Large Insurance Losses Distributions

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Abstract

Large insurance losses happen infrequently, but they happen. In this paper we present the standard distribution models used in fire, wind–storm or flood insurance. We also present the classical Cramér-Lundberg model for the total claim amount and some more recent extensions. The classical insurance risk measure is the ruin probability and we give a full account of the ruin event in such models. Finally, we present some results for an integrated insurance risk model, where also investment risk is taken into account.

Keywords: Cramér-Lundberg model, integrated risk process, integrated tail distribution function, Pollaczek-Khinchine formula, quintuple law, regular variation, renewal measure, risk model, ruin probability, sample path leading to ruin, subexponential distribution

1 Subexponential Distribution Functions

Subexponential distribution functions (d.f.s) are a special class of heavy–tailed d.f.s. The name arises from one of their properties, that their right tail decreases more slowly than any exponential tail; see (1.1). This implies that large values can occur in a sample with non–negligible probability, which proposes the subexponential d.f.s as natural candidates for situations, where extremely large values occur in a sample compared to the mean size of the data. Such a pattern is often seen in insurance data, for instance in fire, wind–storm or flood insurance (collectively known as catastrophe insurance), but also in financial and environmental data. Subexponential claims can account for large fluctuations in the risk process of a company. Moreover, the

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subexponential concept has just the right level of generality for risk measurement in insurance and finance models.

Various review papers have appeared on subexponentials; see e.g. [20, 24, 43]; textbook accounts are in [1, 2, 12, 31, 37, 38]. We present two defining properties of subexponential d.f.s.

Definition 1.1 (Subexponential distribution functions) *Let $(X_i)_{i\in\mathbb{N}}$ be i. i. d. positive random variables with d.f. F such that $F(x) < 1$ for all $x > 0$. Denote by $\overline{F}(x) = 1 - F(x)$ for $x \geq 0$, the tail of F, and for $n \in \mathbb{N}$,*

$$\overline{F^{n*}}(x) = 1 - F^{n*}(x) = \mathbb{P}(X_1 + \cdots + X_n > x)\,, \quad x \geq 0\,,$$

the tail of the n–fold convolution of F. F is a subexponential d.f. *$(F \in \mathcal{S})$ if one of the following equivalent conditions holds:*

(a) $\lim_{x\to\infty} \frac{\overline{F^{n*}}(x)}{\overline{F}(x)} = n$ for some (all) $n \geq 2$,

(b) $\lim_{x\to\infty} \frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(\max(X_1,\ldots,X_n) > x)} = 1$ for some (all) $n \geq 2$.

Remark 1.2 (i) A proof of the equivalence is based on ($\sim$ means that the quotient of the left hand side and the right hand side tends to 1 as $x \to \infty$)

$$\frac{\mathbb{P}(X_1 + \cdots + X_n > x)}{\mathbb{P}(\max(X_1,\ldots,X_n) > x)} \sim \frac{\overline{F^{n*}}(x)}{n\overline{F}(x)} \stackrel{x\to\infty}{\longrightarrow} 1 \quad \Longleftrightarrow \quad F \in \mathcal{S}\,.$$

(ii) In much of the present discussion we are dealing only with the right tail of a d.f. This concept can be formalized by denoting two d.f.s F and G with support unbounded to the right *tail–equivalent* if $\lim_{x\to\infty} \overline{F}(x)/\overline{G}(x) = c \in (0,\infty)$. From Definition (a) and the fact that $\mathcal{S}$ is closed with respect to tail–equivalence follows that $\mathcal{S}$ is closed with respect to taking sums and maxima of i. i. d. random variables. Subexponential d.f.s can also be defined on $\mathbb{R}$ by requiring that F restricted to $(0,\infty)$ is subexponential; see [33].
(iii) The heavy–tailedness of $F \in \mathcal{S}$ is demonstrated by the implications

$$F \in \mathcal{S} \Longrightarrow \lim_{x\to\infty} \frac{\overline{F}(x-y)}{\overline{F}(x)} = 1 \quad \forall\, y \in \mathbb{R} \Longrightarrow \overline{F}(x)/\mathrm{e}^{-\varepsilon x} \stackrel{x\to\infty}{\longrightarrow} \infty \quad \forall\, \varepsilon > 0\,. \qquad (1.1)$$

□

A famous subclass of $\mathcal{S}$ is the class of d.f.s with regularly varying tails; see [4].

