Transmit and Receive Array Gain of Uniform Linear Arrays of Isotrops

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***Abstract*—Both in the signal processing and in the information theory literature, it is common to assume that the array gain of antenna arrays grows linearly with the number of antennas. However, such an assertion is valid only when the antennas are uncoupled. When antenna coupling is present and properly taken into account by the signal processing algorithms, we show that both the transmit and the receive array gain can grow super-linearly with the number of antennas. In special cases, the receive array gain can even grow exponentially with the antenna number. These results are exciting, for they imply that the potential of radio communication systems which use more than one antenna at the receiver or the transmitter, is likely to be much higher than previously reported.**

I. Introduction

When it comes to the analytical treatment of antenna arrays it is common practice, both in the signal processing and in the information theory literature, to assume that the individual antennas of the antenna array are *uncoupled* isotropic radiators (see, e.g. [1], chapter 7). A consequence of this assumption is that the array gain grows linearly with the number of antennas [1]. It has therefore become common knowledge in the information theory literature, that the channel capacity of wireless multi-input single-output (MISO), or single-input multi-output (SIMO) communication systems grows proportional to the logarithm of the number of antennas, when the signal to noise ratio (SNR) is large.

On the other hand, closely spaced antennas will, in general, exhibit some electro-magnetic coupling [2]. It is an interesting and highly relevant question, how such antenna coupling influences the achievable array gain, and hence, the channel capacity of multi-antenna radio communication systems, especially when coupling effects are taken into account by the signal processing algorithms.

In this paper, we demonstrate that both the transmit and the receive array gain may be substantially *increased* when antenna coupling is present and taken into account when computing the respective beamforming vectors. For the case of uniform linear antenna arrays of isotropic radiators, we show that the array gains can grow *super-linearly* with the number of antennas. In case that the receiver noise is of purely thermal origin, it turns out that the receive array gain even grows *exponentially* with the antenna number. Consequently, there are cases where the channel capacity of a wireless SIMO communication system can grow *linearly* with the number of antennas. This is a property that has previously been attributed only to multi-input multi-output (MIMO) communication systems.

These results are exciting, for they show that the potential of multi-antenna wireless communication systems might well be substantially higher than previously reported. Key to this increased potential are the physical coupling effects of antenna elements in the array.

II. Transmit Array

Let us start by focusing on a communication system, where the transmitter is equipped with N isotropic antennas, lined up into a uniform linear array (ULA). The receiver shall possess a single isotropic antenna. The electric currents that flow through the transmit antennas excite an electric field which is turned into an electric voltage by the receive antenna. In the absence of receiver noise, the input-output relationship can therefore be written as

$$v = \boldsymbol{z}^{\mathrm{T}} \boldsymbol{i}, \tag{1}$$

where $v \in \mathbb{C} \cdot \mathrm{V}$, is the narrow-band complex envelope of the receive voltage, while $\boldsymbol{i} \in \mathbb{C}^{N\times 1} \cdot \mathrm{A}$, denotes the vector of the narrow-band complex envelopes of the electric currents flowing through the transmit antennas. The vector $\boldsymbol{z} \in \mathbb{C}^{N\times 1} \cdot \Omega$, is the *transimpedance* vector, which depends on the wave propagation between transmitter and receiver. Herein, the superscript $^{\mathrm{T}}$ denotes transposition, while V, A, and Ω, denote the physical units »Volt«, »Ampere«, and »Ohm«, respectively.

A. Transmit Power

We define as transmit power, the power that is radiated by the antenna array. In the following we assume that the antenna array is lossless, such that the electric input power is equal to the radiated power. As antennas are linear circuit elements, the relationship between the vector $\boldsymbol{u} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, of voltage envelopes at the transmit antennas, and the current envelope vector $\boldsymbol{i}$, is a linear one: $\boldsymbol{u} = \boldsymbol{Z}\boldsymbol{i}$, where $\boldsymbol{Z} \in \mathbb{C}^{N\times N} \cdot \Omega$, is the so-called *impedance matrix* of the antenna array. Since antennas are reciprocal, $\boldsymbol{Z} = \boldsymbol{Z}^{\mathrm{T}}$ holds true. The transmit power is given by the expression $P_{\mathrm{Tx}} = \mathrm{E}\left[\mathrm{Re}\left\{\boldsymbol{u}^{\mathrm{H}}\boldsymbol{i}\right\}\right]$, which can also be written as:

$$P_{\mathrm{Tx}} = \mathrm{E}\left[\boldsymbol{i}^{\mathrm{H}}\,\mathrm{Re}\{\boldsymbol{Z}\}\,\boldsymbol{i}\right]. \tag{2}$$

The symbol E, and the superscript $^{\mathrm{H}}$, hereby denote the expectation operation, and the complex conjugate transpose, respectively. For a ULA of isotropic radiators, the real-part of the impedance matrix can be computed in closed form [3]:

$$\mathrm{Re}\{\boldsymbol{Z}\} = R_0 \cdot \boldsymbol{C} \tag{3}$$

$$(\boldsymbol{C})_{m,n} = \mathrm{sinc}\left(2\pi\frac{\Delta}{\lambda}(m-n)\right), \tag{3a}$$

where $\mathrm{sinc}(x) = \sin(x)/x$, λ is the wavelength, Δ is the distance between neighboring antennas, and $R_0 \in \mathbb{R}_+ \cdot \Omega$, is the radiation resistance. For convenience, a derivation of (3) and (3a) is given in the appendix. As we will see, the transmit array gain does not depend on the value of R_0. What matters is the *relative* antenna coupling matrix: $\boldsymbol{C} \in \mathbb{R}^{N\times N}$. As can be observed from (3a), *only* when the element spacing Δ happens to be an integer multiple of $\lambda/2$, the antennas are uncoupled, because the $\boldsymbol{C}$–matrix becomes the identity. For all other spacings, the antennas are coupled.

