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Minimal Transmit Power in Parallel Vector Broadcast Channels With Linear Precoding

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***Abstract*—We consider the communication over a set of parallel multiple-input single-output (MISO) broadcast channels with separate linear precoding on each of them. In this setup, we study the problem of fulfilling per-user quality of service constraints, expressed in terms of rates, using minimal transmit power. By means of a dual decomposition approach and a branch-and-bound algorithm solving the arising nonconvex subproblems, we find the so far unknown globally optimal solution for the case of linear precoding with time sharing. Although prohibitively complex for implementation in a practical system, the new power minimization method is highly interesting from a theoretical point of view as it can be used to quantify the performance gap between the outcome of heuristic algorithms and the theoretical limitations of the system with linear transceivers. We also extend the approach such that zero-forcing constraints can be handled.**

***Index Terms*—Linear transceivers, multiple-input single-output (MISO), multi-user multi-carrier systems, parallel broadcast channels, power minimization, quality of service.**

I. Introduction

In the last few years, different approaches have been published to minimize the sum transmit power in a multi-user communication system while serving each user at a certain requested rate (e.g., [1]–[12]). Many of them are heuristic algorithms [4]–[12], which aim at a low computational complexity and accept in exchange that a sum transmit power higher than the global optimum is attained. On the other hand, to evaluate the performance of the heuristic approaches, algorithms computing the globally optimal solution are indispensable. For the problem of power minimization in parallel vector broadcast channels, which we consider in this paper, the globally optimal solution is achieved by means of nonlinear dirty paper coding (DPC) and can be computed using one of the algorithms in [1]–[3].

Many practical algorithms for power minimization that are applicable to parallel vector broadcast channels make the assumption of linear transceivers (e.g., [7]–[12]) as well as the assumption that precoding is performed separately on each of the parallel subchannels, i.e., no transmit symbol may be spread across several subchannels (e.g., [8]–[12]). These restrictions can be interpreted as additional constraints to the power minimization problem. In these cases, it is not obvious which part of the performance gap between the suboptimal and the globally optimal solution is inherent to the new constraints and which part results from the incapability of the algorithm to find the globally optimal solution of the problem with the new constraints. Thus, there is a strong interest in also finding the globally optimal solution of the power minimization problem with separate linear precoding on each subchannel. The same is true for the case where zero-forcing constraints are introduced, as done, e.g., in [8]–[12]. The globally optimal solution for both cases will be computed in this paper. Even though the computation of the globally optimal solutions is exponentially complex in the number of users and therefore not applicable for real-time implementation, it is of high theoretical interest. For instance, the results from this paper are necessary for the proof of the suboptimality of separate precoding on each of the parallel subchannels provided in our work [13], which might have an impact on practical design.

The considered system model, which will be introduced in detail in Section II, complies to any practical multi-user system with a multi-antenna base station that uses a set of orthogonal subchannels to serve a set of single-antenna user terminals. The most straightforward application of the considered system model is a multi-carrier system that does not exhibit intercarrier interference, but the concept is not limited to multi-carrier systems. For instance, with slight modifications (introduction of weighting factors), the subchannels can also model time intervals in a fading channel environment (e.g., [2]).

As stated above, we will assume that linear precoding is performed separately on each of the orthogonal subchannels. In [14], such a separate precoding was called carrier-noncooperative transmission. The advantage is a significant reduction of the problem dimension and, consequently, a significant simplification of the optimization procedure. On the other hand, carrier-noncooperative transmission has been shown to be suboptimal in many settings (e.g., [14] and [15]). For the problem of power minimization in parallel vector broadcast channels, carrier-noncooperative transmission has been proven to be optimal for the case where nonlinear precoding is allowed [2], but it is suboptimal for linear transceivers [13]. Nevertheless, such a separate precoding will be assumed throughout this paper since, as explained above, this enables us to find a lower bound on the sum power achievable by any algorithm based on this assumption.

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Apart from the restriction to linear, carrier-noncooperative precoding, we do not impose any further limitations. In particular, we allow time sharing between different transmit strategies. This enables us to apply a dual approach (cf. Section III), even though the corresponding problem without time sharing exhibits a duality gap. This dual approach consists of solving an inner problem, which we study in Section IV, and an outer problem, which is solved in Section V. Finally, the solution of the primal problem has to be found, which can be done with the primal recovery method presented in Section V-B. At the end of the paper, we present numerical results that compare linear precoding with the globally optimal nonlinear strategy, and we show that the results can be used to bound the global optimum of related system assumptions not covered by this paper, such as linear transceivers without time sharing and linear transceivers with carrier cooperation.

Notation: In this paper, vectors are typeset in boldface lowercase letters and matrices in boldface uppercase letters. We write $\mathbf{0}$ for the zero matrix or vector, $\mathbf{I}_\bullet$ for the identity matrix of size $\bullet$, $\mathbf{1}$ for the all-ones vector, and a_i for the ith element of the vector $\boldsymbol{a}$. We use $\bullet^{\mathrm{T}}$ to denote the transpose of a vector or matrix and $\bullet^{\mathrm{H}}$ for the conjugate transpose. The notation $|\bullet|$ is used for the absolute value of a scalar as well as for the cardinality of a set. The order relation $\boldsymbol{x} \geq \boldsymbol{y}$ has to be understood element-wise, and $\mathbb{R}^N_{0,+}$ is the closed positive orthant of the $\mathbb{R}^N$, i.e., $\mathbb{R}^N_{0,+} = \{\boldsymbol{x} \in \mathbb{R}^N : \boldsymbol{x} \geq \mathbf{0}\}$. We use the shorthand notation $(\bullet^{(n)})_{\forall n}$ for $(\bullet^{(1)}, \ldots, \bullet^{(N)})$.

II. System Model and Problem Formulation

We consider a downlink system where the base station is equipped with M antennas while the K receivers are equipped with a single antenna each. The broadcast channel between the transmitting base station and the receivers is assumed to consist of N orthogonal subchannels, e.g., carriers.

The frequency flat vector channel between the base station and user k on subchannel n is denoted by $\boldsymbol{h}_k^{(n),\mathrm{H}} \in \mathbb{C}^{1\times M}$. These channels are assumed to be known and to satisfy the regularity condition [16]

$$\operatorname{Rank}\left[\boldsymbol{H}_{\mathcal{K}}^{(n)}\right] = \min\left(|\mathcal{K}|, M\right) \quad \forall n, \quad \forall \mathcal{K} \subseteq \{1, \ldots, K\} \tag{1}$$

where $\boldsymbol{H}_{\mathcal{K}}^{(n)} \in \mathbb{C}^{|\mathcal{K}|\times M}$ is a matrix whose rows are the channel vectors $\boldsymbol{h}_k^{(n),\mathrm{H}}$ of all users k with $k \in \mathcal{K}$. The additive circularly symmetric complex Gaussian noise $\eta_k^{(n)} \sim \mathcal{CN}(0, \sigma_k^{(n),2})$ is assumed to be independent across users and across subchannels and independent of the transmitted data.

