This full text paper was peer reviewed at the direction of IEEE Communications Society subject matter experts for publication in the IEEE Globecom 2010 proceedings.

MIMO Bidirectional Broadcast Channels with Common Message

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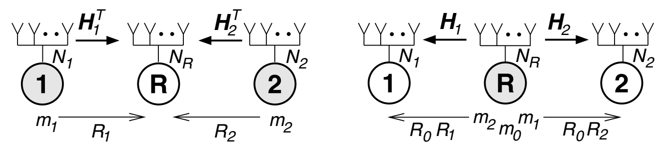
Abstract—**In this work, we study the MIMO Gaussian *bidirectional broadcast channel (BBC) with common message* and characterize the capacity region. Moreover, we show that the transmit covariance matrix optimization problem has the same structure as the corresponding optimization problem of the BBC without common message which leads to the comfortable position to transfer results from one scenario to the other. This problem is motivated by the concept of bidirectional relaying in a three-node network, where a half-duplex relay node establishes a bidirectional communication between two other nodes and thereby adds an own multicast message to the communication.**



(a) MIMO MAC phase (b) MIMO BBC phase

Fig. 1. Decode-and-forward bidirectional relaying with multiple antennas at all nodes. In the initial multiple access (MAC) phase, nodes 1 and 2 transmit their messages m_1 and m_2 with rates R_1 and R_2 to the relay node. In the succeeding bidirectional broadcast (BBC) phase, the relay forwards the messages m_1 and m_2 with rates R_1 and R_2 and adds its own common message m_0 with rate R_0 to the communication.

I. INTRODUCTION

The concept of bidirectional relaying turns out to be a key technique to improve the performance in future wireless networks such as sensor, ad-hoc, and even cellular systems. It applies to three-node networks, where a half-duplex relay node establishes a bidirectional communication between two other nodes. Moreover, since spatial MIMO techniques can improve the performance significantly [1], we assume multiple antennas at all nodes as shown in Figure 1.

In this work we consider the broadcast phase of a two-phase decode-and-forward protocol, where the relay has successfully decoded both messages the nodes have sent in the previous multiple access (MAC) phase. Here, the relay re-encodes and transmits both messages and an additional common message in such a way that each receiving node can decode the other's message and the common message using its own message from the previous phase as side information. This is the *bidirectional broadcast channel (BBC) with common message*.

The problem of jointly broadcasting bidirectional and multicast information arises for example in car-to-car communication. A reasonable situation would be if two cars locally share information among themselves via the relay such as speed and direction, while also receiving globally broadcast information such as traffic or road conditions.

Furthermore, the joint consideration of different types of communication is also motivated by the fact that there is a trend to merge coexisting wireless services such that they work on the same wireless resources. This convergence promises an increased spectral efficiency together with a significantly reduced complexity.

The BBC without common message is widely studied. Capacity achieving strategies can be found, for instance, in [2–5] for discrete memoryless channels and in [6] for MIMO Gaussian channels. The concept of bidirectional relaying and its extensions are subject of further research activities, e.g., confer [7] for a survey of different processing strategies. Optimal beamforming strategies for multi-antenna bidirectional relaying with analogue network coding is analyzed in [8]. In [9] extensions to the case where the relay supports the communication of multiple pairs of users are presented. Bidirectional relaying with an additional private message for the relay node in the MAC phase is addressed in [10]. Some work on the SISO Gaussian broadcast channel with common message and certain side information at the receivers can be found in [11] and [5] where the latter assumes degraded message sets. A general model for multi-user settings with correlated sources is given in [12].[1]

II. MIMO BIDIRECTIONAL BROADCAST PHASE

We assume N_R antennas at the relay node and N_k antennas at node k, $k = 1, 2$, as shown in Figure 1. Then, the discrete-time complex-valued input-output relation between the relay

The work of T. J. Oechtering was supported by the European Commission through the FP7 project FeedNetBack. The work of H. Boche was partly supported by the German Research Foundation (DFG) under Grant BO 1734/25-1.

