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Stability analysis of multiplayer games on adaptive simplicial complexes

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ABSTRACT

We analyze the influence of multiplayer interactions and network adaptation on the stability of equilibrium points in evolutionary games. We consider the Snowdrift game on simplicial complexes. In particular, we consider as a starting point the extension from only two-player interactions to coexistence of two- and three-player interactions. The state of the system and the topology of the interactions are both adaptive through best-response strategies of nodes and rewiring strategies of edges, respectively. We derive a closed set of low-dimensional differential equations using pairwise moment closure, which yields an approximation of the lower moments of the system. We numerically confirm the validity of these moment equations. Moreover, we demonstrate that the stability of the fixed points remains unchanged for the considered adaption process. This stability result indicates that rational best-response strategies in games are very difficult to destabilize, even if higher-order multiplayer interactions are taken into account.

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Social systems are characterized by complex networks with interactions of two or more agents (e.g., individuals, group representatives, companies). In game theoretic models of these systems, agents get a payoff depending on their chosen action and their neighbors' actions. This article extends the well-established class of adaptive two-player games on networks, where the state of the system and the topology of the network change over time, to three-player games on adaptive simplicial complexes. As an example, the Snowdrift game with rational adaption rules is considered. We study the network using direct simulations as well as moment closure, leading to a low-dimensional differential equation. We obtain a stable equilibrium state with coexistence of cooperation and defection, which is remarkably robust across very large ranges of parameters.

I. INTRODUCTION

Complex networks are a well-established tool for the description of dynamical processes with interaction between adjacent nodes/agents. In many real world systems, however, interactions of

more than two nodes take place and have an important impact on the dynamics of the system,¹ such as transmission of diseases from one infected to many others, social science,² biological systems,³ neuroscience,^{4,5} and many more. Here, we are interested in the influence of polyadic (also known as higher-order⁶) interactions in the context of game theory. So far, these multi-player interactions have been considered on fixed hypergraphs, where the structure or topology remains constant.⁷⁻⁹ In contrast, adaptive coevolutionary networks^{10,11} have been successfully applied to many model systems with two-node interactions, e.g., in neural processing,¹² epidemic spreading,¹³ opinion forming,¹⁴ and evolutionary social games.¹⁵⁻²¹ In all of these systems, the network topology changes due to adding, deleting, or rewiring of two-player edges. Evolutionary social games are often characterized by a social dilemma, in which the global payoff is not maximal, if individuals optimize their own payoff, as in the prisoner's dilemma or the Snowdrift game.

Much less is known about adaptive multiplayer games on hypergraphs, where both the hypergraph and the state of the nodes evolve with time. In such systems, the games are not restricted to two-player interactions, governed by usual edges in networks, but also extend to three (or more) player interactions, governed

by higher-dimensional hyperedges. Recently, an extension of the voter model to adaptive simplicial complexes has shown that multiplayer peer pressure interaction can lead to stronger polarization of opinions than expected from classical two-node interactions.^{22,23} In fact, it is understood that polyadic, higher-order coupling for static topologies can already lead to important new effects in many dynamical process; see, e.g., Refs. 24–29. Hence, combining the previously mentioned directions, it becomes a natural question as to how evolutionary social games behave with additional higher-order interactions of more than two players and adaptive topology of the system.

Multiplayer games without adaptivity have been studied as a model class for quite some time. For example, in the multiplayer Snowdrift game, the density of cooperators drops in a replicator dynamics with the number of players and the cost-to-benefit ratio,³⁰ there are multiplayer extensions that are equivalent to a generalized public goods game,³¹ and spatial extensions have been investigated.^{32–34} Yet, it remains unclear how adaptivity of the topology influences the dynamics of multiplayer games; therefore, we have to start from basic models.

