Flavored Model Building

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Abstract

In this thesis we discuss possibilities to solve the family replication problem and to understand the observed strong hierarchy among the fermion masses and the diverse mixing pattern of quarks and leptons. We show that non-abelian discrete symmetries which act non-trivially in generation space can serve as profound explanation. We present three low energy models with the permutation symmetry S_4, the dihedral group D_5 and the double-valued group T' as flavor symmetry. The T' model turns out to be very predictive, since it explains tri-bimaximal mixing in the lepton sector and, moreover, leads to two non-trivial relations in the quark sector, $\sqrt{\frac{m_d}{m_s}} = |V_{us}|$ and $\sqrt{\frac{m_d}{m_s}} = \left|\frac{V_{td}}{V_{ts}}\right|$. The main message of the T' model is the observation that the diverse pattern in the quark and lepton mixings can be well-understood, if the flavor symmetry is not broken in an arbitrary way, but only to residual (non-trivial) subgroups. Apart from leading to deeper insights into the origin of the fermion mixings this idea enables us to perform systematic studies of large classes of discrete groups. This we show in our study of dihedral symmetries D_n and D'_n. As a result we find only five distinct (Dirac) mass matrix structures arising from a dihedral group, if we additionally require partial unification of either left-handed or left-handed conjugate fermions and the determinant of the mass matrix to be non-vanishing. Furthermore, we reveal the ability of dihedral groups to predict the Cabibbo angle θ_C, i.e. $|V_{us(cd)}| = \cos(\frac{3\pi}{7})$, as well as maximal atmospheric mixing, $\theta_{23} = \frac{\pi}{4}$, and vanishing θ_{13} in the lepton sector.

Kurzfassung

In dieser Dissertation diskutieren wir Möglichkeiten, das Problem der Familien-Replikation zu lösen und die beobachtete starke Hierarchie unter den Fermionmassen und das verschiedenartige Mischungsmuster von Quarks and Leptonen zu verstehen. Wir zeigen, daß nicht-abelsche diskrete Symmetrien, welche nicht-trivial im Generationenraum agieren, als tiefere Erklärung dienen können. Wir präsentieren drei Niederenergie-Modelle mit der Permutationssymmetrie S_4, der dihedrischen Gruppe D_5 und der zwei-wertigen Gruppe T' als Generationensymmetrie. Das T' Modell erweist sich als sehr vorhersagekräftig, da es tri-bimaximale Mischung im Leptonensektor erklärt und darüberhinaus zu zwei nicht-trivialen Beziehungen im Quarksektor führt, $\sqrt{\frac{m_d}{m_s}} = |V_{us}|$ und $\sqrt{\frac{m_d}{m_s}} = \left|\frac{V_{td}}{V_{ts}}\right|$. Die Hauptaussage des T' Modells ist die Beobachtung, daß das verschiedenartige Muster in den Quark- und Leptonmischungen wohlverstanden werden kann, falls die Generationensymmetrie nicht in beliebiger Weise gebrochen wird, sondern nur zu verbleibenden (nicht-trivialen) Untergruppen. Abgesehen von tieferen Einblicken in den Ursprung von Fermionmischungen erlaubt es uns diese Idee, systematische Studien von großen Klassen diskreter Gruppen durchzuführen. Dies zeigen wir in unserer Studie über dihedrische Symmetrien D_n und D'_n. Als Ergebnis finden wir nur fünf unterschiedliche (Dirac-) Massenmatrixstrukturen, die von einer dihedrischen Gruppe stammen können, falls wir zusätzlich die partielle Vereinheitlichung von entweder links-händigen oder links-händigen konjugierten Fermionen verlangen sowie erwarten, daß die Determinante der Massenmatrix nicht-verschwindend ist. Desweiteren legen wir die Fähigkeit dihedrischer Gruppen offen, den Cabibbo Winkel θ_C, d.h. $|V_{us(cd)}| = \cos(\frac{3\pi}{7})$, wie auch maximale atmosphärische Mischung, $\theta_{23} = \frac{\pi}{4}$, und verschwindendes θ_{13} im Leptonensektor vorherzusagen.

Contents

Chapter 1

Introduction

The gauge interactions of elementary particles can be well described by the Standard Model (SM) gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$. Furthermore, fundamental properties of the fermions like charge quantization can be understood in the framework of Grand Unified Theories (GUTs) such as $SU(5)$. Supersymmetry (SUSY), the fundamental symmetry of fermions and bosons, is the third ingredient of many successful models. However, there is no convincing model which explains

- the existence of the three generations of fermions,
- the observed strong hierarchy among the charged fermions,
- the fact that quarks have small mixings, while two of the lepton mixing angles are large.

Moreover, if special mixing patterns as $\mu\tau$ symmetry [1–6],

$$\sin^2(\theta_{23}) = \frac{1}{2} \; , \quad \sin^2(\theta_{13}) = 0 \; , \tag{1.1}$$

or tri-bimaximal mixing (TBM) [7–10],

$$\sin^2(\theta_{23}) = \frac{1}{2} \; , \quad \sin^2(\theta_{12}) = \frac{1}{3} \; , \quad \sin^2(\theta_{13}) = 0 \; , \tag{1.2}$$

are realized in the lepton sector, this requires a profound explanation. As the symmetries of the SM turned out to be so successful in describing the gauge interactions, it is tempting to ask whether a symmetry acting on the three generations could be the origin of the different mass and mixing patterns of quarks and leptons. The possible size of the flavor symmetry (frequently also called generation, family or horizontal symmetry) is depending on the choice of the gauge group, i.e. in the SM the maximal possible symmetry is $U(3)^5$ (without left-handed conjugate neutrinos ν^c and $U(3)^6$ with three left-handed conjugate neutrinos), while it is reduced to $U(3)$, if the gauge group is $SO(10)$. However, this constraint on the size of the flavor group is rather loose, since in most of the cases we are dealing with symmetries which can be embedded into $SO(3)$ or $SU(3)$. Furthermore, various properties of this flavor symmetry have to be fixed

- it can be either abelian or non-abelian,
- either discrete or continuous,
- either local or global,
- it can either commute with the gauge group(s) or not.

Abelian symmetries, as $U(1)$ for example, have been shown to be able to explain the observed fermion mass hierarchy, if the charges of the three generations are appropriately chosen. This idea was first proposed by Froggatt and Nielsen in 1979 [11]. However, these symmetries actually cannot give reason to the existence of three generations and cannot predict certain mixing patterns, like TBM, since the irreducible representations of abelian groups are all one-dimensional. Therefore the predictive power of a non-abelian symmetry is in general larger than that of an abelian group. Choosing among a discrete or a continuous group has two main consequences: *a.*) in case of a spontaneously broken symmetry a continuous one leads to the appearance of a Goldstone boson/gauge boson, whereas the breaking of a discrete group does not [1] and *b.*) discrete groups contain several small representations which are all appropriate to describe the three fermion generations, whereas in continuous groups, as $SO(3)$ or $SU(3)$, only a single non-trivial possibility exists to assign the fermion generations. If the symmetry is taken to be local, the requirement of anomaly freedom can pose severe constraints on the assignment of the fermions. Moreover, locality preserves the symmetry from being broken by quantum gravity effects at the Planck scale. Concerning the issue whether the flavor symmetry shall commute with the gauge group(s) or not, we in general assume the simplest case, i.e. these groups commute [2]. This leads to the choice of a non-abelian, discrete flavor symmetry commuting with the gauge group(s) which might also be realized locally. Since we do not observe an intact flavor symmetry at (very) low energies, it must be broken. Thereby, we need to specify the scale of flavor symmetry breaking and whether it should be broken spontaneously or explicitly. As the explicit breaking generally leads to several additional parameters, we confine ourselves to the discussion of spontaneously broken flavor symmetries. In order not to introduce further scales into the theory, the flavor group is broken at the electroweak scale together with the SM gauge group. For this purpose, we have to assume the existence of several copies of the SM Higgs doublet which transform non-trivially under the flavor group. It is well-known that such multi-Higgs doublet models are severely constrained by direct searches for Higgs bosons and even more by indirect bounds from flavor changing neutral currents (FCNCs) and lepton flavor violating processes (LFVs). However, even such semi-realistic models already give insights into certain fundamental features of a discrete flavor symmetry. In Chapter 3, we present two models which are constructed in this manner. In the first model the permutation group S_4 is employed as flavor symmetry, while the flavor structure of the second model is determined by the dihedral symmetry D_5. The fermion assignment under the flavor group is (almost) uniquely determined by additional requirements. In the case of S_4 we demand that the model can be simultaneously embedded into an $SO(10)$ GUT and into a continuous flavor symmetry, $SO(3)_f$ or $SU(3)_f$. The group S_4 can explain the existence of exactly three generations due to its representation structure. In particular, the embedding into a continuous flavor group leads to the existence of several Higgs doublets in this model. Therefore, we pursue a more minimalist approach in the discussion of the D_5 model. The result is a low energy model with four Higgs doublets, which can still be embedded into the Pati-Salam gauge group $SU(4)_C \times SU(2)_L \times SU(2)_R$, but the embedding into a continuous flavor symmetry is not straightforward. The number of Higgs fields is this time bounded from below by the requirements that the Higgs potential has to be free from accidental symmetries and all fermion masses and their mixings ought to be fitted at tree level. In contrast to the first model, only two of the three generations of fermions can be unified into irreducible D_5 representations. These models have in common that they are able to accommodate the experimental data, but cannot make any predictions. The reason for this is twofold: on the one hand there still exist several

[1] The possible problem of domain walls arising from a spontaneously broken discrete group [12] could be solved, for example, by (low scale) inflation.

[2] There exist only very few models in the literature in which the gauge and the flavor group(s) do not commute, for example [13].

possible couplings in the Yukawa sector, albeit the flavor symmetry constrains the matrix structures, and on the other hand the vacuum expectation values (VEVs) of the Higgs fields (appearing in the mass matrices) can in general only be adjusted, but not predicted due to the complicated form of the Higgs potentials. Therefore, we continue in Chapter 4 with a model in which the flavor and the electroweak symmetry breaking scale are disentangled. The flavor group which is the double-valued tetrahedral group T' is broken spontaneously by gauge singlets (flavons) at a scale of $(10^{11}...10^{13})$ GeV. The gauge group of the model is still the SM and supersymmetry is introduced as an additional ingredient. Therefore, the framework of this model is the minimal supersymmetric Standard Model (MSSM). In contrast to the two models presented in Chapter 3, the T' model leads to several predictions, namely TBM in the lepton sector and two relations among $|V_{us}|$, $|V_{td}/V_{ts}|$ and m_d/m_s, $\sqrt{\frac{m_d}{m_s}} = |V_{us}|$ and $\sqrt{\frac{m_d}{m_s}} = \left|\frac{V_{td}}{V_{ts}}\right|$, in the quark sector. According to the group theory of T' and the prediction of TBM this model is intimately connected to approaches [14, 15] [16–21] describing the lepton sector with the help of the alternating group A_4. Since the Higgs fields do not transform under the flavor symmetry T', the Yukawa couplings are in general non-renormalizable operators suppressed by the cutoff scale Λ of the theory. As the leading order result is not completely satisfactory in the quark sector, the main challenge in this model is to generate the Cabibbo angle $\theta_C = \lambda \approx 0.22$ and appropriate masses for the quarks of the first family via next-to-leading order effects only. These have to be kept at a level of λ^2 for leptons in order not to spoil TBM. Apart from its predictive power the model allows for a deeper understanding of the diverse mixing pattern in the quark and the lepton sector, since large mixing angles in the lepton sector are interpreted as the mismatch of two different T' subgroups, Z_3 and Z_4, preserved in the charged lepton and the neutrino sector, while quark mixings turn out to be small, since the preserved subgroups in the up and down quark sector are the same. The fact that T' is not broken in an arbitrary way allows the definite prediction of TBM. The preservation of a residual group of T' can be naturally maintained in the flavon potential due to the usage of flavored gauge singlets and due to the SUSY framework. The actual realization of the model requires three additional symmetries: a Z_3 symmetry in order to separate the sectors in which T' is broken to different subgroups, a $U(1)_{FN}$ in order to fully maintain the mass hierarchy among the fermions and a $U(1)_R$ symmetry which is necessary for the construction of the flavon potential. As the fermion assignment under T' is rather diverse, the model cannot be embedded into a GUT or a continuous flavor symmetry without adding new fields to complete the representations. In the third part of the work we further exploit the idea that a flavor symmetry is not broken in an arbitrary way, but such that one of its subgroups remains conserved, in order to predict a certain pattern in the fermion mixing. We show that a systematic study of a large class of discrete non-abelian symmetries becomes possible with this requirement. As class of groups we choose the dihedral symmetries D_n and D'_n. We arrive at only five distinct (Dirac) mass matrix structures, if either left-handed or left-handed conjugate fermions ought to unify partially and the determinant of the mass matrix has to be non-vanishing. As a result of this general study, we find a new way to predict the element $|V_{us}|$ or $|V_{cd}|$, or equivalently the Cabibbo angle θ_C, if different directions of subgroups remain preserved in the up and down quark sector. In case that the flavor symmetry is D_7 $|V_{us}|$ or $|V_{cd}|$ can be fixed to $\cos(\frac{3\pi}{7}) \approx 0.2225$ which is only 2 % below its best fit value. Thereby, the Cabibbo Kobayashi Maskawa (CKM) matrix element is only determined by fundamental group theoretical quantities, namely the index n of the dihedral group D_n, the index j of the representation $\underline{\mathbf{2}}_{\mathbf{j}}$ under which the quarks transform and indices m_u and m_d which specify the direction of the preserved subgroup. Moreover, we find two neat examples in the literature in which maximal atmospheric mixing and vanishing θ_{13} in the lepton sector result from the preservation of different subgroups of the flavor group, D_4 and D_3, respectively. We analyze these in detail and elucidate their group theoretical background.

The work presented in this thesis has been mainly published in [22], [23], [24], [25] and [26].

The thesis is structured as follows: In Chapter 2 we briefly review the experimental results for the fermion masses and their mixing parameters and present some prominent approaches to explain the observed data. Chapter 3 contains the study of the two low energy models in which the flavor symmetry is spontaneously broken at the electroweak scale. Thereby, we introduce the group theory of each flavor symmetry, present the fermion masses and studies to accommodate the experimental data. In both cases we put much emphasis on a careful study of the Higgs potentials. Furthermore, our conventions of fermion masses and mixings, used throughout this work, can be found in Chapter 3. Chapter 4 is dedicated to the T' model in which the flavor symmetry is broken at high energies by gauge singlets. We perform a detailed analysis of the leading as well as next-to-leading order results in order to show that the model accommodates all data. Thereby, it leads to several predictions. We elucidate the idea of the preservation of different subgroups of the flavor symmetry in different sectors of the theory and how this leads to predictions for the fermion mixing pattern. The scalar potential is of particular interest also in this model, since only in case that the VEV configurations naturally conserve the different subgroups of the flavor group the model can be viable. In Chapter 5 we adopt the idea of preserving certain subgroups of the flavor group in the fermion mass matrices in order to study the series of dihedral symmetries D_n and D'_n. We discuss three examples leading to a prediction of the Cabibbo angle θ_C and $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ in the lepton sector, respectively. Finally, we conclude in Chapter 6 and give a short outlook. Appendix A contains the basic knowledge about group theory of discrete symmetries necessary to follow this work, Appendix B is dedicated to the Kronecker products and Clebsch Gordan coefficients of the various discrete groups (S_4, D_5, T', D_n and D'_n and D_7) used as flavor symmetries. Lastly, the explicit form of the next-to-leading order terms arising in the T' model can be found in Appendix C.

Chapter 2

Experimental and Theoretical Status

In this chapter we briefly review the information about the fermion masses and the mixing parameters, gained in numerous experiments. Furthermore, we mention prominent attempts to explain these observations.

2.1 Experimental Results

In a long series of experiments three generations of elementary particles have been discovered. Each generation consists of two quarks and anti-quarks which are colored, i.e. charged under $SU(3)_C$, one charged lepton and anti-lepton and a neutrino (which could be its own anti-particle). They can be classified according to their transformation properties under the SM gauge group into up quarks u, c and t, down quarks d, s and b, charged leptons, e, μ and τ, and neutrinos ν_e, ν_μ and ν_τ [1]. Experiments showed that the charged fermion masses are strongly hierarchical, i.e. the mass of the first generation is much smaller than the one of the second generation and the third generation is the heaviest one. The masses of the fermions are conveniently given at a common scale μ, which is here taken to be the mass M_Z of the Z boson. The quark masses are given by [27] [2]

$$\begin{aligned} &m_u(M_Z) = (1.7 \pm 0.4)\ \mathrm{MeV}\ , \quad m_c(M_Z) = (0.62 \pm 0.03)\ \mathrm{GeV}\ , \quad m_t(M_Z) = (171 \pm 3)\ \mathrm{GeV}\ , \\ &m_d(M_Z) = (3.0 \pm 0.6)\ \mathrm{MeV}\ , \quad m_s(M_Z) = (54 \pm 8)\ \mathrm{MeV}\ , \quad m_b(M_Z) = (2.87 \pm 0.03)\ \mathrm{GeV}\ . \end{aligned} \tag{2.1}$$

The charged lepton masses are very precisely known and their running masses extrapolated to the mass scale $\mu = M_Z$ read [28]

$$\begin{aligned} &m_e(M_Z) = (0.48684727 \pm 0.00000014)\ \mathrm{MeV}\ , \quad m_\mu(M_Z) = (102.75138 \pm 0.00033)\ \mathrm{MeV}\ , \\ &m_\tau(M_Z) = 1.74669^{+0.00030}_{-0.00027}\ \mathrm{GeV}\ . \end{aligned} \tag{2.2}$$

In the neutrino sector only two mass squared differences have been measured in oscillation experiments [3]. The solar mass squared difference is measured to be

$$\Delta\,\mathrm{m}^2_{21} = (7.6^{+0.5}_{-0.3}) \times 10^{-5}\ \mathrm{eV}^2\ , \tag{2.3}$$

[1]These are the eigenstates of the weak interaction, while the mass eigenstates are denoted by ν_1, ν_2 and ν_3.

[2]Note that the masses of the lighter quarks have rather large errors. Their ratios can be determined more precisely by using chiral perturbation theory.

[3]The LSND experiment [29] indicated the existence of a third independent mass squared difference which could only be explained by the assumption of additional (sterile) neutrinos. However, the data recently published by the MiniBooNE collaboration [30] do not confirm the LSND result.

and the absolute value of the atmospheric mass squared difference is found to be

$$|\Delta\, \mathrm{m}^2_{31}| = (2.4^{+0.3}_{-0.3}) \times 10^{-3}\ \mathrm{eV}^2 \tag{2.4}$$

at $2\,\sigma$ level [31]. Since $\Delta\, \mathrm{m}^2_{31} \lesseqgtr 0$, the neutrinos can have two mass orderings, i.e. they can be normally ordered, $m_1 < m_2 < m_3$, (NH) or their ordering can be inverted (IH), $m_3 < m_1 < m_2$. Furthermore, they can be quasi degenerate (QD), i.e. the measured mass squared differences can be (much) smaller than the absolute mass scale m_0. Several sources provide information about m_0: β-decay experiments which measure the endpoint of the tritium decay [32, 33]

$$m_\beta = \left(\sum_{i=1}^{3} |U^{e\,i}_{MNS}|^2\, m_i^2 \right)^{1/2} \leq 2.2\,\mathrm{eV}\ , \tag{2.5}$$

cosmology which sets a bound on the sum of the neutrino masses [34]

$$\sum_{i=1}^{3} m_i \lesssim 1\ \mathrm{eV}\ , \tag{2.6}$$

as well as the search for neutrinoless double-beta decay ($0\nu\beta\beta$) [35–38] [4]

$$|m_{ee}| = |\sum_{i=1}^{3} \left(U^{e\,i}_{MNS}\right)^2\ m_i| \leq 0.9\,\mathrm{eV}\ . \tag{2.7}$$

In addition, a signal $|m_{ee}| \neq 0$ would show that neutrinos are Majorana particles, unlike the other fermions.
Quarks as well as leptons are known to have non-vanishing mixing. In the quark sector, the mixing is parameterized by the CKM matrix whose entries are well determined [40]

$$|V_{CKM}| = \begin{pmatrix} 0.97383^{+0.00024}_{-0.00023} & 0.2272^{+0.0010}_{-0.0010} & (3.96^{+0.09}_{-0.09}) \times 10^{-3} \\ 0.2271^{+0.0010}_{-0.0010} & 0.97296^{+0.00024}_{-0.00024} & (42.21^{+0.10}_{-0.80}) \times 10^{-3} \\ (8.14^{+0.32}_{-0.64}) \times 10^{-3} & (41.61^{+0.12}_{-0.78}) \times 10^{-3} & 0.999100^{+0.000034}_{-0.000004} \end{pmatrix} \tag{2.8}$$

together with the Jarlskog invariant [41] which measures the CP violation,

$$J_{CP} = (3.08^{+0.16}_{-0.18}) \times 10^{-5}\ . \tag{2.9}$$

The standard parameterization of V_{CKM} is given in terms of the three mixing angles θ_{ij} and the CP phase δ [40]

$$V_{CKM} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}\,\mathrm{e}^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}\,\mathrm{e}^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}\,\mathrm{e}^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}\,\mathrm{e}^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}\,\mathrm{e}^{i\delta} & c_{23}c_{13} \end{pmatrix} \tag{2.10}$$

with $s_{ij} = \sin(\theta_{ij})$ and $c_{ij} = \cos(\theta_{ij})$. The angles are restricted to lie in the first quadrant and δ can take any value between 0 and $2\,\pi$. J_{CP} is related to the mixing angles θ_{ij} and the CP phase δ through

$$J_{CP} = \frac{1}{8}\, \sin(2\,\theta_{12})\, \sin(2\,\theta_{23})\, \sin(2\,\theta_{13})\, \cos(\theta_{13})\, \sin(\delta)\ . \tag{2.11}$$

[4] The claim [39] of $|m_{ee}| \neq 0$ is controversial.

Values of $\sin(\theta_{ij})$ and δ are then

$$s_{12} = 0.2243 \ , \quad s_{23} = 0.0413 \ , \quad s_{13} = 0.0037 \ , \quad \delta = 1.05 \ \text{ radian} \ . \tag{2.12}$$

They are taken from the Particle Data Booklet which appeared in 2004 [42] and cannot be found in the newest version of 2006 [40]. However, we display these values, since it is useful for a comparison to the lepton sector in which it is more convenient to present the sines of the mixing angles. The elements of the CKM matrix are usually denoted by

$$V_{CKM} = \begin{pmatrix} V_{ud} & V_{us} & V_{ub} \\ V_{cd} & V_{cs} & V_{cb} \\ V_{td} & V_{ts} & V_{tb} \end{pmatrix} \ . \tag{2.13}$$

The lepton mixing is encoded in the Maki Nakagawa Sakata (MNS) mixing matrix U_{MNS}. Its parameterization is analogous to the one of the CKM matrix, i.e. it contains three mixing angles θ_{ij} and one Dirac CP phase δ. Additionally, two phases, denoted by ϕ_1 and ϕ_2 and associated with the possible Majorana nature of the neutrinos, can be present. Instead of displaying the elements of U_{MNS} it is more convenient in the lepton sector to show the experimental results of the sines of the mixing angles θ_{ij}. These are measured in (anti-)neutrino oscillations by various experiments which use either natural sources of neutrinos (e.g. the Sun) or artificial neutrino sources (e.g. nuclear power plants). Not all three mixing angles have been measured and their experimental errors are at present still much larger than those of the quark mixings. There exist several global analyses [31,43] using different techniques to extract the sines of the mixing angles from the various data of the very diverse experiments. Here the results of the analysis performed by Maltoni et al. [31] are quoted

$$\sin^2(\theta_{12}) = 0.32^{+0.05}_{-0.04} \ , \ \ \sin^2(\theta_{23}) = 0.50^{+0.13}_{-0.12} \ \ \text{and} \ \ \sin^2(\theta_{13}) = 0.007^{+0.026}_{-0.007} \ \ (\text{at } 2\,\sigma \text{ level}) \ . \tag{2.14}$$

These values can be found in the 2007-update of [31]. As one can see, according to the latest global fits the best fit value of $\sin^2(\theta_{13})$ now deviates from zero [5], while the best fit for $\sin^2(\theta_{23})$ is still maximal mixing, i.e. $\theta_{23} = \frac{\pi}{4}$. In particular, these two values are of theoretical interest, since vanishing mixing as well as maximal mixing do not seem to be accidental results of a tuning of parameters of the theory, but require a more fundamental explanation. According to their observation in (anti-)neutrino experiments θ_{12} is also called solar, θ_{23} atmospheric and θ_{13} reactor mixing angle. Up to now, the Dirac CP phase δ as well as the Majorana phases ϕ_1 and ϕ_2 have not been determined in experiments. Although it is not convenient in neutrino physics, we also display the allowed ranges of the entries of the MNS matrix for comparison with the quark sector. Since the authors of the global analysis [31] do not give U_{MNS} in this form, we show another (rather old) result [44] for illustration purposes

$$|U_{MNS}| = \begin{pmatrix} 0.79 - 0.88 & 0.47 - 0.61 & < 0.20 \\ 0.19 - 0.52 & 0.42 - 0.73 & 0.58 - 0.82 \\ 0.20 - 0.53 & 0.44 - 0.74 & 0.56 - 0.81 \end{pmatrix} \tag{2.15}$$

at $3\,\sigma$ level. Similar to the elements of V_{CKM}, the U_{MNS} elements are also denoted by

$$U_{MNS} = \begin{pmatrix} U_{e1} & U_{e2} & U_{e3} \\ U_{\mu 1} & U_{\mu 2} & U_{\mu 3} \\ U_{\tau 1} & U_{\tau 2} & U_{\tau 3} \end{pmatrix} \ . \tag{2.16}$$

[5]However, it is still compatible with $\sin^2(\theta_{13}) = 0$. At the $2\,\sigma$ level $\sin^2(\theta_{13})$ is bounded from above by 0.033.

For further definitions of the mixing matrices V_{CKM} and U_{MNS} and further conventions see Section 3.1.
Finally, we would like to mention the experimental results of searches for a fourth generation of fermions. The number of light neutrinos is fixed to be $N_\nu = 2.994 \pm 0.012$ by the Z boson decay width [40] [6]. Additionally, this decay width enforces a lower limit on new SM-like fermions of m $>$ 45 GeV. Further lower bounds on the mass of heavy charged leptons are m $>$ 100.8 GeV for the decay into $W\nu$ and m $>$ 102.6 GeV for stable leptons. For stable neutral heavy leptons the bounds read m $>$ 45 GeV for Dirac particles and m $>$ 39.5 GeV for Majorana fermions. In case that these particles are not stable the bounds are about a factor of two larger. For quarks, an analogous bound on quasi-stable b's reads m $>$ 190 GeV. All values are taken from [40]. For further reading we refer to [45].

2.2 Theoretical Approaches

In the following, we review some ideas to describe the observed fermion mass and mixing patterns. A very simple ansatz is to require that certain elements in the fermion mass matrices vanish. These texture zeros are in general not motivated by any further principle, but only by the desire to make the model predictive. The cases of five and six texture zeros [46] and different four zero structures [47, 48] in the up and down quark mass matrix have been extensively studied. Several of them are able to accommodate the fermion mass hierarchy (most of the time with additional assumptions) and some of them can correlate the quark masses with elements of the CKM matrix [49–54] leading to relations, like

$$\left|\frac{V_{td}}{V_{ts}}\right| = \sqrt{\frac{m_d}{m_s}} \quad \text{and} \quad \left|\frac{V_{ub}}{V_{cb}}\right| = \sqrt{\frac{m_u}{m_c}} \,. \tag{2.17}$$

In particular, it is often assumed that the (11) element in the mass matrices vanishes. Since these ansätze turned out to be quite successful in the quark sector, texture zeros have also been used to derive predictions in the lepton sector. Thereby, in most of the studies it is presumed that the mass matrix of the charged leptons is diagonal. As shown in [55–57], the maximum number of independent texture zeros is two in case that neutrinos are Majorana particles. For Dirac neutrinos more texture zeros are allowed [58] due to the fact that a Dirac mass matrix does not have to be symmetric in contrast to a mass matrix for Majorana fermions. However, such studies suffer from the problem that these zeros are in general not protected against corrections. This is especially relevant, if the vanishing of certain elements of the mass matrix is imposed at a scale much higher than the electroweak scale. In case of neutrinos with a quasi degenerate mass spectrum or with an inverted hierarchy renormalization group running effects can drastically change the predictions of a mass matrix with texture zeros, as discussed in [59].
In order to go a step further, one has to search for a profound reason for a certain structure of the fermion mass matrices. As explained in the Introduction, it is reasonable to presume that a new symmetry is responsible for the fermion masses and mixings. In the simplest case it is a $U(1)$ group. If the charges of the left-handed and left-handed conjugate fermions are chosen in an appropriate way, such a symmetry can indeed produce the observed hierarchy among the fermions and also the order of magnitudes of the mixing angles. This idea has been proposed first by Froggatt and Nielsen [11]. However, this approach has two disadvantages: *a.*) since only the left-handed up and down quarks and the left-handed charged leptons and neutrinos are unified into a doublet under $SU(2)_L$ in the SM, we have various possibilities to assign different charges to the

[6] Similar bounds can be derived from cosmology.

different kinds of fermions as well as to the three generations, and *b.*) concerning the mixing angles it is not possible to arrive at a definite prediction, i.e. a certain value of a non-vanishing mixing angle. The prospects to achieve this are fairly better with a non-abelian symmetry. Due to the fact that the mass of the fermions belonging to the third generation is much larger than the masses of the two other generations, the symmetry $U(2)$ under which the three generations transform as $\underline{\mathbf{2}} + \underline{\mathbf{1}}$ has been employed as flavor symmetry [60]. Thereby, $U(2)$ is broken in two steps in order to maintain the hierarchy of the fermion masses. However, in order to understand the existence of exactly three generations only two continuous symmetries are appropriate, namely $SO(3)$ or $SU(3)$. Both of them have been studied in detail in [61] and [62]. The models are successful in describing the fermion mass spectrum and the diverse mixing pattern of quarks and leptons. However, they are rather complicated with regard to the number of additional heavy degrees of freedom and auxiliary symmetries needed to suppress unwanted operators. Furthermore, it turned out that it is non-trivial to combine them with a GUT like $SU(5)$ or $SO(10)$. For a profound understanding of precise values of the mixing angles discrete non-abelian flavor symmetries seem to offer the best prospects, in particular, if they are not broken in an arbitrary way. In this thesis we will present four examples which all allow for the prediction of a special mixing structure: in Chapter 4 we will discuss a model with the flavor symmetry T' which predicts TBM, and in Chapter 5 dihedral groups are employed in order to derive the Cabibbo angle θ_C or $\mu\tau$ symmetry in the lepton sector. In the discussion of these models it will turn out that the discrete non-abelian symmetry alone is not sufficient to fully understand the fermion mass hierarchy in the majority of the cases. For this purpose, the Froggatt-Nielsen mechanism has to be invoked, see Chapter 4.

There are also other approaches which can explain some properties of the fermion mass spectrum and the mixing patterns without invoking an additional horizontal symmetry. For example, it is well-known that in GUTs some features of the fermion masses can be maintained by a proper choice of the Higgs representations. The explanation [63] of

$$m_\tau = m_b \;, \quad m_\mu = 3\, m_s \quad \text{and} \quad m_e = \frac{1}{3}\, m_d \tag{2.18}$$

with the help of a Higgs field transforming as $\underline{\mathbf{45}}$ in $SU(5)$ is maybe the most prominent example. Note that Eq.(2.18) holds at the GUT scale of 10^{16} GeV, but not at low energies. In another recently studied model [64] the fermion mass hierarchy and the size of the mixing angles is also not attributed to a horizontal symmetry, but rather to the fact that additional heavy degrees of freedom are present in the model and that the Pati-Salam gauge group $SU(4)_C \times SU(2)_L \times SU(2)_R$ is broken in a particular way. Although this works well for the second and third generation of fermions, the model cannot explain the features of the first generation. Similar to the models with a $U(1)$ group as flavor symmetry, also in this approach a definite prediction for a certain mixing angle cannot be derived.

Finally, extra-dimensional models can be counted as rather exotic idea to shed light on the fermion mass pattern. They generate the mass hierarchy by the appropriate localization of the three generations in the extra dimension(s), see for instance [65].

Chapter 3

Flavored SM

In this chapter we present two models in which the SM is extended by a discrete flavor symmetry. In the first one the permutation group of four distinct objects, named S_4, is used in order to unify the three generations of fermions into one irreducible representation of the flavor group, whereas in the second model the flavor symmetry is taken to be the dihedral group D_5 which only has one- and two-dimensional irreducible representations. Therefore, the three generations are assigned to $\underline{\mathbf{1}}+\underline{\mathbf{2}}$ under the flavor group in this model. Both assignment structures have appealing features. The unification into one irreducible representation solves the so-called "family replication problem", i.e. it answers the question why we observe exactly three generations of fermions. On the other hand we also have several hints that the assignment $\underline{\mathbf{1}}+\underline{\mathbf{2}}$ might be even more favorable than a complete unification of the generations. The facts that the masses of the charged fermions of the first generation are so much lighter than the masses of the two other ones and the observation of nearly maximal atmospheric mixing, i.e. $\theta_{23} \approx \frac{\pi}{4}$, point towards a combination of the second and third generation into a doublet, while the first one transforms as a singlet. However, also the assignment $\underline{\mathbf{2}}+\underline{\mathbf{1}}$ is reasonable, since only the Cabibbo angle $\theta_C = \theta^q_{12}$ is sizable in the quark sector and the hierarchy among the fermion masses is also in accordance with a scenario in which the masses of the fermions of the third generation originate from a different mechanism. In both models we use additional guidelines to fix the transformation properties of the fermions under the flavor group (almost) uniquely. We work out the phenomenology of both models by studying the resulting mass matrices, analytically and also numerically. In order not to introduce new scales into the theory the flavor symmetry is broken spontaneously at the electroweak scale together with the gauge group of the SM. For this purpose, the models contain several copies of the SM Higgs $SU(2)_L$ doublet field (with hypercharge $Y=-1$) which transform non-trivially under S_4 and D_5, respectively. In case of the S_4 model we deal with six Higgs doublets, while the D_5 model contains four Higgs doublets. In both cases the Higgs potential is quite complicated and needs a careful study to show that the advocated VEV structures, necessary to fit the fermion masses, can be achieved. The two models shown here are published in [22] and [23].
This chapter is organized as follows: Section 3.1 contains our conventions for the fermion mass terms in the SM; Section 3.2 is dedicated to the S_4 model and Section 3.3 treats the D_5 model in detail. In Section 3.2 and Section 3.3, we start by presenting the mathematical structure of the employed flavor symmetry, then give an outline of the model, discuss the fermion mass matrices and the Higgs potential and finally comment on special aspects and problems of the particular model. In the last section, Section 3.4, we point out which lessons can be learnt from the study of these two models for model building with discrete flavor symmetries.

3.1 Conventions

We assume that the reader is familiar with the structure of the SM and therefore only display our specific conventions of Yukawa couplings, fermion masses and mixings. We work with left-handed and left-handed conjugate fields, since this is also common in the context of GUTs. The three generations of fermions transform in the following way

$$Q_i = \begin{pmatrix} u_i \\ d_i \end{pmatrix}_L \sim (\underline{\mathbf{3}}, \underline{\mathbf{2}}, +\frac{1}{3}) \ , \quad L_i = \begin{pmatrix} \nu_i \\ e_i \end{pmatrix}_L \sim (\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$$

$$u^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -\frac{4}{3}) \ , \ c^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -\frac{4}{3}) \ , \ t^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, -\frac{4}{3})$$

$$d^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, +\frac{2}{3}) \ , \ s^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, +\frac{2}{3}) \ , \ b^c \sim (\underline{\overline{\mathbf{3}}}, \underline{\mathbf{1}}, +\frac{2}{3})$$

$$e^c \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, +2) \ , \ \mu^c \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, +2) \ , \ \tau^c \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, +2)$$

$$\left[\nu^c_e \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0) \ , \ \nu^c_\mu \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0) \ , \ \nu^c_\tau \sim (\underline{\mathbf{1}}, \underline{\mathbf{1}}, 0)\right]$$

under $(SU(3)_C, SU(2)_L, U(1)_Y)$. The index i denotes the i^{th} generation. As usual, the electric charge Q is $Q = T_3 + \frac{Y}{2}$ with T_3 being the weak isospin and Y the hypercharge. The left-handed conjugate neutrinos are put in brackets, since, strictly speaking, they do not belong to the SM.
A Yukawa interaction, e.g. for the charged leptons, is then of the form

$$\lambda_{i\,e} L_i^T \epsilon \Phi e^c \ , \tag{3.1}$$

and similarly for the down quarks

$$\lambda_{i\,s} Q_i^T \epsilon \Phi s^c \ . \tag{3.2}$$

For up quarks and neutrinos, however, Φ has to be replaced by its conjugate $\tilde{\Phi} = \epsilon\Phi^\star$, e.g.

$$\lambda_{i\,t} Q_i^T \Phi^\star t^c \quad \text{and} \quad \lambda_{i\,\mu} L_i^T \Phi^\star \nu^c_\mu \ . \tag{3.3}$$

Thereby, the Higgs field $\Phi = (\phi^0, \phi^-)^T$ transforms under the SM gauge group as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ and ϵ is the anti-symmetric two-by-two matrix in $SU(2)_L$ space. When the neutral component of the Higgs field acquires a VEV, $\langle\Phi\rangle \neq 0$, these terms generate masses for the fermions. The VEV of the Higgs field is determined by the electroweak scale, i.e. $\langle\Phi\rangle \approx 174$ GeV. In the case of multi-Higgs doublet models this equation has to be replaced by $\sum\limits_{i=1}^{n} |\langle\Phi_i\rangle|^2 \approx (174 \text{ GeV})^2$ where Φ_i denote the n Higgs doublets. As their VEVs can be complex, the absolute value of $\langle\Phi_i\rangle$ appears in the formula. Note that also here only the neutral component of each Higgs field is assumed to get a non-vanishing VEV. However, this is in case of more than one Higgs doublet in general an assumption which is put in by hand. The resulting mass matrices are in general complex three-by-three matrices given in the basis in which the left-handed fields are on the left-hand side and the left-handed conjugate ones are on the right-hand side. The matrices are denoted by $\mathcal{M}_{u,d,e(l),\nu}$ in the following. All these terms generate Dirac masses, since they connect left-handed and left-handed conjugate fields. Neutrinos could also be Majorana particles and therefore can acquire a non-vanishing mass by interactions connecting either two left-handed or two left-handed conjugate fields. For left-handed fields this interaction needs a Higgs triplet field Ξ transforming as $(\underline{\mathbf{1}}, \underline{\mathbf{3}}, +2)$ under the SM

$$\lambda_{ij} L_i^T \Xi L_j \ , \tag{3.4}$$

where Ξ has the explicit form

$$\Xi = \begin{pmatrix} \xi^0 & -\frac{\xi^+}{\sqrt{2}} \\ -\frac{\xi^+}{\sqrt{2}} & \xi^{++} \end{pmatrix} . \tag{3.5}$$

If (the neutral component of) Ξ acquires a non-vanishing VEV, it generates masses for the light neutrinos. Unlike the VEVs of the Higgs doublets, the VEV of Ξ is strongly constrained by the ρ parameter, i.e. it cannot exceed a value of a few GeV. Thereby, it explains the smallness of the neutrino masses (at least partly). The resulting mass matrix is denoted by M_{LL}. The left-handed conjugate neutrinos can get a direct mass term, e.g. $m_{e\mu}\nu_e^c\,\nu_\mu^c$, since they do not transform under the SM gauge group. Their mass can be very heavy, i.e. around GUT scale, as it is not protected by any symmetry. Their mass matrix will be denoted by M_{RR} in the following. Since a Majorana mass term connects the same fields with each other, M_{LL} and M_{RR} have to be symmetric. We will always omit a possible factor of $\frac{1}{2}$ appearing in the Majorana mass terms in the following. Majorana neutrinos can acquire their mass in three different ways: *a.*) if the model contains Higgs triplets, but no left-handed conjugate neutrinos, M_{LL} is the light neutrino mass matrix; *b.*) if the model contains left-handed conjugate neutrinos, but no Higgs triplets, integrating out the heavy left-handed conjugate neutrinos gives a Majorana mass for the light neutrinos which is approximately given by

$$M_\nu = (-)\mathcal{M}_\nu\, M_{RR}^{-1}\,\mathcal{M}_\nu^T\ . \tag{3.6}$$

This is usually called the type-1 seesaw mechanism [66–70]. It nicely explains the smallness of the neutrino masses compared to the other fermion masses; *c.*) if also Higgs triplets are present in the model, one calls this type-2 seesaw [71, 72] and the terms from the two contributions are simply added

$$M_\nu = M_{LL} - \mathcal{M}_\nu\, M_{RR}^{-1}\,\mathcal{M}_\nu^T\ . \tag{3.7}$$

If no left-handed conjugate neutrinos ν^c and no Higgs triplets exist in the model, the neutrinos can still get mass from the non-renormalizable operator connecting two left-handed lepton doublets and two Higgs doublets of the form $(L_i\,\tilde{\Phi})\,(L_j\,\tilde{\Phi})$. This operator then arises through the mediation of unspecified fields present in the high energy completion of the low energy theory and is suppressed by the mass scale of these fields (or by the cutoff scale of theory).
The mixing matrices of quarks and leptons, i.e. the CKM matrix V_{CKM} and the MNS matrix U_{MNS}, originate from the mismatch of the mass bases of the up quarks (neutrinos) and down quarks (charged leptons). The general complex three-by-three matrices $\mathcal{M}_u$ and $\mathcal{M}_d$ can be diagonalized by a bi-unitary transformation

$$U_u^\dagger \mathcal{M}_u V_u = \mathrm{diag}\,(m_u, m_c, m_t) \quad \text{and} \quad U_d^\dagger \mathcal{M}_d V_d = \mathrm{diag}\,(m_d, m_s, m_b) \tag{3.8}$$

with the unitary matrices $U_{u,d}$ and $V_{u,d}$ acting on the left-handed and the left-handed conjugate fields, respectively. Obviously, m_u, m_c, etc. denotes the mass of the up quark, charm quark, etc.. $U_{u,d}$ can be calculated from the hermitean matrices $\mathcal{M}_{u,d}\,\mathcal{M}_{u,d}^\dagger$, i.e.

$$U_u^\dagger \mathcal{M}_u \mathcal{M}_u^\dagger U_u = \mathrm{diag}\,(m_u^2, m_c^2, m_t^2) \quad \text{and} \quad U_d^\dagger \mathcal{M}_d \mathcal{M}_d^\dagger U_d = \mathrm{diag}\,(m_d^2, m_s^2, m_b^2)\ . \tag{3.9}$$

The CKM matrix is then given as

$$V_{CKM} = U_u^T\, U_d^\star\ , \tag{3.10}$$

since it is defined as the matrix which diagonalizes $\mathcal{M}_d \mathcal{M}_d^\dagger$ in the basis in which the up quark mass matrix is already diagonal. V_{CKM} can also be defined by the weak current interaction $\bar{u}'\gamma_\mu W^{\mu\,+} d'$, if one changes from interaction u',d' to mass eigenstates u, d, then $\bar{u}\gamma_\mu W^{\mu\,+} U_u^T \, U_d^\star \, d \equiv \bar{u}\gamma_\mu W^{\mu\,+} V_{CKM} \, d$. The CKM matrix has three free angles θ_{ij} and one phase δ. θ_{ij} can be chosen to lie in the first quadrant and δ can take any value between 0 and $2\,\pi$. The standard parameterization of V_{CKM} has already been shown in the previous chapter, see Eq.(2.10). Also the experimental results for these quantities can be found there. Analogously, to the quarks the charged lepton mass matrix is diagonalized by a bi-unitary transformation

$$U_l^\dagger \mathcal{M}_l V_l = \mathrm{diag}\,(m_e, m_\mu, m_\tau) \; . \tag{3.11}$$

Again, only the transformation acting on the left-handed fields, namely U_l, is relevant. In case that the neutrinos are also Dirac fermions their mass matrix $\mathcal{M}_\nu$ is diagonalized in completely the same way as for the other fermions via the unitary transformations U_ν and V_ν. However, if they are Majorana fermions additional phases can arise. A general symmetric matrix M_ν can be diagonalized by a unitary matrix U_ν

$$U_\nu^\dagger \, M_\nu \, U_\nu^\star = \mathrm{diag}\,(m_1, m_2, m_3) \tag{3.12}$$

and therefore also $U_\nu^\dagger \, M_\nu M_\nu^\dagger U_\nu = \mathrm{diag}\,(m_1^2, m_2^2, m_3^2)$. Technically, U_ν is calculated through the last equation and then applied to M_ν in order to fix the additional phases such that not only m_i^2, but also m_i are positive definite. Two of these three additional phases are physical and called Majorana phases. The U_{MNS} matrix is then given as

$$U_{MNS} = U_l^T \, U_\nu^\star \; . \tag{3.13}$$

It is defined as the matrix which connects the flavor and mass eigenstates of the neutrinos in the basis in which the charged leptons are diagonal, i.e.

$$\nu_{\alpha\,L} = \sum_{i=1}^{3} U_{MNS}^{\alpha\,i} \, \nu_{i\,L} \quad \text{for} \quad \alpha = e, \mu, \tau \quad \text{and} \quad i = 1, 2, 3 \; . \tag{3.14}$$

U_{MNS} can be parameterized in the same way as the CKM matrix, if the neutrinos are Dirac particles, while in case of Majorana neutrinos one has to multiply the standard parameterization from the right-hand side with a diagonal matrix containing the Majorana phases, i.e.

$$U_{MNS} = V \; \cdot \; \mathrm{diag}\,(\mathrm{e}^{i\,\phi_1}, \mathrm{e}^{i\,\phi_2}, 1) \; , \tag{3.15}$$

where V is in the standard form containing the mixing angles θ_{ij} and the Dirac phase δ and $\phi_{1,2}$ are the Majorana phases. $\phi_{1,2}$ lie in the interval $[0, \pi)$.
In numerical analyses the mixing matrices are in general not in the standard form of Eq.(2.10). The mixing angles θ_{ij} and the CP phase δ, however, can be extracted [73] by

$$\sin(\theta_{13}) = |U_{13}| \; , \quad \tan(\theta_{12}) = \frac{|U_{12}|}{|U_{11}|} \; , \quad \tan(\theta_{23}) = \frac{|U_{23}|}{|U_{33}|} \tag{3.16}$$

$$\delta = -\arg\left(\frac{\frac{U_{11}^\star \, U_{13} \, U_{31} \, U_{33}^\star}{\cos(\theta_{12}) \, \cos^2(\theta_{13}) \, \cos(\theta_{23}) \, \sin(\theta_{13})} + \cos(\theta_{12}) \, \cos(\theta_{23}) \, \sin(\theta_{13})}{\sin(\theta_{12}) \, \sin(\theta_{23})} \right) \tag{3.17}$$

for any unitary matrix U. If present, the Majorana phases are given by

$$\phi_1 = -\arg(\mathrm{e}^{i\,\delta_e} \, U_{11}^\star) \quad \text{and} \quad \phi_2 = -\arg(\mathrm{e}^{i\,\delta_e} \, U_{12}^\star) \quad \text{with} \quad \delta_e = \arg(\mathrm{e}^{i\,\delta} \, U_{13}) \; . \tag{3.18}$$

The measure of CP violation J_{CP}, which is given in Eq.(2.11) in terms of the mixing angles θ_{ij} and the CP phase δ, can be calculated using four elements of U

$$J_{CP} = \mathrm{Im}(U_{11}\, U^{\star}_{12}\, U^{\star}_{21}\, U_{22}) = \mathrm{Im}(U_{11}\, U^{\star}_{13}\, U^{\star}_{31}\, U_{33}) = \mathrm{Im}(U_{22}\, U^{\star}_{23}\, U^{\star}_{32}\, U_{33}) \; . \tag{3.19}$$

3.2 S_4 Model

The model presented in this section is a low energy model in which the SM is extended by the flavor symmetry S_4 [1]. As mentioned, S_4 is only broken spontaneously at the electroweak scale. This enforces us to consider a multi-Higgs doublet model. We need six Higgs doublets in total in this model. S_4 is broken completely in one step by the VEVs of the Higgs doublets which are necessary to fit the fermion masses and mixings. Although we realize this model at low energies, we choose the transformation properties of the fermions such that it can be embedded into $SO(10)$, i.e. we assume that all fermions transform according to the same representation under S_4. Furthermore we are guided by a second idea, namely the embedding of the discrete flavor symmetry into a continuous one at very high energies. This idea is based on two considerations: *a.*) one might think about a kind of "super-GUT" which unifies the gauge and the flavor group into one simple group [2] and *b.*) the proper treatment of anomalies seems to require the embedding into a continuous group [3]. S_4 is a subgroup of $SO(3)$ and $SU(3)$ and we discuss both possibilities. According to the classification of the subgroups of $SU(3)$ [80–84] S_4 is isomorphic to the group $\Delta(24)$ which belongs to the series of groups $\Delta(6\,n^2)$ with $n = 2$. The scale at which this continuous group breaks down to its discrete relic is expected to be above the scale of Grand Unification, i.e. $> 10^{16}$ GeV. Therefore, the low energy model will first be embedded into $SO(10)$ and then into $SO(3)_f$ or $SU(3)_f$. As will be shown below, the embedding of S_4 into $SO(3)_f$ leads to the same constraints on the fermion assignments as its embedding into $SU(3)_f$. Both single out a unique assignment. Accordingly, also the transformation properties of the Higgs fields are constrained. We then discuss the fermion mass matrices arising from this setup and sketch the numerical procedure to fit the data. The potential of the Higgs fields is also calculated and studied in the CP conserving case. The VEV structures which need to be realized for a successful fit of the data are motivated by the potential. Finally, we summarize and comment on the main features and problems of this model.

3.2.1 Introduction to S_4

The group S_4 is the permutation group of four distinct objects and is also isomorphic to the octahedral symmetry O. It has five irreducible representations: two of them are one-dimensional, denoted as $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{1}}_{\mathbf{2}}$, one is two-dimensional, denoted as $\underline{\mathbf{2}}$, and two are three-dimensional, $\underline{\mathbf{3}}_{\mathbf{1}}$ and $\underline{\mathbf{3}}_{\mathbf{2}}$. The one-dimensional representation $\underline{\mathbf{1}}_{\mathbf{1}}$ is the identity/trivial representation which is also called symmetric in the context of permutation groups. Similarly, the representation $\underline{\mathbf{1}}_{\mathbf{2}}$ is sometimes called anti-symmetric or alternating. All representations are real and the two three-dimensional ones are faithful. The order of the group is 24. It is therefore the smallest group (together with the group T' which will be discussed in Chapter 4) which possesses one-, two- and three-dimensional representations. The characters of the representations can be found in Table 3.1. The group is uniquely determined by its generators A and B and their relations [86]

[1]For other models using S_4 as flavor symmetry see [74–76].

[2]However, studies in the 1980s [77, 78] showed that such a "super-GUT" has to be quite large and therefore leads in general to the appearance of additional chiral generations which accompany the three generations of the SM in order to complete the "super-GUT" multiplets.

[3]There is a recent paper by T. Araki in which he claims that anomalies of discrete groups can be calculated with the Fujikawa method independent of the embedding into a continuous group [79].

	classes							
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$			
G	$\mathbb{1}$	A^2	$\mathrm{A\,B}^2$	B	A			
$^\circ\mathcal{C}_i$	1	3	6	8	6			
$^\circ\mathrm{h}_{\ \mathcal{C}_i}$	1	2	2	3	4	$\mathbb{c}^{(\mu)}$	partitions	faithful
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	1	1	1	$[4]$	
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	1	-1	1	-1	1	$[1^4]$	
$\underline{\mathbf{2}}$	2	2	0	-1	0	1	$[2^2]$	
$\underline{\mathbf{3}}_{\mathbf{1}}$	3	-1	1	0	-1	1	$[3\,1]$	$\surd$
$\underline{\mathbf{3}}_{\mathbf{2}}$	3	-1	-1	0	1	1	$[2\,1^2]$	$\surd$
cycles	(1^4)	(2^2)	$(1^2\,2)$	$(1\,3)$	(4)			

Table 3.1: Character table of the group S_4. Since this group is a permutation group we also display the cycle structure of the classes and the partitions for each representation. The notation of the cycle structure $(1^{\alpha_i}\,2^{\alpha_2}...)$ corresponds to α_1 one-cycles (i), α_2 two-cycles (ij), ... with the constraint $\sum_i \alpha_i\, i = 4$. The partitions are correlated to the Young diagrams. Further information on cycles and partitions can be found [85]. The rest of the notation and the mathematics is explained in Appendix A.

$$\mathrm{A}^4 = \mathbb{1} \;\; , \;\; \mathrm{B}^3 = \mathbb{1} \quad \text{and} \quad \mathrm{A\,B^2\,A} = \mathrm{B} \;\; , \;\; \mathrm{A\,B\,A} = \mathrm{B\,A^2\,B}\,. \tag{3.20}$$

One possible realization of generators for the irreducible representations is [86]

$$\text{... for } \;\underline{\mathbf{2}} \quad : \quad \mathrm{A} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = -\frac{1}{2}\begin{pmatrix} 1 & \sqrt{3} \\ -\sqrt{3} & 1 \end{pmatrix} \quad , \tag{3.21}$$

$$\text{... for } \;\underline{\mathbf{3}}_{\mathbf{1}} \quad : \quad \mathrm{A} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad , \tag{3.22}$$

$$\text{... for } \;\underline{\mathbf{3}}_{\mathbf{2}} \quad : \quad \mathrm{A} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \quad \text{and} \quad \mathrm{B} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad . \tag{3.23}$$

For the one-dimensional representations $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{1}}_{\mathbf{2}}$ the generators can be found in the character table, see Table 3.1.
The Kronecker products and Clebsch Gordan coefficients, calculated with this set of generators, are given in Appendix B.1.