B. Optimum Beamforming

The transmitter uses its transmit antennas for beamforming:

$$\boldsymbol{i} = \boldsymbol{w} \cdot s, \tag{4}$$

where $\boldsymbol{w} \in \mathbb{C}^{N\times 1} \times \mathrm{A}$ is the beamforming vector, while $s \in \mathbb{C}$, is a zero-mean, unity variance, random variable, which models the information signal that should be transfered to the receiver. The receive antenna is terminated by a resistive load. Consequently, the receive power is proportional to the average squared magnitude of the receive voltage envelope:

$$P_{\mathrm{Rx}} = \gamma \mathrm{E}\left[|v|^2\right], \tag{5}$$

$$= \gamma \boldsymbol{w}^{\mathrm{H}} \boldsymbol{z}^* \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}, \tag{5a}$$

where $\gamma \in \mathbb{R}_+ \cdot \Omega^{-1}$, is a positive constant, and (5a) follows by substituting (4) into (1), and the latter into (5). Herein, $\boldsymbol{z}$ is assumed to be constant. The optimum beamforming vector, for a given $\boldsymbol{z}$, is the one which maximizes the received signal power for a given transmit power P_{Tx}:

$$\boldsymbol{w}_{\mathrm{opt}} = \arg\max_{\boldsymbol{w}} \boldsymbol{w}^{\mathrm{H}} \boldsymbol{z}^* \boldsymbol{z}^{\mathrm{T}} \boldsymbol{w}, \quad \text{s.t.} \quad R_0 \cdot \boldsymbol{w}^{\mathrm{H}} \boldsymbol{C} \boldsymbol{w} \le P_{\mathrm{Tx}}, \tag{6}$$

where the constraint follows from putting (3) and (4) into (2). It can be shown [3], that the receive power under optimum beamforming is given by:

$$P_{\mathrm{Rx}}^{\max} = \gamma \mathrm{E}\left[\, |v|^2 \,\middle|\, \boldsymbol{w} = \boldsymbol{w}_{\mathrm{opt}} \right]$$

$$= \frac{P_{\mathrm{Tx}} \gamma}{R_0} \boldsymbol{z}^{\mathrm{T}} \boldsymbol{C}^{-1} \boldsymbol{z}^*. \tag{7}$$

C. Transmit Array Gain

If only *one* transmit antenna, say the n-th one, is excited by a non-zero electric current, the receive power becomes:

$$P_{\mathrm{Rx},n} = \frac{P_{\mathrm{Tx}} \gamma}{R_0} |z_n|^2,$$

where z_n is the n-th component of $\boldsymbol{z}$. The average received power over all possible transmit antennas then becomes:

$$P_{\mathrm{Rx,avr}}^{\mathrm{single}} = \frac{P_{\mathrm{Tx}} \gamma}{N R_0} \sum_{n=1}^{N} |z_n|^2, \tag{8}$$

The *transmit array gain* is defined as:

$$A_{\mathrm{Tx}} = \frac{P_{\mathrm{Rx}}^{\max}}{P_{\mathrm{Rx,avr}}^{\mathrm{single}}}. \tag{9}$$

When we substitute (7), and (8) into (9), it follows that

$$A_{\mathrm{Tx}} = N \frac{\boldsymbol{z}^{\mathrm{T}} \boldsymbol{C}^{-1} \boldsymbol{z}^*}{\boldsymbol{z}^{\mathrm{T}} \boldsymbol{z}^*}. \tag{10}$$

Recall that only when the antennas are spaced an integer multiple of half the wavelength apart, the antennas are uncoupled, that is, $\boldsymbol{C} = \mathbf{I}$. For these special spacings only, the transmit array gain equals the number N of antennas. All other antenna spacings make the transmit array gain also depend on the antenna spacing (via $\boldsymbol{C}$), and on the transimpedance vector $\boldsymbol{z}$, that is, on the propagation scenario!

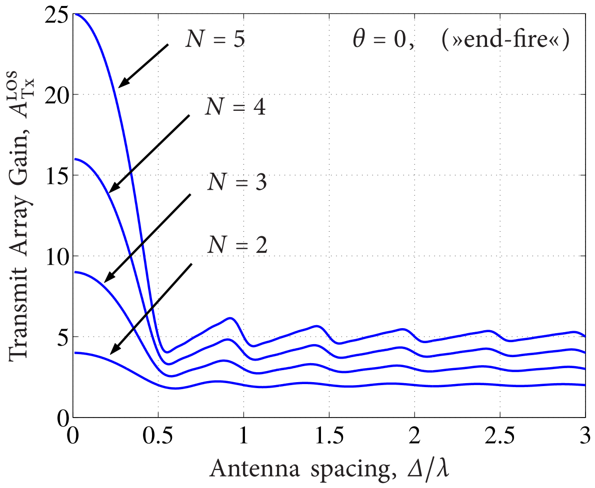

Figure 1: Transmit array gain in »end-fire« LOS ($\theta = 0$), as function of antenna spacing for different antenna numbers.