Throughout the paper, we will assume that no transmit symbol may be spread across various subchannels. Thus, the transmitted data can be written as the symbols $s_k^{(n)} \sim \mathcal{CN}(0,1)$ of user k on subchannel n, and the received signal of user k on subchannel n is given by

$$y_k^{(n)} = \boldsymbol{h}_k^{(n),\mathrm{H}} \sum_{k'=1}^{K} \boldsymbol{t}_{k'}^{(n)} \sqrt{p_{\mathrm{DL},k'}^{(n)}}\, s_{k'}^{(n)} + \eta_k^{(n)} \tag{2}$$

where $p_{\mathrm{DL},k'}^{(n)} \geq 0$ is the transmit power of user k' on subchannel n, and $\boldsymbol{t}_{k'}^{(n)} \in \mathbb{C}^M$ is the corresponding unit norm beamformer.

The optimizations in this paper will be performed in the dual uplink channel [17], where the received signal is

$$\tilde{\boldsymbol{y}}^{(n)} = \sum_{k'=1}^{K} \boldsymbol{g}_{k'}^{(n)} \sqrt{p_{k'}^{(n)}}\, \tilde{s}_{k'}^{(n)} + \boldsymbol{\eta}^{(n)} \tag{3}$$

with the dual uplink channels $\boldsymbol{g}_k^{(n)} = \sigma_k^{(n),-1} \boldsymbol{h}_k^{(n)}$, the noise covariance matrix $\boldsymbol{C}_{\boldsymbol{\eta}^{(n)}} = \mathbf{I}_M$, and the uplink powers $p_k^{(n)}$. On each subchannel of the dual uplink channel, the same rates $r_k^{(n)}$ as in the original downlink channel are achievable with $\sum_{k=1}^K p_k^{(n)} = \sum_{k=1}^K p_{\mathrm{DL},k}^{(n)}$ [17]. The advantage of the dual uplink formulation is that the rates on subchannel n can be expressed as functions of the uplink transmit power vector $\boldsymbol{p}^{(n)} = [p_1^{(n)}, \ldots, p_K^{(n)}]^{\mathrm{T}}$ without a dependence on filter vectors:

$$r_k^{(n)}\left(\boldsymbol{p}^{(n)}\right) = \log_2\left(1 + p_k^{(n)} \boldsymbol{g}_k^{(n),\mathrm{H}} \boldsymbol{R}_k^{(n),-1} \boldsymbol{g}_k^{(n)}\right) \tag{4}$$

with

$$\boldsymbol{R}_k^{(n)} = \mathbf{I}_M + \sum_{k'\neq k} p_{k'}^{(n)} \boldsymbol{g}_{k'}^{(n)} \boldsymbol{g}_{k'}^{(n),\mathrm{H}}. \tag{5}$$

Finally, the total rate r_k of user k is given by $\sum_{n=1}^N r_k^{(n)}(\boldsymbol{p}^{(n)})$.

We focus on finding the minimal total transmit power necessary to fulfill given per-user rate requirements $\boldsymbol{\rho} = [\rho_1, \ldots, \rho_K]^{\mathrm{T}}$. In terms of an optimization problem, this can be written as

$$\begin{aligned} \min_{\left(\lambda_\ell^{(n)} \in \mathbb{R}_{0,+}, \boldsymbol{p}_\ell^{(n)} \in \mathbb{R}^K_{0,+}\right)_{\forall n, \forall \ell}} & \sum_{n=1}^N \sum_{\ell=1}^L \lambda_\ell^{(n)} \mathbf{1}^{\mathrm{T}} \boldsymbol{p}_\ell^{(n)} \\ \text{s.t.} & \sum_{n=1}^N \sum_{\ell=1}^L \lambda_\ell^{(n)} \boldsymbol{r}^{(n)}\left(\boldsymbol{p}_\ell^{(n)}\right) \geq \boldsymbol{\rho} \\ \text{and} & \sum_{\ell=1}^L \lambda_\ell^{(n)} = 1 \quad \forall n \end{aligned} \tag{6}$$

with $\boldsymbol{r}^{(n)}(\boldsymbol{p}_\ell^{(n)}) = [r_1^{(n)}(\boldsymbol{p}_\ell^{(n)}), \ldots, r_K^{(n)}(\boldsymbol{p}_\ell^{(n)})]^{\mathrm{T}}$. This formulation represents time sharing between L different operation points $\boldsymbol{p}_1^{(n)}, \ldots, \boldsymbol{p}_L^{(n)} \in \mathbb{R}^K_{0,+}$ on each carrier n, which means that a transmit strategy on carrier n does not have to consist of a certain power allocation, but it can consist of L such allocations, where the ℓth allocation is applied during a fraction of $\lambda_\ell^{(n)}$ of the total time. The factors $\lambda_\ell^{(n)}$ have to sum up to $\sum_{\ell=1}^L \lambda_\ell^{(n)} = 1\ \forall n$, and the total power as well as the total per-user rates are the weighted averages of the respective quantities in the various strategies. In principle, we allow L to be arbitrary, but due to the Carathéodory theorem [18, Theorem 2.1.6], no more than $L = K+1$ operation points are necessary on each carrier to achieve the same optimum as with any higher L.

Note that the problem (6) with time sharing always has a solution since arbitrary rate requirements are feasible even with linear precoding if time sharing is allowed [15].[1] The reasoning of [15] can also be extended to the case with zero-forcing constraints. Having found the optimal solution of (6), the uplink

[1]Without time sharing, this is only true for nonlinear precoding [19].

receive filters can be optimized for each user and for each operation point separately, and afterwards, the results can be transformed back to the downlink by applying the results from [17] on each subchannel for each operation point.

For later use, we define the single-user rate vector $\boldsymbol{r}_{\mathrm{SU}}^{(n)}(\boldsymbol{p}^{(n)}) = [r_{\mathrm{SU},1}^{(n)}(\boldsymbol{p}^{(n)}), \ldots, r_{\mathrm{SU},K}^{(n)}(\boldsymbol{p}^{(n)})]^{\mathrm{T}}$ with

$$r_{\mathrm{SU},k}^{(n)}\left(\boldsymbol{p}^{(n)}\right) = \log_2\left(1 + p_k^{(n)} \boldsymbol{g}_k^{(n),\mathrm{H}} \boldsymbol{g}_k^{(n)}\right) \tag{7}$$

and the single-user power vector $\boldsymbol{q}_{\mathrm{SU}}^{(n)}(\boldsymbol{\rho}^{(n)}) = [q_{\mathrm{SU},1}^{(n)}(\boldsymbol{\rho}^{(n)}), \ldots, q_{\mathrm{SU},K}^{(n)}(\boldsymbol{\rho}^{(n)})]^{\mathrm{T}}$ with

$$q_{\mathrm{SU},k}^{(n)}\left(\boldsymbol{\rho}^{(n)}\right) = \frac{2^{\rho_k^{(n)}} - 1}{\boldsymbol{g}_k^{(n),\mathrm{H}} \boldsymbol{g}_k^{(n)}}. \tag{8}$$

As the interference is neglected in (7), it always holds that $r_{\mathrm{SU},k}^{(n)}(\boldsymbol{p}^{(n)}) \geq r_k^{(n)}(\boldsymbol{p}^{(n)})$.

