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Characterization of a Class of "Convexificable" Resource Allocation Problems

Holger Boche
Technical University of Berlin
Heinrich Hertz Institute
Einsteinufer 25, 10587 Berlin
holger.boche@mk.tu-berlin.de

Siddharth Naik
Technical University of Berlin
Heinrich Hertz Chair for Mobile Comm.
Einsteinufer 25, 10587 Berlin
naik@hhi.fraunhofer.de

Tansu Alpcan
Technical University of Berlin
Deutsche Telekom Laboratories
Ernst–Reuter Platz 7, 10587 Berlin
alpcan@sec.t-labs.tu-berlin.de

***Abstract*—This paper investigates the possibility of having convex formulations of optimization problems for interference coupled wireless systems. An axiomatic framework for interference functions proposed by Yates in 1995 is used to model interference coupling in our paper. The paper shows, that under certain very natural assumptions – the exponential mapping is the unique transformation (up to a constant), for "convexification" of resource allocation problems for linear interference functions. The paper shows that it is sufficient to check for the joint convexity of the sum of weighted utility functions of inverse signal–to–interference (plus noise)–ratio, if we would like the resulting resource allocation problem to be convex. The paper characterizes the largest class of interference functions, which allow a convex formulation of a problem for interference coupled wireless systems. It extends previous literature on log–convex interference functions and provides boundaries on the class of problems in wireless systems, which are jointly convex and hence can be efficiently solved at least from a numerical perspective.**

I. Introduction

It is strongly believed, that the dividing line between "easy" and "difficult" problems in optimization is convexity [1]. In our paper we attempt to check for joint convexity of functions, which are functions of the inverse signal–to–interference (plus noise) ratio (SINR), which is an important measure for link performance in wireless systems. Such functions are frequently encountered as loss minimization problems in wireless communications, e.g. minimum mean square error (MMSE) and bit error rate (BER). Such wireless systems are often interference coupled. We adopt an axiomatic approach to model interference in our system. An axiomatic approach was proposed by Yates in [2] with extensions in [3], [4]. The Yates framework of *standard interference functions* (discussed in in Section II-B) is general enough to incorporate cross-layer effects and it serves as a theoretical basis for a plethora of algorithms.

Proving inherent boundaries on the problems, which can be characterized as jointly convex problems could help in channelizing future research directions and obtaining practically implementable resource allocation strategies utilizing the wide gamut of convex optimization tools. We focus our attention on a problem, namely that of characterizing the subclass of general interference functions for which we can get a meaningful convex optimization problem with respect to (wrt.) a wireless systems perspective. The answer to this problem shall help us provide another motivation for using the log–scale for measuring power in electrical engineering.

The paper [5] states that there exists no signal–to–(noise) plus interference (SINR) based utility functions, which are convex in the power domain. Furthermore, the sum of such weighted functions can never be convex. In this paper, we attempt to obtain convexity of certain resource allocation problems, under an appropriate transformation. The main contributions of this paper are as follows:

- Linear interference functions are the simplest and most frequently encountered class of interference functions. The paper shows (Theorem 1), that under certain very natural assumptions in interference coupled wireless systems – the transformation $p_k = \exp(\mu s_k)$, $\mu > 0$ where p_k is the power of an arbitrary user k, $s \in \mathbb{R}$ – is the unique transformation for "convexification" of resource allocation problems in interference coupled wireless systems for linear interference function. Such a change of variable is an additional motivation for the utilization of the log–domain in communication systems.
- Theorems 1 and 2 show that under very natural assumptions, it is sufficient to check for the joint convexity of the weighted sum of functions of inverse SINR function wrt. $\boldsymbol{s}$, if we would like the resulting resource allocation problem to be convex.
- Theorems 3 and 4 extend the analysis beyond linear interference functions. Theorems 3 and 4 characterize the largest class of interference functions ($\mathcal{C}$), which allow a problem in interference coupled wireless systems to be formulated as a convex optimization problem. $\mathcal{C}$ interference functions (see Definition 4), include *log–convex* interference functions, extending previous literature on the topic of convex characterization of resource allocation problems in interference coupled wireless systems.

II. Interference Coupled Wireless Systems

In this paper we investigate the case of interference coupled wireless systems. We begin, by providing certain notational

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conventions used in the paper in Section II-A below.

A. Preliminaries and Notation

Matrices and vectors are denoted by bold capital letters and bold lowercase letters, respectively. Let $\boldsymbol{y}$ be a vector, then $y_l = [\boldsymbol{y}]_l$ is the l^{th} component. The notation $\boldsymbol{y} \geq 0$ implies that $y_l \geq 0$ for all components l. $\boldsymbol{x} > \boldsymbol{y}$ implies $x_l > y_l$ for all components l. Similar definitions hold for the reverse directions. $\boldsymbol{x} \neq \boldsymbol{y}$ implies that the vector differs in at least one component. Let $\mathcal{F}$ imply a set, with the exception that $\mathcal{I}$ is used to represent a function. The set of non-negative reals is denoted as $\mathbb{R}_+$. The set of positive reals is denoted as $\mathbb{R}_{++}$. Let $e^{\boldsymbol{y}}$ and $\log(\boldsymbol{y})$ denote component–wise exponential and logarithm, respectively. Let g, f, h and ψ represent functions throughout the paper.