Example 1.3 (Distribution functions with regularly varying tails) For a positive measurable function f we write $f \in \mathcal{R}(\alpha)$ for $\alpha \in \mathbb{R}$ (f is *regularly varying with index* α) if

$$\lim_{x\to\infty} \frac{f(tx)}{f(x)} = t^{\alpha} \quad \forall\, t > 0\,. \tag{1.2}$$

Let $\overline{F} \in \mathcal{R}(-\alpha)$ for $\alpha \geq 0$, then it has the representation

$$\overline{F}(x) = x^{-\alpha}\ell(x)\,, \quad x > 0\,,$$

for some $\ell \in \mathcal{R}(0)$. To check Definition 1.1 (a) for $n = 2$ split the convolution integral and use partial integration to obtain

$$\frac{\overline{F^{2*}}(x)}{\overline{F}(x)} = 2\int_0^{x/2} \frac{\overline{F}(x-y)}{\overline{F}(x)}\, dF(y) + \frac{(\overline{F}(x/2))^2}{\overline{F}(x)}\,, \quad x > 0\,.$$

Immediately, by (1.2), the last term tends to 0. The integrand satisfies $\overline{F}(x-y)/\overline{F}(x) \leq \overline{F}(x/2)/\overline{F}(x)$ for $0 \leq y \leq x/2$; hence, Lebesgue dominated convergence applies and, since F satisfies (1.1), the integral on the right hand side tends to 1 as $x \to \infty$. Examples of d.f.s with regularly varying tail include the Pareto, Burr, transformed beta (also called generalized F), log–gamma and stable d.f.s (see [12]). □

Example 1.4 (Further subexponential distributions) Apart from the d.f.s in Example 1.3 also the lognormal, the two Benktander families and the heavy-tailed Weibull (shape parameter less than 1) belong to $\mathcal{S}$. The integrated tail d.f.s (see (2.2)) of all of these d.f.s are also in $\mathcal{S}$ provided they have a finite mean; see Table 1.2.6 in [12]. □

2 Insurance Risk Models

2.1 The Cramér-Lundberg Model

The classical insurance risk model is the *Cramér-Lundberg model* (cf. [12, 31, 38]), where the *claim times* constitute a Poisson process, i. e. the *interclaim times* $(T_n)_{n\in\mathbb{N}}$ are i. i. d. exponential random variables with parameter $\lambda > 0$. Furthermore, the claim sizes $(X_k)_{k\in\mathbb{N}}$ (independent of the claims arrival process) are i. i. d. positive random variables with d.f. F and $\mathbb{E}(X_1) = \mu < \infty$. The *risk process* is for *initial reserve* $u \geq 0$ and *premium rate* $c > 0$ defined as

$$R(t) = u + ct - \sum_{k=1}^{N(t)} X_k\,, \quad t \geq 0\,, \tag{2.1}$$

where (with the convention $\sum_{i=1}^0 a_i = 0$) $N(0) = 0$ and $N(t) = \sup\{k \geq 0 : \sum_{i=1}^k T_i \leq t\}$ for $t > 0$. We denote the *ruin time* by

$$\tau(u) = \inf\{t > 0 : u + ct - \sum_{k=1}^{N(t)} X_k < 0\}\,, \quad u \geq 0\,.$$