III. Dual Decomposition Approach

As a vehicle to solve the optimization problem (6), we introduce the modified problem

$$\min_{\left(\boldsymbol{p}^{(n)}\right)_{\forall n} \in \mathbb{R}_{0,+}^{KN}} \quad \sum_{n=1}^{N} \mathbf{1}^{\mathrm{T}} \boldsymbol{p}^{(n)} \qquad \text{s.t.} \sum_{n=1}^{N} \boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \geq \boldsymbol{\rho} \tag{9}$$

which is a reformulation of (6) without time sharing, and we apply a dual decomposition approach (e.g., [18]) to problem (9). Therefore, the whole procedure of optimizing the dual function does not explicitly rely on time sharing. However, in general, it is not possible to find a primarily feasible solution of problem (9) achieving a sum power equal to the optimum of the dual problem since (9) exhibits a duality gap. Instead, as will be shown in Section V-B, we can find a feasible solution of problem (6) achieving this value of the sum transmit power. The reason for this is that the nonconvex rate constraints are convexified due to the possibility of time sharing in problem (6), so that the duality gap is closed. A formal proof for this statement could be given by verifying that (6) satisfies the so-called time-sharing condition introduced in [20], which implies that the duality gap vanishes.

The dual decomposition approach derived in the following can also be found in [2], where systems with nonlinear precoding were considered. We dualize the rate constraints of problem (9) so that the dual function reads as

$$\Theta(\boldsymbol{\mu}) = \min_{\left(\boldsymbol{p}^{(n)}\right)_{\forall n} \in \mathbb{R}_{0,+}^{KN}} \quad \Phi\left(\left(\boldsymbol{p}^{(n)}\right)_{\forall n}, \boldsymbol{\mu}\right) \tag{10}$$

where $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_K]^{\mathrm{T}} \in \mathbb{R}_{0,+}^{K}$ is the vector of dual variables, and the Lagrangian function Φ is given by

$$\Phi\left(\left(\boldsymbol{p}^{(n)}\right)_{\forall n}, \boldsymbol{\mu}\right) = \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{\rho} + \sum_{n=1}^{N} \Phi^{(n)}\left(\boldsymbol{p}^{(n)}, \boldsymbol{\mu}\right) \tag{11}$$

with $\Phi^{(n)}$ defined as

$$\Phi^{(n)}\left(\boldsymbol{p}^{(n)}, \boldsymbol{\mu}\right) = \mathbf{1}^{\mathrm{T}} \boldsymbol{p}^{(n)} - \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right). \tag{12}$$

Thus, in order to evaluate the dual function $\Theta(\boldsymbol{\mu})$, we have to solve

$$\min_{\boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^{K}} \quad \mathbf{1}^{\mathrm{T}} \boldsymbol{p}^{(n)} - \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \tag{13}$$

separately on each subchannel n. Due to the decomposition into per-subchannel problems, it is possible to solve the problem for systems with a high number of subchannels by solving the per-subchannel problems in a parallelized manner. Solution methods for the inner problem will be presented in the following section.

Finally, the dual problem

$$\max_{\boldsymbol{\mu} \in \mathbb{R}_{0,+}^{K}} \quad \Theta(\boldsymbol{\mu}) \tag{14}$$

has to be solved in an outer loop in order to find the optimal dual variables $\boldsymbol{\mu}$, and in each iteration, the inner problem has to be solved in order to evaluate the dual function. A method to solve the outer problem will be discussed in Section V.

IV. Solution to the Inner Problem

This section is devoted to finding the globally optimal solution of the inner problem (13) on subchannel n. If nonlinear precoding is allowed, this problem is equivalent to a convex problem [2], which can be solved efficiently. However, if the system is constrained to use linear precoding, the inner problem is nonconvex due to the nonconcavity of the underlying rate equations. Nevertheless, as shown below, the globally optimal solution can be found by means of monotonic optimization.

A. Linear Precoding Without Zero-Forcing Constraints

In order to reveal monotonicity properties of the inner problem, we introduce a rate space formulation, i.e., we use the rates as optimization variables instead of the powers. Let $\mathcal{R}^{(n)} = \{\boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)}) : \boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^{K}\}$ denote the set of rate vectors achievable on subchannel n with finite sum power.[2] As is claimed in Lemma 1 (cf. Appendix A), the inverse function $\boldsymbol{r}^{(n),-1} : \mathcal{R}^{(n)} \mapsto \mathbb{R}_{0,+}^{K}$ of the rate function $\boldsymbol{r}^{(n)} : \boldsymbol{p}^{(n)} \mapsto \boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)})$ exists and can be evaluated by means of any globally optimal power minimization algorithm for vector broadcast channels with linear precoding (e.g., the one proposed in [21]). We define

$$\boldsymbol{q}^{(n)}\left(\boldsymbol{\rho}^{(n)}\right) = \begin{cases} \boldsymbol{r}^{(n),-1}\left(\boldsymbol{\rho}^{(n)}\right), & \text{if } \boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)} \\ [\infty, \ldots, \infty]^{\mathrm{T}}, & \text{otherwise} \end{cases} \tag{15}$$

where $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$ can be checked with the feasibility test proposed in [16] and [22]. With this definition, we get the following rate space formulation of the inner problem (13) on subchannel n:

$$\max_{\boldsymbol{\rho}^{(n)} \in \mathbb{R}_{0,+}^{K}} \quad \boldsymbol{\mu}^{\mathrm{T}} \boldsymbol{\rho}^{(n)} - \mathbf{1}^{\mathrm{T}} \boldsymbol{q}^{(n)}\left(\boldsymbol{\rho}^{(n)}\right). \tag{16}$$

[2]Even though we allow time sharing in this paper, the inner problem is solved without time sharing, and the possibility of time sharing is included in the primal recovery in Section V-B. Therefore, in the inner problem, not all rate vectors are achievable, and $\mathcal{R}^{(n)}$ is a strict subset of $\mathbb{R}_{0,+}^{K}$ in general.

Proposition 1: The rate space formulation (16) is a difference-of-monotonic (DM) problem, i.e., its cost function is the difference of monotonic functions.