D. Scaling Law

Let us now look at a line of sight (LOS) propagation scenario. We assume that the antenna array is centered in the origin and aligned with the z-axis of a Cartesian coordinate system. Because there is just one path connecting receiver and transmitter, and the receive voltage v, is proportional to the electric field strength at the receiver, we see from (1), and (30):

$$\boldsymbol{z} = \gamma' \boldsymbol{a}, \tag{11}$$

where γ', is a constant, and

$$\boldsymbol{a} = \left[\, 1 \;\; \mathrm{e}^{-\mathrm{j}\mu} \;\; \cdots \;\; \mathrm{e}^{-\mathrm{j}(N-1)\mu} \,\right]^{\mathrm{T}} \in \mathbb{C}^{N\times 1}, \tag{12}$$

is the array steering vector, where $\mu = 2\pi(\Delta/\lambda)\cos\theta$. Herein, θ is the elevation angle of departure – that is, the angle between wave propagation towards the receiver, and the array axis (the z-axis of the coordinate system). When we substitute (12) into (11), and the latter into (10), we see that the array gain in a LOS scenario becomes:

$$A_{\mathrm{Tx}}^{\mathrm{LOS}} = \boldsymbol{a}^{\mathrm{T}} \boldsymbol{C}^{-1} \boldsymbol{a}^*. \tag{13}$$

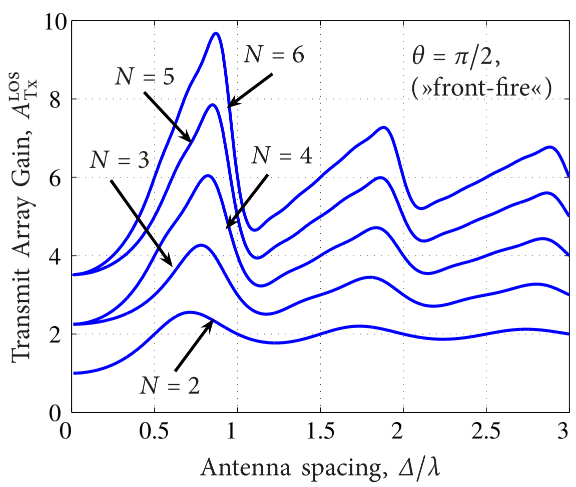

Figure 2: Transmit array gain in the »front-fire« LOS ($\theta = \pi/2$), as function of antenna spacing for different antenna numbers.

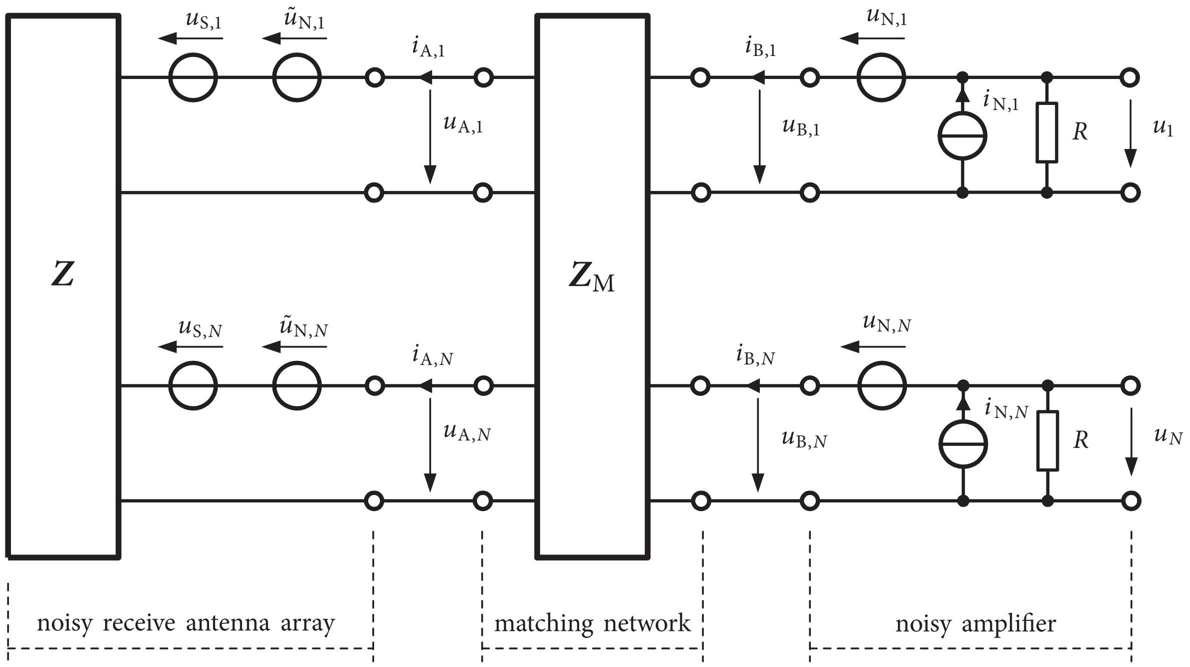

Figure 3: Circuit-theoretic model of a receive antenna array cascaded with an impedance matching network and the inputs of the low-noise amplifiers.