B. Interference Functions

In a wireless system, the users' utilities can strongly depend on the underlying physical layer. Consider K users with transmit powers $\boldsymbol{p} = [p_1, \ldots, p_K]^T$ and $\mathcal{K} := \{1, \ldots, K\}$. The noise power at each receiver is σ^2. Hence, the SINR at each receiver depends on the *extended power vector* $\underline{\boldsymbol{p}} = [\boldsymbol{p}, \sigma^2]^T = [p_1, \ldots, p_K, \sigma^2]^T$. The resulting SINR of user k is $\text{SINR}_k(\underline{\boldsymbol{p}}) = \frac{p_k}{\mathcal{I}_k(\underline{\boldsymbol{p}})} = \gamma_k(\underline{\boldsymbol{p}})$, where $\mathcal{I}_k$ is the interference (plus noise) as a function of $\underline{\boldsymbol{p}}$. In order to model interference coupling, we shall follow the axiomatic approach proposed in [2], [4]. The general interference functions possess the properties of conditional positivity, scale invariance and monotonicity wrt. the power component and strict monotonicity wrt. the noise component. For further details, we refer to the Appendix V.

C. Impact of Interference Coupling

Users in a wireless systems coupled by interference are interested in maximizing their own utility or minimizing their loss. In such interference coupled systems, where at least one user $k \in \mathcal{K}$ sees interference from another user $j \in \mathcal{K}$ and $j \neq k$, i.e. it is not possible to completely orthogonalize all the users in the system, consider the following function, $\log(\mathcal{I}_k(\underline{\boldsymbol{p}})/\underline{p}_k)$. The function

$$f(\underline{\boldsymbol{p}}, \boldsymbol{\omega}) = \sum_{k \in \mathcal{K}} \omega_k \log(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k}) \tag{1}$$

for all weight vectors $\boldsymbol{\omega} > \mathbf{0}$ is not jointly convex with respect to $\underline{\boldsymbol{p}}$. Furthermore, the function (1) is not a convex optimization problem even for fixed linear interference functions, e.g. $\mathcal{I}_k(\underline{\boldsymbol{p}}) = \sum_{j \in \mathcal{K}} v_{kj} \underline{p}_j + \theta_k^2$, where v_{kj} is the link-gain between transmitter j and receiver k.

[5] implies that if u_k is the rate of user k, then the following sum of weighted rate maximization problem cannot be solved by convex optimization techniques in its current form. Similarly if we have that u_k is the MMSE of user k, then the following sum of weighted MMSE minimization problem cannot be solved by convex optimization techniques in its current form. None the less, through appropriate substitution of variables, the above formulation can be converted into a convex optimization problem. Such a change of variable is discussed in Section III.

III. Analysis of Resource Allocation Problems – Linear Interference Functions

In this section, we shall analyze convexity properties of functions of inverse SINR for linear interference functions. Before we delve into our analysis, we shall briefly review the concepts of feasible SINR regions and feasible quality–of–service (QoS) regions. The feasible SINR region $\mathbf{F}$ is the set of all feasible SINR vectors $\bar{\boldsymbol{\gamma}}$, that can be supported for all users by means of power control, with interference being treated as noise. We define a set $\mathcal{P}$ as the set of vectors, which satisfy certain power constraints, e.g.

- for the case of total power constraints: $\mathcal{P} := \{\boldsymbol{p} \mid \boldsymbol{p} = [p_1, \ldots, p_K], \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}\}$, where P_{total} is the total power constraint,
- for the case of individual power constraints: $\mathcal{P} := \{\boldsymbol{p} \mid \boldsymbol{p} \leq \hat{\boldsymbol{p}}\}$, where $\hat{\boldsymbol{p}} = [\hat{p}_1, \ldots, \hat{p}_K]$ are the individual power constraints, and
- for the case of individual and total power constraints: $\mathcal{P} := \{\boldsymbol{p} \mid \boldsymbol{p} = [p_1, \ldots, p_K], \sum_{k \in \mathcal{K}} p_k \leq P_{\text{total}}, \boldsymbol{p} \leq \hat{\boldsymbol{p}}\}$.

The feasible SINR region $\mathbf{F}$ can be written as follows:

$$\mathbf{F} = \{\bar{\boldsymbol{\gamma}} \mid \exists \underline{\boldsymbol{p}} \geq \mathbf{0}, \boldsymbol{p} \in \mathcal{P}, \gamma_k(\underline{\boldsymbol{p}})) \geq \bar{\gamma}_k, \bar{\gamma}_k \in \mathbb{R}_+, \forall k \in \mathcal{K}\} \tag{2}$$

and the corresponding feasible QoS region is

$$\mathbf{U} = \{\bar{\boldsymbol{u}} \mid \exists \gamma \in \mathbf{F}, u_k(\gamma_k(\underline{\boldsymbol{p}})) \geq \bar{u}_k, \bar{u}_k \in \mathbb{R}_+, \forall k \in \mathcal{K}\}. \tag{3}$$

We know from [6] and the references there in, that the feasible SINR region ($\mathbf{F}$) is in general not convex[1]. Furthermore, we also know from [5], that we can never have joint convexity of the inverse SINR in the power domain. Hence, we would like to investigate the possibility of finding a suitable transformation ψ (or ψ^{-1}), which

1) transforms the problem from the power domain to the $\boldsymbol{s}$–domain, i.e. $\psi^{-1} : \mathbb{R}_+ \mapsto S$, where $S = \mathbb{R}$, $s \in S$ and the inverse SINR and functions of inverse SINR are jointly convex wrt. $\boldsymbol{s} = [s_1, \ldots, s_K]$,
2) transforms the feasible SINR region into a convex feasible QoS set $\mathbf{U}$, where $\psi^{-1}(\gamma_k) = u_k$, for all $k \in \mathcal{K}$ and $\boldsymbol{u} \in \mathbf{U}$.