[1]*Notation:* Random variables are denoted by capital letters and their corresponding realizations by lower case letters; bold capital and bold lower case letters denote matrices and vectors; $(\cdot)^{-1}$, $(\cdot)^T$, and $(\cdot)^H$ denote inverse, transpose, and Hermitian transpose, respectively; $\mathrm{tr}(\cdot)$ is the trace of a matrix; $\boldsymbol{Q} \succeq 0$ means the matrix $\boldsymbol{Q}$ is positive semidefinite; $\mathbb{E}\{\cdot\}$ is the expectation.

node and node k, $k = 1, 2$, is given by

$$\boldsymbol{y}_k = \boldsymbol{H}_k \boldsymbol{x} + \boldsymbol{n}_k, \tag{1}$$

where $\boldsymbol{y}_k \in \mathbb{C}^{N_k \times 1}$ denotes the output at node k, $\boldsymbol{H}_k \in \mathbb{C}^{N_k \times N_R}$ the multiplicative channel matrix, $\boldsymbol{x} \in \mathbb{C}^{N_R \times 1}$ the input of the relay node, and $\boldsymbol{n}_k \in \mathbb{C}^{N_k \times 1}$ the independent additive noise according to a circular symmetric complex Gaussian distribution $\mathcal{CN}(\boldsymbol{0}, \sigma^2 \boldsymbol{I}_{N_k})$. We assume perfect channel state information at all nodes and an average transmit power constraint $\text{tr}(\boldsymbol{Q}) \leq P$ with $\boldsymbol{Q} = \mathbb{E}\{\boldsymbol{x}\boldsymbol{x}^H\}$.

If there is no common message for the relay to transmit, then we know from [6] that for a given covariance matrix $\boldsymbol{Q}$, zero-mean circular symmetric complex Gaussian distributed input is optimal, this means $\boldsymbol{x} \sim \mathcal{CN}(\boldsymbol{0}, \boldsymbol{Q})$. A direct consequence is the following theorem.

Theorem 1 ([6]): The capacity region of the MIMO Gaussian BBC without common message and with average power constraint P is given by

$$\bigcup_{\boldsymbol{Q}:\, \text{tr}(\boldsymbol{Q}) \leq P,\, \boldsymbol{Q} \succeq 0} \mathcal{R}'(\boldsymbol{Q})$$

with

$$\mathcal{R}'(\boldsymbol{Q}) := \{(R_1', R_2') \in \mathbb{R}_+^2 : R_k' \leq C_k(\boldsymbol{Q}), k = 1, 2\}$$

and $C_k(\boldsymbol{Q}) = \log\det(\boldsymbol{I}_{N_k} + \frac{1}{\sigma^2}\boldsymbol{H}_k \boldsymbol{Q} \boldsymbol{H}_k^H)$.

III. Enabling Additional Common Message

We consider the standard model with a block code of arbitrary but sufficiently long block length n. Let $\mathcal{M}_k := \{1, ..., M_k^{(n)}\}$ be the individual message set of node k, $k = 1, 2$, which is also known at the relay node. Further, $\mathcal{M}_0 := \{1, ..., M_0^{(n)}\}$ is the common message set of the relay node.

Definition 1: A $(M_0^{(n)}, M_1^{(n)}, M_2^{(n)}, n)$*-code* for the MIMO Gaussian BBC with common message and average power constraint P consists of one encoder at the relay node

$$f : \mathcal{M}_0 \times \mathcal{M}_1 \times \mathcal{M}_2 \to \mathcal{X}^n$$

with $\mathcal{X}^n := \{(\boldsymbol{x}_1, \boldsymbol{x}_2, ..., \boldsymbol{x}_n) \in \mathbb{C}^{N_R \times n} : \frac{1}{n}\sum_{i=1}^n \boldsymbol{x}_i^H \boldsymbol{x}_i \leq P\}$ and corresponding decoders at nodes 1 and 2

$$g_1 : \mathbb{C}^{N_R \times n} \times \mathcal{M}_1 \to \mathcal{M}_0 \times \mathcal{M}_2 \cup \{0\},$$
$$g_2 : \mathbb{C}^{N_R \times n} \times \mathcal{M}_2 \to \mathcal{M}_0 \times \mathcal{M}_1 \cup \{0\}.$$

The element 0 in the definition of the decoder plays the role of an erasure symbol and is included for convenience only.

When the relay has sent the message $m = (m_0, m_1, m_2)$, and nodes 1 and 2 have received $\boldsymbol{y}_1^n$ and $\boldsymbol{y}_2^n$, the decoder at node 1 is in error if $g_1(\boldsymbol{y}_1^n, m_1) \neq (m_0, m_2)$. Accordingly, the decoder at node 2 is in error if $g_2(\boldsymbol{y}_2^n, m_2) \neq (m_0, m_1)$. Then, the average probability of error at node k is given by

$$\mu_k^{(n)} := \frac{1}{M_0^{(n)} M_1^{(n)} M_2^{(n)}} \sum_{m_0=1}^{M_0^{(n)}} \sum_{m_1=1}^{M_1^{(n)}} \sum_{m_2=1}^{M_2^{(n)}} \lambda_k(m_0, m_1, m_2)$$

where $\lambda_k(m_0, m_1, m_2)$ denotes the probability that decoder k decodes incorrectly, $k = 1, 2$.