The *ruin probability in infinite time* is defined as

$$\psi(u) = \mathbb{P}(R(t) < 0 \text{ for some } 0 \leq t < \infty \mid R(0) = u) = \mathbb{P}(\tau(u) < \infty)\,, \quad u \geq 0\,.$$

By definition of the risk process, ruin can occur only at the claim times (see also Figure 1), hence for $u \geq 0$,

$$\begin{aligned} \psi(u) &= \mathbb{P}(R(t) < 0 \text{ for some } t \geq 0 \mid R(0) = u) \\ &= \mathbb{P}\left(u + \sum_{k=1}^n (cT_k - X_k) < 0 \text{ for some } n \in \mathbb{N}\right). \end{aligned}$$

Provided that $\mathbb{E}(cT_1 - X_1) = c/\lambda - \mu > 0$, then $(R(t))_{t\geq 0}$ has a positive drift and hence, $R(t) \to +\infty$ a. s. as $t \to \infty$.

A ladder height analysis shows that the *integrated tail distribution function*

$$F_I(x) = \frac{1}{\mu}\int_0^x \overline{F}(y)dy\,, \quad x \geq 0\,, \tag{2.2}$$

of F is the d.f. of the first undershoot of $R(t) - u$ below 0 under the condition that $R(t)$ falls below u in finite time. Setting $\rho = \lambda\mu/c < 1$, the number $\widetilde{N}$ of times, where $R(t)$ achieves a new local minimum in finite time plus 1, is geometrically distributed with parameter $(1-\rho)$, i. e. $\mathbb{P}(\widetilde{N} = n) = (1-\rho)\rho^n$, $n \in \mathbb{N}_0$. As $(R(t))_{t\geq 0}$ is a Markov process its sample path splits into i. i. d. cycles, each starting with a new partial minimum of the risk process. Combining these findings we obtain the following representation of the ruin probability.

Theorem 2.1 (Pollacek-Khinchine formula) *Consider the risk process (2.1) in the Cramér-Lundberg model with $\rho = \lambda\mu/c < 1$. Let F_I be as in (2.2). Then the ruin probability is given by*

$$\psi(u) = (1-\rho)\sum_{n=1}^{\infty} \rho^n \overline{F_I^{n*}}(u)\,, \quad u \geq 0\,.$$

We are interested in the ruin event represented by a probabilistic description of the last cycle before ruin, i. e. the quantities:

$R(\tau(u))$	the level of the risk process at ruin,
$R(\tau(u)-)$	the level of the risk process just before ruin,
$\inf_{0\leq t<\tau(u)} R(t)$	the infimum of the risk process before ruin.

The following result holds.

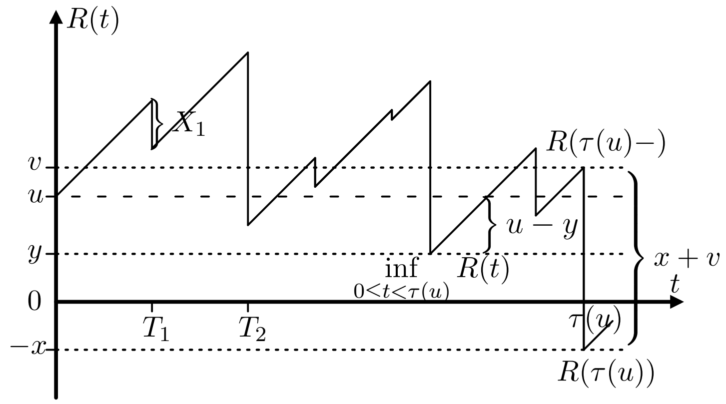

Figure 1: Sample path of a risk process in the Cramér-Lundberg model

Theorem 2.2 *Consider the risk process (2.1) in the Cramér-Lundberg model with $\rho = \lambda\mu/c < 1$. Define the renewal measure of the descending ladder height process of $(ct - \sum_{k=1}^{N(t)} X_k)_{t\geq 0}$ given by*

$$V(u) = \frac{1}{c}\sum_{n=0}^{\infty} \rho^n F_I^{n*}(u)\,, \quad u \geq 0,$$

where F_I is as in (2.2). Then for $x > 0$, $v \geq y$ and $y \in [0, u]$,

$$\mathbb{P}(-R(\tau(u)) \in dx, R(\tau(u)-) \in dv, \inf_{0\leq t<\tau(u)} R(t) \in dy) = \lambda F(dx+v)V(u-dy)\,dv.$$