While finding our transformation ψ^{-1}, we shall assume that: *transformation $\psi(s) = p$ is strictly monotonic increasing and twice continuously differentiable, throughout the paper.*

The feasible SINR region is convex on the logarithmic scale[2].

[1] The feasible SINR region can also be defined as $\{\boldsymbol{\gamma} > \mathbf{0} \mid \rho(\boldsymbol{\gamma}) \leq 1\}$, where $\rho(\boldsymbol{\gamma}) := \rho(\text{diag}\{\boldsymbol{\gamma}\}\boldsymbol{V}_{\text{res}})$ is the spectral radius of the weighted coupling matrix, where $\boldsymbol{V} = [\boldsymbol{V}_{\text{res}}, \mathbf{1}^T]$ and $\boldsymbol{V}_{\text{res}}$ is an irreducible matrix containing the interference coupling coefficients, without the dependency on noise. A detailed explanation of the relationship between $\boldsymbol{V}$ and $\gamma_k(\underline{\boldsymbol{p}})) \geq \bar{\gamma}_k$in (2) can be found in [4, pp. 5–7].

[2] This is due to the fact that the spectral radius $\rho(\boldsymbol{\gamma})$ is *log–convex* after a change of variable $\boldsymbol{\gamma} = e^{\boldsymbol{s}}$, where $\boldsymbol{s} \in \mathbb{R}^{K+1}$ is the logarithmic SINR. Note that $\rho(\boldsymbol{\gamma})$ fulfills the axioms A1-A4, i.e. the spectral radius is a special case of the more general interference functions.

Even though the feasible QoS region $\mathbf{U}$ is a convex set after a logarithmic transformation, the function $\log(1/\gamma_k(\underline{\boldsymbol{p}}))$, for $k \in \mathcal{K}$ is not jointly convex with respect to $\underline{\boldsymbol{p}}$. Since linear interference functions are the simplest type of interference functions and they are frequently encountered in communication systems. Hence, expecting the feasible QoS region $\mathbf{U}$ to be convex (see 2) for all linear interference functions, is a very natural requirement for communication systems.

We now return to problem of finding a suitable transformation ψ. To formalize the conditions 1) and 2) we introduce the following requirement.

Requirement 1. For all linear interference functions, the feasible QoS region $\mathbf{U}$ resulting from the transformation $\gamma_k = \psi(u_k)$, for all users $k \in \mathcal{K}$ without power constraints is convex.

The function $\psi(s) = e^s$ is one such function satisfying requirement 1. We further introduce another requirement, which expects joint convexity of the inverse SINR, which can be thought of as loss minimization in wireless systems and joint convexity of the inverse SINR raised to α for all $\alpha > 0$.

Requirement 2. For all scalars $\alpha > 0$ the function

$$\Big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\Big)^\alpha$$

is jointly convex with respect to $\boldsymbol{s}$.

Expecting the function $(1/\text{SINR})^\alpha$, with $\alpha > 0$ to be convex wrt. $\boldsymbol{s}$, implies that we expect the expression of the α–order diversity of a system with a certain inverse SINR to be convex. [7] provides a characterization of the multiplexing rate tuples of the users as a function of the common diversity gain for each user. It characterizes the diversity multiplexing tradeoff in multiple access channels, when all users have the same diversity requirements.

We now introduce the families of functions $\mathcal{C}onv$ and $\mathcal{EC}onv$ below which will help us introduce our last requirement.

Definition 1. *$\mathcal{C}onv$ is the family of all strictly monotonic increasing, continuous and convex functions g.*

Definition 2. *$\mathcal{EC}onv$ is the family of all strictly monotonic increasing, continuous function g, such that $g(e^x)$ is convex.*

$\mathcal{C}onv \subsetneq \mathcal{EC}onv$. Infact, $\mathcal{EC}onv$ is much larger the $\mathcal{C}onv$. If a utility function g is in the class $\mathcal{EC}onv$, then it has the property that $g(e^x)$ is convex. The example $g(x) = \log x$, which is frequently encountered in wireless communication systems shows that even a concave function could be transformed into a convex function. Hence, we would like to investigate the possibility of ensuring convexity for the class of $\mathcal{EC}onv$ functions. For this purpose we introduce our last requirement, which expects joint convexity of functions of inverse SINR, which are frequently encountered in wireless systems, e.g. MMSE: $g(x) = 1/(1+x)$ and high-SNR approximation of BER $g(x) = x^{-\alpha}$ with diversity order α.