In this work, we define the adaptive simplicial Snowdrift game by combining explicit three-player interactions with the Snowdrift game on an adaptive simplicial complex. We are interested in the question if multiplayer interactions that increase the individual payoff, such as best response and rewiring, can destabilize the steady states of the system. To study this question, we combine analytical and numerical techniques. On the analytical level, we derive low-dimensional moment equations for the lowest moments and their closures based on pairwise approximation. Numerically, we study the dynamics by direct simulation. We observe good agreement with the time evolution of the moments in both approaches. Second, we analyze the steady states of the moment equations. It turns out that the stability properties of the main attracting steady state are robust under adding polyadic, higher-order interactions beyond the classical two-player scenario. In particular, the more classical adaptive Snowdrift model with only two-player interactions behaves very similar to the variant with additional three-player interactions over large robust parameter ranges. One possible interpretation of this result is that as long as agents behave in a sufficiently rational way, polyadic interactions between them do not destroy the stability of the system. Finally, we numerically evaluate how introducing some partial irrational behavior of the agents could destabilize the system. Even in this case, it is very difficult to find instabilities.

There are two important conclusions that can be drawn from these results. First, one may conjecture that many large-scale stable economic systems remain stable even if very complex polyadic interactions are considered as long as the rules the players follow are sufficiently rational. Second, although it has been shown that higher-order multiplayer interactions can change dynamics in several examples for certain parameters, it should be acknowledged that there are also many cases, where the system dynamics can be very robust to adding such interactions between nodes.

This paper is structured as follows. In Sec. II, we introduce the adaptive simplicial Snowdrift model. In Sec. III, we specify the full moment equations for the system and their closure, with a detailed derivation given in Appendixes A and B and present numerical results from simulation. The stability of equilibria of the closed

moment equations is analyzed in Sec. IV. This paper is summarized in Sec. V.

II. ADAPTIVE SIMPLICIAL SNOWDRIFT GAME

A. Multiplayer Snowdrift game

The Snowdrift game is a famous example for a game with a social dilemma.^{35,36} In this game, the players may or may not contribute to some task with a total cost c and a common benefit b . The cost c is split even among all the cooperators (strategy C), but each player, also the defectors (strategy D), get the same benefit, as long as the task is done. This leads to the most commonly used payoff matrix of the two-player Snowdrift game,³²

$$P = \begin{array}{cc} & \begin{array}{c} C \\ D \end{array} \\ \begin{array}{c} C \\ D \end{array} & \begin{pmatrix} b - \frac{c}{2} & b - c \\ b & 0 \end{pmatrix} \end{array}, \quad (1)$$

where the cost c is split among the cooperators, while there is a benefit b if there is at least one cooperator. We assume that the benefit exceeds the cost, $b > c > 0$, since otherwise defection becomes the single dominating strategy;³² i.e., for high costs, $b < c < 2b$, the system is equivalent to the prisoner's dilemma. Choosing $b > c$ implies $P(D, C) > P(C, C) > P(C, D) > P(D, D)$, which is the defining property of Snowdrift games.³² First, this implies that there is no dominated action and that there are two pure Nash equilibria, (C, D) and (D, C), where deviation leads to smaller payoffs. Second, both strategies can invade a population of opposite players such that there is a mixed evolutionary stable state with a fraction $p_C = 1 - \frac{c}{2b-c}$ of cooperating players.³² However, for all $p_C < 1$, the total payoff is smaller than in populations with only cooperators since $p_C^2(b - c/2) + p_C(1 - p_C)(b + b - c) + (1 - p_C)^2 \cdot 0 = p_C(2 - p_C)(b - c/2) \leq b - c/2$. Therefore, the Snowdrift game is a so-called social dilemma,³² where maximizing individual payoffs does not maximize the payoff for the whole society.

There are several ways to extend the Snowdrift game to a multiplayer game.^{30,32–34} We consider the following payoff functions for L players, which are defined for cooperating (C) and defecting (D) players as³⁰

$$P_C(n_C) = \begin{cases} 0 & \text{for } n_C = 0, \\ b - \frac{c}{n_C} & \text{for } n_C \in [1, L], \end{cases} \quad (2)$$

$$P_D(n_C) = \begin{cases} 0 & \text{for } n_C = 0, \\ b & \text{for } n_C \in [1, L - 1], \end{cases} \quad (3)$$

where $0 \leq n_C \leq L$ is the number of cooperating players C. Here, the total cost c of completing the task is shared equally among all cooperating players. For $L = 2$, the payoff matrix in Eq. (1) is recovered.