Definition 2: A rate triple (R_0, R_1, R_2) is said to be *achievable* for the MIMO Gaussian BBC with common message and average power constraint P if for any $\delta > 0$ there exists an $n(\delta) \in \mathbb{N}$ and a sequence of $(M_0^{(n)}, M_1^{(n)}, M_2^{(n)}, n)$-codes satisfying the power constraint P such that for all $n \geq n(\delta)$ we have $\frac{\log M_0^{(n)}}{n} \geq R_0 - \delta$, $\frac{\log M_2^{(n)}}{n} \geq R_1 - \delta$, and $\frac{\log M_1^{(n)}}{n} \geq R_2 - \delta$ while $\mu_1^{(n)}, \mu_2^{(n)} \to 0$ as $n \to \infty$. The set of all achievable rate triples is the *capacity region of the MIMO Gaussian BBC with common message*.

Now, we can state the capacity region of the MIMO Gaussian BBC with common message. Therefore we define

$$\mathcal{R}(\boldsymbol{Q}) := \big\{(R_0, R_1, R_2) \in \mathbb{R}_+^3 : R_0 + R_1 \leq C_1(\boldsymbol{Q}) \tag{2a}$$
$$R_0 + R_2 \leq C_2(\boldsymbol{Q})\big\} \tag{2b}$$

with $C_k(\boldsymbol{Q}) = \log\det(\boldsymbol{I}_{N_k} + \frac{1}{\sigma^2}\boldsymbol{H}_k \boldsymbol{Q} \boldsymbol{H}_k^H)$, $k = 1, 2$.

Remark 1: Clearly, the sum constraints (2a) and (2b) in the definition of $\mathcal{R}(\boldsymbol{Q})$ immediately implies that the rate of the common message has to fulfill $R_0 \leq \min\{C_1(\boldsymbol{Q}), C_2(\boldsymbol{Q})\}$.

Theorem 2: The capacity region of the MIMO Gaussian BBC with common message and average power constraint P is given by

$$\mathcal{C}_{\text{BBC}}^{\text{MIMO}} = \bigcup_{\boldsymbol{Q}:\, \text{tr}(\boldsymbol{Q}) \leq P,\, \boldsymbol{Q} \succeq 0} \mathcal{R}(\boldsymbol{Q}). \tag{3}$$

Since the $\log\det$ function is concave in $\boldsymbol{Q}$, the region in (3) is already convex. And consequently, an additional time-sharing operation will not enlarge the region.

A. Proof of Achievability

It suffices to show that for given covariance matrix $\boldsymbol{Q}$ all rate triples $(R_0, R_1, R_2) \in \mathcal{R}(\boldsymbol{Q})$ as specified in (2a)-(2b) are achievable. Then, (3) follows immediately by taking the union over all covariance matrices that satisfy $\text{tr}(\boldsymbol{Q}) \leq P$.

We extend the proof of achievability for the BBC without common message of Theorem 1, cf. [6, Sec. III-A], to our scenario by using the idea of [13] for the classical broadcast channel as outlined. The relay node encodes the messages (m_0, m_1, m_2) with rates (R_0, R_1, R_2). The decoder at node 1 has m_1 as side information and wants to decode $(m_0, m_2) = m_2'$ with rate $R_2' = R_0 + R_2$ and, similarly, the decoder at node 2 has m_2 and wants $(m_0, m_1) = m_1'$ with rate $R_1' = R_0 + R_1$. Following [6, Sec. III-A] we know that for given covariance matrix $\boldsymbol{Q}$ all rate pairs $(R_1', R_2') \in \mathbb{R}_+^2$ satisfying $R_k' \leq C_k(\boldsymbol{Q})$, $k = 1, 2$, are achievable. Thus, similar to [13], all rate triples $(R_0, R_1' - R_0, R_2' - R_0) = (R_0, R_1, R_2) \in \mathbb{R}_+^3$ with $R_0 \leq \min\{C_1(\boldsymbol{Q}), C_2(\boldsymbol{Q})\}$, cf. Remark 1, are also achievable for the MIMO Gaussian BBC with common message. ■

B. Proof of Weak Converse

We have to show that for any given sequence of $(M_0^{(n)}, M_1^{(n)}, M_2^{(n)}, n)$-codes with $\mu_1^{(n)}, \mu_2^{(n)} \to 0$ there exists a covariance matrix $\boldsymbol{Q}$ satisfying the average power constraint

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$\text{tr}(\boldsymbol{Q}) \leq P$ such that

$$R_0 + R_1 := \liminf_{n\to\infty} \frac{1}{n}(\log M_0^{(n)} + \log M_2^{(n)}) \leq C_1(\boldsymbol{Q})$$
$$R_0 + R_2 := \liminf_{n\to\infty} \frac{1}{n}(\log M_0^{(n)} + \log M_1^{(n)}) \leq C_2(\boldsymbol{Q})$$

are satisfied. For this purpose we need a version of Fano's lemma suitable for the MIMO Gaussian BBC with common message.