Proof of Proposition 1: The first summand is obviously nondecreasing in $\boldsymbol{\rho}^{(n)}$ since $\boldsymbol{\mu} \geq \mathbf{0}$. If the rate vector $\boldsymbol{\rho}^{(n)}$ is achievable with the sum power $\mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$, all rate vectors $\boldsymbol{\rho}'^{(n)} \leq \boldsymbol{\rho}^{(n)}$ are also achievable with this sum power since they are elements of the rate region with sum power $\mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$. Thus, $\mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}'^{(n)}) \leq \mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$. As can be easily verified, this inequality also holds if $\boldsymbol{\rho}^{(n)}$ is not achievable with finite transmit power. Consequently, $\mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$ is nondecreasing in $\boldsymbol{\rho}^{(n)}$. ■

A possible way to find the globally optimal solution of a DM problem is the branch-and-bound (BB) method [23],[3] which successively bounds the optimal value from above by cutting the $\mathbb{R}_{0,+}^{K}$ into boxes $[\boldsymbol{a}, \boldsymbol{b}] := \{\boldsymbol{x} : \boldsymbol{a} \leq \boldsymbol{x} \leq \boldsymbol{b}\} \subset \mathbb{R}_{0,+}^{N}$. We define a set of boxes $\mathbb{B}$ and initialize it with $\mathbb{B} \leftarrow \{\mathcal{B}_0\}$, $\mathcal{B}_0 = [\mathbf{0}, \boldsymbol{b}_0]$, where the vector $\boldsymbol{b}_0$ has to fulfill $\boldsymbol{b}_0 \geq \boldsymbol{\rho}_{\mathrm{opt}}^{(n)}$ with $\boldsymbol{\rho}_{\mathrm{opt}}^{(n)}$ being the unknown optimizer of (16). A method to find such a vector $\boldsymbol{b}_0$ is proposed in Appendix B. By construction, the box $\mathcal{B}_0$ fulfills the nonnegativity constraint of (16) and surely contains the optimal solution.

Due to the monotonicity of the two summands, the following upper bound holds inside any box $\mathcal{B} = [\boldsymbol{a}, \boldsymbol{b}]$:

$$\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho}^{(n)} - \mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}\left(\boldsymbol{\rho}^{(n)}\right) \leq U_{\mathcal{B}} = \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{b} - \mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{a}) \quad \forall \boldsymbol{\rho}^{(n)} \in \mathcal{B}. \tag{17}$$

On the other hand, a lower bound to the maximal value of the objective function inside the box $\mathcal{B}$ is obtained by evaluating the function for an arbitrary $\boldsymbol{\rho}^{(n)} \in \mathcal{B}$. For the sake of small computational complexity, we choose

$$\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho}_{\mathcal{B},\mathrm{opt}}^{(n)} - \mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}\left(\boldsymbol{\rho}_{\mathcal{B},\mathrm{opt}}^{(n)}\right) \geq L_{\mathcal{B}} = \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{a} - \mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{a}) \tag{18}$$

so that we do not have to evaluate $\boldsymbol{q}^{(n)}$ twice.

In order to gradually refine the bounds, in each iteration, the box

$$\mathcal{B}_{\mathrm{branch}} = \arg\max_{\mathcal{B} \in \mathbb{B}} U_{\mathcal{B}} \tag{19}$$

is cut along its longest edge, i.e., $\mathcal{B}_{\mathrm{branch}} = [\boldsymbol{a}_{\mathrm{branch}}, \boldsymbol{b}_{\mathrm{branch}}]$ is cut along direction

$$k^{*} = \arg\max_{k} \quad b_{\mathrm{branch},k} - a_{\mathrm{branch},k}. \tag{20}$$

For cutting the box, the subdivision rule

$$\mathcal{B}_{\mathrm{new}} \leftarrow \left[\boldsymbol{a}_{\mathrm{branch}}, \boldsymbol{b}_{\mathrm{branch}} - \frac{b_{\mathrm{branch},k^*} - a_{\mathrm{branch},k^*}}{2}\boldsymbol{e}_{k^*}\right] \tag{21}$$

$$\mathcal{B}_{\mathrm{branch}} \leftarrow \left[\boldsymbol{a}_{\mathrm{branch}} + \frac{b_{\mathrm{branch},k^*} - a_{\mathrm{branch},k^*}}{2}\boldsymbol{e}_{k^*}, \boldsymbol{b}_{\mathrm{branch}}\right] \tag{22}$$

is used, where $\boldsymbol{e}_i$ is the ith canonical unit vector, which has a one as the ith entry and zeros elsewhere, and the new box $\mathcal{B}_{\mathrm{new}}$ is added to $\mathbb{B}$. After the branch step, new bounds for $\mathcal{B}_{\mathrm{branch}}$ and $\mathcal{B}_{\mathrm{new}}$ have to be computed. Reusing the value $\boldsymbol{q}^{(n)}(\boldsymbol{a}_{\mathrm{branch}})$ already computed earlier, the function $\boldsymbol{q}^{(n)}$ has to be evaluated only once per iteration, which is the most costly part of the algorithm if an efficient way to store the set $\mathbb{B}$ and to search inside it is chosen.

[3]The branch-reduce-and-bound (BRB) algorithm for DM from [23] is more general than the simplified algorithm used in this paper as it also allows for DM constraints while we only have a DM cost function.

Note that all upper bounds from (17) are utopian bounds (i.e., they are better than the actual optimum) while all lower bounds from (18) correspond to achievable values by construction. Thus, in each iteration, the best solution currently known [the current best value (CBV)] is given by the highest lower bound. The algorithm is stopped when the gap between the highest upper bound and the current best value has decreased below a certain desired error tolerance ϵ^{BB}. To reduce complexity and memory consumption, boxes can be dropped whenever their upper bound exceeds the current best value by no more than the desired accuracy. The method is summarized in Algorithm 1. In line (10), $L_{\mathcal{B}_{\mathrm{CBV}}}$ is the lower bound (18) for the box $\mathcal{B}_{\mathrm{CBV}}$, which has already been computed earlier during the execution of the algorithm. A visualization is given in Fig. 1, where a possible set of boxes after the fourth iteration in a two user system is shown.

Algorithm 1: Branch-and-Bound Method (Inner Problem)