3.2.2 Outline of the Model

Since we can choose among one-, two- and three-dimensional representations of S_4, when we want assign the three generations of fermions, we already have some freedom. However, this freedom is drastically reduced, if we take into account the constraint that the model has to be embeddable into $SO(10)$ and $SO(3)_f$ or $SU(3)_f$ at the same time. Taking this as guideline two possible assignments for the three generations of fermions remain: either all of them transform as the trivial representation $\underline{\mathbf{1}}_{\mathbf{1}}$ or they form the irreducible three-dimensional representation $\underline{\mathbf{3}}_{\mathbf{2}}$. Note that the two three-dimensional representations $\underline{\mathbf{3}}_{\mathbf{1}}$ and $\underline{\mathbf{3}}_{\mathbf{2}}$ are not equivalent concerning this point, since

$\mathrm{D}^{(l)}$ of the rotation group	Resolution of $\mathrm{D}^{(l)}$ into representations of S_4
$l = 0$	$\underline{\mathbf{1}}_{\mathbf{1}}$
$l = 1$	$\underline{\mathbf{3}}_{\mathbf{2}}$
$l = 2$	$\underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}}$
$l = 3$	$\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}} + \underline{\mathbf{3}}_{\mathbf{2}}$
$l = 4$	$\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}} + \underline{\mathbf{3}}_{\mathbf{2}}$

$SU(3)$ representations		Resolution into S_4 representations
Dynkin label	Dimension	
(00)	1	$\underline{\mathbf{1}}_{\mathbf{1}}$
(10)	3	$\underline{\mathbf{3}}_{\mathbf{2}}$
(20)	6	$\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}}$
(11)	8	$\underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}} + \underline{\mathbf{3}}_{\mathbf{2}}$
(30)	10	$\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}} + 2\,\underline{\mathbf{3}}_{\mathbf{2}}$

Table 3.2: Breaking sequences $SU(3) \rightarrow S_4$ and $SO(3) \rightarrow S_4$ for the smallest representations. $\mathrm{D}^{(l)}$ is the $(2\,l+1)$-dimensional representation of the rotation group $SO(3)$.

only $\underline{\mathbf{3}}_{\mathbf{2}}$ can be identified with the fundamental representation of $SO(3)$ and $SU(3)$, respectively, whereas $\underline{\mathbf{3}}_{\mathbf{1}}$ is contained in the five-dimensional representation of $SO(3)$ (together with the two-dimensional representation of S_4) and in the six-dimensional one of $SU(3)$ (together with $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}$). Note also that the identification of $\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}$ of S_4 with $\underline{\mathbf{3}}$ of $SO(3)$ or $SU(3)$ is possible, but rather leads to the breaking sequence $SU(3) \rightarrow S_3$ and $SO(3) \rightarrow S_3$, respectively, since $\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}$ is not a faithful representation of S_4, but only of S_3. The embedding schemes of S_4 into $SO(3)$ and $SU(3)$ are shown in Table 3.2.
The unique possibility is therefore to assign the three generations of fermions to the three-dimensional representation $\underline{\mathbf{3}}_{\mathbf{2}}$ of S_4. Assigning of all them to the trivial representation would lead back to the SM (conventional $SO(10)$), since then the flavor symmetry does not have any impact on the mass matrices and mixing structures.
In the next step we have to choose the S_4 representations under which the Higgs fields should transform. According to Appendix B.1 the Kronecker product $\underline{\mathbf{3}}_{\mathbf{2}} \times \underline{\mathbf{3}}_{\mathbf{2}}$ contains the representations $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{2}}$, $\underline{\mathbf{3}}_{\mathbf{1}}$ and $\underline{\mathbf{3}}_{\mathbf{2}}$. As one can deduce from the Clebsch Gordan coefficients (see Appendix B.1), only the Higgs field ϕ_0 transforming as trivial representation under S_4 gives a contribution to the mass matrices which has a non-vanishing trace [4] . Unfortunately, this contribution alone leads to a degenerate mass spectrum and therefore additional fields are needed. Moreover, as it transforms trivially, ϕ_0 cannot break the flavor symmetry. Also, in order to introduce non-vanishing fermion

[4] The requirement that the trace of the mass matrix is non-vanishing is not absolutely necessary. One can think, for example, of the following mass matrix

$$\mathcal{M}_u = \begin{pmatrix} 0 & m_u & 0 \\ 0 & 0 & m_c \\ m_t & 0 & 0 \end{pmatrix} . \tag{3.24}$$

$\mathrm{Tr}(\mathcal{M}_u)$ vanishes, but $\mathrm{Tr}(\mathcal{M}_u\,\mathcal{M}_u^\dagger)$ does not and the quark masses can be explained by this mass matrix. However, the important feature of this matrix is the fact that it is highly non-symmetric, i.e. it basically consists of one large entry in its (31) element. In the discussion of the mass matrices stemming from S_4-invariant Yukawa couplings, however, it becomes clear that a symmetric structure is favorable and therefore they have to have a non-vanishing trace in order to fit the charged fermion masses.

mixing, further Higgs fields have to contribute. For this purpose, we have to include at least three Higgs fields ξ_i which form a triplet under S_4. If they transform as $\underline{\mathbf{3_1}}$, the resulting mass matrix is symmetric, while Higgs fields transforming as $\underline{\mathbf{3_2}}$ alone lead to an anti-symmetric mass matrix. Since the second possibility ($\phi_0 \sim \underline{\mathbf{1_1}}$ and $\xi_i \sim \underline{\mathbf{3_2}}$) leads to mass matrices with no definite symmetry, we discard it. The embedding of S_4 into $SO(3)_f$ or $SU(3)_f$ enforces us then to add two further Higgs fields $\phi_{1,2}$ forming a doublet under S_4, since $\underline{\mathbf{1_1}} + \underline{\mathbf{3_1}}$ cannot be identified with a complete representation of $SO(3)_f$ or $SU(3)_f$. If we do so, the model contains six Higgs fields $\phi_0 \sim \underline{\mathbf{1_1}}$, $\phi_{1,2} \sim \underline{\mathbf{2}}$ and $\xi_i \sim \underline{\mathbf{3_1}}$ in total which correspond to $\underline{\mathbf{1}} + \underline{\mathbf{5}}$ in $SO(3)_f$ and $\underline{\mathbf{6}}$ in $SU(3)_f$, respectively. These break S_4 completely at the electroweak scale. The mass matrices are symmetric, which is welcome, since also the mass matrices originating from $SO(10)$ models with Higgs fields transforming as $\underline{\mathbf{10}}$ and $\overline{\underline{\mathbf{126}}}$ under $SO(10)$ are symmetric (see below). Finally, the left-handed conjugate neutrinos acquire only a direct mass and are therefore degenerate. We exclude the existence of additional gauge singlets which can lead to a non-trivial structure of the Majorana mass matrix for the left-handed conjugate neutrinos, since this leads to a breaking of S_4 at the seesaw scale. Furthermore, we also discard the possibility that the left-handed neutrinos acquire a Majorana mass by the coupling to a Higgs triplet. The following table summarizes the fields present in our model

Field	Q	u^c	d^c	L	e^c	ν^c	ϕ_0	$\phi_{1,2}$	$\xi_{1,2,3}$
S_4	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{3_2}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{3_1}}$

Table 3.3: The particle content and its symmetry properties under S_4. The Higgs fields ϕ_0, $\phi_{1,2}$ and $\xi_{1,2,3}$ are copies of the SM Higgs field, i.e. transform as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

3.2.3 Fermion Masses and Mixings

According to the S_4 transformation properties of the fields shown in Table 3.3 the Yukawa couplings are of the form

$$\begin{aligned}
\mathcal{L}_Y \;=\;& \alpha_0^u\,(Q_1\,u^c + Q_2\,c^c + Q_3\,t^c)\,\tilde{\phi}_0 + \alpha_1^u\,(\sqrt{3}\,(Q_2\,c^c - Q_3\,t^c)\,\tilde{\phi}_1 + (-2\,Q_1\,u^c + Q_2\,c^c + Q_3\,t^c)\,\tilde{\phi}_2)\\
+\;& \alpha_2^u\,((Q_2\,t^c + Q_3\,c^c)\tilde{\xi}_1 + (Q_1\,t^c + Q_3\,u^c)\,\tilde{\xi}_2 + (Q_1\,c^c + Q_2\,u^c)\,\tilde{\xi}_3)\\
+\;& \alpha_0^d\,(Q_1\,d^c + Q_2\,s^c + Q_3\,b^c)\,\phi_0 + \alpha_1^d\,(\sqrt{3}\,(Q_2\,s^c - Q_3\,b^c)\,\phi_1 + (-2\,Q_1\,d^c + Q_2\,s^c + Q_3\,b^c)\,\phi_2)\\
+\;& \alpha_2^d\,((Q_2\,b^c + Q_3\,s^c)\xi_1 + (Q_1\,b^c + Q_3\,d^c)\,\xi_2 + (Q_1\,s^c + Q_2\,d^c)\,\xi_3)\\
+\;& \alpha_0^e\,(L_1\,e^c + L_2\,\mu^c + L_3\,\tau^c)\,\phi_0 + \alpha_1^e\,(\sqrt{3}\,(L_2\,\mu^c - L_3\,\tau^c)\,\phi_1 + (-2\,L_1\,e^c + L_2\,\mu^c + L_3\,\tau^c)\,\phi_2)\\
+\;& \alpha_2^e\,((L_2\,\tau^c + L_3\,\mu^c)\xi_1 + (L_1\,\tau^c + L_3\,e^c)\,\xi_2 + (L_1\,\mu^c + L_2\,e^c)\,\xi_3)\\
+\;& \alpha_0^\nu\,(L_1\,\nu_e^c + L_2\,\nu_\mu^c + L_3\,\nu_\tau^c)\,\tilde{\phi}_0 + \alpha_1^\nu\,(\sqrt{3}\,(L_2\,\nu_\mu^c - L_3\,\nu_\tau^c)\,\tilde{\phi}_1 + (-2\,L_1\,\nu_e^c + L_2\,\nu_\mu^c + L_3\,\nu_\tau^c)\,\tilde{\phi}_2)\\
+\;& \alpha_2^\nu\,((L_2\,\nu_\tau^c + L_3\,\nu_\mu^c)\tilde{\xi}_1 + (L_1\,\nu_\tau^c + L_3\,\nu_e^c)\,\tilde{\xi}_2 + (L_1\,\nu_\mu^c + L_2\,\nu_e^c)\,\tilde{\xi}_3) + \text{h.c.}
\end{aligned} \tag{3.25}$$

They lead to the following mass matrices for $i = u, d, e, \nu$

$$\mathcal{M}_i = \begin{pmatrix} \alpha_0^i\,\phi_0 - 2\,\alpha_1^i\,\phi_2 & \alpha_2^i\,\xi_3 & \alpha_2^i\,\xi_2 \\ \alpha_2^i\,\xi_3 & \alpha_0^i\,\phi_0 + \alpha_1^i\,(\sqrt{3}\,\phi_1 + \phi_2) & \alpha_2^i\,\xi_1 \\ \alpha_2^i\,\xi_2 & \alpha_2^i\,\xi_1 & \alpha_0^i\,\phi_0 + \alpha_1^i\,(-\sqrt{3}\,\phi_1 + \phi_2) \end{pmatrix} \tag{3.26}$$

with the Higgs fields being replaced by their VEVs for the down quarks and the charged leptons and by the complex conjugate of their VEVs for the up quarks and the neutrinos. The sum of the

VEVs has to be equal the electroweak scale. Note that the fields $\phi_{0,1,2}$ only appear in the diagonal entries. As mentioned, the contribution coming from the Higgs field ϕ_0 is proportional to the unit matrix, since ϕ_0 transforms trivially under S_4. The fields ϕ_1 and ϕ_2 on the other hand couple such that their contribution to the mass matrices is traceless. Finally, the fields ξ_i which form a triplet under S_4 induce flavor-changing interactions, i.e. their contributions are encoded in the off-diagonal elements of the mass matrix $\mathcal{M}_i$. Note that the mass matrix $\mathcal{M}_i$ would be of the same form, if the fermion generations transformed as $\underline{\mathbf{3_1}}$ instead of $\underline{\mathbf{3_2}}$. In general, all parameters in Eq.(3.26) can be complex. By comparing the number of parameters with the number of observables, i.e. fermion masses and mixings, we have to fit, one realizes that the numbers are equal in the case of CP conservation. We then deal with twelve Yukawa couplings, five VEVs and the mass scale M_R of the left-handed conjugate neutrinos which will be introduced below. The sixth VEV cannot be counted as a free parameter, since we need to satisfy the constraint that the sum of the squares of the VEVs adds up to the square of the electroweak scale. The observables are the six quark masses, the three charged lepton masses, the masses of the three light neutrinos and the three mixing angles θ_{ij} in the quark as well as in the lepton sector. Not all of the observables have already been measured in experiments: In the neutrino sector only the two mass squared differences are known, but not the absolute mass scale of the light neutrinos, and the leptonic mixing angle θ_{13} is only bounded from above. In the case of CP violation, the model contains more parameters than observables and therefore we cannot expect to make predictions, unless we do not impose further constraints on the parameters, which, for example, could arise from a certain vacuum alignment which sets some of the VEVs to zero or some of them equal. Note that it does not matter whether the CP violation is spontaneous or explicit, since in both cases the number of additional parameters is larger than the number of CP phases (one Dirac phase in the quark sector, one in the lepton sector and in case of Majorana neutrinos two further Majorana phases). Furthermore, the left-handed conjugate neutrinos ν^c acquire a direct mass term

$$\mathcal{L}_{\nu^c} = M_R \left(\nu_e^c \, \nu_e^c + \nu_\mu^c \, \nu_\mu^c + \nu_\tau^c \, \nu_\tau^c\right) + \text{h.c.} \tag{3.27}$$

Therefore the mass matrix of the left-handed conjugate neutrinos is proportional to the unit matrix, i.e. all left-handed conjugate neutrinos are degenerate. This favors the resonant leptogenesis scenario [87–89] in which the left-handed conjugate neutrinos have to have approximately the same mass. The mass matrix of the light neutrinos originating from the type-1 seesaw is of the form

$$M_\nu = (-)\frac{1}{M_R}\mathcal{M}_\nu \mathcal{M}_\nu^T = (-)\frac{1}{M_R}(\mathcal{M}_\nu)^2 \,, \tag{3.28}$$

where the last step is possible, since the Dirac mass matrices of the fermions are symmetric by construction. Since the form of M_{RR} is trivial, it cannot serve as the origin of the large lepton mixings.

In the following, we sketch the idea of the fit procedure. We choose two rank one matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ as starting point for the Dirac mass matrices. Thereby, the large mass of the third generation can already be explained. The mass matrices which fit all data arise through perturbations. Since the Majorana mass matrix of the left-handed conjugate neutrinos is proportional to the unit matrix, we have to expect that the Dirac neutrino mass matrix strongly differs from the two rank one matrices with which we start our numerical procedure. Otherwise, we cannot produce large mixings in the lepton sector.

$\mathcal{M}_1$ is the democratic mass matrix whose entries are all equal, whereas $\mathcal{M}_2$ only consists of a non-vanishing 2-3 block. Both matrices have been widely used in the literature, see for instance [90] and [91], respectively.

$$\mathcal{M}_1 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad \mathcal{M}_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix} \tag{3.29}$$

In the following, we describe in which limit the matrices $\mathcal{M}_1$ and $\mathcal{M}_2$ arise from the general mass matrices shown in Eq.(3.26). In both cases, we need to advocate a certain VEV structure together with a fine-tuning of some Yukawa couplings. The matrix structure $\mathcal{M}_1$ originates from the CP conserving minimum in which the VEVs of the fields ξ_i are equal, the VEVs of $\phi_{1,2}$ vanish and $\langle\phi_0\rangle \neq 0$. We have to assume that the two VEVs $\langle\xi_i\rangle$ and $\langle\phi_0\rangle$ are equal. In addition, we also have to require that the Yukawa couplings α_0^i and α_2^i are equal. These two equalities cannot, unfortunately, motivated any further in the context of this model. However, one can explain $\alpha_{0,2}^i \gg \alpha_1^i$ for $i = u, d, e$ and $\alpha_1^\nu \gg \alpha_{0,2}^\nu$ by an auxiliary Z_2 symmetry. The last inequality is confirmed by the numerical result of the fit. For the second mass matrix structure $\mathcal{M}_2$, only the VEVs of ϕ_0, ϕ_2 and ξ_1 should be non-vanishing and additionally have almost the same value. Furthermore, also here the Yukawa couplings should have a relation, namely $\alpha_0^i : \alpha_1^i : \alpha_2^i \approx 2 : 1 : 3$ for $i = u, d, e\,(,\nu)$, in order to reproduce the dominant 2-3 block.
For the actual numerical values of the parameters and the detailed discussion of the results we refer to [22]. Let us just mention that these two examples can reproduce almost all data successfully. They lead to quark mixings which are in general a bit smaller than the experimental best fit values. However, this might be cured by radiative corrections. Furthermore, the amount of CP violation in the quark sector cannot be correctly reproduced in one of the cases. For leptons we can fit the central values within the error bars for the masses (mass differences) and the known mixing angles θ_{12} and θ_{23}. Thereby, the neutrinos are taken to be normally ordered. The expectation that the Dirac mass matrix of the neutrinos has to have a structure which is completely different from the one of the charged fermion mass matrix is confirmed, since in both examples the coupling to the Higgs fields $\phi_{1,2}$ is large such that $\mathcal{M}_\nu$ has large entries on the diagonal.

3.2.4 Treatment of the Higgs Potential

In this section, we construct the S_4-invariant multi-Higgs doublet potential of the fields ϕ_0, $\phi_{1,2}$ and $\xi_{1,2,3}$ and argue that the VEV structure postulated in Section 3.2.3 can, indeed, be achieved by this potential. We concentrate on the case of CP conservation, although in the fits all parameters in the fermion mass matrices are taken to be complex. The mass terms of the Higgs fields turn out to have a simple form due to the flavor symmetry. However, there are numerous quartic couplings. The potential V reads

$$
\begin{aligned}
V \;=\;& -\mu_1^2\,(\phi_0^\dagger\phi_0) - \mu_2^2\sum_{j=1}^{2}\phi_j^\dagger\phi_j - \mu_3^2\sum_{i=1}^{3}\xi_i^\dagger\xi_i \qquad (3.30)\\
+\;& \lambda_0\,(\phi_0^\dagger\phi_0)^2 + \lambda_1\,(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2)^2 \; + \lambda_2\,(\phi_1^\dagger\phi_2-\phi_2^\dagger\phi_1)^2\\
+\;& \lambda_3\left[(\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1)^2+(\phi_1^\dagger\phi_1-\phi_2^\dagger\phi_2)^2\right]\\
+\;& \sigma_1\,(\phi_0^\dagger\phi_0)(\phi_1^\dagger\phi_1+\phi_2^\dagger\phi_2)+\left\{\sigma_2\left[(\phi_0^\dagger\phi_1)^2+(\phi_0^\dagger\phi_2)^2\right]+\text{h.c.}\right\}\\
+\;& \tilde\sigma_2\left[|\phi_0^\dagger\phi_1|^2+|\phi_0^\dagger\phi_2|^2\right]+\left\{\sigma_3\left[(\phi_0^\dagger\phi_1)(\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1)+(\phi_0^\dagger\phi_2)(\phi_1^\dagger\phi_1-\phi_2^\dagger\phi_2)\right]+\text{h.c.}\right\}\\
+\;& \lambda_1^\xi\left(\sum_{i=1}^{3}\xi_i^\dagger\xi_i\right)^2+\lambda_2^\xi\left[3\,(\xi_2^\dagger\xi_2-\xi_3^\dagger\xi_3)^2+(-2\,\xi_1^\dagger\xi_1+\xi_2^\dagger\xi_2+\xi_3^\dagger\xi_3)^2\right]\\
+\;& \lambda_3^\xi\left[(\xi_2^\dagger\xi_3+\xi_3^\dagger\xi_2)^2+(\xi_1^\dagger\xi_3+\xi_3^\dagger\xi_1)^2+(\xi_1^\dagger\xi_2+\xi_2^\dagger\xi_1)^2\right]\\
+\;& \lambda_4^\xi\left[(\xi_2^\dagger\xi_3-\xi_3^\dagger\xi_2)^2+(\xi_1^\dagger\xi_3-\xi_3^\dagger\xi_1)^2+(\xi_1^\dagger\xi_2-\xi_2^\dagger\xi_1)^2\right]\\
+\;& \tau_1\,(\phi_0^\dagger\phi_0)\left(\sum_{i=1}^{3}\xi_i^\dagger\xi_i\right)+\tau_2\left(\sum_{j=1}^{2}\phi_j^\dagger\phi_j\right)\left(\sum_{i=1}^{3}\xi_i^\dagger\xi_i\right)\\
+\;& \tau_3\left[\sqrt{3}\,(\phi_1^\dagger\phi_2+\phi_2^\dagger\phi_1)(\xi_2^\dagger\xi_2-\xi_3^\dagger\xi_3)+(\phi_1^\dagger\phi_1-\phi_2^\dagger\phi_2)(-2\,\xi_1^\dagger\xi_1+\xi_2^\dagger\xi_2+\xi_3^\dagger\xi_3)\right]\\
+\;& \left\{\kappa_1\left[4\,(\phi_2^\dagger\xi_1)^2+(\sqrt{3}\,\phi_1^\dagger\xi_2+\phi_2^\dagger\xi_2)^2+(\sqrt{3}\,\phi_1^\dagger\xi_3-\phi_2^\dagger\xi_3)^2\right]+\text{h.c.}\right\}\\
+\;& \tilde\kappa_1\left[4\,|\phi_2^\dagger\xi_1|^2+|\sqrt{3}\,\phi_1^\dagger\xi_2+\phi_2^\dagger\xi_2|^2+|\sqrt{3}\,\phi_1^\dagger\xi_3-\phi_2^\dagger\xi_3|^2\right]\\
+\;& \left\{\kappa_2\left[4\,(\phi_1^\dagger\xi_1)^2+(\sqrt{3}\,\phi_2^\dagger\xi_2-\phi_1^\dagger\xi_2)^2+(\sqrt{3}\,\phi_2^\dagger\xi_3+\phi_1^\dagger\xi_3)^2\right]+\text{h.c.}\right\}\\
+\;& \tilde\kappa_2\left[4\,|\phi_1^\dagger\xi_1|^2+|\sqrt{3}\,\phi_2^\dagger\xi_2-\phi_1^\dagger\xi_2|^2+|\sqrt{3}\,\phi_2^\dagger\xi_3+\phi_1^\dagger\xi_3|^2\right]\\
+\;& \Big\{\kappa_3\left[2\,(\phi_2^\dagger\xi_1)(\xi_2^\dagger\xi_3+\xi_3^\dagger\xi_2)-(\sqrt{3}\,\phi_1^\dagger\xi_2+\phi_2^\dagger\xi_2)(\xi_1^\dagger\xi_3+\xi_3^\dagger\xi_1)+(\sqrt{3}\,\phi_1^\dagger\xi_3-\phi_2^\dagger\xi_3)(\xi_1^\dagger\xi_2+\xi_2^\dagger\xi_1)\right]\\
+\;& \text{h.c.}\}\\
+\;& \Big\{\kappa_4\left[2\,(\phi_1^\dagger\xi_1)(\xi_3^\dagger\xi_2-\xi_2^\dagger\xi_3)+(\sqrt{3}\,\phi_2^\dagger\xi_2-\phi_1^\dagger\xi_2)(\xi_1^\dagger\xi_3-\xi_3^\dagger\xi_1)-(\sqrt{3}\,\phi_2^\dagger\xi_3+\phi_1^\dagger\xi_3)(\xi_2^\dagger\xi_1-\xi_1^\dagger\xi_2)\right]\\
+\;& \text{h.c.}\}\\
+\;& \left\{\kappa_5\left[(\phi_0^\dagger\xi_1)^2+(\phi_0^\dagger\xi_2)^2+(\phi_0^\dagger\xi_3)^2\right]+\text{h.c.}\right\}+\tilde\kappa_5\left[|\phi_0^\dagger\xi_1|^2+|\phi_0^\dagger\xi_2|^2+|\phi_0^\dagger\xi_3|^2\right]\\
+\;& \left\{\kappa_6\left[(\phi_0^\dagger\xi_1)(\xi_2^\dagger\xi_3+\xi_3^\dagger\xi_2)+(\phi_0^\dagger\xi_2)(\xi_1^\dagger\xi_3+\xi_3^\dagger\xi_1)+(\phi_0^\dagger\xi_3)(\xi_1^\dagger\xi_2+\xi_2^\dagger\xi_1)\right]+\text{h.c.}\right\}\\
+\;& \left\{\omega_1\left[\sqrt{3}\,(\phi_0^\dagger\phi_1)(\xi_2^\dagger\xi_2-\xi_3^\dagger\xi_3)+(\phi_0^\dagger\phi_2)(-2\,\xi_1^\dagger\xi_1+\xi_2^\dagger\xi_2+\xi_3^\dagger\xi_3)\right]+\text{h.c.}\right\}\\
+\;& \left\{\omega_2\left[2(\phi_0^\dagger\xi_1)(\phi_2^\dagger\xi_1)-(\phi_0^\dagger\xi_2)(\sqrt{3}\,\phi_1^\dagger\xi_2+\phi_2^\dagger\xi_2)+(\phi_0^\dagger\xi_3)(\sqrt{3}\,\phi_1^\dagger\xi_3-\phi_2^\dagger\xi_3)\right]+\text{h.c.}\right\}\\
+\;& \left\{\omega_3\left[2(\xi_1^\dagger\phi_0)(\phi_2^\dagger\xi_1)-(\xi_2^\dagger\phi_0)(\sqrt{3}\,\phi_1^\dagger\xi_2+\phi_2^\dagger\xi_2)+(\xi_3^\dagger\phi_0)(\sqrt{3}\,\phi_1^\dagger\xi_3-\phi_2^\dagger\xi_3)\right]+\text{h.c.}\right\}
\end{aligned}
$$

The parameters in curly brackets are in general complex, while the rest is real. The Higgs potential has 30 parameters in total. 27 of them are quartic couplings out of which 11 are complex. It is interesting to notice that the potential is invariant under the following transformation

$$\phi_1 \quad\rightarrow\quad -\phi_1 \quad \text{and} \quad \xi_2 \quad\leftrightarrow\quad \xi_3 \qquad (3.31)$$

and the fields ϕ_0, ϕ_2 and ξ_1 remain unchanged. Note that the VEV structures used to arrive at the matrix structures $\mathcal{M}_1$ and $\mathcal{M}_2$ also respect this symmetry. If all VEVs are real, they can be parameterized as

$$\begin{aligned}
\langle\phi_0\rangle &= v_0 \ , \qquad \langle\phi_1\rangle = u\,\cos(\alpha) \ , \qquad \langle\phi_2\rangle = u\,\sin(\alpha) \ , \\
\langle\xi_1\rangle &= v\,\cos(\beta) \ , \ \langle\xi_2\rangle = v\,\sin(\beta)\,\cos(\gamma) \ , \ \langle\xi_3\rangle = v\,\sin(\beta)\,\sin(\gamma) \ .
\end{aligned} \tag{3.32}$$

The potential V_{min} at the minimum takes the form

$$\begin{aligned}
V_{min} &= -\mu_1^2\,v_0^2 - \mu_2^2\,u^2 - \mu_3^2\,v^2 + \lambda_0\,v_0^4 + (\lambda_1+\lambda_3)\,u^4 \\
&+ \left(\lambda_1^\xi + 4\,\lambda_3^\xi\,\sin^2(\beta)\right)\,v^4 + \lambda_2^\xi\,\left[\left(2-3\,\sin^2(\beta)\right)^2 + 3\,\sin^4(\beta)\,\cos^2(2\,\gamma)\right]\,v^4 \\
&- \lambda_3^\xi\,\left(3+\cos^2(2\,\gamma)\right)\,\sin^4(\beta)\,v^4 + (\sigma_1 + 2\,\mathrm{Re}(\sigma_2) + \tilde{\sigma}_2)\,v_0^2\,u^2 + 2\,\mathrm{Re}(\sigma_3)\,\sin(3\alpha)\,v_0\,u^3 \\
&+ (\tau_1 + 2\,\mathrm{Re}(\kappa_5) + \tilde{\kappa}_5)\,v_0^2\,v^2 + (4\,\mathrm{Re}(\kappa_1+\kappa_2) + 2\,(\tilde{\kappa}_1+\tilde{\kappa}_2) + \tau_2)\,u^2\,v^2 \\
&+ (2\,\mathrm{Re}(\kappa_1-\kappa_2) + \tilde{\kappa}_1 - \tilde{\kappa}_2 + \tau_3)\,\left[-\cos(2\,\alpha)\,\left(2-3\,\sin^2(\beta)\right)\right. \\
&+ \left.\sqrt{3}\,\sin(2\,\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,u^2\,v^2 \\
&+ 3\,\mathrm{Re}(\kappa_6)\,\sin(\beta)\,\sin(2\,\beta)\,\sin(2\,\gamma)\,v_0\,v^3 \\
&+ 2\,\mathrm{Re}(\omega_2+\omega_3-\omega_1)\,\left[\sin(\alpha)\,\left(2-3\,\sin^2(\beta)\right) - \sqrt{3}\,\cos(\alpha)\,\sin^2(\beta)\,\cos(2\gamma)\right]\,u\,v_0\,v^2
\end{aligned} \tag{3.33}$$

The minimization conditions are the derivatives of V_{min} with respect to the angles α, β and γ and the parameters v, v_0, u.

$$\begin{aligned}
\frac{\partial V_{min}}{\partial\alpha} &= 2\,u\,\left[v^2 v_0\,\left(\cos(\alpha)\,\left(2-3\,\sin^2(\beta)\right) + \sqrt{3}\,\sin(\alpha)\sin^2(\beta)\,\cos(2\,\gamma)\right)\,y_1\right. \\
&+ u\,v^2\,\left(\sin(2\,\alpha)\,\left(2-3\,\sin^2(\beta)\right) + \sqrt{3}\cos(2\,\alpha)\sin^2(\beta)\cos(2\,\gamma)\right)\,y_2 \\
&+ \left. 3\,v_0\,u^2\,\cos(3\,\alpha)\,\mathrm{Re}(\sigma_3)\right]
\end{aligned} \tag{3.34a}$$

$$\begin{aligned}
\frac{\partial V_{min}}{\partial\beta} &= v^2\,\left[-2\,\sqrt{3}\,u\,v_0\,\left(\sqrt{3}\,\sin(\alpha) + \cos(\alpha)\,\cos(2\,\gamma)\right)\,\sin(2\,\beta)\,y_1\right. \\
&+ \sqrt{3}\,u^2\,\left(\sqrt{3}\,\cos(2\,\alpha) + \sin(2\,\alpha)\,\cos(2\,\gamma)\right)\,\sin(2\,\beta)\,y_2 \\
&+ 2\,v^2\,\left(\sin(4\,\beta) + \sin^2(\beta)\sin(2\,\beta)\sin^2(2\,\gamma)\right)\,y_3 \\
&+ \left.\frac{3}{2}\,v\,v_0\,\left(3\,\sin(3\,\beta) - \sin(\beta)\right)\,\sin(2\,\gamma)\,\mathrm{Re}(\kappa_6)\right]
\end{aligned} \tag{3.34b}$$

$$\begin{aligned}
\frac{\partial V_{min}}{\partial\gamma} &= 2\,v^2\,\sin^2(\beta)\,\left[2\,\sqrt{3}\,u\,v_0\,\cos(\alpha)\,\sin(2\,\gamma)\,y_1\right. \\
&- \sqrt{3}\,u^2\,\sin(2\,\alpha)\,\sin(2\,\gamma)\,y_2 \\
&+ v^2\,\sin^2(\beta)\,\sin(4\,\gamma)\,y_3 \\
&+ \left. 6\,v\,v_0\,\cos(\beta)\,\cos(2\,\gamma)\,\mathrm{Re}(\kappa_6)\right]
\end{aligned} \tag{3.34c}$$

$$\begin{aligned}
\frac{\partial V_{min}}{\partial v} &= 2\,v\,\left(-\mu_3^2 + 2\,(\lambda_1^\xi + 4\,\lambda_2^\xi)\,v^2\right) + 2\,v\,v_0^2\,(2\,\mathrm{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) \\
&+ 2\,v\,u^2\,(4\,\mathrm{Re}(\kappa_1+\kappa_2) + 2\,(\tilde{\kappa}_1+\tilde{\kappa}_2) + \tau_2) + 9\,v_0\,v^2\,\mathrm{Re}(\kappa_6)\,\sin(\beta)\,\sin(2\,\beta)\,\sin(2\,\gamma) \\
&+ 4\,u\,v_0\,v\,\left[\sin(\alpha)\,\left(2-3\,\sin^2(\beta)\right) - \sqrt{3}\,\cos(\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,y_1 \\
&+ 2\,u^2\,v\,\left[-\cos(2\,\alpha)\,(2-3\,\sin^2(\beta)) + \sqrt{3}\,\sin(2\,\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,y_2 \\
&+ 4\,v^3\,\left[4\,\sin^2(\beta) - (3+\cos^2(2\,\gamma))\,\sin^4(\beta)\right]\,y_3
\end{aligned} \tag{3.35a}$$

$$\begin{aligned}\frac{\partial V_{min}}{\partial v_0} &= 2\,v_0\,\left(-\mu_1^2+2\,\lambda_0\,v_0^2\right)+2\,v_0\,u^2\,(\sigma_1+2\,\mathrm{Re}(\sigma_2)+\tilde{\sigma}_2)+2\,u^3\,\mathrm{Re}(\sigma_3)\,\sin(3\,\alpha) \quad (3.35\text{b})\\ &+ 2\,v_0\,v^2\,(2\,\mathrm{Re}(\kappa_5)+\tilde{\kappa}_5+\tau_1)+3\,v^3\,\mathrm{Re}(\kappa_6)\,\sin(\beta)\,\sin(2\,\beta)\,\sin(2\,\gamma)\\ &+ 2\,u\,v^2\,\left[\sin(\alpha)\,(2-3\,\sin^2(\beta))-\sqrt{3}\,\cos(\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,y_1\end{aligned}$$

$$\begin{aligned}\frac{\partial V_{min}}{\partial u} &= 2\,u\,\left(-\mu_2^2+2\,(\lambda_1+\lambda_3)\,u^2\right)+2\,u\,v_0^2\,(\sigma_1+2\,\mathrm{Re}(\sigma_2)+\tilde{\sigma}_2)+6\,v_0\,u^2\,\mathrm{Re}(\sigma_3)\,\sin(3\,\alpha)\\ &+ 2\,(4\,\mathrm{Re}(\kappa_1+\kappa_2)+2\,(\tilde{\kappa}_1+\tilde{\kappa}_2)+\tau_2)\,u\,v^2\\ &+ 2\,v_0\,v^2\,\left[\sin(\alpha)\,(2-3\,\sin^2(\beta))-\sqrt{3}\,\cos(\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,y_1\\ &+ 2\,u\,v^2\,\left[-\cos(2\,\alpha)\,(2-3\,\sin^2(\beta))+\sqrt{3}\,\sin(2\,\alpha)\,\sin^2(\beta)\,\cos(2\,\gamma)\right]\,y_2 \qquad (3.35\text{c})\end{aligned}$$

with y_i being defined as

$$y_1 = \mathrm{Re}(\omega_2+\omega_3-\omega_1) \qquad (3.36\text{a})$$

$$y_2 = 2\,\mathrm{Re}(\kappa_1-\kappa_2)+\tilde{\kappa}_1-\tilde{\kappa}_2+\tau_3 \qquad (3.36\text{b})$$

$$y_3 = \lambda_3^{\xi}-3\,\lambda_2^{\xi} \qquad (3.36\text{c})$$

All derivatives have to vanish. However, since we do not want to rely on some restrictions on the parameters of the Higgs potential which might cause the appearance of accidental symmetries [5] we enforce each term in the equations $\frac{\partial V_{min}}{\partial \alpha}=0$, $\frac{\partial V_{min}}{\partial \beta}=0$ and $\frac{\partial V_{min}}{\partial \gamma}=0$ to vanish separately. The remaining conditions $\frac{\partial V_{min}}{\partial v}=0$, $\frac{\partial V_{min}}{\partial v_0}=0$ and $\frac{\partial V_{min}}{\partial u}=0$ are used to determine the VEVs as functions of the parameters of the potential.
The VEV configuration $\langle\phi_0\rangle=v_0$, $\langle\phi_{1,2}\rangle=0$ and $\langle\xi_i\rangle=\frac{v}{\sqrt{3}}$ is used to reproduce the democratic mass matrix $\mathcal{M}_1$. Thereby, the angles β and γ are taken to be $\beta=\arccos(\frac{1}{\sqrt{3}})$ and $\gamma=\frac{\pi}{4}$, while α is no longer a variable, since u vanishes. As one can see, with this choice each term in $\frac{\partial V_{min}}{\partial \alpha}=0$, $\frac{\partial V_{min}}{\partial \beta}=0$ and $\frac{\partial V_{min}}{\partial \gamma}=0$ vanishes separately, as demanded above. Furthermore, we arrive at the following equations for v_0 and v

$$v\,(2\,\sqrt{3}\,v\,v_0\,\mathrm{Re}(\kappa_6)+\frac{2}{3}\,(3\,\lambda_1^{\xi}+4\,\lambda_3^{\xi})\,v^2-\mu_3^2+(2\,\mathrm{Re}(\kappa_5)+\tilde{\kappa}_5+\tau_1)\,v_0^2) = 0 \qquad (3.37)$$

$$\frac{2}{\sqrt{3}}\,\mathrm{Re}(\kappa_6)\,v^3+2\,\lambda_0\,v_0^3-v_0\,\mu_1^2+v^2\,v_0\,(2\,\mathrm{Re}(\kappa_5)+\tilde{\kappa}_5+\tau_1) = 0 \qquad (3.38)$$

As mentioned, the equality of $\langle\phi_0\rangle$ and $\langle\xi_i\rangle$ is an additional assumption which is not a direct consequence of these minimization conditions. However, if all quartic couplings have approximately the same value and also the mass parameters are equal, the VEVs are nearly the same.
For the second advocated VEV structure, $\langle\phi_0\rangle=v_0$, $\langle\phi_2\rangle=u$, $\langle\xi_1\rangle=v$ and a vanishing VEV for the rest of the fields, the angles α and β read $\alpha=\frac{\pi}{2}$ and $\beta=0$, while the angle γ is irrelevant. Also this configuration leads to a vanishing coefficient for each of the terms in $\frac{\partial V_{min}}{\partial \alpha}=0$, $\frac{\partial V_{min}}{\partial \beta}=0$ and $\frac{\partial V_{min}}{\partial \gamma}=0$. The equations which determine the VEVs v_0, v and u are

[5] This is one of the major concerns in the next model.

$$\begin{aligned} v\,(2\,v^2\,(\lambda_1^\xi + 4\,\lambda_2^\xi) - \mu_3^2 + v_0^2\,(2\,\mathrm{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) + u^2\,(4\,(2\,\mathrm{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2\,\tau_3) & \\ +4\,u\,v_0\,\mathrm{Re}(\omega_2 + \omega_3 - \omega_1)) &= 0 \quad (3.39) \\ 2\,u^3\,(\lambda_1 + \lambda_3) - 3\,u^2\,v_0\,\mathrm{Re}(\sigma_3) - \mu_2^2\,u + v_0^2\,u\,(\sigma_1 + 2\,\mathrm{Re}(\sigma_2) + \tilde{\sigma}_2) & \\ +u\,v^2\,(4\,(2\,\mathrm{Re}(\kappa_1) + \tilde{\kappa}_1) + \tau_2 + 2\,\tau_3) + 2\,v^2\,v_0\,\mathrm{Re}(\omega_2 + \omega_3 - \omega_1) &= 0 \quad (3.40) \\ 2\,v_0^3\,\lambda_0 - \mu_1^2\,v_0 + v_0\,u^2\,(\sigma_1 + 2\,\mathrm{Re}(\sigma_2) + \tilde{\sigma}_2) - u^3\,\mathrm{Re}(\sigma_3) + v^2\,v_0\,(2\,\mathrm{Re}(\kappa_5) + \tilde{\kappa}_5 + \tau_1) & \\ +2\,v^2\,u\,\mathrm{Re}(\omega_2 + \omega_3 - \omega_1) &= 0 \quad (3.41) \end{aligned}$$

Also here, the equality of the VEVs, as required by the numerical fit, is not a generic result of the Higgs potential and its minimization conditions.
As studied in [22] the requirement that the parameters of the Higgs potential, $\mathrm{Re}(\sigma_3)$ and $\mathrm{Re}(\kappa_6)$, and the combinations y_i, which appear in the derivatives above, should not vanish, is reasonable, since, if y_i, $\mathrm{Re}(\kappa_6)$ and $\mathrm{Re}(\sigma_3)$ are zero, both minima break accidental symmetries such that one finds additional Goldstone bosons in the Higgs mass spectrum. The general formulae of the Higgs masses at the two minima are also given in [22]. One can check that the minima are degenerate, if the mass parameters μ_i are taken to be equal.
The actual VEV structures used in the fit are expected to arise as perturbations from the ones discussed here, if the parameters of the potential are smoothly varied. Two crucial issues have not been studied here, namely *a.*) the question whether the minima are global or only local and *b.*) the stability of the potential as a whole. Both issues are essential in a realistic theory, but we believe that due to the number of free parameters in this potential this can be maintained in at least one point of the parameter space.

3.2.5 Embedding of the Model into $SO(10) \times SO(3)_f$ or $SO(10) \times SU(3)_f$

In this section we show how one can promote this model to a complete GUT model. Concerning the Yukawa couplings, their number is reduced, since in $SO(10)$ all fermions are unified into a $\underline{\mathbf{16}}$, i.e. all Yukawa couplings $\alpha_i^{u,d,e,\nu}$ have to be replaced by one coupling α_i. However, a Higgs doublet in the low energy model can originate from $\underline{\mathbf{10}}$ and $\overline{\underline{\mathbf{126}}}$ in the complete $SO(10)$ model and thereby new couplings are allowed, i.e. the Yukawa couplings $\alpha_i^{u,d,e,\nu}$ are replaced by two couplings $\alpha_{i,\underline{\mathbf{10}}}$ and $\alpha_{i,\overline{\underline{\mathbf{126}}}}$ in case that there exists one $\underline{\mathbf{10}}$ and one $\overline{\underline{\mathbf{126}}}$ for every Higgs doublet. The Higgs doublets cannot be identified with a $\underline{\mathbf{120}}$, since the $SO(10)$ structure demands the mass matrix to be anti-symmetric, whereas it should be symmetric according to the flavor symmetry. Therefore, such a contribution will vanish in general. The left-handed conjugate neutrinos acquire direct mass terms in the low energy model. In order to reproduce this in an $SO(10)$ GUT a $\overline{\underline{\mathbf{126}}}$ has to exist which transforms trivially under S_4. For the Dirac mass matrices the Higgs doublets contained in $\underline{\mathbf{10}}$ and $\overline{\underline{\mathbf{126}}}$ have to transform non-trivially under S_4. Furthermore, none of the Higgs triplet components of the $\overline{\underline{\mathbf{126}}}$ should acquire a non-vanishing VEV, since the low energy model does not contain Higgs triplets which also would contribute to the light neutrino mass matrix. A minimal choice of Higgs fields contributing to fermion masses in a complete $SO(10)$ model are six $\underline{\mathbf{10}}$s transforming as $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{2}}$ and $\underline{\mathbf{3}}_{\mathbf{1}}$ under S_4 and one $\overline{\underline{\mathbf{126}}}$ being invariant under S_4. However, since the Higgs fields transforming as $\underline{\mathbf{10}}$ under $SO(10)$ cannot differentiate among down quarks and charged leptons, the fields $\overline{\underline{\mathbf{126}}}$ also have to contribute to the Dirac mass matrices of the charged fermions. With only one $\overline{\underline{\mathbf{126}}}$ which is invariant under S_4 we might not be able to explain the different spectrum of down quarks and charged leptons and therefore all Higgs doublets ought to be promoted to a $\underline{\mathbf{10}}$ and a $\overline{\underline{\mathbf{126}}}$ in $SO(10)$. Thereby, the number of VEVs of Higgs doublets is increased compared to the low energy model. In order not to break S_4 at a high energy scale only the Higgs field $\overline{\underline{\mathbf{126}}} \sim \underline{\mathbf{1}}_{\mathbf{1}}$

should get a GUT scale VEV. In addition to the Higgs fields needed for the mass matrices further Higgs fields, like $\underline{\mathbf{210}}$ have to exist in order to employ the breaking of $SO(10)$ down to the SM. These should be neutral under S_4 in order not to break it at high energies. This excessive number of Higgs fields usually causes several severe problems: first of all, such a large number of fields leads to a blow-up of the gauge coupling at a scale which is only slightly above the GUT scale; secondly, as we used in our low energy model six Higgs fields which originate in an $SO(10)$ model from several Higgs fields transforming as $\underline{\mathbf{10}}$ and $\overline{\underline{\mathbf{126}}}$, this leads to the so-called doublet-doublet splitting problem, i.e. the fact that only certain combinations of the Higgs doublets of $\underline{\mathbf{10}}$ and $\overline{\underline{\mathbf{126}}}$ shall appear at low energies; thirdly, similar to the doublet-doublet splitting problem there exists the doublet-triplet splitting problem, i.e. the separation of the $SU(2)_L$ doublet component of the GUT Higgs field from the colored triplet component; fourthly, the Higgs potential which now has to have $SO(10)$ invariance might not be of a simpler form than the one discussed above. Therefore also here the issue of the vacuum alignment has to be carefully studied.

Similar to the changes occurring, if the model is embedded into $SO(10)$, also the embedding of S_4 into $SO(3)_f$ or $SU(3)_f$ will reduce the number of Yukawa couplings, since, for example, the Higgs fields transforming as $\underline{\mathbf{2}}$ and $\underline{\mathbf{3_1}}$ are unified into a single representation in $SO(3)_f$ or $SU(3)_f$. In this case the Yukawa couplings α_1^i and α_2^i have to be equal. There is one caveat concerning the Majorana masses of the left-handed conjugate neutrinos. If S_4 is embedded into $SO(3)_f$, the direct mass term is still allowed, since also in $SO(3)_f$ the product of two fundamental representations contains a group invariant. However, if S_4 is embedded into $SU(3)_f$, the real three-dimensional representation $\underline{\mathbf{3_2}}$ under which all the fermions transform is identified with the fundamental representation $\underline{\mathbf{3}}$ of $SU(3)_f$ which is complex. Therefore the product $\underline{\mathbf{3}} \times \underline{\mathbf{3}}$ no longer contains an invariant and the left-handed conjugate neutrinos cannot acquire a direct mass in this way. In order to solve this problem one has to introduce gauge singlets transforming as $\underline{\mathbf{6}}$ under $SU(3)_f$ and one has to require that they possess an S_4-invariant VEV, i.e. only the combination of fields transforming as trivial representation under S_4 should have a non-vanishing VEV. In this way the low energy model presented here can be reproduced from a high energy model with an $SU(3)_f$ flavor symmetry. From this discussion one might conclude that an embedding of S_4 into $SO(3)_f$ is preferred. However, combining the embedding of S_4 into $SU(3)_f$ and of the SM into $SO(10)$ changes the situation again, since then also the $SO(10)$ structure does not allow a direct mass term for the left-handed conjugate neutrinos and this has to be generated by, for example, a renormalizable coupling to Higgs fields transforming as $\overline{\underline{\mathbf{126}}}$.

If the model is appropriately extended to a real $SO(10) \times SO(3)_f$ or $SO(10) \times SU(3)_f$ model, we still have to consider the breaking of the continuous flavor symmetry down to S_4 in order to arrive at $SO(10) \times S_4$ at the GUT scale. For this purpose, we need to add scalar singlets under $SO(10)$ which transform according to $SO(3)_f$ ($SU(3)_f$) representations containing a trivial representation of S_4. The smallest representations of $SO(3)$ ($SU(3)$) which can do the job are $\underline{\mathbf{9}}$ of $SO(3)$ and $\underline{\mathbf{6}}$ of $SU(3)$, as can be inferred from Table 3.2.

3.2.6 Summary and Conclusions

The low energy model presented here can explain the existence of the three generations of fermions and leads to mass matrices which are phenomenologically viable apart from small deviations in the quark sector. This is one of the rare models which has the feature to be embeddable into an $SO(10)$ GUT and simultaneously into the continuous flavor symmetry $SO(3)_f$ or $SU(3)_f$. Although its Higgs sector is too complicated to be solved in general, we studied it in certain limits in order to show that the VEV structures, advocated in the fits of the fermion masses and mixings, discussed in [22], can be produced. However, it is not possible to identify these with preferred minima of the potential. Unfortunately, the model cannot make any predictions due to the large number of parameters in the Yukawa as well as the Higgs sector. Moreover, in our numerical analysis we had to assume some further restrictions on the Yukawa couplings which can be maintained by an additional symmetry in only one of the two cases. The embedding of this model into a larger framework, as shown in Section 3.2.5, might solve this problem (at least partly). The difference between the quark and lepton mixings has to stem from the different structure of the Dirac mass matrices of charged and neutral fermions, since M_{RR} is proportional to the unit matrix in this model.

Concerning the Higgs sector we do not only have to face the problem that its potential cannot be treated in a general way and therefore the study of VEV configurations is only possible in certain limits, but also the presence of more than one light Higgs doublet causes several flavor changing neutral current processes which are severely constrained by experiments. Generally, the masses of the Higgs particles turn out to be very low [6] in case of a natural choice of parameters, i.e. for mass parameters around the electroweak scale and quartic couplings in the perturbative range. This problem tends to persist, if the model shall be embedded into a GUT and/or into a continuous flavor symmetry. We will comment on this issue further in Section 3.4.

In summary, this model can lead to viable fermion masses and mixings, but it might only serve as a first step towards more realistic models with the flavor symmetry S_4.

3.3 D_5 Model

In this section, we discuss another low energy model with a discrete flavor symmetry. The gauge group is again the one of the SM and the dihedral group D_5 serves as flavor symmetry. It is broken only spontaneously at the electroweak scale by VEVs of Higgs doublets. D_5 is the smallest group which possesses two irreducible inequivalent two-dimensional representations which are called $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ in the following. The purpose of this study is to explore the opportunities for model building resulting from the presence of these two inequivalent representations. Thereby, the aim is to arrive at the minimal phenomenological viable model which makes use of this feature of D_5. In particular, we require the number of Higgs fields to be as small as possible. The model is phenomenological viable, if all fermion masses and mixings can be accommodated without additional contributions from, for example, higher-dimensional operators. Again, we carefully investigate the Higgs potential. This time the major concern is an acceptable Higgs mass spectrum, i.e. one which does not contain uneaten Goldstone bosons arising from the spontaneous breaking of accidental symmetries of the potential. As will be shown, this poses a rather strong constraint on our model, since the minimal setup of fields whose potential is free from accidental symmetries consists of four Higgs doublets which have to transform as the two inequivalent two-dimensional representations of D_5. Additionally, we do not want to completely give up the idea of GUTs, i.e. the model ought to be embeddable at least into the Pati-Salam group $SU(4)_C \times SU(2)_L \times SU(2)_R$. In order to fully exploit the non-abelian structure of D_5 we demand that all left-handed and left-handed conjugate fermions transform as $\underline{\mathbf{1}}+\underline{\mathbf{2}}$. The one- and two-dimensional representations used for left-handed and left-handed conjugate fields, however, do not need to be equivalent. Taking into account these conditions leads to the model presented below in which the left-handed fermions transform as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}}$ and left-handed conjugate ones as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}}$. The resulting mass matrices are studied in a similar fashion as done above. This time, however, we also considered the case of Dirac neutrinos in the

[6] Some of them are even below the LEP bound [92].

numerical examples shown in [23] [7]. All fits are done for neutrinos with normal mass ordering. Much emphasis is put on the analysis of the Higgs sector, especially the group theoretical reasons for the appearance of accidental symmetries are elucidated. A numerical treatment of the Higgs mass spectrum can be found in [23]. Finally, we summarize and comment on several aspects of the model.

3.3.1 Introduction to D_5

We briefly introduce the group D_5, although we will discuss the series of single- and double-valued dihedral groups D_n and D'_n in general below in Chapter 5. Its irreducible representations are $\underline{\mathbf{1_1}}$ (trivial representation), $\underline{\mathbf{1_2}}$, $\underline{\mathbf{2_1}}$ and $\underline{\mathbf{2_2}}$. All of them are real and both two-dimensional representations are faithful. The order of D_5 is 10. The generators A and B fulfill the relations [86]

$$\mathrm{A}^5 = \mathbb{1} \quad , \quad \mathrm{B}^2 = \mathbb{1} \quad , \quad \mathrm{A\,B\,A} = \mathrm{B}\ . \tag{3.42}$$

A and B can be chosen as [86]

$$\text{... for } \underline{\mathbf{2_1}} \quad : \quad \mathrm{A} = \left(\begin{array}{cc} \mathrm{e}^{\frac{2\pi i}{5}} & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi i}{5}} \end{array} \right) \quad , \quad \mathrm{B} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) , \tag{3.43}$$

$$\text{... for } \underline{\mathbf{2_2}} \quad : \quad \mathrm{A} = \left(\begin{array}{cc} \mathrm{e}^{\frac{4\pi i}{5}} & 0 \\ 0 & \mathrm{e}^{-\frac{4\pi i}{5}} \end{array} \right) \quad , \quad \mathrm{B} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) . \tag{3.44}$$

For the one-dimensional representations the generators A and B can be found in the character table, see Table 3.4. Note that we decided to use complex generators for the two-dimensional representations, although they are real. Therefore, we can find a similarity transformation U which connects the representation matrices and their complex conjugates

$$U = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \quad \text{with} \quad U\,U^T = U^T\,U = \mathbb{1}\ , \tag{3.45}$$

so that $\mathrm{A}^\star = U^T\,\mathrm{A}\,U$ and $\mathrm{B}^\star = U^T\,\mathrm{B}\,U = \mathrm{B}$. For $\left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) \sim \underline{\mathbf{2_i}}$ then $\left(\begin{array}{c} a_2^\star \\ a_1^\star \end{array} \right)$ transforms as $\underline{\mathbf{2_i}}$.
For completeness, we also show a possible choice of real generators [86]

$$\text{... for } \underline{\mathbf{2_1}} \quad : \quad \mathrm{A_r} = \left(\begin{array}{cc} \cos\left(\frac{2\pi}{5}\right) & -\sin\left(\frac{2\pi}{5}\right) \\ \sin\left(\frac{2\pi}{5}\right) & \cos\left(\frac{2\pi}{5}\right) \end{array} \right) \quad , \quad \mathrm{B_r} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) , \tag{3.46}$$

$$\text{... for } \underline{\mathbf{2_2}} \quad : \quad \mathrm{A_r} = \left(\begin{array}{cc} \cos\left(\frac{4\pi}{5}\right) & -\sin\left(\frac{4\pi}{5}\right) \\ \sin\left(\frac{4\pi}{5}\right) & \cos\left(\frac{4\pi}{5}\right) \end{array} \right) \quad , \quad \mathrm{B_r} = \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) . \tag{3.47}$$

They are linked to the complex ones via the similarity transformation V

$$V = \frac{1}{\sqrt{2}} \left(\begin{array}{cc} 1 & i \\ 1 & -i \end{array} \right) , \tag{3.48}$$

i.e. the generator sets $\{\mathrm{A}, \mathrm{B}\}$ and $\{\mathrm{A_r}, \mathrm{B_r}\}$ fulfill

$$\mathrm{A_r} = V^\dagger\,\mathrm{A}\,V \quad \text{and} \quad \mathrm{B_r} = V^\dagger\,\mathrm{B}\,V\ . \tag{3.49}$$

Note that V is unitary, i.e. $V^\dagger\,V = V\,V^\dagger = \mathbb{1}$.
In the following, we stick to the complex generators for $\underline{\mathbf{2_i}}$. Further details of the group theory can be found in Appendix B.2.

[7] This raises the typical problem that the masses of the neutrinos have to be strongly suppressed compared to the ones of the charged fermions, i.e. their Yukawa couplings have to be extremely small.

	classes					
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$		
G	$\mathbb{1}$	B	A	A^2		
$^\circ\mathcal{C}_i$	1	5	2	2		
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	2	5	5	$\mathbb{c}^{(\mu)}$	faithful
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	1	1	
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	-1	1	1	1	
$\underline{\mathbf{2}}_{\mathbf{1}}$	2	0	α	β	1	$\surd$
$\underline{\mathbf{2}}_{\mathbf{2}}$	2	0	β	α	1	$\surd$

Table 3.4: Character table of the group D_5. α and β are given as $\alpha = \frac{1}{2}\left(-1+\sqrt{5}\right) = 2\cos(\frac{2\pi}{5})$ and $\beta = \frac{1}{2}\left(-1-\sqrt{5}\right) = 2\cos(\frac{4\pi}{5})$ and therefore $\alpha + \beta = -1$. For further explanations see Appendix A.

3.3.2 Outline of the Model

As mentioned, our main motivation here is to study the changes in the opportunities to build viable low energy models, if the flavor symmetry contains more than one irreducible two-dimensional representation. For this purpose, it is sufficient to stick to the smallest symmetry which offers this possibility. This is the introduced group D_5. The three non-abelian symmetries having a smaller order than D_5 are D_3 ($\cong S_3$) , D_4 and D'_2 (sometimes called Q). All of them only possess one two-dimensional representation and two (or four) one-dimensional ones. They have been studied quite extensively in the literature, for instance [93–95]. In contrast to this the group D_5 has been rarely used as flavor symmetry [96].

Our requirement that all particles have to transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ and not only, for example, the left-handed leptons reduces the number of possible assignments and also the number of uncorrelated elements in the fermion mass matrices. Further constraints arise from the fact that the model should be embeddable into the Pati-Salam gauge group [8]. We find two possible assignments which fulfill these conditions [9]

$$Q_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}\,,\; \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}\,,\; u^c\,, d^c \sim \underline{\mathbf{1}}_{\mathbf{1}} \;\text{ and }\; \begin{pmatrix} c^c \\ t^c \end{pmatrix}, \begin{pmatrix} s^c \\ b^c \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}\,, \tag{3.50}$$

$$L_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}\,,\; \begin{pmatrix} L_2 \\ L_3 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}}\,,\; e^c\,, \nu^c_e \sim \underline{\mathbf{1}}_{\mathbf{1}} \;\text{ and }\; \begin{pmatrix} \mu^c \\ \tau^c \end{pmatrix}, \begin{pmatrix} \nu^c_\mu \\ \nu^c_\tau \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{i}} \tag{3.51}$$

with $\mathrm{i} = 1, 2$ and

$$Q_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}\,,\; \begin{pmatrix} Q_2 \\ Q_3 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}\,,\; u^c\,, d^c \sim \underline{\mathbf{1}}_{\mathbf{1}} \;\text{ and }\; \begin{pmatrix} c^c \\ t^c \end{pmatrix}, \begin{pmatrix} s^c \\ b^c \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}\,, \tag{3.52}$$

$$L_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}\,,\; \begin{pmatrix} L_2 \\ L_3 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}\,,\; e^c\,, \nu^c_e \sim \underline{\mathbf{1}}_{\mathbf{1}} \;\text{ and }\; \begin{pmatrix} \mu^c \\ \tau^c \end{pmatrix}, \begin{pmatrix} \nu^c_\mu \\ \nu^c_\tau \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}\,. \tag{3.53}$$

[8] An embedding of the model into $SU(5)$ might also be interesting, but is not considered in the following.

[9] These assignments do not necessarily imply that the first generation of fermions is assigned to $\underline{\mathbf{1}}$ and the other ones form a doublet under D_5, since we are free to permute them. Since left-handed fields form doublets under $SU(2)_L$ up and down quarks as well as charged leptons and neutrinos are permuted in the same way, so that the permutation matrix drops out in the calculation of V_{CKM} and U_{MNS}, see [23] for details. As they are also not relevant in the computation of the masses, we use the shown assignments without loss of generality.

In the first case, Higgs fields transforming as any D_5 representation can couple directly to the fermions, whereas in the second one a Higgs field assigned to $\underline{\mathbf{1}}_{\mathbf{2}}$ under D_5 does not contribute to the fermion masses. Note that in each of the assignments we could replace the one-dimensional representation $\underline{\mathbf{1}}_{\mathbf{1}}$ by $\underline{\mathbf{1}}_{\mathbf{2}}$ and would get analogous results.
As all Higgs fields transform as $SU(2)_L$ doublets, they have to pass strong constraints from experiments -from direct searches at LEP as well as from FCNCs. Therefore, it is advantageous to keep the number of Higgs fields as small as possible. However, this has to be reconciled with the requirement to fit the fermion masses and mixings at tree-level and to keep the Higgs sector free from accidental symmetries. This is necessary in order to ensure that the VEVs of the Higgs fields can be chosen arbitrarily so that the fermion masses and mixings can be accommodated without breaking such a symmetry. One way to circumvent this is the introduction of soft breaking terms of mass dimension two into the Higgs potential which break the accidental, but also the flavor symmetry itself explicitly. We will disregard this possibility [10], since it only leads to further parameters in the model which are not constrained by symmetries apart from the SM gauge group. As will be shown below (see Section 3.3.4), the minimal potential includes four Higgs fields transforming as the two inequivalent doublets under D_5. A computation of the mass matrices for the two different assignments reveals that the couplings to these Higgs fields give either rise to a matrix structure with three zeros or one with one texture zero. The three texture zeros appear, if left-handed and left-handed conjugate fields are assigned to the same D_5 representations. According to the Clebsch Gordan coefficients presented in Appendix B.2 the (11), (23) and (32) element of each matrix vanish. A study [48] performed for hermitean quark mass matrices showed that this form cannot accommodate the quark masses and mixings. In general, our mass matrices do not need to be hermitean, however, we take this result as a strong indication that the mass matrix structure with three texture zeros is disfavored. For the second assignment we only encounter one texture zero in the (11) element of the mass matrix, while all other elements are non-vanishing, since the Kronecker product of $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ is decomposed into $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ (see Appendix B.2). Mass matrices with a texture zero in the (11) element have been studied frequently in the literature, see for instance [98], and have proven to be useful to fit the fermion masses and mixings.
As in the S_4 model the mass matrix of the left-handed conjugate neutrinos, if allowed, only stems from direct mass terms. Therefore it has a very simple form and two of the left-handed conjugate neutrinos are degenerate. Note that the form of the mass matrix is the same for both assignments discussed above. For the sake of minimality, we again dismiss the possible existence of gauge singlets as well as Higgs triplets. The assignment of the fields of the minimal model is summarized in Table 3.5.

