Karamata's Tauberian theorem (Theorem 1.7.1' in Bingham, Goldie and Teugels [4]) and $\rho(h) = \int_0^\infty e^{-hr}\, \widetilde{\pi}(dr)$; cf. (2.7). Furthermore, if (2.11) holds, then by Proposition 2.1 (b) and $\widetilde{\pi}(dr) = \lambda r^{-1}\pi(dr)$, Karamata's theorem (Theorem 1.5.11 in [4]) yields

$$\widetilde{\pi}(r) \;=\; \lambda \int_0^r s^{-1}\, \pi(ds) = \lambda r^{-1}\pi(r) + \lambda \int_{r^{-1}}^\infty \pi(s^{-1})\, ds \sim \lambda(2H+1)(2H)^{-1} r^{-1}\pi(r)$$

for $r \to 0$. Hence, statements (2.9) and (2.10) follow.

If $r^{-1}\pi'(r)$ is monotone on $(0, r_0)$ for some $r_0 > 0$, and invoking the monotone density theorem (Theorem 1.7.2b in [4]), we get from (2.9)

$$r^{-1}\pi'(r) \sim \lambda^{-1} l(r^{-1}) r^{2H-1} \quad \text{for } r \to 0.$$

Hence, Theorem 1.6.1 in [4] yields $\pi(r) \sim \lambda^{-1}(2H+1)^{-1} l(r^{-1}) r^{2H+1}$ for $r \to 0$. □

Example 2.7 A typical example of π to generate long range dependence in a supOU process is a gamma distribution with density $\pi(dr) = \Gamma(2H+1)^{-1} r^{2H} e^{-r}\, dr$ for $r > 0$ and $H > 0$. Then $\lambda = 2H$ and

$$\rho(h) = \Gamma(2H)^{-1} \int_0^\infty r^{2H-1} e^{-r(h+1)}\, dr = (h+1)^{-2H} \quad \text{for} \quad h \geq 0.$$

Remark 2.8 CARMA processes as reviewed by Brockwell [5] can be interpreted as a superposition of OU-type processes. These models correspond to linear combinations of OU processes driven by one single Lévy process. This mechanism creates only processes with asymptotically exponentially decreasing correlation functions.

2.3 Positive shot noise process

The structure of a supOU process can be well understood when considering the following example.

Let Λ be a *positive compound Poisson random measure* in the sense that it has generating quadruple $(\mu\mathbb{P}_F((0,1]), 0, \mu\mathbb{P}_F, \pi)$, where $\mu > 0$, $\mathbb{P}_F$ is a probability measure on $\mathbb{R}_+$ with corresponding d. f. F, and π is a probability measure on $\mathbb{R}_+$ with $\lambda^{-1} := \int_{\mathbb{R}_+} r^{-1}\,\pi(dr) < \infty$. Then Λ has the representation

$$\Lambda(A) = \sum_{k=-\infty}^{\infty} Z_k\,\mathbf{1}_{\{(R_k,\Gamma_k)\in A\}} \qquad \text{for } A \in \mathcal{T}, \tag{2.12}$$

where $(\Gamma_k)_{k\in\mathbb{Z}}$ constitute the jump times of a Poisson process $N = (N_t)_{t\in\mathbb{R}}$ on $\mathbb{R}$ with intensity $\mu > 0$. The process N is independent of the i. i. d. sequence of positive r. v. s $(Z_k)_{k\in\mathbb{Z}}$ with d. f. F. Finally, the i. i. d. sequence $(R_k)_{k\in\mathbb{Z}}$ with distribution π is independent of all other quantities.

The resulting supOU process is then the positive shot noise process

$$V_t = \int_{\mathbb{R}_+\times\mathbb{R}} \mathrm{e}^{-r(t-s)}\,\mathbf{1}_{[0,\infty)}(t-s)\,d\,\Lambda(r,\lambda s) = \sum_{k=-\infty}^{N_{\lambda t}} \mathrm{e}^{-R_k(t-\Gamma_k/\lambda)} Z_k \qquad \text{for } t \geq 0, \tag{2.13}$$

and from (2.6) we get, if $\mathbb{E}\log(1+Z_1) < \infty$ (which is the analogue of (2.3) in this model),

$$\nu_V\,[x,\infty) = \mu \int_x^\infty y^{-1}\overline{F}(y)\,dy \qquad \text{for } x > 0$$

and a stationary version of V exists.

The qualitative extreme behavior of this supOU process can be seen in Figure 1 in detail. The supOU jumps upwards, whenever $(N_{\lambda t})_{t\geq 0}$ jumps and decreases continuously between two jumps. This means in particular that V has local suprema exactly at the jump times Γ_k/λ (and $t = 0$). Consequently, it is the discrete-time skeleton of V at points Γ_k/λ that determines the extreme behavior of the shot noise process. Although the underlying driving Lévy process L of the supOU process as given in (2.2) and the driving Lévy process of the OU-type process are the same, we see the influence of $(R_k)_{k\in\mathbb{N}}$ on the exponential decrease of V for the simple OU-type process, which governs the memory of the supOU process.