Lemma 1: For our context we have the following versions of Fano's inequality

$$H(M_0, M_2|\boldsymbol{Y}_1^n, M_1) \leq \mu_1^{(n)} \log(M_0^{(n)} M_2^{(n)}) + 1 = n\epsilon_1^{(n)},$$
$$H(M_0, M_1|\boldsymbol{Y}_2^n, M_2) \leq \mu_2^{(n)} \log(M_0^{(n)} M_1^{(n)}) + 1 = n\epsilon_2^{(n)}$$

with $\epsilon_1^{(n)} = \frac{\log(M_0^{(n)} M_2^{(n)})}{n}\mu_1^{(n)} + \frac{1}{n} \to 0$ and $\epsilon_2^{(n)} = \frac{\log(M_0^{(n)} M_1^{(n)})}{n}\mu_2^{(n)} + \frac{1}{n} \to 0$ for $n \to \infty$ as $\mu_1^{(n)}, \mu_2^{(n)} \to 0$.

The lemma can easily be shown using standard arguments as in [6, Lemma 2] but now including the common message.

With this, we can bound $H(M_0) + H(M_2)$ as follows

$$\begin{aligned} H(M_0) + H(M_2) &= H(M_0|M_1, M_2) + H(M_2|M_1) \\ &= H(M_0, M_2|M_1) \\ &= I(M_0, M_2; \boldsymbol{Y}_1^n|M_1) + H(M_0, M_2|\boldsymbol{Y}_1^n, M_1) \\ &\leq I(M_0, M_2; \boldsymbol{Y}_1^n|M_1) + n\epsilon_1^{(n)} \\ &\leq I(M_0, M_1, M_2; \boldsymbol{Y}_1^n) + n\epsilon_1^{(n)} \\ &\leq I(\boldsymbol{X}^n; \boldsymbol{Y}_1^n) + n\epsilon_1^{(n)} \end{aligned} \tag{4}$$

where the equalities and inequalities follow from the independence of M_0, M_1, and M_2, the chain rule for entropy, the definition of mutual information, Lemma 1, the chain rule for mutual information, the positivity of mutual information, and the data processing inequality. Accordingly, using the same arguments we also obtain

$$H(M_0) + H(M_1) \leq I(\boldsymbol{X}^n; \boldsymbol{Y}_2^n) + n\epsilon_2^{(n)}. \tag{5}$$

Note that (4) and (5) immediately imply that $H(M_0) \leq \min\{I(\boldsymbol{X}^n; \boldsymbol{Y}_1^n) + n\epsilon_1^{(n)}, I(\boldsymbol{X}^n; \boldsymbol{Y}_2^n) + n\epsilon_2^{(n)}\}$, cf. also Remark 1.

The rest of the proof is almost identical to [6, Sec. III-B] and follows from standard arguments. It only remains to bound the term $I(\boldsymbol{X}^n; \boldsymbol{Y}_k^n)$, $k = 1, 2$, in such a way that we obtain the well known and expected $\log\det$ expression. Exactly as in [6, Lemma 3] it can be shown that $\frac{1}{n}I(\boldsymbol{X}^n; \boldsymbol{Y}_k^n) \leq \log\det\big(\boldsymbol{I}_{N_k} + \frac{1}{\sigma^2}\boldsymbol{H}_k\big(\frac{1}{n}\sum_{i=1}^n \boldsymbol{Q}_i\big)\boldsymbol{H}_k^H\big)$. Following [6, Sec. III-B] this immediately leads to

$$R_0 + R_k \leq \log\det(\boldsymbol{I}_{N_k} + \tfrac{1}{\sigma^2}\boldsymbol{H}_k\boldsymbol{Q}\boldsymbol{H}_k^H), \quad k = 1, 2$$

which proves the weak converse of Theorem 2. ∎

C. Example

As an example, we consider the SIMO Gaussian BBC with common message, which means that the relay node is equipped with a single antenna, i.e., $N_R = 1$, while the two other nodes still have multiple antennas. Figure 2 depicts the

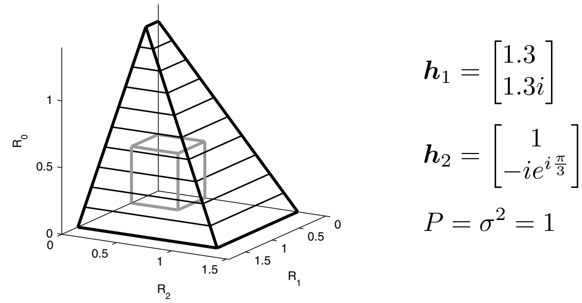


Fig. 2. Capacity region of the SIMO Gaussian BBC with common message (black) and a comparable TDMA approach (gray) with $N_R = 1$ and $N_1 = N_2 = 2$.

capacity region of the SIMO Gaussian BBC with common message and illustrates how the optimal strategy outperforms the simple TDMA approach which realizes the same routing task with three orthogonal time slots.