2.2 The Lévy Risk Model

A natural and useful generalization of the Cramér-Lundberg model as in (2.1) is to replace the compound Poisson process $(\sum_{k=1}^{N(t)} X_k)_{t\geq 0}$ by a general subordinator $(S(t))_{t\geq 0}$. A *subordinator* is an increasing Lévy process, i. e. it starts in 0, has independent and stationary increments, and has a. s. increasing sample paths. A subordinator can be characterized by its *Lévy-Khinchine representation* $\mathbb{E}(\exp(ivS(t))) = \exp(t\varphi(v))$ for $t \geq 0$, $v \in \mathbb{R}$ with

$$\varphi(v) = \int_{\mathbb{R}_+} \left(\mathrm{e}^{ivx} - 1\right)\, \nu(dx),$$

where ν is a measure on $\mathbb{R}_+ := (0, \infty)$, called *Lévy measure*, satisfying $\int_{\mathbb{R}_+}(1 \wedge |x|)\,\nu(dx) < \infty$; see the monographs [3, 29, 39, 42] for more details on Lévy processes. The upward movement of the subordinator represents the total claim amount process, extending the compound Poisson model by modelling many small claims between more severe claims at Poisson times.

On top of this already rather general model, uncertainty in the aggregated claims and in the premium income can be modelled by a Brownian motion $(B(t))_{t\geq 0}$ giving the risk process

$$R(t) = u + ct - S(t) + \sigma B(t)\,, \quad t \geq 0, \tag{2.3}$$

where u and c are as in the Cramér-Lundberg model and $\sigma \geq 0$. More general Lévy insurance models are investigated in [7,22,25,26]; see also the monograph [29]. More about the Cramér-Lundberg model perturbed by a Brownian motion can be found in [8, 16–18, 44] and perturbed by an α-stable Lévy motion in [14, 40]. The class of Γ-subordinators with and without perturbation by a Brownian motion have been investigated in [9, 46].

In this model the ruin time is defined as

$$\tau(u) = \inf\{t > 0 : u + ct - S(t) + \sigma B(t) < 0\}\,, \quad u \geq 0\,,$$

and the ruin probability by $\psi(u) = \mathbb{P}(\tau(u) < \infty)$.

Again we have to assume that $R(t) \to +\infty$ a. s. as $t \to \infty$, which is implied by $\rho := \mathbb{E}(S(1))/c < 1$, i. e. the premium income outweighs the expected total claims.

Analogously to (2.2) we define the integrated tail d.f. of ν as

$$\nu_I(x) = \frac{1}{\mathbb{E}(S(1))} \int_0^x \nu(y, \infty)\, dy, \qquad x \geq 0. \tag{2.4}$$

In the Cramér-Lundberg model $\nu(y, \infty) = \lambda \overline{F}(y)$ and $\mathbb{E}(S(1)) = \lambda\mu$, hence we get back $\nu_I = F_I$ and $\rho = \lambda\mu/c$.