2.4 Convolution equivalent distributions

We aim at an extreme value analysis of supOU processes, where a first step always concerns the tail behavior of the model. To relate the tail behavior of the underlying driving Lévy process, represented by the tail of L_1 as in (2.2), and the tail of

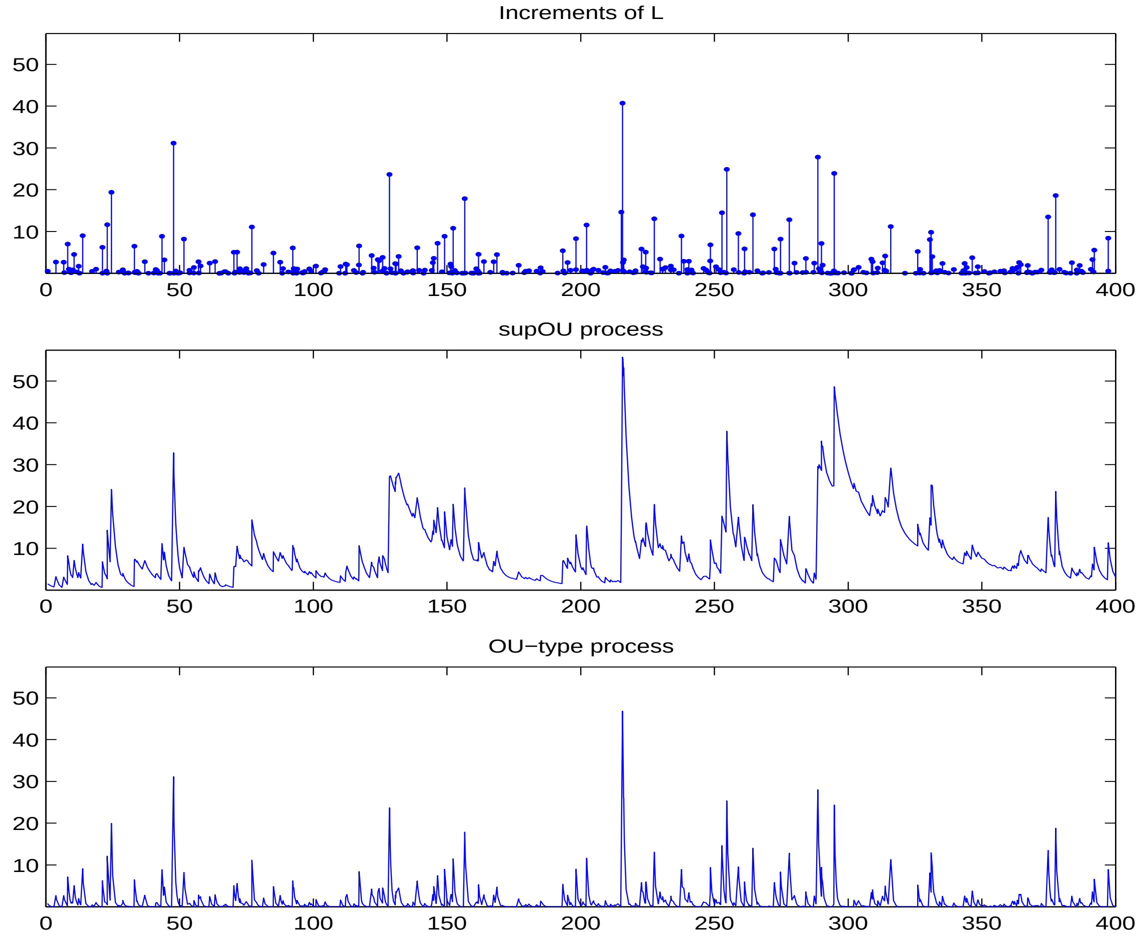

Figure 1: Sample path of a supOU process $V_t = \sum_{k=-\infty}^{N_{\lambda t}} \mathrm{e}^{-R_k(t-\Gamma_k/\lambda)} Z_k$ as in Section 2.3 and, for comparison, the OU-type process $V_t = \sum_{k=-\infty}^{N_{\lambda t}} \mathrm{e}^{-\lambda t+\Gamma_k} Z_k$ for $0 \le t \le 400$, with $\lambda = 1/3$, $\mu = 1/3$, $F(x) = 1 - \exp(-x^{1/2})$ for $x > 0$ and $\pi(r) = r^{3/2}$ for $r \in (0,1)$. In the first plot we show the increments of the underlying driving Lévy process $L_{\lambda t} = \sum_{k=1}^{N_{\lambda t}} Z_k$ for $0 \le t \le 400$.

the stationary process given by V_0 we shall invoke relation (2.6) between the Lévy measures.

The *convolution equivalent distributions* play a prominent role here, where we distinguish different classes.

Definition 2.9

(a) A d.f. F on $\mathbb{R}$ with $F(x) < 1$ for all $x \in \mathbb{R}$ belongs to the class of convolution equivalent distributions denoted by $\mathcal{S}(\gamma)$ for some $\gamma \ge 0$, if the following conditions hold:

(i) F belongs to the class $\mathcal{L}(\gamma)$, i. e. for all $y \in \mathbb{R}$ locally uniformly

$$\lim_{x\to\infty} \overline{F}(x+y)/\overline{F}(x) = \exp(-\gamma y).$$

(ii) $\lim_{x\to\infty} \overline{F*F}(x)/\overline{F}(x)$ exists and is finite.

If Z is a r. v. with d. f. $F \in \mathcal{S}(\gamma)$, then we also write $Z \in \mathcal{S}(\gamma)$.