IV. Optimization Problem

Since the capacity region $\mathcal{C}_{\text{BBC}}^{\text{MIMO}}$ is convex, the rate triples on the dominant surface characterize the capacity region completely. Therefore, one is interested in finding the optimal transmit covariance matrices that achieve the rate triples on the dominant surface. Since such a rate triple is a solution of a weighted rate sum problem, we consider the corresponding convex optimization problem

$$R_\Sigma(\boldsymbol{w}) = \max_{R_0,R_1,R_2} \sum_{k=0}^{2} w_k R_k \tag{6a}$$
$$\text{s.t. } R_0 + R_k \leq C_k(\boldsymbol{Q}),\ k = 1, 2 \tag{6b}$$
$$R_k \geq 0,\ k = 0, 1, 2 \tag{6c}$$
$$\text{tr}(\boldsymbol{Q}) \leq P,\ \boldsymbol{Q} \succeq 0, \tag{6d}$$

with $\boldsymbol{w} = (w_0, w_1, w_2) \in \mathbb{R}_+^3$ the weight vector and $C_k(\boldsymbol{Q}) = \log\det(\boldsymbol{I}_{N_k} + \frac{1}{\sigma^2}\boldsymbol{H}_k\boldsymbol{Q}\boldsymbol{H}_k^H)$, $k = 1, 2$ in the following.

Obviously, in the optimum of (6) the constraints (6b) will be satisfied with equality. Since otherwise, if $R_0 + R_k < C_k(\boldsymbol{Q})$, we can increase the rate R_k up to the point where we have equality, i.e., $R_0 + R_k = C_k(\boldsymbol{Q})$, without affecting the other rates and therewith increasing the weighted rate sum $R_\Sigma(\boldsymbol{w})$. Consequently, (6) can be rewritten as

$$\max_{\boldsymbol{Q}, R_0} \quad (w_0 - w_1 - w_2)R_0 + w_1 C_1(\boldsymbol{Q}) + w_2 C_2(\boldsymbol{Q}) \tag{7}$$
$$\text{s.t.} \quad 0 \leq R_0 \leq C_k(\boldsymbol{Q}),\ k = 1, 2, \quad \text{tr}(\boldsymbol{Q}) \leq P,\ \boldsymbol{Q} \succeq 0.$$

Then, the Lagrangian for this optimization problem is

$$\begin{aligned} L(\boldsymbol{Q}, R_0, \boldsymbol{\nu}, \xi, \mu, \boldsymbol{\Psi}) = &-(w_0 - w_1 - w_2)R_0 - \sum_{k=1}^{2} w_k C_k(\boldsymbol{Q}) \\ &+ \nu_1\big(R_0 - C_1(\boldsymbol{Q})\big) + \nu_2\big(R_0 - C_2(\boldsymbol{Q})\big) \\ &- \xi R_0 + \mu\big(\text{tr}(\boldsymbol{Q}) - P\big) - \text{tr}(\boldsymbol{Q}\boldsymbol{\Psi}) \end{aligned}$$

with Lagrange multipliers $\xi, \mu \in \mathbb{R}$, $\boldsymbol{\nu} = (\nu_1, \nu_2) \in \mathbb{R}^2$, and $\boldsymbol{\Psi} \in \mathbb{C}^{N_R \times N_R}$, from which we get the Karush-Kuhn-Tucker conditions with $C'_k(\boldsymbol{Q}) = \boldsymbol{H}_k^H(\sigma^2\boldsymbol{I}_{N_k} + \boldsymbol{H}_k\boldsymbol{Q}\boldsymbol{H}_k^H)^{-1}\boldsymbol{H}_k$, $k = 1, 2$, as

$$\mu\boldsymbol{I}_{N_R} - \boldsymbol{\Psi} = (w_1+\nu_1)C'_1(\boldsymbol{Q})+(w_2+\nu_2)C'_2(\boldsymbol{Q}) \quad \text{(8a)}$$

$$w_0 = w_1 + w_2 + \nu_1 + \nu_2 - \xi \quad \text{(8b)}$$

$$R_0 \leq C_k(\boldsymbol{Q}),\ k = 1, 2 \quad \text{(8c)}$$

$$\boldsymbol{Q} \succeq 0, \qquad \text{tr}(\boldsymbol{Q}) \leq P \quad \text{(8d)}$$