The properties of the multiplayer Snowdrift game are similar to the two-player game. There are no dominated actions. Best response against all-defecting opponents is to choose cooperation since the payoff increases, $P_C(n_C = 1) = b - c > 0 = P_D(n_C = 0)$. On the other hand, if there are already $l \geq 1$ cooperating opponents, best response is to choose defection because $P_D(n_C = l) = b > b - \frac{c}{l+1} = P_C(n_C = l + 1) \forall l \in [1, L - 1]$. Again, this implies

that all pure strategy profiles, in which exactly one player cooperates, $n_C = 1$, are pure Nash equilibria and that there is some mixed evolutionary stable strategy. Note that in general, multiplayer games with $L > 2$ are not reducible to multiple two-player encounters.²⁴

B. Adaptive networks

We start by defining a model class for evolutionary dynamics of the Snowdrift game on adaptive hypergraphs. A hypergraph $\mathcal{G} = (\mathcal{N}, \mathcal{E})$ with up to d -player interactions consists of a discrete set of nodes \mathcal{N} and a set of (hyper)edges $\mathcal{E} = \bigcup_{n=2}^d \mathcal{E}_n$ connecting the nodes. Here, $\mathcal{E}_n \subset \mathcal{N}^n$ is the set of n -dimensional hyperedges. The nodes $i \in \mathcal{N}$ represent individual players, and we assign a state (or label) $X: \mathcal{N} \rightarrow \mathcal{A}$ to each node, which for the Snowdrift game is either cooperation or defection, $X(i) \in \{C, D\}$. In ordinary graphs, only edges $e = (i, j) \in \mathcal{E}_2$ are present and represent possible two-player interactions between the corresponding nodes. In hypergraphs also higher-dimensional edges are allowed, e.g., three-node hyperedges $s = (i, j, k)$. Thus, this framework naturally allows one to generalize two-player games on ordinary networks/graphs to multiplayer games on hypergraphs.

This framework allows for two different possibilities for coevolutionary (also known as adaptive) network dynamics.³⁷ The first is the evolution of the node state, while the set of edges remains constant. The second is the evolution of the set of edges through creation and deletion, while the state of nodes remains unchanged. These two types are called *dynamic on* and *of* the network, respectively, and in adaptive networks, both dynamics take place.¹¹

In this work, we restrict the class of hypergraphs to simplicial complexes to keep the analytical complexity of the models more tractable, but our approach will be extended in future work. From the viewpoint of applications, the simplicial complex restriction is applicable to systems, in which interaction of more than two agents implies that each subset of these agents also interacts with each other, e.g., in friendship groups.

A simplicial complex \mathcal{S} is a set of simplices with

- (i) $s \in \mathcal{S} \Rightarrow \partial s \subset \mathcal{S}$ and
- (ii) for simplices $s_1, s_2 \in \mathcal{S}$, the intersection satisfies $s_1 \cap s_2 = \emptyset$ or $s_1 \cap s_2 \in \partial s_1, \partial s_2$,

where ∂s is the set of faces of simplex s . Thus, the first condition means that every face of simplices in \mathcal{S} is again in \mathcal{S} . A simplicial k -complex is a simplicial complex \mathcal{S} with $\dim(\mathcal{S}) := \sup_{s \in \mathcal{S}} \dim(s) = k$; i.e., the maximal dimension of simplices in \mathcal{S} is k and there is at least one such simplex. The first step to a generalized multiplayer Snowdrift game is to include interactions of three players.^{1,38} This means that the coevolutionary network dynamics is realized on a simplicial two-complex consisting of nodes $i \in \mathcal{N}$, edges $(i, j) \in \mathcal{E}_2$, and simplicial triangles $(i, j, k) \in \mathcal{E}_3$. The corresponding hypergraph is then given by $\mathcal{G} = (\mathcal{N}, \mathcal{E}_2 \cup \mathcal{E}_3)$. As before, the nodes correspond to the agents, while the edges and triangles represent two-player and three-player interactions, respectively. Let us remark that in ordinary graphs, one usually calls a set of three nodes $i, j, k \in \mathcal{N}$ a triangle if each is connected to the others directly, $\{(i, j), (j, k), (k, i)\} \subset \mathcal{E}$. In contrast, a simplicial triangle is a tuple $(i, j, k) \in \mathcal{E}_3$. For simplicity and better distinction, we will call the elements of \mathcal{E}_3 just *simplex* in the following and leave the term triangle for the case of