Field	Q_1	$Q_{2,3}$	u^c	(c^c, t^c)	d^c	(s^c, b^c)	L_1	$L_{2,3}$	e^c	(μ^c, τ^c)	ν_e^c	(ν_μ^c, ν_τ^c)	$\chi_{1,2}$	$\psi_{1,2}$
D_5	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{1}}$	$\underline{\mathbf{2}}_{\mathbf{2}}$

Table 3.5: The particle content and its symmetry properties under D_5. The Higgs fields $\chi_{1,2}$ and $\psi_{1,2}$ are copies of the SM Higgs field, i.e. transform as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

3.3.3 Fermion Masses and Mixings

Choosing the D_5 transformation properties of the fermions and the four Higgs fields $\chi_{1,2}$ and $\psi_{1,2}$ according to Table 3.5, we arrive at Yukawa couplings of the form

[10] However, in many models soft breaking terms are introduced into the potential, see for example [94, 97].

$$
\begin{aligned}
\mathcal{L}_Y \;=\;& \alpha_0^u \, (Q_2 \, t^c \, \tilde{\chi}_1 + Q_3 \, c^c \, \tilde{\chi}_2) + \alpha_1^u \, (Q_3 \, t^c \, \tilde{\psi}_1 + Q_2 \, c^c \, \tilde{\psi}_2) + \alpha_2^u \, (Q_1 \, c^c \, \tilde{\chi}_1 + Q_1 \, t^c \, \tilde{\chi}_2) \\
+\;& \alpha_3^u \, (Q_2 \, u^c \, \tilde{\psi}_1 + Q_3 \, u^c \, \tilde{\psi}_2) \\
+\;& \alpha_0^d \, (Q_2 \, b^c \, \chi_2 + Q_3 \, s^c \, \chi_1) + \alpha_1^d \, (Q_3 \, b^c \, \psi_2 + Q_2 \, s^c \, \psi_1) + \alpha_2^d \, (Q_1 \, s^c \, \chi_2 + Q_1 \, b^c \, \chi_1) \\
+\;& \alpha_3^d \, (Q_2 \, d^c \, \psi_2 + Q_3 \, d^c \, \psi_1) \\
+\;& \alpha_0^l \, (L_2 \, \tau^c \, \chi_2 + L_3 \, \mu^c \, \chi_1) + \alpha_1^l \, (L_3 \, \tau^c \, \psi_2 + L_2 \, \mu^c \, \psi_1) + \alpha_2^l \, (L_1 \, \mu^c \, \chi_2 + L_1 \, \tau^c \, \chi_1) \\
+\;& \alpha_3^l \, (L_2 \, e^c \, \psi_2 + L_3 \, e^c \, \psi_1) \\
+\;& \alpha_0^\nu \, (L_2 \, \nu_\tau^c \, \tilde{\chi}_1 + L_3 \, \nu_\mu^c \, \tilde{\chi}_2) + \alpha_1^\nu \, (L_3 \, \nu_\tau^c \, \tilde{\psi}_1 + L_2 \, \nu_\mu^c \, \tilde{\psi}_2) + \alpha_2^\nu \, (L_1 \, \nu_\mu^c \, \tilde{\chi}_1 + L_1 \, \nu_\tau^c \, \tilde{\chi}_2) \\
+\;& \alpha_3^\nu \, (L_2 \, \nu_e^c \, \tilde{\psi}_1 + L_3 \, \nu_e^c \, \tilde{\psi}_2) + \text{h.c.}
\end{aligned} \tag{3.54}
$$

The resulting Dirac mass matrices are

$$
\mathcal{M}_{u,\nu} = \left(\begin{array}{ccc} 0 & \alpha_2^{u,\nu} \, \langle \chi_1 \rangle^\star & \alpha_2^{u,\nu} \, \langle \chi_2 \rangle^\star \\ \alpha_3^{u,\nu} \, \langle \psi_1 \rangle^\star & \alpha_1^{u,\nu} \, \langle \psi_2 \rangle^\star & \alpha_0^{u,\nu} \, \langle \chi_1 \rangle^\star \\ \alpha_3^{u,\nu} \, \langle \psi_2 \rangle^\star & \alpha_0^{u,\nu} \, \langle \chi_2 \rangle^\star & \alpha_1^{u,\nu} \, \langle \psi_1 \rangle^\star \end{array} \right) , \; \mathcal{M}_{d,l} = \left(\begin{array}{ccc} 0 & \alpha_2^{d,l} \, \langle \chi_2 \rangle & \alpha_2^{d,l} \, \langle \chi_1 \rangle \\ \alpha_3^{d,l} \, \langle \psi_2 \rangle & \alpha_1^{d,l} \, \langle \psi_1 \rangle & \alpha_0^{d,l} \, \langle \chi_2 \rangle \\ \alpha_3^{d,l} \, \langle \psi_1 \rangle & \alpha_0^{d,l} \, \langle \chi_1 \rangle & \alpha_1^{d,l} \, \langle \psi_2 \rangle \end{array} \right) \; , \tag{3.55}
$$

where $\langle \psi_i \rangle$ and $\langle \chi_i \rangle$ denote the VEVs of the fields ψ_i and χ_i. The VEVs and Yukawa couplings $\alpha_i^{u,d,l,\nu}$ are in general complex. The (11) element of the mass matrices is zero, since there is no Higgs field transforming trivially under D_5. Similar to the S_4 model, there are more parameters in our model than observables to fit. One can think of ways to reduce the number of parameters by either invoking correlations among the independent Yukawa couplings $\alpha_i^{u,d,l,\nu}$ or by assuming a certain VEV configuration with several VEVs being zero. However, the possibility to constrain the otherwise uncorrelated couplings will be discarded. If we have a closer look at the second possibility, we see that for two or more VEVs equal to zero, either some fermions remain massless or there is no CP violation. Both results would have to be corrected by higher-dimensional operators, radiative corrections, etc. and therefore will not be discussed in further detail. Furthermore, a study of the potential reveals that setting two VEVs to zero enforces in some cases constraints on the parameters in the potential which increase its symmetry. The weaker constraint that either only one VEV vanishes or two VEVs are equal may lead to viable fits, but will also not be used, since it actually does not reduce the number of parameters much, since the sixteen Yukawa couplings provide the majority of the free parameters. A third possibility to reduce the number of parameters in the fermion sector is the assumption of spontaneous CP violation. As the numerical study in [23] showed, successful fits can be found under this constraint. However, in the Higgs potential we need to assume that CP is explicitly broken.

If we consider the neutrinos to be Majorana particles, we can add a direct mass term for the left-handed conjugate neutrinos, as done in the S_4 model. It is of the form

$$
\mathcal{L}_{\nu^c} = M_1 \, \nu_e^c \, \nu_e^c + M_2 \, \left(\nu_\mu^c \, \nu_\tau^c + \nu_\tau^c \, \nu_\mu^c \right) + \text{h.c.} \tag{3.56}
$$

so that two of the three left-handed conjugate neutrinos are degenerate, if no flavored gauge singlets are present in the model. Again, this mass degeneracy could allow the resonant leptogenesis mechanism [87–89] to work. The light neutrino mass matrix arises, as usual, from the type-1 seesaw.

As starting point for our numerical study, however, we consider a matrix which is simpler than the one with only one texture zero. In the course of the fit this matrix is perturbed and we find a set of Yukawa couplings and VEVs which allows to fit all data to the experimental best fit values. This approach is therefore very similar to the one used in the numerical search within the S_4 model. We choose the matrix

$$\mathcal{M} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & a & b \\ 0 & b & a \end{pmatrix} \tag{3.57}$$

as starting point. It allows masses for the second and third generation. We identify $a - b$ with the mass of the third and $a + b$ with the one of the second generation. Therefore a and b have to have almost the same absolute value, but an opposite sign. The vanishing of the mass of the first generation is a valid approximation. Concerning the mixing angles, notice that the matrix $\mathcal{M}$ leads to large mixing. However, the small quark mixings can be reproduced, since the large mixings in the up and down quark sector cancel. The matrix $\mathcal{M}$ arises from the general matrices Eq.(3.55) in the limit of vanishing Yukawa couplings $\alpha^i_{2,3}$ and VEVs which are pairwise equal, i.e. $\langle\chi_1\rangle = \langle\chi_2\rangle$ and $\langle\psi_1\rangle = \langle\psi_2\rangle$. However, in order to maintain the smallness of the second generation we have to ensure that $|a| \approx |b|$ which implies that the two uncorrelated Yukawa couplings α^i_0 and α^i_1 have nearly the same absolute value and all VEVs have to be equal. In the framework of the D_5 model we cannot find a further explanation for the equality of the Yukawa couplings and have to accept this as a certain fine-tuning [11]. Concerning the equality of all VEVs, this is allowed by the minimization conditions of the potential. In order to explain the hierarchy between the Yukawa couplings $\alpha^i_{2,3}$ involving the first generation and $\alpha^i_{0,1}$ an additional $U(1)$ symmetry can be employed. The fermions of the second and third generation have a vanishing $U(1)$ charge, while it is non-vanishing for the first generation. Note that the second and third generation have to have the same charge under the additional $U(1)$, since otherwise this $U(1)$ would not commute with the flavor group D_5. The results of such fits are presented in [23]. The neutrinos are taken to be either Dirac particles or Majorana fermions. In both cases, they are normally ordered. Furthermore, one can infer from the given numbers of Yukawa couplings and VEVs that the resulting mass matrices for the charged fermions which fit all data are quite close to the initial matrix given in Eq.(3.57).

3.3.4 Treatment of the Higgs Potential

Potential of the Presented Model

We discuss the D_5-invariant potential of the four Higgs fields $\chi_{1,2}$ and $\psi_{1,2}$ which form the two two-dimensional representations of D_5, $\underline{\mathbf{2_1}}$ and $\underline{\mathbf{2_2}}$. As usual, the mass terms have a simple structure due to the flavor symmetry, and the number of quartic couplings is limited, but still several of them exist. The complete potential is of the form

[11] Nevertheless, one could also take up the position that this is not the only possible solution which fits all data and that one might find other configurations of parameters, Yukawa couplings and VEVs, which are more natural in that sense.

$$\begin{aligned}
V_4(\chi_i,\psi_i) &= -\mu_1^2\sum_{i=1}^{2}\chi_i^\dagger\,\chi_i - \mu_2^2\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i + \lambda_1\left(\sum_{i=1}^{2}\chi_i^\dagger\,\chi_i\right)^2 + \tilde{\lambda}_1\left(\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i\right)^2 \\
&+ \lambda_2\left(\chi_1^\dagger\,\chi_1 - \chi_2^\dagger\,\chi_2\right)^2 + \lambda_3\,|\chi_1^\dagger\,\chi_2|^2 + \tilde{\lambda}_2\left(\psi_1^\dagger\,\psi_1 - \psi_2^\dagger\,\psi_2\right)^2 + \tilde{\lambda}_3\,|\psi_1^\dagger\,\psi_2|^2 \\
&+ \sigma_1\left(\sum_{i=1}^{2}\chi_i^\dagger\,\chi_i\right)\left(\sum_{j=1}^{2}\psi_j^\dagger\,\psi_j\right) + \sigma_2\left(\chi_1^\dagger\,\chi_1 - \chi_2^\dagger\,\chi_2\right)\left(\psi_1^\dagger\,\psi_1 - \psi_2^\dagger\,\psi_2\right) \\
&+ \left\{\tau_1\left(\chi_1^\dagger\,\psi_1\right)\left(\chi_2^\dagger\,\psi_2\right) + \mathrm{h.c.}\right\} + \left\{\tau_2\left(\chi_1^\dagger\,\psi_2\right)\left(\chi_2^\dagger\,\psi_1\right) + \mathrm{h.c.}\right\} \\
&+ \left\{\kappa_1\left[\left(\chi_1^\dagger\,\chi_2\right)\left(\chi_1^\dagger\,\psi_2\right) + \left(\chi_2^\dagger\,\chi_1\right)\left(\chi_2^\dagger\,\psi_1\right)\right] + \mathrm{h.c.}\right\} \\
&+ \left\{\kappa_2\left[\left(\psi_1^\dagger\,\psi_2\right)\left(\chi_2^\dagger\,\psi_2\right) + \left(\psi_2^\dagger\,\psi_1\right)\left(\chi_1^\dagger\,\psi_1\right)\right] + \mathrm{h.c.}\right\} \\
&+ \kappa_3\left[|\chi_1^\dagger\,\psi_1|^2 + |\chi_2^\dagger\,\psi_2|^2\right] + \kappa_4\left[|\chi_1^\dagger\,\psi_2|^2 + |\chi_2^\dagger\,\psi_1|^2\right]
\end{aligned} \tag{3.58}$$

where the couplings $\tau_{1,2}$ and $\kappa_{1,2}$ are in general complex. We checked that this potential does not have any accidental (global) symmetries. Assuming that the fields $\chi_{1,2}$, $\psi_{1,2}$ transform in the following way

$$\chi_1 \;\rightarrow\; \chi_1\,\mathrm{e}^{i\,\alpha}\;,\;\; \chi_2 \;\rightarrow\; \chi_2\,\mathrm{e}^{i\,\beta}\;,\;\; \psi_1 \;\rightarrow\; \psi_1\,\mathrm{e}^{i\,\gamma}\;,\;\; \psi_2 \;\rightarrow\; \psi_2\,\mathrm{e}^{i\,\delta}\;, \tag{3.59}$$

we can derive conditions for the phases α, β, γ and δ

$$\alpha + \beta - \gamma - \delta = 0\;, \tag{3.60a}$$

$$2\,\alpha - \beta - \delta = 0 \quad \text{and} \quad 2\,\beta - \alpha - \gamma = 0\;, \tag{3.60b}$$

$$2\,\delta - \beta - \gamma = 0 \quad \text{and} \quad 2\,\gamma - \alpha - \delta = 0\;. \tag{3.60c}$$

The first condition is imposed by the couplings τ_1 and τ_2 so that both of them leave an $U(1)^3$ symmetry invariant. The first set of two conditions is derived from the κ_1 term and the second set from the κ_2 term. Therefore, each of them leaves one $U(1)^2$ unbroken. The rest of the couplings leaves the complete $U(1)^4$ symmetry invariant. This shows that at least two couplings are needed to break all accidental symmetries. Furthermore, one sees that each of the two sets derived from κ_1 and κ_2, respectively, implies the condition imposed by the couplings τ_1 and τ_2 so that, for example, the combination of the terms τ_1 and κ_1 does not break all accidental symmetries. Only the combination of the two terms κ_1 and κ_2 can break $U(1)^4$ down to $U(1)_Y$. This has to be emphasized, since it occurs quite frequently that the minimization conditions for certain VEV structures can only be fulfilled, if some of the parameters of the potential are set to zero. For instance, if one of the conditions read $\kappa_1 = 0$, we would have to dismiss this VEV configuration, since it enforces an accidental symmetry to appear in the potential which is likely to be broken by these VEVs. It is interesting to ask for the origin of the two couplings, κ_1 and κ_2, which are necessary to arrive at a potential only invariant under the SM gauge group. They have in common that they combine three fields belonging to one D_5 doublet with one field belonging to the other D_5 doublet, namely $(\chi_1^\dagger\,\chi_2)\,(\chi_1^\dagger\,\psi_2)$ in the κ_1 term and $(\psi_1^\dagger\,\psi_2)\,(\chi_2^\dagger\,\psi_2)$ in the κ_2 term. This is possible, since the Kronecker products of the two-dimensional representations read, $\underline{\mathbf{2_1}} \times \underline{\mathbf{2_1}} = \underline{\mathbf{1_1}} + \underline{\mathbf{1_2}} + \underline{\mathbf{2_2}}$, $\underline{\mathbf{2_2}} \times \underline{\mathbf{2_2}} = \underline{\mathbf{1_1}} + \underline{\mathbf{1_2}} + \underline{\mathbf{2_1}}$, and $\underline{\mathbf{2_1}} \times \underline{\mathbf{2_2}} = \underline{\mathbf{2_1}} + \underline{\mathbf{2_2}}$.

In the following, an additional symmetry is imposed on the Higgs potential in order to maintain the equality of all VEVs. As we took a matrix as given in Eq.(3.57) with $|a| \approx |b|$ as starting point for our fit procedure, we had to invoke that the VEVs of the four Higgs doublets are almost equal. Even without studying the minimization conditions deduced from V_4 in detail, we can conclude that this is not a natural result, since, for example, the mass parameters μ_1 and μ_2 as well as λ_i and $\tilde{\lambda}_i$ can have completely different values so that the fields $\chi_{1,2}$ and $\psi_{1,2}$ can have completely different couplings. Therefore, we search for an additional (simple) symmetry, denoted by T, which allows us to further correlate the parameters $\mu_{1,2}$ and also the quartic couplings. The simplest one, we found, is a combination of an exchange symmetry among the fields χ_i and ψ_i and the components of one of the D_5 doublets, for example,

$$\chi_i \quad \leftrightarrow \quad \psi_i \quad \text{and} \quad \chi_1 \quad \leftrightarrow \quad \chi_2 \; . \tag{3.61}$$

The first part of the symmetry is necessary to enforce the equality of the two mass parameters and the quartic couplings λ_i and $\tilde{\lambda}_i$, while the second one prevents the couplings κ_1 and κ_2 from being set to zero. Two aspects are noteworthy: *a.*) we can interchange the actions of Eq.(3.61) and *b.*) we can achieve the same result, if we replace the exchange of the components χ_1 and χ_2 by the exchange of ψ_1 and ψ_2. The constraints on all couplings are

$$\mu_1 = \mu_2 \; , \quad \lambda_i = \tilde{\lambda}_i \; , \quad \sigma_2 = 0 \; , \quad \tau_1 = \tau_2^\star \; , \quad \kappa_1 = \kappa_2^\star \; , \quad \kappa_3 = \kappa_4 \; . \tag{3.62}$$

The fact that σ_2 is required to vanish does not cause the appearance of accidental symmetries, as σ_2 leaves $U(1)^4$ invariant. Assuming real VEVs for the fields $\chi_{1,2}$ and $\psi_{1,2}$, these can be parameterized as

$$\langle\chi_1\rangle = \frac{v}{\sqrt{2}}\cos(\alpha) \; , \quad \langle\chi_2\rangle = \frac{v}{\sqrt{2}}\sin(\alpha) \; , \quad \langle\psi_1\rangle = \frac{u}{\sqrt{2}}\cos(\beta) \quad \text{and} \quad \langle\psi_2\rangle = \frac{u}{\sqrt{2}}\sin(\beta) \; . \tag{3.63}$$

The potential, invariant under the additional symmetry T, then reads at the minimum

$$\begin{aligned} V_{4\,T\,min} &= -\frac{1}{2}\mu_1^2\,(u^2+v^2) + \frac{1}{32}\,(u^4+v^4)\,(8\,\lambda_1 + 4\,\lambda_2 + \lambda_3) + \frac{1}{4}\,u^2\,v^2\,(\sigma_1+\kappa_3) \\ &+ \frac{1}{32}\,(v^4\,\cos(4\,\alpha) + u^4\,\cos(4\,\beta))\,(4\,\lambda_2 - \lambda_3) + \frac{1}{4}\,u\,v\,\big[u^2\,\cos(\alpha-\beta)\,\sin(2\,\beta) \\ &+ v^2\,\sin(2\,\alpha)\,\sin(\alpha+\beta)\big]\,\mathrm{Re}(\kappa_1) + \frac{1}{4}\,u^2\,v^2\,\sin(2\,\alpha)\,\sin(2\,\beta)\,\mathrm{Re}(\tau_1) \end{aligned} \tag{3.64}$$

We can derive the minimization conditions

$$\begin{aligned} \frac{\partial V_{4\,T\,min}}{\partial\alpha} &= -\frac{1}{8}\,v^4\,\sin(4\,\alpha)\,y + \frac{1}{2}\,u^2\,v^2\,\cos(2\,\alpha)\,\sin(2\,\beta)\,\mathrm{Re}(\tau_1) \\ &+ \frac{1}{4}\,u\,v\,\big[v^2\,(\cos(2\,\alpha)\,\sin(\alpha+\beta) + \sin(3\,\alpha+\beta)) - u^2\,\sin(\alpha-\beta)\,\sin(2\,\beta)\big]\,\mathrm{Re}(\kappa_1) \end{aligned} \tag{3.65a}$$

$$\begin{aligned} \frac{\partial V_{4\,T\,min}}{\partial\beta} &= -\frac{1}{8}\,u^4\,\sin(4\,\beta)\,y + \frac{1}{2}\,u^2\,v^2\,\sin(2\,\alpha)\,\cos(2\,\beta)\,\mathrm{Re}(\tau_1) \\ &+ \frac{1}{4}\,u\,v\,\big[u^2\,(\cos(\alpha-\beta)\,\cos(2\,\beta) + \cos(\alpha-3\,\beta)) + v^2\,\sin(2\,\alpha)\,\cos(\alpha+\beta)\big]\,\mathrm{Re}(\kappa_1) \end{aligned} \tag{3.65b}$$

where $y = 4\,\lambda_2 - \lambda_3$. The two minimization conditions for α and β are fulfilled for $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$. Each of the terms in Eq.(3.65a) and Eq.(3.65b) then vanishes separately, especially there is no constraint on $\mathrm{Re}(\tau_1)$, $\mathrm{Re}(\kappa_1)$ or the combination y. This is important, since demanding these parameters to vanish could lead to accidental symmetries. $V_{4\,T\,min}$ at $\alpha = \frac{\pi}{4}$ and $\beta = \frac{\pi}{4}$ is given by

$$\begin{aligned} V_{4\,T\,min}\left(\alpha = \frac{\pi}{4}, \beta = \frac{\pi}{4}\right) &= -\frac{1}{2}\mu_1^2\,(u^2+v^2) + \frac{1}{16}\,(u^4+v^4)\,(4\,\lambda_1 + \lambda_3) \\ &+ \frac{1}{4}\,u^2\,v^2\,(\sigma_1 + \kappa_3 + \mathrm{Re}(\tau_1)) + \frac{1}{4}\,u\,v\,(u^2+v^2)\,\mathrm{Re}(\kappa_1) \end{aligned} \tag{3.66}$$

Since this expression is symmetric in u and v, the solution $u = v$ is at least favored by this.

Unfortunately, the potential V_{4T} does not allow the VEV configurations used in the numerical examples. Abandoning the symmetry T solves this problem. However, the hope that one might be able to construct a model in which CP is broken only spontaneously is not fulfilled, since for the numerical values of the VEVs (which do not show any symmetry) the minimization conditions can be solved in this case, only if the parameters of V_4 are constrained in a way that accidental symmetries appear in the Higgs sector. Therefore, we have to study the general D_5-invariant potential with complex parameters which indeed allows to find a minimum with the appropriate VEV configuration. However, the phenomenological problem of too low Higgs masses exists, if all parameters of the potential are chosen to be of natural order. For further details we refer to [23]. Finally, in order to judge whether the minima, found in the numerical analysis, are global or only local, we calculated the potential along the directions of the eigenvectors of the Higgs mass matrices. It turned out that none of the other minima, found in this search, is deeper than the one in which the VEV configuration, invoked by the fermion fits, is realized. Additionally, we performed a scan of the potential in order to check whether it is stable for large values of the Higgs fields. We did not find any indication that the potential is unstable. Since all these results have been achieved by numerical studies rather than by analytical arguments, they do not have to be correct for the points in field space which have not been included in the numerics.

Three and Other Four Higgs Potentials

For comparison, we show the three and the other four Higgs potentials which are invariant under D_5. The Higgs fields are required to have the following transformation properties: *a.*) at least two fields should form a doublet under D_5 and *b.*) we exclude the case in which two Higgs fields have exactly the same transformation properties under the flavor symmetry. These two requirements are connected to the choice of the fermion transformation properties under D_5 which are assumed to be $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ for left-handed and left-handed conjugate fields. The inclusion of a doublet is then necessary in order to allow non-vanishing couplings between the generation which transforms as $\underline{\mathbf{1}}$ and the two other fermion generations forming a doublet. It is therefore responsible for the non-zero mixing between $\underline{\mathbf{1}}$ and $\underline{\mathbf{2}}$. It is sufficient to include (at maximum) one field with a certain transformation property, since the existence of more than one Higgs field transforming according to the representation μ under the flavor group does not lead to new mass matrix structures for the fermions compared to the case in which only one Higgs field $\sim \mu$ is present. Apart from the possibility studied before two further configurations fulfill these requirements: a potential with either three Higgs fields transforming as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under D_5 (where it is irrelevant as which one- and two-dimensional representation they exactly transform) or with four Higgs fields which transform as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{i}}$ (i $= 1, 2$ possible). As we will show in a moment both potentials contain an accidental $U(1)$ symmetry which is necessarily broken by VEVs which lead to realistic fermion masses.

The three Higgs potential of the fields ϕ, ψ_1 and ψ_2, with $\phi \sim \underline{\mathbf{1}}_{\mathbf{i}}$ and $\begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}}$, reads

$$
\begin{aligned}
V_3(\phi, \psi_i) &= -\mu_1^2 \, \phi^\dagger \phi - \mu_2^2 \sum_{i=1}^{2} \psi_i^\dagger \psi_i + \lambda_s \left(\phi^\dagger \phi\right)^2 + \lambda_1 \left(\sum_{i=1}^{2} \psi_i^\dagger \psi_i\right)^2 \\
&+ \lambda_2 \left(\psi_1^\dagger \psi_1 - \psi_2^\dagger \psi_2\right)^2 + \lambda_3 \, |\psi_1^\dagger \psi_2|^2 \\
&+ \sigma_1 \left(\phi^\dagger \phi\right) \left(\sum_{i=1}^{2} \psi_i^\dagger \psi_i\right) + \left\{\sigma_2 \left(\phi^\dagger \psi_1\right) \left(\phi^\dagger \psi_2\right) + \text{h.c.}\right\} + \sigma_3 \sum_{i=1}^{2} |\phi^\dagger \psi_i|^2
\end{aligned}
\tag{3.67}
$$

where only σ_2 is complex. It can be made real by appropriate redefinition of the field ϕ, for example. In order to reveal the existence of the accidental $U(1)$ symmetry, let the Higgs fields ϕ and ψ_i transform as

$$\phi \;\to\; \phi\,\mathrm{e}^{i\,\alpha}\,, \quad \psi_1 \;\to\; \psi_1\,\mathrm{e}^{i\,\beta}\,, \quad \psi_2 \;\to\; \psi_2\,\mathrm{e}^{i\,\gamma}\,. \tag{3.68}$$

The only non-trivial condition for the phases α, β and γ arises from σ_2

$$2\,\alpha - \beta - \gamma = 0\,, \tag{3.69}$$

i.e. $\alpha = \frac{1}{2}(\beta+\gamma)$. Hence, we find two independent $U(1)$ symmetries, $U(1)_\beta$ and $U(1)_\gamma$. The fields ϕ, ψ_1 and ψ_2 have the following charges: $Q(\phi;\beta) = \frac{1}{2}$, $Q(\psi_1,\beta) = +1$ and $Q(\psi_2;\beta) = 0$ under $U(1)_\beta$, while $Q(\phi;\gamma) = \frac{1}{2}$, $Q(\psi_2;\gamma) = +1$ and $Q(\psi_1;\gamma) = 0$ under $U(1)_\gamma$. In order to recover $U(1)_Y$, we recall that all Higgs fields have hypercharge $Y = -1$, i.e. $U(1)_Y$ is equivalent to the negative of the sum of $Q(\cdot;\beta)$ and $Q(\cdot;\gamma)$. We can choose the difference of the charges $Q(\cdot;\beta)$ and $Q(\cdot;\gamma)$ as orthogonal combination and call this symmetry $U(1)_X$. The Higgs fields have charges $Q(\phi;X) = 0$, $Q(\psi_1;X) = +1$ and $Q(\psi_2;X) = -1$ under $U(1)_X$. As one can see, a non-vanishing VEV for any of the two components of the D_5 doublet ψ_i breaks this symmetry spontaneously and therefore will lead to the appearance of a Goldstone boson which is not eaten by a gauge boson. Taking another definition of $U(1)_X$ does not change the situation. For example, $U(1)_X$ can also be defined so that ψ_1 is neutral, the charge of ϕ is -1 and ψ_2 carries the charge -2 [12]. Again, only one Higgs field is uncharged under the additional $U(1)$ symmetry. However, with only one non-vanishing VEV we cannot fit the fermion masses and mixings at tree level. As already mentioned, we do not discuss the case in which this accidental symmetry is broken explicitly by additional terms in the potential which break the flavor symmetry D_5 at the same time. In [23] we compared this potential to other three Higgs potentials with fields transforming as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under other small dihedral symmetries such as D_3 and D_4. We analyzed the differences among these symmetries and found the mathematical reason for the appearance of an accidental $U(1)$ symmetry. It is the fact that D_5 does not allow a coupling between three Higgs fields being part of the D_5 doublet and the Higgs field which transforms as $\underline{\mathbf{1}}$ under D_5, i.e. $(\phi^\dagger\,\psi_i)\,(\psi_j^\dagger\,\psi_k)$ is not part of a D_5-invariant. One can show this with the help of the Kronecker products displayed in Appendix B.2: It holds $\underline{\mathbf{1}} \times \underline{\mathbf{2}} = \underline{\mathbf{2}}$ and at the same time $\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}} + \underline{\mathbf{1}}' + \underline{\mathbf{2}}'$ with $\underline{\mathbf{2}}$ and $\underline{\mathbf{2}}'$ being inequivalent two-dimensional representations. This fact indicates (together with the observation that the four Higgs potential of the two doublets $\chi_{1,2}$ and $\psi_{1,2}$ is only free of accidental symmetries due to the two couplings κ_1 and κ_2, containing $(\chi_1^\dagger\,\chi_2)\,(\chi_1^\dagger\,\psi_2)$ and $(\psi_1^\dagger\,\psi_2)\,(\chi_2^\dagger\,\psi_2)$) that couplings involving three fields stemming from the same D_5-multiplet are necessary for breaking all accidental symmetries. This study shows that a thorough investigation of the Higgs potential is mandatory in all cases, since overlooking accidental symmetries causes severe phenomenological problems.

The alternative four Higgs potential is maintained by augmenting the discussed three Higgs potential by a Higgs field χ. This field is assigned to the one-dimensional representation, which is inequivalent to the one under which the field ϕ transforms. We arrive at

[12] Note that there is a misprint in the first version of the published paper [23].

$$
\begin{aligned}
V_{4,alt}(\phi,\chi,\psi_i) &= -\mu_1^2\,\phi^\dagger\,\phi-\mu_2^2\,\chi^\dagger\,\chi-\mu_3^2\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i+\lambda_s^1\,\left(\phi^\dagger\,\phi\right)^2+\lambda_s^2\,\left(\chi^\dagger\,\chi\right)^2 \\
&+ \lambda_1\,\left(\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i\right)^2+\lambda_2\,\left(\psi_1^\dagger\,\psi_1-\psi_2^\dagger\,\psi_2\right)^2+\lambda_3\,\left|\psi_1^\dagger\,\psi_2\right|^2 \\
&+ \sigma_{s\,1}\,\left(\phi^\dagger\,\phi\right)\,\left(\chi^\dagger\,\chi\right)+\left\{\sigma_{s\,2}\,\left(\phi^\dagger\,\chi\right)^2+\text{h.c.}\right\}+\sigma_{s\,3}\,|\phi^\dagger\,\chi|^2 \\
&+ \sigma_1\,\left(\phi^\dagger\,\phi\right)\,\left(\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i\right)+\sigma_2\,\left(\chi^\dagger\,\chi\right)\,\left(\sum_{i=1}^{2}\psi_i^\dagger\,\psi_i\right)+\left\{\sigma_3\,\left(\phi^\dagger\,\chi\right)\,\left(\psi_1^\dagger\,\psi_1-\psi_2^\dagger\,\psi_2\right)+\text{h.c.}\right\} \\
&+ \left\{\sigma_4\,\left(\phi^\dagger\,\psi_1\right)\,\left(\phi^\dagger\,\psi_2\right)+\text{h.c.}\right\}+\sigma_5\,\left[\left|\phi^\dagger\,\psi_1\right|^2+\left|\phi^\dagger\,\psi_2\right|^2\right] \\
&+ \left\{\sigma_6\,\left(\chi^\dagger\,\psi_1\right)\,\left(\chi^\dagger\,\psi_2\right)+\text{h.c.}\right\}+\sigma_7\,\left[\left|\chi^\dagger\,\psi_1\right|^2+\left|\chi^\dagger\,\psi_2\right|^2\right] \\
&+ \left\{\sigma_8\,\left[\left(\phi^\dagger\,\psi_1\right)\,\left(\chi^\dagger\,\psi_2\right)-\left(\phi^\dagger\,\psi_2\right)\,\left(\chi^\dagger\,\psi_1\right)\right]+\text{h.c.}\right\} \\
&+ \left\{\sigma_9\,\left[\left(\phi^\dagger\,\psi_1\right)\,\left(\psi_1^\dagger\,\chi\right)-\left(\phi^\dagger\,\psi_2\right)\,\left(\psi_2^\dagger\,\chi\right)\right]+\text{h.c.}\right\}
\end{aligned} \tag{3.70}
$$

Without loss of generality one can assume that $\phi \sim \underline{\mathbf{1}}_{\mathbf{1}}$ and $\chi \sim \underline{\mathbf{1}}_{\mathbf{2}}$. For definiteness, we assume that $\psi_i \sim \underline{\mathbf{2}}_{\mathbf{1}}$. The complex parameters are $\sigma_{s\,2}$, σ_3, σ_4, σ_6, σ_8 and σ_9. According to the method above we find the following constraints on the phases of the Higgs fields ϕ, χ and $\psi_{1,2}$

$$\alpha-\beta=0\;, \tag{3.71a}$$
$$2\,\alpha-\gamma-\delta=0\;, \tag{3.71b}$$
$$2\,\beta-\gamma-\delta=0\;, \tag{3.71c}$$
$$\alpha+\beta-\gamma-\delta=0\;, \tag{3.71d}$$

with

$$\phi\;\;\rightarrow\;\;\phi\,\mathrm{e}^{i\,\alpha}\;,\;\;\chi\;\;\rightarrow\;\;\chi\,\mathrm{e}^{i\,\beta}\;,\;\;\psi_1\;\;\rightarrow\;\;\psi_1\,\mathrm{e}^{i\,\gamma}\;,\;\;\psi_2\;\;\rightarrow\;\;\psi_2\,\mathrm{e}^{i\,\delta}\;. \tag{3.72}$$

The first condition stems from the couplings $\sigma_{s\,2}$, σ_3 and σ_9, whereas the second and third condition originate from σ_4 and σ_6, respectively. Finally, the σ_8 term gives rise to the last condition. All other coupling terms do not lead to constraints on the phases α, β, γ and δ and therefore leave a $U(1)^4$ symmetry invariant. Out of the conditions Eq.(3.71) only two are independent. They are solved, for example, for $\beta=\alpha$ and $\delta=2\,\alpha-\gamma$ which shows that there are two independent $U(1)$ symmetries, $U(1)_\alpha$ and $U(1)_\gamma$, present in the potential. The fields ϕ, χ and $\psi_{1,2}$ carry the following charges: $Q(\phi;\alpha)=+1$, $Q(\chi;\alpha)=+1$, $Q(\psi_1;\alpha)=0$ and $Q(\psi_2;\alpha)=+2$ under $U(1)_\alpha$ and $Q(\phi;\gamma)=0$, $Q(\chi;\gamma)=0$, $Q(\psi_1;\gamma)=+1$ and $Q(\psi_2;\gamma)=-1$ under $U(1)_\gamma$. In order to recover the usual $U(1)_Y$ and an additional $U(1)_X$ symmetry one can redefine the charges by taking $-\left[Q(\cdot;\alpha)+Q(\cdot;\gamma)\right]$ and $\left(Q(\cdot;\alpha)-Q(\cdot;\gamma)\right)$. Then all fields have charge -1 under $U(1)_{-[\alpha+\gamma]}$ and under $U(1)_X\equiv U(1)_{[\alpha-\gamma]}$: $Q(\phi;X)=+1$, $Q(\chi;X)=+1$, $Q(\psi_1;X)=-1$ and $Q(\psi_2;X)=+3$. As one can see, all Higgs fields are charged under the additional $U(1)_X$ and therefore any non-trivial VEV configuration breaks this additional symmetry leading to a massless Goldstone boson in the Higgs spectrum. Also in this case, the definition of the additional symmetry $U(1)_X$ is not unique, since it can, for example, be defined as $U(1)_\gamma$. Then only two of the four Higgs fields, namely ψ_1 and ψ_2, are charged under the additional symmetry. However, this also does not allow the Higgs fields to have an arbitrary VEV configuration as needed for viable fermion masses and mixings.

3.3.5 Summary and Conclusions

In this section, we presented a second low energy model in which the flavor symmetry is broken at the electroweak scale. Thereby, the dihedral group D_5 plays the role of the flavor symmetry. As D_5 only contains one- and two-dimensional representations, the fermions are assigned to $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ in a way that the model can be embedded into the Pati-Salam group. The viability of the model has been shown with a numerical fit, whose results can be found in [23]. Thereby, neutrinos can be either Dirac or Majorana particles. In both cases the (light) neutrinos are normally ordered. For Majorana neutrinos, two of the left-handed conjugate neutrinos are degenerate. We concentrated on building the most economical model which fits all data. Especially, we tried to construct the model with the least number of new (Higgs) fields. As discussed in detail, the crucial issue is the Higgs potential, since the simplest D_5-invariant potentials contain accidental symmetries. These are necessarily broken by VEV structures advocated by successful fits of the fermion masses and mixings. Investigating three potentials invariant under D_5 we were able to figure out the origin of these accidental symmetries [13]. Similar to the S_4 model, it turned out that the VEV configuration advocated by the fermion fits can be adjusted, but not predicted in this model. Moreover, a numerical analysis of the potential showed that the typical Higgs masses are rather low, i.e. some of them are even below the LEP bound. As this also arose as a problem in the S_4 model, this seems to be a generic feature of these low energy models. As mentioned in Section 3.2.6, FCNC processes mediated by these additional Higgs fields pose even stronger limits on the Higgs masses than the direct searches. In the specific example given here, FCNCs are expected to be suppressed at least for the first two generations due to the small values of the Yukawa couplings [14].

In the S_4 model one important selection criterion for the fermion assignment was the possibility to embed the discrete group into a continuous one, like $SO(3)$ and $SU(3)$. One can ask whether this could be applied also here in the D_5 model. Similar to S_4, D_5 is a subgroup of $SO(3)$ and $SU(3)$. As one can read off from Table 3.6, our chosen assignment does not allow such an embedding, since we cannot identify the three generations transforming as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{i}}$ with the fundamental representation of $SO(3)$ or $SU(3)$. Such an identification would require that $\underline{\mathbf{1}}_{\mathbf{1}}$ has to be replaced by the non-trivial one-dimensional representation $\underline{\mathbf{1}}_{\mathbf{2}}$. However, even then we could either identify the three left-handed or the three left-handed conjugate fermion generations with $\underline{\mathbf{3}}$ of $SO(3)$ ($SU(3)$), since they transform under inequivalent two-dimensional representations of D_5. The other one then has to be placed into a larger representation of the continuous group. This, however, demands the existence of additional fields to complete this representation. An embedding therefore prefers the assignment in which left-handed and left-handed conjugate fields transform in the same way, i.e. as $\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{i}}$ ($\mathrm{i} = 1, 2$). As argued above, such a choice would enforce a larger number of Higgs fields in order to arrive at viable results for the fermion and the Higgs sector and therefore has not been studied here.

In summary, also this model might not serve as a fully realistic theory, but it is worth to be investigated, since it shows apart from some new assignment structures and interesting mass matrices for fermions that the Higgs sector which is the most complicated part of the model needs a careful study. Especially, this has been overlooked in several other low energy models with discrete flavor symmetries, see for example [99].

[13] We also studied potentials invariant under D_3, D_4 and D_6 in the paper [23].

[14] For the actual numerical values we refer to [23]. But one can already infer from the fact that the mass matrices of the charged fermions are close to the matrix given in Eq.(3.57) that the Yukawa couplings involving the first generation are small.

$\mathrm{D}^{(l)}$ of the rotation group	Resolution of $\mathrm{D}^{(l)}$ into representations of D_5
$l=0$	$\underline{\mathbf{1}}_{\mathbf{1}}$
$l=1$	$\underline{\mathbf{1}}_{\mathbf{2}}+\underline{\mathbf{2}}_{\mathbf{1}}$
$l=2$	$\underline{\mathbf{1}}_{\mathbf{1}}+\underline{\mathbf{2}}_{\mathbf{1}}+\underline{\mathbf{2}}_{\mathbf{2}}$
$l=3$	$\underline{\mathbf{1}}_{\mathbf{2}}+\underline{\mathbf{2}}_{\mathbf{1}}+2\,\underline{\mathbf{2}}_{\mathbf{2}}$
$l=4$	$\underline{\mathbf{1}}_{\mathbf{1}}+2\,\underline{\mathbf{2}}_{\mathbf{1}}+2\,\underline{\mathbf{2}}_{\mathbf{2}}$

$SU(3)$ representations		Resolution into D_5 representations
Dynkin Label	Dimension	
(00)	1	$\underline{\mathbf{1}}_{\mathbf{1}}$
(10)	3	$\underline{\mathbf{1}}_{\mathbf{2}}+\underline{\mathbf{2}}_{\mathbf{1}}$
(20)	6	$2\,\underline{\mathbf{1}}_{\mathbf{1}}+\underline{\mathbf{2}}_{\mathbf{1}}+\underline{\mathbf{2}}_{\mathbf{2}}$
(11)	8	$\underline{\mathbf{1}}_{\mathbf{1}}+\underline{\mathbf{1}}_{\mathbf{2}}+2\,\underline{\mathbf{2}}_{\mathbf{1}}+\underline{\mathbf{2}}_{\mathbf{2}}$
(30)	10	$2\,\underline{\mathbf{1}}_{\mathbf{2}}+2\,\underline{\mathbf{2}}_{\mathbf{1}}+2\,\underline{\mathbf{2}}_{\mathbf{2}}$

Table 3.6: Breaking sequences $SU(3) \rightarrow D_5$ and $SO(3) \rightarrow D_5$ for the smallest representations. $\mathrm{D}^{(l)}$ is the $(2\,l+1)$-dimensional representation of the rotation group. Note that there exists another possibility to embed D_5 which arises from this one, if the two-dimensional representations $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ are interchanged throughout in the tables.

3.4 Comments on the Two Models

Summarizing the lessons that can be learnt from these two low energy models one can say the following: Both of them are successful in describing the observed mass and mixing patterns of the fermions. However, the main problem is caused by the fact that, although in both models the fermion assignment is determined (almost) uniquely by using additional guidelines such as the embedding into $SO(10) \times SO(3)_f$ ($SO(10) \times SU(3)_f$) or the requirement to include the least number of new fields, the number of parameters, Yukawa couplings and VEVs, is very large, such that none of the two models can make testable predictions [15] [16]. There are two ways to reduce the number of parameters: either one constrains the Yukawa couplings by additional symmetries or one expects a certain vacuum alignment to be realized which, for example, equates the VEVs. The first possibility is not applicable in all models, especially, if the couplings shall fulfill certain non-trivial relations. Furthermore, it generally increases the complexity to the model. Invoking a certain VEV structure on the other hand is very appealing, if it is a direct consequence of the minimization of the potential. However, due to the complicated structure of the multi-Higgs doublet potentials this is in general not the case. A fairly better chance to successfully maintain a vacuum alignment arises, if the Higgs doublets are replaced by gauge singlets. Additionally, supersymmeterizing the theory allows to construct potentials which lead to the correct alignment via the vanishing of F-terms. A prominent example for this mechanism will be discussed in detail in Chapter 4. In this model the alignment is crucial, in particular for the prediction of TBM in the lepton sector. Furthermore, the replacement of Higgs doublets by gauge singlets elegantly solves the problems of too light scalars which would mediate FCNC processes. In some cases these might be suppressed by very small Yukawa couplings. However, not only the severe bounds on FCNCs are hardly fulfilled in models

[15] We do not expect that the prediction of degenerate left-handed conjugate neutrinos will be testable.

[16] Of course, one can find models in the literature, for example [93], which can make testable predictions in the framework of a low energy theory with the flavor symmetry being (spontaneously) broken at the electroweak scale by flavored Higgs doublets.

with several Higgs doublets, but also the limits coming from the non-observation of a Higgs particle challenge them, since for Higgs mass parameters around the electroweak scale and quartic couplings in the perturbative range the lowest Higgs mass is around 50 GeV. Apart from this, multi-Higgs doublet potentials which are invariant under discrete symmetries tend to suffer from accidental symmetries. These lead to additional Goldstone bosons in the Higgs spectrum, if the VEVs are chosen freely in order to fit the fermion masses and mixings. One example has been discussed in detail in the D_5 model. The frequently used argument that according to the number of Higgs fields one can expect the couplings in the potential to be non-invariant under additional symmetries is proven to be wrong in this case. Also the general assumption that in a complicated potential arbitrary VEV structures can be adjusted by an appropriate set of parameters does not hold.

In addition, models with gauge singlets offer better possibilities to combine a conventional GUT, like $SO(10)$ with a flavor symmetry. As shown explicitly in the S_4 model, if the $SO(10)$ Higgs fields are required to transform non-trivially under the flavor group, this causes the existence of several large representations, $\underline{\mathbf{10}}$ and $\underline{\overline{\mathbf{126}}}$, in the model. These, however, lead to severe problems in general and therefore such models can hardly be realistic. With gauge singlets on the other hand the number of Higgs fields which transform under $SO(10)$ will be the same as in the conventional GUT scenarios, for example one $\underline{\mathbf{10}}$ and one $\underline{\overline{\mathbf{126}}}$ [17] and therefore problems, enumerated in Section 3.2.5, do not arise.

All this seems to favor the replacement of the Higgs doublets by flavored gauge singlets. Thereby, the flavor group is broken independently from the electroweak symmetry, presumably at a much higher energy scale. The particle spectrum at low energies only contains the SM particles. The Yukawa structure of such a model differs from the ones shown above, since the terms consist of two fermions, the usual Higgs doublet and a suitable number of gauge singlets needed to form invariants under the flavor symmetry. In general, these terms are therefore non-renormalizable and are appropriately suppressed by the cutoff scale of the theory. In a high energy completion they arise from new, heavy degrees of freedom (vector-like fermions as well as scalar fields). These are integrated out in order to arrive at the effective low energy theory. This is exactly how the model presented in the next chapter is implemented.

[17] Recent studies [100, 101] which tried to fit all fermion masses and mixings in models with only one $\underline{\mathbf{10}}$ and one $\underline{\overline{\mathbf{126}}}$ indicate that this scenario might be disfavored and has to be extended by, for example, additional Higgs fields transforming as $\underline{\mathbf{120}}$ under $SO(10)$ [102]. Other models [103–105] with Higgs fields in smaller $SO(10)$ representations might solve this problem in a better way. However, they are in general non-renormalizable.

Chapter 4

Flavored MSSM

In this chapter we discuss a flavored extension of the MSSM. The flavor symmetry is the double-valued tetrahedral group T' [1]. In contrast to the two low energy models presented in the preceding chapter, the flavor symmetry breaking scale is not the electroweak scale, but rather a high energy scale close to the GUT scale. For this reason, additional gauge singlets, called flavons, transforming only under T', are introduced, while the MSSM Higgs doublets, h_u and h_d, are singlets under T'. Due to this the model can, for example, still predict gauge coupling unification. The solution of the vacuum alignment problem (up to a small number of degeneracies) is attributed to the facts, that the model is supersymmetric and that the flavor symmetry is broken at high energies by gauge singlets. We can obtain several predictions in this model

- TBM in the lepton sector, i.e.

$$\sin^2(\theta_{12}^{TBM}) = \frac{1}{3} \; , \quad \sin^2(\theta_{23}^{TBM}) = \frac{1}{2} \; , \quad \sin^2(\theta_{13}^{TBM}) = 0 \; , \tag{4.1}$$

  which matches the experimental best fit values within $\sim 1\,\sigma$.

- two non-trivial relations among $|V_{us}|$, $|V_{td}/V_{ts}|$ and m_d/m_s

$$\sqrt{\frac{m_d}{m_s}} = |V_{us}| + \mathcal{O}(\lambda^2) \;\text{ and }\; \sqrt{\frac{m_d}{m_s}} = \left|\frac{V_{td}}{V_{ts}}\right| + \mathcal{O}(\lambda^2) \;\text{ (due to }\; |V_{ub}| \sim \mathcal{O}(\lambda^4)\;) \; . \tag{4.2}$$

  These relations are well-known and usually result from the assumption of certain texture zeros in the quark mass matrices [49–54].

This model serves as an extension of the successful A_4 models [14,15] [16–21], since A_4 is isomorphic to the tetrahedral group T whose double group T' is. Hence the T' model can produce TBM in the same manner as the A_4 models and, furthermore, is able to describe the quark sector correctly due to the existence of additional representations, not present in A_4. In order to show the viability of the model, not only the leading order is considered, but also the next-to-leading order terms are calculated in each sector. They arise through two flavon insertions and correct the masses matrices and the vacuum alignment. These effects have an important impact in this model, since they are necessary to give masses to the first generation of quarks and produce non-vanishing mixing angles $\theta_C \equiv \theta_{12}^q$ and θ_{13}^q. However, at the same time they could spoil the successful prediction of the TBM in the lepton sector. In order to prevent this, the corrections should be kept small, at a level of

[1] Several synonyms for T' can be found in the literature, e.g. the binary tetrahedral group [106], ${}^{(d)}T$ [107], $SL_2(F_3)$ [108], Type 24/13 [109].

$\lambda^2 \approx 0.04$. In the subsequent sections we will show that this requirement can indeed be fulfilled. Apart from its successful predictions, this model allows for a deeper understanding of the diverse mixing pattern of quarks and leptons. The charged fermion and neutrino sector couple to distinct flavons at the leading order level. The set of fields coupling to the charged fermions breaks T' down to a Z_3 subgroup, while the set coupling to neutrinos breaks it to a Z_4 group [2]. Hence, the small quark mixings arise through the breaking of the flavor symmetry down to the same subgroup, whereas large lepton mixing angles result from the mismatch of the different subgroups, Z_3 and Z_4, in the charged lepton and neutrino sector. To be precise, the definite prediction of the very specific pattern of TBM and not only of large mixings, is intimately connected to the preservation of (different) T' subgroups in the charged lepton and neutrino sector. The fact that the vacuum alignment problem can be solved in this model, is also correlated to the aspect that the flavor symmetry is not broken in an arbitrary way, since VEV configurations corresponding to preserved subgroups seem to be favored by the minimization conditions in a certain class of potentials.
The actual implementation of the model enforces the existence of additional symmetries: *a.*) a Z_3 symmetry is necessary in order to separate the charged fermion and neutrino sector -at least- at leading order, *b.*) an additional $U(1)$ symmetry is used to explain the fermion mass hierarchy, and *c.*) the construction of the flavon potential is realized with the help of another $U(1)$ symmetry, denoted as $U(1)_R$, which extends the well-known R parity.
This chapter is structured as follows: In Section 4.1 the new features which arise from supersymmeterizing the SM are reviewed; Section 4.2 contains the group theory of T' and Section 4.3 is dedicated to the outline of the model. Thereby, the structure of the preserved subgroups is explained. The fermion mass matrices at leading and next-to-leading order are given in Section 4.4. The flavon potential and the question of the vacuum alignment are discussed in detail in Section 4.5. Finally, some comments on the model including a short outlook can be found in Section 4.6.

4.1 Basic Ingredients for Model Building in the MSSM

In this section we collect the facts about the MSSM and SUSY model building which are necessary to understand the construction of the T' model. Therefore, this section does not provide a review of SUSY nor of the MSSM. For this purpose, we refer to, for example, [110].
In SUSY model building the notion of superfields and the superpotential w are of particular importance. In the MSSM all SM fields are promoted to superfields which contain the SM particles, i.e. fermions, gauge bosons and the Higgs field, and their superpartners, i.e. sfermions, gauginos and Higgsinos. In the literature, they are conveniently denoted with the same symbol as the SM fields which then carries a hat, e.g. $\hat{Q}$ stands for a superfield which consists of an $SU(2)_L$ quark doublet and its superpartners. The superpotential w is renormalizable, gauge invariant and has mass dimension three, if one replaces the superfields by their scalar components, e.g. a term in the superpotential can be quadratic in the fields with a coupling of mass dimension one or it can be trilinear in the fields and the corresponding coupling is dimensionless. A special feature of w is the fact that it is holomorphic, i.e. only the fields themselves, but not their complex conjugates are allowed to appear in w. In the following, we only display the superpotentials w. Thereby, we omit the hats over the superfields in general.
The Yukawa couplings are contained in w and are of the same form as in the SM with the slight difference that due to holomorphy of w two Higgs fields h_u and h_d have to exist in the MSSM. Thereby, h_u gives masses to the up quarks, if it acquires a VEV, while h_d is responsible for down quark and charged lepton masses. If left-handed conjugate neutrinos exist, a VEV of h_u also gives

[2]In the A_4 model, the flavons coupling to neutrinos at leading order break A_4 down to a Z_2 group, see below.

rise to Dirac masses for neutrinos. The VEVs of h_u and h_d are denoted by v_u and v_d. Their sum equals the electroweak scale v, i.e. $v_u^2 + v_d^2 = v^2$. Usually, v_u and v_d are parameterized in terms of v and $\tan(\beta) = \frac{v_u}{v_d}$, i.e. $v_u = v\,\sin(\beta)$ and $v_d = v\,\cos(\beta)$. Small values of $\tan(\beta)$ then correspond to $\mathcal{O}(v_u) \approx \mathcal{O}(v_d)$, while large values of $\tan(\beta)$ indicate that $v_d \ll v_u$. Generally, $\tan(\beta)$ has to be larger than one and smaller than 50 or 60. Compared to the SM, h_d plays the role of the SM Higgs doublet, i.e. transforms as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$, whereas h_u, which is identified with the conjugate of the usual SM Higgs doublet, transforms as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, +1)$. A second reason for the existence of these two fields, h_u and h_d, is the fact that the theory would be anomalous, if only one of the fields was present. This is different from the SM, since in a supersymmetric theory the Higgs scalars are accompanied by Higgsinos, i.e. spin-$\frac{1}{2}$ particles, which induce anomalies similar to the SM fermions. The appearance of the mass matrices deduced from the superpotential w is exactly the same as in the SM, i.e. all formulae shown in Section 3.1 can also be applied here.
According to gauge invariance under $SU(3)_C \times SU(2)_L \times U(1)_Y$ additional terms are allowed in the MSSM superpotential w, which are not present in the SM Lagrangian. These terms, for example, can mediate proton decay and are therefore severely constrained. They can be eliminated by a simple additional symmetry, called R parity, under which all SM particles have charge $+1$, while all superparticles have charge -1. Equivalently, one can define matter parity under which all quark and lepton supermultiplets have charge -1 and Higgs supermultiplets as well as gauge bosons and gauginos have charge $+1$. In the model below this symmetry is extended to a continuous one, called $U(1)_R$ symmetry.
Potentials of scalar fields originate from two sources in a supersymmetric theory, F-terms and D-terms. The F-terms are derivatives of the superpotential w with respect to a (super-)field, while D-terms are only present, if the fields transform non-trivially under a gauge group (factor), like the Higgs fields h_u and h_d. In the T' model we only have to consider gauge singlets, i.e. fields which transform trivially under the SM gauge group, and hence all D-terms vanish. Note that SUSY is unbroken, if all F- and D-terms vanish independently. However, in nature SUSY cannot be unbroken, since otherwise fermions and sfermions would have the same mass. Without constructing a complete theory which explains the spontaneous breaking of SUSY, we can parameterize this by the soft SUSY breaking terms. These include soft masses as well as trilinear couplings, so-called A-terms. Note that these soft terms do not need to be holomorphic in the fields. In the MSSM more than a hundred independent parameters are present in these terms. An important feature of these terms is the fact that they do not lead to quadratic divergences and therefore still allow the hierarchy problem to be solved.
As a last feature, we would like to mention that the invariance of the superpotential w under a $U(1)$ symmetry does not necessarily mean that the total $U(1)$ charge of each term has to vanish, but rather that all terms in w have to acquire the same $U(1)$ charge, if the symmetry is applied to w, i.e. $w \xrightarrow{U(1)} \mathrm{e}^{i\,\alpha}\, w$ with arbitrary α. This holds, since the superpotential itself actually does not appear in the Lagrangian, but only, for example, in the form of an F-term $|\frac{\partial w}{\partial \phi}|^2$.

4.2 Group Theory of T'

T' is the double-valued group of the tetrahedral symmetry T which is isomorphic to A_4, the group of the even permutations of four objects. The group order of T' is therefore twice the one of A_4, i.e. it is 24. T' contains apart from the representations present in A_4, i.e. the three inequivalent one-dimensional representations, called 1, $1'$ and $1''$, and the three-dimensional one, 3, further three two-dimensional representations, 2, $2'$ and $2''$. Out of these seven representations two are real (1 and 3), one is pseudo-real (2) and the other four ones are complex (conjugated to each other), $1'$ to

	classes								
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$	$\mathcal{C}_6$	$\mathcal{C}_7$		
G	$\mathbb{1}$	$\mathbb{R}$	S	$ST\mathbb{R}$	T^2	T	$(ST)^2\mathbb{R}$		
$^\circ\mathcal{C}_i$	1	1	6	4	4	4	4		
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	2	4	6	3	3	6	$\mathbb{c}^{(\mu)}$	faithful
1	1	1	1	1	1	1	1	1	
$1'$	1	1	1	ω	ω^2	ω	ω^2	0	
$1''$	1	1	1	ω^2	ω	ω^2	ω	0	
2	2	-2	0	1	-1	-1	1	-1	√
$2'$	2	-2	0	ω	$-\omega^2$	$-\omega$	ω^2	0	√
$2''$	2	-2	0	ω^2	$-\omega$	$-\omega^2$	ω	0	√
3	3	3	-1	0	0	0	0	1	

Table 4.1: Character table of the group T' taken from [111, 112]. ω is the third root of unity, i.e. $\omega = \mathrm{e}^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ and $1 + \omega + \omega^2 = 0$ holds. For further explanations consult Appendix A.

$1''$ and $2'$ to $2''$. Clearly, only the two-dimensional representations are faithful. As usual, 1 is the trivial representation of the group. The pseudo-reality of 2 mirrors the fact that T' is a subgroup of the (continuous) group $SU(2)$ whose fundamental representation $\underline{\mathbf{2}}$ is also pseudo-real. Since 1, $1'$, $1''$ and 3 also exist in A_4 they are called single-valued representations, while the additional two-dimensional ones are called double-valued. It is important to notice that A_4 (T) is not a subgroup of T', although they are closely related. The character table is shown in Table 4.1. One can choose the following generators S and T for the irreducible representations [111, 112]

$$\text{... for } 2 \;:\quad S = -\frac{1}{\sqrt{3}}\begin{pmatrix} i & \sqrt{2}\,\mathrm{e}^{\frac{\pi i}{12}} \\ -\sqrt{2}\,\mathrm{e}^{-\frac{\pi i}{12}} & -i \end{pmatrix} \;,\quad T = \omega \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{4.3}$$

$$\text{... for } 2' \;:\quad S = -\frac{1}{\sqrt{3}}\begin{pmatrix} i & \sqrt{2}\,\mathrm{e}^{\frac{\pi i}{12}} \\ -\sqrt{2}\,\mathrm{e}^{-\frac{\pi i}{12}} & -i \end{pmatrix} \;,\quad T = \omega^2 \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{4.4}$$

$$\text{... for } 2'' \;:\quad S = -\frac{1}{\sqrt{3}}\begin{pmatrix} i & \sqrt{2}\,\mathrm{e}^{\frac{\pi i}{12}} \\ -\sqrt{2}\,\mathrm{e}^{-\frac{\pi i}{12}} & -i \end{pmatrix} \;,\quad T = \begin{pmatrix} \omega & 0 \\ 0 & 1 \end{pmatrix} \,, \tag{4.5}$$

$$\text{... for } 3 \;:\quad S = \frac{1}{3}\begin{pmatrix} -1 & 2\,\omega & 2\,\omega^2 \\ 2\,\omega^2 & -1 & 2\,\omega \\ 2\,\omega & 2\,\omega^2 & -1 \end{pmatrix} \;,\quad T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \,. \tag{4.6}$$

As usual the generators S and T of the one-dimensional representations can be found in the character table. S and T fulfill the relations

$$S^2 = \mathbb{R},\;\; T^3 = \mathbb{1},\;\; (S\,T)^3 = \mathbb{1},\;\; \mathbb{R}^2 = \mathbb{1}, \tag{4.7}$$

with $\mathbb{R} = \mathbb{1}$ for the one-dimensional and the three-dimensional representations (i.e. the representations which T' shares with A_4) and $\mathbb{R} = -\mathbb{1}$ for the two-dimensional ones 2, $2'$ and $2''$. As $\mathbb{R} = \pm\mathbb{1}$, it commutes with all elements of the group.
Since the generators of all representations are chosen to be complex, there exist similarity transformations U which connect them with their complex conjugates, i.e. $U^T\,S\,U = S^\star$ and $U^T\,T\,U = T^\star$.