$$\boldsymbol{\Psi} \succeq 0, \qquad \nu_1, \nu_2, \xi, \mu \geq 0 \quad \text{(8e)}$$

$$\text{tr}(\boldsymbol{Q}\boldsymbol{\Psi}) = 0, \qquad \mu(\text{tr}(\boldsymbol{Q}) - P) = 0 \quad \text{(8f)}$$

$$\xi R_0 = 0, \qquad \nu_k\big(R_0 - C_k(\boldsymbol{Q})\big) = 0,\ k = 1, 2 \quad \text{(8g)}$$

with primal, dual, and complementary slackness conditions (8c)-(8d), (8e), and (8f)-(8g) respectively.

Although the optimization problem (6) is a convex optimization problem and can therefore be efficiently solved using interior point method, it is worth to study its structure in more detail.

Theorem 3: Let $\boldsymbol{w} = (w_0, w_1, w_2) \in \mathbb{R}_+^3$ be any weight vector for the MIMO Gaussian BBC with common message. If $w_0 < w_1+w_2$, then $R_0 = 0$ and the optimal transmit covariance matrix $\boldsymbol{Q}_{\text{opt}}$ is given by the solution of the corresponding optimization problem for the BBC without common message.

Proof: Already the formulation (7) of the optimization problem indicates that if $w_0 < w_1 + w_2$, for the weighted rate sum it is optimal to set $R_0 = 0$. Otherwise with increasing common rate, the weighted rate sum decreases. More precisely, since $\nu_1, \nu_2 \geq 0$, cf. (8e), (8b) shows that for $w_0 < w_1 + w_2$ we must have $\xi > 0$ which indeed implies $R_0 = 0$ by (8g).

Consequently, for $w_0 < w_1 + w_2$ the problem reduces to weighted rate sum optimization problem of the Gaussian BBC without common message as studied in [14] and [15]. ■

Since for $w_0 < w_1 + w_2$ Theorem 3 implies $R_0 = 0$, we assume $w_0 \geq w_1+w_2$ in the following. Next, we want to know when a given transmit covariance matrix, which is optimal for the BBC without common message, is also optimal for the BBC with common message, i.e., $R_0 > 0$.

Theorem 4: Let $R_0 > 0$ and $\boldsymbol{Q}_{\text{opt}}$ be the optimal transmit covariance matrix for the MIMO Gaussian BBC without common message for weight vector $\boldsymbol{w}' = (w'_1, w'_2) \in \mathbb{R}_+^2$. If $C_1(\boldsymbol{Q}_{\text{opt}}) < C_2(\boldsymbol{Q}_{\text{opt}})$, then $\boldsymbol{Q}_{\text{opt}}$ is also the optimal transmit covariance matrix for the MIMO Gaussian BBC with common message for all weight vectors $\boldsymbol{w} = (w_0, w_1, w_2) \in \mathbb{R}_+^3$ satisfying

$$w_0 = w'_1 + w'_2, \quad w_1 \leq w'_1, \quad \text{and} \quad w_2 = w'_2.$$

If $C_1(\boldsymbol{Q}_{\text{opt}}) > C_2(\boldsymbol{Q}_{\text{opt}})$, the assertion remains valid with

$$w_0 = w'_1 + w'_2, \quad w_1 = w'_1, \quad \text{and} \quad w_2 \leq w'_2.$$

If $C_1(\boldsymbol{Q}_{\text{opt}}) = C_2(\boldsymbol{Q}_{\text{opt}})$, the assertion remains valid with

$$w_0 = w'_1 + w'_2, \quad w_1 \leq w'_1, \quad \text{and} \quad w_2 \leq w'_2$$

and, obviously, $w_k = w'_k$, $k = 1, 2$ if $R_0 < C_k(\boldsymbol{Q}_{\text{opt}})$.

Proof: We prove the assertion for $C_1(\boldsymbol{Q}_{\text{opt}}) < C_2(\boldsymbol{Q}_{\text{opt}})$. Then, the case $C_1(\boldsymbol{Q}_{\text{opt}}) > C_2(\boldsymbol{Q}_{\text{opt}})$ follows accordingly using the same arguments.