Requirement 3. For all functions $g \in \mathcal{EC}onv$, the corresponding function

$$\sum_{k\in\mathcal{K}} \omega_k g\Big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\Big),$$

where $\sum_{k\in\mathcal{K}} \omega_k = 1$ is jointly convex wrt. $\boldsymbol{s}$.

The function $\psi(s) = e^s$ is one such function satisfying requirement 3. We now present a result, which shows the only transformation (from the s–domain to power and from the utility domain to SINR) permitted under certain conditions, if we expect the supportable QoS region to be convex for all linear interference functions.

Theorem 1. *Function ψ satisfies requirements 1 and 2, if and only if there exists a $\mu, c > 0$ such that $\psi(s_k) = c\exp(\mu s_k)$, $s \in \mathbb{R}$, for $1 \le k \le K+1$.*

Proof: "$\Longleftarrow$": This direction can be easily verified.

"$\Longrightarrow$": From the assumptions of the theorem, we have that for all linear interference functions, the supportable QoS region (without power control) should be convex wrt. $\boldsymbol{s}$, i.e. $\big(\frac{1}{\gamma_k}\big)^\alpha = \big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\big)^\alpha$, is a convex function wrt. $\boldsymbol{s} = [s_1, \dots, s_K]$. We shall investigate the 2 user case, without any loss of generality. Therefore, we check for the convexity of $\big(\frac{\psi(s_2)}{\psi(s_1)}\big)^\alpha$, for a certain fixed $\alpha > 0$. We fix the power of user 2. Hence, we fix the value s_2 and check for the convexity of $1/(\psi(s_1))^\alpha$, for all $\alpha > 0$. Then, we have that

$$(\psi'(s_1))^2 - \frac{1}{\alpha+1}\psi''(s_1)\psi(s_1) \ge 0. \tag{4}$$

Taking the limit of $\alpha \to 0$, we obtain

$$(\psi'(s_1))^2 - \psi''(s_1)\psi(s_1) \ge 0. \tag{5}$$

From requirement 1 we have that the feasible QoS region is convex for all linear interference functions. Then, from Theorem 1 in [6] we have that ψ is log–convex and

$$(\psi'(s_1))^2 - \psi''(s_1)\psi(s_1) \le 0. \tag{6}$$

From (5) and (6) we have that

$$(\psi'(s))^2 - \psi''(s)\psi(s) = 0. \tag{7}$$

If ψ is a solution of (7), with $\psi(s) > 0$ for $s \in \mathbf{S}$, then we have that $(\psi'(s))^2 - \psi''(s)\psi(s)/(\psi(s))^2 = 0$. This gives us $\frac{d}{ds}\big(\frac{\psi'(s)}{\psi(s)}\big) = 0$, i.e. $\frac{\psi'(s)}{\psi(s)} = \mu$, with $\mu > 0$. Therefore, $\frac{d\psi}{\psi} = \mu ds$, i.e. $\psi(s) = c\exp(\mu s)$. Furthermore, we have that $\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)} = \frac{c\mathcal{I}_k(\exp(\mu\boldsymbol{s}))}{c\exp(\mu s_k)} = \frac{\mathcal{I}_k(\exp(\mu\boldsymbol{s}))}{\exp(\mu s_k)}$, so we can choose $c = 1$. ■

Historically, there have been a plethora of different motivations for utilizing the log–scale for measuring power in communication systems, e.g.. the logarithmic nature means that a very large range of ratios can be represented by a convenient number, e.g. Bode Plot. Theorem 1, provides another reason as to why it is advantageous to work in the log–domain, instead of the power domain.

We have proved Theorem 1, when we can scale the noise. Similarly, we can prove for the noise free case. We shall now analyze the case, when we have noise and we do not scale the noise. Consider the function

$$\frac{\sum_{j\in\mathcal{K}\setminus k} v_{kj}\psi(s_j)+v_{k(K+1)}\sigma^2}{\psi(s_k)} \tag{8}$$

As $v_{k(K+1)} \rightarrow 0$, (8) tends to function in the noise free case. We have already shown, that with the above proof technique we can prove the Theorem for the noise free case. Furthermore, we know that the limit function of a sequence of convex functions is convex, giving us our desired result. We prove all theorems throughout the paper, with noise scaling w.l.o.g. A function f is convex, if and only if the function $-f$ is concave. Hence, it is important to check for the convexity of the SINR. Since, it is different as checking for the convexity of SINR and the concatenation of a convex and a concave function need not be convex.

Theorem 1 presents a result, from the perspective an arbitrary user k. We now extend the result to a system level perspective in the Theorem 2 below.

Theorem 2. *Function ψ satisfies requirements 1 and 3, if and only if there exists scalars $c,\mu>0$ such that $\psi(s_k)=c\exp(\mu s_k)$, where $s\in\mathbb{R}$, for $1\le k\le K+1$.*

Proof: We know that under requirement 1, that for all $\alpha>0$ the function $\big(\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)\big)^\alpha$ is convex, if and only if $\psi(s_k)=c\exp(\mu s_k)$ for $\mu>0$ (from Theorem 1). Therefore, it is sufficient to prove that for all $g\in\mathcal{EC}onv$ the function $\sum_{k\in\mathcal{K}}\omega_k g(\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k))$ with $\sum_{k\in\mathcal{K}}\omega_k=1$ is convex, if and only if $\big(\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)\big)^\alpha$ for $\alpha>0$ is convex.