To present the results in this general set-up we also require the exponential d.f.

$$G(x) = 1 - \exp(-2cx/\sigma^2)\, \mathbf{1}_{\{\sigma>0\}}\,, \quad x \geq 0\,, \tag{2.5}$$

which is the d.f. of $(\sup_{t\geq 0}\{-ct - \sigma B(t)\})_{t\geq 0}$. Finally, we have to build convolutions of ν_I with itself and with convolution powers of G. As an explaining example we recall that

$$G * \nu_I(x) = \int_0^x G(x-y) d\nu_I(y) = \frac{1}{\mathbb{E}(S(1))} \int_0^x G(x-y)\nu(y,\infty)dy\,, \quad x \geq 0\,.$$

Then the ruin probability satisfies the following equation (see [22], Theorem 3.1).

Theorem 2.3 (Pollaczek-Khinchine Formula) *Consider the risk process (2.3) in the Lévy risk model with $\rho = \mathbb{E}(S(1))/c < 1$. Let ν_I be as in (2.4) and G as in (2.5). Then the ruin probability can be represented as*

$$\psi(u) = (1-\rho) \sum_{n=1}^{\infty} \rho^n\, \overline{G^{(n+1)*} * \nu_I^{n*}}(u)\,, \quad u \geq 0\,.$$

The quintuple-law in [7] is again based on a ladder height analysis:

Theorem 2.4 *Consider the risk process (2.3) in the Lévy risk model with* $\rho = \mathbb{E}(S(1))/c < 1$. *Define the renewal measure of the descending ladder height process of* $(ct - S(t) + \sigma B(t))_{t\geq 0}$ *given by*

$$V(u) = \frac{1}{c}\sum_{n=0}^{\infty} \rho^n \,(G^{(n+1)*} * \nu_I^{n*})(u)\,, \quad u \geq 0\,,$$

where ν_I *is as in* (2.4) *and* G *as in (2.5). Then for* $x > 0$, $v \geq y$ *and* $y \in [0, u]$,

$$\mathbb{P}(-R(\tau(u)) \in dx, R(\tau(u)-) \in dv, \inf_{0\leq t<\tau(u)} R(t) \in dy) = \nu(dx + v)V(u - dy)\, dv.$$

3 Risk Theory in the Presence of Heavy Tails

3.1 Asymptotic Ruin Theory

In this section we investigate the occurrence of large, possibly ruinous claims modelled by $\nu_I \in \mathcal{S}$, which translates in the Cramér-Lundberg model to $F_I \in \mathcal{S}$. We refer to [25, 26] for more details on the results of this section; these papers include also more general Lévy insurance risk models. The results for the Cramér-Lundberg model can also be found in [1].

The following part (a) of Proposition 3.1 is [11], Proposition 1. The result (b) can partly be found already in [2]. Its present form goes back to [10, 11] and [6].

Proposition 3.1 *(a) Let* $H = G * F$ *be the convolution of two d.f.s on* $(0, \infty)$. *If* $F \in \mathcal{S}$ *and* $\overline{G}(x) = o(\overline{F}(x))$ *as* $x \to \infty$, *then* $H \in \mathcal{S}$.

(b) Suppose $(p_n)_{n\geq 0}$ *defines a probability measure on* $\mathbb{N}_0$ *such that* $\sum_{n=0}^{\infty} p_n(1 + \varepsilon)^n < \infty$ *for some* $\varepsilon > 0$ *and* $p_k > 0$ *for some* $k \geq 2$. *Let*

$$K(x) = \sum_{n=0}^{\infty} p_n H^{n*}(x)\,, \quad x \geq 0\,.$$

Then

$$H \in \mathcal{S} \iff \lim_{x\to\infty} \frac{\overline{K}(x)}{\overline{H}(x)} = \sum_{n=1}^{\infty} n p_n \iff K \in \mathcal{S} \text{ and } \overline{H}(x) \neq o(\overline{K}(x))\,.$$

For the ruin probability this Proposition implies with the Pollaczek-Khinchine formula (Theorem 2.3) the following result.