(b) The class $\mathcal{S}(0) = \mathcal{S}$ is called subexponential distributions.

Most of the literature on this topic is formulated for positive r. v. s, which extend to r. v. s on $\mathbb{R}$, when considering $Z \in \mathcal{S}(\gamma)$ if and only if $Z^+ \in \mathcal{S}(\gamma)$. Important properties of $\mathcal{S}(\gamma)$ can be found in Theorem A.3.

Subexponential distributions are heavy-tailed in the sense that no exponential moments exist. $\mathcal{S}$ contains all d. f. s F with *regularly varying tails* (Definition A.2), denoted by $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$, but is much larger. Distribution functions in $\mathcal{S}(\gamma)$ for some $\gamma > 0$ have exponential tails, hence are lighter tailed than subexponential distributions.

Next we present two different regimes governed by extreme value theory, which classifies distributions according to their *maximum domain of attraction*. The maximum domain of attraction condition is an assumption on the tail behavior of a d. f. F. Suppose we can find sequences of real numbers $a_n > 0$ and $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} n\overline{F}(a_n x + b_n) = -\log G(x) \qquad \text{for } x \in \mathbb{R}\,,$$

for some non-degenerate d. f. G. Then we say F is in the maximum domain of attraction of G ($F \in \mathrm{MDA}(G)$). The Fisher-Tippett Theorem A.1 says that G is either a Fréchet (Φ_α, $\alpha > 0$), Gumbel (Λ) or Weibull (Ψ_α, $\alpha > 0$) distribution. Convolution equivalent distributions can be in two different maximum domains of attraction, since they have unbounded support to the right (thus excluding the Weibull distribution). All d. f. s such that $\overline{F} \in \mathcal{R}_{-\alpha}$ for some $\alpha > 0$ are subexponential and belong to $\mathrm{MDA}(\Phi_\alpha)$. Other convolution equivalent distributions may belong to $\mathrm{MDA}(\Lambda)$.

Example 2.10 Typical examples of d. f. s in $\mathcal{S} \cap \mathrm{MDA}(\Lambda)$ have density functions

$$g(x) \sim \text{const.}\, x^{\beta} \mathrm{e}^{-x^{\alpha}} \qquad \text{for } x \to \infty$$

for some $\beta \in \mathbb{R}$, $\alpha \in (0,1)$, like the heavy-tailed Weibull distributions. Distribution functions, whose probability density satisfies

$$g(x) \sim \text{const.}\, x^{\beta-1} \mathrm{e}^{-\gamma x} \qquad \text{for } x \to \infty \tag{2.14}$$

for $\beta < 0$ are an important subclass of $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$. The papers of Cline [7] and Goldie and Resnick [13] investigate criteria for d. f. s to be in $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$.

We present here some important examples satisfying (2.14), which are also used for financial modelling; we refer to Schoutens [20] for an overview of these d. f. s.

(a) $GIG(\beta, \delta, \gamma)$ (generalized inverse Gaussian distribution) with $\beta < 0$, $\delta > 0$ and $\gamma \geq 0$, is in $\mathcal{S}(\gamma^2/2)$ with probability density

$$g(x) = \text{const.}\, x^{\beta-1} \exp\left(-\left(\delta^2 x^{-1} + \gamma^2 x\right)/2\right) \quad \text{for } x > 0.$$

A special case is for $\beta = -1/2$ the inverse Gaussian distribution $IG(\delta, \gamma)$.

(b) $NIG(\alpha, \beta, \delta, \mu)$ (normal inverse Gaussian distribution) is for $\beta, \delta, \mu \in \mathbb{R}$ and $\alpha > |\beta|$ in $\mathcal{S}(\alpha - \beta)$ and

$$g(x) \sim \text{const.}\, x^{-3/2} \exp(-x(\alpha - \beta)) \quad \text{as } x \to \infty.$$

(c) $GH(\alpha, \beta, \delta, \mu, \gamma)$ (generalized hyperbolic distribution) is for $\beta, \delta, \mu \in \mathbb{R}$, $\alpha > |\beta|$, $\gamma < 0$ in $\mathcal{S}(\alpha - \beta)$ and

$$g(x) \sim \text{const.}\, x^{\gamma-1} \exp(-x(\alpha - \beta)) \quad \text{as } x \to \infty.$$

For $\gamma = -1/2$ the GH distribution is the NIG distribution, while the hyperbolic distribution occurs for $\gamma = 1$.

(d) $CGMY(C, G, M, Y)$ for $C, M, G > 0$, $Y \in (-\infty, 2]$, introduced by Carr, Geman, Madan and Yor [6]. For $0 < Y < 2$ it belongs to $\mathcal{S}(M)$ with Lévy density

$$\nu(dx) = C|x|^{-1-Y} \exp\left(\frac{G-M}{2}x - \frac{G+M}{2}|x|\right) \quad \text{for } x \in \mathbb{R}\backslash\{0\}.$$

All these distributions are selfdecomposable, which means that they are possible stationary distributions of OU-type processes and, hence, also of supOU processes. We summarize in Proposition A.5 necessary and sufficient conditions of d. f. s to be selfdecomposable.

3 Tail behavior

We use extensively the fact that for every infinitely divisible convolution equivalent distribution the tail of the distribution function and the tail of its Lévy measure are asymptotically equivalent; see Theorem A.3 *(i)*.