"$\Longleftarrow$": This direction can be easily verified. Hence, we skip the proof.

"$\Longrightarrow$": We know that requirements 1 and 3 are satisfied. We can choose $g(x)=x^\alpha$. Then, $\sum_{k\in\mathcal{K}}\omega_k\big(\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)\big)^\alpha$, with $\sum_{k\in\mathcal{K}}\omega_k=1$ and $\boldsymbol{\omega}>\mathbf{0}$ is jointly convex wrt. $\boldsymbol{s}$, for all $\alpha>0$ Let us choose weight vectors as follows:

$$\omega_k^{(n)}=\begin{cases}1-\frac{1}{n} & k=j\\ \frac{1}{(K-1)n} & k\neq j\end{cases}$$

Taking the limit as n tends to ∞, we obtain

$$\Big(\frac{\mathcal{I}_j(\psi(\boldsymbol{s}))}{\psi(s_j)}\Big)^\alpha=\lim_{n\to\infty}\sum_{k\in\mathcal{K}}\omega_k^{(n)}\Big(\frac{\mathcal{I}_k(\psi(\boldsymbol{s}))}{\psi(s_k)}\Big)^\alpha.$$

The limit function of a sequence of convex function is convex. Therefore, $\big(\mathcal{I}_j(\psi(\boldsymbol{s}))/\psi(s_j)\big)^\alpha$ is jointly convex in $\boldsymbol{s}$. Therefore, requirements 1 and 2 are satisfied. Then, from Theorem 1 we have our desired result. ∎

From Theorems 1 and 2, we have an equivalence between requirements 2 and 3. We have established, that for linear interference functions the unique transformation that allows us to obtain convex optimization problems is the exponential function. Furthermore, we have shown that the exponential function is the unique mapping if we would like the the very natural and practical requirement (3) to be satisfied. We would now like to extend certain of our developed intuition to beyond the framework of linear interference functions.

IV. Analysis of Resource Allocation Problems – Beyond Linear Interference Functions

In this section we shall extend certain results obtained for linear interference functions to a larger class of interference functions. We are interested in finding the largest class of interference functions, which allow us to apply convex optimization techniques to certain non–convex problems. We tackle the following problem.

What should the structure of the interference functions $\mathcal{I}_k(e^{\boldsymbol{s}})$ embody for all $k\in\mathcal{K}$, such that for all convex, continuous and increasing functions $g_1,\dots,g_K$ and for all weight vectors $\boldsymbol{\omega}>\mathbf{0}$, the function

$$\sum_{k\in\mathcal{K}}\omega_k g_k\big(\frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}\big) \tag{9}$$

is jointly convex wrt. $\boldsymbol{s}\in\mathbb{R}^{K+1}$?

The function (9) is a weighted sum of functions of inverse SINR in the log–domain. Hence it plays the role of a loss function in wireless systems. Intuitively, while tackling such a problem we would like to minimize such a function so as to optimize the satisfaction of the users in the system. We now present a result, which inform us as to when, such a function can be optimized by means of a convex optimization techniques.

Theorem 3. *The function (9) is jointly convex wrt. $\boldsymbol{s}\in\mathbb{R}^{K+1}$ for all weight vectors $\boldsymbol{\omega}>\mathbf{0}$ and for all convex, continuous and increasing functions $g_1,\dots,g_K$, if and only if the functions $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ for all $k\in\mathcal{K}$ are jointly convex wrt. $\boldsymbol{s}\in\mathbb{R}^{K+1}$.*

Proof: "$\Longleftarrow$": This direction can be easily verified. We know that $g(\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k})$ is convex, when g and $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ are convex functions. Furthermore, since the weighted sum of convex functions is convex, we can obtain our desired result.

"$\Longrightarrow$": We have that the function (9) is convex for all weight vectors $\boldsymbol{\omega}>\mathbf{0}$ and for all convex, continuous and increasing functions $g_1,\dots,g_K$. Choose $g_k(x)=x$ for all $k\in\mathcal{K}$. Let us choose weight vectors as in the proof of Theorem 2 and following the same direction as in the proof of Theorem 2 we obtain our desired result. ∎

The largest class of interference functions, resulting in convex resource allocation problems, would be equal to or larger, than *log–convex* interference functions. *Log–convexity* is a useful property that allows one to apply convex optimization techniques to certain non–convex problems.

Definition 3. *Log–convex* interference function: An interference function $\mathcal{I}:\mathbb{R}_+^{K+1}\mapsto\mathbb{R}_+$ is said to be a *log–convex* interference function if $A1-A4$ are fulfilled and $\mathcal{I}(\exp\{\boldsymbol{s}\})$ is log-convex on $\mathbb{R}^{K+1}$.