In Figure 1, the array gain that is achievable in the direction $\theta = 0$, is depicted for different N, as a function of the antenna spacing. We can observe that the array gain raises quickly as we reduce the antenna spacing below $\lambda/2$. The maximum array gain happens to be equal to N^2. That is, the transmit array gain can grow *super-linearly* with the number of antennas! Figure 2 shows the achievable array gain in the direction $\theta = \pi/2$. For beamforming in this direction, one has to put the antennas considerably apart, in order to effectively use the array for beamforming. The optimum antenna spacing is larger than $\lambda/2$, but smaller than λ. In this setup, the array gain grows linearly with the number N of antennas, yet its peak value is larger than N.

III. Receive Antenna Array

The receive array gain achievable by optimum beamforming, critically depends on the spatial covariance of receiver noise. In radio communication systems, it makes sense to distinguish between *intrinsic* noise, and *extrinsic* noise. While the latter comes from background radiation received by the antennas, the intrinsic noise originates from components of the radio frequency front end, most importantly, from the first stage of the low noise amplifier (LNA). When intrinsic noise is the sole source of noise, it is the state of the art to conjecture that the receiver noise is spatially white, or at least uncorrelated. However, such a conjecture is, in general, not correct! While it is true that the noise sources inside the LNAs do generate noise independently, the individual noise contributions may superimpose because of coupling effects. Coupling occurs in the antenna array because of the proximity of closely spaced antennas, and because of an impedance matching network that can frequently be found connected between the antenna array and the input of the LNAs.

Every electronic LNA generates two kinds of noise: *voltage* noise, and *current* noise [4]. It turns out that current noise contributes quite differently to the noise covariance than does voltage noise. Therefore, the noise covariance depends on the relative intensity of these two kinds of noise contributions. It is specified by the so-called *noise-resistance*, which is defined as the ratio of the root-mean-square (RMS) noise voltage and the RMS noise current [4]. We show that the lower the noise-resistance, the higher the receive array gain becomes. A low noise resistance may therefore be even more important than a low noise figure! For the case of noise which is purely of intrinsic origin, a noise-resistance of zero turns out to let the receive array gain grow *exponentially* with the number of antennas. This result is exciting, because it demonstrates the fact that it is possible for a SIMO radio communication system, to have a channel capacity that grows *linearly* with the number of antennas, in the high SNR regime! Such a linear growth of channel capacity has previously been reported only for MIMO communications.

A. Circuit-Theoretic System Model

Let us start from the circuit theoretic system model that is shown in Figure 3. It consists of the following three blocks: the antenna array, the impedance matching network, and the inputs of the LNAs. The N–antenna receive array is described as a passive, linear N–port. The received signals are taken care of by two voltage sources per port: the voltages $u_{S,i}$, model the desired received signal, while $\tilde{u}_{N,i}$ model the received background noise (extrinsic noise). The impedance matching network is described by:

$$\begin{bmatrix} \boldsymbol{u}_A \\ \boldsymbol{u}_B \end{bmatrix} = \begin{bmatrix} -\mathrm{j}\,\mathrm{Im}\{\boldsymbol{Z}\} & \mathrm{j}\,\mathrm{Re}\{\boldsymbol{Z}\} \\ \mathrm{j}\,\mathrm{Re}\{\boldsymbol{Z}\} & \mathbf{O} \end{bmatrix} \begin{bmatrix} -\boldsymbol{i}_A \\ \boldsymbol{i}_B \end{bmatrix}. \tag{14}$$

Herein, $\boldsymbol{Z} \in \mathbb{C}^{N\times N} \cdot \Omega$, is the impedance matrix of the antenna array. The two vectors $\boldsymbol{u}_\mathrm{A} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, and $\boldsymbol{u}_\mathrm{B} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, contain the complex envelopes of the voltages at the input and the output of the matching network, respectively, while the vectors $\boldsymbol{i}_\mathrm{A} \in \mathbb{C}^{N\times 1} \cdot \mathrm{A}$, and $\boldsymbol{i}_\mathrm{B} \in \mathbb{C}^{N\times 1} \cdot \mathrm{A}$, contain the respective complex current envelopes. The rationale behind the specific choice of matching network in (14) becomes apparent from:

$$\boldsymbol{u}_\mathrm{B} \;=\; \mathrm{Re}\{\boldsymbol{Z}\}\, \boldsymbol{i}_\mathrm{B} + \mathrm{j}\left(\boldsymbol{u}_\mathrm{S} + \tilde{\boldsymbol{u}}_\mathrm{N}\right), \tag{15}$$

where we have stacked the desired signal voltage envelopes $u_{\mathrm{S},i}$, and the extrinsic noise voltage envelopes $\tilde{u}_{\mathrm{N},i}$ into the vectors $\boldsymbol{u}_\mathrm{S} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, and $\tilde{\boldsymbol{u}}_\mathrm{N} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, respectively. As we can see from (15), the coupling of the *matched* antenna array now depends only on the real-part of the impedance matrix of the antenna array. In this way, we can directly make use of (3), and (3a). The voltage envelopes in $\boldsymbol{u}_\mathrm{B}$ are then fed to the inputs of the LNAs. The latter are modeled by a noiseless resistor R, and two sources which model the amplifier's current and voltage noise envelopes, $u_{\mathrm{N},i}$, and $i_{\mathrm{N},i}$, respectively. Collecting these envelopes into respective vectors $\boldsymbol{u}_\mathrm{N} \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}$, and $\boldsymbol{i}_\mathrm{N} \in \mathbb{C}^{N\times 1} \cdot \mathrm{A}$, we find the following system equation:

$$\boldsymbol{u} \;=\; \boldsymbol{Q}\left(\mathrm{j}\boldsymbol{u}_\mathrm{S} + \mathrm{Re}\{\boldsymbol{Z}\}\, \boldsymbol{i}_\mathrm{N} + \boldsymbol{u}_\mathrm{N} + \mathrm{j}\tilde{\boldsymbol{u}}_\mathrm{N}\right). \tag{16}$$

where $\boldsymbol{u} = \left[\, u_1 \;\, u_2 \;\, \cdots \;\, u_N \,\right]^\mathrm{T}$, is the vector of the output voltage envelopes, and $\boldsymbol{Q} = \left(\mathbf{I}_N + \mathrm{Re}\{\boldsymbol{Z}\}/R\right)^{-1}$. Clearly, the amplifier noise current $\boldsymbol{i}_\mathrm{N}$ contributes to the noise portion of $\boldsymbol{u}$ quite differently than the amplifier noise voltage $\boldsymbol{u}_\mathrm{N}$, or the background noise $\tilde{\boldsymbol{u}}_\mathrm{N}$.

B. Signal Model

Let us focus again on LOS propagation. The desired received signal is given by: $\boldsymbol{u}_\mathrm{S} = u_\mathrm{S} \cdot \boldsymbol{a}$, where $\boldsymbol{a}$, is the array steering vector from (12), and $u_\mathrm{S} \in \mathbb{C} \cdot \mathrm{V}$, is the information bearing signal voltage envelope. By defining the receiver noise voltage vector as the vector of output voltage envelopes for $\boldsymbol{u}_\mathrm{S} = \mathbf{0}$:

$$\boldsymbol{\eta} \;=\; \boldsymbol{Q}\left(\mathrm{Re}\{\boldsymbol{Z}_\mathrm{A}\}\, \boldsymbol{i}_\mathrm{N} + \boldsymbol{u}_\mathrm{N} + \mathrm{j}\tilde{\boldsymbol{u}}_\mathrm{N}\right) \in \mathbb{C}^{N\times 1} \cdot \mathrm{V}, \tag{17}$$

we can rewrite the system equation (16) as:

$$\boldsymbol{u} \;=\; \boldsymbol{h} \cdot u_\mathrm{S} + \boldsymbol{\eta}, \tag{18}$$

where

$$\boldsymbol{h} \;=\; \mathrm{j}\boldsymbol{Q}\boldsymbol{a} \;\in \mathbb{C}^{N\times 1}. \tag{19}$$

In this paper, we assume that all LNAs have the same current noise intensity, but the current noises of different LNAs are uncorrelated, such that $\mathrm{E}\left[\boldsymbol{i}_\mathrm{N}\boldsymbol{i}_\mathrm{N}^\mathrm{H}\right] = \beta\mathbf{I}_N$, where $\beta \in \mathbb{R}_+ \cdot \mathrm{A}^2$, is a constant, specific to the LNA, and proportional to the noise bandwidth. Similarly, $\mathrm{E}\left[\boldsymbol{u}_\mathrm{N}\boldsymbol{u}_\mathrm{N}^\mathrm{H}\right] = \beta' R^2\mathbf{I}_N$, where $\beta' \in \mathbb{R}_+ \cdot \mathrm{A}^2$, is a constant, that is specific to the LNA, and proportional to the noise bandwidth. For the reason of brevity, we limit the discussion to the case where there is no background noise, hence: $\mathrm{E}\left[\tilde{\boldsymbol{u}}_\mathrm{N}\tilde{\boldsymbol{u}}_\mathrm{N}^\mathrm{H}\right] = \mathbf{O}$. When we assume that all noise sources are uncorrelated, we find that the receiver noise covariance $\boldsymbol{R}_{\boldsymbol{\eta}} = \mathrm{E}\left[\boldsymbol{\eta}\boldsymbol{\eta}^\mathrm{H}\right]$, can be written as:

$$\boldsymbol{R}_{\boldsymbol{\eta}} = \beta R^2 \boldsymbol{Q}\underbrace{\left(\frac{1}{R^2}\left(\mathrm{Re}\{\boldsymbol{Z}\}\right)^2 + \frac{R_\mathrm{N}^2}{R^2}\mathbf{I}_N\right)}_{\boldsymbol{\Upsilon}}\boldsymbol{Q}, \tag{20}$$

where the *noise-resistance* is defined as:

$$R_\mathrm{N} = \sqrt{\frac{\mathrm{E}\left[|u_{\mathrm{N},i}|^2\right]}{\mathrm{E}\left[|i_{\mathrm{N},i}|^2\right]}} = R\sqrt{\frac{\beta'}{\beta}}. \tag{21}$$