Proposition 3.1 (Tail behavior of V)
Let V be a stationary supOU process as in (1.1) and L be the underlying driving Lévy process (2.2).

(a) Then $L_1 \in \mathcal{R}_{-\alpha}$ *if and only if* $V_0 \in \mathcal{R}_{-\alpha}$. *In this case*

$$\mathbb{P}(V_0 > x) \sim \alpha^{-1}\mathbb{P}(L_1 > x) \quad \text{for } x \to \infty.$$

(b) If $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ *with tail representation as given in* (A.1)*, then also* $V_0 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$,

$$\mathbb{P}(V_0 > x) \sim \frac{a(x)}{x} \frac{\mathbb{E}\mathrm{e}^{\gamma V_0}}{\mathbb{E}\mathrm{e}^{\gamma L_1}} \mathbb{P}(L_1 > x) \quad \text{for } x \to \infty,$$

and $\mathbb{P}(V_0 > x) = o(\mathbb{P}(L_1 > x))$ *for* $x \to \infty$.

Proof. Recall from Remark 2.2 that the stationary distribution of a supOU process driven by an i. d. i. s. r. m. with generating quadruple (m, σ^2, ν, π) coincides with the stationary distribution of an OU-type process (1.2) driven by the Lévy process L with generating triplet (m, σ^2, ν). Thus, applying Proposition 3.2 and Proposition 3.9 in Fasen et al. [12] we obtain sufficiency in (a) and (b). To prove the converse of (a) assume that $V_0 \in \mathcal{R}_{-\alpha}$. Since $\nu_V(x, \infty) = \int_x^\infty y^{-1}\nu(y, \infty)\, dy$ for $x > 0$, and $\nu_V(x, \infty) \sim \mathbb{P}(V_0 > x)$ for $x \to \infty$, we obtain by Bingham et al. [4], Theorem 1.7.2, that $\nu(x, \infty) \sim \alpha\nu_V(x, \infty)$ for $x \to \infty$. Hence, by Theorem A.3 (i) we conclude

$$\mathbb{P}(L_1 > x) \sim \alpha\mathbb{P}(V_0 > x) \quad \text{for } x \to \infty. \qquad \square$$

Lemma 3.2 *Let* V *be a stationary supOU process as in (1.1) with absolutely continuous Lévy density* $\nu_V(dx) = u(x)\, dx$*, where*

$$u(x) \sim \mathit{const.}\, x^{\beta-1}\mathrm{e}^{-\gamma x} \quad \text{for } x \to \infty$$

for $\gamma > 0$*, and let* L *be the underlying driving Lévy process (2.2). Then* $V_0 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ *if and only if* $\beta < 0$*, and* $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ *if and only if* $\beta < -1$.

Proof. Using (2.6) we obtain $\nu(x, \infty) = xu(x)$ for $x > 0$. Thus,

$$\frac{\nu(dx)}{dx} = -u(x) - xu'(x) \sim \mathrm{const.}\, \gamma\, x^{\beta}\mathrm{e}^{-\gamma x} \quad \text{for } x \to \infty.$$

The result follows then from Rootzén [18], Lemma 7.1, and Theorem A.3 *(i)*. $\square$

The next proposition follows from Fasen [11], Proposition 3.3, and [10], Theorem 3.3.

Proposition 3.3 (Tail behavior of $M(h)$)
Let V *be a supOU process and define* $M(h) = \sup_{0 \le t \le h} V_t$ *for* $h > 0$.

(a) If $L_1 \in \mathcal{R}_{-\alpha}$, then also $M(h) \in \mathcal{R}_{-\alpha}$ and

$$\mathbb{P}(M(h) > x) \sim \left(\lambda h + \alpha^{-1}\right) \mathbb{P}(L_1 > x) \quad \text{for } x \to \infty.$$

(b) If $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$, then also $M(h) \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ and

$$\mathbb{P}(M(h) > x) \sim \lambda h \frac{\mathbb{E}e^{\gamma V_0}}{\mathbb{E}e^{\gamma L_1}} \mathbb{P}(L_1 > x) \quad \text{for } x \to \infty.$$

Remark 3.4 *(i)* From Lemma 3.2 follows immediately that for $\beta \in [-1, 0)$, $V_0 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ but $L_1 \notin \mathcal{S}(\gamma)$.

(ii) Proposition 3.3 implies that the tail of the maximum of a supOU process driven by an i. d. i. s. r. m. with generating quadruple (m, σ^2, ν, π) behaves like the tail of the maximum of an OU-type process driven by a Lévy process with generating triplet (m, σ^2, ν). From this we conclude immediately that the long memory property of supOU processes does not affect the tail behavior of $M(h)$.

4 Extremal behavior of supOU processes

For a general i. d. i. s. r. m. Λ we decompose

$$\Lambda = \Lambda^{(1)} + \Lambda^{(2)} \tag{4.1}$$

into two independent i. d. i. s. r. m. s.

$\Lambda^{(1)}$ has only jumps greater than 1; i.e. it has generating quadruple $(0, 0, \nu_1, \pi)$ with $\nu_1(x, \infty) = \nu(1 \vee x, \infty)$ for $x > 0$ and $\nu_1(-\infty, 1] = 0$. Consequently, $\Lambda^{(1)}$ is a positive compound Poisson random measure with representation (2.12) whose underlying driving Lévy process $L^{(1)}$ is a compound Poisson process with intensity $\nu(1, \infty)$, jump times $-\infty < \cdots < \Gamma_{-1} < \Gamma_0 < 0 < \Gamma_1 < \cdots < \infty$ and jump sizes Z_k with probability measure $\nu_1/\nu(1, \infty)$.