First, note that we have $\xi = 0$ by (8g) since $R_0 > 0$ as assumed. If $C_1(\boldsymbol{Q}_{\text{opt}}) < C_2(\boldsymbol{Q}_{\text{opt}})$, then from (8c) follows that $R_0 < C_2(\boldsymbol{Q}_{\text{opt}})$ which immediately implies together with (8g) that $\nu_2 = 0$. With this, (8a) reads as

$$\mu\boldsymbol{I}_{N_R} - \boldsymbol{\Psi} = (w_1 + \nu_1)C'_1(\boldsymbol{Q}_{\text{opt}}) + w_2C'_2(\boldsymbol{Q}_{\text{opt}}). \quad (9)$$

We observe that the structure of (9) equals the structure of the MIMO Gaussian BBC without common message, cf. [15, Eq. (2a)], so that the optimization problem of the BBC with common message reduces to the BBC without common message but modified individual weights $w'_1 = w_1 + \nu_1 = w_0 - w_2$ and $w'_2 = w_2$. This implies that any solution for the weighted rate sum optimization problem of the BBC without common message and individual weights $\boldsymbol{w}' = (w'_1, w'_2)$ is also a solution for the corresponding problem of the BBC with common message and all weight vectors $\boldsymbol{w} = (w_0, w_1, w_2)$ that satisfy $w_0 = w'_1 + w'_2$, $w_1 \leq w'_1$, and $w_2 = w'_2$.

The third case follows immediately from (8a) and (8b) and $\nu_k \geq 0$, $k = 1, 2$. Further, $R_0 < C_k(\boldsymbol{Q}_{\text{opt}})$ immediately implies $w_k = w'_k$, $k = 1, 2$, since $\nu_k = 0$ by (8g). ■

These results can further be used to characterize the weighted rate sum optimal rate triples in detail.

V. Applications

Since Theorem 3 and Theorem 4 reveal a direct connection between the BBC with common message and the BBC without common message, it is possible to obtain transmit strategies for one case from the other. This is demonstrated in the following.

A. Egalitarian Solution for the MISO Gaussian BBC

Next, we consider the MISO case, where the relay node is equipped with multiple antennas, while the two other nodes each have a single antenna. Then, the input-output relation between the relay node and node k can be expressed as $y_k = \boldsymbol{x}^T\boldsymbol{h}_k+n_k$, $k = 1, 2$, with $\boldsymbol{x} \in \mathbb{C}^{N_R\times 1}$ the vector-valued input, $\boldsymbol{h}_k \in \mathbb{C}^{N_R\times 1}$ the channel, and y_k the scalar-valued output. Note that the notation has slightly changed for convenience.

In the following, we are interested in the transmit strategy $\boldsymbol{Q}_{\text{eq}}$ that realizes equal sum rates for the BBC with common message, i.e., $R_0 + R_1 = R_0 + R_2$, since this characterizes the achievable rate region for the XOR coding approach.

Theorem 5: Let $\boldsymbol{h}_k = \|\boldsymbol{h}_k\|\boldsymbol{u}_k$, $k = 1, 2$, $\rho = |\rho|e^{i\varphi} = \boldsymbol{u}_1^H\boldsymbol{u}_2$. Then the transmit covariance matrix $\boldsymbol{Q}_{\text{eq}} = P\boldsymbol{q}(t_{\text{eq}})\boldsymbol{q}^H(t_{\text{eq}})$ with normalized beamforming vector

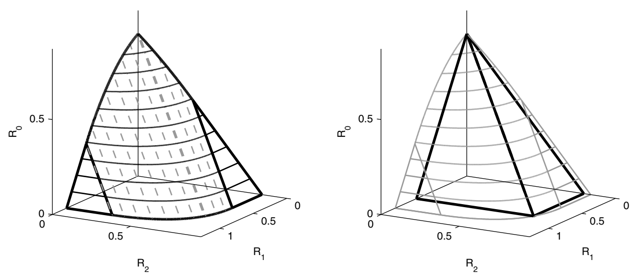
$$\boldsymbol{q}(t) := \frac{t\boldsymbol{u}_1 + (1-t)e^{-i\varphi}\boldsymbol{u}_2}{\|t\boldsymbol{u}_1 + (1-t)e^{-i\varphi}\boldsymbol{u}_2\|} \quad (10)$$

and

$$t_{\text{eq}} = \frac{\|\boldsymbol{h}_2\|(1-|\rho|)}{\|\boldsymbol{h}_1\|(1-|\rho|) + \|\boldsymbol{h}_2\|(1-|\rho|)} \quad (11)$$

achieves equal sum rates, i.e., $R_0 + R_1 = R_0 + R_2$.