For the two-dimensional representation 2 U reads

$$U = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad U^T\, U = U\, U^T = \mathbb{1}\ , \tag{4.8}$$

and for the three-dimensional representation we find

$$U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad \text{with} \quad U^T\, U = U\, U^T = \mathbb{1}\ . \tag{4.9}$$

The representations $2'$ and $2''$ are complex conjugated to each other. However, this is not true for their representation matrices, i.e. also here we have to apply a similarity transformation V which transforms $S^\star$ and $T^\star$ of $2'$ into S and T of $2''$. It is given by

$$V = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad \text{with} \quad V^T\, V = V\, V^T = \mathbb{1}\ , \tag{4.10}$$

then $V^T\, S^\star_{2'}\, V = S_{2''}$ and $V^T\, T^\star_{2'}\, V = T_{2''}$.
Since the model which will be discussed in the following is an extension of the A_4 model presented in [15], we elucidate the links between the chosen generators here and in [15]. Thereby, one has to notice that the generator T in [15] actually equals T^2 here, $T^2 = \omega^2$ for $1'$ equals $T = \omega^2$ for $1'$ in [15], similarly for $1''$ and also for the diagonal generator T of 3. Apart from this we need to employ the similarity transformation W

$$W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix} \tag{4.11}$$

so that

$$W^\dagger\, S\, W = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} \quad \text{and} \quad W^\dagger\, T^2\, W = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega^2 & 0 \\ 0 & 0 & \omega \end{pmatrix}\ , \tag{4.12}$$

which are the generators S and T presented in [15]. However, all this does not affect the Clebsch Gordan coefficients. They are shown in Appendix B.3 together with the Kronecker products.

4.3 Outline of the Model

In this section we give an overview over the model. First, we discuss the assignments of leptons and quarks under T'. In order to arrive at the same result for the lepton sector as the A_4 model we have to assign the left-handed lepton doublets to the three-dimensional representation of T', whereas the left-handed conjugate leptons e^c, μ^c and τ^c transform as the three inequivalent one-dimensional representations. As explained in Section 4.2, these representations can also be found in the group A_4, since T' is its double-covering. Similar to the A_4 model, we need an additional Z_3 symmetry which enables us to separate the charged lepton and the neutrino sector in the following discussion. The charged lepton masses arise as always from the Yukawa couplings connecting l and e^c (μ^c, τ^c), while the neutrinos are assumed to get masses from the dimension five operator $(l\, h_u)\,(l\, h_u)$ which is suppressed by the cutoff scale Λ. Due to the separating Z_3 symmetry there can exist two sets of flavon fields, $\{\varphi_T\}$ and $\left\{\varphi_S, \xi, \tilde{\xi}\right\}$, which only couple to charged leptons and neutrinos (at leading order), respectively. According to the choice of the generators S and T the charged lepton mass matrix is diagonal, if the VEV structure of φ_T is proportional to $(1, 0, 0)$. As will be shown below this configuration preserves the Z_3 symmetry generated by T. However, T' cannot explain the hierarchy among the charged leptons and therefore an additional $U(1)_{FN}$ symmetry is advocated under which the first and second generation of left-handed conjugate charged leptons, e^c and μ^c, transform non-trivially. To implement the Froggatt-Nielsen mechanism [11] we then have to assume the existence of a further gauge singlet θ which only carries a non-vanishing $U(1)_{FN}$ charge which can be chosen as -1 without loss of generality [3]. The mass matrix of the neutrinos leads to TBM (independent of the exact value of the eigenvalues), if $\langle\varphi_S\rangle = v_S\,(1, 1, 1)$ holds and at least one of the trivially transforming flavons ξ and $\tilde{\xi}$ has a non-vanishing VEV [4]. T' is thereby broken down to a Z_4 symmetry which will become clear in a moment. The lepton mixings are then tri-bimaximal. In the quark sector a similar assignment of left-handed and left-handed conjugate fields as used for the charged leptons cannot produce a Cabibbo angle of order $\lambda \approx 0.22$. This has been discussed at length for the A_4 model in [15] and the same result holds for the equivalent T' model, as T' and A_4 share the one- and three-dimensional representations. Moreover, this solution suffers from the fact that all quark masses including the top quark mass are only produced by non-renormalizable terms, if we do not assume that there exist copies of the MSSM Higgs doublets h_u and h_d which also transform non-trivially under the flavor symmetry. Therefore, we choose different representations for the quarks, i.e. we assign the first two generations of the quarks to a two-dimensional representation of T' and the third one transforms trivially. By choosing the Z_3 charges of the quarks properly they can couple to the flavon fields φ_T which already give masses to the charged leptons. Since the quarks transform in a different way under T' than the charged leptons, their mass matrices have a different form. Furthermore, we need to introduce flavons η and ξ'' transforming as doublet and non-trivial singlet under T', respectively. They have the same Z_3 charge as the fields φ_T and therefore couple at leading order to charged fermions only. However, due to T' invariance of the Yukawa couplings they only contribute to the quark masses and not to the charged lepton mass matrix. With $\langle\eta\rangle = (v_1, 0)$ and $\langle\xi''\rangle = 0$ also these fields preserve the Z_3 subgroup generated by T which is left unbroken by $\langle\varphi_T\rangle \propto (1, 0, 0)$. The resulting quark mass matrices only have a non-vanishing 2-3 subblock. The (33) entry dominates in the up as well as the down quark mass matrix, since it arises from a renormalizable coupling for the up quarks and from a coupling involving only the FN field θ, but no flavon in case of the down quarks [5]. Obviously, this does not describe all quark masses and mixings correctly. Therefore, the inclusion of subleading terms is mandatory. In general, these terms have two main effects: $a.$) new operators containing two flavons arise which, for example, couple φ_T also to the neutrino sector such that the separation of the charged fermion and neutrino sector is not rigid anymore and $b.$) subleading terms in the flavon potential will correct the VEV alignment, for example, $\langle\varphi_S\rangle = v_S\,(1, 1, 1)$ is replaced

[3] If $U(1)_{FN}$ is not broken explicitly in the theory, its spontaneous breaking will cause the existence of a massless Goldstone boson.

[4] Otherwise, the mass matrix of the light neutrinos has one vanishing and two degenerate eigenvalues and therefore does not allow a unique determination of the eigenvectors (up to phases), see below Eq.(4.27) and Eq.(4.29) for $a = 0$.

[5] Note that this is slightly different in the model presented in [24]. The reason for this is the fact that the left-handed conjugate down quarks do not transform under $U(1)_{FN}$ in [24]. However, this raises the problem that the mass of the bottom quark, which is generated at the renormalizable level, cannot be of the same order as the mass of the τ lepton which acquires an additional suppression factor λ^2, since it stems from a non-renormalizable Yukawa coupling involving one flavon. The problem is solved in a very simple way, if all left-handed conjugate down quarks carry charge $+1$ under $U(1)_{FN}$. Then all Yukawa couplings leading to masses for down quarks are non-renormalizable, i.e. need an insertion of the FN field θ, and are also suppressed by at least λ^2. This only corrects the absolute mass scale of the down quarks, while all other results of the original model presented in [24] still hold.

by $\langle\varphi_S\rangle = (v_S + \delta v_{S\,1}, v_S + \delta v_{S\,2}, v_S + \delta v_{S\,3})$. The hierarchy among the quark masses is mainly due to two facts: *a.*) the masses for the first generation of quarks are not generated at the leading order level, i.e. they are suppressed compared to the others and *b.*) the left-handed conjugate quarks are also charged under the $U(1)_{FN}$ symmetry, which already introduces the hierarchy among the charged leptons. The main challenge in this model then arises from the fact that we have to generate the Cabibbo angle (and the masses m_u and m_d) via subleading effects only, while keeping these corrections at a level which does not spoil the nice result of the TBM in the lepton sector. This can be successfully implemented, as will be shown in Section 4.4. There is one caveat to mention, the solution is possible and well motivated, however, contains a certain (small) tuning of two parameters which will be discussed in Section 4.4.2. All the information given here about the transformation properties of fermions and flavons under the symmetries of the model, T', Z_3 and $U(1)_{FN}$, are collected in Table 4.2.

Field	l	e^c	μ^c	τ^c	$Q_{1,2}$	(u^c, c^c)	(d^c, s^c)	Q_3	t^c	b^c	$h_{u,d}$	φ_T	φ_S	$\xi, \tilde{\xi}$	η	ξ''	θ
T'	3	1	$1''$	$1'$	$2''$	$2''$	$2''$	1	1	1	1	3	3	1	$2'$	$1''$	1
Z_3	ω	ω^2	ω^2	ω^2	ω	ω^2	ω^2	ω	ω^2	ω^2	1	1	ω	ω	1	1	1
$U(1)_{FN}$	0	2	1	0	0	1	1	0	0	1	0	0	0	0	0	0	-1

Table 4.2: The transformation rules of the fields under the symmetries associated to the groups T', Z_3 and $U(1)_{FN}$. Note that only the scalar fields h_u and h_d transform non-trivially under the SM gauge group, whereas the flavon fields φ_T, φ_S, ξ, $\tilde{\xi}$, η and ξ'' are gauge singlets. The FN field θ is only charged under $U(1)_{FN}$. Again, ω equals $e^{\frac{2\pi i}{3}} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$.
In contrast to the model presented in [24] all left-handed conjugate down quarks have a non-vanishing $U(1)_{FN}$ charge.

The fermion masses in this model originate from the superpotential w which consists of three parts

$$w = w_l + w_q + w_d \; . \tag{4.13}$$

Thereby, w_l and w_q contain the Yukawa couplings of leptons and quarks, respectively, while the couplings involving only flavons are collected in w_d.
In the following we want to explain in more detail why the presented VEV structures of the flavons preserve a Z_3 or a Z_4 subgroup. This can be done with the help of the matrix forms of the generators S and T shown in Section 4.2. For the set of flavons $\{\varphi_T, \eta, \xi''\}$ coupling only to charged fermions at leading order, we advocated the structure

$$\langle\varphi_T\rangle = (v_T, 0, 0) \; , \;\; \langle\eta\rangle = (v_1, 0) \; , \;\; \langle\xi''\rangle = 0 \; . \tag{4.14}$$

Inspecting the generators S and T of the representations 3, $2'$ and $1''$, one sees that these VEV structures are eigenvectors of the generator T belonging the eigenvalue one, i.e.

$$\text{... for } \;\; 1'' \quad : \qquad T \cdot 0 = \omega^2 \cdot 0 = 1 \cdot 0 \; , \tag{4.15}$$

$$\text{... for } \;\; 2' \quad : \qquad T \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} \; , \tag{4.16}$$

$$\text{... for } \;\; 3 \quad : \qquad T \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = 1 \cdot \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \; . \tag{4.17}$$

T generates a Z_3 group, as $T^3 = \mathbb{1}$ holds. The fact that the VEV structures, shown in Eq.(4.14), are proportional to the eigenvector belonging to the eigenvalue one of the generator T implies that these structures only allow non-vanishing VEVs for components which transform trivially (as 1) -and not as ω or ω^2- under the Z_3 subgroup of T'. As in general fields which are invariant under a symmetry cannot break this symmetry, φ_T, η and ξ'' acquiring VEVs of the form Eq.(4.14) leave the subgroup Z_3 unbroken. Similarly, we can investigate the second VEV structure

$$\langle \varphi_S \rangle = (v_S, v_S, v_S) \; , \quad \langle \xi \rangle = u \; , \quad \langle \tilde{\xi} \rangle = 0 \; . \tag{4.18}$$

Here the generator of the relevant Z_4 subgroup can be written as

$$T\,S\,T^2 = \frac{1}{3} \left(\begin{array}{ccc} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{array} \right) \tag{4.19}$$

for the three-dimensional representation. Obviously, the vector with equal entries is an eigenvector to the eigenvalue one and the VEV structure of φ_S is proportional to this. For the two flavons ξ and $\tilde{\xi}$ which transform trivially under T' a non-vanishing VEV is always allowed [6], since the trivial representation of a group is also always a trivial representation in any of its subgroups. $\langle \tilde{\xi} \rangle = 0$ is therefore not induced by the requirement of preserving a certain subgroup of T', but is attributed to the details of the flavon potential, as will be explained below in Section 4.5. Note that for the two-dimensional representations $T\,S\,T^2$ does not possess an eigenvalue $+1$, i.e. these representations do not contain a component transforming trivially under the residual Z_4 group. The fact that $T\,S\,T^2$ generates a Z_4 subgroup can be seen by using the generator relations Eq.(4.7)

$$(T\,S\,T^2)^2 = T\,S\,T^3\,S\,T^2 \overset{T^3=\mathbb{1}}{=} T\,S^2 T^2 \overset{S^2=\mathbb{R}}{=} T\,\mathbb{R}\,T^2 \overset{T^3=\mathbb{1}\;,\;[T,\mathbb{R}]=0}{=} \mathbb{R} = \pm\mathbb{1} \quad \Longrightarrow \quad (T\,S\,T^2)^4 = \mathbb{1} \; . \tag{4.20}$$

Since $\mathbb{R} = +\mathbb{1}$ for the one- and three-dimensional representations, $T\,S\,T^2$ generates in this case only a Z_2 group which corresponds to the preserved Z_2 group in the A_4 models.

4.4 Results for Fermion Masses and Mixings

In this section we write down the invariant Yukawa couplings and mass matrices for the fermions, study the fermion masses and their mixings and investigate the terms appearing at subleading order.

4.4.1 Leading Order Results

At leading order, the Yukawa couplings arising from the two parts w_l and w_q of the superpotential can be written in shortform as [7]

$$\begin{aligned} w_l &= \frac{y_e}{\Lambda^3} \, (\varphi_T \, l) \, e^c \, \theta^2 \, h_d + \frac{y_\mu}{\Lambda^2} \, (\varphi_T \, l)' \, \mu^c \, \theta \, h_d + \frac{y_\tau}{\Lambda} \, (\varphi_T \, l)'' \, \tau^c \, h_d \\ &+ \frac{1}{\Lambda^2} \, (x_a \, \xi + \tilde{x}_a \, \tilde{\xi}) \, (l \, l) \, h_u \, h_u + \frac{x_b}{\Lambda^2} \, (\varphi_S \, l \, l) \, h_u \, h_u + \text{h.o.} \end{aligned} \tag{4.21}$$

[6]The same holds in this case for flavons transforming as non-trivial one-dimensional representations. We comment on this in Section 4.6.

[7]Note that we present the couplings in the basis $L\,L^c$ and not $L^c\,L$ as done in [24]. Note further that we display the FN field explicitly in the terms. Hence all Yukawa couplings are of order one in our notation and do not include the additional suppression factors stemming from the Froggatt-Nielsen mechanism. This is different in the published work [24].

for the leptons and for the quarks we find

$$\begin{aligned} w_q &= y_t\,(Q_3\,t^c)\;h_u + \frac{y_b}{\Lambda}\;(Q_3\,b^c)\;\theta\,h_d \\ &+ \frac{y_1}{\Lambda^2}\,(\varphi_T\,D_q\,D_u^c)\,\theta\,h_u + \frac{y_5}{\Lambda^2}\,(\varphi_T\,D_q\,D_d^c)\,\theta\,h_d \\ &+ \frac{y_2}{\Lambda^2}\,\xi''\,(D_q\,D_u^c)'\,\theta\,h_u + \frac{y_6}{\Lambda^2}\,\xi''\,(D_q\,D_d^c)'\,\theta\,h_d \\ &+ \frac{1}{\Lambda}\,\left[y_3\,(\eta\,D_q)\,t^c + \frac{y_4}{\Lambda}\,Q_3\,(D_u^c\,\eta)\,\theta\right]\;h_u + \frac{1}{\Lambda^2}\;[y_7\,(\eta\,D_q)\,b^c + y_8\,Q_3\,(D_d^c\,\eta)]\;\theta\,h_d + \text{h.o.} \end{aligned} \tag{4.22}$$

where D_q is given by $(Q_1, Q_2)^t$, D_u^c by $(u^c, c^c)^t$ and D_d^c by $(d^c, s^c)^t$. The term +h.o. in w_l and w_q indicates the higher-order contributions.
The explicit form of the couplings is

$$\begin{aligned} w_l &= \frac{y_e}{\Lambda^3}\,(\varphi_{T\,1}\,l_1 + \varphi_{T\,2}\,l_3 + \varphi_{T\,3}\,l_2)\,e^c\,\theta^2\,h_d \\ &+ \frac{y_\mu}{\Lambda^2}\,(\varphi_{T\,3}\,l_3 + \varphi_{T\,1}\,l_2 + \varphi_{T\,2}\,l_1)\,\mu^c\,\theta\,h_d \\ &+ \frac{y_\tau}{\Lambda}\,(\varphi_{T\,2}\,l_2 + \varphi_{T\,1}\,l_3 + \varphi_{T\,3}\,l_1)\,\tau^c\,h_d \\ &+ \frac{1}{\Lambda^2}\,(x_a\,\xi + \tilde{x}_a\,\tilde{\xi})\,(l_1\,l_1 + l_2\,l_3 + l_3\,l_2)\,h_u\,h_u \\ &+ \frac{1}{3}\,\frac{x_b}{\Lambda^2}\,[\varphi_{S\,1}\,(2\,l_1\,l_1 - l_2\,l_3 - l_3\,l_2) + \varphi_{S\,2}\,(2\,l_2\,l_2 - l_1\,l_3 - l_3\,l_1) \\ &\qquad + \varphi_{S\,3}\,(2\,l_3\,l_3 - l_1\,l_2 - l_2\,l_1)]\,h_u\,h_u + \text{h.o.} \end{aligned} \tag{4.23}$$

and

$$\begin{aligned} w_q &= y_t\,(Q_3\,t^c)\;h_u + \frac{y_b}{\Lambda}\;(Q_3\,b^c)\;\theta\,h_d \\ &+ \frac{y_1}{\Lambda^2}\,(\varphi_{T\,1}\,Q_2\,c^c + i\,\varphi_{T\,2}\,Q_1\,u^c + \left(\frac{1-i}{2}\right)\;\varphi_{T\,3}\,(Q_1\,c^c + Q_2\,u^c))\,\theta\,h_u \\ &+ \frac{y_5}{\Lambda^2}\,(\varphi_{T\,1}\,Q_2\,s^c + i\,\varphi_{T\,2}\,Q_1\,d^c + \left(\frac{1-i}{2}\right)\;\varphi_{T\,3}\,(Q_1\,s^c + Q_2\,d^c))\,\theta\,h_d \\ &+ \frac{y_2}{\Lambda^2}\,\xi''\,(Q_1\,c^c - Q_2\,u^c)\,\theta\,h_u + \frac{y_6}{\Lambda^2}\,\xi''\,(Q_1\,s^c - Q_2\,d^c)\,\theta\,h_d \\ &+ \frac{1}{\Lambda}\,\left[y_3\,(\eta_1\,Q_2 - \eta_2\,Q_1)\,t^c + \frac{y_4}{\Lambda}\,Q_3\,(c^c\,\eta_1 - u^c\,\eta_2)\,\theta\right]\;h_u \\ &+ \frac{1}{\Lambda^2}\;[y_7\,(\eta_1\,Q_2 - \eta_2\,Q_1)\,b^c + y_8\,Q_3\,(s^c\,\eta_1 - d^c\,\eta_2)]\;\theta\,h_d + \text{h.o.} \end{aligned} \tag{4.24}$$

Inserting the VEVs

$$\langle\varphi_S\rangle = (v_S, v_S, v_S)\;,\quad \langle\xi\rangle = u\;,\quad \langle\tilde{\xi}\rangle = 0\;, \tag{4.25a}$$
$$\langle\varphi_T\rangle = (v_T, 0, 0)\;,\quad \langle\eta\rangle = (v_1, 0)\;,\quad \langle\xi''\rangle = 0\;, \tag{4.25b}$$

which preserve the subgroups Z_4 and Z_3 according to Section 4.3 and

$$\langle h_u\rangle = v_u \quad \text{and} \quad \langle h_d\rangle = v_d\;, \tag{4.26}$$

the mass matrices have the following appearance [8]

$$\mathcal{M}_l = \begin{pmatrix} y_e \frac{\langle\theta\rangle^2}{\Lambda^2} & 0 & 0 \\ 0 & y_\mu \frac{\langle\theta\rangle}{\Lambda} & 0 \\ 0 & 0 & y_\tau \end{pmatrix} \frac{v_T}{\Lambda} v_d \,, \quad M_\nu = \begin{pmatrix} a + \frac{2}{3}b & -\frac{1}{3}b & -\frac{1}{3}b \\ -\frac{1}{3}b & \frac{2}{3}b & a - \frac{1}{3}b \\ -\frac{1}{3}b & a - \frac{1}{3}b & \frac{2}{3}b \end{pmatrix} \frac{v_u^2}{\Lambda} \,, \tag{4.27}$$

$$\mathcal{M}_u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_1 \frac{v_T}{\Lambda}\frac{\langle\theta\rangle}{\Lambda} & y_3 \frac{v_1}{\Lambda} \\ 0 & y_4 \frac{v_1}{\Lambda}\frac{\langle\theta\rangle}{\Lambda} & y_t \end{pmatrix} v_u \,, \quad \mathcal{M}_d = \begin{pmatrix} 0 & 0 & 0 \\ 0 & y_5 \frac{v_T}{\Lambda} & y_7 \frac{v_1}{\Lambda} \\ 0 & y_8 \frac{v_1}{\Lambda} & y_b \end{pmatrix} \frac{\langle\theta\rangle}{\Lambda} v_d \,. \tag{4.28}$$

We introduced the abbreviation $a = x_a \frac{u}{\Lambda}$ and $b = x_b \frac{v_S}{\Lambda}$ into the neutrino mass matrix M_ν. M_ν is diagonalized by

$$U_\nu = \begin{pmatrix} \frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{so that} \quad U_\nu^\dagger\, M_\nu\, U_\nu^\star = \frac{v_u^2}{\Lambda} \begin{pmatrix} a+b & 0 & 0 \\ 0 & a & 0 \\ 0 & 0 & -a+b \end{pmatrix} \,. \tag{4.29}$$

Since the charged lepton mass matrix is diagonal, U_ν equals U_{MNS} and therefore lepton mixings are tri-bimaximal. However, in order to ensure that this result is not completely destroyed by subleading corrections, we have to calculate them and show that they do not exceed the order λ^2. A study of the mass spectrum and $|m_{ee}|$ can be found in [113]. Its results also hold, if A_4 is replaced by T', since the lepton sector coincides in both models. The ratios $\frac{u}{\Lambda}$ and $\frac{v_S}{\Lambda}$ turn out to be of the order λ^2. The cutoff scale Λ of the theory is expected to be around $(10^{13}...10^{15})$ GeV. It is therefore between the seesaw and the GUT scale. The VEVs of the flavon fields are then of the order $(10^{11}...10^{13})$ GeV, i.e. T' is broken far above the electroweak scale.
The hierarchy among the charged leptons

$$m_e = \left| y_e \frac{\langle\theta\rangle^2}{\Lambda^2} \frac{v_T}{\Lambda} v_d \right| \,, \quad m_\mu = \left| y_\mu \frac{\langle\theta\rangle}{\Lambda} \frac{v_T}{\Lambda} v_d \right| \quad \text{and} \quad m_\tau = \left| y_\tau \frac{v_T}{\Lambda} v_d \right| \tag{4.30}$$

can be maintained by the Froggatt-Nielsen mechanism. It is sufficient to assume that $\frac{\langle\theta\rangle}{\Lambda}$ is of order λ^2 for reproducing the correct hierarchy, i.e. $m_e : m_\mu : m_\tau \approx \lambda^4 : \lambda^2 : 1$. For the absolute mass scale, we arrive at m_τ of the order of a GeV, if $v_d \sim \mathcal{O}(100 \text{ GeV})$ and $\frac{v_T}{\Lambda} \sim \lambda^2$. Since v_d is large, $\tan(\beta)$ is small in this model.
In the quark sector the masses of the second and third generation are reproduced

$$m_c \approx \left| y_1 \frac{v_T}{\Lambda} \frac{\langle\theta\rangle}{\Lambda} v_u \right| \,, \quad m_t \approx |y_t\, v_u| \,, \tag{4.31}$$

$$m_s \approx \left| y_5 \frac{v_T}{\Lambda} \frac{\langle\theta\rangle}{\Lambda} v_d \right| \,, \quad m_b \approx \left| y_b \frac{\langle\theta\rangle}{\Lambda} v_d \right| \tag{4.32}$$

as well as the CKM element

$$V_{cb} \approx \left(\frac{y_7}{y_b} - \frac{y_3}{y_t} \right) \frac{v_1}{\Lambda} \,. \tag{4.33}$$

[8] We display the mass matrices in the basis in which the left-handed fields are on the left-hand side and the left-handed conjugate ones are on the right-hand side. This is different from [24] and therefore our mass matrices are the transposes of the ones shown in [24].

All quantities involving the first generation, i.e. the masses of the up and down quark as well as the CKM elements V_{ud}, V_{us}, V_{ub}, etc., vanish at this order. They have to be generated by next-to-leading order contributions to the mass matrices. m_t can be naturally large for $v_u \approx \mathcal{O}(100\ \mathrm{GeV})$, while m_b is suppressed compared to m_t by the Froggatt-Nielsen mechanism so that it is of the same order as m_τ. Since we already chose $\frac{v_T}{\Lambda} \approx \lambda^2$, m_s is in the range of the muon mass and $m_s : m_b \approx \lambda^2 : 1$. The mass of the charm quark is $m_c \approx \lambda^4\, v_u$ and therefore $m_c : m_t \approx \lambda^4 : 1$ holds. Finally, V_{cb} is around λ^2 for $\frac{v_1}{\Lambda} \sim \lambda^2$ which fits the experimental results quite well.

4.4.2 Next-to-Leading Order Results

In this section we show how the next-to-leading order modifies the results of the leading order. As already mentioned, there are two different sources of corrections: *a.*) additional operators arising from the insertion of two flavon fields instead of only one and *b.*) corrections to the vacuum alignment, induced by four flavon terms in w_d, which lead to shifts in the VEVs and thereby also contribute to the fermion masses.

In the following, we display all operators with two flavons correcting the up and down quark mass matrices

$$\begin{aligned}
\Delta w_q &= \frac{1}{\Lambda^2}\,(\varphi_T\,\varphi_T)\,Q_3\,t^c\,h_u + \frac{1}{\Lambda^2}\,(D_q\,\eta\,\varphi_T)\,t^c\,h_u \qquad (4.34)\\
&+ \frac{1}{\Lambda^3}\,Q_3\,(\eta\,\varphi_T\,D_u^c)\,\theta\,h_u + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_T)''\,(D_q\,D_u^c)'\,\theta\,h_u + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_T)_S\,(D_q\,D_u^c)_3\,\theta\,h_u\\
&+ \frac{1}{\Lambda^3}\,\xi''\,\varphi_T\,(D_q\,D_u^c)_3\,\theta\,h_u + \frac{1}{\Lambda^3}\,(\eta\,\eta)_3\,(D_q\,D_u^c)_3\,\theta\,h_u\\
&+ \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_T)\,Q_3\,b^c\,\theta\,h_d + \frac{1}{\Lambda^3}\,(D_q\,\eta\,\varphi_T)\,b^c\,\theta\,h_d\\
&+ \frac{1}{\Lambda^3}\,Q_3\,(\eta\,\varphi_T\,D_d^c)\,\theta\,h_d + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_T)''\,(D_q\,D_d^c)'\,\theta\,h_d + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_T)_S\,(D_q\,D_d^c)_3\,\theta\,h_d\\
&+ \frac{1}{\Lambda^3}\,\xi''\,\varphi_T\,(D_q\,D_d^c)_3\,\theta\,h_d + \frac{1}{\Lambda^3}\,(\eta\,\eta)_3\,(D_q\,D_d^c)_3\,\theta\,h_d
\end{aligned}$$

Here we omit order one coefficients. The explicit form of the terms can be found in Appendix C. Analogously, the lepton sector is subject to corrections from additional operators

$$\begin{aligned}
\Delta\,w_l &= \frac{1}{\Lambda^4}\,\big((\varphi_T\,\varphi_T)_S + \xi''\,\varphi_T + (\eta\,\eta)_3\big)\,(l\,e^c)\,\theta^2\,h_d + \frac{1}{\Lambda^3}\,\big((\varphi_T\,\varphi_T)_S + \xi''\,\varphi_T + (\eta\,\eta)_3\big)\,(l\,\mu^c)\,\theta\,h_d\\
&+ \frac{1}{\Lambda^2}\,\big((\varphi_T\,\varphi_T)_S + \xi''\,\varphi_T + (\eta\,\eta)_3\big)\,(l\,\tau^c)\,h_d \qquad (4.35)\\
&+ \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)\,(l\,l)\,h_u^2 + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)'\,(l\,l)''\,h_u^2 + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)''\,(l\,l)'\,h_u^2\\
&+ \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)_S\,(l\,l)_S\,h_u^2 + \frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)_A\,(l\,l)_S\,h_u^2\\
&+ \frac{1}{\Lambda^3}\,(\xi\,[\tilde{\xi}]\,\varphi_T)\,(l\,l)_S\,h_u^2 + \frac{1}{\Lambda^3}\,(\xi''\,\varphi_S)\,(l\,l)_S\,h_u^2 + \frac{1}{\Lambda^3}\,\xi\,[\tilde{\xi}]\,\xi''\,(l\,l)'\,h_u^2
\end{aligned}$$

Again, we left out all order one coefficients. The explicit form of the corrections is shown in Appendix C. Note that the notation $\xi[\tilde{\xi}]$ indicates that either the field ξ or the field $\tilde{\xi}$ is involved in the coupling. Secondly, note that not all the terms displayed in the equations have to be linearly independent. However, this fact is not relevant for our purposes. The contributions to the fermion mass matrices arise, if the scalar fields are replaced by their VEVs, as given in Eq.(4.25) and Eq.(4.26). Therefore, all terms containing the fields $\tilde{\xi}$ and ξ'' vanish. As one can see, all corrections to the charged fermion masses can be absorbed into the leading order structure by

redefining the Yukawa couplings $y_{e,\mu,\tau}$, $y_{1,3,4}$, $y_{5,7,8}$, y_t and y_b, i.e. the results given in Eq.(4.27) and Eq.(4.28) are not affected. Only the mass matrix M_ν of the light neutrinos gets corrections which change the leading order result. For example, the term $\frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)'\,(l\,l)''\,h_u^2$ leads to a contribution $\frac{1}{\Lambda^3}\,v_T\,v_S\,(l_2\,l_2+l_1\,l_3+l_3\,l_1)\,v_u^2$ which cannot be reconciled with the TBM structure of M_ν. Therefore, we expect that the actual lepton mixing angles deviate from the ones predicted at leading order.
The second source of corrections to the mass matrices originates from shifts of the VEVs due to four flavon operators in w_d. The shifts of the VEVs are parameterized in general by

$$\begin{aligned}&\langle\varphi_S\rangle=(v_S+\delta v_{S\,1},v_S+\delta v_{S\,2},v_S+\delta v_{S\,3})\;,\;\;\langle\varphi_T\rangle=(v_T+\delta v_{T\,1},\delta v_{T\,2},\delta v_{T\,3})\;,\\&\langle\eta\rangle=(v_1+\delta v_1,\delta v_2)\;,\langle\xi\rangle=u\;,\langle\tilde{\xi}\rangle=\delta\tilde{u}\;,\langle\xi''\rangle=\delta u''\;.\end{aligned}\tag{4.36}$$

As will be shown below, the order of the shifts is generically $\lambda^4\,\Lambda$ in case that the VEVs themselves are of order $\lambda^2\,\Lambda$. Note that the VEV of ξ is not shifted, since it remains undetermined at leading order, see below. Note further that we expect all shifts to be independent. To find the corrections to the mass matrices, we have to insert the shifted VEVs into the leading order terms, Eq.(4.21) and Eq.(4.22). We arrive at

$$\delta\mathcal{M}_l \;=\; \begin{pmatrix} y_e\,\delta v_{T\,1}\,\frac{\langle\theta\rangle^2}{\Lambda^2} & y_\mu\,\delta v_{T\,2}\,\frac{\langle\theta\rangle}{\Lambda} & y_\tau\,\delta v_{T\,3}\\ y_e\,\delta v_{T\,3}\,\frac{\langle\theta\rangle^2}{\Lambda^2} & y_\mu\,\delta v_{T\,1}\,\frac{\langle\theta\rangle}{\Lambda} & y_\tau\,\delta v_{T\,2}\\ y_e\,\delta v_{T\,2}\,\frac{\langle\theta\rangle^2}{\Lambda^2} & y_\mu\,\delta v_{T\,3}\,\frac{\langle\theta\rangle}{\Lambda} & y_\tau\,\delta v_{T\,1}\end{pmatrix}\frac{v_d}{\Lambda} \tag{4.37}$$

$$\delta M_\nu \;=\; \begin{pmatrix} \tilde{x}_a\,\frac{\delta\tilde{u}}{\Lambda}+\frac{2}{3}\,b\,\frac{\delta v_{S\,1}}{v_S} & -\frac{1}{3}\,b\,\frac{\delta v_{S\,3}}{v_S} & -\frac{1}{3}\,b\,\frac{\delta v_{S\,2}}{v_S}\\ -\frac{1}{3}\,b\,\frac{\delta v_{S\,3}}{v_S} & \frac{2}{3}\,b\,\frac{\delta v_{S\,2}}{v_S} & \tilde{x}_a\,\frac{\delta\tilde{u}}{\Lambda}-\frac{1}{3}\,b\,\frac{\delta v_{S\,1}}{v_S}\\ -\frac{1}{3}\,b\,\frac{\delta v_{S\,2}}{v_S} & \tilde{x}_a\,\frac{\delta\tilde{u}}{\Lambda}-\frac{1}{3}\,b\,\frac{\delta v_{S\,1}}{v_S} & \frac{2}{3}\,b\,\frac{\delta v_{S\,3}}{v_S}\end{pmatrix}\frac{v_u^2}{\Lambda}\;, \tag{4.38}$$

$$\delta\mathcal{M}_u \;=\; \begin{pmatrix} i\,y_1\,\frac{\delta v_{T\,2}}{\Lambda}\,\frac{\langle\theta\rangle}{\Lambda} & \left(\frac{1-i}{2}\,y_1\,\frac{\delta v_{T\,3}}{\Lambda}+y_2\,\frac{\delta u''}{\Lambda}\right)\frac{\langle\theta\rangle}{\Lambda} & -y_3\,\frac{\delta v_2}{\Lambda}\\ \left(\frac{1-i}{2}\,y_1\,\frac{\delta v_{T\,3}}{\Lambda}-y_2\,\frac{\delta u''}{\Lambda}\right)\frac{\langle\theta\rangle}{\Lambda} & y_1\frac{\delta v_{T\,1}}{\Lambda}\,\frac{\langle\theta\rangle}{\Lambda} & y_3\,\frac{\delta v_1}{\Lambda}\\ -y_4\,\frac{\delta v_2}{\Lambda}\,\frac{\langle\theta\rangle}{\Lambda} & y_4\,\frac{\delta\,v_1}{\Lambda}\,\frac{\langle\theta\rangle}{\Lambda} & 0\end{pmatrix} v_u\;, \tag{4.39}$$

$$\delta\mathcal{M}_d \;=\; \begin{pmatrix} i\,y_5\,\frac{\delta v_{T\,2}}{\Lambda} & \frac{1-i}{2}\,y_5\,\frac{\delta v_{T\,3}}{\Lambda}+y_6\,\frac{\delta u''}{\Lambda} & -y_7\,\frac{\delta v_2}{\Lambda}\\ \frac{1-i}{2}\,y_5\,\frac{\delta v_{T\,3}}{\Lambda}-y_6\,\frac{\delta u''}{\Lambda} & y_5\frac{\delta v_{T\,1}}{\Lambda} & y_7\,\frac{\delta v_1}{\Lambda}\\ -y_8\,\frac{\delta v_2}{\Lambda} & y_8\,\frac{\delta\,v_1}{\Lambda} & 0\end{pmatrix}\frac{\langle\theta\rangle}{\Lambda}\,v_d\;. \tag{4.40}$$

As one can see, the dominating effects for the charged fermions have to originate from the shifts of the VEVs, since only these contributions change the structure of the mass matrices.
We can now analyze the impact of the next-to-leading order corrections on the fermion masses and mixings. Regarding the lepton sector we observe that the discussion is exactly the same as performed in [15]. This has two main reasons: *a.*) the structure of the VEV shifts [9] is clearly the same as in the A_4 model and therefore leads to the same corrections and *b.*) the two flavon

[9]To be correct, in [15] $\delta v_{T\,2}$ and $\delta v_{T\,3}$ turn out to be the same, while they are independent in our model here. However, this does not really matter for the discussion of the corrections to TBM.

insertions either coincide with the ones of the A_4 model (e.g. $\frac{1}{\Lambda^3}\,(\varphi_T\,\varphi_S)'\,(l\,l)''\,h_u^2$), are unimportant, since they can be absorbed into the leading order (e.g. $\frac{1}{\Lambda^4}\,(\eta\,\eta)_3\,(l\,e^c)\,\theta^2\,h_d$) or vanish, since the VEV of the flavon field vanishes (e.g. $\frac{1}{\Lambda^4}\,\xi''\,\varphi_T\,(l\,e^c)\,\theta^2\,h_d$ and $\frac{1}{\Lambda^3}\,\xi\,[\tilde{\xi}]\,\xi''\,(l\,l)'\,h_u^2$). Therefore, we can refer to [15] for the analysis of the lepton sector. Its result shows that all deviations of the lepton mixing angles from TBM can be kept small, i.e. at the order λ^2 in case that all flavon VEVs are of the order $\lambda^2\,\Lambda$ and their shifts are of the order $\lesssim \lambda^4\,\Lambda$.
The quark sector needs a more careful study. If we simply plug in the generic order of the VEVs and their shifts, we arrive at the following mass matrix structures

$$\mathcal{M}_u \sim \mathcal{O}\left(\left(\begin{array}{ccc} \lambda^6 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^6 & \lambda^4 & 1 \end{array}\right) v_u\right) \; , \quad \mathcal{M}_d \sim \mathcal{O}\left(\left(\begin{array}{ccc} \lambda^4 & \lambda^4 & \lambda^4 \\ \lambda^4 & \lambda^2 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{array}\right) \lambda^2\, v_d\right) \; . \tag{4.41}$$

As one can see, this raises the problem that the up quark mass turns out to be too large. Especially the (11) element is too large and should rather be of the order λ^8 and not λ^6. For this purpose, we have to demand that the shift of the second field contained in the triplet φ_T, i.e. $\delta v_{T\,2}$, is not of the order $\lambda^4\,\Lambda$, but $\lambda^6\,\Lambda$ [10]. However, then the mass of the down quark is expected to be too small. We can cure this problem by assuming that the coupling y_6 is slightly larger than its natural value, i.e. $y_6 \sim \frac{1}{\lambda}$. In this way the orders of the mass matrix elements turn out to be

$$\mathcal{M}_u \sim \mathcal{O}\left(\left(\begin{array}{ccc} \lambda^8 & \lambda^6 & \lambda^4 \\ \lambda^6 & \lambda^4 & \lambda^2 \\ \lambda^6 & \lambda^4 & 1 \end{array}\right) v_u\right) \; , \quad \mathcal{M}_d \sim \mathcal{O}\left(\left(\begin{array}{ccc} \lambda^6 & \lambda^3 & \lambda^4 \\ \lambda^3 & \lambda^2 & \lambda^2 \\ \lambda^4 & \lambda^2 & 1 \end{array}\right) \lambda^2\, v_d\right) \; . \tag{4.42}$$

We have to make one comment: Since the (11) element of the up and down quark mass matrix has to be tuned to $\lambda^8\,v_u$ and $\lambda^8\,v_d$, respectively, we now also have to take into account the contributions from three flavon insertions which are generically at least of order λ^6, since they contain three powers of the ratio flavon VEV over cutoff scale. They are, similar to the other operators, appropriately suppressed by λ^2 stemming from the Froggatt-Nielsen mechanism. Therefore, their contributions to all elements are negligible apart from the (11) element, where they appear at the same order as the first non-vanishing term coming from the shifts in the VEVs. Diagonalizing the mass matrices $\mathcal{M}_u$ and $\mathcal{M}_d$ which contain all leading and next-to-leading order contributions we arrive at the following expressions for quark masses and mixings

$$m_u \approx \left| y_1\, v_u\, \frac{\langle\theta\rangle}{\Lambda} \left\{ i\, \frac{\delta v_{T\,2}}{\Lambda} - \left[\left(\frac{1-i}{2}\right)^2 \frac{\delta v_{T\,3}^2}{v_T\,\Lambda} - \frac{y_2^2}{y_1^2}\, \frac{\delta u''^2}{v_T\,\Lambda} \right] \right\} + ... \right| \; , \tag{4.43}$$

$$m_d \approx \left| v_d\, \frac{\langle\theta\rangle}{\Lambda}\, \frac{y_6^2}{y_5}\, \frac{\delta u''^2}{v_T\,\Lambda} \right| \; , \quad m_c \approx \left| y_1\, v_u\, \frac{\langle\theta\rangle}{\Lambda}\, \frac{v_T}{\Lambda} \right| \; , \quad m_s \approx \left| y_5\, v_d\, \frac{\langle\theta\rangle}{\Lambda}\, \frac{v_T}{\Lambda} \right| \; , \tag{4.44}$$

$$m_t \approx |y_t\, v_u| \; , \quad m_b \approx \left| y_b\, v_d\, \frac{\langle\theta\rangle}{\Lambda} \right| \; , \tag{4.45}$$

[10] We will show below that this assumption can be made without spoiling the order of the other shifts.

$$V_{ud} \approx V_{cs} \approx 1 \; , \;\; V_{tb} \approx 1 \; , \tag{4.46}$$

$$V^{*}_{us} \approx -V_{cd} \approx -\frac{y_6}{y_5} \frac{\delta u''}{v_T} - \left[\left(\frac{1-i}{2} \right) \frac{\delta v_{T\,3}}{v_T} - \frac{y_2}{y_1} \frac{\delta u''}{v_T} \right] \; , \tag{4.47}$$

$$V^{*}_{ub} \approx - \left(\frac{y_7}{y_b} - \frac{y_3}{y_t} \right) \left\{ \frac{\delta v_2}{\Lambda} + \frac{v_1}{v_T} \left[\left(\frac{1-i}{2} \right) \frac{\delta v_{T\,3}}{\Lambda} - \frac{y_2}{y_1} \frac{\delta u''}{\Lambda} \right] \right\} \; , \tag{4.48}$$

$$V^{*}_{cb} \approx -V_{ts} \approx \left(\frac{y_7}{y_b} - \frac{y_3}{y_t} \right) \frac{v_1}{\Lambda} \; , \tag{4.49}$$

$$V_{td} \approx -\frac{y_6}{y_5} \left(\frac{y_7}{y_b} - \frac{y_3}{y_t} \right) \frac{v_1 \, \delta u''}{v_T \, \Lambda} + \left(\frac{y_7}{y_b} - \frac{y_3}{y_t} \right) \frac{\delta v_2}{\Lambda} \; . \tag{4.50}$$

All quark masses are then of the correct order, i.e.

$$m_u \sim \mathcal{O}(\lambda^8 \, v_u) \; , \;\; m_c \sim \mathcal{O}(\lambda^4 \, v_u) \; , \;\; m_t \sim \mathcal{O}(v_u) \; , \tag{4.51}$$

$$m_d \sim \mathcal{O}(\lambda^6 \, v_d) \; , \;\; m_s \sim \mathcal{O}(\lambda^4 \, v_d) \; , \;\; m_b \sim \mathcal{O}(\lambda^2 \, v_d) \; . \tag{4.52}$$

The diagonal elements of V_{CKM} equal one at leading order. V_{us} and V_{cd} are of order λ, since $y_6 \sim \frac{1}{\lambda}$, while V_{cb} and V_{ts} are of order λ^2. Finally, the elements V_{td} and V_{ub} are of order λ^3 and λ^4, respectively. Therefore, V_{ub} is rather small, but still this is allowed. Since the Yukawa couplings and VEVs as well as their shifts can be complex, J_{CP} can also be accommodated in this model.
In summary, the phenomenology of the quark sector can well be described in this model. Thereby, the quantities involving the first generation are only generated by subleading effects. Apart from two fine-tunings, i.e. $\delta v_{T\,2} \sim \lambda^6 \, \Lambda$ instead of $\lambda^4 \, \Lambda$ and $y_6 \sim \frac{1}{\lambda}$, all other parameters can be of their natural order. Additionally, one can show that the following two equations are fulfilled up to corrections of order λ^2

$$\sqrt{\frac{m_d}{m_s}} = |V_{us}| + \mathcal{O}(\lambda^2) \; , \tag{4.53}$$

$$\sqrt{\frac{m_d}{m_s}} = \left| \frac{V_{td}}{V_{ts}} \right| + \mathcal{O}(\lambda^2) \; . \tag{4.54}$$

Thereby, the second relation can be deduced from the first one by using the unitarity of the CKM matrix, i.e. $V_{ud} \, V^{\star}_{td} + V_{us} \, V^{\star}_{ts} + V_{ub} \, V^{\star}_{tb} = 0$, while keeping in mind that $V_{ub} \sim \lambda^4$ (and $V_{ud} \approx 1$, $V_{tb} \approx 1$) in our model.

4.5 Treatment of the Flavon Potential

In this section we analyze the problem of the vacuum alignment in detail, since all results of the model crucially depend on the fact whether we can achieve the advocated VEV structures in a natural way, i.e. without any further assumptions on the potential and tunings of its parameters. As already experienced in the models presented in Chapter 3, in the framework of the SM and with flavored $SU(2)_L$ Higgs doublets this task turns out to be hardly solvable. Therefore, the T' model is implemented in the MSSM and furthermore the breaking of the electroweak and the flavor symmetry are disentangled, i.e. the flavor symmetry is now broken by gauge singlets only. Two important aspects then allow us to solve the technical problem of the VEV alignment: *a.*) the potential of fields transforming as gauge singlets is much simpler than the corresponding one in which Higgs doublets are used and *b.*) since we now break the flavor symmetry at a high energy scale and not at the electroweak scale, we can minimize and analyze the gauge singlet potential in

the SUSY limit, i.e. F- and D-terms have to vanish. As the fields are gauge singlets, they do not possess D-terms, so we only have to consider their F-terms.
The actual construction of the flavon potential responsible for the VEV alignment is however more complicated. It turns out that according to the studies presented in [14, 15] the simplest implementation needs two further ingredients

- the introduction of so-called driving fields, denoted with an upper index 0, which are gauge singlets transforming non-trivially under the flavor symmetry and the additional Z_3 symmetry. Their F-terms are the determining equations for the VEVs of the flavons φ_S, φ_T, ξ, $\tilde{\xi}$, η and ξ''. The driving fields themselves, however, do not acquire a VEV. All driving fields are collected in Table 4.3.

Field	φ_T^0	φ_S^0	ξ^0	η^0	ξ'^0
T'	3	3	1	$2''$	$1'$
Z_3	1	ω	ω	1	1

Table 4.3: The transformation rules of the driving fields under the symmetries T' and Z_3.

- an additional $U(1)$ symmetry, called $U(1)_R$, which is an extension of the well-known R parity. All fermions have $U(1)_R$ charge +1, the scalar fields which have a non-vanishing VEV are not charged under $U(1)_R$, whereas the driving fields have charge +2. All terms in w_l and w_q then have a $U(1)_R$ charge of +2. Demanding this also for the terms contained in w_d enforces them to be linear in the driving fields. Additionally, $U(1)_R$ does not allow a direct coupling between SM fermions and driving fields. The $U(1)_R$ charges of all types of fields are summarized in Table 4.4.

Field	Fermions	Scalars with VEV$\neq 0$	Scalars without VEV
	l, e^c, μ^c, τ^c,	Flavons, Higgs doublets, FN field	Driving fields
	$Q_{1,2,3}$, u^c, c^c, t^c, d^c, s^c, b^c	h_u, h_d, φ_T, φ_S, ξ, $\tilde{\xi}$, η, ξ'', θ	φ_T^0, φ_S^0, ξ^0, η^0, ξ'^0
$U(1)_R$	+1	0	+2

Table 4.4: $U(1)_R$ charges of the fields of the model.

4.5.1 Leading Order Results

The renormalizable part of the superpotential w_d containing flavon and driving fields is of the form

$$\begin{aligned} w_d &= M\,(\varphi_T^0\,\varphi_T) + g\,(\varphi_T^0\,\varphi_T\,\varphi_T) + g_7\,\xi''\,(\varphi_T^0\,\varphi_T)' + g_8\,(\varphi_T^0\,\eta\,\eta) \\ &+ g_1\,(\varphi_S^0\,\varphi_S\,\varphi_S) + g_2\,\tilde{\xi}\,(\varphi_S^0\,\varphi_S) \\ &+ g_3\,\xi^0\,(\varphi_S\,\varphi_S) + g_4\,\xi^0\,\xi^2 + g_5\,\xi^0\,\xi\,\tilde{\xi} + g_6\,\xi^0\,\tilde{\xi}^2 \\ &+ M_\eta\,(\eta\,\eta^0) + g_9\,(\varphi_T\,\eta\,\eta^0) \\ &+ M_\xi\,\xi''\,\xi'^0 + g_{10}\,\xi'^0\,(\varphi_T\,\varphi_T)'' + \text{h.o.} \end{aligned} \tag{4.55}$$

As one can see, the T' and Z_3 assignment of the driving fields allows the existence of three mass terms with masses M, M_η and M_ξ, while the couplings $g_{(i)}$ are dimensionless. The ordering of the terms in Eq.(4.55) is such that the terms in the first line involve the driving fields φ_T^0, the ones in the second line only the fields φ_S^0, etc.. The notation of the couplings is according to the one chosen in the A_4 model [15] whose extension this T' model is. As usual +h.o. indicates the existence of higher-dimensional operators which will be discussed below. Since the fields ξ and $\tilde{\xi}$ transform in exactly the same way under T' and Z_3, we can redefine them such that only $\tilde{\xi}$ couples to $\varphi_S^0\,\varphi_S$ (see coupling g_2). As we will see below, the inclusion of the two fields ξ and $\tilde{\xi}$ is necessary, because the field coupled to $\varphi_S^0\,\varphi_S$ has to have a vanishing VEV [14, 15, 18]. Note that neither the field θ nor the MSSM Higgs fields h_u and h_d appear in the superpotential w_d, since all flavon and driving fields are uncharged under $U(1)_{FN}$ and transform non-trivially under T' or Z_3. The explicit form of the T' contractions is given by

$$\begin{aligned}
w_d \;=\;& M\,(\varphi^0_{T\,1}\,\varphi_{T\,1}+\varphi^0_{T\,2}\,\varphi_{T\,3}+\varphi^0_{T\,3}\,\varphi_{T\,2}) \\
+\;& \frac{2}{3}\,g\,(\varphi^0_{T\,1}\,(\varphi^2_{T\,1}-\varphi_{T\,2}\,\varphi_{T\,3})+\varphi^0_{T\,2}\,(\varphi^2_{T\,2}-\varphi_{T\,1}\,\varphi_{T\,3})+\varphi^0_{T\,3}\,(\varphi^2_{T\,3}-\varphi_{T\,1}\,\varphi_{T\,2})) \\
+\;& g_7\,\xi''\,(\varphi^0_{T\,1}\,\varphi_{T\,2}+\varphi^0_{T\,2}\,\varphi_{T\,1}+\varphi^0_{T\,3}\,\varphi_{T\,3}) \\
+\;& g_8\,(i\,\varphi^0_{T\,1}\,\eta_1^2+(1-i)\,\varphi^0_{T\,2}\,\eta_1\,\eta_2+\varphi^0_{T\,3}\,\eta_2^2) \\
+\;& \frac{2}{3}\,g_1\,(\varphi^0_{S\,1}\,(\varphi^2_{S\,1}-\varphi_{S\,2}\,\varphi_{S\,3})+\varphi^0_{S\,2}\,(\varphi^2_{S\,2}-\varphi_{S\,1}\,\varphi_{S\,3})+\varphi^0_{S\,3}\,(\varphi^2_{S\,3}-\varphi_{S\,1}\,\varphi_{S\,2})) \\
+\;& g_2\,\tilde{\xi}\,(\varphi^0_{S\,1}\,\varphi_{S\,1}+\varphi^0_{S\,2}\,\varphi_{S\,3}+\varphi^0_{S\,3}\,\varphi_{S\,2}) \\
+\;& g_3\,\xi^0\,(\varphi^2_{S\,1}+2\,\varphi_{S\,2}\,\varphi_{S\,3}) \\
+\;& g_4\,\xi^0\,\xi^2+g_5\,\xi^0\,\xi\,\tilde{\xi}+g_6\,\xi^0\,\tilde{\xi}^2 \\
+\;& M_\eta\,(\eta_1\,\eta_2^0-\eta_2\,\eta_1^0) \\
+\;& g_9\,(\eta_1^0\,((1-i)\,\eta_1\,\varphi_{T\,3}-\eta_2\,\varphi_{T\,1})-\eta_2^0\,((1+i)\,\eta_2\,\varphi_{T\,2}+\eta_1\,\varphi_{T\,1})) \\
+\;& M_\xi\,\xi''\,\xi'^0 \\
+\;& g_{10}\,\xi'^0\,(\varphi^2_{T\,2}+2\,\varphi_{T\,1}\,\varphi_{T\,3})
\end{aligned} \tag{4.56}$$

Calculating the F-terms for the driving fields leads to two sets of equations

$$\frac{\partial\,w}{\partial\,\varphi^0_{S\,1}}=g_2\,\tilde{\xi}\,\varphi_{S1}+\frac{2\,g_1}{3}\,(\varphi_{S1}^2-\varphi_{S2}\,\varphi_{S3})=0 \tag{4.57a}$$

$$\frac{\partial\,w}{\partial\,\varphi^0_{S\,2}}=g_2\,\tilde{\xi}\,\varphi_{S3}+\frac{2\,g_1}{3}\,(\varphi_{S2}^2-\varphi_{S1}\,\varphi_{S3})=0 \tag{4.57b}$$

$$\frac{\partial\,w}{\partial\,\varphi^0_{S\,3}}=g_2\,\tilde{\xi}\,\varphi_{S2}+\frac{2\,g_1}{3}\,(\varphi_{S3}^2-\varphi_{S1}\,\varphi_{S2})=0 \tag{4.57c}$$

$$\frac{\partial\,w}{\partial\,\xi^0}=g_4\,\xi^2+g_5\,\xi\,\tilde{\xi}+g_6\,\tilde{\xi}^2+g_3\,(\varphi_{S1}^2+2\,\varphi_{S2}\,\varphi_{S3})=0 \tag{4.57d}$$

and

$$\frac{\partial w}{\partial \varphi_{T1}^0} = M\,\varphi_{T1} + \frac{2\,g}{3}\,(\varphi_{T1}^2 - \varphi_{T2}\,\varphi_{T3}) + g_7\,\xi''\,\varphi_{T2} + i\,g_8\,\eta_1^2 = 0 \tag{4.58a}$$

$$\frac{\partial w}{\partial \varphi_{T2}^0} = M\,\varphi_{T3} + \frac{2\,g}{3}\,(\varphi_{T2}^2 - \varphi_{T1}\,\varphi_{T3}) + g_7\,\xi''\,\varphi_{T1} + (1-i)\,g_8\,\eta_1\,\eta_2 = 0 \tag{4.58b}$$

$$\frac{\partial w}{\partial \varphi_{T3}^0} = M\,\varphi_{T2} + \frac{2\,g}{3}\,(\varphi_{T3}^2 - \varphi_{T1}\,\varphi_{T2}) + g_7\,\xi''\,\varphi_{T3} + g_8\,\eta_2^2 = 0 \tag{4.58c}$$

$$\frac{\partial w}{\partial \eta_1^0} = -M_\eta\,\eta_2 + g_9\,((1-i)\,\eta_1\,\varphi_{T3} - \eta_2\,\varphi_{T1}) = 0 \tag{4.58d}$$

$$\frac{\partial w}{\partial \eta_2^0} = M_\eta\,\eta_1 - g_9\,((1+i)\,\eta_2\,\varphi_{T2} + \eta_1\,\varphi_{T1}) = 0 \tag{4.58e}$$

$$\frac{\partial w}{\partial \xi'^0} = M_\xi\,\xi'' + g_{10}\,(\varphi_{T2}^2 + 2\,\varphi_{T1}\,\varphi_{T3}) = 0 \tag{4.58f}$$

Since the superpotential w_d is only linear in the driving fields, these do not appear in their F-terms. The first set of equations Eq.(4.57) only contains the fields $\varphi_{S\,i}$, ξ and $\tilde{\xi}$. The minimization conditions can be solved analytically and allow only a finite set of distinct solutions. In order to select the correct alignment, i.e. the one in which a Z_4 subgroup of T' is preserved, we have to demand that the VEV of $\tilde{\xi}$ vanishes. The equivalent configurations are then [11]

$$\langle\varphi_{S1}\rangle = \langle\varphi_{S2}\rangle = \langle\varphi_{S3}\rangle = v_S\;,\quad \langle\xi\rangle = u \quad\text{and}\quad v_S^2 = -\frac{g_4}{3\,g_3}\,u^2 \tag{4.59a}$$

$$\langle\varphi_{S1}\rangle = \omega\,v_S\;,\quad \langle\varphi_{S2}\rangle = \omega^2\,v_S\;,\quad \langle\varphi_{S3}\rangle = v_S\;,\quad \langle\xi\rangle = u \quad\text{and}\quad v_S^2 = -\frac{g_4}{3\,g_3}\,\omega\,u^2 \tag{4.59b}$$

$$\langle\varphi_{S1}\rangle = \omega^2\,v_S\;,\quad \langle\varphi_{S2}\rangle = \omega\,v_S\;,\quad \langle\varphi_{S3}\rangle = v_S\;,\quad \langle\xi\rangle = u \quad\text{and}\quad v_S^2 = -\frac{g_4}{3\,g_3}\,\omega^2\,u^2 \tag{4.59c}$$

Thereby, $\langle\xi\rangle = u$ remains undetermined. They break to different directions of Z_4 groups. Without loss of generality, we can confine ourselves to the first configuration which has been used in the study of the fermion mass matrices [15]. If $\langle\tilde{\xi}\rangle \neq 0$ was allowed, for example, also the solution

$$\langle\varphi_{S1}\rangle = -\frac{3\,g_2}{2\,g_1}\,\langle\tilde{\xi}\rangle\;,\quad \langle\varphi_{S2}\rangle = \langle\varphi_{S3}\rangle = 0 \tag{4.60}$$

with $\langle\xi\rangle \neq 0$, being calculable with the help of Eq.(4.57d), would arise from Eq.(4.57). This VEV configuration does not preserve a Z_4, but rather a Z_6 subgroup generated by the elements T and $\mathbb{R}$, for example. The neutrino mass matrix derived with this configuration, however, does not lead to Eq.(4.27). A way to achieve $\langle\tilde{\xi}\rangle = 0$ is to demand that the soft SUSY breaking mass $m_{\tilde{\xi}}^2$ is larger than zero. As a last feature of Eq.(4.57), we have to mention that these lead to flat directions in the supersymmetric limit. They can be removed by including soft SUSY breaking terms into the potential and possibly also by the next-to-leading order corrections to the flavon superpotential.