$\Lambda^{(2)}$ summarizes all other features of the model; i.e. it has generating quadruple $(m, \sigma^2, \nu_2, \pi)$ with $\nu_2(-\infty, -x) = \nu(-\infty, -x)$ and $\nu_2(x, \infty) = \nu(1 \wedge x, 1]$ for $x > 0$. This means that all the small positive jumps, the negative jumps, the Gaussian component and the drift are summarized in $\Lambda^{(2)}$.

For $d \in \mathbb{N}_0$ let $t_1, \ldots, t_d \geq 0$, and define

$$M_k = \sup_{t \in [\Gamma_k/\lambda, \Gamma_{k+1}/\lambda)} V_t \quad \text{and} \quad \mathbf{V}(\Gamma_k) = (V_{\Gamma_k + t_1}, \ldots, V_{\Gamma_k + t_d}) \quad \text{for } k \in \mathbb{N}.$$

For a Radon measure ϑ we write PRM(ϑ) for a *Poisson random measure* with intensity measure ϑ, see Definition A.7. In our set-up ϑ will be a Radon measure on either of the spaces $S_F = [0,\infty) \times (0,\infty] \times [-\infty,\infty]^d$ or $S_G = [0,\infty) \times (-\infty,\infty] \times [-\infty,\infty]^d$, and $M_P(S_F)$ and $M_P(S_G)$ will denote the spaces of all point measures on S_F and S_G, respectively. For details on point processes see Resnick [17].

The following proposition is a consequence of Fasen [9], Theorem 2.5.1 and [10], Theorem 4.1.

Proposition 4.1 (Point process behavior)
Let V be a stationary supOU process as in (1.1) *and L be the underlying driving Lévy process* (2.2)*. Decompose Λ as in (4.1).*

(a) Let $L_1 \in \mathcal{R}_{-\alpha}$ with norming constants $a_T > 0$ such that

$$\lim_{T\to\infty} T\mathbb{P}(L_1 > a_T x) = x^{-\alpha} \quad \text{for } x > 0.$$

Suppose $\sum_{k=1}^{\infty} \varepsilon_{(s_k,P_k)}$ is a $\mathrm{PRM}(\vartheta)$ with $\vartheta(dt \times dx) = dt \times \alpha x^{-\alpha-1}\,dx$ independent of the i. i. d. sequences $(\Gamma_{k,j})_{j\in\mathbb{N}}$ for $k \in \mathbb{N}$ with $(\Gamma_{k,j})_{j\in\mathbb{N}} \stackrel{d}{=} (\Gamma_j)_{j\in\mathbb{N}}$ and independent of the i. i. d. sequence $(R_k)_{k\in\mathbb{N}}$ with probability distribution π. Define $\Gamma_{k,0} = 0$ for $k \in \mathbb{N}$. Then, in the space $M_P(S_F)$,

$$\sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k/(\lambda n), a_{\lambda n}^{-1} M_k, a_{\lambda n}^{-1}\mathbf{V}(\Gamma_k/\lambda))} \stackrel{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty}\sum_{j=0}^{\infty} \varepsilon_{(s_k, P_k \mathrm{e}^{-R_k\Gamma_{k,j}/\lambda}, P_k(\mathrm{e}^{-R_k(\Gamma_{k,j}/\lambda+t_1)},\ldots,\mathrm{e}^{-R_k(\Gamma_{k,j}/\lambda+t_d)}))}.$$

(b) Let $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ with norming constants $a_T > 0$ and $b_T \in \mathbb{R}$ such that

$$\lim_{T\to\infty} T\mathbb{P}(L_1 > a_T x + b_T) = \exp(-x) \quad \text{for } x \in \mathbb{R}.$$

Suppose $\sum_{k=1}^{\infty} \varepsilon_{(s_k,P_k)}$ is a $\mathrm{PRM}(\vartheta)$ with $\vartheta(dt \times dx) = dt \times [\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}\mathrm{e}^{-x}\,dx$ independent of the i. i. d. sequence $(R_k)_{k\in\mathbb{N}}$ with probability distribution π. Then, in the space $M_P(S_G)$,

$$\sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k/(\lambda n), a_{\lambda n}^{-1}(M_k - b_{\lambda n}), b_{\lambda n}^{-1}\mathbf{V}(\Gamma_k/\lambda))} \stackrel{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{(s_k, P_k, (\mathrm{e}^{-R_k t_1},\ldots,\mathrm{e}^{-R_k t_d}))}.$$

We give an interpretation of the point process results. In both parts of Proposition 4.1 the limit relations of the first two components show that the local suprema M_k of V around Γ_k/λ, normalized by the constants determined via L_1, converge weakly to the same extreme value distribution as L_1. The third vector component indicates that for instance for $d = 1$ and $t_1 = 0$ that the second and third component have the same limiting behavior; i. e. the M_k behave like $V_{\Gamma_k/\lambda}$. The results show also that local extremes of V on high levels happen at the jump times Γ/λ of the Lévy process $(L^{(1)}_{\lambda t})_{t\geq 0}$. Thus, the various features of L, which are modelled in Λ_2, have no influence on the location of local extremes on high levels. Moreover, the third vector component indicates that, if the supOU process has an exceedance over a high threshold, then it decreases after this event exponentially fast with a random rate R_k and the distribution π of R_k governs the short/long range dependence of the model.