three pairwise connected nodes. Note that within a simplicial complex, each simplex $(i, j, k) \in \mathcal{E}_3$ implies the existence of the triangle $\{(i, j), (j, k), \text{ and } (k, i)\}$ since each face of (i, j, k) must be contained, but not vice versa.

The state $X(i) \in \{C, D\}$ of the nodes $i \in \mathcal{N}$ is used to label the elements of the graph as follows. First, we identify C with label 0 and D with label 1 such that 0-nodes are cooperating and 1-nodes are defecting. The number of C-nodes in the simplicial complex is n_0 , while the number of D-nodes is called n_1 . Together, they satisfy $N := |\mathcal{N}| = n_0 + n_1$.

Similarly, the edges are classified into 00-, 01-, and 11-edges. Again, the number of xy -edges is denoted n_{xy} , and they are counted as $n_{xy} = \sum_{i,j=1}^N A_{ji} \delta_{X(i),x} \delta_{X(j),y}$, where $A_{ji} = 1$ if $(j, i) \in \mathcal{E}_2$ and zero else and $\delta_{a,b}$ is the Kronecker delta. Under the dynamics we specify below, the total number of edges will satisfy a conservation law $M := |\mathcal{E}_2| = n_{00} + n_{01} + n_{10} + n_{11} = n_{00} + 2n_{01} + n_{11} = k_{\text{deg}} N$, where k_{deg} is the average degree of nodes in the network.

The simplices are characterized by their number of defecting nodes, and we call $s = (i, j, k) \in \mathcal{E}_3$ a simplex of type I if it contains exactly I defecting nodes; i.e., $X(i) + X(j) + X(k) = I$. The set of all I -type simplices is denoted by \mathcal{S}_I . Similarly, triangles $\{(i, j), (j, k), (k, l)\} \subset \mathcal{E}$ are of type I if they consist of exactly I defecting nodes.

In general, dynamical systems on adaptive networks (or hypergraphs) are characterized by dynamics on the network, i.e., update rules of node status, and dynamics of the network, i.e., creation or deletion of nodes, edges, or simplices. Assuming a fixed set of nodes \mathcal{N} , this means that their dynamics takes place on a time-dependent simplicial hypergraph. In the following, we specify a set of dynamical rules, which define the adaptive simplicial Snowdrift game.

C. Adaptive simplicial Snowdrift model

The adaptive simplicial Snowdrift model is based on the adaptive simplex voter model.^{22,23} We consider the generalization to two-simplices, i.e., additional three-player interactions, which often is sufficient for interesting dynamics, such as de-stabilization.^{24–29} We assume that each player has perfect knowledge about the currently chosen actions of the other players. Based on the structure of the simplicial two-complex, we consider three different update rules on the node status and the set of edges. For all operations, one specific node $i \in \mathcal{N}$ (corresponding to a player) is selected. This player faces either one opponent along an edge $(i, j) \in \mathcal{E}_2$ or two opponents through a simplex $(i, j, k) \in \mathcal{E}_3$ if this exists. The considered strategies are *rewiring*, *best response on edge*, and *best response on simplex*.

Rewiring is a dynamic of the network topology and changes the simplicial complex. This is well established for networks with only two-player interactions.^{10–21} The selected player i removes the currently chosen edge (i, j) and establishes a new edge (i, k) with uniform probability to some other player k with node status $X(k) = C$; see Fig. 1(a). Rewiring from D to C strictly increases the payoff for player i since $P(X(i), C) > P(X(i), D)$ for $X(i) \in \{C, D\}$. If the removed edge (i, j) was part of a simplex (i, j, m) , then this simplex is removed as well, and in order to keep the number of simplices constant, a new simplex is created on top of one existing triangle.