The second set of equations Eq.(4.58) determines the VEV configuration of the flavons, which are responsible for the charged fermion masses at leading order. Again, these equations can be solved analytically and have only a finite number of solutions. Apart from the trivial case, in which all VEVs vanish, the VEV configurations can be divided into three classes, i.e. the first class preserves a Z_4 subgroup, the second one leaves a Z_6 group invariant, while the third contains all configurations

[11] We omit the trivial solution, in which all VEVs vanish.

in which the residual group is a Z_3. Examples for the three classes are

$$\langle \xi'' \rangle = -\frac{M}{g_7}\ , \ \ \langle \eta \rangle = (0,0)\ , \ \ \langle \varphi_T \rangle = (v_T, v_T, v_T) \quad \text{with} \quad v_T^2 = \frac{M\, M_\xi}{3\, g_7\, g_{10}} \tag{4.61a}$$

$$\langle \xi'' \rangle = 0\ , \ \ \langle \eta \rangle = (0,0)\ , \ \ \langle \varphi_T \rangle = (v_T, 0, 0) \quad \text{with} \quad v_T = -\frac{3\,M}{2\,g} \tag{4.61b}$$

$$\langle \xi'' \rangle = 0\ , \ \ \langle \eta \rangle = \pm(v_1,0)\ , \ \ \langle \varphi_T \rangle = (v_T, 0, 0) \tag{4.61c}$$

$$\text{with} \quad v_1 = \frac{1}{g_9\,\sqrt{3\,g_8}}\sqrt{i\,(2\,M_\eta^2\,g + 3\,M\,M_\eta\,g_9)} \quad \text{and} \quad v_T = \frac{M_\eta}{g_9}$$

Note the characteristic features of the three different classes: VEV configurations belonging to the first class do not allow a VEV for the flavons $\eta_{1,2}$ which transform as doublet under T', while configurations belonging to the second class additionally require that the VEV of ξ'' vanishes. Finally, the third class allows a non-vanishing VEV for the flavon η, but no VEV for the field ξ''. All these vacua are degenerate in our model in the limit of unbroken SUSY. However, by choosing the signs of the soft masses of the flavons properly we can single out the third class of solutions, i.e. we have to demand that $m^2_{\xi''} > 0$ and $m^2_\eta < 0$.

Before we discuss the next-to-leading order corrections to the superpotential w_d, we also show the F-terms of the flavons, since they determine the VEVs of the driving fields

$$\begin{aligned}\frac{\partial\, w_d}{\partial\, \varphi_{T\,1}} &= M\,\varphi^0_{T\,1} + \frac{2\,g}{3}\,(2\,\varphi^0_{T\,1}\,\varphi_{T\,1} - \varphi^0_{T\,2}\,\varphi_{T\,3} - \varphi^0_{T\,3}\,\varphi_{T\,2}) + g_7\,\varphi^0_{T\,2}\,\xi'' \\ &\quad - g_9\,(\eta^0_1\,\eta_2 + \eta^0_2\,\eta_1) + 2\,g_{10}\,\xi'^0\,\varphi_{T\,3} = 0\end{aligned} \tag{4.62a}$$

$$\begin{aligned}\frac{\partial\, w_d}{\partial\, \varphi_{T\,2}} &= M\,\varphi^0_{T\,3} + \frac{2\,g}{3}\,(2\,\varphi^0_{T\,2}\,\varphi_{T\,2} - \varphi^0_{T\,1}\,\varphi_{T\,3} - \varphi^0_{T\,3}\,\varphi_{T\,1}) + g_7\,\varphi^0_{T\,1}\,\xi'' \\ &\quad - (1+i)\,g_9\,\eta^0_2\,\eta_2 + 2\,g_{10}\,\xi'^0\,\varphi_{T\,2} = 0\end{aligned} \tag{4.62b}$$

$$\begin{aligned}\frac{\partial\, w_d}{\partial\, \varphi_{T\,3}} &= M\,\varphi^0_{T\,2} + \frac{2\,g}{3}\,(2\,\varphi^0_{T\,3}\,\varphi_{T\,3} - \varphi^0_{T\,1}\,\varphi_{T\,2} - \varphi^0_{T\,2}\,\varphi_{T\,1}) + g_7\,\varphi^0_{T\,3}\,\xi'' \\ &\quad + (1-i)\,g_9\,\eta^0_1\,\eta_1 + 2\,g_{10}\,\xi'^0\,\varphi_{T\,1} = 0\end{aligned} \tag{4.62c}$$

$$\frac{\partial\, w_d}{\partial\, \eta_1} = M_\eta\,\eta^0_2 + g_8\,(2\,i\,\varphi^0_{T\,1}\,\eta_1 + (1-i)\,\varphi^0_{T\,2}\,\eta_2) + g_9\,((1-i)\,\eta^0_1\,\varphi_{T\,3} - \eta^0_2\,\varphi_{T\,1}) = 0 \tag{4.63a}$$

$$\frac{\partial\, w_d}{\partial\, \eta_2} = -M_\eta\,\eta^0_1 + g_8\,((1-i)\,\varphi^0_{T\,2}\,\eta_1 + 2\,\varphi^0_{T\,3}\,\eta_2) - g_9\,(\eta^0_1\,\varphi_{T\,1} + (1+i)\,\eta^0_2\,\varphi_{T\,2}) = 0 \tag{4.63b}$$

$$\frac{\partial\, w_d}{\partial\, \xi''} = M_\xi\,\xi'^0 + g_7\,(\varphi^0_{T\,2}\,\varphi_{T\,1} + \varphi^0_{T\,1}\,\varphi_{T\,2} + \varphi^0_{T\,3}\,\varphi_{T\,3}) = 0 \tag{4.63c}$$

$$\frac{\partial\, w_d}{\partial\, \varphi_{S\,1}} = \frac{2\,g_1}{3}\,(2\,\varphi^0_{S\,1}\,\varphi_{S\,1} - \varphi^0_{S\,2}\,\varphi_{S\,3} - \varphi^0_{S\,3}\,\varphi_{S\,2}) + g_2\,\varphi^0_{S\,1}\,\tilde{\xi} + 2\,g_3\,\xi^0\,\varphi_{S\,1} = 0 \tag{4.64a}$$

$$\frac{\partial\, w_d}{\partial\, \varphi_{S\,2}} = \frac{2\,g_1}{3}\,(2\,\varphi^0_{S\,2}\,\varphi_{S\,2} - \varphi^0_{S\,1}\,\varphi_{S\,3} - \varphi^0_{S\,3}\,\varphi_{S\,1}) + g_2\,\varphi^0_{S\,3}\,\tilde{\xi} + 2\,g_3\,\xi^0\,\varphi_{S\,3} = 0 \tag{4.64b}$$

$$\frac{\partial\, w_d}{\partial\, \varphi_{S\,3}} = \frac{2\,g_1}{3}\,(2\,\varphi^0_{S\,3}\,\varphi_{S\,3} - \varphi^0_{S\,1}\,\varphi_{S\,2} - \varphi^0_{S\,2}\,\varphi_{S\,1}) + g_2\,\varphi^0_{S\,2}\,\tilde{\xi} + 2\,g_3\,\xi^0\,\varphi_{S\,2} = 0 \tag{4.64c}$$

$$\frac{\partial w_d}{\partial \xi} = 2\, g_4\, \xi^0\, \xi + g_5\, \xi^0\, \tilde{\xi} = 0 \tag{4.65a}$$

$$\frac{\partial w_d}{\partial \tilde{\xi}} = g_2\, (\varphi^0_{S\,1}\, \varphi_{S\,1} + \varphi^0_{S\,3}\, \varphi_{S\,2} + \varphi^0_{S\,2}\, \varphi_{S\,3}) + g_5\, \xi^0\, \xi + 2\, g_6\, \xi^0\, \tilde{\xi} = 0 \tag{4.65b}$$

Other parts of the F-terms involving squarks and sleptons do not contribute to the minimization of the potential, since their VEVs have to vanish in order to preserve $SU(3)_C$ and $U(1)_{em}$. As one can see, every term contains one driving field so that all equations are fulfilled, if all VEVs of these fields vanish and no additional constraints are imposed on the VEVs of the flavons φ_S, ξ, $\tilde{\xi}$, φ_T, η and ξ''.

4.5.2 Next-to-Leading Order Results

The next-to-leading order corrections to w_d consist of all terms which are invariant under all symmetries of the model and made up of one driving field and three flavons. These terms are all non-renormalizable and suppressed by the cutoff scale Λ. They perturb the vacuum alignment of the flavons, i.e.

$$\langle \varphi_S \rangle = (v_S, v_S, v_S)\ ,\ \ \langle \varphi_T \rangle = (v_T, 0, 0)\ ,\ \ \langle \eta \rangle = (v_1, 0)\ ,\ \ \langle \xi \rangle = u\ ,\ \ \langle \tilde{\xi} \rangle = 0\ , \langle \xi'' \rangle = 0\ , \tag{4.66}$$

are shifted into

$$\begin{aligned} &\langle \varphi_S \rangle = (v_S + \delta v_{S\,1}, v_S + \delta v_{S\,2}, v_S + \delta v_{S\,3})\ ,\ \ \langle \varphi_T \rangle = (v_T + \delta v_{T\,1}, \delta v_{T\,2}, \delta v_{T\,3})\ , \\ &\langle \eta \rangle = (v_1 + \delta v_1, \delta v_2)\ , \langle \xi \rangle = u\ , \langle \tilde{\xi} \rangle = \delta \tilde{u}\ , \langle \xi'' \rangle = \delta u''\ . \end{aligned} \tag{4.67}$$

Thereby, the corrections $\delta v_{T\,i}$, $\delta v_{S\,i}$, δv_i, $\delta \tilde{u}$ and $\delta u''$ are independent from each other. Note that there might also be a correction to the VEV u, but we do not have to indicate this explicitly by adding a term δu, since u is undetermined at leading order. As we have seen above, the symmetry Z_3 is able to separate the two sets of flavons $\{\varphi_T\ , \eta\ , \xi''\}$ and $\left\{\varphi_S\ , \xi\ , \tilde{\xi}\right\}$ at the leading order. However, a complete separation at next-to-leading order is not possible. Therefore, we expect several terms mixing the fields $\{\varphi_T\ , \eta\ , \xi''\}$ and $\left\{\varphi_S\ , \xi\ , \tilde{\xi}\right\}$. We find that there are 43 independent terms in total contributing at this order

$$\Delta w_{d\,1} = \frac{1}{\Lambda} \left(\sum_{i=3}^{13} t_i\, I^T_i + \sum_{i=1}^{12} s_i\, I^S_i + \sum_{i=1}^{3} x_i\, I^X_i \right) \tag{4.68a}$$

$$\Delta w_{d\,2} = \frac{1}{\Lambda} \left(\sum_{i=14}^{18} t_i\, I^T_i + \sum_{i=13}^{15} s_i\, I^S_i + x_4\, I^X_4 + \sum_{i=1}^{4} n_i\, I^N_i + \sum_{i=1}^{4} y_i\, I^Y_i \right) \tag{4.68b}$$

Their explicit form can be found in Appendix C. The contributions are split up into two classes, $\Delta w_{d\,1}$ and $\Delta w_{d\,2}$. $\Delta w_{d\,1}$ only contains corrections which are already present in the original A_4 model [15], whereas $\Delta w_{d\,2}$ consists of all terms which include at least one driving or flavon field which is only present in the T' model. In this way a comparison of the results here and those found in case of the A_4 model [15] is more transparent. Before presenting the results, we change the notation of the parameters of w_d a bit,

$$g_3 \equiv 3\, \tilde{g}_3^2\ ,\ \ g_4 \equiv -\tilde{g}_4^2 \quad \text{and} \quad g_8 \equiv i\, \tilde{g}_8^2\ , \tag{4.69}$$

such that the VEVs read

$$v_S = \frac{\tilde{g}_4}{3\, \tilde{g}_3}\, u\ ,\ \ v_T = \frac{M_\eta}{g_9} \quad \text{and} \quad v_1 = \frac{1}{\sqrt{3}\, \tilde{g}_8\, g_9} \sqrt{2\, g\, M_\eta^2 + 3\, g_9\, M\, M_\eta} \tag{4.70}$$

where we have chosen the "+" sign for the VEV v_1. In order to calculate the corrections to the VEVs we work along the lines of [15], i.e. we only take into account terms which are at most linear in the shifts of the VEVs and no terms of the order $\mathcal{O}(\frac{\delta \mathrm{VEV}}{\Lambda})$, where Λ is the cutoff scale. If we plug in the VEVs v_T and v_1, the linearized equations for the shifts of the VEVs take the form

$$\frac{\tilde{g}_4\, u^3}{3\,\tilde{g}_3\,\Lambda}\left(t_{11}+\frac{\tilde{g}_4^2}{3\,\tilde{g}_3^2}\,(t_6+t_7+t_8)\right)+\frac{t_3}{\Lambda}\,v_T^3+(1-i)\,\frac{t_{16}}{\Lambda}\,v_1^2\,v_T-2\,v_T\left(\frac{2\,g\,v_T}{3}+M\right)\frac{\delta v_1}{v_1} \tag{4.71a}$$
$$+\left(M+\frac{4\,g\,v_T}{3}\right)\delta v_{T\,1}=0$$

$$\frac{\tilde{g}_4\, u^3}{3\,\tilde{g}_3\,\Lambda}\left(t_{11}+\frac{\tilde{g}_4^2}{3\,\tilde{g}_3^2}\,(t_6+t_7+t_8)\right)+\left(M-\frac{2\,g\,v_T}{3}\right)\delta v_{T\,2}=0 \tag{4.71b}$$

$$\frac{\tilde{g}_4\, u^3}{3\,\tilde{g}_3\,\Lambda}\left(t_{11}+\frac{\tilde{g}_4^2}{3\,\tilde{g}_3^2}\,(t_6+t_7+t_8)\right)+g_7\,v_T\,\delta u''+(1+i)\,v_T\left(\frac{2\,g\,v_T}{3}+M\right)\frac{\delta v_2}{v_1} \tag{4.71c}$$
$$+\left(M-\frac{2\,g\,v_T}{3}\right)\delta v_{T\,3}=0$$

$$\left(\frac{9\,\tilde{g}_3\,s_{10}}{\tilde{g}_4}+\frac{3\,\tilde{g}_4\,s_3}{\tilde{g}_3}+2\,s_6\right)\frac{v_T\,u}{\Lambda}+3\,g_2\,\delta\tilde{u}+2\,g_1\,(2\,\delta v_{S\,1}-\delta v_{S\,2}-\delta v_{S\,3})=0 \tag{4.71d}$$

$$\left(\frac{3\,\tilde{g}_4\,s_4}{\tilde{g}_3}-s_6-\frac{3}{2}\,s_8\right)\frac{v_T\,u}{\Lambda}+3\,g_2\delta\tilde{u}+2\,g_1\,(2\,\delta v_{S\,2}-\delta v_{S\,1}-\delta v_{S\,3})=0 \tag{4.71e}$$

$$\left(\frac{3\,\tilde{g}_4\,s_5}{\tilde{g}_3}-s_6+\frac{3}{2}\,s_8\right)\frac{v_T\,u}{\Lambda}+3\,g_2\,\delta\tilde{u}+2\,g_1\,(2\,\delta v_{S\,3}-\delta v_{S\,1}-\delta v_{S\,2})=0 \tag{4.71f}$$

$$\frac{x_2\,v_T\,u}{3\,\tilde{g}_3\,\Lambda}+\frac{g_5}{\tilde{g}_4}\,\delta\tilde{u}+2\,\tilde{g}_3\,(\delta v_{S\,1}+\delta v_{S\,2}+\delta v_{S\,3})=0 \tag{4.71g}$$

$$v_T\,\delta v_2-\frac{1}{2}(1-i)\,v_1\,\delta v_{T\,3}=0 \tag{4.71h}$$

$$-\frac{1}{2\,\Lambda}\,(1+i)\,n_4\,v_1^2+\frac{n_1}{\Lambda}\,v_T^2+g_9\,\delta v_{T\,1}=0 \tag{4.71i}$$

$$\frac{\tilde{g}_4^2\,y_3\,u^3}{3\,\tilde{g}_3^2\,\Lambda}+M_\xi\,\delta u''+2\,g_{10}\,v_T\,\delta v_{T\,3}=0 \tag{4.71j}$$

As one can see, Eq.(4.71d),Eq.(4.71e), Eq.(4.71f) and Eq.(4.71g) do not receive a contribution from the terms of $\Delta w_{d\,2}$, i.e. the shifts $\delta v_{S\,i}$ and δu are the same as in the A_4 model. Eq.(4.71a), Eq.(4.71b) and Eq.(4.71c) are also correlated to the analogous equations in the A_4 model. In order to see this, one has to set the couplings appearing in $\Delta w_{d\,2}$ to zero and take into account that $v_T = -\frac{3\,M}{2\,g}$ in the A_4 model so that $-2\,v_T\,(\frac{2}{3}\,g\,v_T+M)\,\frac{\delta v_1}{v_1}$ vanishes and expressions like $(M+\frac{4\,g\,v_T}{3})\,\delta v_{T\,1}$ are reduced to $-M\,\delta v_{T\,1}$. Taking this into account Eq.(4.71a), Eq.(4.71b) and Eq.(4.71c) fully coincide with the equations found in [15]. The last three equations are not present in case of A_4 and they simply vanish, if the couplings and the VEVs of the fields only present in case of T' and not A_4 are set to zero. The generic order of all shifts is the square of a VEV of a flavon field over the cutoff scale Λ which is $\lambda^4\,\Lambda$, if the ratio VEV over Λ is around λ^2. Hence, the relative size of a shift compared to a non-vanishing VEV is λ^2. Thereby, it is assumed that all masses, M, M_ξ and M_η, are of the order of the VEVs. This is reasonable, since they are (at least partly) correlated to the VEVs, as one can read off Eq.(4.70). However, the analysis of the quark masses showed that we need to fine-tune the shift of the VEV of $\varphi_{T\,2}$, i.e. $\delta v_{T\,2}$, so that it is of order $\lambda^6\,\Lambda$ instead of order $\lambda^4\,\Lambda$, while all other shifts of the VEVs shall still be of the generic order $\lambda^4\,\Lambda$. First of all, notice that the fine-tuning of $\delta v_{T\,2}$, is just a mild version of the extreme case that the four couplings $t_{6,7,8,11}$ vanish. Then Eq.(4.71b) leads to $\delta v_{T\,2}=0$ [12]. Eq.(4.71i) shows that

[12] The case, in which the bracket in front of $\delta v_{T\,2}$ vanishes, is highly tuned, since then several uncorrelated param-

$\delta v_{T\,1}$ is of the order $n_1\,\mathcal{O}\left(\frac{v_T^2}{\Lambda}\right) + n_4\,\mathcal{O}\left(\frac{v_1^2}{\Lambda}\right)$ and without any accidental cancellations $\delta v_{T\,1}$ will be of order $\frac{v_T^2}{\Lambda} \sim \lambda^4\,\Lambda$. Plugging Eq.(4.71i) into Eq.(4.71a) one arrives at terms of the order

$$t_3\,\mathcal{O}\left(\frac{v_T^3}{\Lambda}\right) + t_{16}\,\mathcal{O}\left(\frac{v_1^2\,v_T}{\Lambda}\right) + \mathcal{O}\left(v_T\right)\,\delta v_1 + \mathcal{O}\left(\frac{v_T^3}{\Lambda}\right) + \mathcal{O}\left(\frac{v_1^2\,v_T}{\Lambda}\right) = 0 \tag{4.72}$$

so that δv_1 is naturally of the order $\mathcal{O}\left(\frac{v_T^2}{\Lambda}\right)$. From Eq.(4.71h) we can read off that δv_2 is proportional to $\delta v_{T\,3}$. Furthermore, Eq.(4.71j) tells us that $\delta u''$ is of the order $y_3\,\mathcal{O}\left(\frac{u^3}{M_\xi\,\Lambda}\right) + \mathcal{O}\left(\frac{v_T}{M_\xi}\right)\,\delta v_{T\,3}$ so that its natural order is $\mathcal{O}\left(\frac{v_T^2}{\Lambda}\right)$ in case that the order of $\delta v_{T\,3}$ is $\mathcal{O}\left(\frac{v_T^2}{\Lambda}\right)$ which we will show in a moment. Plugging Eq.(4.71h) and Eq.(4.71j) into Eq.(4.71c) we can determine the natural size of the VEV $\delta v_{T\,3}$ by

$$y_3\,\mathcal{O}\left(\frac{u^3\,v_T}{M_\xi\,\Lambda}\right) + \mathcal{O}\left(M\right)\,\delta v_{T\,3} + \mathcal{O}\left(\frac{v_T^2}{M_\xi}\right)\,\delta v_{T\,3} = 0 \tag{4.73}$$

to be $\delta v_{T\,3} \sim \mathcal{O}\left(\frac{v_T^2}{\Lambda}\right)$, if none of the couplings is fine-tuned such that some of the terms cancel. Additionally, Eq.(4.71d), Eq.(4.71e), Eq.(4.71f) and Eq.(4.71g) are not influenced by the fine-tuning of $\delta v_{T\,2}$, since they neither contain $\delta v_{T\,2}$ nor the parameters $t_{6,7,8,11}$.

4.6 Conclusions and Comments

In this chapter we augmented the MSSM by the flavor symmetry T' . This model has several salient features

- It predicts TBM in the lepton sector

$$\sin^2(\theta_{12}^{TBM}) = \frac{1}{3}\;, \quad \sin^2(\theta_{23}^{TBM}) = \frac{1}{2}\;, \quad \sin^2(\theta_{13}^{TBM}) = 0\;. \tag{4.74}$$

- It also predicts two non-trivial relations among $|V_{us}|$, $|V_{td}/V_{ts}|$ and m_d/m_s

$$\sqrt{\frac{m_d}{m_s}} = |V_{us}| + \mathcal{O}(\lambda^2) \;\text{ and }\; \sqrt{\frac{m_d}{m_s}} = \left|\frac{V_{td}}{V_{ts}}\right| + \mathcal{O}(\lambda^2) \;\text{ (due to }\; |V_{ub}| \sim \mathcal{O}(\lambda^4)\;)\;. \tag{4.75}$$

- It connects the large lepton and small quark mixings with the breaking of the flavor symmetry, i.e. lepton mixings turn out to be large, since the subgroups which are preserved in the neutrino and charged lepton sector are different, while the flavor symmetry T' is broken to the same subgroup in the up and down quark sector. The necessary separation of the neutral and charged fermion sector can be maintained by a Z_3 symmetry.

- The problem of the vacuum alignment is (almost) solved (up to a small number of degeneracies) by a suitable construction of the flavon potential. This includes the introduction of additional gauge singlet fields as well as of an additional $U(1)$ symmetry, called $U(1)_R$.

- Compared to the two models presented in Chapter 3 it does not suffer from the problem that additional Higgs doublets can cause large FCNCs, since the flavor symmetry is only broken by gauge singlets.

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eters of w_d have to conspire.

We worked out the fermion masses and mixings at leading as well as next-to-leading order and showed that TBM -as a result of the leading order- is only slightly corrected by next-to-leading order effects, while features not maintained by the leading order, i.e. the masses of the quarks of the first generation and the Cabibbo angle, can be generated at next-to-leading order. Up to a minor fine-tuning of two parameters this model is free from any special parameter choices and can naturally accommodate all data. Thereby, the hierarchy among the fermion masses is not completely reproduced by the discrete non-abelian flavor symmetry, but rather by the Froggatt-Nielsen mechanism which involves an additional $U(1)$ group, denoted by $U(1)_{FN}$. One may argue that these additional symmetries make the model complicated, however one has to keep in mind that they are not simply chosen in order to suppress a certain coupling compared to another, but act in a well-defined way, i.e. T' leads to the mixing structure, Z_3 achieves the separation of the neutral and charged fermion sector, $U(1)_{FN}$ allows all fermion mass hierarchies to be realized, and $U(1)_R$ is employed for the vacuum alignment.
Since this model is successful in describing quarks and leptons, one may search for further possible signatures. These include the study of LFVs as well as FCNCs which do not arise in this model via the mediation of additional Higgs fields, but through the superpartners of the SM fermions. It is very interesting to investigate how powerful the flavor symmetry is in suppressing these effects which turn out to be very large in generic MSSM models and which are far above the experimental bounds, if no special assumptions, such as mSUGRA initial conditions, are made. Since the gauge singlet potential contains flat directions in the supersymmetric limit, one can also pose the question whether one combination of the gauge singlets could play the role of an inflaton. To find additional signatures is also relevant in order to differentiate among the existing models which all share the feature that they can (more or less) explain or accommodate the fermion masses and their mixings.

However, the model is not perfect. In the following, we mention some of the unsolved issues. The model predicts TBM due to the fact that different subgroups in the neutrino and the charged lepton sector are preserved. But, we additionally need to choose the transformation properties of the gauge singlets properly [15, 24], which couple to neutrinos at leading order. To be precise, we have to exclude the existence of gauge singlets transforming as non-trivial singlets under T'. If the model also contained gauge singlets χ' and χ'' transforming as $(1', \omega, 0)$ and $(1'', \omega, 0)$ under $(T', Z_3, U(1)_{FN})$, there would exist two further contributions to the light neutrino mass matrix originating from the terms $\frac{z'}{\Lambda^2}\,\chi'\,(l_2\,l_2 + l_1\,l_3 + l_3\,l_1)\,h_u\,h_u$ and $\frac{z''}{\Lambda^2}\,\chi''\,(l_3\,l_3 + l_1\,l_2 + l_2\,l_1)\,h_u\,h_u$ so that $\widetilde{M_\nu}$ would read

$$\widetilde{M_\nu} = \begin{pmatrix} a + \frac{2}{3}\,b & -\frac{1}{3}\,b + z''\,\frac{\langle\chi''\rangle}{\Lambda} & -\frac{1}{3}\,b + z'\,\frac{\langle\chi'\rangle}{\Lambda} \\ -\frac{1}{3}\,b + z''\,\frac{\langle\chi''\rangle}{\Lambda} & \frac{2}{3}\,b + z'\,\frac{\langle\chi'\rangle}{\Lambda} & a - \frac{1}{3}\,b \\ -\frac{1}{3}\,b + z'\,\frac{\langle\chi'\rangle}{\Lambda} & a - \frac{1}{3}\,b & \frac{2}{3}\,b + z''\,\frac{\langle\chi''\rangle}{\Lambda} \end{pmatrix} \frac{v_u^2}{\Lambda}\,. \tag{4.76}$$

$\widetilde{M_\nu}$ does not lead to TBM in general, but also this matrix preserves the Z_4 group generated by $T\,S\,T^2$, since $T\,S\,T^2 = 1$ for the representations $1'$ and $1''$. Therefore, the requirement to preserve this Z_4 group is not sufficient to explain TBM. Since the lepton sector of the T' model is an exact copy of the one of the A_4 model, also the A_4 model suffers from this disadvantage. As our model is an effective theory, we might not be able to reduce the theoretical uncertainties in the predictions, shown above, without constructing a high energy completion of the model in which all non-renormalizable couplings arise at a renormalizable level. $U(1)_R$ forbids a μ term in this model and couplings of h_u and h_d to the driving fields do not lead to an effective μ term, since the VEVs of these fields vanish. However, electroweak symmetry breaking may be induced radiatively. Since the fermions transform in different ways under T', the model can neither be embedded into a GUT, like $SU(5)$ or $SO(10)$, nor can the flavor symmetry T' be embedded into a continuous group, like $SU(2)_f$ or $SU(3)_f$ without additional fields which complete the multiplets. Especially, the lepton assignment necessary to arrive at TBM can hardly be reconciled with the embedding into $SU(2)_f$ or $SU(3)_f$. This is quite different compared to the two models presented in Chapter 3 in which the embedding into a GUT has been one of the selection criteria for the fermion assignments.

From the viewpoint of flavor model building the T' model (and the A_4 model [15, 20]) teach(es) us two important lessons

- the actual prediction of a certain mixing angle is intimately related to the preservation of certain subgroups of a flavor symmetry,
- the vacuum alignment plays a crucial role for the preservation of the subgroups and can be implemented, as shown here, with several flavored gauge singlets and a $U(1)_R$ symmetry.

The first aspect triggers three important questions which are tackled in the next chapter

- Are the group T' and its single-valued group A_4 the only symmetries which allow for such an interpretation of the fermion mixings ?
- Can we systematically study discrete non-abelian groups as flavor symmetries, if we adopt the concept of preserved subgroups ?
- Can we also predict other mixing angles, e.g. the Cabibbo angle θ_C in the quark sector ? Can we explain that in the lepton sector θ_{23} is maximal and $\theta_{13} = 0$ without constraining θ_{12} ? [13]

[13] This is exactly the result of $\mu\tau$ symmetric neutrino mass matrices in the basis in which the charged leptons are diagonal.

Chapter 5

Studies of Dihedral Flavor Symmetries

In this chapter we systematically study the fermion mass matrix structures which arise from a class of discrete groups with the requirement that the flavor symmetry is not broken in an arbitrary way, but (different) subgroups have to be preserved in all cases [1]. This requirement is inspired by the success of the T' model which has been presented in the preceding chapter. The class of discrete groups, which we are going to investigate, are the dihedral groups D_n and their double-valued counterparts D'_n. These groups exist for all $n \in \mathbb{N}$ and therefore constitute an infinite series. They share several properties, e.g. they only contain one- and two-dimensional irreducible representations. D_n is the symmetry group of a regular planar n-gon and well-known in solid state as well as molecular physics. In order to perform such a study we first need to discuss the group theory of D_n and D'_n groups including general formulae for Kronecker products and Clebsch Gordan coefficients as well as the investigation of subgroup structures. In a second step the Dirac mass matrix structures can be calculated and classified according to five basic forms. Similarly, this can be done for the case of Majorana fermions. In the following, we introduce the mathematics of dihedral groups in Section 5.1 and then present in Section 5.2 the general study of the mass matrix structures. Thereby, we will not repeat the detailed discussion laid out in the recently published paper [25], but will rather give a summary of the results found there and elucidate the methods which have been employed. Complementarily to [25], we will investigate three examples in detail in Section 5.3 and Section 5.4 which have been mentioned very briefly in [25]. The first one is the explanation of the Cabibbo angle θ_C with the dihedral group D_7, which has also been published only recently [26] [2]. In the second and third example we analyze existing models [94,116] which use the flavor symmetry D_4 and D_3, respectively, to predict maximal atmospheric mixing and vanishing θ_{13} in the lepton sector. Finally, we summarize and give a short outlook in Section 5.5.

5.1 Group Theory of Dihedral Symmetries

5.1.1 Group Theory of D_n

In the series of dihedral groups D_n all groups with n equal or larger three are non-abelian. The abelian groups D_1 and D_2 are isomorphic to the groups Z_2 and the Klein group $Z_2 \times Z_2$, respec-

[1] The preservation of a certain subgroup is equivalent to demanding that the mass matrix stays invariant, if a certain transformation is applied. This has also been discussed in [114].

[2] The author of [115] made a similar comment on the possible origin of the Cabibbo angle.

tively. The smallest non-abelian D_n group is at the same time the smallest non-abelian group among all groups. The order of D_n is $2\,n$. It only contains real one- and two-dimensional representations. For n even, four representations are one-dimensional, $\underline{\mathbf{1}}_{\mathbf{i}}$ with $\mathrm{i} = 1, .., 4$, and $\frac{n}{2} - 1$ are two-dimensional, $\underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{j} = 1, ..., \frac{n}{2} - 1$. For n odd, each group D_n possesses two one-dimensional, $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{1}}_{\mathbf{2}}$, and $\frac{n-1}{2}$ two-dimensional representations $\underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{j} = 1, ..., \frac{n-1}{2}$. For arbitrary n, the group D_n can be described by two generators, A and B, and their relations

$$\mathrm{A}^n = \mathbb{1} \quad , \quad \mathrm{B}^2 = \mathbb{1} \quad , \quad \mathrm{A\,B\,A} = \mathrm{B}\,. \tag{5.1}$$

A convenient choice [86] of the generators A and B for the two-dimensional representations $\underline{\mathbf{2}}_{\mathbf{j}}$ is

$$\mathrm{A} = \left(\begin{array}{cc} \mathrm{e}^{\left(\frac{2\pi i}{n}\right)\mathrm{j}} & 0 \\ 0 & \mathrm{e}^{-\left(\frac{2\pi i}{n}\right)\mathrm{j}} \end{array} \right) \; , \; \mathrm{B} = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \tag{5.2}$$

with $\mathrm{j} = 1, \dots, \frac{n}{2} - 1$ for n even and $\mathrm{j} = 1, \dots, \frac{n-1}{2}$ for n odd. The generators A and B for the one-dimensional representations can be found in the character tables, which are displayed in a general form in Appendix B.4.1. Since we choose complex generators for real representations, the complex conjugates of A and B are linked to A and B by a similarity transformation U

$$U = \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) \,. \tag{5.3}$$

As a consequence, for all $\left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{j}}$ the combination $\left(\begin{array}{c} a_2^\star \\ a_1^\star \end{array} \right)$ transforms as $\underline{\mathbf{2}}_{\mathbf{j}}$.

In Section 3.3.1 and Section 5.3.1 the group theory of the dihedral symmetries D_5 and D_7 is explicitly shown. All statements given there are only special cases of the statements made here for dihedral groups D_n with arbitrary index n.

Kronecker products and Clebsch Gordan coefficients can be found in Appendix B.4.2 and Appendix B.4.3.

5.1.2 Group Theory of D'_n

D'_n are the double-valued counterparts of D_n. Apart from D'_1 which is isomorphic to the cyclic group Z_4, all other groups of the series D'_n are non-abelian. The group D'_2 is also called quaternion group and therefore abbreviated by Q. According to this, the other groups D'_n are sometimes denoted by $Q_{2\,n}$. Since they are double-valued, the order of D'_n is $4\,n$. Similar to the single-valued groups D_n, they only contain one- and two-dimensional irreducible representations. For all n, D'_n has four one- and $n-1$ two-dimensional representations. For n even, the one-dimensional representations and the representations $\underline{\mathbf{2}}_{\mathbf{j}}$ with j even are real, whereas $\underline{\mathbf{2}}_{\mathbf{j}}$ with j odd are pseudo-real. In case of n odd, the one-dimensional representations $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{1}}_{\mathbf{2}}$ are real and $\underline{\mathbf{1}}_{\mathbf{3}}$ and $\underline{\mathbf{1}}_{\mathbf{4}}$ are complex (conjugated). Similar to n even, the two-dimensional representations with an even index are real and the ones, whose index is odd, are pseudo-real. Therefore, adding the pseudo-real and complex representations to the groups D_n leads to the groups D'_n. Accordingly, the real representations of D'_n are sometimes called even or single-valued, whereas the pseudo-real and complex representations are denoted as odd or double-valued. Note in this context that only a pseudo-real two-dimensional representation can be faithful in the group D'_n [3]. The generators A and B of D'_n fulfill relations being very similar to the ones for the generators of D_n

$$\mathrm{A}^n = \mathbb{R} \quad , \quad \mathrm{B}^2 = \mathbb{R} \quad , \mathbb{R}^2 = \mathbb{1} \quad , \quad \mathrm{A\,B\,A} = \mathrm{B}\,, \tag{5.4}$$

[3] In case of the abelian group D'_1 the two complex representations, $\underline{\mathbf{1}}_{\mathbf{3}}$ and $\underline{\mathbf{1}}_{\mathbf{4}}$, are faithful, whereas the real ones, $\underline{\mathbf{1}}_{\mathbf{1}}$ and $\underline{\mathbf{1}}_{\mathbf{2}}$, are not.

with $\mathbb{R}$ being $\mathbb{1}$ in case of an even representation and $-\mathbb{1}$ for an odd one. For the two-dimensional representations A and B can be chosen as

$$\mathrm{A} = \begin{pmatrix} \mathrm{e}^{\left(\frac{\pi i}{n}\right)\mathrm{j}} & 0 \\ 0 & \mathrm{e}^{-\left(\frac{\pi i}{n}\right)\mathrm{j}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad \mathrm{j} \quad \text{even} \tag{5.5}$$

and

$$\mathrm{A} = \begin{pmatrix} \mathrm{e}^{\left(\frac{\pi i}{n}\right)\mathrm{j}} & 0 \\ 0 & \mathrm{e}^{-\left(\frac{\pi i}{n}\right)\mathrm{j}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \quad \text{for} \quad \mathrm{j} \quad \text{odd} \; . \tag{5.6}$$

As usual, the generators for the one-dimensional representations are displayed in the character tables which can be found in Appendix B.4.1. Note that the generator B has an additional factor i for odd representations compared to even ones. For the two-dimensional representations which are either real or pseudo-real, but not complex, we can again find a similarity transformation U, connecting $\mathrm{A}^\star$, $\mathrm{B}^\star$ with A, B. For a representation $\underline{\mathbf{2}}_{\mathbf{j}}$ with j even, it is the same as in the case of the groups D_n, i.e.

$$U = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad \mathrm{j} \quad \text{even} \; , \tag{5.7}$$

while we have to use

$$U = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{for} \quad \mathrm{j} \quad \text{odd} \; . \tag{5.8}$$

Hence, for $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}}$ the combination $\begin{pmatrix} a_2^\star \\ a_1^\star \end{pmatrix}$ transforms as $\underline{\mathbf{2}}_{\mathbf{j}}$, if j is even, and $\begin{pmatrix} -a_2^\star \\ a_1^\star \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}}$, if j is odd.
Kronecker products and Clebsch Gordan coefficients can be found in Appendix B.4.2 and Appendix B.4.4.

5.2 General Results

We present the general results of the systematic study of Dirac mass matrix structures which can arise from a dihedral flavor symmetry, if one of its subgroups remains preserved by the VEVs of the scalar fields. We impose the following constraints in our study

1. At least two of the left-handed or left-handed conjugate fermions have to form an irreducible two-dimensional representation of the dihedral group. Only these assignments are able to reveal the non-abelian nature of the group. Cases in which all fermions are assigned to one-dimensional representations can always be reproduced with an abelian symmetry, like Z_2. Essentially, two different assignment structures have to be taken into account

   a.) The left-handed fermions transform as $\underline{\mathbf{1}}_{\mathbf{i}} + \underline{\mathbf{2}}_{\mathbf{j}}$ and the left-handed conjugate fermions as $\underline{\mathbf{1}}_{\mathbf{l}} + \underline{\mathbf{2}}_{\mathbf{k}}$, where the one- and two-dimensional representations can be inequivalent, i.e. $\mathrm{i} \neq \mathrm{l}$ and $\mathrm{j} \neq \mathrm{k}$ is allowed. We call this assignment the two doublet structure.

   b.) The left-handed fermions transform in the same way as in *a.*), i.e. $L \sim \underline{\mathbf{1}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{j}}$, but the left-handed conjugate fields do not unify under the flavor symmetry and are assigned to three one-dimensional representations which may or may not be equivalent, $\underline{\mathbf{1}}_{\mathrm{i}_1} + \underline{\mathbf{1}}_{\mathrm{i}_2} + \underline{\mathbf{1}}_{\mathrm{i}_3}$ [4]. In the following this is called the three singlet structure.

The analogous assignment in which the left-handed fields do not unify, but the left-handed conjugate ones are assigned to the representation structure $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ is implicitly included in this study, since the exchange of the transformation properties of left-handed and left-handed conjugate fields leads to a transposition of the resulting mass matrix. This transposition, however, does not change the group theoretical part of the analysis, but can have phenomenological implications on the fermion mixing matrices. Similarly, permutations of the three generations do not change anything in the group theoretical part of the discussion.

2. The determinant of the mass matrix has to be non-vanishing. This is required, since the number of distinct matrix structures is efficiently reduced by this constraint so that a complete study becomes possible. Furthermore, this assumption is accordance with phenomenology, as we know that the masses of all charged fermions are non-vanishing. In case of neutrinos this is no longer true, because the data still allow one of the (light) neutrinos to be massless.

3. All Higgs fields which are allowed to have a non-vanishing VEV, since this VEV (structure) preserves the subgroup, are included into the model. In this way the mass matrix structures are only determined by the assignments of the fermions and the group theory of the dihedral symmetries, but not by the arbitrary choice of scalar fields [5]. However, Higgs fields can be easily eliminated from the model by setting their VEVs to zero. For the calculation of the forms of the mass matrices it is sufficient to encounter one Higgs field which transforms under a certain representation of the dihedral group, i.e. we do not consider the case in which two Higgs fields have exactly the same transformation properties under the flavor group [6].

4. The framework in which the mass matrices are calculated is the SM. Thereby, we assume that all Higgs fields in the model are copies of the SM Higgs doublet, i.e. transform as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under $SU(3)_C \times SU(2)_L \times U(1)_Y$. For this reason, the mass matrix structures shown in the following arise for down quarks and charged leptons which couple to the Higgs field itself. Since the mass matrices for the up quark and Dirac neutrinos arise from a coupling to the conjugated Higgs field, we have to take into account slight changes due to the fact that, for example, the representation matrices of $\underline{\mathbf{2}}_{\mathbf{j}}$ of D_n and D'_n are chosen to be complex.

We find only five distinct Dirac mass matrix structures [7] [8]

$$M_1 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & 0 \\ 0 & 0 & C \end{pmatrix} , \quad M_2 = \begin{pmatrix} A & 0 & 0 \\ 0 & 0 & B \\ 0 & C & 0 \end{pmatrix} , \tag{5.9}$$

[4] Note that we choose the assignment as given in the second work on dihedral symmetries [26]. In the assignment presented in [25] the left-handed fields transform as one-dimensional representations and the left-handed conjugate fermions are partially unified. As argued above, this does not change the results.

[5] In case of the T' model presented in Chapter 4 this choice is crucial in order to arrive at the TBM.

[6] However, in a complete model this might be necessary, see, for example, Chapter 4. In this case the flavon potential enforces the existence of two fields with exactly the same transformation properties under T' and Z_3.

[7] As explained in detail in [25] there exist cases which lead to mass matrices with an arbitrary structure. However, a careful analysis shows that in all these cases the flavor symmetry can be reduced to a smaller group which is actually fully broken and therefore, strictly speaking, these matrices do not result from the preservation of a subgroup.

[8] Other mass matrices which do not result from the preservation of a subgroup might also have a certain structure with several elements being correlated, see, for example, Chapter 3. However, these correlations are then due to the fact that the discrete group used as flavor symmetry is non-abelian.

$$M_3 = \begin{pmatrix} A & 0 & 0 \\ 0 & B & C \\ 0 & D & E \end{pmatrix} , \tag{5.10}$$

$$M_4 = \begin{pmatrix} 0 & A & B \\ C & D & E \\ -C\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & D\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & E\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} \end{pmatrix} \quad \text{and} \quad M_5 = \begin{pmatrix} A & C & C\,\mathrm{e}^{-i\,\phi\,\mathrm{k}} \\ B & D & E \\ B\,\mathrm{e}^{-i\,\phi\,\mathrm{j}} & E\,\mathrm{e}^{-i\,\phi\,(\mathrm{j}-\mathrm{k})} & D\,\mathrm{e}^{-i\,\phi\,(\mathrm{j}+\mathrm{k})} \end{pmatrix} \tag{5.11}$$

where A, B, C, D, E are complex numbers which are products of Yukawa couplings and VEVs, $\phi = \frac{2\,\pi}{n}\,m$ (n: index of the dihedral group, m: index of the breaking direction) and j, k are the indices of the representations $\underline{\mathbf{2}}_{\mathbf{j}}$ and $\underline{\mathbf{2}}_{\mathbf{k}}$.
The changes which have to be encountered for up quarks and Dirac neutrinos lead to slightly different forms of the matrix structures

$$M_4 = \begin{pmatrix} 0 & A & B \\ C\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & D\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & E\,\mathrm{e}^{i\,\phi\,\mathrm{j}} \\ -C & D & E \end{pmatrix} \quad \text{and} \quad M_5 = \begin{pmatrix} A & C\,\mathrm{e}^{i\,\phi\,\mathrm{k}} & C \\ B\,\mathrm{e}^{i\,\phi\,\mathrm{j}} & D\,\mathrm{e}^{i\,\phi\,(\mathrm{j}+\mathrm{k})} & E\,\mathrm{e}^{i\,\phi\,(\mathrm{j}-\mathrm{k})} \\ B & E & D \end{pmatrix} . \tag{5.12}$$

M_1, M_2 and M_3, i.e. the diagonal, the semi-diagonal and the block matrix structure, arise quite frequently and can be found for different preserved subgroups. Thereby, the entries A, B, ... of the mass matrices can be correlated depending on the fermion assignment and the subgroup to which the dihedral symmetry is broken. For example, in the semi-diagonal matrix B and C have to be equal in several cases. In contrast to this, the structures M_4 and M_5 can only be found, if the preserved subgroup is of the form $Z_2 =< \mathrm{B}\,\mathrm{A}^m >$, where m is encoded in ϕ. More precisely, M_4 stems from the three singlet structure, while M_5 is a result of the two doublet structure.
The results for Majorana mass matrix structures are very similar apart from four differences

a.) Since Majorana mass terms correlate either left-handed or left-handed conjugate fields, only mass matrix structures, which correspond to forms, resulting from the two doublet structure with equivalent one- and two-dimensional representations, are allowed.

b.) We have to take into account the case in which all fermions involved in the mass terms transform as one-dimensional representations under the dihedral group. We can, for example, imagine the case in which we assign the left-handed and left-handed conjugate neutrinos to the three singlet structure so that a Majorana mass term for the left-handed conjugate neutrinos actually stems from an assignment in which all fields transform as singlets. These additional mass matrix structures can be either diagonal, can have block structure, can be completely arbitrary, i.e. all mass matrix entries are non-vanishing and not correlated, or can be semi-diagonal, if the dihedral group is D'_n with n odd. As one can see, all four types of structures are not new in the sense that they already arose in the discussion of the Dirac mass matrix structures.

c.) Since Majorana mass terms correlate the same fields with each other, these terms have to be symmetric. This leads to the fact that in some cases contributions from Higgs fields, which are allowed a VEV according to the preservation of the subgroup, vanish, since they are anti-symmetric in flavor space.

d.) Obviously, the Higgs fields involved in Majorana mass terms for left-handed and left-handed conjugate neutrinos cannot transform as a copy of the SM Higgs doublet. As explained in

Section 3.1, they must be either Higgs triplets or gauge singlets. In the case of the left-handed conjugate neutrinos a direct mass term is allowed, if it is also invariant under the dihedral symmetry.

As discussed in detail in [25], there is no difference between the groups D_n and their double-valued counterparts D'_n [9] concerning the mass matrix structures which can be produced, if we demand a subgroup of the dihedral group to be preserved.

In the following we elucidate the methods to arrive at these five distinct mass matrix structures.

- In a first step the general structure of the subgroups of D_n and D'_n has to be determined. This is done with the help of the generators A and B given in Section 5.1. As can be shown, all elements of a dihedral group can be written in the form A^x or $\mathrm{B}\,\mathrm{A}^y$. We can then find eigenvalues and eigenvectors of all elements of the group. Thereby, only the case in which the element has an eigenvalue one is interesting. A subgroup consists of all elements which have an eigenvalue one for a certain eigenvector. Concerning the one-dimensional representations we only have to collect the elements whose character is one. These automatically form a subgroup. In case of a two-dimensional representation we actually have to find the representation matrices for all elements and calculate the eigenvalues and eigenvectors of the two-by-two matrices. In general, only two possible forms of eigenvectors arise

$$v \propto \begin{pmatrix} \mathrm{e}^{-\frac{4\pi i \mathrm{j} m}{g}} \\ 1 \end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \quad \text{with} \quad x_1, x_2 \quad \text{arbitrary} . \tag{5.13}$$

  g denotes the order of the dihedral group ($g = 2\,n$ for D_n and $g = 4\,n$ for D'_n), j is the representation index of $\underline{\mathbf{2}}_{\mathbf{j}}$ and m indicates that the first eigenvector belongs to the element $\mathrm{B}\,\mathrm{A}^m$ with m being an integer. It is responsible for the special structure of the mass matrix forms M_4 and M_5. The second eigenvector is a vector with arbitrary entries, i.e. it can only belong to the unit matrix. This eigenvector only arises in the case of a so-called unfaithful representation in which apart from the identity element E also non-trivial elements of the group are represented by the unit matrix. These elements always form a group. By combining at most two representations we can find all possible subgroups of D_n and D'_n. These turn out to be either dihedral groups themselves or cyclic groups. The smallest subgroup is in general a Z_2 group. The structure of all subgroups is given in terms of the generators A and B of the original group. Note that only the two subgroup structures $Z_2 =< \mathrm{B}\,\mathrm{A}^m >$ and $D_{\mathrm{j}} =< \mathrm{A}^{\frac{n}{\mathrm{j}}}, \mathrm{B}\,\mathrm{A}^m >$ for D_n and $Z_4 =< \mathrm{B}\,\mathrm{A}^m >$ and $D'_{\frac{\mathrm{j}}{2}} =< \mathrm{A}^{\frac{2n}{\mathrm{j}}}, \mathrm{B}\,\mathrm{A}^m >$ for D'_n are compatible with the first eigenvector structure.

- The next step is to find the decomposition of all representations of the dihedral groups into irreducible representations of all their subgroups. Thereby, one has to note that the two-dimensional representations of the original group break up into one-dimensional ones of the subgroups in several cases. A complete list can be found in [25].

- The physical interpretation of these results is then: All Higgs fields transforming according to a representation which contains a trivial representation of a certain subgroup can preserve this subgroup, if their VEV is of the form that only the combination of components which

[9] The only slight difference found in case of Majorana mass matrix structures is not relevant, since the new structure, which is allowed, if the flavor group is a D'_n group and not a D_n group, can arise with another fermion assignment also from a D_n group and therefore is not a unique result of a D'_n group.

transforms trivially under the subgroup gets a non-vanishing VEV. As one can show, this is equivalent to enforce the structure of the VEV to be proportional to the eigenvector of the eigenvalue one of the generators of the subgroups. For Higgs fields transforming as an one-dimensional representation this corresponds to the fact that they are allowed to have a non-vanishing VEV, if the characters belonging to the generators of the subgroup are one. If the Higgs fields form on the other hand an irreducible doublet under the dihedral group, it is important whether the VEVs of the Higgs field ψ_1, being the upper, and the field ψ_2, being the lower component of the doublet, fulfill a relation or not: *a.*) If they are independent, this configuration corresponds to an arbitrary eigenvector; *b.*) If $<\psi_1>=<\psi_2>\mathrm{e}^{-\frac{4\pi i \mathrm{j} m}{g}}$ (j: representation index, g: group order of the dihedral group, m: index of the preserved direction) holds, the structure is related to the first eigenvector of Eq.(5.13). A complete list of the subgroups of dihedral symmetries can be found in [25] together with an enumeration of the representations which are allowed to have a non-vanishing VEV in order to preserve a certain subgroup.

- The final step is then the calculation of all mass matrices with the two possible assignment structures of the fermions and for the different possible VEV structures of the Higgs fields which preserve the different subgroups.

All these steps have been performed in [25] with the result shown above. Since we do not want to repeat the details of these calculations, we instead concentrate on three interesting applications of these results: in the first one we derive an expression for the Cabibbo angle θ_C with the help of the flavor symmetry D_7, while the two other ones are models found in the literature which predict $\theta_{23}=\frac{\pi}{4}$ and $\theta_{13}=0$ in the lepton sector. Thereby, we will explain the group theoretical background in detail and perform several of the above mentioned steps explicitly.

In order to show that the Cabibbo angle can be deduced from a dihedral group, the two matrices M_4 and M_5 deserve a further investigation. Therefore, we calculate $M_i\,M_i^\dagger$, $i=4,5$, which can be written in the general form

$$\left(\begin{array}{ccc} a & b\,\mathrm{e}^{i\,\beta} & b\,\mathrm{e}^{i\,(\beta+\phi\mathrm{j})} \\ b\,\mathrm{e}^{-i\,\beta} & c & d\,\mathrm{e}^{i\,\phi\mathrm{j}} \\ b\,\mathrm{e}^{-i\,(\beta+\phi\mathrm{j})} & d\,\mathrm{e}^{-i\,\phi\mathrm{j}} & c \end{array}\right) \tag{5.14}$$

where a, b, c, d and β are real functions of A, B, C, D and E. β lies in the interval $[0,2\,\pi)$. Since we work in the basis in which the left-handed fields are on the left-hand side and the left-handed conjugate fields on the right-hand side (see Section 3.1), the unitary matrix diagonalizing $M_i\,M_i^\dagger$ acts on the left-handed fields. The three eigenvalues are given as $(c-d)$, $\frac{1}{2}\,(a+c+d-\sqrt{(a-c-d)^2+8\,b^2})$ and $\frac{1}{2}\,(a+c+d+\sqrt{(a-c-d)^2+8\,b^2})$. Assuming this ordering of the eigenvalues the mixing matrix U which fulfills $U^\dagger\,M_i\,M_i^\dagger\,U=\mathrm{diag}$ is of the form

$$U=\left(\begin{array}{ccc} 0 & \cos(\theta)\,\mathrm{e}^{i\,\beta} & \sin(\theta)\,\mathrm{e}^{i\,\beta} \\ -\frac{1}{\sqrt{2}}\,\mathrm{e}^{i\,\phi\mathrm{j}} & -\frac{\sin(\theta)}{\sqrt{2}} & \frac{\cos(\theta)}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{\sin(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\mathrm{j}} & \frac{\cos(\theta)}{\sqrt{2}}\,\mathrm{e}^{-i\,\phi\mathrm{j}} \end{array}\right). \tag{5.15}$$

The angle θ is determined to be

$$\tan(2\,\theta)=\frac{2\,\sqrt{2}\,b}{c+d-a} \tag{5.16}$$

and therefore lies in the interval $[0, \frac{\pi}{2})$. Note that at least for charged fermions the ordering of the two eigenvalues unequal to $(c-d)$ is fixed, since the one with the positive sign in front of the square root is necessarily larger than the one with the negative sign.
Assuming that up quark and down quark, or equivalently (Dirac) neutrino and charged lepton, mass matrices are of the form $M_{4,5}$, the CKM matrix, or equivalently the MNS matrix, is given as a product of two matrices which equal the matrix U up to permutations of the columns depending on the ordering of the eigenvalues. According to Section 3.1 the relation between V_{CKM} and U_u and U_d is $V_{CKM} = U_u^T\, U_d^\star$ and, similarly, for U_{MNS} and the unitary matrices U_ν, U_l it is $U_{MNS} = U_l^T\, U_\nu^\star$. Therefore, the general form is $V_{mix} = W_1^T\, W_2^\star$ with $W_i \equiv U(\phi_i(m_i), \theta_i, \beta_i)$. Multiplying the two variants of U, W_1 and W_2, generates one element in the matrix V_{mix} which only depends on the difference of the two group theoretical phases ϕ_1 and ϕ_2 and the index j of the representation $\underline{\mathbf{2}}_{\mathbf{j}}$ under which two of the three left-handed fields transform. Its absolute value is

$$\frac{1}{2}\left|1 + \mathrm{e}^{i\,(\phi_1-\phi_2)\,\mathrm{j}}\right| = \left|\cos((\phi_1 - \phi_2)\,\frac{\mathrm{j}}{2})\right| = \left|\cos(\frac{\pi}{n}\,(m_1 - m_2)\,\mathrm{j})\right| \ . \tag{5.17}$$

The origin of the element is the product of the two eigenvectors corresponding to the eigenvalue $(c-d)$ in the up and down quark sector (neutrino and charged lepton sector). Therefore the ordering of the eigenvalues, i.e. their association with the masses of the fermions, determines the position of this element in the mixing matrix. Note that Eq.(5.17) already shows that non-trivial mixing forces $m_1 \neq m_2$, i.e. non-trivial mixing angles only arise, if the flavor symmetry is broken to two different (directions of) subgroups. The size of the mixing angle is then determined by the fact how large this mismatch actually is. In particular, it is interesting to notice that this formula allows for the prediction of non-trivial values of the mixing angles, i.e. not only rather special values as 0 or $\frac{\pi}{4}$, but also the Cabibbo angle can be explained, see Eq.(5.20) and Eq.(5.21). The other elements involve the two angles θ_1 and θ_2 as well as the difference α of the phases β_1 and β_2. The mixing matrix V_{mix}, for example, is given as

$$V_{mix} = \frac{1}{2}\begin{pmatrix} (\mathrm{e}^{i\,\phi_1\,\mathrm{j}} - \mathrm{e}^{i\,\phi_2\,\mathrm{j}})\,s_2 & 1 + \mathrm{e}^{i\,(\phi_1-\phi_2)\,\mathrm{j}} & -(\mathrm{e}^{i\,\phi_1\,\mathrm{j}} - \mathrm{e}^{i\,\phi_2\,\mathrm{j}})\,c_2 \\ (1 + \mathrm{e}^{-i\,(\phi_1-\phi_2)\,\mathrm{j}})\,s_1\,s_2 + 2\,\mathrm{e}^{i\,\alpha}\,c_1\,c_2 & -(\mathrm{e}^{-i\,\phi_1\,\mathrm{j}} - \mathrm{e}^{-i\,\phi_2\,\mathrm{j}})\,s_1 & -(1 + \mathrm{e}^{-i\,(\phi_1-\phi_2)\,\mathrm{j}})\,s_1\,c_2 + 2\,\mathrm{e}^{i\,\alpha}\,c_1\,s_2 \\ -(1 + \mathrm{e}^{-i\,(\phi_1-\phi_2)\,\mathrm{j}})\,c_1\,s_2 + 2\,\mathrm{e}^{i\,\alpha}\,s_1\,c_2 & (\mathrm{e}^{-i\,\phi_1\,\mathrm{j}} - \mathrm{e}^{-i\,\phi_2\,\mathrm{j}})\,c_1 & (1 + \mathrm{e}^{-i\,(\phi_1-\phi_2)\,\mathrm{j}})\,c_1\,c_2 + 2\,\mathrm{e}^{i\,\alpha}\,s_1\,s_2 \end{pmatrix} \tag{5.18}$$

for $(c-d)$ being identified with the first generation of up quarks (charged leptons) and the second generation of down quarks ((Dirac) neutrinos) [10]. Thereby, $\sin(\theta_i)$ and $\cos(\theta_i)$ are abbreviated by s_i and c_i, respectively. The Jarlskog invariant J_{CP} reads

$$J_{CP}(\mathrm{j}, \phi_1, \phi_2; \theta_1, \theta_2, \alpha) = \frac{1}{8}\,\sin((\phi_1 - \phi_2)\,\mathrm{j})\,\sin(\frac{1}{2}\,(\phi_1 - \phi_2)\,\mathrm{j})\,\sin(2\,\theta_1)\,\sin(2\,\theta_2)\,\sin(\frac{1}{2}\,(\phi_1 - \phi_2)\,\mathrm{j} + \alpha) \ . \tag{5.19}$$

It can be calculated according to Eq.(3.19). For the other possible identifications of the eigenvalues $(c-d)$ the resulting mixing matrix has a similar form up to permutations of rows and/or columns. J_{CP} equals the expression found in Eq.(5.19) up to a possible sign arising from the permutations of rows and columns.
Now we are prepared to derive the Cabibbo angle θ_C from D_7.

[10] The ordering of the two eigenvalues involving the square root is assumed to be the one, as described above.