Require: $(\boldsymbol{\mu}, (\boldsymbol{g}_1^{(n)}, \ldots, \boldsymbol{g}_K^{(n)})_{\forall n}, \epsilon^{\mathrm{BB}})$
(1) **for** $n \in \{1, \ldots, N\}$ **do**
(2) $\quad \mathbb{B} \leftarrow \{\mathcal{B}_0\}$, where $\mathcal{B}_0 \leftarrow [\mathbf{0}, \boldsymbol{b}_0]$ with $\boldsymbol{b}_0$ from Appendix B
(3) $\quad$ compute $U_{\mathcal{B}_0}$ and $L_{\mathcal{B}_0}$ from (17), (18)
(4) $\quad \boldsymbol{\rho}_{\mathrm{CBV}}^{(n)} \leftarrow \mathbf{0}$, $\mathcal{B}_{\mathrm{CBV}} \leftarrow \mathcal{B}_0$
(5) $\quad$ **while** $\mathbb{B} \neq \emptyset$ **do**
(6) $\qquad \mathcal{B}_{\mathrm{branch}} = \arg\max_{\mathcal{B} \in \mathbb{B}} U_{\mathcal{B}}$
(7) $\qquad$ cut $\mathcal{B}_{\mathrm{branch}}$ into $\mathcal{B}_{\mathrm{new}}$ and $\mathcal{B}_{\mathrm{branch}}$ using (20)–(22)
(8) $\qquad$ compute $U_{\mathcal{B}_{\mathrm{branch}}}$, $U_{\mathcal{B}_{\mathrm{new}}}$, $L_{\mathcal{B}_{\mathrm{branch}}}$, $L_{\mathcal{B}_{\mathrm{new}}}$ from (17), (18)
(9) $\qquad \mathbb{B} \leftarrow \mathbb{B} \cup \{\mathcal{B}_{\mathrm{new}}\}$
(10) $\qquad$ **if** $\max_{\mathcal{B} \in \mathbb{B}} L_{\mathcal{B}} > L_{\mathcal{B}_{\mathrm{CBV}}}$ **then**
(11) $\qquad\quad \boldsymbol{\rho}_{\mathrm{CBV}}^{(n)} \leftarrow$ lower corner of $\mathcal{B}_{\mathrm{CBV}} \leftarrow \arg\max_{\mathcal{B} \in \mathbb{B}} L_{\mathcal{B}}$
(12) $\qquad$ **end if**
(13) $\qquad \mathbb{B} \leftarrow \mathbb{B} \setminus \{\mathcal{B} \in \mathbb{B} : U_{\mathcal{B}} - \epsilon^{\mathrm{BB}} \leq L_{\mathcal{B}_{\mathrm{CBV}}}\}$
(14) $\quad$ **end while**
(15) $\quad \boldsymbol{p}_{\mathrm{opt}}^{(n)} \leftarrow \boldsymbol{q}^{(n)}(\boldsymbol{\rho}_{\mathrm{CBV}}^{(n)})$
(16) **end for**
(17) **return** $(\boldsymbol{p}_{\mathrm{opt}}^{(n)})_{\forall n}$

Apart from the conditions that the subdivision rule is a bisectional rectangular subdivision (e.g., [24]) and that the upper bound becomes tight when a box converges to a singleton, which are both fulfilled in our case, the convergence proof of the branch-and-bound method given in [23] requires that the two parts of the objective function are continuous. In the problem at hand, $\mathbf{1}^{\mathrm{T}}\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$ is only continuous at rate vectors $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$ (cf. Lemma 1 in Appendix A). However, as the upper bound $U_{\mathcal{B}}$ takes the value $-\infty$ whenever $\boldsymbol{q}^{(n)}(\boldsymbol{\rho}^{(n)})$

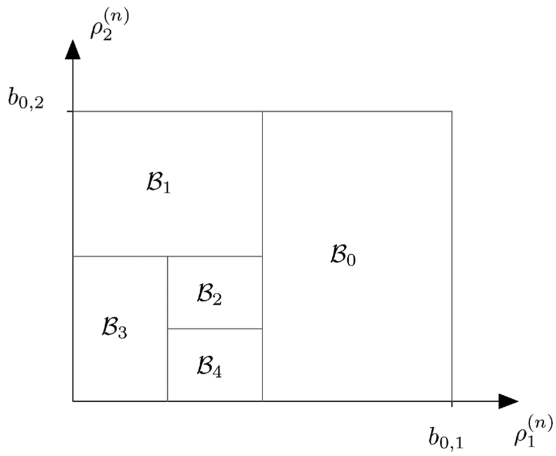

Fig. 1. Schematic of the branch-and-bound method.

is evaluated at a point $\boldsymbol{\rho}^{(n)} \notin \mathcal{R}^{(n)}$, so that the corresponding boxes are no longer considered as the algorithm proceeds, convergence is ensured nevertheless.

Unfortunately, [25, Theorem 4] implies that the worst-case complexity for finding an ϵ^{BB}-optimal solution with the branch-and-bound algorithm is $\mathcal{O}\left(\left(\frac{c_1}{\epsilon^{\mathrm{BB}}}\right)^{\frac{K}{c_2}}\right)$, where c_1 and c_2 are constants that depend on properties of the objective function. However, due to the nonconvex nature of the problem, no globally optimal solution with lower complexity is known. Moreover, the complexity does not grow extraordinarily in the other system variables. Due to the decomposition in per-subchannel problems (13), the overall complexity of the inner problem is linear in the number of subchannels N. An influence of the number of base station antennas M can only be found within the solver evaluating the function $\boldsymbol{q}^{(n)}$, whose complexity is polynomial in M.

Remark 1: Using a more sophisticated bound than the canonical bound for DM problems (17), which is not very tight in general [23], the speed of convergence could possibly be increased. However, this would not change the exponential complexity order of the algorithm.

Remark 2: As the objective of (13) is the difference of sum power and weighted sum rate, it would also be possible to reuse an existing globally optimal algorithm for the weighted sum rate maximization problem with given sum transmit power, e.g., based on a polyblock approach [26] as proposed in [27] and [28], and to optimize the sum power in an outer iteration. However, this approach also has exponential complexity and did not seem to have any advantages with regard to execution time in our numerical simulations.

B. Linear Precoding With Zero-Forcing Constraints

In the following, we will discuss the case where zero-forcing constraints are imposed. By not dualizing these new constraints, they become constraints of the inner problem (13), which is consequently turned into a combinatorial problem.

In the zero-forcing case, a set of users $\mathcal{K}^{(n)}$ with $|\mathcal{K}^{(n)}| \leq \min\{K, M\}$ has to be selected to be served on subchannel n, where M is the number of base station antennas and K is the number of users. We define the joint channel matrix of the active users $\boldsymbol{G}^{(n)} \in \mathbb{C}^{M \times |\mathcal{K}^{(n)}|}$, whose columns are the uplink channel vectors $\boldsymbol{g}_k^{(n)}$ of the users $k \in \mathcal{K}^{(n)}$. For a given subset of active users $\mathcal{K}^{(n)}$, the channel coefficients $\gamma_k^{(n)}$ of the resulting interference-free scalar channels are explicitly given by (e.g., [29])

$$\gamma_k^{(n)} = \begin{cases} \left[\left(\boldsymbol{G}^{(n),\mathrm{H}}\boldsymbol{G}^{(n)}\right)^{-1}\right]_k^{-2}, & \text{if } k \in \mathcal{K}^{(n)} \\ 0, & \text{otherwise} \end{cases} \quad (23)$$

where $[\bullet]_k$ is used to denote the diagonal element corresponding to user k. Thus, the optimal powers can be found by solving

$$\max_{\boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^K} \sum_{k=1}^{K} \left(\mu_k \log\left(1 + \left|\gamma_k^{(n)}\right|^2 p_k^{(n)}\right) - p_k^{(n)}\right). \quad (24)$$

The explicit solution is given by

$$p_k^{(n)} = \max\left\{\mu_k - \left|\gamma_k^{(n)}\right|^{-2}, 0\right\} \quad (25)$$

and the solution to the overall problem can be found by comparing the resulting value of the objective for all $\sum_{|\mathcal{K}^{(n)}|=1}^{\min\{M,K\}} \binom{K}{|\mathcal{K}^{(n)}|}$ possible sets of scheduled users.[4] Clearly, the complexity grows extraordinarily in K, but unlike in the case without zero-forcing, an exact solution can be obtained with a finite number of operations.