C. Optimum Receive Beamforming

The receiver performs linear beamforming, according to:

$$s \;=\; \boldsymbol{w}^\mathrm{H}\boldsymbol{u} \;=\; \boldsymbol{w}^\mathrm{H}\boldsymbol{h} \cdot u_\mathrm{S} + \boldsymbol{w}^\mathrm{H}\boldsymbol{\eta}, \tag{22}$$

where $\boldsymbol{w} \in \mathbb{C}^{N\times 1}$ is the beamforming vector, and $s \in \mathbb{C} \cdot \mathrm{V}$ is the resulting scalar voltage envelope. The SNR becomes:

$$\mathrm{SNR} \;=\; \mathrm{E}\left[|u_\mathrm{S}|^2\right] \frac{\boldsymbol{w}^\mathrm{H}\boldsymbol{h}\boldsymbol{h}^\mathrm{H}\boldsymbol{w}}{\boldsymbol{w}^\mathrm{H}\boldsymbol{R}_{\boldsymbol{\eta}}\boldsymbol{w}}. \tag{23}$$

It is not too difficult to derive the maximum achievable SNR:

$$\mathrm{SNR}_\mathrm{max} \;=\; \mathrm{E}\left[|u_\mathrm{S}|^2\right] \boldsymbol{h}^\mathrm{H}\boldsymbol{R}_{\boldsymbol{\eta}}^{-1}\boldsymbol{h}. \tag{24}$$

When we substitute (19) and (20) into (24) we arrive at:

$$\mathrm{SNR}_\mathrm{max} \;=\; \mathrm{E}\left[|u_\mathrm{S}|^2\right] \frac{\boldsymbol{a}^\mathrm{H}\boldsymbol{\Upsilon}^{-1}\boldsymbol{a}}{\beta R^2}, \tag{25}$$

where the matrix $\boldsymbol{\Upsilon} \in \mathbb{C}^{N\times N}$ is defined in (20).

D. Receive Array Gain

If only a single isotropic radiator is present, we see by setting $N = 1$ in (25), that the SNR becomes

$$\mathrm{SNR}_\mathrm{single} \;=\; \frac{\mathrm{E}\left[|u_\mathrm{S}|^2\right]}{\beta\left(R_0^2 + R_\mathrm{N}^2\right)}, \tag{26}$$

as from (20), $\Upsilon = (R_0^2 + R_\mathrm{N}^2)/R^2$, for $N = 1$. The receive array gain A_Rx, quantifies how much more SNR we can obtain by using all antennas of the array, compared to a single antenna:

$$A_\mathrm{Rx} \;=\; \frac{\mathrm{SNR}_\mathrm{max}}{\mathrm{SNR}_\mathrm{single}} \tag{27}$$

$$\;=\; \boldsymbol{a}^\mathrm{H}\boldsymbol{\Upsilon}^{-1}\boldsymbol{a}\,\frac{R_0^2 + R_\mathrm{N}^2}{R^2}. \tag{27a}$$

In the following, we set the input resistance R of the LNAs equal to the radiation resistance R_0 of the isotropic antenna. As furthermore $\boldsymbol{a}^\mathrm{H}\boldsymbol{a} = N$, we find for the receive array gain:

$$A_\mathrm{Rx} \;=\; N\frac{\boldsymbol{a}^\mathrm{H}\boldsymbol{\Upsilon}^{-1}\boldsymbol{a}}{\boldsymbol{a}^\mathrm{H}\boldsymbol{a}}\left(1 + \frac{R_\mathrm{N}^2}{R_0^2}\right). \tag{28}$$

Note that $\boldsymbol{\Upsilon}$ depends on the antenna spacing Δ/λ, by virtue of the matrix $\boldsymbol{Z}$. It also depends on the noise resistance R_N.

E. The Two-Antenna Array

Let us now have a look how much receive array gain can be obtained by a ULA of isotropic radiators. Let us fix the angle θ of the beamforming to the value of $\theta = 0$, which corresponds to the direction of the ULA line-up – the so-called »end-fire« direction. For a small array of $N = 2$ isotrops, we see in Figure 4, the receive array gain in dB, as function of antenna separation. As was pointed out earlier, current noise contributes differently to the overall receiver noise than does

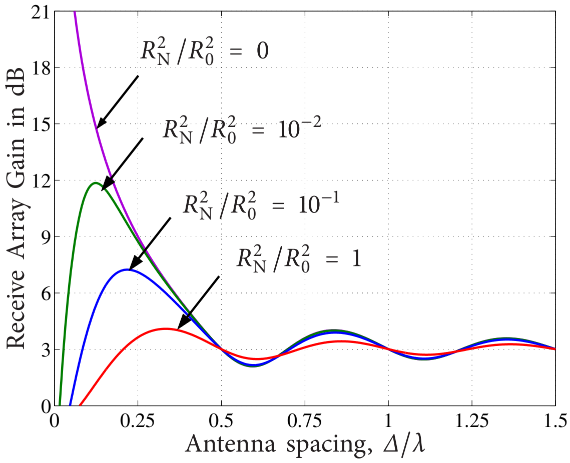

Figure 4: Receive array gain in dB in »end-fire« direction, for two antennas, as function of antenna spacing.