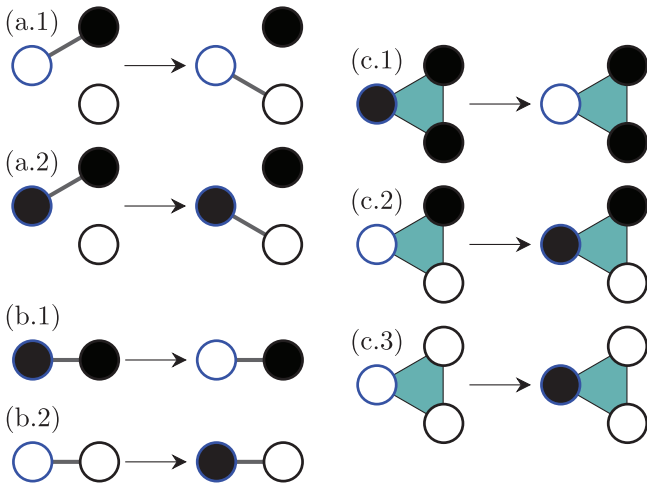


FIG. 1. Sketch of dynamical rules of evolution between cooperating (white) and defecting (black) nodes. (a) Rewiring from D to C neighbors. (b) Best response on edges. (c) Best response on simplices. In all cases, the left node (blue border) takes the considered action. Configurations without change are not shown.

Hence, this action directly influences the types of existing edges and simplices in the simplicial hypergraph.

Best response describes a change of node states and thus is a dynamic on the network. In *best response on edges*, the selected player i adjusts its own action to the best response against the chosen opponent's action in a two-player Snowdrift game. Here, best response to each action is the opposite action; see Sec. II A. In particular, if the opposing action is C (D), the state of the selected player becomes $X(i) = D$ ($X(i) = C$); see Fig. 1(b). In *best response on simplices*, the selected player i adjusts its own action to the best response against the two opponents on a simplex (i, j, k) for the three-player Snowdrift game. This means that if one of the opponents cooperates, $C \in \{X(j), X(k)\}$, the selected player chooses to defect and the state becomes $X(i) = D$. On the other hand, if both opponents defect, $X(j) = X(k) = D$, the selected player chooses cooperation, $X(i) = C$; see Fig. 1(c). Thus, if the state $X(i)$ is different from the best-response action, it changes, and otherwise, it stays constant. If it changes, not only the label of the chosen node changes, but also the types of all edges and all simplices, which this node is part of. Note that best-response strategies consider the optimal choice of a single agent based on knowledge of the opponents' strategy and is thus one of the simplest non-trivial update rules. A more elaborate approach would be to consider so-called Fermi-update rules^{15,18,39} where a probability to adapt the neighbors' strategy is given based on the current payoffs of the respective agents.

Based on these three operations, the evolution of the multi-player Snowdrift game on an adaptive simplicial complex is modeled as follows. Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}_2 \cup \mathcal{E}_3)$ be a simplicial complex as defined above. We define the probability $\rho \in [0, 1]$ to execute best response on simplices and the probability $\phi \in [0, 1]$ to execute rewiring of edges in \mathcal{E}_2 . First, an edge $(i, j) \in \mathcal{E}_2$ is randomly chosen in the network.⁴⁰ The first node in the edge performs the following operations. If the chosen edge (i, j) is not part of any simplex, then with

probability ϕ rewiring and with probability $1 - \phi$, best response on edges is executed by the chosen node i . On the other hand, if the chosen edge is part of at least one simplex with probability ρ , best response in simplices is executed by the chosen node i , and with probability $1 - \rho$, an operation on the edge (i, j) is executed. In the first case, one simplex is chosen uniformly from all simplices of which the edge is part of and node i performs best response on this simplex. In the second case, the edge operation is again either rewiring of (i, j) with probability ϕ or best response on the edge (i, j) with probability $1 - \phi$.