As for OU-type processes there is an essential difference between the models (a) and (b). In the second component and the third vector component of the limit point process in *(a)* all points $\Gamma_{k,j}/\lambda$ influence the limit, whereas in *(b)* only $\Gamma_{k,0}=0$ does. This phenomenon certainly originates in the very large jumps caused by regular variation of the underlying driving Lévy process. Even though the behavior of the supOU between the large jumps has the tendency to decrease exponentially fast (this comes from the shot-noise process generated by $\Lambda^{(1)}$ and may be overlaid by small positive jumps, negative jumps, a drift and a Gaussian component), huge positive jumps can have a long lasting influence on excursions above high thresholds. This is in contrast to the semi-heavy tailed case in *(b)*.

Result *(b)* can be interpreted that local extremes of models in $\mathcal{S}(\gamma)\cap \mathrm{MDA}(\Lambda)$ show no extremal clusters. The constant $[\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}$ in the intensity of the Poisson random measure, which is 1 for $\gamma=0$, reflects that for $\gamma>0$ the small jumps of L have a certain influence on the size of the local extremes of V, which is in contrast to subexponential models in *(a)* and *(b)* with $\gamma=0$. Although $(V_{\Gamma_k/\lambda})_{k\in\mathbb{N}}$ is not a stationary sequence $V_{\Gamma_k/\lambda}\overset{k\to\infty}{\Longrightarrow} V_0+Z_1$ (recall that Z_1 has d. f. $\nu_1/\nu(1,\infty)$). Furthermore,

$$\nu(1,\infty)\mathbb{P}(V_0+Z_1>x)\sim [\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}\mathbb{P}(L_1>x) \quad \text{for } x\to\infty.$$

Thus *(b)* implies that the exceedances of $(V_{\Gamma_k/\lambda})_{k\in\mathbb{N}}$ at times $(\Gamma_k/\lambda)_{k\in\mathbb{N}}$ behave like those of an i. i. d. sequence with distribution V_0+Z_1. We have seen this constant $[\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}$ already earlier in Proposition 3.3.

Corollary 4.2 (Point process of exceedances)
Let V satisfy the assumptions of Proposition 4.1 and decompose Λ as in (4.1).

(a) Let $L_1\in\mathcal{R}_{-\alpha}$. Suppose $(s_k)_{k\in\mathbb{N}}$ are the jump times of a Poisson process with intensity $x^{-\alpha}$ for fixed $x>0$. Let $(\zeta_k)_{k\in\mathbb{N}}$ be i. i. d. discrete r. v. s, independent of $(s_k)_{k\in\mathbb{N}}$, with probability distribution

$$q_k=\mathbb{P}(\zeta_1=k)=\mathbb{E}\exp(-\alpha R_0\Gamma_k/\lambda)-\mathbb{E}\exp(-\alpha R_0\Gamma_{k+1}/\lambda) \quad \text{for } k\in\mathbb{N}.$$

Then

$$\sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k/(\lambda n), a_{\lambda n}^{-1} M_k)}(\cdot \times (x,\infty)) \stackrel{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \zeta_k \varepsilon_{s_k} \quad \textit{in } M_P([0,\infty)).$$

(b) Let $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$. *Suppose* $(s_k)_{k\in\mathbb{N}}$ *are the jump times of a Poisson process with intensity* $[\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}\,\mathrm{e}^{-x}$ *for fixed* $x \in \mathbb{R}$. *Then*

$$\sum_{k=1}^{\infty} \varepsilon_{(\Gamma_k/(\lambda n), a_{\lambda n}^{-1}(M_k - b_{\lambda n}))}(\cdot \times (x,\infty)) \stackrel{n\to\infty}{\Longrightarrow} \sum_{k=1}^{\infty} \varepsilon_{s_k} \quad \textit{in } M_P([0,\infty)).$$

Again the qualitative difference of the two regimes is visible. For a regularly varying underlying driving Lévy process L the limiting process is a compound Poisson process, where at each Poisson point a cluster appears, whose size is random with distribution $(q_k)_{k\in\mathbb{N}}$. In contrast to this, in the $\mathrm{MDA}(\Lambda)$ case, the limit process is simply a homogeneous Poisson process; no clusters appear in the limit.

The next proposition follows immediately from Proposition 4.1.

Proposition 4.3 (Running maxima) *Let* V *be a stationary supOU process as in* (1.1) *and* L *the underlying driving Lévy process* (2.2). *Define* $M(T) = \sup_{0\le t\le T} V_t$ *for* $T > 0$.