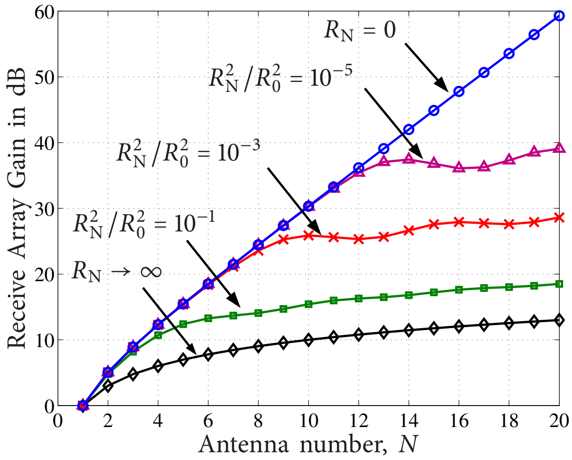

Figure 5: Receive array gain in dB in »end-fire« direction, as function of the antenna number, for $\Delta = 0.4\lambda$.

voltage noise. Figure 4 therefore contains four different curves, which correspond to different values of the noise resistance. A noise resistance of zero, means that the noise originates solely from current noise. This is the case for thermal noise of a resistor. One can therefore argue that zero noise resistance corresponds to the case of having an *ideal* amplifier, which does not introduce any additional noise besides the inescapable thermal noise of the real-part of its input impedance. Hence, as can be seen from the top most curve in Figure 4, with ideal amplifiers, the receive array gain grows unboundedly, as the distance of the antennas is reduced towards zero. However, as soon as the noise resistance is even slightly greater than zero, the receive array gain first grows towards a maximum value as the antenna separation is reduced, and then begins to drop. We can see from Figure 4, that the peak receive array gain increases with decreasing noise resistance. For high array gain it is therefore important, that the LNAs are designed such that current noise dominates over voltage noise. A low noise resistance may be become more important than a low noise figure! Note that for an antenna separation of half of the wavelength, or integer multiples thereof, the receive array gain always equals the number of antennas. This effect comes about, because the antennas are uncoupled for these separations, as we have pointed out already in Section II-C.

F. The Scaling Law

The noise-resistance also plays a key role in how the receive array gain scales with the number of antennas. In Figure 5, we see the receive array gain in dB for the direction of beamforming fixed to $\theta = 0$. For a fixed antenna spacing of $\Delta = 0.4\lambda$, we have drawn five different curves, each of which corresponds to a different noise-resistance. The best and most remarkable performance is achieved as the noise-resistance is reduced to zero. As we have discussed in Section III-E, this case models the ideal amplifier, which only generates thermal noise inside the real-part of its input impedance. As we can see from the top-most curve in Figure 5, the receive array gain grows *exponentially* with the number of antennas. Because at high SNR the channel capacity grows proportional to the logarithm of SNR, the exponential growth of the receive array gain with the number of antennas leads to a *linear* growth of the channel capacity with the antenna number. Such a linear growth has previously been attributed only to MIMO communication systems, which can exploit spatial multiplexing, that is, can transfer several data streams at the same time using the same band of frequencies. However, as we have demonstrated, it is possible to achieve a linear growth of channel capacity with respect to the antenna number, already with a wireless SIMO system, that is, a radio communication system which employs multiple antennas only at the receiver.

When the noise-resistance is increased, we observe from Figure 5, that the exponential growth of the receive array gain with respect to N, starts to flatten out as N is increased over a certain limit, which depends on the noise-resistance. In the limit of an infinite noise-resistance – which just means that there is only voltage noise present at the receiver –, there is no longer any exponential growth, but the receive array gain merely increases linearly with N, just as if the antennas were uncoupled. In order to enjoy high performance it is necessary to make sure that we have both a low noise-resistance, and coupled antennas. The former is a matter of high-frequency engineering of the LNA, while the latter can be achieved by using antenna spacings below half the wave-length. As can also be observed from Figure 5, it is by no means necessary to space the antennas very closely, since a rather moderate spacing of $\Delta = 0.4\lambda$, already shows significant increase of the receive array gain compared to the »vanilla«, half-wavelength spaced antenna array.

For finite noise-resistance which is larger than zero, it turns out that the receive array gain does not increase monotonically with the number of antennas, as can be observed from two of the curves in Figure 5. That is, it can happen that increasing the number of antennas actually leads to a lower receive array gain. At first glance, this effect may look surprising, or even strange. However, it can be understood from the fact that adding one antenna to an array, changes the way how the other antennas perceive the channel. To be more specific, the channel vector from (19), usually changes in all of its components, when we add one more antenna to the array, because the new antenna is located in the near-field of its neighbors, and therefore, usually changes the electro-magnetic field perceived by the remaining antennas. This is the reason behind the fact that adding one antenna may not necessarily improve performance.

It is interesting to note that there is a fundamental difference between receive and transmit array gain, since the transmit array gain only may grow quadratic with N, in a line of sight propagation scenario. Therefore, it usually *does* matter whether the multiple antennas are located at the transmitting or the receiving end of the link. Especially, note that when $R_{\mathrm{N}} = 0$, we have from (20) that $\boldsymbol{Y} = \boldsymbol{C}^2$, for $R = R_0$, such that from (28) and $\boldsymbol{a}^{\mathrm{H}}\boldsymbol{a} = N$, follows

$$A_{\mathrm{Rx}} = \boldsymbol{a}^{\mathrm{H}}\boldsymbol{C}^{-2}\boldsymbol{a}.$$

Comparison with (13) shows the fundamental difference in the two array gains explicitly, because the coupling matrix $\boldsymbol{C}$ appears in the second inverse power for the receive array gain, but only in its first inverse power for the transmit array gain. Only when $\boldsymbol{C} = \mathbf{I}$, there is no difference.