5.3 D_7 can explain θ_C

After presenting the general mass matrix structures which can be achieved with a dihedral flavor symmetry which is broken to non-trivial subgroups, we apply these results in order to predict the CKM element $|V_{us}|$ or $|V_{cd}|$ and, thereby, the Cabibbo angle θ_C. We show that it only depends on group theoretical quantities, namely the index n of the dihedral group D_n, the representation index j of $\underline{\mathbf{2}}_{\mathbf{j}}$ under which two of the three left-handed quark doublets transform and the indices m_u and m_d denoting the preserved subgroups $Z_2 =< \mathrm{B\,A}^{m_u} >$ and $Z_2 =< \mathrm{B\,A}^{m_d} >$. The general formula is according to Eq.(5.17)

$$|V_{us\,(cd)}| = \left|\cos\left(\frac{\pi\,(m_u - m_d)\,\mathrm{j}}{n}\right)\right| \tag{5.20}$$

For example, for $n = 7$, $\mathrm{j} = 1$, $m_u = 3$ and $m_d = 0$, we arrive at

$$|V_{us\,(cd)}| = \left|\cos\left(\frac{\pi\,(3-0)\,1}{7}\right)\right| = \left|\cos\left(\frac{3\,\pi}{7}\right)\right| \approx 0.2225 \tag{5.21}$$

which is only 2% below the experimental best fit value of $0.2272(1)^{+0.0010}_{-0.0010}$ (see Section 2.1). Since we break to two different subgroups of D_n in the up and the down quark sector, D_n is completely broken in the whole Lagrangian -as it should be, since we do not observe an unbroken flavor symmetry at low energies. Due to the fact that none of the subgroups is preserved in the whole Lagrangian we expect corrections from higher-dimensional operators which mix the two different sectors, similar to the mixing of the Z_3 and Z_4 group conserving parts in the T' model, discussed in Chapter 4. Using the results of the T' model as a rough estimate of the generic size of such corrections, namely λ^2, we see that the value of $|V_{us\,(cd)}|$ then can be in full accordance with the experimental result. The separation of the two sectors has to be maintained by an additional ${Z_n}^{(aux)}$ symmetry, like in the T' model. As will be shown below, in case of D_7 a ${Z_2}^{(aux)}$ symmetry is sufficient.
As argued above the exact position of the element, which is explained by group theoretical quantities, is not fixed a priori, but only by the ordering of the eigenvalues.
In the following, we show a low energy model which implements this idea with the flavor symmetry $D_7 \times {Z_2}^{(aux)}$ [11]. As done in the models presented in Chapter 3 we break the flavor symmetry (and the auxiliary symmetry) only spontaneously at the electroweak scale. Therefore, all scalars appearing the model below are copies of the SM Higgs doublet.
We introduce the basics of the group theory of D_7, discuss the structure of the preserved Z_2 subgroups and present two realizations of fermion assignments which both can lead to $|V_{us\,(cd)}| = \cos(\frac{3\pi}{7})$. Thereby, we quote the numerical results given in [26]. For the discussion of the Higgs sector we also refer to [26]. This Higgs potential has features very similar to the ones discussed in the S_4 model and the D_5 model in Chapter 3. In particular, it shares the unpleasant feature that its simplest realization suffers from an accidental symmetry. Furthermore, also here the VEV structures can only be adjusted, but not predicted, and also the Higgs mass spectrum contains rather light particles.

5.3.1 Introduction to D_7

Since the group theory of a general dihedral group D_n has already been presented at length, we only briefly describe the specific features of the group D_7. Its group order is 14 and it has five

[11] D_7 has already been used as flavor symmetry in [117] in order to produce certain mass matrix textures in the quark sector.

	classes						
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$		
G	$\mathbb{1}$	A	A^2	A^3	B		
${}^\circ\mathcal{C}_i$	1	2	2	2	7		
${}^\circ\mathrm{h}_{\mathcal{C}_i}$	1	7	7	7	2	$\mathbb{c}^{(\mu)}$	faithful
$\underline{\mathbf{1_1}}$	1	1	1	1	1	1	
$\underline{\mathbf{1_2}}$	1	1	1	1	-1	1	
$\underline{\mathbf{2_1}}$	2	$2\cos(\varphi)$	$2\cos(2\varphi)$	$2\cos(3\varphi)$	0	1	$\surd$
$\underline{\mathbf{2_2}}$	2	$2\cos(2\varphi)$	$2\cos(4\varphi)$	$2\cos(6\varphi)$	0	1	$\surd$
$\underline{\mathbf{2_3}}$	2	$2\cos(3\varphi)$	$2\cos(6\varphi)$	$2\cos(9\varphi)$	0	1	$\surd$

Table 5.1: Character table of the group D_7. φ is $\frac{2\pi}{7}$. For further explanations see Appendix A.

irreducible representations: $\underline{\mathbf{1_1}}$ (trivial), $\underline{\mathbf{1_2}}$, $\underline{\mathbf{2_1}}$, $\underline{\mathbf{2_2}}$ and $\underline{\mathbf{2_3}}$. All two-dimensional representations are faithful in this group. The generators A and B and their relations can be deduced from the general formulae given in Section 5.1.1

$$\text{... for } \underline{\mathbf{2_1}} \quad : \quad \mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{2\pi i}{7}} & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi i}{7}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.22}$$

$$\text{... for } \underline{\mathbf{2_2}} \quad : \quad \mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{4\pi i}{7}} & 0 \\ 0 & \mathrm{e}^{-\frac{4\pi i}{7}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.23}$$

$$\text{... for } \underline{\mathbf{2_3}} \quad : \quad \mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{6\pi i}{7}} & 0 \\ 0 & \mathrm{e}^{-\frac{6\pi i}{7}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} , \tag{5.24}$$

which fulfill

$$\mathrm{A}^7 = \mathbb{1} \; , \quad \mathrm{B}^2 = \mathbb{1} \; , \quad \mathrm{A\,B\,A} = \mathrm{B} \; . \tag{5.25}$$

As usual the generators of $\underline{\mathbf{1_1}}$ and $\underline{\mathbf{1_2}}$ can be found in the character table displayed in Table 5.1. Again, this is only a special case of the general character table for a dihedral group D_n with an odd index n, as found in Appendix B.4.1. Needless to say that the choice of complex representation matrices for the real representations $\underline{\mathbf{2_j}}$ causes the existence of a similarity transformation U which links $\mathrm{A}^\star$, $\mathrm{B}^\star$ and A, B. It is of the form as shown in Eq.(5.3).
Kronecker products and Clebsch Gordan coefficients of the group D_7 can be found in Appendix B.5.

5.3.2 Study of Subgroups of D_7

In the cases below we always intend to preserve a Z_2 subgroup generated by $\mathrm{B\,A}^m$ ($m = 0, 1, ..., 6$). Thereby, A and B are the generators of the original D_7 group. For this purpose, we have to have a closer look at the decomposition of the D_7 representations under the Z_2 subgroup. In order not to break Z_2 only representations/combinations of components which transform trivially under it are allowed to have non-vanishing VEV. We use the notation $\underline{\mathbf{1_1}}$ for the trivial representation of Z_2 whose generator is $+1$ and $\underline{\mathbf{1_2}}$ for the non-trivial representation which acquires a sign when the Z_2

D_7 representation	Z_2 representation	VEV allowed
$\underline{\mathbf{1_1}}$	$\underline{\mathbf{1_1}}$	$\surd$
$\underline{\mathbf{1_2}}$	$\underline{\mathbf{1_2}}$	$-$
$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2_1}}$	$e^{\frac{2\pi i}{7} m} a_1 + a_2 \sim \underline{\mathbf{1_1}}$, $e^{\frac{2\pi i}{7} m} a_1 - a_2 \sim \underline{\mathbf{1_2}}$	$\propto \begin{pmatrix} e^{-\frac{2\pi i}{7} m} \\ 1 \end{pmatrix}$, i.e. $\langle a_1 \rangle = e^{-\frac{2\pi i}{7} m} \langle a_2 \rangle$
$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2_2}}$	$e^{\frac{4\pi i}{7} m} a_1 + a_2 \sim \underline{\mathbf{1_1}}$, $e^{\frac{4\pi i}{7} m} a_1 - a_2 \sim \underline{\mathbf{1_2}}$	$\propto \begin{pmatrix} e^{-\frac{4\pi i}{7} m} \\ 1 \end{pmatrix}$, i.e. $\langle a_1 \rangle = e^{-\frac{4\pi i}{7} m} \langle a_2 \rangle$
$\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2_3}}$	$e^{\frac{6\pi i}{7} m} a_1 + a_2 \sim \underline{\mathbf{1_1}}$, $e^{\frac{6\pi i}{7} m} a_1 - a_2 \sim \underline{\mathbf{1_2}}$	$\propto \begin{pmatrix} e^{-\frac{6\pi i}{7} m} \\ 1 \end{pmatrix}$, i.e. $\langle a_1 \rangle = e^{-\frac{6\pi i}{7} m} \langle a_2 \rangle$

Table 5.2: Breaking of D_7 down to Z_2 which is generated by $\mathrm{B\,A}^m$ $(m = 0, 1, ..., 6)$. The third column indicates whether a VEV for a scalar transforming under this particular D_7 representation is allowed and which form it has to have in case that the representation is two-dimensional.

transformation is applied, i.e. its generator is -1 [12].
Obviously, the trivial representation of D_7 is identified with the trivial representation $\underline{\mathbf{1_1}}$ of any of its subgroups. The D_7 representation $\underline{\mathbf{1_2}}$ transforms non-trivially under the residual Z_2 group and hence is identified with the non-trivial representation of Z_2, as $\mathrm{B\,A}^m = -1$ for all m. The irreducible two-dimensional representations $\underline{\mathbf{2_j}}$ split up into the one-dimensional representations of the abelian group Z_2, i.e. one combination of the components of $\underline{\mathbf{2_j}}$ transforms as $\underline{\mathbf{1_1}}$ and the other one as $\underline{\mathbf{1_2}}$ under Z_2. The actual combinations are found by looking at the matrix form of the generator $\mathrm{B\,A}^m$ of Z_2

$$\mathrm{B\,A}^m = \begin{pmatrix} 0 & e^{-\frac{2\pi i}{7} \mathrm{j}\, m} \\ e^{\frac{2\pi i}{7} \mathrm{j}\, m} & 0 \end{pmatrix} \quad \text{for} \quad \underline{\mathbf{2_j}} \; . \tag{5.26}$$

The eigenvalues are $+1$ and -1, where $+1$ stands for the trivial representation, while -1 is the generator of $\underline{\mathbf{1_2}}$ of Z_2. The eigenvectors are

$$v_{+1} \propto \begin{pmatrix} e^{-\frac{2\pi i}{7} \mathrm{j}\, m} \\ 1 \end{pmatrix} \quad \text{and} \quad v_{-1} \propto \begin{pmatrix} e^{-\frac{2\pi i}{7} \mathrm{j}\, m} \\ -1 \end{pmatrix} \; . \tag{5.27}$$

Therefore the combination of components $a_{1,2}$ of a two-dimensional representation $\underline{\mathbf{2_j}}$ which transforms trivially under the Z_2 group is

$$e^{\frac{2\pi i}{7} \mathrm{j}\, m} a_1 + a_2 \sim \underline{\mathbf{1_1}} \; , \tag{5.28}$$

while

$$e^{\frac{2\pi i}{7} \mathrm{j}\, m} a_1 - a_2 \sim \underline{\mathbf{1_2}} \; . \tag{5.29}$$

These results are collected in Table 5.2.

[12] Note that this notation slightly deviates from the one used for Z_n representations in the general analysis of dihedral symmetries and their preserved subgroups [25].

5.3.3 D_7 Model - Realization I

In this section, we show one possible assignment of fermions and scalars which allows to predict $|V_{us}| = \cos(\frac{3\pi}{7})$. Only two of the left-handed quark doublets are unified into a two-dimensional representation of D_7, which we choose for simplicity to be $\underline{\mathbf{2_1}}$. The remaining generation of left-handed quark doublets, which will be the first generation in our case, transforms trivially under D_7. The second and third generation of the left-handed conjugate quarks also transform trivially, whereas the first one has to transform as the non-trivial singlet of D_7. According to the Kronecker products shown in Appendix B.5 only Higgs fields $\sim \underline{\mathbf{1_1}}$, $\sim \underline{\mathbf{1_2}}$ or $\underline{\mathbf{2_1}}$ can form invariant Yukawa couplings. However, since we want to preserve a Z_2 subgroup generated by $\mathrm{B\,A}^m$ we cannot choose the VEVs of the Higgs fields in an arbitrary way, i.e. the VEVs of Higgs fields $\sim \underline{\mathbf{1_2}}$ have to vanish and the ones of fields $H_{1,2}$ forming a $\underline{\mathbf{2_1}}$ under D_7 have to be correlated, $\langle H_1 \rangle = \mathrm{e}^{-\frac{2\pi i}{7} m} \langle H_2 \rangle$, where m is the index of the direction of the preserved Z_2 group. To arrive at a non-trivial value of $|V_{us\,(cd)}|$ m has to be distinct in the up and down quark sector. As indicated above, we can choose $m_u = 3$ and $m_d = 0$ in case that j = 1 (as it is here). $m_u \neq m_d$ requires that different Higgs fields couple to up and down quarks. In the SM we can easily maintain this by an additional ${Z_2}^{(aux)}$ symmetry under which the left-handed conjugate down quarks acquire a sign. Higgs fields which do not transform under this symmetry then automatically only couple to up quarks, while Higgs fields, which do transform, only couple to down quarks. As the Higgs fields, which couple to up quarks, shall preserve the generator $\mathrm{B\,A}^3$, their VEVs have to have the form

$$\langle H^u_s \rangle > 0 \; , \;\; \langle H^u_1 \rangle = \mathrm{e}^{-\frac{6\pi i}{7}} \langle H^u_2 \rangle \; , \tag{5.30}$$

for Higgs fields

$$H^u_s \sim \underline{\mathbf{1_1}} \quad \text{and} \quad \begin{pmatrix} H^u_1 \\ H^u_2 \end{pmatrix} \sim \underline{\mathbf{2_1}} \; . \tag{5.31}$$

For the fields

$$H^d_s \sim \underline{\mathbf{1_1}} \quad \text{and} \quad \begin{pmatrix} H^d_1 \\ H^d_2 \end{pmatrix} \sim \underline{\mathbf{2_1}} \tag{5.32}$$

responsible for the masses of the down quarks, the VEV configuration has to read

$$\langle H^d_s \rangle > 0 \; , \;\; \langle H^d_1 \rangle = \langle H^d_2 \rangle \; , \tag{5.33}$$

in order to conserve the Z_2 subgroup generated by B ($m_d = 0$). In Table 5.3 the fields of the model and their transformation properties under D_7 and ${Z_2}^{(aux)}$ are summarized. In the following, we

Field	Q_1	$Q_{2,3}$	u^c	c^c	t^c	d^c	s^c	b^c	H^u_s	$H^u_{1,2}$	H^d_s	$H^d_{1,2}$
D_7	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{2_1}}$	$\underline{\mathbf{1_2}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{1_2}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{2_1}}$	$\underline{\mathbf{1_1}}$	$\underline{\mathbf{2_1}}$
${Z_2}^{(aux)}$	$+$	$+$	$+$	$+$	$+$	$-$	$-$	$-$	$+$	$+$	$-$	$-$

Table 5.3: The particle content and its symmetry properties under $D_7 \times {Z_2}^{(aux)}$ for Realization I. Since we present a realization of the prediction of $|V_{us\,(cd)}|$ in a low energy model, the Higgs fields $H^{u,d}_s$ and $H^{u,d}_{1,2}$ are copies of the SM Higgs field, i.e. transform as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

parameterize the VEVs $\langle H_2^{u,d} \rangle$ by $v_{u,d} > 0$

$$\langle H_2^u \rangle = v_u \, \mathrm{e}^{\frac{3\pi i}{7}} \quad \text{and} \quad \langle H_2^d \rangle = v_d \; . \tag{5.34}$$

In this way the VEVs of the upper and lower component of a D_7 doublet have an opposite phase and apart from this the VEVs can be real. This parameterization considerably simplifies the analysis of the Higgs potential, see [26]. The Yukawa couplings are of the form

$$\begin{aligned} \mathcal{L}_Y \;\; = & \;\; y_1^u \, Q_1 \, c^c \, \tilde{H}_s^u + y_2^u \, Q_1 \, t^c \, \tilde{H}_s^u + y_3^u \, (Q_2 \, u^c \, \tilde{H}_1^u - Q_3 \, u^c \, \tilde{H}_2^u) \\ + & \;\; y_4^u \, (Q_2 \, c^c \, \tilde{H}_1^u + Q_3 \, c^c \, \tilde{H}_2^u) + y_5^u \, (Q_2 \, t^c \, \tilde{H}_1^u + Q_3 \, t^c \, \tilde{H}_2^u) \\ + & \;\; y_1^d \, Q_1 \, s^c \, H_s^d + y_2^d \, Q_1 \, b^c \, H_s^d + y_3^d \, (Q_2 \, d^c \, H_2^d - Q_3 \, d^c \, H_1^d) \\ + & \;\; y_4^d \, (Q_2 \, s^c \, H_2^d + Q_3 \, s^c \, H_1^d) + y_5^d \, (Q_2 \, b^c \, H_2^d + Q_3 \, b^c \, H_1^d) + \text{h.c.} \end{aligned} \tag{5.35}$$

Then the form of the mass matrices is

$$\mathcal{M}_u = \begin{pmatrix} 0 & y_1^u \, \langle H_s^u \rangle & y_2^u \, \langle H_s^u \rangle \\ y_3^u \, v_u \, \mathrm{e}^{\frac{3\pi i}{7}} & y_4^u \, v_u \, \mathrm{e}^{\frac{3\pi i}{7}} & y_5^u \, v_u \, \mathrm{e}^{\frac{3\pi i}{7}} \\ -y_3^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} & y_4^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} & y_5^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} \end{pmatrix} \quad \text{and} \quad \mathcal{M}_d = \begin{pmatrix} 0 & y_1^d \, \langle H_s^d \rangle & y_2^d \, \langle H_s^d \rangle \\ y_3^d \, v_d & y_4^d \, v_d & y_5^d \, v_d \\ -y_3^d \, v_d & y_4^d \, v_d & y_5^d \, v_d \end{pmatrix} \tag{5.36}$$

The masses of the up quarks are

$$2 \, |y_3^u|^2 \, v_u^2 \; , \tag{5.37}$$

$$\frac{1}{2} \, (|y_1^u|^2 + |y_2^u|^2) \, \langle H_s^u \rangle^2 + (|y_4^u|^2 + |y_5^u|^2) \, v_u^2 \pm \frac{1}{2} \sqrt{[(|y_1^u|^2 + |y_2^u|^2) \, \langle H_s^u \rangle^2 - 2 \, (|y_4^u|^2 + |y_5^u|^2) \, v_u^2]^2 + 8 \, \langle H_s^u \rangle^2 \, v_u^2 \, |y_1^u \, (y_4^u)^\star + y_2^u \, (y_5^u)^\star|^2} \tag{5.38}$$

and the down quark masses read

$$2 \, |y_3^d|^2 \, v_d^2 \; , \tag{5.39}$$

$$\frac{1}{2} \, (|y_1^d|^2 + |y_2^d|^2) \, \langle H_s^d \rangle^2 + (|y_4^d|^2 + |y_5^d|^2) \, v_d^2 \pm \frac{1}{2} \sqrt{[(|y_1^d|^2 + |y_2^d|^2) \, \langle H_s^d \rangle^2 - 2 \, (|y_4^d|^2 + |y_5^d|^2) \, v_d^2]^2 + 8 \, \langle H_s^d \rangle^2 \, v_d^2 \, |y_1^d \, (y_4^d)^\star + y_2^d \, (y_5^d)^\star|^2} \tag{5.40}$$

In order to arrive at $|V_{us}| = \cos(\frac{3\pi}{7})$ we have to identify the up quark mass m_u with $\sqrt{2} \, |y_3^u| \, v_u$ and the strange quark mass m_s with $\sqrt{2} \, |y_3^d| \, v_d$. According to above, the lighter one of the remaining two generations, m_c and m_d, respectively, is identified with the eigenvalue in which a minus sign appears in front of the square root. The CKM matrix is then of the form

$$|V_{CKM}| = \begin{pmatrix} \cos(\frac{\pi}{14}) \, s_d & \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14}) \, c_d \\ \frac{1}{2} \, |2 \, \mathrm{e}^{i\,\alpha} \, c_d \, c_u + (1 + \mathrm{e}^{-\frac{6\pi i}{7}}) \, s_d \, s_u| & \cos(\frac{\pi}{14}) \, s_u & \frac{1}{2} \, |2 \, \mathrm{e}^{i\,\alpha} \, c_u \, s_d - (1 + \mathrm{e}^{-\frac{6\pi i}{7}}) \, c_d \, s_u| \\ \frac{1}{2} \, |2 \, \mathrm{e}^{i\,\alpha} \, c_d \, s_u - (1 + \mathrm{e}^{-\frac{6\pi i}{7}}) \, c_u \, s_d| & \cos(\frac{\pi}{14}) \, c_u & \frac{1}{2} \, |2 \, \mathrm{e}^{i\,\alpha} \, s_d \, s_u + (1 + \mathrm{e}^{-\frac{6\pi i}{7}}) \, c_d \, c_u| \end{pmatrix} \tag{5.41}$$

with $s_{d,u} = \sin(\theta_{d,u})$, $c_{d,u} = \cos(\theta_{d,u})$ and the phase $\alpha = \beta_u - \beta_d$. The Jarlskog invariant J_{CP} reads

$$J_{CP} = \frac{1}{8} \, \sin\left(\frac{3\pi}{7}\right) \, \sin\left(\frac{6\pi}{7}\right) \, \sin(2\,\theta_d) \, \sin(2\,\theta_u) \, \sin\left(\frac{3\pi}{7} + \alpha\right) \; . \tag{5.42}$$

As indicated by the indices, θ_u and β_u are associated with the mixing matrix U_u which diagonalizes $\mathcal{M}_u \mathcal{M}_u^\dagger$, while θ_d and β_d stem from U_d. They are rather non-trivial functions of the entries of the mass matrices. In [26] a numerical example is given which is able to fit all quark masses and mixing parameters quite well with the element $|V_{us}|$ being fixed to $\cos(\frac{3\pi}{7})$. All quark masses can be fitted to the central values, while the CKM matrix is of the form

$$|V_{CKM}| = \left(\begin{array}{ccc} 0.97492 & 0.2225 & 3.95 \times 10^{-3} \\ 0.2224 & 0.97404 & 42.23 \times 10^{-3} \\ 8.11 \times 10^{-3} & 41.64 \times 10^{-3} & 0.9991 \end{array} \right) \tag{5.43}$$

with $J_{CP} = 3.09 \times 10^{-5}$. In order to match the above results we display the explicit form of $A_{d,u}$, $B_{d,u}$, ... for the considered case

$$A_u = y_1^u \langle H_s^u \rangle \ , \ \ B_u = y_2^u \langle H_s^u \rangle \ , \ \ C_u = y_3^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} \ , \ \ D_u = y_4^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} \ , \ \ E_u = y_5^u \, v_u \, \mathrm{e}^{-\frac{3\pi i}{7}} \ , \tag{5.44}$$

$$A_d = y_1^d \langle H_s^d \rangle \ , \ \ B_d = y_2^d \langle H_s^d \rangle \ , \ \ C_d = y_3^d \, v_d \ , \ \ D_d = y_4^d \, v_d \ , \ \ E_d = y_5^d \, v_d \ . \tag{5.45}$$

The group theoretical phases $\phi_{u,d}$ are $\phi_u = \frac{6\pi}{7}$, since the Z_2 subgroup is generated by $\mathrm{B}\,\mathrm{A}^3$ in the up quark sector, and $\phi_d = 0$, as $Z_2 =< \mathrm{B} >$ is the residual group in the down quark sector. The representation index j equals one, because the left-handed quark doublets of the second and third generation transform as $\underline{\mathbf{2}}_{\mathbf{1}}$ under D_7.

5.3.4 D_7 Model - Realization II

The other possible fermion assignment which leads to the prediction of one element of V_{CKM} has the advantage that two generations of left-handed as well as left-handed conjugate quarks are unified into a doublet under D_7. We choose these two generations to be the second and third one and choose the doublet to be $\underline{\mathbf{2}}_{\mathbf{1}}$ in both cases. In Eq.(5.20) the representation index is again $\mathrm{j} = 1$. The first generation is chosen to transform trivially under D_7. According to the Kronecker products shown in Appendix B.5 Higgs fields transforming as $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$, $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ can couple to the quarks to form D_7 invariants. However, in order to preserve the Z_2 subgroups, also here we cannot choose the VEVs arbitrarily: fields $\sim \underline{\mathbf{1}}_{\mathbf{2}}$ should not get a non-vanishing VEV and the VEVs of Higgs fields, which form a doublet under D_7, have to have the same absolute value and have to be correlated by a phase. As in realization I, we will choose $m_u = 3$ and $m_d = 0$ in order to arrive at $\cos(\frac{3\pi}{7})$ for $|V_{us\,(cd)}|$. At this point we also introduce the same additional $Z_2{}^{(aux)}$ symmetry as in realization I. According to Table 5.2 the VEV structures have to be of the form

$$\langle H_s^u \rangle > 0 \ , \ \ \langle H_1^u \rangle = \mathrm{e}^{-\frac{6\pi i}{7}} \langle H_2^u \rangle \ , \ \ \langle h_1^u \rangle = \mathrm{e}^{-\frac{12\pi i}{7}} \langle h_2^u \rangle \tag{5.46}$$

for Higgs fields

$$H_s^u \sim \underline{\mathbf{1}}_{\mathbf{1}} \ , \ \ \left(\begin{array}{c} H_1^u \\ H_2^u \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{1}} \ \ \text{and} \ \ \left(\begin{array}{c} h_1^u \\ h_2^u \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{2}} \tag{5.47}$$

coupling to up quarks only and for a similar set of Higgs fields

$$H_s^d \sim \underline{\mathbf{1}}_{\mathbf{1}} \ , \ \ \left(\begin{array}{c} H_1^d \\ H_2^d \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{1}} \ \ \text{and} \ \ \left(\begin{array}{c} h_1^d \\ h_2^d \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{2}} \tag{5.48}$$

responsible for the masses of the down quarks, the VEV configuration shall read

$$\langle H_s^d \rangle > 0 \ , \ \ \langle H_1^d \rangle = \langle H_2^d \rangle \ , \ \ \langle h_1^d \rangle = \langle h_2^d \rangle \ . \tag{5.49}$$

In Table 5.4 the fields of the model and their transformation properties under D_7 and $Z_2{}^{(aux)}$ are summarized. In the following, we will parameterize the VEVs $\langle H_2^{u,d}\rangle$ and $\langle h_2^{u,d}\rangle$ by $v_{u,d} > 0$ and

Field	Q_1	$Q_{2,3}$	u^c	(c^c, t^c)	d^c	(s^c, b^c)	H_s^u	$H_{1,2}^u$	$h_{1,2}^u$	H_s^d	$H_{1,2}^d$	$h_{1,2}^d$
D_7	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_2$	$\underline{\mathbf{1}}_1$	$\underline{\mathbf{2}}_1$	$\underline{\mathbf{2}}_2$
$Z_2{}^{(aux)}$	$+$	$+$	$+$	$+$	$-$	$-$	$+$	$+$	$+$	$-$	$-$	$-$

Table 5.4: The particle content and its symmetry properties under $D_7\times\, Z_2{}^{(aux)}$ for Realization II. Since we present a realization of the prediction of $|V_{us\,(cd)}|$ in a low energy model, the Higgs fields $H_s^{u,d}$, $H_{1,2}^{u,d}$ and $h_{1,2}^{u,d}$ are copies of the SM Higgs field, i.e. transform as $(\underline{\mathbf{1}}, \underline{\mathbf{2}}, -1)$ under the SM gauge group $SU(3)_C \times SU(2)_L \times U(1)_Y$.

$w_{u,d} > 0$

$$\langle H_2^u\rangle = v_u\, \mathrm{e}^{\frac{3\,\pi\, i}{7}}\ ,\ \ \langle H_2^d\rangle = v_d\ ,\ \ \langle h_2^u\rangle = w_u\, \mathrm{e}^{\frac{6\,\pi\, i}{7}} \quad \text{and} \quad \langle h_2^d\rangle = w_d\ . \tag{5.50}$$

Also here we adopted the parameterization, which leads to opposite phases for the VEVs of upper and lower components of D_7 doublets. Apart from this the VEVs are real. The mass matrices for up and down quarks read

$$\mathcal{M}_u = \begin{pmatrix} y_1^u\,\langle H_s^u\rangle^\star & y_2^u\,\langle H_1^u\rangle^\star & y_2^u\,\langle H_2^u\rangle^\star \\ y_3^u\,\langle H_1^u\rangle^\star & y_5^u\,\langle h_1^u\rangle^\star & y_4^u\,\langle H_s^u\rangle^\star \\ y_3^u\,\langle H_2^u\rangle^\star & y_4^u\,\langle H_s^u\rangle^\star & y_5^u\,\langle h_2^u\rangle^\star \end{pmatrix} \quad \text{and} \quad \mathcal{M}_d = \begin{pmatrix} y_1^d\,\langle H_s^d\rangle & y_2^d\,\langle H_2^d\rangle & y_2^d\,\langle H_1^d\rangle \\ y_3^d\,\langle H_2^d\rangle & y_5^d\,\langle h_2^d\rangle & y_4^d\,\langle H_s^d\rangle \\ y_3^d\,\langle H_1^d\rangle & y_4^d\,\langle H_s^d\rangle & y_5^d\,\langle h_1^d\rangle \end{pmatrix} \tag{5.51}$$

and with VEVs according to Eq.(5.46), Eq.(5.49) and Eq.(5.50) the form is

$$\mathcal{M}_u = \begin{pmatrix} y_1^u\,\langle H_s^u\rangle & y_2^u\, v_u\, \mathrm{e}^{\frac{3\,\pi\, i}{7}} & y_2^u\, v_u\, \mathrm{e}^{-\frac{3\,\pi\, i}{7}} \\ y_3^u\, v_u\, \mathrm{e}^{\frac{3\,\pi\, i}{7}} & y_5^u\, w_u\, \mathrm{e}^{\frac{6\,\pi\, i}{7}} & y_4^u\,\langle H_s^u\rangle \\ y_3^u\, v_u\, \mathrm{e}^{-\frac{3\,\pi\, i}{7}} & y_4^u\,\langle H_s^u\rangle & y_5^u\, w_u\, \mathrm{e}^{-\frac{6\,\pi\, i}{7}} \end{pmatrix} \quad \text{and} \quad \mathcal{M}_d = \begin{pmatrix} y_1^d\,\langle H_s^d\rangle & y_2^d\, v_d & y_2^d\, v_d \\ y_3^d\, v_d & y_5^d\, w_d & y_4^d\,\langle H_s^d\rangle \\ y_3^d\, v_d & y_4^d\,\langle H_s^d\rangle & y_5^d\, w_d \end{pmatrix}\ . \tag{5.52}$$

The eigenvalues of the up and down quark mass matrices are given by

$$|y_4^u\,\langle H_s^u\rangle - y_5^u\, w_u|^2\ , \tag{5.53}$$

$$(|y_2^u|^2 + |y_3^u|^2)\, v_u^2 + \frac{1}{2}\,(|y_1^u|^2\,\langle H_s^u\rangle^2 + |y_4^u\,\langle H_s^u\rangle + y_5^u\, w_u|^2) \tag{5.54}$$

$$\pm\tfrac{1}{2}\,\sqrt{[2\,(|y_3^u|^2 - |y_2^u|^2)\, v_u^2 - |y_1^u|^2\,\langle H_s^u\rangle^2 + |y_4^u\,\langle H_s^u\rangle + y_5^u\, w_u|^2]^2 + 8\, v_u^2\, |y_1^u\,(y_3^u)^\star\,\langle H_s^u\rangle + y_2^u\,((y_4^u)^\star\,\langle H_s^u\rangle + (y_5^u)^\star\, w_u)|^2}$$

and for $\mathcal{M}_d$

$$|y_4^d\,\langle H_s^d\rangle - y_5^d\, w_d|^2\ , \tag{5.55}$$

$$(|y_2^d|^2 + |y_3^d|^2)\, v_d^2 + \frac{1}{2}\,(|y_1^d|^2\,\langle H_s^d\rangle^2 + |y_4^d\,\langle H_s^d\rangle + y_5^d\, w_d|^2) \tag{5.56}$$

$$\pm\tfrac{1}{2}\,\sqrt{[2\,(|y_3^d|^2 - |y_2^d|^2)\, v_d^2 - |y_1^d|^2\,\langle H_s^d\rangle^2 + |y_4^d\,\langle H_s^d\rangle + y_5^d\, w_d|^2]^2 + 8\, v_d^2\, |y_1^d\,(y_3^d)^\star\,\langle H_s^d\rangle + y_2^d\,((y_4^d)^\star\,\langle H_s^d\rangle + (y_5^d)^\star\, w_d)|^2}\ .$$

The position of the group theoretically determined element in the mixing matrix is given by the fact whether $|y_4^u\,\langle H_s^u\rangle - y_5^u\, w_u|$ is assigned to m_u, m_c or m_t and $|y_4^d\,\langle H_s^d\rangle - y_5^d\, w_d|$ to m_d, m_s or

m_b. In the numerical example shown in [26] it turned out that $|V_{cd}|$ equals $\cos(\frac{3\pi}{7})$ which reveals that m_c is associated with $|y_4^u \langle H_s^u \rangle - y_5^u w_u|$ and m_d with $|y_4^d \langle H_s^d \rangle - y_5^d w_d|$. The other elements of V_{CKM} have the following form

$$|V_{CKM}| = \begin{pmatrix} \cos(\frac{\pi}{14})\, s_u & \frac{1}{2}\, |2\, \mathrm{e}^{i\,\alpha}\, c_d\, c_u + (1 + \mathrm{e}^{-\frac{6\,\pi\,i}{7}})\, s_d\, s_u| & \frac{1}{2}\, |2\, \mathrm{e}^{i\,\alpha}\, c_u\, s_d - (1 + \mathrm{e}^{-\frac{6\,\pi\,i}{7}})\, c_d\, s_u| \\ \cos(\frac{3\pi}{7}) & \cos(\frac{\pi}{14})\, s_d & \cos(\frac{\pi}{14})\, c_d \\ \cos(\frac{\pi}{14})\, c_u & \frac{1}{2}\, |2\, \mathrm{e}^{i\,\alpha}\, c_d\, s_u - (1 + \mathrm{e}^{-\frac{6\,\pi\,i}{7}})\, c_u\, s_d| & \frac{1}{2}\, |2\, \mathrm{e}^{i\,\alpha}\, s_d\, s_u + (1 + \mathrm{e}^{-\frac{6\,\pi\,i}{7}})\, c_d\, c_u| \end{pmatrix} \tag{5.57}$$

and

$$J_{CP} = \frac{1}{8}\, \sin\left(\frac{3\,\pi}{7}\right)\, \sin\left(\frac{6\,\pi}{7}\right)\, \sin(2\,\theta_d)\, \sin(2\,\theta_u)\, \sin\left(\frac{3\,\pi}{7} + \alpha\right)\, . \tag{5.58}$$

Here we used the same abbreviations as above in Eq.(5.41). Note that the formula for J_{CP} coincides with the one given above, see Eq.(5.42). In the numerical example presented in [26] the CKM matrix is of the form

$$|V_{CKM}| = \begin{pmatrix} 0.97489 & 0.2226 & 3.95\, \times\, 10^{-3} \\ 0.2225 & 0.97401 & 42.23\, \times\, 10^{-3} \\ 8.11\, \times\, 10^{-3} & 41.64\, \times\, 10^{-3} & 0.9991 \end{pmatrix} \tag{5.59}$$

and the value of J_{CP} is $3.09\, \times\, 10^{-5}$. The explicit form of $A_{d,u}$, $B_{d,u}$, ... reads

$$A_u = y_1^u \langle H_s^u \rangle\, , \quad B_u = y_3^u\, v_u\, \mathrm{e}^{-\frac{3\,\pi\,i}{7}}\, , \quad C_u = y_2^u\, v_u\, \mathrm{e}^{-\frac{3\,\pi\,i}{7}}\, , \quad D_u = y_5^u\, w_u\, \mathrm{e}^{-\frac{6\,\pi\,i}{7}}\, , \quad E_u = y_4^u \langle H_s^u \rangle\, , \tag{5.60}$$

$$A_d = y_1^d \langle H_s^d \rangle\, , \quad B_d = y_3^d\, v_d\, , \quad C_d = y_2^d\, v_d\, , \quad D_d = y_5^d\, w_d\, , \quad E_d = y_4^d \langle H_s^d \rangle \tag{5.61}$$

together with $\phi_u = \frac{6\pi}{7}$ and $\phi_d = 0$ which correspond to $Z_2 =< \mathrm{B}\, \mathrm{A}^3 >$ and $Z_2 =< \mathrm{B} >$ as preserved subgroups for up and down quarks, respectively. Since all generations transform as $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}}$ in this setup, the representation indices j and k are both equal to one.

5.3.5 Summary and Comments

As a first application of the general results found in [25], we have presented a way to predict the Cabibbo angle θ_C in terms of group theoretical quantities only. Thereby, we discussed the group theoretical background of the model in detail and showed how the different Z_2 subgroups of D_7 can be preserved. As already mentioned in Section 5.2, two different mass matrix structures, called M_4 and M_5 above, allow one element of the mixing matrix to be determined only by fundamental group theoretical quantities. We constructed two realizations, one in which the quark mass matrices are of the form of M_4 and one in which the mass matrices have the same structure as M_5. In the numerical examples, shown in [26], it turns out that in the first realization the element $|V_{us}|$ is determined by group theory only, while in the second one it is $|V_{cd}|$ which is fixed to $\cos(\frac{3\pi}{7})$. A discussion of the Higgs potential for the first realization can be found in [26]. Since this is again a multi-Higgs doublet potential, it suffers from the same problems as the potentials studied in the S_4 and D_5 model. Especially, the fact that also here the VEVs can only be adjusted, but by no means predicted is unpleasant, since in this D_7 model the fact that $|V_{us\,(cd)}|$ are determined by group theory crucially depends on the VEV configuration.

Concerning the fermion assignment shown in Section 5.3.3 and Section 5.3.4, one should notice that it is not the only possible one for the quarks which leads to a prediction of the CKM element $|V_{us\,(cd)}|$ to be $\cos(\frac{3\pi}{7})$. There exist several other possibilities, for example, the representation index of the doublet under which the left-handed quarks transform can be chosen as j = 3 and the Z_2

group to which D_7 is then broken in the up quark sector should be replaced by $Z_2 =< \mathrm{B\,A} >$, i.e. $m_u = 1$. Furthermore, the flavor group does not necessarily have to be D_7. Taking D_{14} works as well with the according changes of the representation index j and the indices of the preserved subgroups, m_u and m_d.
In the work published [26] we also presented ways to generate only the Cabibbo angle θ_C, while the other two mixing angles θ^q_{13} and θ^q_{23} vanish. Again, D_7 and D_{14} can be chosen as flavor symmetry. The subgroups which are preserved in each sector can then not only be of the form $Z_2 =< \mathrm{B\,A}^m >$, but also $D_2 =< \mathrm{A}^7, \mathrm{B\,A}^m >$, if we assume D_{14} as flavor group. Apart from studies concerned with the quark sector we also performed a numerical analysis of the MNS matrix in [26] in order to show that the lepton mixing can be explained with the help of a dihedral group which is broken to different (directions of) subgroups. As the elements of U_{MNS} are much less constrained than the ones of V_{CKM}, we find multiple solutions which allow good fits of the lepton mixing parameters.
As we used the embedding into GUTs (and continuous flavor symmetries) as a guideline for the construction of the models analyzed in Chapter 3, it is legitimate to ask whether one could also embed the low energy models with the flavor symmetry D_7 into a larger framework. As we need to introduce an additional symmetry $Z_2^{(aux)}$ to separate the up and down quark sector, it is clear that we cannot combine this setup with an $SO(10)$ GUT. However, it is still possible to embed the second realization into $SU(5)$, since the left-handed and left-handed conjugate up quarks transform in the same way under $D_7 \times Z_2^{(aux)}$. For the embedding into a continuous group, like $SO(3)$, we expect that this is possible in general, since, for example, we could identify $\underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{1}}$ with the fundamental representation $\underline{\mathbf{3}}$ of $SO(3)$.
Concerning the problem of the Higgs potential, realizations, in which the Higgs doublets are replaced by gauge singlets, as done in the T' model in Chapter 4, can again offer better opportunities to control the vacuum alignment. However, we probably have to tackle another problem, if we try to achieve the proper vacuum alignment, namely the fact that in this model the flavor group D_7 is not broken to two different subgroup structures, but rather to two different directions of subgroups. In the up as well as the down quark sector we preserve a Z_2 group which is generated by $\mathrm{B\,A}^m$. The only difference lies in the fact that m_u is not equal to m_d. This has to be contrasted with the T' (A_4) models in which the symmetry is broken to a Z_3 group in the charged fermion (lepton) and to a Z_4 (Z_2) group in the neutrino sector. In the realization of the vacuum alignment presented in the preceding chapter it turned out that vacua preserving subgroups with the same subgroup structure are degenerate. This shows that it might not be straightforward to extend the construction of the vacuum alignment used in Chapter 4 to the case here in which the preserved subgroups have the same group structure, but are generated by different group elements of the original group. On the other hand, the D_7 model has an advantage over the T' model, since we do not choose the representations under which the Higgs/flavon fields transform which couple directly to the fermions. Therefore, the structure of the mass matrices and the form of the mixing matrix (up to permutations which correspond to the ordering of the eigenvalues in the up and down quark sector) are completely determined by the group theory of the flavor symmetry and the choice of the fermion assignment.

5.4 Preserved Subgroups Explain $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ for Leptons

The discussion of the T' model in Chapter 4 and the fact that we can predict the Cabibbo angle θ_C with the help of the dihedral flavor symmetry D_7 already showed that the study of models in which the flavor symmetry (independent of its nature) is not broken in an arbitrary way, but only down to (different) subgroups might be the key feature in order to make clear predictions, especially,

	classes						
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	$\mathcal{C}_4$	$\mathcal{C}_5$		
G	$\mathbb{1}$	$(g\,h)^2$	g	h	$g\,h$		
$^\circ\mathcal{C}_i$	1	1	2	2	2		
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	2	2	2	4	$\mathbb{c}^{(\mu)}$	faithful
$\underline{\mathbf{1}}_{++}$	1	1	1	1	1	1	
$\underline{\mathbf{1}}_{+-}$	1	1	1	-1	-1	1	
$\underline{\mathbf{1}}_{-+}$	1	1	-1	1	-1	1	
$\underline{\mathbf{1}}_{--}$	1	1	-1	-1	1	1	
$\underline{\mathbf{2}}$	2	-2	0	0	0	1	$\surd$

Table 5.5: Character table of the group D_4. The notation of the representations is according to [94]. This table can be also found in [95]. For further explanations see Appendix A.

for the fermion mixing angles. In the literature there exist two further neat examples which also use a dihedral group as flavor symmetry and which can predict maximal atmospheric mixing and vanishing θ_{13}. In the first model [94] the flavor symmetry is D_4 and in the second one [116] it is D_3 (which is isomorphic to S_3). It is very interesting to see that these two groups which belong to the smallest non-abelian discrete symmetries turn out to be so useful in this context, although they only have a very limited number of representations and their structure is very simple. In both cases the flavor symmetry has to be accompanied by an additional $Z_2{}^{(aux)}$ symmetry which allows the separation of the different sectors which preserve different subgroups of the flavor symmetry. We discuss both models in detail and show which are the preserved subgroups in the different sectors of the theory.

5.4.1 $D_4 \times Z_2^{(aux)}$ Model

In the following, we explain the structure of the D_4 model in detail and explicitly show how the preservation of subgroups leads to the prediction of $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ in the lepton sector.
Since the authors of [94] work in another group basis than we do here we first have to introduce their basis of the representation matrices and notations for the representations. The representations are denoted by $\underline{\mathbf{1}}_{++}$, $\underline{\mathbf{1}}_{+-}$, $\underline{\mathbf{1}}_{-+}$, $\underline{\mathbf{1}}_{--}$ and $\underline{\mathbf{2}}$. The generators are called g and h and given by

$$g = +1 \quad , \quad h = +1 \quad \text{... for } \underline{\mathbf{1}}_{++} \tag{5.62}$$
$$g = +1 \quad , \quad h = -1 \quad \text{... for } \underline{\mathbf{1}}_{+-} \tag{5.63}$$
$$g = -1 \quad , \quad h = +1 \quad \text{... for } \underline{\mathbf{1}}_{-+} \tag{5.64}$$
$$g = -1 \quad , \quad h = -1 \quad \text{... for } \underline{\mathbf{1}}_{--} \tag{5.65}$$
$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad , \quad h = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{... for } \underline{\mathbf{2}}\,. \tag{5.66}$$

They fulfill the relations

$$g^2 = \mathbb{1}\ ,\ \ h^2 = \mathbb{1}\ ,\ \ (g\,h)^4 = \mathbb{1}\ . \tag{5.67}$$

From the generators g and h the character table of the group can be deduced, see Table 5.5. The

Kronecker products of the representations read

$$\underline{\mathbf{1}}_{++} \times \mu = \mu \quad \forall \quad \mu \tag{5.68a}$$
$$\underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{+-} = \underline{\mathbf{1}}_{++} \; , \;\; \underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{-+} = \underline{\mathbf{1}}_{--} \; , \;\; \underline{\mathbf{1}}_{+-} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{-+} \; , \tag{5.68b}$$
$$\underline{\mathbf{1}}_{-+} \times \underline{\mathbf{1}}_{-+} = \underline{\mathbf{1}}_{++} \; , \;\; \underline{\mathbf{1}}_{-+} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{+-} \; , \;\; \underline{\mathbf{1}}_{--} \times \underline{\mathbf{1}}_{--} = \underline{\mathbf{1}}_{++} \tag{5.68c}$$
$$\underline{\mathbf{1}}_{+-} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \; , \;\; \underline{\mathbf{1}}_{-+} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \; , \;\; \underline{\mathbf{1}}_{--} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \; , \tag{5.68d}$$
$$\underline{\mathbf{2}} \times \underline{\mathbf{2}} = \underline{\mathbf{1}}_{++} + \underline{\mathbf{1}}_{+-} + \underline{\mathbf{1}}_{-+} + \underline{\mathbf{1}}_{--} \tag{5.68e}$$

The non-trivial Clebsch Gordan coefficients read

$$\begin{pmatrix} A\, a_1 \\ A\, a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \;\; \begin{pmatrix} B\, a_1 \\ -B\, a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \;\; \begin{pmatrix} C\, a_2 \\ C\, a_1 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \;\; \begin{pmatrix} D\, a_2 \\ -D\, a_1 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \tag{5.69}$$
$$a_1\, a_1' + a_2\, a_2' \sim \underline{\mathbf{1}}_{++} \; , \;\; a_1\, a_1' - a_2\, a_2' \sim \underline{\mathbf{1}}_{+-} \; , \;\; a_1\, a_2' + a_2\, a_1' \sim \underline{\mathbf{1}}_{-+} \; , \;\; a_1\, a_2' - a_2\, a_1' \sim \underline{\mathbf{1}}_{--} \; , \tag{5.70}$$

for

$$A \sim \underline{\mathbf{1}}_{++} \; , \;\; B \sim \underline{\mathbf{1}}_{+-} \; , \;\; C \sim \underline{\mathbf{1}}_{-+} \; , \;\; D \sim \underline{\mathbf{1}}_{--} \quad \text{and} \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \sim \underline{\mathbf{2}} \; . \tag{5.71}$$

The authors assign the left-handed and right-handed [13] fermions to the D_4 representations $\underline{\mathbf{1}}_{++} + \underline{\mathbf{2}}$, i.e. the first generation transforms trivially and the second and third one are unified into a doublet. They include three copies of the SM Higgs doublet into their model which transform as singlets under D_4, $\phi_{1,2} \sim \underline{\mathbf{1}}_{++}$ and $\phi_3 \sim \underline{\mathbf{1}}_{+-}$. The Higgs doublet ϕ_1 contributes to the Dirac mass matrix of the neutrinos and the electron mass, while $\phi_{2,3}$ determine the masses m_μ and m_τ. This separation is possible due to the additional $Z_2^{(aux)}$ under which the particles transform according to Table 5.6. Additionally, they introduce two gauge singlets $\chi_{1,2}$ which form a doublet under D_4 which only couples to the right-handed neutrinos. With this knowledge we can reproduce the Yukawa

Field	D_e	(D_μ, D_τ)	e_R	(μ_R, τ_R)	$\nu_{e\,R}$	$(\nu_{\mu\,R}, \nu_{\tau\,R})$	ϕ_1	ϕ_2	ϕ_3	$\chi_{1,2}$
D_4	$\underline{\mathbf{1}}_{++}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_{++}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_{++}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_{++}$	$\underline{\mathbf{1}}_{++}$	$\underline{\mathbf{1}}_{+-}$	$\underline{\mathbf{2}}$
${Z_2}^{(aux)}$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$

Table 5.6: The particle content and its symmetry properties under $D_4 \times {Z_2}^{(aux)}$. We adopted the notation of [94], where the left-handed lepton doublets are $D_\alpha = (\nu_\alpha, \alpha)^t$ for $\alpha = e, \mu, \tau$, the right-handed charged leptons are e_R, μ_R, τ_R and three right-handed neutrinos exist, $\nu_{e\,R}$, $\nu_{\mu\,R}$, $\nu_{\tau\,R}$. The three scalar fields ϕ_i are copies of the SM Higgs doublet and the fields $\chi_{1,2}$ are gauge singlets, which only contribute to the Majorana mass matrix of the right-handed neutrinos.

[13] Unlike we, the authors work with left-handed and right-handed instead of left-handed and left-handed conjugate fields. The appearance of the Dirac and Majorana mass terms therefore slightly changes. However, the physical results stay the same. Since we assume that the reader is familiar with the SM, we do not discuss the mass terms in the basis of left- and right-handed fields at length, as done in Section 3.1 for those in the basis of left-handed and left-handed conjugate fields.

couplings and coupling terms of the right-handed neutrinos shown in [94]. They read [14]

$$\begin{aligned}
\mathcal{L}_Y &= y_1\, \overline{D}_e\, \nu_{e\,R}\, \tilde{\phi}_1 + y_2\, (\overline{D}_\mu\, \nu_{\mu\,R} + \overline{D}_\tau\, \nu_{\tau\,R})\, \tilde{\phi}_1 && (5.72)\\
&+ y_3\, \overline{D}_e\, e_R\, \phi_1 + y_4\, (\overline{D}_\mu\, \mu_R + \overline{D}_\tau\, \tau_R)\, \phi_2 + y_5\, (\overline{D}_\mu\, \mu_R - \overline{D}_\tau\, \tau_R)\, \phi_3 + \text{h.c.}\\
\mathcal{L}_{\nu_R} &= M\, \nu_{e\,R}\, \nu_{e\,R} + M'\, (\nu_{\mu\,R}\, \nu_{\mu\,R} + \nu_{\tau\,R}\, \nu_{\tau\,R}) && (5.73)\\
&+ y_\chi\, (\nu_{e\,R}\, \nu_{\mu\,R}\, \chi_1 + \nu_{e\,R}\, \nu_{\tau\,R}\, \chi_2) + y_\chi\, (\nu_{\mu\,R}\, \nu_{e\,R}\, \chi_1 + \nu_{\tau\,R}\, \nu_{e\,R}\, \chi_2) + \text{h.c.}
\end{aligned}$$

and lead to the following mass matrices [15]

$$\mathcal{M}_\nu = \frac{v_1^\star}{\sqrt{2}}\, \mathrm{diag}\, (y_1, y_2, y_2)\ , \tag{5.74}$$

$$\mathcal{M}_l = \frac{1}{\sqrt{2}}\, \mathrm{diag}\, (y_3\, v_1, y_4\, v_2 + y_5\, v_3, y_4\, v_2 - y_5\, v_3)\ , \tag{5.75}$$

$$M_{RR} = \begin{pmatrix} M & y_\chi\, W\, \cos(\gamma) & y_\chi\, W\, \sin(\gamma) \\ y_\chi\, W\, \cos(\gamma) & M' & 0 \\ y_\chi\, W\, \sin(\gamma) & 0 & M' \end{pmatrix} \tag{5.76}$$

with the VEVs of the scalars

$$\langle \phi_i \rangle = \frac{v_i}{\sqrt{2}} \quad (i = 1, 2, 3)\ , \quad \langle \chi_1 \rangle = W\, \cos(\gamma) \quad \text{and} \quad \langle \chi_2 \rangle = W\, \sin(\gamma) \qquad (W > 0). \tag{5.77}$$

For $\gamma = \frac{\pi}{4}$ the VEVs of the gauge singlets are equal, $\langle \chi_1 \rangle = \langle \chi_2 \rangle = \frac{W}{\sqrt{2}}$, and the Majorana mass matrix for the right-handed neutrinos is $\mu\tau$ symmetric

$$M_{RR} = \begin{pmatrix} M & M_\chi & M_\chi \\ M_\chi & M' & 0 \\ M_\chi & 0 & M' \end{pmatrix} \quad \text{with} \quad M_\chi = \frac{y_\chi\, W}{\sqrt{2}}\ . \tag{5.78}$$

Since the Dirac mass matrix of the neutrinos is also $\mu\tau$ symmetric, the light neutrino mass matrix derived from the type-1 seesaw shares this property. Therefore, this model predicts $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ for leptons, while θ_{12} is not constrained. Due to the choice of the group basis $\mathcal{M}_l$ and $\mathcal{M}_\nu$ are diagonal and the lepton mixings solely originate from the right-handed Majorana mass matrix M_{RR}. The authors of [94] show that the assumption $\gamma = \frac{\pi}{4}$ can be derived from the minimization of the scalar potential for $W \sim |M|, |M'| \gg v$, where $v \approx v_i$ is the electroweak scale.

In the following, we analyze the mathematical structure of D_4 and show that $\mathcal{M}_l$, $\mathcal{M}_\nu$ and M_{RR} conserve different subgroups of D_4. We find that $\mathcal{M}_l$ conserves a D_2 subgroup, $\mathcal{M}_\nu$ does not break D_4 at all and M_{RR} preserves a Z_2 subgroup.

We start with the simplest observation, namely $\mathcal{M}_\nu$ does not break D_4: This is easy to see, since the Higgs field giving a Dirac mass to the neutrinos transforms as $\underline{\mathbf{1}}_{++}$ under D_4 which is the invariant under D_4 and therefore cannot induce any breaking of the symmetry. Now we turn the charged lepton mass matrix: We show that the Higgs fields coupling to the charged leptons preserve the $D_2 \cong Z_2 \times Z_2$ group generated by g and $(g\, h)^2$. From the generator relations it is clear that g and

[14]Note the additional overall sign which the authors of [94] include into the Dirac mass terms and the additional factor of $\frac{1}{2}$ they induce into the mass terms/couplings of the right-handed neutrinos.

[15]The mass matrices are given in the basis which is used in the terms in the Lagrangian. They therefore differ from the ones shown in [94] which are rather given in the basis $\bar{R}\, L$ for the Dirac mass terms and $\bar{R}\, \bar{R}$ for the right-handed neutrinos. This mainly induces a complex conjugation of the parameters of the model. Due to this choice of basis the type-1 seesaw formula also deviates from the one which is used in the majority of the publications. However, the physical results are the same independent of the choice of basis.

$(g\,h)^2$ both generate a Z_2 group. Furthermore it is obvious that they do not coincide. Additionally, one has to show that g and $(g\,h)^2$ commute

$$g\,(g\,h)^2 = g\,g\,h\,g\,h \overset{g^2=\mathbb{1}}{=} h\,g\,h\,g\,g = (h\,g)^2\,g \overset{(\star)}{=} (g\,h)^2\,g \quad \Rightarrow \quad g\;,\;(g\,h)^2 \quad \text{commute}\,. \tag{5.79}$$

with $(\star)$ being

$$\begin{aligned} & (g\,h)^2\,(g\,h)^2 = \mathbb{1} \\ \Leftrightarrow\quad & (g\,h)^2 = (g\,h)^{-2} = (h^{-1}\,g^{-1})^2 \\ \Leftrightarrow\quad & (g\,h)^2 = (h\,g)^2\;,\;\text{ since }\; g = g^{-1} \;\text{ and }\; h = h^{-1}\;. \end{aligned} \tag{5.80}$$

Therefore g and $(g\,h)^2$ generate a group $Z_2 \times Z_2$ which is isomorphic to D_2. We now have to check how the D_4 representations transform under the D_2 subgroup. From the generators given above, we see that g and $(g\,h)^2$ have the following form for the representations

$$g = +1 \quad,\quad (g\,h)^2 = +1 \qquad \text{... for } \underline{\mathbf{1}}_{++} \tag{5.81}$$

$$g = +1 \quad,\quad (g\,h)^2 = +1 \qquad \text{... for } \underline{\mathbf{1}}_{+-} \tag{5.82}$$

$$g = -1 \quad,\quad (g\,h)^2 = +1 \qquad \text{... for } \underline{\mathbf{1}}_{-+} \tag{5.83}$$

$$g = -1 \quad,\quad (g\,h)^2 = +1 \qquad \text{... for } \underline{\mathbf{1}}_{--} \tag{5.84}$$

$$g = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad,\quad (g\,h)^2 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \qquad \text{... for } \underline{\mathbf{2}}\,. \tag{5.85}$$

Only the two one-dimensional representations $\underline{\mathbf{1}}_{++}$ and $\underline{\mathbf{1}}_{+-}$ preserve the D_2 subgroup, since only in their case both generators of D_2, g and $(g\,h)^2$ equal $+1$, i.e. $\underline{\mathbf{1}}_{++}$ and $\underline{\mathbf{1}}_{+-}$ do not transform, when g and $(g\,h)^2$ are applied. For the two-dimensional representation we see that the generator $(g\,h)^2$ does not have an eigenvalue $+1$ and so none of the combinations of the upper and lower components of a D_4 doublet transforms as invariant under the D_2 subgroup. Since D_2 is abelian, $\underline{\mathbf{2}}$ of D_4 splits up into two one-dimensional representations which can be easily read off from the two generators g and $(g\,h)^2$, since these are simultaneously diagonal. They indicate that the upper component transforms as $+1$ under g and -1 under $(g\,h)^2$, while the lower one transforms as -1 under both generators g and $(g\,h)^2$. In the model of [94] the charged leptons acquire masses through the coupling to the Higgs fields $\phi_{1,2} \sim \underline{\mathbf{1}}_{++}$ and $\phi_3 \sim \underline{\mathbf{1}}_{+-}$ under D_4. As shown before, if only these fields get a VEV they preserve a D_2 subgroup. Finally, we show that the right-handed Majorana mass sector conserves a Z_2 group which is generated by h alone. According to the generators above the one-dimensional representations $\underline{\mathbf{1}}_{++}$ and $\underline{\mathbf{1}}_{-+}$ transform as $+1$ under h and therefore preserve this generator. For the two-dimensional representation we have to calculate the eigenvalues and eigenvectors of h in order to find the combination of upper and lower components of the doublet which preserves h. The eigenvalues are $+1$ and -1 and the corresponding eigenvectors are

$$v_{+1} \propto \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \text{and} \quad v_{-1} \propto \begin{pmatrix} 1 \\ -1 \end{pmatrix}\,. \tag{5.86}$$

Therefore

$$a_1 + a_2 \tag{5.87}$$

transforms trivially under the remaining Z_2 group, while

$$a_1 - a_2 \tag{5.88}$$

gets a sign, if $a_{1,2}$ are the upper and lower component of a doublet $\underline{\mathbf{2}}$. The VEV structure which does not break the Z_2 group is therefore $\langle a_1 \rangle = \langle a_2 \rangle$. Looking at the mass matrix structure for the right-handed neutrinos we recognize that the two gauge singlets $\chi_{1,2} \sim \underline{\mathbf{2}}$ exactly preserve this Z_2 subgroup, since the VEVs of χ_1 and χ_2 are equal. The direct mass terms do not break the flavor symmetry anyway. Therefore M_{RR} conserves the Z_2 group. One can then ask two questions

a.) Does the model significantly change, if we include an additional gauge singlet ψ which transforms as $\underline{\mathbf{1}}_{-+}$ and which is also allowed to have a non-vanishing VEV, since it does not transform under the generator h ?

b.) Since we do not include a scalar field for all representations which transform trivially under the Z_2 subgroup, found in the right-handed Majorana neutrino sector, do we maybe preserve a larger group ?