V. Solution to the Dual Problem

As pointwise minimum of a family of affine functions, the dual function (10) is concave in its variables $\boldsymbol{\mu}$, so that the dual problem (14) is a convex problem [18]. Instead of the ellipsoid method, which was applied in [2] to solve the dual problem, we use a so-called multiple cuts [30] version of the cutting plane method [18], [31]. There are several reasons for this: First of all, the algorithm impresses with its simplicity and its fast convergence, and it provides an easy way to perform the primal recovery. Secondly, it allows us to exploit the structure of the dual decomposition approach to speed up the solution of the outer problem. Finally, it can be easily extended to deal with inaccurate evaluations of the dual function.

A. Outer Approximation

We choose the following reformulation of the dual problem, which relies on the decomposability of the Lagrangian function:

$$\begin{aligned} \max_{\boldsymbol{\mu} \in \mathbb{R}_{0,+}^K, \boldsymbol{z} \in \mathbb{R}^N} \quad & \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho} + \sum_{n=1}^{N} z_n \\ \text{s.t.}\, z_n \leq\, & \mathbf{1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}) - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}^{(n)}\left(\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu})\right) \quad \forall n \end{aligned} \quad (26)$$

where the powers $\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu})$ are the optimizers of (13) for a given $\boldsymbol{\mu}$. Due to the optimality of $\boldsymbol{p}_{\mathrm{opt}}^{(n)}$, the constraints on z_n are equivalent to

$$z_n \leq \mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)} - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \quad \forall n, \quad \forall \boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^K. \quad (27)$$

[4]If needed, the case of nonlinear DPC zero-forcing [29] could be solved in a similar manner: the effective channel coefficients $\gamma_k^{(n)}$ would be obtained from the QR decomposition of the joint channel matrix, and the encoding order would have to be optimized by comparing the $|\mathcal{K}^{(n)}|!$ possible encoding orders of each subset of users.

We introduce vectors $\boldsymbol{\mu}_\ell \in \mathbb{R}_{0,+}^K, \ell = 1, \ldots, L$, which are constants, so that the powers $\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)$ and rates $\boldsymbol{r}^{(n)}(\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell))$ are constants as well. Then, we relax problem (26) to

$$\max_{\boldsymbol{\mu}\in\mathbb{R}_{0,+}^K, \boldsymbol{z}\in\mathbb{R}^N} \quad \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho} + \sum_{n=1}^N z_n$$
$$\text{s.t. } z_n \leq \mathbf{1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell) - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}^{(n)}\left(\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)\right) \forall n,\ \forall \ell \leq L. \quad (28)$$

Note that the constraints in (28) are NL linear constraints, so that (28) is a linear program, which can be efficiently solved. The solution gives an upper bound to the optimum of (26).

Just like in the standard cutting plane algorithm, the optimizer $(\boldsymbol{\mu}^*, \boldsymbol{z}^*)$ of the relaxed problem (28) violates the original constraint set (27) as long as the algorithm has not yet converged. A refined outer approximation can then be obtained by setting $\boldsymbol{\mu}_{L+1} = \boldsymbol{\mu}^*$ and increasing L by one. Repeating this procedure iteratively, the optima of the approximated problems form a decreasing sequence that converges to the global optimum. The procedure can be interpreted in a graphical manner as an outer approximation by tangent hyperplanes in the points $\boldsymbol{\mu}_\ell$, which cut the $\mathbb{R}^{K+N}$ into halfspaces. As N new hyperplanes are added in each iteration, this multicut version refines the outer approximation faster than the standard cutting plane algorithm.

B. Primal Recovery

As in [18], the primal recovery can be performed by means of the dual problem of (28), which reads

$$\min_{\left(\lambda_\ell^{(n)}\in\mathbb{R}_{0,+}\right)_{\forall n,\forall \ell}} \sum_{n=1}^N \sum_{\ell=1}^L \lambda_\ell^{(n)} \mathbf{1}^{\mathrm{T}}\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)$$
$$\text{s.t. } \sum_{n=1}^N \sum_{\ell=1}^L \lambda_\ell^{(n)} \boldsymbol{r}^{(n)}\left(\boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)\right) \geq \boldsymbol{\rho}$$
$$\text{and } \sum_{\ell=1}^L \lambda_\ell^{(n)} = 1 \quad \forall n \quad (29)$$

where the variables $\lambda_\ell^{(n)}$ are the dual variables associated with the inequality constraints of problem (28). Note that most algorithms for solving linear programs (e.g., the simplex method and primal-dual interior-point methods [18]) return a primal-dual optimal pair, i.e., when solving the linear program (28), not only the optimizer $(\boldsymbol{\mu}^*, \boldsymbol{z}^*)$, but also the optimal dual variables $\lambda_\ell^{(n),*} \forall n, \forall \ell$ are obtained. Thus, no additional effort is needed to solve problem (29).

It remains to be shown that the sequence of primal solutions obtained from (29) converges to the globally optimal solution of (6) as the cutting plane algorithm proceeds. Comparing (29) to (6), it becomes clear that the optimal solution $\lambda_\ell^{(n),*}$ of (29) yields a feasible strategy for (6) with $\boldsymbol{p}_\ell^{(n)} = \boldsymbol{p}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)$ and time-sharing weights $\lambda_\ell^{(n)} = \lambda_\ell^{(n),*}$. Since linear programs do not exhibit a duality gap [18], the minimum of (29) is equal to the maximum of (28), which converges to the optimal value of the dual problem (14). Consequently, the optimum of (14) is an achievable value of (6). On the other hand, the optimum of the dual problem (14) is a lower bound for the optimum of the primal problem (6) [18]. Therefore, both optima are equal, and the duality gap is zero as claimed in Section III.

C. Convergence Criterion and Accuracy

From solving the inner problem in each iteration of the cutting plane method, we know the values of $\Theta(\boldsymbol{\mu}_\ell)$. The difference between the latest upper bound and the current best value $\max_{\ell\in\{1,\ldots,L\}} \Theta(\boldsymbol{\mu}_\ell)$ can be used to test for convergence, i.e., the algorithm terminates when

$$\left(\boldsymbol{\mu}^{*,\mathrm{T}}\boldsymbol{\rho} + \mathbf{1}^{\mathrm{T}}\boldsymbol{z}^*\right) - \max_{\ell\in\{1,\ldots,L\}} \Theta(\boldsymbol{\mu}_\ell) \leq \epsilon^{\mathrm{CP}} \quad (30)$$

where ϵ^{CP} is the error tolerance corresponding to the desired accuracy. However, the accuracy of the final solution does not only depend on ϵ^{CP}, but also on the error tolerance ϵ^{BB} of the solver of the inner problem. Therefore, we study the effect of an inaccurately solved inner problem on the solution of the dual problem.