Let $f(\boldsymbol{s}):=\mathcal{I}(\exp\{\boldsymbol{s}\})$. The function $f:\mathbb{R}^{K+1}\mapsto\mathbb{R}_+$ is *log–convex* on $\mathbb{R}^K$ if and only if $\log f$ is convex or equivalently $f(\boldsymbol{s}(\lambda))\le f(\boldsymbol{s}^{(1)})^{1-\lambda}f(\boldsymbol{s}^{(2)})^{1-\lambda}$, for all $\lambda\in(0,1)$, $\boldsymbol{s}^{(1)},\boldsymbol{s}^{(2)}\in\mathbb{R}^K$, where $\boldsymbol{s}(\lambda)=(1-\lambda)\boldsymbol{s}^{(1)}+\lambda\boldsymbol{s}^{(2)}$, $\lambda\in(0,1)$. Note that the *log–convexity* in Definition 3 is based on a change of variable $\underline{\boldsymbol{p}} = \exp\{\boldsymbol{s}\}$ (component–wise exponential). Such a technique has been previously used to exploit a "hidden convexity" of functions, which are otherwise non-convex.

If interference functions $\mathcal{I}_1, \dots, \mathcal{I}_K$ are *log–convex* interference functions, then the function $\tilde{F}$ satisfies $\tilde{F}(\boldsymbol{s}, \boldsymbol{\omega}) = \sum_{k\in\mathcal{K}} \omega_k \log \frac{e^{s_k}}{\mathcal{I}_k(e^{\boldsymbol{s}})} = -\sum_{k\in\mathcal{K}} \omega_k \log \frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}$ and is concave. We now present a result for log–convex interference functions. Let $\mathcal{I}$ be a log–convex interference function. Then for all weight vectors $\boldsymbol{\omega} \geq \boldsymbol{0}$ let

$$f_{\mathcal{I}}(\boldsymbol{\omega}) = \inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \frac{\mathcal{I}(\underline{\boldsymbol{p}})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l}}. \quad (10)$$

If $f_{\mathcal{I}}(\boldsymbol{\omega}) > 0$, then $\sum_{l\in\mathcal{K}} \omega_l = 1$ [8, Lemma 3, pp. 5474]. We have that $\mathcal{I}(\underline{\boldsymbol{p}}) = \sup_{\boldsymbol{\omega}: f_{\mathcal{I}}(\boldsymbol{\omega})>0} f_{\mathcal{I}}(\boldsymbol{\omega}) \prod_{l\in\mathcal{K}} (\underline{p}_l)^{\omega_l}$. Furthermore, when $\mathcal{I}_k$ is a log–convex interference function for all $k \in \mathcal{K}$ then $\frac{1}{\gamma_k(e^{\boldsymbol{s}})} = \frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}$ is convex wrt. $s \in \mathbb{R}^K$.

From Theorems 1 and 2, we know that for all strictly monotonic increasing, continuous and convex functions g and for all weight vectors $\boldsymbol{\omega} > \boldsymbol{0}$ – it is sufficient to check for the joint convexity of the function $\sum_{k\in\mathcal{K}} \omega_k g\big(\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)\big)$ wrt. $\boldsymbol{s}$. Hence, we define a new class of interference functions below:

Definition 4. $\mathcal{C}$ *interference functions*: A general interference function $\mathcal{I}_k$ is said to a be $\mathcal{C}$ interference function if the function $\mathcal{I}_k(\psi(\boldsymbol{s}))/\psi(s_k)$ is jointly convex wrt. $\boldsymbol{s}$, where $\psi(s_k) = e^{\mu s_k}$, for $k \in \mathcal{K}$.

$\mathcal{C}$ is the largest class of interference functions, permitting the use of convex optimization techniques to solve certain non–convex problems. This class of interference functions is a larger class than log–convex interference functions. The inclusion order of the families of interference functions is as follows: Convex $\subset$ Log–Convex $\subset \mathcal{C} \subset$ General Interference Functions. Furthermore, *log–convex* interference functions are not dense with respect to the set of $\mathcal{C}$ *interference functions.* We now define the following function, which we shall utilize in analyzing the convexity of the function $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$.

$$\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) := \inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l}}. \quad (11)$$

Lemma 1. *The function $\tilde{f}_{\mathcal{I}_k}$ defined by (11) is log–concave, for all $k \in \mathcal{K}$.*

Proof: Let $\boldsymbol{\omega}^{(1)}, \boldsymbol{\omega}^{(2)}$ be arbitrary weight vectors, such that $\boldsymbol{\omega}(\lambda) = (1-\lambda)\boldsymbol{\omega}^{(1)} + \lambda\boldsymbol{\omega}^{(2)}$. Then, we have that

$$\begin{aligned}
\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}(\lambda)) &= \inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\big(\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(1)}}\big)^{1-\lambda}\big(\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(2)}}\big)^{\lambda}} \\
&= \inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \Big(\frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(1)}}}\Big)^{1-\lambda}\Big(\frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(2)}}}\Big)^{\lambda} \\
&\geq \underbrace{\Big(\inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}}\Big(\frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(1)}}}\Big)\Big)}_{\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}^{(1)})}{}^{1-\lambda} \\
&\qquad \underbrace{\Big(\inf_{\underline{\boldsymbol{p}}>\boldsymbol{0}}\Big(\frac{\exp(\frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k})}{\prod_{l=1}^K (\underline{p}_l)^{\omega_l^{(2)}}}\Big)\Big)}_{\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}^{(2)})}{}^{\lambda}.
\end{aligned}$$