IV. Conclusion

Receive array gain depends on four factors: 1) the number of antennas, their radiation characteristics and spacing, 2) the noise-resistance of the receive amplifiers, 3) the direction of beamforming, and 4) the properties of background radiation. In order to obtain large array gain, the amplifier current noise should significantly dominate over its voltage noise and the received background radiation. Therefore, a low noise-resistance may be even more important than low noise-figure! Large receive array gains are possible, even increasing exponentially with the number of antennas. In theory, the channel capacity of wireless SIMO systems can therefore grow linearly with the number of antennas – a property which is usually ascribed only to multi-input multi-output (MIMO) systems.

On the other hand, the transmit array gain depends on two factors: 1) the number of antennas, their radiation characteristics and spacing, and 2) the direction of beamforming. In a line-of-sight scenario, the transmit array gain can come arbitrarily close to the square of the number of transmit antennas. For this to happen, it is essential that the antenna spacing is below half the wavelength. Exponential growth of transmit array gain is, however, not possible in a line of sight propagation scenario. Consequently, there is, in general, a fundamental difference between wireless MISO and SIMO communication systems, which only vanishes in case that the antennas that form the array are uncoupled. In case of uniform linear arrays of isotropic radiators, this happens when the antenna spacing is chosen to be an integer multiple of half of the wavelength.

Antenna array based wireless communication systems may possess a much higher potential than previously reported. The key to this increased potential lies in the antenna coupling of closely spaced antenna arrays, that has to be known and taken into account by transmit and receive signal processing.

It should be noted, that even though we have used isotropic radiators for ease of development, the results also hold qualitatively for more realistic antennas, as is demonstrated by the authors in [5], for arrays of Hertzian dipoles.

Appendix

The complex envelope of the electric field strength in a distance r and elevation angle θ, in the far field of a ULA of isotropic radiators, located in the origin and aligned with the z-axis, can be written as (see e.g., [6] on pp 250 and 258):

$$E = \alpha \frac{\mathrm{e}^{-\mathrm{j}2\pi r/\lambda}}{r} \sum_{n=1}^{N} \mathrm{e}^{-\mathrm{j}2\pi\frac{\Delta}{\lambda}(n-1)\cos\theta} \cdot i_n, \tag{29}$$

where i_n is the complex envelope of the current flowing into the n-th radiator, λ is the wavelength, and α is a constant. Defining the array steering vector $\boldsymbol{a} = \begin{bmatrix} 1 & \mathrm{e}^{-\mathrm{j}\mu} & \cdots & \mathrm{e}^{-\mathrm{j}(N-1)\mu} \end{bmatrix}^{\mathrm{T}}$, where $\mu = 2\pi(\Delta/\lambda)\cos\theta$, we can rewrite (29) as:

$$E = \alpha \frac{\mathrm{e}^{-\mathrm{j}2\pi r/\lambda}}{r} \boldsymbol{a}^{\mathrm{T}}\boldsymbol{i}, \tag{30}$$

where $\boldsymbol{i} = \begin{bmatrix} i_1 & i_2 & \cdots & i_N \end{bmatrix}^{\mathrm{T}}$. The power density is then given by (see e.g., [7], eq. (8) on page 571):

$$\begin{aligned} S &= \alpha' \,\mathrm{E}\left[|E|^2\right] \\ &= \frac{\alpha' |\alpha|^2}{r^2} \mathrm{E}\left[\boldsymbol{i}^{\mathrm{H}} \boldsymbol{a}^*(\theta)\, \boldsymbol{a}^{\mathrm{T}}(\theta)\, \boldsymbol{i}\right], \end{aligned} \tag{31}$$

where $\alpha' > 0$ is another constant. The radiated power is then obtained by integrating the power density over a sphere in the far-field around the array:

$$\begin{aligned} P_{\mathrm{rad}} &= \iint_{\mathrm{sphere}} S \,\mathrm{d}A \\ &= \underbrace{4\pi\alpha' |\alpha|^2}_{R_0} \mathrm{E}\left[\boldsymbol{i}^{\mathrm{H}} \underbrace{\left(\frac{1}{2}\int_0^{\pi} \boldsymbol{a}^*(\theta)\, \boldsymbol{a}^{\mathrm{T}}(\theta) \sin(\theta)\,\mathrm{d}\theta\right)}_{\boldsymbol{C}} \boldsymbol{i}\right]. \end{aligned} \tag{32}$$

Assuming that the radiators are lossless, the radiated power equals the electric input power from (2), such that the real-part of the array impedance matrix can be readily obtained from (32):

$$\mathrm{Re}\{\boldsymbol{Z}\} = R_0 \cdot \boldsymbol{C}, \tag{33}$$

where the radiation resistance $R_0 \in \mathbb{R}_+ \cdot \Omega$, and the coupling matrix $\boldsymbol{C} \in \mathbb{R}^{N\times N}$, are defined in (32). The result (3a) then follows from (32) and (12) by standard integration:

$$(\boldsymbol{C})_{m,n} = \frac{1}{2}\int_0^{\pi} \mathrm{e}^{\mathrm{j}(m-n)2\pi\frac{\Delta}{\lambda}\cos\theta} \sin(\theta)\,\mathrm{d}\theta \tag{34}$$

$$= \operatorname{sinc}\left(2\pi\frac{\Delta}{\lambda}(m-n)\right). \tag{34a}$$

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