Let $\tilde{\boldsymbol{p}}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell)$ be an approximated optimizer of (13) for a given $\boldsymbol{\mu}_\ell$, which is used to add a new constraint to the approximated constraint set in (28). As long as the rates $\boldsymbol{r}^{(n)}(\tilde{\boldsymbol{p}}_{\mathrm{opt}}^{(n)}(\boldsymbol{\mu}_\ell))$ are computed exactly using (4), the new hyperplane is still a valid outer approximation of (27), but potentially no tangent to the exact feasible set. Thus, it might happen that all new constraints are inactive in an iteration. Whenever this is the case, the error tolerance ϵ^{BB} can be decreased in order to obtain a tighter approximation in the next iteration.

However, inaccurate evaluation of the dual function might lead to approximated values $\tilde{\Theta}(\boldsymbol{\mu}_\ell)$ that are higher than the actual optimum, leading to a premature termination of the algorithm. Therefore, when successively increasing the accuracy, we exclude all indices ℓ that have not been processed with the hitherto lowest ϵ^{BB} from the maximization in (30). When this modified condition (30) is fulfilled, we either decrease ϵ^{BB} or, if ϵ^{BB} has already reached a certain final error tolerance, we let the algorithm terminate.

As solving the inner problem with high accuracy yields a significant increase in complexity, such a successive reduction of the error tolerance makes sense, especially since the inner solver turned out to converge very slowly for dual variables $\boldsymbol{\mu}$ that are far from being optimal. As the proposed method is robust against high initial error tolerances, the initial ϵ^{BB} can be magnitudes higher than the final one.

D. Initialization

For problem (28) to be bounded in the first iterations of the cutting plane algorithm, we add the initial constraints

$$z_n \leq P_{\mathrm{init}}^{(n)} - \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho}_{\mathrm{init}}^{(n)} \quad \forall n \quad (31)$$

where $P_{\mathrm{init}}^{(n)}$ is a sum power on subchannel n which is sufficient to achieve some arbitrarily chosen rates $\boldsymbol{\rho}_{\mathrm{init}}^{(n)}$ with $\sum_{n=1}^N \boldsymbol{\rho}_{\mathrm{init}}^{(n)} \geq \boldsymbol{\rho}$. With these constraints, the problem is bounded since

$$\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{\rho} + \sum_{n=1}^N z_n \leq \boldsymbol{\mu}^{\mathrm{T}}\left(\boldsymbol{\rho} - \sum_{n=1}^N \boldsymbol{\rho}_{\mathrm{init}}^{(n)}\right) + \sum_{n=1}^N P_{\mathrm{init}}^{(n)} \leq \sum_{n=1}^N P_{\mathrm{init}}^{(n)}. \quad (32)$$

To find sufficient transmit powers $P_{\mathrm{init}}^{(n)}$, we use time sharing between single-user points, i.e., data for user k is transmitted in an interference-free manner with K times the average data rate during a fraction of $\frac{1}{K}$ of the total time. Then, the powers $P_{\mathrm{init}}^{(n)}$ are the averages

$$P_{\mathrm{init}}^{(n)} = \frac{1}{K}\mathbf{1}^{\mathrm{T}}\boldsymbol{q}_{\mathrm{SU}}^{(n)}\left(K\boldsymbol{\rho}_{\mathrm{init}}^{(n)}\right) \tag{33}$$

with $\boldsymbol{q}_{\mathrm{SU}}^{(n)}$ defined in (8). Note that this initialization also fulfills possible zero-forcing constraints.

VI. Discussion and Numerical Results

For parallel vector broadcast channels with separate linear precoding on each subchannel, we have presented globally optimal solutions of two different scenarios: with and without zero-forcing constraints (ZF). In addition to these two solutions employing time sharing (TS), we also present numerical results for the optimal zero-forcing solution without time sharing, which can be obtained by an exhaustive search. Furthermore, we include the globally optimal solution employing DPC, which is the ultimate minimum of the sum transmit power in parallel vector broadcast channels with per-user rate constraints.

The simulations of Fig. 2 ($M = 2$ transmit antennas, $K = 4$ users, and $N \in \{2,3,4,5\}$ subchannels) have been performed with i.i.d. circularly symmetric complex Gaussian channel coefficients with zero mean and unit variance and per-user rate requirements that are the absolute values of i.i.d. real Gaussian random variables with zero mean and unit variance. The noise power has been fixed to $\sigma_k^{(n),2} = 1 \forall k, \forall n$. The resulting powers are averaged over 1000 realizations of the involved random variables by means of the geometric mean (equivalent to the arithmetic mean in the decibel domain). Independent of the number of subchannels, linear precoding with time sharing has the lowest sum power while linear zero-forcing without time sharing has the highest. This was to be expected since adding a new constraint can never decrease the minimal value of the transmit power. In the example with two subchannels, the number of degrees of freedom (product MN of the number of transmit antennas M and the number of subchannels N) is equal to the number of users, resulting in a very high transmit power, especially for linear zero-forcing without time sharing, which is the strategy with the most restrictions. For $N > 2$, the number of degrees of freedom as well as the available bandwidth is higher, resulting in lower sum transmit powers for all strategies and in a smaller gap between the linear techniques and the optimal DPC.

In Fig. 3 ($M = 2$ transmit antennas, $K = 4$ users, and $N = 2$ subchannels), we have kept the channels, but we have replaced the random rate requirements by $\rho_k = 2\rho_0$ for half of the users and by $\rho_k = \rho_0$ for the rest. Here, the transmit power increases with the rate requirements, and the curves seem to converge to linear asymptotes for high data rates. In fact, this was to be expected since the diagram can be qualitatively interpreted as the reflection of a classical rate-over-SNR diagram across its diagonal.

Note that the low number of subchannels used in the numerical simulations is not necessary for the algorithm to converge in reasonable time since the complexity of the approach is exponential only in the number of users. However, when increasing the number of subchannels while keeping the number of users constant, the system has a high number of available degrees of freedom so that the power gap between the strategies presented in the plots is less pronounced. Therefore, we do not include plots for higher numbers of subchannels.

A case of practical interest which is not covered by the algorithm in this paper is the case of parallel vector broadcast channels with linear precoding where neither zero-forcing is assumed nor time sharing is applied. To find the minimum transmit power for a given set of per-user rate constraints in such a system, a nonconvex optimization problem in much more variables than problem (13) would have to be solved [32] because a dual decomposition approach is no longer possible. However, using the results of this paper, the optimal value of this problem can be bounded. While any feasible solution of the problem, e.g., the globally optimal solution of the stricter problem with additional zero-forcing constraints (*Lin. ZF* in the plots), can be used as upper bound to the optimal transmit power, a lower bound is given by the globally optimal solution of the relaxed problem with time sharing (*Lin. TS*). Whenever

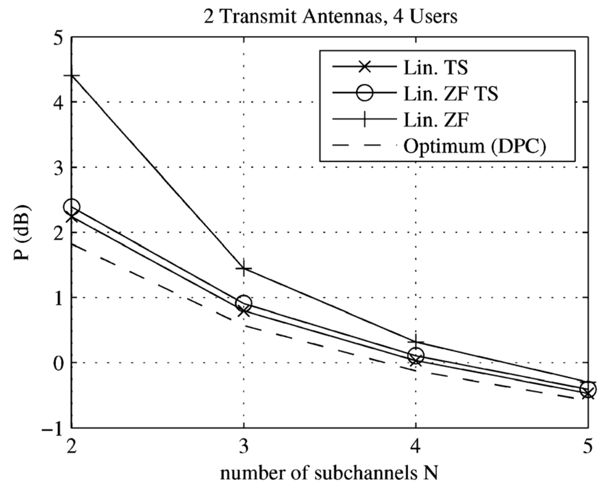

Fig. 2. Transmit power for different numbers of subchannels.