■

Lemma 2. *If the function $\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) > 0$, where $\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})$ is defined in (11), then $\sum_{l\in\mathcal{K}} \omega_l = 0$.*

Proof: We continue from the proof of Lemma 1 and analyze the following expression.

$$\begin{aligned}
\hat{g}_k(\boldsymbol{\omega}) &= \sup_{\boldsymbol{s}\in\mathbb{R}^K} \Big(\sum_{l\in\mathcal{K}} \omega_l s_l - \frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}\Big) \\
&= \sup_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \Big(\sum_{l\in\mathcal{K}} \omega_l \log \underline{p}_l - \frac{\mathcal{I}_k(\underline{\boldsymbol{p}})}{\underline{p}_k}\Big) \\
&= \sup_{\underline{\boldsymbol{p}}>\boldsymbol{0}} \Big(\log \frac{\prod_{l=1}^K (\underline{p}_l)^{\omega_l}}{\exp(\mathcal{I}_k(\underline{\boldsymbol{p}})/\underline{p}_k)}\Big) \\
&= \log \frac{1}{\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})}. \qquad (12)
\end{aligned}$$

Then we have that $\tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) > 0$ if and only if $\hat{g}_k(\boldsymbol{\omega}) < +\infty$. Let us assume that $\sum_{l\in\mathcal{K}} \omega_l > 0$. There exists an arbitrary $\hat{\boldsymbol{s}} \in \mathbb{R}^{K+1}$ and a scalar $\lambda \in \mathbb{R}$ arbitrarily chosen such that

$$\begin{aligned}
\hat{g}_k(\boldsymbol{\omega}) &\geq \sum_{l\in\mathcal{K}} \omega_l(\hat{s}_l + \lambda) - \big(\mathcal{I}_k(e^{(\hat{\boldsymbol{s}}+\lambda\boldsymbol{1})})/e^{\hat{s}_k}e^{\lambda}\big) \\
&= \Big(\lambda \underbrace{\sum_{l\in\mathcal{K}} \omega_l}_{c_1} + \sum_{l\in\mathcal{K}} \omega_l \hat{s}_l - \big(\mathcal{I}_k(e^{\hat{\boldsymbol{s}}})/e^{\hat{s}_k}\big)\Big).
\end{aligned}$$

Furthermore, we have that

$$\hat{g}_k(\boldsymbol{\omega}) \geq \sup_{\lambda>0} \big(\lambda c_1 + \sum_{l\in\mathcal{K}} \omega_l \hat{s}_l - \big(\mathcal{I}_k(e^{\hat{\boldsymbol{s}}})/e^{\hat{s}_k}\big)\big) = +\infty.$$

With analogous arguments for $\sum_{k\in\mathcal{K}} \omega_k < 0$ we can conclude that $\hat{g}_k(\boldsymbol{\omega}) \geq \sup_{\lambda<0} \big(\lambda c_1 + \sum_{l\in\mathcal{K}} \omega_l \hat{s}_l - \big(\mathcal{I}_k(e^{\hat{\boldsymbol{s}}})/e^{\hat{s}_k}\big)\big) = +\infty$. We have our required contradiction, proving the desired result. ■

As can be seen from Lemma 2 the weight vectors $\boldsymbol{\omega}$ can be less than zero. Although this might seem surprising at first glance it is clearly justified from the fact that we are now concerned with the optimization of function $\tilde{f}_{\mathcal{I}}(\boldsymbol{\omega})$, which has the SINR as an inverse argument. We are investigating the case in the $\boldsymbol{s}$–domain ($\boldsymbol{p} = e^{\boldsymbol{s}}$). Hence, a more negative weight implies a larger SINR in the power domain.

Theorem 4. *For all* $s \in \mathbb{R}^{K+1}$ *the function* $g_k(\boldsymbol{s}) := \mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ *is convex, if and only if*

$$\mathcal{I}_k(e^{\boldsymbol{s}}) = e^{s_k} \log h_{\mathcal{I}_k}(\boldsymbol{s}), \quad k \in \mathcal{K}, where \tag{13}$$

$$h_{\mathcal{I}_k}(\boldsymbol{s}) := \sup_{\boldsymbol{\omega}: \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})>0} \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) \prod_{l \in \mathcal{K}} (e^{s_l \omega_l}). \tag{14}$$

Proof: "$\Longleftarrow$": Let us assume that (13) holds. Then, we have that $g_k(\boldsymbol{s}) = \mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k} = \log h_{\mathcal{I}_k}(\boldsymbol{s})$ is convex.
"$\Longrightarrow$": We have chosen in the proof of Lemma 2 that $\hat{g}_k(\boldsymbol{\omega}) = \sup_{s \in \mathbb{R}^K}(\sum_{l \in \mathcal{K}} \omega_l s_l - g_k(\boldsymbol{s}))$. Then from duality results in [9] we have that $g_k(\boldsymbol{s}) = \sup_{\boldsymbol{\omega}: \hat{g}_k(\boldsymbol{\omega})<\infty}(\sum_{l \in \mathcal{K}} \omega_l s_l - \hat{g}_k(\boldsymbol{\omega})) = \mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$. Then, we have that