Proof: From [14, Sec. IV-C] we know that for the MISO Gaussian BBC without common message there exists always an optimal transmit strategy of rank 1 that achieves capacity, i.e., $\boldsymbol{Q} = \boldsymbol{q}\boldsymbol{q}^H$ with $\text{tr}(\boldsymbol{Q}) = P$. Moreover, it is shown that the normalized beamforming vector is given by (10)



(a) Capacity region of the MISO Gaussian BBC with common message

(b) Achievable rate region for the XOR coding approach (black)

Fig. 3. MISO Gaussian BBC with common message with $N_R = 2$, $N_1 = N_2 = 1$ for $\boldsymbol{h}_1 = [1.3\ \ 1.3i]^T$, $\boldsymbol{h}_2 = [1\ \ -ie^{i\frac{\pi}{3}}]^T$, $P = 1$, and $\sigma^2 = 1$.

which achieves for $t \in [0, 1]$ the rate pair $(R_1(t), R_2(t))$ with $R_k(t) := \log(1 + \frac{P}{\sigma^2}|\boldsymbol{h}_k^2 \boldsymbol{q}(t)|^2)$, $k = 1, 2$.

From [14, Prop. 4] we know that the transmit strategy $\boldsymbol{Q}_{\text{eq}} = \boldsymbol{q}(t_{\text{eq}})\boldsymbol{q}^H(t_{\text{eq}})$ with t_{eq} as given in (11) characterizes the egalitarian solution, i.e., $R_1(t_{\text{eq}}) = R_2(t_{\text{eq}})$.

Since each transmit strategy for the BBC without common message is also a transmit strategy for the corresponding BBC with common message, this immediately characterizes the transmit strategy where both nodes have equal sum rates, $R_0 + R_1 = \log(1 + \frac{P}{\sigma^2}|\boldsymbol{h}_1^H \boldsymbol{q}(t_{\text{eq}})|^2) = \log(1 + \frac{P}{\sigma^2}|\boldsymbol{h}_2^H \boldsymbol{q}(t_{\text{eq}})|^2) = R_0 + R_2$, which proves the theorem. ■

Figure 3 depicts the capacity region of the MISO Gaussian BBC with common message and the corresponding rate region which is achievable using the XOR coding approach.

B. MIMO Gaussian BBC with Parallel Channels

Here, we consider the MIMO Gaussian BBC with parallel channels. Let $\boldsymbol{H}_k \boldsymbol{H}_k^H = \boldsymbol{W}_k \boldsymbol{S}_k \boldsymbol{W}_k^H$ with $\boldsymbol{S}_k = \text{diag}(s_{k,1}, s_{k,2}, ..., s_{k,N_R}) \succeq 0$ be the eigenvalue decomposition of the channel k, $k = 1, 2$. Since the channels are parallel, the unitary matrices $\boldsymbol{W}_1 = \boldsymbol{W}_2 = \boldsymbol{W}$ are equal.

Proposition 1: For the MIMO Gaussian BBC with common message and parallel channels, the optimal transmit strategy $\boldsymbol{Q}$ has the eigenvalue decomposition

$$\boldsymbol{Q} = \boldsymbol{W} \boldsymbol{\Sigma}_{\boldsymbol{Q}} \boldsymbol{W}^H \tag{12}$$

with $\boldsymbol{\Sigma}_{\boldsymbol{Q}} = \text{diag}(\lambda_1, \lambda_2, ..., \lambda_{N_R}) \succeq 0$.

Similar to the case without common message [15, Sec. IV], it is optimal to transmit into the direction of the eigenmodes of the channels. Moreover, it shows that the channels separate into N_R parallel channels so that it remains to determine the optimal power allocation between the modes.

Theorem 6: For given weight vector $\boldsymbol{w}$ and power constraint P the optimal transmit covariance matrix $\boldsymbol{Q}$ is given by (12) with activated eigenmodes $\lambda_{k_1(\boldsymbol{w})}, \lambda_{k_2(\boldsymbol{w})}, ..., \lambda_{k_n(\boldsymbol{w})}$. Thereby, the number of activated eigenmodes $k_n(\boldsymbol{w})$ depends on the channels and the transmit power, while the choice which eigenmodes are chosen is determined by the solution of (8a).

Proof: The proof follows immediately from Theorem 3, Theorem 4, Proposition 1, and [15, Sec. IV]. ■

Remark 2: The results also apply in a single-antenna OFDM system where the unitary matrix $\boldsymbol{W}$ equals the IDFT-matrix so that Theorem 6 characterizes the optimal power allocation.

For the more interesting case with non-parallel channels, further results for the high SNR regime can also be obtained from [15] but are omitted due to space constraints.

VI. Conclusion

In this work we studied the bidirectional broadcast channel where the relay adds an own common message to the communication. We characterized the capacity achieving transmit strategy and revealed interesting connections to BBC without common message which allows to obtain optimal transmit strategies for the BBC with common message from the case where the relay has no common message to transmit.

For the convergence of wireless services it shows that bidirectional relaying is a promising approach since it allows to integrate different services efficiently on the same resources.

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