(a) Let $L_1 \in \mathcal{R}_{-\alpha}$ *with norming constants* $a_T > 0$ *such that*

$$\lim_{T\to\infty} T\mathbb{P}(L_1 > a_T x) = x^{-\alpha} \quad \textit{for } x > 0.$$

Then

$$\lim_{T\to\infty} \mathbb{P}(a_{\lambda T}^{-1} M(T) \le x) = \exp(-x^{-\alpha}) \quad \textit{for } x > 0.$$

(b) Let $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ *with norming constants* $a_T > 0$ *and* $b_T \in \mathbb{R}$, *such that*

$$\lim_{T\to\infty} T\mathbb{P}(L_1 > a_T x + b_T) = \exp(-x) \quad \textit{for } x \in \mathbb{R}.$$

Then

$$\lim_{T\to\infty} \mathbb{P}(a_{\lambda T}^{-1}(M(T) - b_{\lambda T}) \le x) = \exp(-[\mathbb{E}\mathrm{e}^{\gamma L_1}]^{-1}\mathbb{E}\mathrm{e}^{\gamma V_0}\mathrm{e}^{-x}) \quad \textit{for } x \in \mathbb{R}.$$

Definition 4.4 (Extremal index function) *Let* $(V_t)_{t\ge 0}$ *be a stationary process. Define the sequence* $M_k(h) = \sup_{(k-1)h\le t\le kh} V_t$ *for* $k \in \mathbb{N}$, $h > 0$. *Let* $\theta(h)$ *be the extremal index (Definition A.8) of the sequence* $(M_k(h))_{k\in\mathbb{N}}$. *Then we call the function* $\theta : (0,\infty) \to [0,1]$ extremal index function.

The idea is to divide the positive real line into blocks of length h. By taking local suprema of the process over these blocks the natural dependence of the continuous-time process is weakened, in certain cases it even disappears. However, for fixed h the extremal index function is a measure for the expected cluster sizes among these blocks. For an extended discussion on the extremal index in the context of discrete- and continuous-time processes see Fasen [9], pp. 83.

Corollary 4.5 (Extremal index function) *Let V be a stationary supOU process as in* (1.1) *and L the underlying driving Lévy process* (2.2)*.*

(a) If $L_1 \in \mathcal{R}_{-\alpha}$, then $\theta(h) = \lambda h\alpha/(\lambda h\alpha + 1)$ for $h > 0$.
(b) If $L_1 \in \mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$, then $\theta(h) = 1$ for $h > 0$.

Regularly varying supOU processes exhibit clusters among blocks, since $\theta(h) < 1$. So they have the potential to model both features: heavy tails and high level clusters. This is in contrast to supOU processes in $\mathcal{S}(\gamma)\cap\mathrm{MDA}(\Lambda)$, where no extremal clusters occur.

5 Conclusion

In this paper we have investigated the extremal behavior of supOU processes, whose underlying driving Lévy process is in the class of convolution equivalent distributions. In contrast to OU-type and COGARCH processes (cf. [14]), regardless of the driving Lévy process they can model long memory. We have concentrated on models with tails ranging from exponential to regularly varying; i. e. tails as they are found in empirical volatility. The stochastic quantities characterizing the extreme behavior for such models, which we have derived in this paper, include

- the tail of the stationary distribution of the supOU process V_0 and $M(h) = \sup_{0\le t\le h} V_t$, and the relation to the tail of the distribution governing the extreme behavior,
- the asymptotic distribution of the running maxima, i. e. their MDA and the norming constants,
- the cluster behavior of the model on high levels.

We want to indicate that long memory of a supOU process represented by π has no influence on the existence of extremal clusters, only on the cluster sizes. SupOU processes in $\mathcal{S}(\gamma) \cap \mathrm{MDA}(\Lambda)$ cannot model clusters on high levels. In contrast to that, regularly varying supOU processes exhibit extremal clusters, which can be described quite precisely by the distribution of the cluster sizes, which depends on

π; see Corollary 4.2. In terms of the tail behavior of V_0, $M(h)$ and the running maxima the results for a supOU process coincide with the results of an OU-type process. Again they are not affected by the long memory property.

Appendix

A Basic notation and definition

We summarize some definitions and concepts used throughout the paper. For details and further references see Embrechts, Klüppelberg and Mikosch [8].

The following is the fundamental theorem in extreme value theory.

Theorem A.1 (Fisher-Tippett Theorem)
Let $(X_n)_{n\in\mathbb{N}}$ be an i. i. d. sequence with d. f. F and denote $M_n = \max_{k=1,\dots,n} X_k$. Suppose we can find sequences of real numbers $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} \mathbb{P}(a_n^{-1}(M_n - b_n) \le x) = \lim_{n\to\infty} F^n(a_n x + b_n) = G(x) \quad \text{for } x \in \mathbb{R}$$

and some non-degenerate d. f. G (we say F is in the maximum domain of attraction *of G and write $F \in \mathrm{MDA}(G)$). Then there are $a > 0$, $b \in \mathbb{R}$ such that $x \mapsto G(ax+b)$ is one of the following three extreme value d. f. s:*

- *Fréchet:* $\Phi_\alpha(x) = G(ax+b) = \begin{cases} 0, & x \le 0, \\ \exp\left(-x^{-\alpha}\right), & x > 0, \end{cases} \quad \text{for} \quad \alpha > 0.$
- *Gumbel:* $\Lambda(x) = G(ax+b) = \exp\left(-\mathrm{e}^{-x}\right), \quad x \in \mathbb{R}\,.$
- *Weibull:* $\Psi_\alpha(x) = G(ax+b) = \begin{cases} \exp\left(-\left(-x\right)^{\alpha}\right), & x \le 0, \\ 1, & x > 0, \end{cases} \quad \text{for} \quad \alpha > 0\,.$