Concerning *a.*) we simply calculate the additional term which is allowed, if a gauge singlet $\psi \sim \underline{\mathbf{1}}_{-+}$ with charge $+1$ under $Z_2^{(aux)}$ exists

$$\mathcal{L}'_{\nu_R} = y_\psi \left(\nu_{\mu\, R}\, \nu_{\tau\, R} + \nu_{\tau\, R}\, \nu_{\mu\, R} \right) \psi \; . \tag{5.89}$$

The mass matrix $\widetilde{M}_{RR}$ reads

$$\widetilde{M}_{RR} = \left(\begin{array}{ccc} M & M_\chi & M_\chi \\ M_\chi & M' & y_\psi \langle \psi \rangle \\ M_\chi & y_\psi \langle \psi \rangle & M' \end{array} \right) \tag{5.90}$$

and therefore is still $\mu\tau$ symmetric and leads again to a mass matrix for the light neutrinos which is also $\mu\tau$ symmetric and hence predicts $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ for leptons. However, the texture zero in M_{RR} leads to additional predictions not correlated to the mixing angles. For example, the authors of [94] showed that the neutrino mass spectrum is normally ordered in their model due to $(M_{RR})_{23} = 0$.

Concerning *b.*) the easiest way to answer this question is to calculate the eight distinct representation matrices of the representation $\underline{\mathbf{2}}$ and check how many of these possess an eigenvalue $+1$ with eigenvector $\left(\begin{array}{c} 1 \\ 1 \end{array} \right)$. The eight matrices read

$$\begin{array}{c} \left(\begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right) , \left(\begin{array}{cc} 1 & 0 \\ 0 & -1 \end{array} \right) , \left(\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array} \right) , \left(\begin{array}{cc} -1 & 0 \\ 0 & -1 \end{array} \right) , \\ \left(\begin{array}{cc} -1 & 0 \\ 0 & 1 \end{array} \right) , \left(\begin{array}{cc} 0 & -1 \\ -1 & 0 \end{array} \right) , \left(\begin{array}{cc} 0 & -1 \\ 1 & 0 \end{array} \right) , \left(\begin{array}{cc} 0 & 1 \\ -1 & 0 \end{array} \right) . \end{array} \tag{5.91}$$

Only the first and the third matrix have $\left(\begin{array}{c} 1 \\ 1 \end{array} \right)$ as eigenvector for the eigenvalue $+1$. The first matrix is the identity and the third one is the generator h. Therefore the maximally preserved subgroup is the Z_2 group generated by h.

5.4.2 $D_3 \times Z_2^{(aux)}$ Model

In this section we analyze the $D_3 \times Z_2^{(aux)}$ model in the same fashion as the $D_4 \times Z_2^{(aux)}$ model. We explain that maximal atmospheric mixing and vanishing θ_{13} stem from the preservation of a Z_3 subgroup in the charged lepton sector, a Z_2 subgroup in the right-handed Majorana neutrino

sector and an unbroken D_3 group in the Dirac neutrino mass matrix. The results of the model are very similar to the ones of the D_4 model. However, we think that it is useful to also discuss this model in detail in order to shed light on the group theoretical reasons for the prediction of maximal atmospheric mixing and vanishing θ_{13}.
We start with the mathematics of D_3. Since the authors of [116] work in the same group basis as we do, the generators A and B are special cases of the general ones shown in Section 5.1.1. They read

$$\mathrm{A} = +1 \quad , \quad \mathrm{B} = +1 \quad \text{... for } \underline{\mathbf{1_1}} \tag{5.92}$$

$$\mathrm{A} = +1 \quad , \quad \mathrm{B} = -1 \quad \text{... for } \underline{\mathbf{1_2}} \tag{5.93}$$

$$\mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{2\pi i}{3}} & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi i}{3}} \end{pmatrix} \quad , \quad \mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{... for } \underline{\mathbf{2}} \, . \tag{5.94}$$

and fulfill

$$\mathrm{A}^3 = \mathbb{1} \; , \;\; \mathrm{B}^2 = \mathbb{1} \; , \;\; \mathrm{A\,B\,A} = \mathrm{B} \tag{5.95}$$

which is a special case of Eq.(5.1). The character table is given in Table A.1. The Kronecker products read

$$\underline{\mathbf{1_1}} \times \mu = \mu \quad \forall \quad \mu \; , \;\; \underline{\mathbf{1_2}} \times \underline{\mathbf{1_2}} = \underline{\mathbf{1_1}} \; , \;\; \underline{\mathbf{1_2}} \times \underline{\mathbf{2}} = \underline{\mathbf{2}} \; , \tag{5.96a}$$

$$[\underline{\mathbf{2}} \times \underline{\mathbf{2}}] = \underline{\mathbf{1_1}} + \underline{\mathbf{2}} \; , \;\; \{\underline{\mathbf{2}} \times \underline{\mathbf{2}}\} = \underline{\mathbf{1_2}} \; , \tag{5.96b}$$

and for the Clebsch Gordan coefficients we find

$$\begin{pmatrix} A\,a_1 \\ A\,a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \;\; \begin{pmatrix} B\,a_1 \\ -B\,a_2 \end{pmatrix} \sim \underline{\mathbf{2}} \; , \tag{5.97}$$

$$a_1\,a'_2 + a_2\,a'_1 \sim \underline{\mathbf{1_1}} \; , \;\; a_1\,a'_2 - a_2\,a'_1 \sim \underline{\mathbf{1_2}} \; , \;\; \begin{pmatrix} a_2\,a'_2 \\ a_1\,a'_1 \end{pmatrix} \sim \underline{\mathbf{2}}$$

with

$$A \sim \underline{\mathbf{1_1}} \; , \;\; B \sim \underline{\mathbf{1_2}} \; , \;\; \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} , \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}} \, . \tag{5.98}$$

The fact that the authors work with left- and right-handed fields requires to know that for $\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} \sim \underline{\mathbf{2}}$ it is $\begin{pmatrix} a_2^\star \\ a_1^\star \end{pmatrix}$ which transforms as $\underline{\mathbf{2}}$.
The assignment of the fields is very similar to the one in the D_4 model, i.e. the second and third generation are unified into a doublet of the flavor symmetry and the first one transforms trivially. Again, there exist three copies of the SM Higgs doublet, called ϕ_i ($i = 1, 2, 3$), which transform as $\underline{\mathbf{1_1}}$ and $\underline{\mathbf{1_2}}$. ϕ_1 gives mass to the electron and generates the Dirac mass term of the neutrinos, while $\phi_{2,3}$ are responsible for μ and τ mass. The right-handed Majorana mass matrix stems from direct D_3-invariant mass terms and from the coupling to a complex gauge singlet χ which forms a doublet under D_3 together with its complex conjugate. The additional $Z_2^{(aux)}$ which constrains the couplings of the fields ϕ_i is the same as above. All this is collected in Table 5.7. Given the particle content of the model we can write down the Yukawa couplings and the mass terms for the

Field	D_e	(D_μ, D_τ)	e_R	(μ_R, τ_R)	$\nu_{e\,R}$	$(\nu_{\mu\,R}, \nu_{\tau\,R})$	ϕ_1	ϕ_2	ϕ_3	$(\chi,\, \chi^\star)$
D_3	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{2}}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{1}$	$\underline{\mathbf{1}}_\mathbf{2}$	$\underline{\mathbf{2}}$
${Z_2}^{(aux)}$	$+$	$+$	$-$	$+$	$-$	$-$	$-$	$+$	$+$	$+$

Table 5.7: The particle content and its symmetry properties under $D_3 \times\, {Z_2}^{(aux)}$. The notation is the same as in Table 5.6. The three scalar fields ϕ_i are copies of the SM Higgs doublet and the field χ is a complex gauge singlet, which only contributes to the Majorana mass matrix of the right-handed neutrinos.

right-handed neutrinos [16]

$$\begin{aligned}
\mathcal{L}_Y &= y_1\, \overline{D}_e\, \nu_{e\,R}\, \tilde{\phi}_1 + y_2\, (\overline{D}_\mu\, \nu_{\mu\,R} + \overline{D}_\tau\, \nu_{\tau\,R})\, \tilde{\phi}_1 && (5.99)\\
&+ y_3\, \overline{D}_e\, e_R\, \phi_1 + y_4\, (\overline{D}_\mu\, \mu_R + \overline{D}_\tau\, \tau_R)\, \phi_2 + y_5\, (\overline{D}_\mu\, \mu_R - \overline{D}_\tau\, \tau_R)\, \phi_3 + \text{h.c.}\\
\mathcal{L}_{\nu_R} &= M\, \nu_{e\,R}\, \nu_{e\,R} + M'\, (\nu_{\mu\,R}\, \nu_{\tau\,R} + \nu_{\tau\,R}\, \nu_{\mu\,R}) && (5.100)\\
&+ y_\chi\, (\nu_{e\,R}\, \nu_{\mu\,R}\, \chi^\star + \nu_{e\,R}\, \nu_{\tau\,R}\, \chi) + y_\chi\, (\nu_{\mu\,R}\, \nu_{e\,R}\, \chi^\star + \nu_{\tau\,R}\, \nu_{e\,R}\, \chi)\\
&+ z_\chi\, (\nu_{\mu\,R}\, \nu_{\mu\,R}\, \chi + \nu_{\tau\,R}\, \nu_{\tau\,R}\, \chi^\star) + \text{h.c.}
\end{aligned}$$

These lead to the following mass matrices [17]

$$\begin{aligned}
\mathcal{M}_\nu &= v_1^\star\, \text{diag}\,(y_1, y_2, y_2)\;, && (5.101)\\
\mathcal{M}_l &= \text{diag}\,(y_3\, v_1, y_4\, v_2 + y_5\, v_3, y_4\, v_2 - y_5\, v_3)\;, && (5.102)\\
M_{RR} &= \begin{pmatrix} M & y_\chi\, |W|\, \mathrm{e}^{-i\,\alpha} & y_\chi\, |W|\, \mathrm{e}^{i\,\alpha} \\ y_\chi\, |W|\, \mathrm{e}^{-i\,\alpha} & z_\chi\, |W|\, \mathrm{e}^{i\,\alpha} & M' \\ y_\chi\, |W|\, \mathrm{e}^{i\,\alpha} & M' & z_\chi\, |W|\, \mathrm{e}^{-i\,\alpha} \end{pmatrix} && (5.103)
\end{aligned}$$

with the VEVs of the scalars

$$\langle \phi_i \rangle = v_i \quad (i = 1, 2, 3)\;,\;\; \langle \chi \rangle = |W|\, \mathrm{e}^{i\,\alpha} \quad \text{and} \quad \langle \chi^\star \rangle = |W|\, \mathrm{e}^{-i\,\alpha}\;. \tag{5.104}$$

Note that the resulting mass matrices are similar to the ones derived with the help of the flavor symmetry $D_4 \times Z_2^{(aux)}$. Especially, the basis is again chosen in such a way that $\mathcal{M}_l$ and $\mathcal{M}_\nu$ are diagonal and the lepton mixing solely stems from the Majorana mass matrix of the right-handed neutrinos. As shown by the authors of [116] M_{RR} can be written as

$$M_{RR} = \begin{pmatrix} M & y_\chi\, |W| & y_\chi\, |W| \\ y_\chi\, |W| & z_\chi\, |W|\, \mathrm{e}^{3\,i\,\alpha} & M' \\ y_\chi\, |W| & M' & z_\chi\, |W|\, \mathrm{e}^{-3\,i\,\alpha} \end{pmatrix} \tag{5.105}$$

if the fermion fields are rephased. This form of M_{RR} clearly shows that $\mu\tau$ symmetry is maintained, if $\mathrm{e}^{3\,i\,\alpha} = \pm 1$. This leads to the allowed values of $\alpha = 0, \pm\frac{\pi}{3}, \pm\frac{2\pi}{3}, \pm\pi$ and therefore requires the VEV configuration of the D_3 doublet, consisting of χ and $\chi^\star$, to be proportional to

$$\begin{pmatrix} \langle \chi \rangle \\ \langle \chi^\star \rangle \end{pmatrix} = \begin{pmatrix} |W|\, \mathrm{e}^{i\,\alpha} \\ |W|\, \mathrm{e}^{-i\,\alpha} \end{pmatrix} = |W|\, \mathrm{e}^{-i\,\alpha} \begin{pmatrix} \mathrm{e}^{2\,i\,\alpha} \\ 1 \end{pmatrix} \propto \underbrace{\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{\alpha=0,\pm\pi}, \;\; \underbrace{\begin{pmatrix} \mathrm{e}^{\frac{2\,\pi\,i}{3}} \\ 1 \end{pmatrix}}_{\alpha=\frac{\pi}{3},-\frac{2\,\pi}{3}}, \;\; \underbrace{\begin{pmatrix} \mathrm{e}^{-\frac{2\,\pi\,i}{3}} \\ 1 \end{pmatrix}}_{\alpha=-\frac{\pi}{3},\frac{2\,\pi}{3}}\;. \tag{5.106}$$

[16] For reasons for the differences in the appearance of the Yukawa terms and Majorana masses see footnote above.
[17] For explanation of differences in mass matrices see footnote above.

As the authors showed in [116] the potential of the scalar fields actually enforces such configurations. Therefore $\mu\tau$ symmetry in the neutrino mass matrix and consequently $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ are predictions of this model. Compared to the model above, note that all elements in M_{RR} are non-vanishing and hence its form is the general one which is $\mu\tau$ symmetric. Due to this no further predictions, for example on the neutrino mass spectrum, can be made.
After presenting this model we analyze the group theory behind it. Therefore we enumerate the elements of the group D_3 and explicitly show under which conditions which subgroup remains unbroken. The six distinct elements of D_3 can be expressed as E, A, A^2, B, B A and $\mathrm{B\,A}^2$. For the representations $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$ and $\underline{\mathbf{2}}$ they read

$$\text{... for } \underline{\mathbf{1}}_{\mathbf{1}} \qquad E = +1\ , \ \ \mathrm{A} = +1\ , \ \ \mathrm{A}^2 = +1\ , \ \ \mathrm{B} = +1\ , \ \ \mathrm{B\,A} = +1\ , \ \ \mathrm{B\,A}^2 = +1 \tag{5.107}$$

$$\text{... for } \underline{\mathbf{1}}_{\mathbf{2}} \qquad E = +1\ , \ \ \mathrm{A} = +1\ , \ \ \mathrm{A}^2 = +1\ , \ \ \mathrm{B} = -1\ , \ \ \mathrm{B\,A} = -1\ , \ \ \mathrm{B\,A}^2 = -1 \tag{5.108}$$

$$\text{... for } \underline{\mathbf{2}} \qquad E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\ , \ \ \mathrm{A} = \begin{pmatrix} \mathrm{e}^{\frac{2\pi i}{3}} & 0 \\ 0 & \mathrm{e}^{-\frac{2\pi i}{3}} \end{pmatrix}\ , \ \ \mathrm{A}^2 = \begin{pmatrix} \mathrm{e}^{-\frac{2\pi i}{3}} & 0 \\ 0 & \mathrm{e}^{\frac{2\pi i}{3}} \end{pmatrix}, \tag{5.109}$$

$$\mathrm{B} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\ , \ \ \mathrm{B\,A} = \begin{pmatrix} 0 & \mathrm{e}^{-\frac{2\pi i}{3}} \\ \mathrm{e}^{\frac{2\pi i}{3}} & 0 \end{pmatrix}\ , \ \ \mathrm{B\,A}^2 = \begin{pmatrix} 0 & \mathrm{e}^{\frac{2\pi i}{3}} \\ \mathrm{e}^{-\frac{2\pi i}{3}} & 0 \end{pmatrix}.$$

According to the generator relations, shown in Eq.(5.95), one finds that there exist four distinct subgroups: three Z_2 groups and one Z_3 group. The three Z_2 groups are generated by B, B A and $\mathrm{B\,A}^2$, respectively. The fact that $(\mathrm{B\,A})^2 = \mathbb{1}$ and $(\mathrm{B\,A}^2)^2 = \mathbb{1}$ is a direct consequence of the generator relations. The Z_3 group contains the elements E, A and A^2. In the following we discuss how these subgroups can be preserved. We start with the Z_3 group and see that both one-dimensional representations transform as the trivial representation of Z_3, while the two-dimensional one splits up into the two complex conjugated non-trivial singlets of Z_3. Therefore, demanding that this group is preserved allows non-vanishing VEVs for scalars transforming as $\underline{\mathbf{1}}_{\mathbf{1}}$ or $\underline{\mathbf{1}}_{\mathbf{2}}$, but not as $\underline{\mathbf{2}}$. A common feature, when preserving a Z_2 subgroup, is the fact that fields in D_3 representations $\underline{\mathbf{1}}_{\mathbf{2}}$ are not allowed to have a VEV, since $\underline{\mathbf{1}}_{\mathbf{2}}$ transforms as non-trivial singlet under all three possible Z_2 groups. Apart from the trivial representation $\underline{\mathbf{1}}_{\mathbf{1}}$, always a certain combination of the upper and lower components of the D_3 doublet transforms trivially under the Z_2 subgroup. For the Z_2 generated by B the combination reads

$$a_1 + a_2\ , \tag{5.110}$$

while $a_1 - a_2$ transforms non-trivially under the Z_2 subgroup. Therefore, $\langle a_1 \rangle = \langle a_2 \rangle$ leaves this group invariant for a_i being the upper and lower component of a D_3 doublet. For $Z_2 =< \mathrm{B\,A} >$ the combination is

$$\mathrm{e}^{\frac{2\pi i}{3}}\, a_1 + a_2 \tag{5.111}$$

and $\mathrm{e}^{\frac{2\pi i}{3}}\, a_1 - a_2$ picks up a sign. The VEV configuration which does not break this subgroup is then $\langle a_1 \rangle = \mathrm{e}^{-\frac{2\pi i}{3}} \langle a_2 \rangle$. And similarly, for the third Z_2 group, generated by $\mathrm{B\,A}^2$, the invariant combination is

$$\mathrm{e}^{-\frac{2\pi i}{3}}\, a_1 + a_2\ , \tag{5.112}$$

whereas $\mathrm{e}^{-\frac{2\pi i}{3}}\, a_1 - a_2$ acquires a sign under the Z_2 subgroup. Hence, $\langle a_1 \rangle = \mathrm{e}^{\frac{2\pi i}{3}} \langle a_2 \rangle$ is the VEV correlation which keeps this Z_2 subgroup intact for $a_{1,2}$ being the components of a doublet. These results should be compared to the discussion of the D_7 model in which the subgroups Z_2 generated by $\mathrm{B}\,\mathrm{A}^m$ $(m = 0, ..., 6)$ played an essential role for the prediction of the CKM element $|V_{us\,(cd)}|$. Applying the group theoretical insights to the D_3 model, we arrive at the conclusions: the Dirac neutrino mass matrix which solely stems from the VEV of the Higgs field ϕ_1 preserves the whole D_3 group, since ϕ_1 does not transform under D_3. The charged leptons are coupled to the three Higgs doublets $\phi_{1,2,3}$ which all transform according to one-dimensional representations of D_3. Therefore the charged lepton sector leaves the Z_3 subgroup invariant. And finally, the right-handed neutrinos get direct mass terms which are D_3-invariant and masses from the VEV of the complex gauge singlet χ whose form is fixed in a way that Eq.(5.106) holds. Hence, the VEV coincides with the Z_2 group preserving VEVs, i.e. the residual group in the right-handed Majorana neutrino sector is a Z_2 symmetry. Obviously, all three sectors together break D_3 completely - as it should be. Due to the different possible VEVs for $\langle\chi\rangle$ all Z_2 directions of the group D_3 are explored and not only one, as in the $D_4 \times Z_2^{(aux)}$ model.

5.4.3 Comments on the $D_4 \times Z_2^{(aux)}$ and $D_3 \times Z_2^{(aux)}$ Model

As discussed in much detail, the prediction of $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ is in both models [94] and [116] intimately related to the fact that the flavor symmetry (D_4 or D_3) is not broken in an arbitrary way, but always to a residual subgroup. The structure of the two models is very similar, i.e. the group basis is in both cases chosen in a way that the charged lepton mass matrix $\mathcal{M}_l$ and the Dirac mass matrix of the neutrinos $\mathcal{M}_\nu$ are diagonal. Only the right-handed Majorana mass matrix has a non-trivial structure. This is achieved by including flavored gauge singlets into the model which transform as a doublet under the flavor group. In order to arrive at a $\mu\tau$ symmetric mass matrix for the light neutrinos, these fields cannot acquire an arbitrary VEV, but only certain subgroup preserving structures are allowed. These can be maintained via the potential in both models, as shown by the authors of [94,116]. The mass matrix of the charged leptons results from the coupling to three Higgs fields ϕ_i which transform like the SM Higgs doublet. They lead to a diagonal matrix whose entries are determined by three independent Yukawa couplings and VEVs $\langle\phi_i\rangle$. Out of these three fields only one should contribute to the Dirac mass matrix of the neutrinos. For this purpose, an additional $Z_2^{(aux)}$ symmetry is introduced in both models. Since the Higgs doublet, coupling to the neutrinos, is invariant under the flavor group, the Dirac neutrino sector does not break this symmetry. The matrix $\mathcal{M}_\nu$ is also $\mu\tau$ symmetric, if left- and right-handed neutrinos transform as $\underline{\mathbf{1}} + \underline{\mathbf{2}}$ under the dihedral flavor symmetry.
As the mass matrix of the charged leptons is diagonal, no FCNCs are induced by the additional Higgs doublets at tree level. Otherwise, these would strongly constrain the model. However, as mentioned in [94, 118], these effects are generated at loop level.
In both models scalar fields exist which acquire VEVs at very different scales, i.e. the Higgs doublets ϕ_i have to have a VEV around the electroweak scale $v \approx 174$ GeV, while the VEVs of the gauge singlets are expected to be of the order of the seesaw scale 10^{13} GeV. This leads to a problem, since the potential contains quartic couplings, $\left(\phi_i^\dagger\,\phi_i\right)\,(\chi_1^2 + \chi_2^2)$ and $\left(\phi_i^\dagger\,\phi_i\right)\,(\chi\,\chi^\star)$, respectively, connecting these fields. In order to make the models viable these terms have to have extremely small coefficients. As the authors mention is [94], the introduction of SUSY can solve this problem.
Finally, we would like to point out that possible extensions to the quark sector as well as the implementation of the models in a GUT framework could be very interesting, since these models have the rare feature that they naturally lead to maximal atmospheric mixing and $\theta_{13} = 0$ in the lepton sector.

5.5 Summary and Outlook

In the first part of this chapter we studied the general mass matrix structures induced by a dihedral flavor symmetry [25]. Thereby, we revealed that the number of distinct structures which we encounter in case that a dihedral flavor symmetry is broken spontaneously to one of its subgroups, is very limited, if we additionally require that the determinant of the mass matrix is non-vanishing and either left-handed or left-handed conjugate fermions have to be partially unified. The five distinct structures have been presented in Eq.(5.9), Eq.(5.10) and Eq.(5.11) (Eq.(5.12)). As an application of these results, we presented a way to predict one element of the quark mixing matrix in terms of group theoretical quantities only. For example, it turned out that $|V_{us\,(cd)}|$, i.e. the Cabibbo angle θ_C, can be explained with the help of the group D_7, if we preserve different Z_2 subgroups in the up and down quark sector [26]. Furthermore, we showed that in two models [94,116], which can successfully explain $\theta_{23} = \frac{\pi}{4}$ and $\theta_{13} = 0$ for leptons, these predictions are based on the fact that the flavor symmetries D_4 and D_3 are only broken to non-trivial subgroups in the different sectors (lepton, Dirac neutrino and Majorana neutrino sector). In a next step, it might be worth studying how these results can be combined to construct a model with a dihedral symmetry which can predict the main features of the quark and lepton mixings simultaneously. A solution which additionally allows an embedding into a GUT (and maybe also into a continuous flavor symmetry) would be even more appealing.

For all results and models which have been presented in this chapter we assumed the framework of the SM. It is hence interesting to ask whether these results are also applicable to other frameworks, such as supersymmetric theories and GUTs. As already discussed in Section 4.1 the Yukawa couplings in the MSSM are of the same form as the ones in the SM apart from the fact that in the MSSM two Higgs fields h_u and h_d produce Dirac masses for the fermions. For this reason, all mass matrices will be of the form as shown in Eq.(5.9), Eq.(5.10) and Eq.(5.11) and additional changes arising from the fact that up quarks and neutrinos only couple to the conjugated Higgs field are obsolete. In general, changes occurring due to the embedding into a GUT can only restrict the freedom to assign the different fermions to different representations of the flavor symmetry. For example, the embedding into $SU(5)$ requires that all fermions, which are unified into a $\underline{\mathbf{10}}$ of $SU(5)$, transform in the same way under the flavor group. Similarly, all fermions, which are unified into $\underline{\overline{\mathbf{5}}}$, have to have the same transformation properties. The Dirac mass matrices in $SU(5)$ stem from the coupling $\underline{\mathbf{10}}\,\underline{\mathbf{10}}$ and $\underline{\overline{\mathbf{5}}}\,\underline{\mathbf{10}}$ for up quarks and down quarks and charged leptons, respectively. This implies that the form of the up quark mass matrix is restricted to one originating from the two doublet structure in which one- and two-dimensional representations are equivalent, while the mass matrices of the other charged fermions can be of a form resulting either from the two doublet or from the three singlet structure. Moreover, it is reasonable to ask whether we could replace the Higgs doublets which transform under the flavor symmetry by gauge singlets and thereby disentangle the electroweak and flavor symmetry breaking, as successfully implemented in the T' model discussed in Chapter 4. Actually, we can do so, since we only have to replace the flavored Higgs fields by one Higgs field, which is neutral under the flavor group, and a suitable combination of gauge singlets, which allows us to form an invariant coupling under the flavor group. Hence, the mass matrix structures are expected to be the same. However, one has to keep in mind that in case of flavored Higgs doublets all couplings are renormalizable, while this is in general not the case, if flavored gauge singlets are involved (see, for example, Yukawa couplings in Chapter 4). Thereby, a hierarchy among the couplings is introduced. This leads to the conclusion that in certain cases the implementation with flavored Higgs doublets is favorable, while in others the structure arising from couplings to flavored gauge singlets can be advantageous.

Finally, one might ask the question, why we chose to study the dihedral groups and not another series of groups, like the permutation groups, S_n and A_n, or the series of $SU(3)$ subgroups [80–84], $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$. The permutation groups, however, are not suitable for such a systematic study, since only small groups, S_2, S_3, S_4 and A_3, A_4, A_5, possess non-trivial representations with dimensions lower or equal three. The groups $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ on the other hand are more interesting, since they contain several three-dimensional representations, so that they can explain the existence of three generations. However, since the groups $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ are less known in particle physics, we decided to discuss the dihedral groups instead. Nevertheless, it is interesting to notice that some groups belonging to the series of $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ are known. For example, A_4, the group which is able to predict TBM in the lepton sector, is actually isomorphic to $\Delta(3\,n^2)$ for $n = 2$. As mentioned in Section 3.2 also the group S_4 is isomorphic to a Δ group, namely $\Delta(6\,n^2)$ with $n = 2$. Therefore, we expect that the groups $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ have the ability to also lead to very interesting mixing patterns, if we demand that certain subgroups remain preserved. In addition, the mixing patterns will in general be different from the ones found in case of a dihedral flavor symmetry, since $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ have completely different group structures compared to the D_n and D'_n groups.

Chapter 6

Conclusions and Outlook

Models in particle theory turned out to be very successful in describing the gauge interactions of the three generations of fermions and basic properties like charge quantization. However, none of them is able to explain the existence of exactly three generations, the hierarchical fermion masses or the diverse mixings of quarks and leptons. As we argued in the Introduction, invoking an additional symmetry, which now acts on the three fermion generations, can shed light on these open questions. In the context of the presented thesis this symmetry is always discrete and non-abelian.

We showed several examples of models with discrete non-abelian flavor groups. The two simplest models, discussed in Chapter 3, augment the SM by the permutation group S_4 and by the dihedral symmetry D_5, respectively. These are only spontaneously broken at the electroweak scale. The fermion assignment can be uniquely determined, if additional requirements such as the embedding into GUTs and/or continuous flavor groups are imposed. However, due to the large number of Yukawa couplings and the complicated structure of the Higgs sector, both models can only fit the fermion masses and the mixing patterns successfully, but are not predictive. Since such models have to contain several flavored Higgs doublets, additional phenomenological problems arise, like large FCNCs and LFVs mediated by the further Higgs particles which are generically rather light. Moreover, several of these potentials possess accidental (continuous) symmetries, albeit their complicated structure. In order to improve this situation and find a predictive model, we studied another class of models in which the electroweak and the flavor symmetry breaking scale are disentangled. As a consequence these models only contain the Higgs fields, which are usually present to break the gauge group and to give masses to the fermions. Additional gauge singlets are then responsible for the flavor symmetry breaking. The actual realization, we presented here, is an extension of the MSSM. The role of the flavor group is thereby played by the double-valued tetrahedral group T'. The model has the salient features to explain TBM in the lepton sector as well as to predict relations among $|V_{us}|$, $|V_{td}/V_{ts}|$ and m_d/m_s. Moreover, it allows a deeper understanding of the diverse mixing pattern observed in the quark and lepton sector: The up and down quark sector preserve the same Z_3 group of T' and therefore lead to small mixing angles. In contrast to this, the subgroups conserved in the neutrino and charged lepton sector do not coincide, i.e. the neutrino mass matrix originates from couplings to fields whose VEVs break T' down to Z_4, whereas the charged leptons preserve the same subgroup as the quarks. The actual prediction of TBM is thereby intimately correlated to the fact that T' is not allowed to be broken in an arbitrary way. An additional Z_3 symmetry is used for the separation of the different T' breaking sectors. A careful study of the flavon potential is mandatory in order to show that the advocated VEV structures can be realized. This is indeed the case thanks to two basic ingredients of the model, namely the flavor symmetry breaking via gauge singlets and the supersymmetric framework. Additionally,

further fields and an extra $U(1)$ symmetry have to be introduced to arrive at the actual potential. The fact that the model does not contain additional Higgs doublets also solves the problem with the FCNCs and LFVs, which would otherwise be mediated. Since it turned out that not all properties of the quarks can be explained with the leading order results, the next-to-leading order has been studied carefully. The main challenge is to accommodate the size of the Cabibbo angle $\theta_C = \lambda \approx 0.22$ and at the same time not to spoil the TBM in the lepton sector. Fortunately, this can be maintained (almost) without any further assumptions on the parameters of the model. The appealing interpretation of the different mixing pattern of quarks and leptons in terms of the distinct breaking of the flavor symmetry in these sectors is a main message of this model. In order to show that T' is not the only symmetry with which this idea can be implemented, we studied the mass matrix structures originating from a large class of discrete symmetries in the third part of the work. As class of symmetries we chose the dihedral groups D_n and their double-valued counterparts D'_n. This systematic study revealed that only five (Dirac) mass matrix structures can arise, if the dihedral symmetry is only broken in a non-trivial way. As additional constraints we required that the determinant of the resulting mass matrices should be non-vanishing and at least two generations of left-handed or left-handed conjugate fermions have to be unified into an irreducible representation. Apart from this, the mass matrix structures have the unique feature that they are only determined by the choice of the fermion representations and the structure of the dihedral group, but not by the choice of the transformation properties of the scalar (Higgs/flavon) fields. Subsequently, we presented three examples of models with dihedral flavor symmetries which make clear predictions for the fermion mixings due to the preservation of non-trivial subgroups. In the first example the mismatch of different directions of residual Z_2 groups in the up and down quark sector gives rise to the Cabibbo angle θ_C, i.e. leads to the prediction of the CKM element $|V_{us}|$ or $|V_{cd}|$ to be $\cos(\frac{3\pi}{7}) \approx 0.2225$, independent of the choice of arbitrarily tunable parameters, like Yukawa couplings. The second and third example show that maximal atmospheric mixing and vanishing θ_{13} can originate from the dihedral groups D_4 and D_3 in case that they are not broken in an arbitrary way in the charged lepton, Dirac neutrino and Majorana neutrino sector. Since these results crucially depend on the VEV configuration, a careful study of the Higgs/flavon potential is obligatory. A first realization of the first model employed flavored Higgs doublets and therefore only could adjust, but not predict the advocated VEV structures. In contrast to this, it can be shown that in the second and third model the subgroup preserving VEVs are natural solutions of the scalar potentials without additional assumptions.
The prospects for the study of flavor symmetries are therefore the following:

- construction of flavored GUT models,
- search for further imprints of the flavor symmetry,
- systematic study of further classes of discrete groups.

Concerning the first item there are several reasons to pursue this aim: *a.*) GUTs themselves have many salient features such as the possibility to explain charge quantization, *b.*) the existing models which can successfully describe the flavor sector can hardly be embedded into a GUT, since the fermion assignment under the flavor symmetry is not compatible with the GUT representations (see, for example, the T' model in Chapter 4) and *c.*) some of the generic problems in GUTs such as how to reconcile the strong hierarchy in the up quark sector with the very mild hierarchy among the (light) neutrinos, might be elegantly solved with a flavor symmetry.
By finding a convincing model which describes the fermion masses and their mixings a first goal is reached. Nevertheless, a flavor symmetry always leads to additional experimental signatures which

have not been considered in most of the models discussed in the literature. Further useful information about the flavor structure can be extracted from constraints on $K^0 - \bar{K}^0$, $B^0_{d,s} - \bar{B}^0_{d,s}$ mixing, rare K and B decays and the non-observation of LFV processes, like $\mu \to e\gamma$, $\tau \to \mu\gamma$, $\mu \to 3e$ and $\mu - e$ conversion in nuclei. In the framework of low energy models, in which the flavor symmetry is broken spontaneously at the electroweak scale, these further signatures arise through the existence of several $SU(2)_L$ Higgs doublets and strongly constrain the models. Therefore, most of the models constructed in this way are only semi-realistic [1]. In supersymmetric models additional imprints emerge. These result from the fact that the flavor symmetry in general constrains all terms in a theory and therefore also the soft SUSY breaking terms, i.e. soft masses of the superpartners and the so-called A-terms. It is well-known that in generic MSSM models the rates of FCNC and LFV processes turn out to be much larger than the experimental values/bounds, as long as no special assumptions on the origin of the soft SUSY parameters are made, like mSUGRA initial conditions. Flavor symmetries could have an important impact on these effects [2]. Furthermore, these effects -once they are observed- allow us to differentiate among the numerous models found in the literature which all have the ability to explain/accommodate the fermion masses and mixings. In particular, the process $\mu \to e\gamma$ is an ideal candidate for this purpose, as the MEG experiment [120] is supposed to deliver its first results in the end of 2008.

Apart from the imprints directly correlated to the flavor sector other possible signatures are worth to be explored. For example, in the T' model presented in Chapter 4 the flavon potential contains flat directions in the supersymmetric limit. These could offer interesting connections to cosmology, since a combination of the flavon fields might be a viable candidate for an inflaton.

In contrast to Lie groups which describe the gauge interactions and which are well-classified, up to now no complete survey of discrete groups as flavor symmetries exists. However, as we have shown by our systematic study of the infinite series of D_n and D'_n groups, this might become possible, if we adopt the concept of the breaking to conserved subgroups instead of allowing a flavor symmetry to be broken in an arbitrary way. Furthermore, as demonstrated by four examples, this seems to be the key to a deeper understanding of the mixing patterns of quarks and leptons and to a precise prediction of the mixing parameters. Therefore, it is very interesting to investigate which mathematical group structure can lead to which mixings. Apart from the D_n and D'_n groups the series $\Delta(3\,n^2)$ and $\Delta(6\,n^2)$ for $n \in \mathbb{N}$ offer very interesting opportunities. These have several properties in common regarding the possible dimension of their representations and at least one of them, namely $\Delta(12)$ which is isomorphic to A_4, already turned out to be able to successfully explain TBM in the lepton sector. By studying them we expect to find new mixing patterns, not found in case of a dihedral group, since their group structure is completely different.

Finally, we could think of other topics related to flavor symmetries which are worth to be investigated such as anomalies of discrete groups or their origin in a complete high energy theory. Thereby, string theory might also offer a possibility. In this case the discrete group does not need to be embedded into a continuous one, but it is an outcome of the string theory construction [3].

[1] In the very improbable case that several distinct Higgs particles will be observed at the Large Hadron Collider (LHC) these models could again be very interesting.

[2] Studies of the effects in models with an $SU(3)$ flavor symmetry can be found in [119].

[3] For example, it has been shown in [121] that D_4 and the group $\Delta(54)$ can be present in a certain class of models.

Appendix A

General Remarks on Discrete Groups

In this Appendix we collect the basic knowledge about discrete groups. For further reading we reference [85]. The general **definition of a group** is: A group G is a set of elements R,S,T etc. for which a **law of composition**, i. e. "multiplication", is given so that the product of any two elements RS is well defined and fulfills the following conditions:

(1) If $R\,,S \in G$, then $RS \in G$

(2) The multiplication is **associative**, i. e. $(RS)T = R(ST)$

(3) There exists a unique element E so that for every $R \in G$: $RE = ER = R$. E is called the **identity**.

(4) For every element $R \in G$ exists a unique element S so that $RS = SR = E$. S is called the **inverse** and usually denoted by $S = R^{-1}$.

Regardless of the law of composition, every element of the group commutes with itself. Clearly, the inverse of R^{-1} is R itself. If all elements of the group G commute, this group is called **abelian**. Otherwise it is **non-abelian**. Two groups G and G' are **isomorphic** ($G \cong G'$), if there exists a one-to-one correspondence between the elements of G and G' which preserves the law of composition and the image of G in G' is G'. These groups then have the same structure. A group is **finite/discrete**, if the number of distinct elements of the group is finite. The **order $^\circ\mathrm{G}$ of $\boldsymbol{G}$** is the number of distinct elements in this group. H is called a **subgroup** of G, if $H \subset G$ and H forms a group under the same law of composition, as G does. The **improper subgroups** of G are $\{E\}$ and G itself. Otherwise the subgroup is called **proper**. The order of H has to fulfill : $^\circ\mathrm{H} \mid {}^\circ\mathrm{G}$ (Lagrange's theorem). The **index** of a subgroup H of G is $\frac{^\circ\mathrm{G}}{^\circ\mathrm{H}}$. Clearly, G cannot be isomorphic to any of its proper subgroups. The **order $^\circ\mathrm{h}$ of the element $\boldsymbol{R}$** of G is the smallest integer h for which $R^{\mathrm{h}} = E$ holds. The elements of a group G are divided into **(conjugate) classes** $\mathcal{C}_i$ which consist of all elements R,S of G which are related by $T \in G$ so that $R = T^{-1}\,S\,T$. Note that elements of the same class $\mathcal{C}_i$ have the same order $^\circ\mathrm{h}_{\mathcal{C}_i}$. Per definitionem, $\mathcal{C}_1$ contains the identity of the group which forms a class on its own. The **order $^\circ\boldsymbol{\mathcal{C}_i}$ of a class $\boldsymbol{\mathcal{C}_i}$** is the number of distinct elements in this class. All classes of G are disjoint and therefore $\sum\limits_{\mathcal{C}_i} {}^\circ\mathcal{C}_i \;=\; {}^\circ\mathrm{G}$. Furthermore it holds that $^\circ\mathcal{C}_i \mid {}^\circ\mathrm{G}$. Trivially, the order of the class $\mathcal{C}_1$ is always one. A subset of the elements of G from which all other elements of G can be formed by multiplication is a set of **generators** of G. Note that it is not uniquely determined and also the number of generators can vary. In particular, if the whole group is generated by only one generator, the group must be abelian. If at least two of the generators do not commute, the group is non-abelian. The generators have to fulfill certain

generator relations which determine the group structure. A **representation** μ of a group G is in our case a set of a N_μ-dimensional squared matrices over $\mathbb{R}$ or $\mathbb{C}$, $\mu = \left\{\mathrm{D}^{(\mu)}(R)\right\}$, which fulfill

$$\mathrm{D}(E) = \mathbb{1} \tag{A.1a}$$
$$\mathrm{D}(R^{-1}) = \mathrm{D}(R)^{-1} \tag{A.1b}$$
$$\mathrm{D}(RS) = \mathrm{D}(R)\mathrm{D}(S) \text{ especially } \mathrm{D}(R)^2 = \mathrm{D}(R^2) \tag{A.1c}$$

For $N_\mu = 1$ these are real or complex numbers. The law of composition is the ordinary (matrix) multiplication. Representations are denoted according to their dimension N_μ, e.g. $\underline{\mathbf{1}}_{\mathbf{i}}$, $\underline{\mathbf{2}}_{\mathbf{i}}$, etc.. The attached index i can be omitted, if there exists only one (irreducible) representation of this dimension in the group under discussion. Sometimes this index is replaced by an appropriate number of primes $'$. For any finite group G the representation matrices can always be chosen to be unitary, i. e. $\mathrm{D}^\dagger(R)\mathrm{D}(R) = \mathrm{D}(R)\mathrm{D}^\dagger(R) = \mathbb{1} \quad \forall \quad R \in G$. Therefore, the representations of finite groups are said to be **unitary**. A representation μ of G is **irreducible**, if it cannot be decomposed into other (smaller) representations of G. Two representations $\mu = \left\{\mathrm{D}^{(\mu)}(R)\right\}$ and $\nu = \left\{\mathrm{D}^{(\nu)}(R)\right\}$ are **equivalent**, if there exists a similarity transformation C such that $\mathrm{D}^{(\mu)}(R) = C\mathrm{D}^{(\nu)}(R)C^{-1} \quad \forall \quad R \in G$. Then, they have the same structure. For a **faithful** representation the number of distinct representation matrices equals the order of the group. The products of a faithful representation with itself contain all other representations of the group. If a group is finite, all representations μ are also finite, i.e. have a finite dimension N_μ. Their dimensions N_μ are related to the order of the group by $\sum_\mu N_\mu^2 = \ ^\circ\mathrm{G}$ and $N_\mu | \ ^\circ\mathrm{G}$. The number of classes equals the number of irreducible representations of the group. If $N_\mu > 1$ for one μ, the group is non-abelian. The smallest non-abelian group is the permutation group of three distinct objects which is isomorphic to the dihedral group of order three. It is called Type 6/2 in a mathematical classification [109]. It has six distinct elements. An overview over many discrete groups is given in [85]. The **character table** of a group contains all traces of the representation matrices of all representations μ. Since the trace of a matrix is invariant under similarity transformations, all elements of one class $\mathcal{C}_i$ have the same **character** χ_i. For one-dimensional representations the characters coincide with the (one-dimensional) representation matrices. Since the representation matrices have to be invertible, all characters have to be unequal zero for one-dimensional representations. All characters of the **trivial** representation, $\underline{\mathbf{1}}_{\mathbf{1}}$ or 1, of the group are equal to one. The character $\chi_1^{(\mu)}$ belonging to $\mathcal{C}_1$, i.e. the class which contains the identity element, equals the dimension N_μ of the representation μ. A representation with real characters is **real**, if its representation matrices can be brought into a real form. If the representation has real characters only, but its representation matrices cannot be brought into real form, it is called **pseudo-real**. If the representation has complex characters, it is called **complex**. Then also its representation matrices are complex. In all groups the number of complex representations is even, since each complex representation μ has its complex conjugate $\bar{\mu}$. The representation matrices of $\bar{\mu}$ are the complex conjugated ones of μ (up to a similarity transformation). The $\mathbb{c}^{(\mu)}$ number of a representation μ indicates whether it is real ($\mathbb{c}^{(\mu)} = 1$), pseudo-real ($\mathbb{c}^{(\mu)} = -1$) or complex ($\mathbb{c}^{(\mu)} = 0$). Pseudo-real representations are usually found in double(-valued) groups which are generically subgroups of $SU(2)$. For illustration, the character table of the smallest non-abelian group is given in Table A.1. The **Kronecker products** $\mu \times \nu$ of all representations μ and ν of the group can be calculated with the help of the character table. They can be uniquely decomposed into the irreducible representations of the group. For the product of μ with itself we define the symmetric, $[\mu \times \mu]$, and anti-symmetric part, $\{\mu \times \mu\}$, of the product. It always holds that $\mu \times \nu = \nu \times \mu$ and $\underline{\mathbf{1}}_{\mathbf{1}} \times \mu = \mu$. In order to find the explicit form of the covariants (in a Kronecker product) which have a well-defined transformation behavior under the

	classes				
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$		
G	$\mathbb{1}$	B	A		
$^\circ\mathcal{C}_i$	1	3	2		
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	2	3	$\mathbb{c}^{(\mu)}$	faithful
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	1	
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	-1	1	1	
$\underline{\mathbf{2}}$	2	0	-1	1	$\surd$

Table A.1: Character table of $S_3 \cong D_3 \cong$ Type 6/2. $^\circ\mathcal{C}_i$ denotes the order of the class and $^\circ\mathrm{h}_{\mathcal{C}_i}$ the order of the elements in $\mathcal{C}_i$. G is a representative of the class, given in terms of the generators A and B which fulfill the relations $\mathrm{A}^3 = \mathbb{1}$, $\mathrm{B}^2 = \mathbb{1}$ and $\mathrm{A\,B\,A} = \mathrm{B}$. $\mathbb{c}^{(\mu)} = 1$ indicates that all representations of this group are real. Furthermore, one can read off the table that only the two-dimensional representation is faithful.

group, one has to calculate the **Clebsch Gordan coefficients** from a certain set of representation matrices. Therefore, their actual appearance is basis-dependent, in contrast to, for example, the results of the Kronecker products which are computed from the characters of the representations. However, if everything is done consistently, the physical results have to be the same in all bases. All formulae, i.e. for the calculation of the Kronecker products, the Clebsch Gordan coefficients, the $\mathbb{c}^{(\mu)}$ numbers and so forth, can be found in [85]. Furthermore, the methods to calculate the embedding schemes of discrete groups into the continuous groups $SO(3)$, $SU(2)$ and $SU(3)$ are explained in [85]. Some material concerning the correlation tables, i.e. the breaking sequences of discrete groups down to their subgroups, can also be found there.

Appendix B

Details of the Presented Groups

In this Appendix we present the Kronecker products and Clebsch Gordan coefficients of the groups we used in the models shown in this work.
For notations and conventions as well as the references concerning the calculations see Appendix A.

B.1 Group Theory of S_4 Model

The Kronecker products can be calculated from the above given character table, see Table 3.1.

$$\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{1}}_{(\mathrm{i+j})\ \mathrm{mod}\ 2\ +1} \quad \forall\ \mathrm{i\ and\ j} \tag{B.1a}$$
$$\underline{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{i}} = \underline{\mathbf{2}} \quad \forall\ \mathrm{i} \tag{B.1b}$$
$$\underline{\mathbf{3}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{3}}_{(\mathrm{i+j})\ \mathrm{mod}\ 2\ +1} \quad \forall\ \mathrm{i\ and\ j} \tag{B.1c}$$
$$\underline{\mathbf{3}}_{\mathbf{i}} \times \underline{\mathbf{2}} = \underline{\mathbf{3}}_{\mathbf{1}} + \underline{\mathbf{3}}_{\mathbf{2}} \quad \forall\ \mathrm{i} \tag{B.1d}$$
$$\underline{\mathbf{3}}_{\mathbf{1}} \times \underline{\mathbf{3}}_{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}} + \underline{\mathbf{3}}_{\mathbf{2}} \tag{B.1e}$$
$$[\underline{\mathbf{2}} \times \underline{\mathbf{2}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}\ , \quad \{\underline{\mathbf{2}} \times \underline{\mathbf{2}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.1f}$$
$$\left[\underline{\mathbf{3}}_{\mathbf{i}} \times \underline{\mathbf{3}}_{\mathbf{i}}\right] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}} + \underline{\mathbf{3}}_{\mathbf{1}}\ , \quad \left\{\underline{\mathbf{3}}_{\mathbf{i}} \times \underline{\mathbf{3}}_{\mathbf{i}}\right\} = \underline{\mathbf{3}}_{\mathbf{2}} \quad \forall\ \mathrm{i} \tag{B.1g}$$

The Clebsch Gordan coefficients can be calculated with the given representation matrices, see Section 3.2.1, for

$$B \sim \underline{\mathbf{1}}_{\mathbf{2}}\ , \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}\ , \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix}, \begin{pmatrix} b'_1 \\ b'_2 \\ b'_3 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{1}}\ , \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \begin{pmatrix} c'_1 \\ c'_2 \\ c'_3 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{2}}\ .$$

They are trivial for the one-dimensional representations as well as for the products $\underline{\mathbf{1}}_{\mathbf{1}} \times \mu$ of any representation μ with the total singlet $\underline{\mathbf{1}}_{\mathbf{1}}$. The ones for the products $\underline{\mathbf{1}}_{\mathbf{2}} \times \mu$ are almost trivial:

$$\begin{pmatrix} -B\,a_2 \\ B\,a_1 \end{pmatrix} \sim \underline{\mathbf{2}} \quad , \quad \begin{pmatrix} B\,b_1 \\ B\,b_2 \\ B\,b_3 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{2}} \quad , \quad \begin{pmatrix} B\,c_1 \\ B\,c_2 \\ B\,c_3 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{1}}\ .$$

The Clebsch Gordan coefficients for $\mu \times \mu$ have the form:
... for $\underline{\mathbf{2}}$:

$$a_1 a'_1 + a_2 a'_2 \sim \underline{\mathbf{1}}_{\mathbf{1}},$$
$$-a_1 a'_2 + a_2 a'_1 \sim \underline{\mathbf{1}}_{\mathbf{2}},$$
$$\begin{pmatrix} a_1 a'_2 + a_2 a'_1 \\ a_1 a'_1 - a_2 a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}$$

... for $\underline{\mathbf{3_1}}$:

$$\sum_{j=1}^{3} b_j b'_j \sim \underline{\mathbf{1_1}},$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}}(b_2 b'_2 - b_3 b'_3) \\ \frac{1}{\sqrt{6}}(-2 b_1 b'_1 + b_2 b'_2 + b_3 b'_3) \end{pmatrix} \sim \underline{\mathbf{2}},$$
$$\begin{pmatrix} b_2 b'_3 + b_3 b'_2 \\ b_1 b'_3 + b_3 b'_1 \\ b_1 b'_2 + b_2 b'_1 \end{pmatrix} \sim \underline{\mathbf{3_1}} \ ,$$
$$\begin{pmatrix} b_3 b'_2 - b_2 b'_3 \\ b_1 b'_3 - b_3 b'_1 \\ b_2 b'_1 - b_1 b'_2 \end{pmatrix} \sim \underline{\mathbf{3_2}}$$

... for $\underline{\mathbf{3_2}}$:

$$\sum_{j=1}^{3} c_j c'_j \sim \underline{\mathbf{1_1}},$$
$$\begin{pmatrix} \frac{1}{\sqrt{2}}(c_2 c'_2 - c_3 c'_3) \\ \frac{1}{\sqrt{6}}(-2 c_1 c'_1 + c_2 c'_2 + c_3 c'_3) \end{pmatrix} \sim \underline{\mathbf{2}},$$
$$\begin{pmatrix} c_2 c'_3 + c_3 c'_2 \\ c_1 c'_3 + c_3 c'_1 \\ c_1 c'_2 + c_2 c'_1 \end{pmatrix} \sim \underline{\mathbf{3_1}},$$
$$\begin{pmatrix} c_3 c'_2 - c_2 c'_3 \\ c_1 c'_3 - c_3 c'_1 \\ c_2 c'_1 - c_1 c'_2 \end{pmatrix} \sim \underline{\mathbf{3_2}}.$$

Note here that the parts belonging to the symmetric part of the product $\mu \times \mu$ are symmetric under the exchange of unprimed and primed whereas the ones belonging to the anti-symmetric part change sign, i.e. are anti-symmetric. Note also that for our choice of generators the Clebsch Gordan coefficients for $\underline{\mathbf{3_1}} \times \underline{\mathbf{3_1}}$ and $\underline{\mathbf{3_2}} \times \underline{\mathbf{3_2}}$ turn out to be the same.
For the couplings $\underline{\mathbf{2}} \times \underline{\mathbf{3_i}}$ and $\underline{\mathbf{3_1}} \times \underline{\mathbf{3_2}}$ we get:

... for $\underline{\mathbf{2}} \times \underline{\mathbf{3_1}}$:

$$\begin{pmatrix} a_2 b_1 \\ -\frac{1}{2}(\sqrt{3} a_1 b_2 + a_2 b_2) \\ \frac{1}{2}(\sqrt{3} a_1 b_3 - a_2 b_3) \end{pmatrix} \sim \underline{\mathbf{3_1}},$$
$$\begin{pmatrix} a_1 b_1 \\ \frac{1}{2}(\sqrt{3} a_2 b_2 - a_1 b_2) \\ -\frac{1}{2}(\sqrt{3} a_2 b_3 + a_1 b_3) \end{pmatrix} \sim \underline{\mathbf{3_2}}$$

... for $\underline{\mathbf{2}} \times \underline{\mathbf{3_2}}$:

$$\begin{pmatrix} a_1 c_1 \\ \frac{1}{2}(\sqrt{3} a_2 c_2 - a_1 c_2) \\ -\frac{1}{2}(\sqrt{3} a_2 c_3 + a_1 c_3) \end{pmatrix} \sim \underline{\mathbf{3_1}},$$
$$\begin{pmatrix} a_2 c_1 \\ -\frac{1}{2}(\sqrt{3} a_1 c_2 + a_2 c_2) \\ \frac{1}{2}(\sqrt{3} a_1 c_3 - a_2 c_3) \end{pmatrix} \sim \underline{\mathbf{3_2}}$$

... for $\underline{\mathbf{3}}_{\mathbf{1}} \times \underline{\mathbf{3}}_{\mathbf{2}}$:

$$\sum_{j=1}^{3} b_j c_j \sim \underline{\mathbf{1}}_{\mathbf{2}},$$

$$\begin{pmatrix} \frac{1}{\sqrt{6}}(2b_1c_1 - b_2c_2 - b_3c_3) \\ \frac{1}{\sqrt{2}}(b_2c_2 - b_3c_3) \end{pmatrix} \sim \underline{\mathbf{2}},$$

$$\begin{pmatrix} b_3c_2 - b_2c_3 \\ b_1c_3 - b_3c_1 \\ b_2c_1 - b_1c_2 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{1}},$$

$$\begin{pmatrix} b_2c_3 + b_3c_2 \\ b_1c_3 + b_3c_1 \\ b_1c_2 + b_2c_1 \end{pmatrix} \sim \underline{\mathbf{3}}_{\mathbf{2}}.$$

Since we choose all the representation matrices to be real, the displayed Clebsch Gordan coefficients are the same even if the representations are conjugated.