Theorem 3.2 (Ruin Probability) *Consider the risk process (2.3) in the Lévy risk model with $\rho = \mathbb{E}(S(1))/c < 1$. If $\nu_I \in \mathcal{S}$, then*

$$\psi(u) \sim \frac{\rho}{1-\rho}\overline{\nu}_I(u) = \frac{1}{c - \mathbb{E}(S(1))}\int_u^\infty \nu(y,\infty)dy\,, \qquad \text{as } u \to \infty.$$

3.2 Sample Path Leading to Ruin

Not surprisingly, the asymptotic behavior of the quantities in Theorem 2.4 are consequences of extreme value theory; see [12] or any other book on extreme value theory for background. As shown in [21], under weak regularity conditions a subexponential d.f. F belongs to the *maximum domain of attraction* of an extreme value d.f. G_α, $\alpha \in (0,\infty]$ (we write $F \in \mathrm{MDA}(G_\alpha)$), where

$$G_\alpha(x) = \Phi_\alpha(x) = \exp(-x^{-\alpha})\,\mathbf{1}_{\{x>0\}}\,,\ \ \alpha < \infty\,,\ \text{ and } G_\infty(x) = \Lambda(x) = \exp(-\mathrm{e}^{-x})\,.$$

The meaning of $F \in \mathrm{MDA}(G_\alpha)$ is that there exist sequences of constants $a_n > 0$, $b_n \in \mathbb{R}$ such that

$$\lim_{n\to\infty} n\overline{F}(a_n x + b_n) = -\log G_\alpha(x) \quad \forall\, x \in \begin{cases} (0,\infty) & \text{if } \alpha \in (0,\infty), \\ \mathbb{R} & \text{if } \alpha = \infty. \end{cases}$$

The following result describes the behavior of the process at ruin, and the upcrossing event itself; see [25].

Theorem 3.3 (Sample Path Leading to Ruin) *Consider the risk process (2.3) in the Lévy risk model with $\rho = \mathbb{E}(S(1))/c < 1$. Define $\widetilde{\nu}(x) := \nu(1,x)/\nu(1,\infty)$ for $x > 1$ and assume that $\widetilde{\nu} \in \mathrm{MDA}(G_{\alpha+1})$ for $\alpha \in (0,\infty]$. Define $a(\cdot) = \nu_I(\cdot,\infty)/\nu(\cdot,\infty)$. Then, in $\mathbb{P}(\cdot \mid \tau(u) < \infty)$–distribution,*

$$\left(\frac{R(\tau(u)-) - u}{a(u)}, \frac{-R(\tau(u))}{a(u)}\right) \longrightarrow (V_\alpha, T_\alpha)\ , \qquad \text{as } u \to \infty.$$

V_α and T_α are positive random variables with d.f. satisfying for $x, y > 0$

$$\mathbb{P}(V_\alpha > x, T_\alpha > y) = \begin{cases} \left(1 + \dfrac{x+y}{\alpha}\right)^{-\alpha} & \text{if } \alpha \in (0,\infty)\,, \\ \mathrm{e}^{-(x+y)} & \text{if } \alpha = \infty. \end{cases}$$

Remark 3.4 Extreme value theory and Theorem 2.4 is the basis of this result: recall first that $\widetilde{\nu} \in \mathrm{MDA}(\Phi_{\alpha+1})$ is equivalent to $\nu(\cdot,\infty) \in \mathcal{R}(-(\alpha+1))$ and, hence, to $\nu_I \in \mathrm{MDA}(\Phi_\alpha)$ by Karamata's theorem. The normalizing function $a(u)$ tends in the subexponential case to infinity as $u \to \infty$. For $\nu(\cdot,\infty) \in \mathcal{R}(-(\alpha+1))$ Karamata's theorem gives $a(u) \sim u/\alpha$ as $u \to \infty$. $\square$

4 Insurance Risk Models with Investment

The following extension of the insurance risk process has attracted attention over the last years. It is based on the simple fact that an insurance company not only deals with the risk coming from insurance claims, but also invests capital at a large scale into financial markets.