Definition A.2 *A positive measurable function $u : \mathbb{R} \to \mathbb{R}_+$ is called* regularly varying with index α, *denoted by $u \in \mathcal{R}_\alpha$ for $\alpha \in \mathbb{R}$, if*

$$\lim_{t\to\infty} \frac{u(tx)}{u(t)} = x^\alpha \quad \text{for } x > 0\,.$$

The function u is said to be slowly varying *if $\alpha = 0$.*

Theorem A.3 *Let F be a d. f. with $F(x) < 1$ for all $x \in \mathbb{R}$ and $\widehat{f}(\gamma) = \int_{-\infty}^{\infty} \mathrm{e}^{\gamma x}\, F(dx)$.*

(i) *Let F be infinitely divisible with Lévy measure ν and $\gamma \geq 0$. Then*

$$F \in \mathcal{S}(\gamma) \quad \Longleftrightarrow \quad \nu(1,\cdot]/\nu(1,\infty) \in \mathcal{S}(\gamma).$$

(ii) *Suppose $F \in \mathcal{S}(\gamma)$, $\lim_{x\to\infty} \overline{G}(x)/\overline{F}(x) = q \geq 0$ and $\widehat{f}_G(\gamma) < \infty$. Then*

$$\lim_{x\to\infty} \frac{\overline{F * G}(x)}{\overline{F}(x)} = \widehat{f}_2(\gamma) + q\widehat{f}_1(\gamma)$$

and $F * G \in \mathcal{S}(\gamma)$. If $q > 0$, then also $G \in \mathcal{S}(\gamma)$.

(iii) *$F \in \mathcal{L}(\gamma)$, $\gamma \geq 0$, has the representation*

$$\overline{F}(x) = c(x) \exp\left[-\int_0^x \frac{1}{a(y)}\,dy\right] \qquad \text{for } x > 0, \tag{A.1}$$

where $a, c : \mathbb{R}_+ \to \mathbb{R}_+$ and $\lim_{x\to\infty} c(x) = c > 0$ and a is absolutely continuous with $\lim_{x\to\infty} a(x) = \gamma^{-1}$ and $\lim_{x\to\infty} a'(x) = 0$.

The following concept has proved useful in comparing tails.

Definition A.4 (Tail-equivalence)
Two d. f. s F and G (or two measures μ and ν) are called tail-equivalent *if both have support unbounded to the right and there exists some $c > 0$ such that*

$$\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = c \quad \text{or} \quad \lim_{x\to\infty} \nu(x,\infty)/\mu(x,\infty) = c\,.$$

For two tail-equivalent d. f. s in MDA(G) for some G one can choose the same norming constants.

Proposition A.5 *Let X be a r. v. The following conditions are equivalent:*

(a) *X is selfdecomposable.*

(b) *There exists a Lévy process L such that $X \stackrel{d}{=} \int_0^\infty \mathrm{e}^{-s}\,dL_s$.*

(c) *X is infinitely divisible with absolutely continuous Lévy measure given by*

$$\nu(dx) = \frac{k(x)}{|x|}\,dx \qquad \text{for } x \in \mathbb{R}\backslash\{0\},$$

$k(x) \geq 0$, and $k(x)$ is increasing on $(-\infty, 0)$ and decreasing on $(0,\infty)$.

Remark A.6 The integral in (b) exists if and only if (2.3) holds. The above proposition is presented and discussed in Barndorff-Nielsen and Shephard [2], where also further references can be found. It can also be found e.g. in Sato [19], Cor. 15.11 and Theorem 17.5.

Definition A.7 (Poisson random measure)
Let $(A, \mathcal{A}, \vartheta)$ be a measurable space, where ϑ is σ-finite, and $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A Poisson random measure *N with intensity measure ϑ, denoted by* PRM(ϑ), *is a collection of r. v. s $(N(A))_{A\in\mathcal{A}}$, where $N(A) : (\Omega, \mathcal{F}, \mathbb{P}) \to (\mathbb{N}_0, \mathcal{B}(N_0))$, with $N(\emptyset) = 0$, such that:*

(a) *Given any sequence $(A_n)_{n\in\mathbb{N}}$ of mutually disjoint sets in $\mathcal{A}$:*

$$N\Big(\bigcup_{n\in\mathbb{N}} A_n\Big) = \sum_{n\in\mathbb{N}} N(A_n) \qquad \text{a. s.}$$

(b) *$N(A)$ is Poisson distributed with intensity $\vartheta(A)$ for every $A \in \mathcal{A}$.*

(c) *For mutually disjoint sets $A_1, \ldots, A_n \in \mathcal{A}$, $n \in \mathbb{N}$, the r. v. s $N(A_1), \ldots, N(A_n)$ are independent.*

Definition A.8 (Extremal index)
Let $X = (X_n)_{n\in\mathbb{Z}}$ be a strictly stationary sequence and $\theta \geq 0$. If for every $x > 0$ there exists a sequence $u_n(x)$ with

$$\lim_{n\to\infty} n\mathbb{P}(X_1 > u_n(x)) = x \quad \text{and} \quad \lim_{n\to\infty} \mathbb{P}(\max_{k=1,\ldots,n} X_n \leq u_n(x)) = \exp(-\theta x),$$

then θ is called the extremal index *of X and has value in $[0, 1]$.*

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