$$\begin{aligned}
\mathcal{I}_k(\underline{\boldsymbol{p}}) &= \underline{p}_k \sup_{\boldsymbol{\omega}: \hat{g}_k(\boldsymbol{\omega})<+\infty} (\sum_{l \in \mathcal{K}} \omega_l \log \underline{p}_l - \hat{g}_k(\boldsymbol{\omega})) \\
&= \underline{p}_k \sup_{\boldsymbol{\omega}: \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})>0} (\sum_{l \in \mathcal{K}} \omega_l \log \underline{p}_l + \log \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})) \\
&= \underline{p}_k \log(\sup_{\boldsymbol{\omega}: \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega})>0} \tilde{f}_{\mathcal{I}_k}(\boldsymbol{\omega}) \prod_{l=1}^{K} (\underline{p}_l)^{\omega_l}).
\end{aligned}$$

After transformation to the $\boldsymbol{s}$–domain, we have our desired result. ■

We notice in Theorem 4, that $\mathcal{I}_k(e^{\boldsymbol{s}})$ is convex, which is stronger than the condition that it is log–convex, since $\log h_{\mathcal{I}_k}(\boldsymbol{s})$ is convex. We see that log–convexity plays a significant role in the analysis.

Theorem 4 has characterized the class of interference functions $\mathcal{C}$, which leads to the inverse SINR function $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ to be convex in the $\boldsymbol{s}$–domain, for all users $k \in \mathcal{K}$. This class of interference functions is much broader than the class of *log–convex* interference functions. $\tilde{f}_{\mathcal{I}}$ has been introduced for the purpose of investigating the convexity properties of SINR. It should be noted that the function $\tilde{f}_{\mathcal{I}}$ is inherently based on the properties of the underlying interference functions.

From Theorems 3 and 4 we can deduce that *the function (9) is jointly convex wrt.* $\boldsymbol{s} \in \mathbb{R}^{K+1}$ *for all weight vectors* $\boldsymbol{\omega} > \boldsymbol{0}$ *and for all convex, continuous and increasing functions* $g_1, \ldots, g_K$*, if and only if the functions* $\mathcal{I}_k(e^{\boldsymbol{s}})/e^{s_k}$ *for all* $k \in \mathcal{K}$ *possess the structure defined by (13).*

Example 1. Consider the function $q_\alpha(x) = x$, for $\alpha \geq 1$. Consider the case, when we are interested in minimizing the function $\sum_{k \in \mathcal{K}} \omega_k \big(\frac{\mathcal{I}_k(e^{\boldsymbol{s}})}{e^{s_k}}\big)^{\alpha_k}$, with $\alpha_k \geq 1$, for all users $k \in \mathcal{K}$. Such a problem is met in the form of minimizing the weighted probability of errors. Here, the probability of error for user k, $k \in \mathcal{K}$.with diversity order α_k for user k can be approximated as $1/\big(e^{s_k}/\mathcal{I}_k(e^{\boldsymbol{s}})\big)^{\alpha_k}$. We see an example of such a function in [10]. A strategy for system resources by joint optimization of transmit powers and beamformers for minimizing the sum of weighted inverse SIR was considered. It was shown, how the weighting factors should be chosen so that the sum optimization approach achieves optimal max–min fairness [7].

V. Appendix: Interference Functions

Definition 5. *Interference functions*: We say that $\mathcal{I} : \mathbb{R}_+^{K+1} \mapsto \mathbb{R}_+$ is an *interference function* if the following axioms are fulfilled:

$A1$	conditional positivity $\mathcal{I}(\underline{\boldsymbol{p}}) > 0$ if $\underline{\boldsymbol{p}} > \boldsymbol{0}$
$A2$	scale invariance $\mathcal{I}(\alpha \underline{\boldsymbol{p}}) = \alpha \mathcal{I}(\underline{\boldsymbol{p}}), \forall \alpha \in \mathbb{R}_+$
$A3$	monotonicity $\mathcal{I}(\underline{\boldsymbol{p}}) \geq \mathcal{I}(\underline{\hat{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \underline{\hat{\boldsymbol{p}}}$
$A4$	strict monotonicity $\mathcal{I}(\underline{\boldsymbol{p}}) > \mathcal{I}(\underline{\hat{\boldsymbol{p}}})$ if $\underline{\boldsymbol{p}} \geq \underline{\hat{\boldsymbol{p}}}$, $\underline{p}_{K+1} > \underline{\hat{p}}_{K+1}$.

Note that we require that $\mathcal{I}(\underline{\boldsymbol{p}})$ is *strictly monotone* wrt. the last component $\underline{p}_{K+1}$. An example is $\mathcal{I}(\underline{\boldsymbol{p}}) = \boldsymbol{v}^T \underline{\boldsymbol{p}} + \sigma^2$, where $\boldsymbol{v} \in \mathbb{R}_+^K$ is a vector of interference coupling coefficients. The axiomatic framework A1-A4 is connected with the framework of *standard interference functions* [2]. The details about the relationship between the model A1-A4 and Yates' *standard interference functions* were discussed in [4] and further investigated in [11]. For the purpose of this paper it is sufficient to be aware that there exists a connection between these two models and the results of this paper are applicable to *standard interference functions*.

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