To keep the level of sophistication moderate we assume that the insurance risk process $(R(t))_{t\geq 0}$ is the Cramér-Lundberg model as defined in (2.1). We suppose that the insurance company invests its reserve into a Black-Scholes type market consisting of a *riskless bond* and a *risky stock* modelled by an exponential Lévy process. Their price processes follow the equations

$$P_0(t) = \mathrm{e}^{\delta t} \quad \text{and} \quad P_1(t) = \mathrm{e}^{L(t)}\,, \quad t \geq 0.$$

The constant $\delta > 0$ is the *riskless interest rate*, $(L(t))_{t\geq 0}$ denotes a Lévy process characterized by its Lévy-Khinchine representation $\mathbb{E}(\exp(ivL(t))) = \exp(t\varphi(v))$ for $t \geq 0$, $v \in \mathbb{R}$, with

$$\varphi(v) = ivm - \frac{1}{2}v^2\sigma^2 + \int_{\mathbb{R}} \left(\mathrm{e}^{ivx} - 1 - ivx\,\mathbf{1}_{[-1,1]}(x)\right)\,\nu(dx).$$

The quantities (m, σ^2, ν) are called the *generating triplet* of the Lévy process L. Here $m \in \mathbb{R}$, $\sigma^2 \geq 0$ and ν is a Lévy measure satisfying $\nu(\{0\}) = 0$ and $\int_{\mathbb{R}}(1\wedge|x|^2)\,\nu(dx) < \infty$. We assume that

$$0 < \mathbb{E}(L(1)) < \infty \quad \text{and} \quad \text{either } \sigma > 0 \text{ or } \nu(-\infty, 0) > 0 \tag{4.1}$$

such that $L(t)$ is negative with positive probability and hence, $P_1(t)$ is less than one with positive probability.

We denote by $\theta \in (0, 1]$ the *investment strategy*, which is the fraction of the reserve invested into the risky asset. For details and more background on this model we refer to [27]. Then the *investment process* is the solution of the stochastic differential equation

$$dP_\theta(t) = P_\theta(t-)\,d((1-\theta)\delta t + \theta\widehat{L}(t))\,, \quad t \geq 0, \tag{4.2}$$

where $(\widehat{L}(t))_{t\geq 0}$ is a Lévy process satisfying $\mathcal{E}(\widehat{L}(t)) = \exp(L(t))$ and $\mathcal{E}$ denotes the stochastic exponential (cf. [36]). The solution of (4.2) is given by

$$P_\theta(t) = \mathrm{e}^{L_\theta(t)}\,, \quad t \geq 0,$$

where L_θ is such that $\mathcal{E}((1-\theta)\delta t + \theta\widehat{L}(t)) = \exp(L_\theta(t))$.

The *integrated risk process* of the insurance company is given by

$$I_\theta(t) = \mathrm{e}^{L_\theta(t)}\left(u + \int_0^t \mathrm{e}^{-L_\theta(v)}(c\,dv - dS(v))\right), \quad t \geq 0. \tag{4.3}$$

In this model we are interested in the ruin probability

$$\psi_\theta(u) = \mathbb{P}(I_\theta(t) < 0 \text{ for some } t \geq 0 \mid I_\theta(0) = u), \quad u \geq 0.$$

To find the asymptotic ruin probability for such models there exist two approaches. The first one is analytic and derives an integro-differential equation for the ruin probability, whose asymptotic solution can in certain cases be found. This works in particular for the case, where the investment process is a geometric Brownian motion with drift, and has been considered in [13, 15, 41, 45]; see also [34] for an overview.

This method breaks down for general exponential Lévy investment processes. But equation (4.3) can be viewed as a continuous time random recurrence equation, often called a generalized Ornstein-Uhlenbeck process, cf. [30] and references therein. Splitting up the integral in an appropriate way, asymptotic theory for discrete random recurrence equations can be applied. The theoretical basis for this approach can be found in [32] and is based on the seminal paper [19]. This approach is applied in [23, 35].