B.2 Group Theory of D_5 Model

The Kronecker products $\mu \times \nu$ among the representations $\underline{\mathbf{1}}_{\mathbf{1}}$, $\underline{\mathbf{1}}_{\mathbf{2}}$, $\underline{\mathbf{2}}_{\mathbf{1}}$ and $\underline{\mathbf{2}}_{\mathbf{2}}$ read:

$$\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{1}}_{(\mathrm{i}+\mathrm{j}) \bmod 2 + 1} \quad \text{for} \quad \{\mathrm{i},\mathrm{j}\} \in \{1,2\} \tag{B.2a}$$

$$\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathrm{j}} \quad \text{for} \quad \{\mathrm{i},\mathrm{j}\} \in \{1,2\} \tag{B.2b}$$

$$[\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}} \ , \ \ \{\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.2c}$$

$$[\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}} \ , \ \ \{\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.2d}$$

$$\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{2}} = \underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}} \tag{B.2e}$$

They are only special cases of the general formulae shown in Appendix B.4. Similarly, also the following Clebsch Gordan coefficients are only special cases. However, for the reader, unfamiliar with the group structure of the dihedral groups D_n with arbitrary index n, they are explicitly listed here, for

$$B \sim \underline{\mathbf{1}}_{\mathbf{2}} \ , \ \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \ , \ \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b'_1 \\ b'_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \ .$$

As usual, the Clebsch Gordan coefficients for $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}}$ and $\underline{\mathbf{1}}_{\mathbf{1}} \times \mu$ are trivial, whereas a non-trivial sign appears in $\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{i}}$:

$$\begin{pmatrix} B\, a_1 \\ -B\, a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \quad \text{and} \quad \begin{pmatrix} B\, b_1 \\ -B\, b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \ .$$

The Clebsch Gordan coefficients for $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ are:

... for $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}$:

$$a_1\, a'_2 + a_2\, a'_1 \sim \underline{\mathbf{1}}_{\mathbf{1}},$$
$$a_1\, a'_2 - a_2\, a'_1 \sim \underline{\mathbf{1}}_{\mathbf{2}},$$
$$\begin{pmatrix} a_1\, a'_1 \\ a_2\, a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}$$

... for $\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}$:

$$b_1\, b'_2 + b_2\, b'_1 \sim \underline{\mathbf{1}}_{\mathbf{1}},$$
$$b_1\, b'_2 - b_2\, b'_1 \sim \underline{\mathbf{1}}_{\mathbf{2}},$$
$$\begin{pmatrix} b_2\, b'_2 \\ b_1\, b'_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}$$

... for $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{2}}$:

$$\begin{pmatrix} a_2\, b_1 \\ a_1\, b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}},$$
$$\begin{pmatrix} a_2\, b_2 \\ a_1\, b_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}.$$

Note here that due to the usage of complex matrices for the real representations $\underline{\mathbf{2}}_{\mathbf{1,2}}$ the Clebsch Gordan coefficients for $\underline{\mathbf{2}}^{\star}_{\mathbf{i}}$ differ from the shown ones. As explained in Section 3.3.1 they are connected via the similarity transformation U, e.g. the trivial representation contained in the product $\underline{\mathbf{2}}^{\star}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}$ is

$$a^{\star}_1\, a'_1 + a^{\star}_2\, a'_2 \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad \text{for} \quad \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a'_1 \\ a'_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \ .$$

B.3 Group Theory of T' Model

The Kronecker products are the following ones for the representations 1, $1'$, $1''$, 2, $2'$, $2''$ and 3:

$$1 \times \mu = \mu \quad \forall \quad \mu \tag{B.3a}$$
$$1' \times 1' = 1'' \; , \;\; 1'' \times 1'' = 1' \; , \;\; 1' \times 1'' = 1 \tag{B.3b}$$
$$1' \times 2 = 2' \; , \;\; 1' \times 2' = 2'' \; , \;\; 1' \times 2'' = 2 \tag{B.3c}$$
$$1'' \times 2 = 2'' \; , \;\; 1'' \times 2' = 2 \; , \;\; 1'' \times 2'' = 2' \tag{B.3d}$$
$$1 \times 3 = 3 \; , \;\; 1' \times 3 = 3 \; , \;\; 1'' \times 3 = 3 \tag{B.3e}$$
$$[2 \times 2] = 3 \; , \;\; \{2 \times 2\} = 1 \tag{B.3f}$$
$$\left[2' \times 2'\right] = 3 \; , \;\; \left\{2' \times 2'\right\} = 1'' \tag{B.3g}$$
$$\left[2'' \times 2''\right] = 3 \; , \;\; \left\{2'' \times 2''\right\} = 1' \tag{B.3h}$$
$$2 \times 2' = 1' + 3 \; , \;\; 2 \times 2'' = 1'' + 3 \; , \;\; 2' \times 2'' = 1 + 3 \tag{B.3i}$$
$$2 \times 3 = 2 + 2' + 2'' \; , \;\; 2' \times 3 = 2 + 2' + 2'' \; , \;\; 2'' \times 3 = 2 + 2' + 2'' \tag{B.3j}$$
$$[3 \times 3] = 1 + 1' + 1'' + 3 \; , \;\; \{3 \times 3\} = 3 \tag{B.3k}$$

In general the products of two single-valued or two double-valued representations decompose into single-valued representations, whereas the products of one single- and one double-valued representation split up into irreducible double-valued representations.
In the following we display the Clebsch Gordan coefficients for

$$A \sim 1 \; , A' \sim 1' \; , A'' \sim 1'' \; , \left(\begin{array}{c} a_1 \\ a_2 \end{array}\right) , \left(\begin{array}{c} a'_1 \\ a'_2 \end{array}\right) \sim 2 \; , \left(\begin{array}{c} b_1 \\ b_2 \end{array}\right) , \left(\begin{array}{c} b'_1 \\ b'_2 \end{array}\right) \sim 2' \; , \left(\begin{array}{c} c_1 \\ c_2 \end{array}\right) , \left(\begin{array}{c} c'_1 \\ c'_2 \end{array}\right) \sim 2'' \; ,$$
$$\left(\begin{array}{c} d_1 \\ d_2 \\ d_3 \end{array}\right) , \left(\begin{array}{c} d'_1 \\ d'_2 \\ d'_3 \end{array}\right) \sim 3 \; .$$

The Clebsch Gordan coefficients for the products of the form $1^{(\prime\prime)} \times 1^{(\prime\prime)}$ and $1^{(\prime\prime)} \times 2^{(\prime\prime)}$ are trivial. $1^{(\prime\prime)} \times 3$ are the first non-trivial products, since the entries of the three-dimensional representation are permuted:

$$\left(\begin{array}{c} A\, d_1 \\ A\, d_2 \\ A\, d_3 \end{array}\right) \sim 3 \; , \quad \left(\begin{array}{c} A'\, d_3 \\ A'\, d_1 \\ A'\, d_2 \end{array}\right) \sim 3 \; , \quad \left(\begin{array}{c} A''\, d_2 \\ A''\, d_3 \\ A''\, d_1 \end{array}\right) \sim 3 \; .$$

The Clebsch Gordan coefficients for $\mu \times \mu$ have the form:

... for 2:

$$a_1\, a'_2 - a_2\, a'_1 \sim 1,$$
$$\left(\begin{array}{c} \frac{1-i}{2}\,(a_1\, a'_2 + a_2\, a'_1) \\ i\, a_1\, a'_1 \\ a_2\, a'_2 \end{array}\right) \sim 3$$

... for $2'$:

$$b_1\, b'_2 - b_2\, b'_1 \sim 1'',$$
$$\left(\begin{array}{c} i\, b_1\, b'_1 \\ b_2\, b'_2 \\ \frac{1-i}{2}\,(b_1\, b'_2 + b_2\, b'_1) \end{array}\right) \sim 3$$

... for $2''$:

$$c_1\, c'_2 - c_2\, c'_1 \sim 1',$$
$$\left(\begin{array}{c} c_2\, c'_2 \\ \frac{1-i}{2}\,(c_1\, c'_2 + c_2\, c'_1) \\ i\, c_1\, c'_1 \end{array}\right) \sim 3$$

... for 3:

$$d_1\,d'_1 + d_2\,d'_3 + d_3\,d'_2 \sim 1,$$
$$d_3\,d'_3 + d_1\,d'_2 + d_2\,d'_1 \sim 1',$$
$$d_2\,d'_2 + d_1\,d'_3 + d_3\,d'_1 \sim 1'',$$
$$\frac{1}{3}\begin{pmatrix} 2\,d_1\,d'_1 - d_2\,d'_3 - d_3\,d'_2 \\ 2\,d_3\,d'_3 - d_1\,d'_2 - d_2\,d'_1 \\ 2\,d_2\,d'_2 - d_1\,d'_3 - d_3\,d'_1 \end{pmatrix} \sim 3,$$
$$\frac{1}{2}\begin{pmatrix} d_2\,d'_3 - d_3\,d'_2 \\ d_1\,d'_2 - d_2\,d'_1 \\ d_3\,d'_1 - d_1\,d'_3 \end{pmatrix} \sim 3.$$

and for the mixed products we find:

... for $2 \times 2'$:

$$a_1\,b_2 - a_2\,b_1 \sim 1',$$
$$\begin{pmatrix} a_2\,b_2 \\ \frac{1-i}{2}\,(a_1\,b_2 + a_2\,b_1) \\ i\,a_1\,b_1 \end{pmatrix} \sim 3$$

... for $2 \times 2''$:

$$a_1\,c_2 - a_2\,c_1 \sim 1'',$$
$$\begin{pmatrix} i\,a_1\,c_1 \\ a_2\,c_2 \\ \frac{1-i}{2}\,(a_1\,c_2 + a_2\,c_1) \end{pmatrix} \sim 3$$

... for $2' \times 2''$:

$$b_1\,c_2 - b_2\,c_1 \sim 1,$$
$$\begin{pmatrix} \frac{1-i}{2}\,(b_1\,c_2 + b_2\,c_1) \\ i\,b_1\,c_1 \\ b_2\,c_2 \end{pmatrix} \sim 3$$

... for 2×3:

$$\begin{pmatrix} (1+i)\,a_2\,d_2 + a_1\,d_1 \\ (1-i)\,a_1\,d_3 - a_2\,d_1 \end{pmatrix} \sim 2,$$
$$\begin{pmatrix} (1+i)\,a_2\,d_3 + a_1\,d_2 \\ (1-i)\,a_1\,d_1 - a_2\,d_2 \end{pmatrix} \sim 2',$$
$$\begin{pmatrix} (1+i)\,a_2\,d_1 + a_1\,d_3 \\ (1-i)\,a_1\,d_2 - a_2\,d_3 \end{pmatrix} \sim 2''$$

... for $2' \times 3$:

$$\begin{pmatrix} (1+i)\,b_2\,d_1 + b_1\,d_3 \\ (1-i)\,b_1\,d_2 - a_2\,d_3 \end{pmatrix} \sim 2,$$
$$\begin{pmatrix} (1+i)\,b_2\,d_2 + b_1\,d_1 \\ (1-i)\,b_1\,d_3 - b_2\,d_1 \end{pmatrix} \sim 2',$$
$$\begin{pmatrix} (1+i)\,b_2\,d_3 + b_1\,d_2 \\ (1-i)\,b_1\,d_1 - b_2\,d_2 \end{pmatrix} \sim 2''$$

... for $2'' \times 3$:

$$\begin{pmatrix} (1+i)\,c_2\,d_3 + c_1\,d_2 \\ (1-i)\,c_1\,d_1 - c_2\,d_2 \end{pmatrix} \sim 2,$$
$$\begin{pmatrix} (1+i)\,c_2\,d_1 + c_1\,d_3 \\ (1-i)\,c_1\,d_2 - c_2\,d_3 \end{pmatrix} \sim 2',$$
$$\begin{pmatrix} (1+i)\,c_2\,d_2 + c_1\,d_1 \\ (1-i)\,c_1\,d_3 - c_2\,d_1 \end{pmatrix} \sim 2''$$

All these results coincide with the ones shown in [111, 112].

B.4 Group Theory of D_n and D'_n Groups

In this section we present the character tables of the dihedral groups D_n and D'_n, general formulae for the Kronecker products as well as for the Clebsch Gordan coefficients.

B.4.1 Character Tables

D_n,	classes					
n odd	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	...	$\mathcal{C}_{m+1}$	$\mathcal{C}_{m+2}$
G	$\mathbb{1}$	A	A^2	...	A^m	B
$^\circ\mathcal{C}_i$	1	2	2	...	2	n
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	n	n	...	n	2
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	...	1	1
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	1	1	...	1	-1
$\underline{\mathbf{2}}_{\mathbf{1}}$	2	$2\cos(\varphi)$	$2\cos(2\varphi)$	...	$2\cos(m\varphi)$	0
$\underline{\mathbf{2}}_{\mathbf{2}}$	2	$2\cos(2\varphi)$	$2\cos(4\varphi)$	...	$2\cos(2m\varphi)$	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{\mathbf{2}}_m$	2	$2\cos(m\varphi)$	$2\cos(2m\varphi)$	...	$2\cos(m^2\varphi)$	0

Table B.1: Character table of the group D_n with n odd. m denotes $\frac{n-1}{2}$ and φ is $\frac{2\pi}{n}$.

D_n,	classes							
n even	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	...	$\mathcal{C}_m$	$\mathcal{C}_{m+1}$	$\mathcal{C}_{m+2}$	$\mathcal{C}_{m+3}$
G	$\mathbb{1}$	A	A^2	...	A^{m-1}	A^m	B	A B
$^\circ\mathcal{C}_i$	1	2	2	...	2	1	$\frac{n}{2}$	$\frac{n}{2}$
$^\circ\mathrm{h}_{\mathcal{C}_i}$	1	n	$\frac{n}{2}$	...	$n\ [\frac{n}{2}]$	2	2	2
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	...	1	1	1	1
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	1	1	...	1	1	-1	-1
$\underline{\mathbf{1}}_{\mathbf{3}}$	1	-1	1	...	$(-1)^{m-1}$	$(-1)^m$	1	-1
$\underline{\mathbf{1}}_{\mathbf{4}}$	1	-1	1	...	$(-1)^{m-1}$	$(-1)^m$	-1	1
$\underline{\mathbf{2}}_{\mathbf{1}}$	2	$2\cos(\varphi)$	$2\cos(2\varphi)$	...	$2\cos((m-1)\varphi)$	$2\cos(m\varphi)$	0	0
$\underline{\mathbf{2}}_{\mathbf{2}}$	2	$2\cos(2\varphi)$	$2\cos(4\varphi)$	...	$2\cos(2(m-1)\varphi)$	$2\cos(2m\varphi)$	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{\mathbf{2}}_{m-1}$	2	$2\cos((m-1)\varphi)$	$2\cos(2(m-1)\varphi)$	...	$2\cos((m-1)^2\varphi)$	$2\cos((m-1)m\varphi)$	0	0

Table B.2: Character table of the group D_n with n even. m denotes $\frac{n}{2}$ and φ is $\frac{2\pi}{n}$. Note that $^\circ\mathrm{h}_{\mathcal{C}_i}$ depends on m for A^{m-1}, i.e. it is n for m being even (n is then divisible by four) and it is $\frac{n}{2}$ for m being odd (n is then divisible by two, but not by four).

D'_n, n odd	classes							
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	...	$\mathcal{C}_n$	$\mathcal{C}_{n+1}$	$\mathcal{C}_{n+2}$	$\mathcal{C}_{n+3}$
G	$\mathbb{1}$	A	A^2	...	A^{n-1}	A^n	B	A B
$°\mathcal{C}_i$	1	2	2	...	2	1	n	n
$°\mathrm{h}_{\mathcal{C}_i}$	1	$2\,n$	n	...	n	2	4	4
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	...	1	1	1	1
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	1	1	...	1	1	-1	-1
$\underline{\mathbf{1}}_{\mathbf{3}}$	1	-1	1	...	1	-1	$-i$	i
$\underline{\mathbf{1}}_{\mathbf{4}}$	1	-1	1	...	1	-1	i	$-i$
$\underline{\mathbf{2}}_{\mathbf{1}}$	2	$2\cos(\varphi)$	$2\cos(2\,\varphi)$	...	$2\cos((n-1)\,\varphi)$	-2	0	0
$\underline{\mathbf{2}}_{\mathbf{2}}$	2	$2\cos(2\,\varphi)$	$2\cos(4\,\varphi)$	...	$2\cos(2\,(n-1)\,\varphi)$	2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{\mathbf{2}}_{n-1}$	2	$2\cos((n-1)\,\varphi)$	$2\cos(2\,(n-1)\,\varphi)$	...	$2\cos((n-1)^2\,\varphi)$	2	0	0

Table B.3: Character table of the group D'_n with n odd. φ is $\frac{\pi}{n}$.

D'_n, n even	classes							
	$\mathcal{C}_1$	$\mathcal{C}_2$	$\mathcal{C}_3$	...	$\mathcal{C}_n$	$\mathcal{C}_{n+1}$	$\mathcal{C}_{n+2}$	$\mathcal{C}_{n+3}$
G	$\mathbb{1}$	A	A^2	...	A^{n-1}	A^n	B	A B
$°\mathcal{C}_i$	1	2	2	...	2	1	n	n
$°\mathrm{h}_{\mathcal{C}_i}$	1	$2\,n$	n	...	$2\,n$	2	4	4
$\underline{\mathbf{1}}_{\mathbf{1}}$	1	1	1	...	1	1	1	1
$\underline{\mathbf{1}}_{\mathbf{2}}$	1	1	1	...	1	1	-1	-1
$\underline{\mathbf{1}}_{\mathbf{3}}$	1	-1	1	...	-1	1	1	-1
$\underline{\mathbf{1}}_{\mathbf{4}}$	1	-1	1	...	-1	1	-1	1
$\underline{\mathbf{2}}_{\mathbf{1}}$	2	$2\cos(\varphi)$	$2\cos(2\,\varphi)$	...	$2\cos((n-1)\,\varphi)$	-2	0	0
$\underline{\mathbf{2}}_{\mathbf{2}}$	2	$2\cos(2\,\varphi)$	$2\cos(4\,\varphi)$	...	$2\cos(2\,(n-1)\,\varphi)$	2	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\underline{\mathbf{2}}_{n-1}$	2	$2\cos((n-1)\,\varphi)$	$2\cos(2\,(n-1)\,\varphi)$	...	$2\cos((n-1)^2\,\varphi)$	-2	0	0

Table B.4: Character table of the group D'_n with n even. φ is $\frac{\pi}{n}$.

B.4.2 Kronecker Products of D_n and D'_n

The products of the one-dimensional representations of D_n are:

$\times$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$
$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$
$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$
$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$
$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$

where the representations $\underline{\mathbf{1}}_{\mathbf{3,4}}$ only exists in groups D_n with an even index n. The products $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ transform as:

$$\underline{\mathbf{1}}_{\mathbf{1,2}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{j}}$$

and for n even there are also:

$$\underline{\mathbf{1}}_{\mathbf{3,4}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{k}} \quad \text{with} \quad \mathrm{k} = \frac{n}{2} - \mathrm{j}$$

If 4 is a divisor of n, the products of the representation $\underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{j} = \frac{n}{4}$ with any one-dimensional representation of the group also transform as $\underline{\mathbf{2}}_{\mathbf{j}}$.

The products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}}$ are of the form $\underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{j} = \min(2\,\mathrm{i}, n - 2\,\mathrm{i})$. In case that the group D_n has an index n which is divisible by four one also finds the structure $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \sum_{\mathrm{j}=1}^{4} \underline{\mathbf{1}}_{\mathbf{j}}$ for $\mathrm{i} = \frac{n}{4}$. This shows that there is at most one representation in each group D_n with this property. The mixed products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ can have two structures: *a.*) $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{l}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$ and $\mathrm{l} = \min(\mathrm{i}+\mathrm{j}, n - (\mathrm{i}+\mathrm{j}))$ and *b.*) $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}} + \underline{\mathbf{2}}_{\mathbf{k}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$ for $\mathrm{i}+\mathrm{j} = \frac{n}{2}$.

For D'_n with n even the one-dimensional representations have the same product structure as for D_n while for n being odd they are:

$\times$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$
$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$
$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$
$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$
$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{4}}$	$\underline{\mathbf{1}}_{\mathbf{3}}$	$\underline{\mathbf{1}}_{\mathbf{1}}$	$\underline{\mathbf{1}}_{\mathbf{2}}$

due to the fact that the two one-dimensional representations $\underline{\mathbf{1}}_{\mathbf{3}}$ and $\underline{\mathbf{1}}_{\mathbf{4}}$ are complex conjugated to each other.
The rest of the formulae for the different product structures are the same as in the case of $D_{2\,n}$, i.e. in each formula above which is given for D_n one has to replace n by $2\,n$.

The Kronecker products can also be found in [122].

B.4.3 Clebsch Gordan Coefficients of D_n

Here we display the Clebsch Gordan coefficients for the Kronecker products $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}}$, $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ and $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$. Since we discuss the groups D_n independent from their index n, we present the Clebsch Gordan coefficients in a slightly more general notation than above. This will be explained with two examples in the following.
For $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{1}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{k}}$ the Clebsch Gordan coefficient is trivially one. For $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ the Clebsch Gordan coefficients are:

for $\mathrm{i} = 1$

$$\left(\begin{array}{c} (\ 1 \quad 0\ ) \\ (\ 0 \quad 1\ ) \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{j}}$$

for $\mathrm{i} = 2$

$$\left(\begin{array}{c} (\ 1 \quad 0\ ) \\ (\ 0 \quad -1\ ) \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{j}}$$

I.e. for $A \sim \underline{\mathbf{1}}_{\mathbf{1}}$ and $\left(\begin{array}{c} a_1 \\ a_2 \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{j}}$ $\left(\begin{array}{c} A\,a_1 \\ A\,a_2 \end{array} \right)$ transforms as $\underline{\mathbf{2}}_{\mathbf{j}}$ and for $B \sim \underline{\mathbf{1}}_{\mathbf{2}}$ it is $\left(\begin{array}{c} B\,a_1 \\ -B\,a_2 \end{array} \right)$ which transforms as $\underline{\mathbf{2}}_{\mathbf{j}}$.
If the index n of D_n is even, the group has two further one-dimensional representations $\underline{\mathbf{1}}_{\mathbf{3,4}}$ whose products with $\underline{\mathbf{2}}_{\mathbf{j}}$ are of the form:

for $\mathrm{i} = 3$

$$\left(\begin{array}{c} (\ 0 \quad 1\ ) \\ (\ 1 \quad 0\ ) \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{k}}$$

for $\mathrm{i} = 4$

$$\left(\begin{array}{c} (\ 0 \quad 1\ ) \\ (\ -1 \quad 0\ ) \end{array} \right) \sim \underline{\mathbf{2}}_{\mathbf{k}} \quad \text{with} \quad \mathrm{k} = \frac{n}{2} - \mathrm{j}$$

$\mathrm{k} = \mathrm{j}$ holds in the case that $\mathrm{k} = \mathrm{j} = \frac{n}{4}$, i.e. four has to be a divisor of n.
For the products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}}$ the covariant combinations are:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{2}}$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}} \text{ for } \mathrm{j} = 2\,\mathrm{i} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}} \text{ for } \mathrm{j} = n - 2\,\mathrm{i}$$

Then, for example, the invariant reads $a_1'\,a_2 + a_2'\,a_1 \sim \underline{\mathbf{1}}_{\mathbf{1}}$ for $a_1^{(\prime)}$ and $a_2^{(\prime)}$ being the upper and lower component of the two-dimensional representation $\underline{\mathbf{2}}_{\mathbf{i}}$.
If the index n of D_n is even and $\mathrm{i} = \frac{n}{4}$ (4 has to be a divisor of n), there is a second possibility: $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \sum_{\mathrm{j}=1}^{4} \underline{\mathbf{1}}_{\mathbf{j}}$. The Clebsch Gordan coefficients are

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad , \quad \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{2}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{3}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{4}} \,.$$

For the products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{i} \neq \mathrm{j}$ there are the two structures $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{l}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$ and $\mathrm{l} = \min(\mathrm{i} + \mathrm{j}, n - (\mathrm{i} + \mathrm{j}))$ or $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}} + \underline{\mathbf{2}}_{\mathbf{k}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$, if $\mathrm{i} + \mathrm{j} = \frac{n}{2}$ (obviously n has to be even). The Clebsch Gordan coefficients for $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{l}}$ are:

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \text{ for } \mathrm{k} = \mathrm{i} - \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \text{ for } \mathrm{k} = \mathrm{j} - \mathrm{i}$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{l}} \text{ for } \mathrm{l} = \mathrm{i} + \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{l}} \text{ for } \mathrm{l} = n - (\mathrm{i} + \mathrm{j})$$

For the structure $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}} + \underline{\mathbf{2}}_{\mathbf{k}}$ with $\mathrm{i} + \mathrm{j} = \frac{n}{2}$ the Clebsch Gordan coefficients are

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{3}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{4}}$$

and

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \text{ for } \mathrm{k} = \mathrm{i} - \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \text{ for } \mathrm{k} = \mathrm{j} - \mathrm{i}$$

B.4.4 Clebsch Gordan Coefficients of D'_n

For n even the Clebsch Gordan coefficients for the products $\underline{\mathbf{1}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{2}}_{\mathbf{k}}$ are the same as in the case of $D_{2\,n}$, i.e. for $\mathrm{i} = 3, 4$ the condition for k is $\mathrm{j} + \mathrm{k} = n$ instead of $\frac{n}{2}$.
If n is odd, the same holds for j odd whereas for j even the Clebsch Gordan coefficients of the products $\underline{\mathbf{1}}_{\mathbf{3}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ and $\underline{\mathbf{1}}_{\mathbf{4}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ have to be interchanged.
The Clebsch Gordan coefficients for the products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}}$ are the same as for $D_{2\,n}$, if i is even. Similarly, the ones of $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{i} \neq \mathrm{j}$ are the same, if i, j are both even or one is even and one is odd, if n is even. For n being odd the only difference is that in the case that the product is of the form $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}} = \underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}} + \underline{\mathbf{2}}_{\mathbf{k}}$ the Clebsch Gordan coefficients for the covariant combination transforming as $\underline{\mathbf{1}}_{\mathbf{3}}$ and $\underline{\mathbf{1}}_{\mathbf{4}}$ are interchanged.
Concerning the structure of the products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{1}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{j}}$ with $\mathrm{j} = \min(2\,\mathrm{i}, 2\,n - 2\,\mathrm{i})$ for i odd, one finds the following:

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{2}}$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}} \;\text{ for } \mathrm{j} = 2\,\mathrm{i} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{j}} \;\text{ for } \mathrm{j} = 2\,n - 2\,\mathrm{i}$$

If $\mathrm{i} = \frac{n}{2}$ (n even), then one has $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \sum_{\mathrm{j}=1}^{4} \underline{\mathbf{1}}_{\mathbf{j}}$. The Clebsch Gordan coefficients are

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{1}} \quad , \quad \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{2}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{3}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{4}} \,.$$

$\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ for i, j being odd is either $\underline{\mathbf{2}}_{\mathbf{k}} + \underline{\mathbf{2}}_{\mathbf{l}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$ and $\mathrm{l} = \min(\mathrm{i} + \mathrm{j}, 2\,n - (\mathrm{i} + \mathrm{j}))$ or $\underline{\mathbf{1}}_{\mathbf{3}} + \underline{\mathbf{1}}_{\mathbf{4}} + \underline{\mathbf{2}}_{\mathbf{k}}$ with $\mathrm{k} = |\mathrm{i} - \mathrm{j}|$, if $\mathrm{i} + \mathrm{j} = n$. The Clebsch Gordan coefficients in the first case are:

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \;\text{ for } \mathrm{k} = \mathrm{i} - \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \;\text{ for } \mathrm{k} = \mathrm{j} - \mathrm{i}$$

and

$$\begin{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ 0 & -1 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{l}} \;\text{ for } \mathrm{l} = \mathrm{i} + \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \\ \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{l}} \;\text{ for } \mathrm{l} = 2\,n - (\mathrm{i} + \mathrm{j})$$

In the second one the Clebsch Gordan coefficients are:

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{3}} \quad , \quad \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \sim \underline{\mathbf{1}}_{\mathbf{4}}$$

and

$$\begin{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \;\text{ for } \mathrm{k} = \mathrm{i} - \mathrm{j} \quad \text{or} \quad \begin{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\ \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{k}} \;\text{ for } \mathrm{k} = \mathrm{j} - \mathrm{i}\,.$$

B.5 Group Theory of D_7 Model

Although we presented the whole group theory of a dihedral group D_n with arbitrary index n in Appendix B.4, we show for the reader, who is unfamiliar with the group theory of discrete groups, the Kronecker products as well as the Clebsch Gordan coefficients explicitly.
The Kronecker products are

$$\underline{\mathbf{1}}_{\mathbf{1}} \times \mu = \mu \;, \;\; \underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{2}} = \underline{\mathbf{1}}_{\mathbf{1}} \tag{B.4a}$$
$$\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{i}} = \underline{\mathbf{2}}_{\mathbf{i}} \;\forall\, \mathrm{i} \tag{B.4b}$$
$$[\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}} \;, \;\; \{\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{1}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.4c}$$
$$[\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{3}} \;, \;\; \{\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{2}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.4d}$$
$$[\underline{\mathbf{2}}_{\mathbf{3}} \times \underline{\mathbf{2}}_{\mathbf{3}}] = \underline{\mathbf{1}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{1}} \;, \;\; \{\underline{\mathbf{2}}_{\mathbf{3}} \times \underline{\mathbf{2}}_{\mathbf{3}}\} = \underline{\mathbf{1}}_{\mathbf{2}} \tag{B.4e}$$
$$\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{2}} = \underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{3}} \;, \;\; \underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{3}} = \underline{\mathbf{2}}_{\mathbf{2}} + \underline{\mathbf{2}}_{\mathbf{3}} \;, \;\; \underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{3}} = \underline{\mathbf{2}}_{\mathbf{1}} + \underline{\mathbf{2}}_{\mathbf{2}} \;. \tag{B.4f}$$

The Clebsch Gordan coefficients are shown for

$$B \sim \underline{\mathbf{1}}_{\mathbf{2}} \;, \; \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}, \begin{pmatrix} a_1' \\ a_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \;, \; \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \begin{pmatrix} b_1' \\ b_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \;, \; \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}, \begin{pmatrix} c_1' \\ c_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \;.$$

They are trivial for $\underline{\mathbf{1}}_{\mathbf{1}} \times \mu$ and $\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{1}}_{\mathbf{2}}$. For $\underline{\mathbf{1}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{i}}$ a non-trivial sign appears

$$\begin{pmatrix} B\,a_1 \\ -B\,a_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \;, \; \begin{pmatrix} B\,b_1 \\ -B\,b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \;, \; \begin{pmatrix} B\,c_1 \\ -B\,c_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \;.$$

The Clebsch Gordan coefficients for $\mu \times \mu$ have the form:

... for $\underline{\mathbf{2}}_{\mathbf{1}}$:
$$\begin{gathered} a_1\,a_2' + a_2\,a_1' \sim \underline{\mathbf{1}}_{\mathbf{1}} \\ a_1\,a_2' - a_2\,a_1' \sim \underline{\mathbf{1}}_{\mathbf{2}} \\ \begin{pmatrix} a_1\,a_1' \\ a_2\,a_2' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}} \end{gathered}$$

... for $\underline{\mathbf{2}}_{\mathbf{2}}$:
$$\begin{gathered} b_1\,b_2' + b_2\,b_1' \sim \underline{\mathbf{1}}_{\mathbf{1}} \\ b_1\,b_2' - b_2\,b_1' \sim \underline{\mathbf{1}}_{\mathbf{2}} \\ \begin{pmatrix} b_2\,b_2' \\ b_1\,b_1' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \end{gathered}$$

... for $\underline{\mathbf{2}}_{\mathbf{3}}$:
$$\begin{gathered} c_1\,c_2' + c_2\,c_1' \sim \underline{\mathbf{1}}_{\mathbf{1}} \\ c_1\,c_2' - c_2\,c_1' \sim \underline{\mathbf{1}}_{\mathbf{2}} \\ \begin{pmatrix} c_2\,c_2' \\ c_1\,c_1' \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}} \end{gathered}$$

For the rest of the products $\underline{\mathbf{2}}_{\mathbf{i}} \times \underline{\mathbf{2}}_{\mathbf{j}}$ we get:

... for $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{2}}$:
$$\begin{gathered} \begin{pmatrix} a_2\,b_1 \\ a_1\,b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}}, \\ \begin{pmatrix} a_1\,b_1 \\ a_2\,b_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \end{gathered}$$

... for $\underline{\mathbf{2}}_{\mathbf{1}} \times \underline{\mathbf{2}}_{\mathbf{3}}$:
$$\begin{gathered} \begin{pmatrix} a_2\,c_1 \\ a_1\,c_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}, \\ \begin{pmatrix} a_2\,c_2 \\ a_1\,c_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{3}} \end{gathered}$$

... for $\underline{\mathbf{2}}_{\mathbf{2}} \times \underline{\mathbf{2}}_{\mathbf{3}}$:

$$\begin{pmatrix} b_2\, c_1 \\ b_1\, c_2 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{1}},$$
$$\begin{pmatrix} b_2\, c_2 \\ b_1\, c_1 \end{pmatrix} \sim \underline{\mathbf{2}}_{\mathbf{2}}.$$

Appendix C

Next-to-Leading Order Terms in the T' Model

In this Appendix we present the explicit form of the next-to-leading order terms contributing to the fermion masses and the flavon potential. For the quarks the terms read

$$(\varphi_T\,\varphi_T)\,Q_3\,t^c = (\varphi_{T\,1}^2 + 2\,\varphi_{T\,2}\,\varphi_{T\,3})\,Q_3\,t^c \tag{C.1a}$$
$$(D_q\,\eta\,\varphi_T)\,t^c = ((1-i)\,\eta_1\,\varphi_{T\,3} - \eta_2\,\varphi_{T\,1})\,Q_1\,t^c - ((1+i)\,\eta_2\,\varphi_{T\,2} + \eta_1\,\varphi_{T\,1})\,Q_2\,t^c \tag{C.1b}$$
$$Q_3\,(\eta\,\varphi_T\,D_u^c) = ((1-i)\,\eta_1\,\varphi_{T\,3} - \eta_2\,\varphi_{T\,1})\,Q_3\,u^c - ((1+i)\,\eta_2\,\varphi_{T\,2} + \eta_1\,\varphi_{T\,1})\,Q_3\,c^c \tag{C.1c}$$
$$(\varphi_T\,\varphi_T)''\,(D_q\,D_u^c)' = (\varphi_{T\,2}^2 + 2\,\varphi_{T\,1}\,\varphi_{T\,3})\,(Q_1\,c^c - Q_2\,u^c) \tag{C.1d}$$
$$\begin{aligned}(\varphi_T\,\varphi_T)_S\,(D_q\,D_u^c)_3 = {} & (\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,Q_2\,c^c + \left(\frac{1-i}{2}\right)(\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,(Q_1\,c^c + Q_2\,u^c) \\ & + i\,(\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,Q_1\,u^c\end{aligned} \tag{C.1e}$$
$$\xi''\,\varphi_T\,(D_q\,D_u^c)_3 = \xi''\left(\varphi_{T\,2}\,Q_2\,c^c + \left(\frac{1-i}{2}\right)\varphi_{T\,1}\,(Q_1\,c^c + Q_2\,u^c) + i\,\varphi_{T\,3}\,Q_1\,u^c\right) \tag{C.1f}$$
$$(\eta\,\eta)_3\,(D_q\,D_u^c)_3 = \eta_1^2\,Q_2\,c^c - \eta_1\,\eta_2\,(Q_1\,c^c + Q_2\,u^c) + \eta_2^2\,Q_1\,u^c \tag{C.1g}$$
$$(\varphi_T\,\varphi_T)\,Q_3\,b^c = (\varphi_{T\,1}^2 + 2\,\varphi_{T\,2}\,\varphi_{T\,3})\,Q_3\,b^c \tag{C.1h}$$
$$(D_q\,\eta\,\varphi_T)\,b^c = ((1-i)\,\eta_1\,\varphi_{T\,3} - \eta_2\,\varphi_{T\,1})\,Q_1\,b^c - ((1+i)\,\eta_2\,\varphi_{T\,2} + \eta_1\,\varphi_{T\,1})\,Q_2\,b^c \tag{C.1i}$$
$$Q_3\,(\eta\,\varphi_T\,D_d^c) = ((1-i)\,\eta_1\,\varphi_{T\,3} - \eta_2\,\varphi_{T\,1})\,Q_3\,d^c - ((1+i)\,\eta_2\,\varphi_{T\,2} + \eta_1\,\varphi_{T\,1})\,Q_3\,s^c \tag{C.1j}$$
$$(\varphi_T\,\varphi_T)''\,(D_q\,D_d^c)' = (\varphi_{T\,2}^2 + 2\,\varphi_{T\,1}\,\varphi_{T\,3})\,(Q_1\,s^c - Q_2\,d^c) \tag{C.1k}$$
$$\begin{aligned}(\varphi_T\,\varphi_T)_S\,(D_q\,D_d^c)_3 = {} & (\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,Q_2\,s^c + \left(\frac{1-i}{2}\right)(\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,(Q_1\,s^c + Q_2\,d^c) \\ & + i\,(\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,Q_1\,d^c\end{aligned} \tag{C.1l}$$
$$\xi''\,\varphi_T\,(D_q\,D_d^c)_3 = \xi''\left(\varphi_{T\,2}\,Q_2\,s^c + \left(\frac{1-i}{2}\right)\varphi_{T\,1}\,(Q_1\,s^c + Q_2\,d^c) + i\,\varphi_{T\,3}\,Q_1\,d^c\right) \tag{C.1m}$$
$$(\eta\,\eta)_3\,(D_q\,D_d^c)_3 = \eta_1^2\,Q_2\,s^c - \eta_1\,\eta_2\,(Q_1\,s^c + Q_2\,d^c) + \eta_2^2\,Q_1\,d^c \tag{C.1n}$$

For the leptons the explicit structure is given by

$$(\varphi_T\,\varphi_T)_S\,(l\,e^c) = ((\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,l_1 + (\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,l_2 + (\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,l_3)\,e^c \tag{C.2a}$$
$$\xi''\,\varphi_T\,(l\,e^c) = \xi''\,(\varphi_{T\,2}\,l_1 + \varphi_{T\,1}\,l_2 + \varphi_{T\,3}\,l_3)\,e^c \tag{C.2b}$$
$$(\eta\,\eta)_3(l\,e^c) = (i\,\eta_1^2\,l_1 + (1-i)\,\eta_1\,\eta_2\,l_2 + \eta_2^2\,l_3)\,e^c \tag{C.2c}$$
$$(\varphi_T\,\varphi_T)_S\,(l\,\mu^c) = ((\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,l_2 + (\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,l_3 + (\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,l_1)\,\mu^c \tag{C.2d}$$
$$\xi''\,\varphi_T\,(l\,\mu^c) = \xi''\,(\varphi_{T\,2}\,l_2 + \varphi_{T\,1}\,l_3 + \varphi_{T\,3}\,l_1)\,\mu^c \tag{C.2e}$$
$$(\eta\,\eta)_3\,(l\,\mu^c) = (i\,\eta_1^2\,l_2 + (1-i)\,\eta_1\,\eta_2\,l_3 + \eta_2^2\,l_1)\,\mu^c \tag{C.2f}$$
$$(\varphi_T\,\varphi_T)_S\,(l\,\tau^c) = ((\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,l_3 + (\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,l_1 + (\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,l_2)\,\tau^c \tag{C.2g}$$
$$\xi''\,\varphi_T\,(l\,\tau^c) = \xi''\,(\varphi_{T\,2}\,l_3 + \varphi_{T\,1}\,l_1 + \varphi_{T\,3}\,l_2)\,\tau^c \tag{C.2h}$$
$$(\eta\,\eta)_3\,(l\,\tau^c) = (i\,\eta_1^2\,l_3 + (1-i)\,\eta_1\,\eta_2\,l_1 + \eta_2^2\,l_2)\,\tau^c \tag{C.2i}$$
$$(\varphi_T\,\varphi_S)\,(l\,l) = (\varphi_{T\,1}\,\varphi_{S\,1} + \varphi_{T\,2}\,\varphi_{S\,3} + \varphi_{T\,3}\,\varphi_{S\,2})\,(l_1\,l_1 + l_2\,l_3 + l_3\,l_2) \tag{C.2j}$$
$$(\varphi_T\,\varphi_S)'\,(l\,l)'' = (\varphi_{T\,3}\,\varphi_{S\,3} + \varphi_{T\,1}\,\varphi_{S\,2} + \varphi_{T\,2}\,\varphi_{S\,1})\,(l_2\,l_2 + l_1\,l_3 + l_3\,l_1) \tag{C.2k}$$
$$(\varphi_T\,\varphi_S)''\,(l\,l)' = (\varphi_{T\,2}\,\varphi_{S\,2} + \varphi_{T\,1}\,\varphi_{S\,3} + \varphi_{T\,3}\,\varphi_{S\,1})\,(l_3\,l_3 + l_1\,l_2 + l_2\,l_1) \tag{C.2l}$$
$$\begin{aligned}(\varphi_T\,\varphi_S)_S\,(l\,l)_S = {} & (2\,\varphi_{T\,1}\,\varphi_{S\,1} - \varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2})\,(2\,l_1\,l_1 - l_2\,l_3 - l_3\,l_2)\\ & + (2\,\varphi_{T\,2}\,\varphi_{S\,2} - \varphi_{T\,1}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,1})\,(2\,l_3\,l_3 - l_1\,l_2 - l_2\,l_1)\\ & + (2\,\varphi_{T\,3}\,\varphi_{S\,3} - \varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1})\,(2\,l_2\,l_2 - l_1\,l_3 - l_3\,l_1)\end{aligned} \tag{C.2m}$$
$$\begin{aligned}(\varphi_T\,\varphi_S)_A\,(l\,l)_S = {} & (\varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2})\,(2\,l_1\,l_1 - l_2\,l_3 - l_3\,l_2)\\ & + (\varphi_{T\,3}\,\varphi_{S\,1} - \varphi_{T\,1}\,\varphi_{S\,3})\,(2\,l_3\,l_3 - l_1\,l_2 - l_2\,l_1)\\ & + (\varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1})\,(2\,l_2\,l_2 - l_1\,l_3 - l_3\,l_1)\end{aligned} \tag{C.2n}$$
$$\begin{aligned}(\xi\,[\tilde{\xi}]\,\varphi_T)\,(l\,l)_S = {} & \xi\,[\tilde{\xi}]\,(\varphi_{T\,1}\,(2\,l_1\,l_1 - l_2\,l_3 - l_3\,l_2) + \varphi_{T\,3}\,(2\,l_3\,l_3 - l_1\,l_2 - l_2\,l_1)\\ & + \varphi_{T\,2}\,(2\,l_2\,l_2 - l_1\,l_3 - l_3\,l_1))\end{aligned} \tag{C.2o}$$
$$\begin{aligned}(\xi''\,\varphi_S)\,(l\,l)_S = {} & \xi''\,(\varphi_{S\,2}\,(2\,l_1\,l_1 - l_2\,l_3 - l_3\,l_2) + \varphi_{S\,1}\,(2\,l_3\,l_3 - l_1\,l_2 - l_2\,l_1)\\ & + \varphi_{S\,3}\,(2\,l_2\,l_2 - l_1\,l_3 - l_3\,l_1))\end{aligned} \tag{C.2p}$$
$$\xi\,[\tilde{\xi}]\,\xi''\,(l\,l)' = \xi\,[\tilde{\xi}]\,\xi''\,(l_3\,l_3 + l_1\,l_2 + l_2\,l_1) \tag{C.2q}$$

As in the main part of the text, we use the notation $\xi\,[\tilde{\xi}]$ in order to indicate that either the field ξ or $\tilde{\xi}$ is involved in the coupling.

For the flavon potential the explicit form of the terms contained in Δw_{d1} and Δw_{d2} reads

$$
\begin{aligned}
I_3^T &= (\varphi_T^0\,\varphi_T)\,(\varphi_T\,\varphi_T) = (\varphi_{T\,1}^0\,\varphi_{T\,1} + \varphi_{T\,2}^0\,\varphi_{T\,3} + \varphi_{T\,3}^0\,\varphi_{T\,2})\,(\varphi_{T\,1}^2 + 2\,\varphi_{T\,2}\,\varphi_{T\,3}) && \text{(C.3a)}\\
I_4^T &= (\varphi_T^0\,\varphi_T)'\,(\varphi_T\,\varphi_T)'' = (\varphi_{T\,3}^0\,\varphi_{T\,3} + \varphi_{T\,1}^0\,\varphi_{T\,2} + \varphi_{T\,2}^0\,\varphi_{T\,1})\,(\varphi_{T\,2}^2 + 2\,\varphi_{T\,1}\,\varphi_{T\,3}) && \text{(C.3b)}\\
I_5^T &= (\varphi_T^0\,\varphi_T)''\,(\varphi_T\,\varphi_T)' = (\varphi_{T\,2}^0\,\varphi_{T\,2} + \varphi_{T\,1}^0\,\varphi_{T\,3} + \varphi_{T\,3}^0\,\varphi_{T\,1})\,(\varphi_{T\,3}^2 + 2\,\varphi_{T\,1}\,\varphi_{T\,2}) && \text{(C.3c)}\\
I_6^T &= (\varphi_T^0\,\varphi_S)\,(\varphi_S\,\varphi_S) = (\varphi_{T\,1}^0\,\varphi_{S\,1} + \varphi_{T\,2}^0\,\varphi_{S\,3} + \varphi_{T\,3}^0\,\varphi_{S\,2})\,(\varphi_{S\,1}^2 + 2\,\varphi_{S\,2}\,\varphi_{S\,3}) && \text{(C.3d)}\\
I_7^T &= (\varphi_T^0\,\varphi_S)'\,(\varphi_S\,\varphi_S)'' = (\varphi_{T\,3}^0\,\varphi_{S\,3} + \varphi_{T\,1}^0\,\varphi_{S\,2} + \varphi_{T\,2}^0\,\varphi_{S\,1})\,(\varphi_{S\,2}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,3}) && \text{(C.3e)}\\
I_8^T &= (\varphi_T^0\,\varphi_S)''\,(\varphi_S\,\varphi_S)' = (\varphi_{T\,2}^0\,\varphi_{S\,2} + \varphi_{T\,1}^0\,\varphi_{S\,3} + \varphi_{T\,3}^0\,\varphi_{S\,1})\,(\varphi_{S\,3}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,2}) && \text{(C.3f)}\\
I_9^T &= (\varphi_T^0\,(\varphi_S\,\varphi_S)_S)\,\xi && \text{(C.3g)}\\
&= \frac{2}{3}\,(\varphi_{T\,1}^0\,(\varphi_{S\,1}^2 - \varphi_{S\,2}\,\varphi_{S\,3}) + \varphi_{T\,2}^0\,(\varphi_{S\,2}^2 - \varphi_{S\,1}\,\varphi_{S\,3}) + \varphi_{T\,3}^0\,(\varphi_{S\,3}^2 - \varphi_{S\,1}\,\varphi_{S\,2}))\,\xi\\
I_{10}^T &= (\varphi_T^0\,(\varphi_S\,\varphi_S)_S)\,\tilde{\xi} && \text{(C.3h)}\\
&= \frac{2}{3}\,(\varphi_{T\,1}^0\,(\varphi_{S\,1}^2 - \varphi_{S\,2}\,\varphi_{S\,3}) + \varphi_{T\,2}^0\,(\varphi_{S\,2}^2 - \varphi_{S\,1}\,\varphi_{S\,3}) + \varphi_{T\,3}^0\,(\varphi_{S\,3}^2 - \varphi_{S\,1}\,\varphi_{S\,2}))\,\tilde{\xi}\\
I_{11}^T &= (\varphi_T^0\,\varphi_S)\,\xi^2 = (\varphi_{T\,1}^0\,\varphi_{S\,1} + \varphi_{T\,2}^0\,\varphi_{S\,3} + \varphi_{T\,3}^0\,\varphi_{S\,2})\,\xi^2 && \text{(C.3i)}\\
I_{12}^T &= (\varphi_T^0\,\varphi_S)\,\xi\,\tilde{\xi} = (\varphi_{T\,1}^0\,\varphi_{S\,1} + \varphi_{T\,2}^0\,\varphi_{S\,3} + \varphi_{T\,3}^0\,\varphi_{S\,2})\,\xi\,\tilde{\xi} && \text{(C.3j)}\\
I_{13}^T &= (\varphi_T^0\,\varphi_S)\,\tilde{\xi}^2 = (\varphi_{T\,1}^0\,\varphi_{S\,1} + \varphi_{T\,2}^0\,\varphi_{S\,3} + \varphi_{T\,3}^0\,\varphi_{S\,2})\,\tilde{\xi}^2 && \text{(C.3k)}\\
I_1^S &= (\varphi_S^0\,\varphi_T)_S\,(\varphi_S\,\varphi_S)_S && \text{(C.3l)}\\
&= \frac{2}{9}\,((2\,\varphi_{S\,1}^0\,\varphi_{T\,1} - \varphi_{S\,2}^0\,\varphi_{T\,3} - \varphi_{S\,3}^0\,\varphi_{T\,2})\,(\varphi_{S\,1}^2 - \varphi_{S\,2}\,\varphi_{S\,3})\\
&\qquad + (2\,\varphi_{S\,3}^0\,\varphi_{T\,3} - \varphi_{S\,1}^0\,\varphi_{T\,2} - \varphi_{S\,2}^0\,\varphi_{T\,1})\,(\varphi_{S\,2}^2 - \varphi_{S\,1}\,\varphi_{S\,3})\\
&\qquad + (2\,\varphi_{S\,2}^0\,\varphi_{T\,2} - \varphi_{S\,1}^0\,\varphi_{T\,3} - \varphi_{S\,3}^0\,\varphi_{T\,1})\,(\varphi_{S\,3}^2 - \varphi_{S\,1}\,\varphi_{S\,2}))\\
I_2^S &= (\varphi_S^0\,\varphi_T)_A\,(\varphi_S\,\varphi_S)_S && \text{(C.3m)}\\
&= \frac{1}{3}\,((\varphi_{S\,2}^0\,\varphi_{T\,3} - \varphi_{S\,3}^0\,\varphi_{T\,2})\,(\varphi_{S\,1}^2 - \varphi_{S\,2}\,\varphi_{S\,3})\\
&\qquad + (\varphi_{S\,1}^0\,\varphi_{T\,2} - \varphi_{S\,2}^0\,\varphi_{T\,1})\,(\varphi_{S\,2}^2 - \varphi_{S\,1}\,\varphi_{S\,3})\\
&\qquad + (\varphi_{S\,3}^0\,\varphi_{T\,1} - \varphi_{S\,1}^0\,\varphi_{T\,3})\,(\varphi_{S\,3}^2 - \varphi_{S\,1}\,\varphi_{S\,2}))\\
I_3^S &= (\varphi_S^0\,\varphi_T)\,(\varphi_S\,\varphi_S) = (\varphi_{S\,1}^0\,\varphi_{T\,1} + \varphi_{S\,2}^0\,\varphi_{T\,3} + \varphi_{S\,3}^0\,\varphi_{T\,2})\,(\varphi_{S\,1}^2 + 2\,\varphi_{S\,2}\,\varphi_{S\,3}) && \text{(C.3n)}\\
I_4^S &= (\varphi_S^0\,\varphi_T)'\,(\varphi_S\,\varphi_S)'' = (\varphi_{S\,3}^0\,\varphi_{T\,3} + \varphi_{S\,1}^0\,\varphi_{T\,2} + \varphi_{S\,2}^0\,\varphi_{T\,1})\,(\varphi_{S\,2}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,3}) && \text{(C.3o)}\\
I_5^S &= (\varphi_S^0\,\varphi_T)''\,(\varphi_S\,\varphi_S)' = (\varphi_{S\,2}^0\,\varphi_{T\,2} + \varphi_{S\,1}^0\,\varphi_{T\,3} + \varphi_{S\,3}^0\,\varphi_{T\,1})\,(\varphi_{S\,3}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,2}) && \text{(C.3p)}\\
I_6^S &= (\varphi_S^0\,(\varphi_T\,\varphi_S)_S)\,\xi && \text{(C.3q)}\\
&= \frac{1}{3}\,(\varphi_{S\,1}^0\,(2\,\varphi_{T\,1}\,\varphi_{S\,1} - \varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2}) + \varphi_{S\,2}^0\,(2\,\varphi_{T\,2}\,\varphi_{S\,2} - \varphi_{T\,1}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,1})\\
&\qquad + \varphi_{S\,3}^0\,(2\,\varphi_{T\,3}\,\varphi_{S\,3} - \varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1}))\,\xi\\
I_7^S &= (\varphi_S^0\,(\varphi_T\,\varphi_S)_S)\,\tilde{\xi} && \text{(C.3r)}\\
&= \frac{1}{3}\,(\varphi_{S\,1}^0\,(2\,\varphi_{T\,1}\,\varphi_{S\,1} - \varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2}) + \varphi_{S\,2}^0\,(2\,\varphi_{T\,2}\,\varphi_{S\,2} - \varphi_{T\,1}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,1})\\
&\qquad + \varphi_{S\,3}^0\,(2\,\varphi_{T\,3}\,\varphi_{S\,3} - \varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1}))\,\tilde{\xi}\\
I_8^S &= (\varphi_S^0\,(\varphi_T\,\varphi_S)_A)\,\xi && \text{(C.3s)}\\
&= \frac{1}{2}\,(\varphi_{S\,1}^0\,(\varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2}) + \varphi_{S\,2}^0\,(\varphi_{T\,3}\,\varphi_{S\,1} - \varphi_{T\,1}\,\varphi_{S\,3}) + \varphi_{S\,3}^0\,(\varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1}))\,\xi\\
I_9^S &= (\varphi_S^0\,(\varphi_T\,\varphi_S)_A)\,\tilde{\xi} && \text{(C.3t)}\\
&= \frac{1}{2}\,(\varphi_{S\,1}^0\,(\varphi_{T\,2}\,\varphi_{S\,3} - \varphi_{T\,3}\,\varphi_{S\,2}) + \varphi_{S\,2}^0\,(\varphi_{T\,3}\,\varphi_{S\,1} - \varphi_{T\,1}\,\varphi_{S\,3}) + \varphi_{S\,3}^0\,(\varphi_{T\,1}\,\varphi_{S\,2} - \varphi_{T\,2}\,\varphi_{S\,1}))\,\tilde{\xi}\\
I_{10}^S &= (\varphi_S^0\,\varphi_T)\,\xi^2 = (\varphi_{S\,1}^0\,\varphi_{T\,1} + \varphi_{S\,2}^0\,\varphi_{T\,3} + \varphi_{S\,3}^0\,\varphi_{T\,2})\,\xi^2 && \text{(C.3u)}\\
I_{11}^S &= (\varphi_S^0\,\varphi_T)\,\xi\tilde{\xi} = (\varphi_{S\,1}^0\,\varphi_{T\,1} + \varphi_{S\,2}^0\,\varphi_{T\,3} + \varphi_{S\,3}^0\,\varphi_{T\,2})\,\xi\,\tilde{\xi} && \text{(C.3v)}\\
I_{12}^S &= (\varphi_S^0\,\varphi_T)\,\tilde{\xi}^2 = (\varphi_{S\,1}^0\,\varphi_{T\,1} + \varphi_{S\,2}^0\,\varphi_{T\,3} + \varphi_{S\,3}^0\,\varphi_{T\,2})\,\tilde{\xi}^2 && \text{(C.3w)}\\
I_1^X &= \xi^0\,(\varphi_T\,(\varphi_S\,\varphi_S)_S) && \text{(C.3x)}\\
&= \frac{2}{3}\,\xi^0\,(\varphi_{T\,1}\,(\varphi_{S\,1}^2 - \varphi_{S\,2}\,\varphi_{S\,3}) + \varphi_{T\,2}\,(\varphi_{S\,2}^2 - \varphi_{S\,1}\,\varphi_{S\,3}) + \varphi_{T\,3}\,(\varphi_{S\,3}^2 - \varphi_{S\,1}\,\varphi_{S\,2}))\\
I_2^X &= \xi^0\,(\varphi_T\,\varphi_S)\,\xi = \xi^0\,(\varphi_{T\,1}\,\varphi_{S\,1} + \varphi_{T\,2}\,\varphi_{S\,3} + \varphi_{T\,3}\,\varphi_{S\,2})\,\xi && \text{(C.3y)}\\
I_3^X &= \xi^0\,(\varphi_T\,\varphi_S)\,\tilde{\xi} = \xi^0\,(\varphi_{T\,1}\,\varphi_{S\,1} + \varphi_{T\,2}\,\varphi_{S\,3} + \varphi_{T\,3}\,\varphi_{S\,2})\,\tilde{\xi} && \text{(C.3z)}
\end{aligned}
$$

Thereby we arrive to the same result as [15] concerning the number of invariants and structure of invariants contained in $\Delta w_{d\,1}$. The additional terms present in $\Delta w_{d\,2}$ are of the form

$$\begin{aligned}
I_{14}^T &= (\varphi_T^0\,\varphi_T)''\,\xi''\,\xi'' = (\varphi_{T\,2}^0\,\varphi_{T\,2} + \varphi_{T\,1}^0\,\varphi_{T\,3} + \varphi_{T\,3}^0\,\varphi_{T\,1})\,\xi''\,\xi'' && \text{(C.4a)}\\
I_{15}^T &= (\varphi_T\,\eta)\,(\varphi_T^0\,\eta) && \text{(C.4b)}\\
&= ((1+i)\,\varphi_{T\,1}\,\eta_2 + \varphi_{T\,3}\,\eta_1)\,((1-i)\,\varphi_{T\,2}^0\,\eta_1 - \varphi_{T\,3}^0\,\eta_2)\\
&\quad - ((1-i)\,\varphi_{T\,2}\,\eta_1 - \varphi_{T\,3}\,\eta_2)\,((1+i)\,\varphi_{T\,1}^0\,\eta_2 + \varphi_{T\,3}^0\,\eta_1)\\
I_{16}^T &= (\varphi_T\,\eta)'\,(\varphi_T^0\,\eta)'' && \text{(C.4c)}\\
&= ((1+i)\,\varphi_{T\,2}\,\eta_2 + \varphi_{T\,1}\,\eta_1)\,((1-i)\,\varphi_{T\,1}^0\,\eta_1 - \varphi_{T\,2}^0\,\eta_2)\\
&\quad - ((1-i)\,\varphi_{T\,3}\,\eta_1 - \varphi_{T\,1}\,\eta_2)\,((1+i)\,\varphi_{T\,3}^0\,\eta_2 + \varphi_{T\,2}^0\,\eta_1)\\
I_{17}^T &= (\eta\,\xi'')\,(\varphi_T^0\,\eta) = \xi''\,(\eta_1\,((1-i)\,\varphi_{T\,2}^0\,\eta_1 - \varphi_{T\,3}^0\,\eta_2) - \eta_2\,((1+i)\,\varphi_{T\,1}^0\,\eta_2 + \varphi_{T\,3}^0\,\eta_1)) && \text{(C.4d)}\\
I_{18}^T &= (\varphi_T\,\varphi_T)_S\,(\varphi_T^0\,\xi'') && \text{(C.4e)}\\
&= \frac{2}{3}\,((\varphi_{T\,1}^2 - \varphi_{T\,2}\,\varphi_{T\,3})\,\varphi_{T\,2}^0 + (\varphi_{T\,3}^2 - \varphi_{T\,1}\,\varphi_{T\,2})\,\varphi_{T\,1}^0 + (\varphi_{T\,2}^2 - \varphi_{T\,1}\,\varphi_{T\,3})\,\varphi_{T\,3}^0)\,\xi''\\
I_{13}^S &= (\varphi_S^0\,\varphi_S)'\,\xi\,\xi'' = (\varphi_{S\,3}^0\,\varphi_{S\,3} + \varphi_{S\,1}^0\,\varphi_{S\,2} + \varphi_{S\,2}^0\,\varphi_{S\,1})\,\xi\,\xi'' && \text{(C.4f)}\\
I_{14}^S &= (\varphi_S^0\,\varphi_S)'\,\tilde{\xi}\,\xi'' = (\varphi_{S\,3}^0\,\varphi_{S\,3} + \varphi_{S\,1}^0\,\varphi_{S\,2} + \varphi_{S\,2}^0\,\varphi_{S\,1})\,\tilde{\xi}\,\xi'' && \text{(C.4g)}\\
I_{15}^S &= (\varphi_S\,\xi'')\,(\varphi_S^0\,\varphi_S)_S && \text{(C.4h)}\\
&= \frac{1}{3}\,\xi''\,(\varphi_{S\,2}\,(2\,\varphi_{S\,1}^0\,\varphi_{S\,1} - \varphi_{S\,2}^0\,\varphi_{S\,3} - \varphi_{S\,3}^0\,\varphi_{S\,2}) + \varphi_{S\,3}\,(2\,\varphi_{S\,2}^0\,\varphi_{S\,2} - \varphi_{S\,1}^0\,\varphi_{S\,3} - \varphi_{S\,3}^0\,\varphi_{S\,1})\\
&\quad + \varphi_{S\,1}\,(2\,\varphi_{S\,3}^0\,\varphi_{S\,3} - \varphi_{S\,1}^0\,\varphi_{S\,2} - \varphi_{S\,2}^0\,\varphi_{S\,1}))\\
I_4^X &= (\varphi_S\,\varphi_S)'\,\xi^0\,\xi'' = (\varphi_{S\,3}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,2})\,\xi^0\,\xi'' && \text{(C.4i)}
\end{aligned}$$

and furthermore the structures involving the driving fields $\eta_{1,2}^0$ and ξ'^0 read

$$\begin{aligned}
I_1^N &= (\varphi_T\,\varphi_T)\,(\eta^0\,\eta) = (\varphi_{T\,1}^2 + 2\,\varphi_{T\,2}\,\varphi_{T\,3})\,(\eta_1^0\,\eta_2 - \eta_2^0\,\eta_1) && \text{(C.5a)}\\
I_2^N &= (\varphi_T\,\eta)\,(\eta^0\,\varphi_T) && \text{(C.5b)}\\
&= ((1+i)\,\varphi_{T\,1}\,\eta_2 + \varphi_{T\,3}\,\eta_1)\,((1-i)\,\eta_1^0\,\varphi_{T\,1} - \eta_2^0\,\varphi_{T\,2})\\
&\quad - ((1-i)\,\varphi_{T\,2}\,\eta_1 - \varphi_{T\,3}\,\eta_2)\,((1+i)\,\eta_2^0\,\varphi_{T\,3} + \eta_1^0\,\varphi_{T\,2})\\
I_3^N &= (\eta\,\xi'')\,(\eta^0\,\varphi_T) = \xi''\,(\eta_1\,((1-i)\,\eta_1^0\,\varphi_{T\,1} - \eta_2^0\,\varphi_{T\,2}) - \eta_2\,((1+i)\,\eta_2^0\,\varphi_{T\,3} + \eta_1^0\,\varphi_{T\,2})) && \text{(C.5c)}\\
I_4^N &= (\eta\,\eta)_3\,(\eta^0\,\eta)_3 = \frac{1}{2}\,(1+i)\,\eta_1^2\,(\eta_1^0\,\eta_2 + \eta_2^0\,\eta_1) + \eta_2^3\,\eta_2^0 + (1+i)\,\eta_1^2\,\eta_2\,\eta_1^0 && \text{(C.5d)}\\
I_1^Y &= (\varphi_T\,\varphi_T)\,\xi'^0\,\xi'' = (\varphi_{T\,1}^2 + 2\,\varphi_{T\,2}\,\varphi_{T\,3})\,\xi'^0\,\xi'' && \text{(C.5e)}\\
I_2^Y &= (\varphi_T\,\eta)'\,(\xi'^0\,\eta)'' = ((1+i)\,\varphi_{T\,2}\,\eta_2 + \varphi_{T\,1}\,\eta_1)\,\xi'^0\,\eta_2 - ((1-i)\,\varphi_{T\,3}\,\eta_1 - \varphi_{T\,1}\,\eta_2)\,\xi'^0\,\eta_1 && \text{(C.5f)}\\
I_3^Y &= (\varphi_S\,\varphi_S)''\,\xi'^0\,\xi = (\varphi_{S\,2}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,3})\,\xi'^0\,\xi && \text{(C.5g)}\\
I_4^Y &= (\varphi_S\,\varphi_S)''\,\xi'^0\,\tilde{\xi} = (\varphi_{S\,2}^2 + 2\,\varphi_{S\,1}\,\varphi_{S\,3})\,\xi'^0\,\tilde{\xi} && \text{(C.5h)}
\end{aligned}$$

List of Tables

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