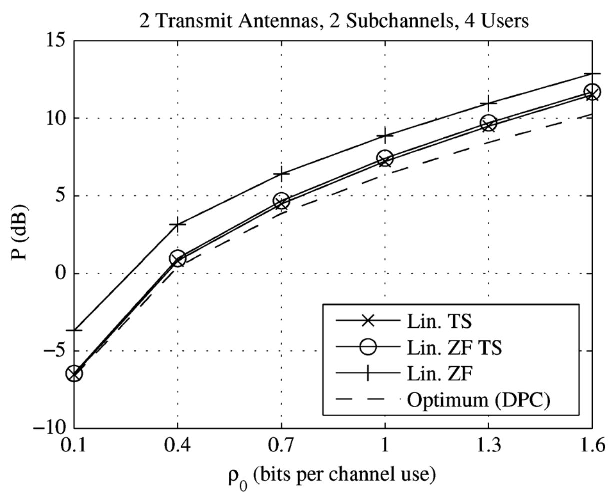

Fig. 3. Transmit power for different per-user rate requirements.

an algorithm for linear precoding without zero-forcing performs close to this lower bound in a certain channel model, it is clear that it also performs close to the globally optimal linear solution without time sharing. In Fig. 2, it can be seen that the two bounds get closer to each other for an increasing number of subchannels. This is in compliance with the fact that the duality gap of the problem without time sharing vanishes with increasing numbers of subchannels (cf. [20]), yielding a global optimum closer to the optimal time-sharing solution. In a similar way, using the curves *Lin. TS* and *Optimum (DPC)*, we can bound the globally optimal solution for systems with linear transceivers that allow using time sharing as well as spreading transmit symbols across subchannels.

Appendix A
Inverse of the Rate Function

Lemma 1: The inverse function $\boldsymbol{r}^{(n),-1} : \mathcal{R}^{(n)} \mapsto \mathbb{R}_{0,+}^K$ of $\boldsymbol{r}^{(n)} : \boldsymbol{p}^{(n)} \mapsto \boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)})$ exists and is continuous at all $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$. Evaluating $\boldsymbol{r}^{(n),-1}(\boldsymbol{\rho}^{(n)})$ for some $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$ is equivalent to solving the optimization problem

$$\min_{\boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^K} \mathbf{1}^{\mathrm{T}} \boldsymbol{p}^{(n)} \quad \text{s.t. } \boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \geq \boldsymbol{\rho}^{(n)}. \tag{34}$$

Proof of Lemma 1: According to [7], the global optimizer of (34) is the unique fixed point of a certain fixed point iteration, and this fixed point exists for all $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$. By construction, all rate constraints of (34) are active in the unique fixed point, i.e., there is a unique power allocation $\boldsymbol{p}^{(n)} = \boldsymbol{r}^{(n),-1}(\boldsymbol{\rho}^{(n)})$ achieving $\boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)}) = \boldsymbol{\rho}^{(n)}$. This allocation can be obtained by finding the global optimum of (34) by any means.

When restricted to a compact domain $\mathcal{Q} = \{\boldsymbol{p}^{(n)} \in \mathbb{R}_{0,+}^K : \mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)} \leq Q_{\max}\}$ with $Q_{\max} > \mathbf{1}^{\mathrm{T}}\boldsymbol{r}^{(n),-1}(\boldsymbol{\rho}^{(n)})$ for a given $\boldsymbol{\rho}^{(n)}, \boldsymbol{r}^{(n)} : \mathcal{Q} \mapsto \boldsymbol{r}^{(n)}(\mathcal{Q})$ is a continuous, bijective function defined on a compact domain, which implies that its inverse function is continuous (e.g., [33, Theorem 17.14]). Thus, $\boldsymbol{r}^{(n),-1}$ is continuous at any $\boldsymbol{\rho}^{(n)} \in \mathcal{R}^{(n)}$. ∎

Appendix B
Upper Bound to the Optimal Rate Vector

As zero is an achievable value of problem (16), all $\boldsymbol{\rho}^{(n)} = \boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)})$ with

$$\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) - \mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)} < 0 \tag{35}$$

cannot be optimal. With the single-user rates defined in (7), we have

$$\boldsymbol{r}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \leq \boldsymbol{r}_{\mathrm{SU}}^{(n)}\left(\boldsymbol{p}^{(n)}\right) \leq \boldsymbol{r}_{\mathrm{SU}}^{(n)}\left(\mathbf{1}\mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)}\right). \tag{36}$$

Thus, a sufficient condition for (35) is

$$\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}_{\mathrm{SU}}^{(n)}\left(\mathbf{1}P^{(n)}\right) < P^{(n)} \tag{37}$$

with $P^{(n)} = \mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)}$. Due to the strict concavity of $\boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}_{\mathrm{SU}}^{(n)}(\mathbf{1}P^{(n)})$, there exists a value $P_{\max}^{(n)}$ such that (37) holds for all $P^{(n)} > P_{\max}^{(n)}$. The smallest possible value of $P_{\max}^{(n)}$ fulfills $P_{\max}^{(n)} = \boldsymbol{\mu}^{\mathrm{T}}\boldsymbol{r}_{\mathrm{SU}}^{(n)}(\mathbf{1}P_{\max}^{(n)})$ and can be found, e.g., by means of a fixed point iteration. Therefore, it suffices to consider the vectors $\boldsymbol{\rho}^{(n)} = \boldsymbol{r}^{(n)}(\boldsymbol{p}^{(n)}) \leq \boldsymbol{r}_{\mathrm{SU}}^{(n)}(\mathbf{1}\mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)})$ with $\mathbf{1}^{\mathrm{T}}\boldsymbol{p}^{(n)} \leq P_{\max}^{(n)}$, i.e., we choose $\boldsymbol{b}_0 = \boldsymbol{r}_{\mathrm{SU}}^{(n)}(\mathbf{1}P_{\max}^{(n)})$. If the weights μ_k are small, we find $P_{\max}^{(n)} = 0$, and the optimizer of (16) is $\boldsymbol{\rho}^{(n)} = \mathbf{0}$.

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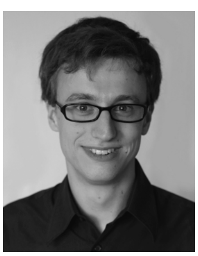

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