Common starting point is the process $(I_\theta(t))_{t\geq 0}$ as defined in (4.3), which is the same as the process (2.1) in [35]. We present Theorem 3.2 (a) of [35] below and formulate the necessary conditions in our terminology.

The Laplace exponent of L_θ is denoted by $\phi_\theta(v) = \log \mathbb{E}(\mathrm{e}^{-vL_\theta(1)})$. If we assume that

$$\mathcal{V}_\infty := \{v \geq 0 : \mathbb{E}(\mathrm{e}^{-vL_1}) < \infty\} \quad \text{is right open,} \tag{4.4}$$

then (cf. e. g. Lemma 4.1 of [27]) there exists a unique $\kappa(\theta)$ such that

$$\phi_\theta(\kappa(\theta)) = 0. \tag{4.5}$$

Theorem 4.1 (Ruin probability) *We consider the integrated risk process $(I_\theta(t))_{t\geq 0}$ as in (4.3) satisfying (4.1) and (4.4). Furthermore, we assume for the claim size variable X that $\mathbb{E}(X^{\max\{1,\kappa(\theta)+\epsilon\}}) < \infty$ for some $\epsilon > 0$, where $\kappa(\theta)$ is given in (4.5), and we assume that $\phi_\theta(2) < \infty$. Let U be uniformly distributed on $[0, 1]$ and independent of $(L_\theta(t))_{t\geq 0}$. Assume that the d.f. of $L_\theta(U)$ has an absolutely continuous component. Then there exists a constant $C_1 > 0$ such that*

$$\psi_\theta(u) \sim C_1 u^{-\kappa(\theta)}, \quad \text{as } u \to \infty.$$

Remark 4.2 If the insurance company invests its money in a classical Black-Scholes model, where $L(t) = \gamma t + \sigma^2 W(t)$, $t \geq 0$, is a Brownian motion with drift $\gamma > 0$, variance $\sigma^2 > 0$ and $(W(t))_{t\geq 0}$ is a standard Brownian motion, then $(L_\theta(t))_{t\geq 0}$ is again a Brownian motion with drift $\gamma_\theta = \theta\gamma + (1-\theta)(\delta + \frac{\sigma^2}{2}\theta)$ and variance $\sigma_\theta^2 = \theta^2\sigma^2$. Since

$$\phi_\theta(s) = -\gamma_\theta s + \frac{\sigma_\theta^2}{2} s^2, \quad s \geq 0\,,$$

we obtain $\kappa(\theta) = 2\gamma_\theta/\sigma_\theta^2$. Hence, if $\mathbb{E}(X^{\max\{1,\kappa(\theta)+\epsilon\}}) < \infty$ for some $\epsilon > 0$, Theorem 4.1 applies. □

Whereas the ruin probability is indeed the classical insurance risk measure, other risk measures have been suggested for investment risk. The *discounted net loss process*

$$V_\theta(t) = \int_0^t \mathrm{e}^{-L_\theta(v)}\,(dS(v) - c dv)\,, \quad t \geq 0\,, \tag{4.6}$$

converges a.s. to V_θ^* when $\phi_\theta(1) < \lambda$, and measures the risk of the insurance company, since, in particular, the relation

$$\mathbb{P}(I_\theta(t) < 0 \mid I_\theta(0) = u) = \mathbb{P}(V_\theta(t) > u), \quad t, u \geq 0\,,$$

holds. This fact has been exploited in [27] to review this model from the point of view of the investing risk manager. Invoking the Value-at-Risk as alternative risk measure to the ruin probability they derived the tail behavior of V_θ^*. For heavy-tailed as well as for light-tailed claim sizes V_θ^* has right Pareto tail indicating again the riskiness of investment. See also [5, 28] for further insight into this problem.

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