III. MOMENT EQUATIONS

In order to analyze the dynamics of the system, we derive a set of moment equations for the lowest moments n_0 , n_{00} , and n_{11} . The other lowest moments follow from the conservation rule for nodes, $N = n_0 + n_1$, and for edges, $M = n_{00} + 2n_{01} + n_{11}$. For an intuitive understanding, first, we consider the system with only two-player interactions. Second, we present the corresponding moment equations for the adaptive three-player Snowdrift game, derived in detail in Appendix A. The resulting moment equations depend on higher-order moments, such as the distribution of triples, triangles, and simplices.

A. Adaptive two-player game

First, we consider the proposed model, including only two-player interactions. For this, let $\mathcal{G} = (\mathcal{N}, \mathcal{E}_2)$ be a graph with nodes \mathcal{N} and edges \mathcal{E}_2 . First, an edge $(i, j) \in \mathcal{E}_2$ is chosen at random. With probability ϕ , the node i rewires to a cooperating node. With probability $1 - \phi$, it performs best response on the chosen edge.

The number of 0-nodes n_0 increases (decreases) due to best response on 00-edges (11-edges), which implies

$$\frac{d}{dt} n_0 = - \underbrace{\frac{n_{00}}{M} (1 - \phi)}_{\text{b.r. 00-edge}} + \underbrace{\frac{n_{11}}{M} (1 - \phi)}_{\text{b.r. 11-edge}}. \quad (4)$$

Here, n_{00}/M and n_{11}/M are the probabilities that the initial edge is of type 00 and 11, respectively, while $1 - \phi$ is the probability of best response.

The number of 00-edges n_{00} increases due to rewiring when the initial edge is a 01-edge. Note that each 00-edge is counted twice, giving a factor of 2. Furthermore, it decreases (increases) if best response takes place on a 00-edge (11-edge). For a 00-edge, we have to consider the direct contribution from the edge itself plus all additional 0-type neighbors. For this, we define the excess degree $E_0^{(0,0)}(1)$ as the expected number of 0-type neighbors to a 0-type node being already connected to 1 other 0-node; see Appendix B 1. The total contribution of 00-edges is then given by the probability $\frac{n_{00}}{M} (1 - \phi)$ to perform best response on any 00-edge multiplied with $1 + E_0^{(0,0)}(1)$, where the +1 accounts for the direct contribution of the 00-edge itself. Similarly, best response on 11-edges leads to $\frac{n_{11}}{M} (1 - \phi)$ multiplied with the excess degree $E_1^{(1,0)}(1)$, which is the expected number of 0-type neighbors to a 1-type node being already

connected to one 1-type node. This leads to

$$\frac{d}{dt}n_{00} = 2 \left[\underbrace{\frac{n_{01}}{M}\phi}_{\text{rew. 01}} - \underbrace{\frac{n_{00}}{M}(1-\phi)}_{\text{b.r. 00-edge}} \left(1 + E_0^{(0,0)}(1)\right) + \underbrace{\frac{n_{11}}{M}(1-\phi)}_{\text{b.r. 11-edge}} E_1^{(1,0)}(1) \right]. \quad (5)$$

Pairwise approximation gives $E_0^{(0,0)}(1) \approx \frac{n_{00}}{n_0}$ and $E_1^{(1,0)}(1) \approx \frac{n_{01}}{n_1}$, see Appendix B 1, such that we obtain the pairwise closed equation

$$\frac{d}{dt}n_{00} = 2 \left[\frac{n_{01}}{M}\phi - \frac{n_{00}}{M}(1-\phi) \left(1 + \frac{n_{00}}{n_0}\right) + \frac{n_{11}}{M}(1-\phi) \frac{n_{01}}{n_1} \right]. \quad (6)$$

Finally, the number n_{11} of 11-edges decreases due to rewiring of 11-edges, while best response increases (decreases) n_{11} when taking place on a 00-edge (11-edge). The corresponding contributions of best response on edges are given by the probabilities $\frac{n_{00}}{M}(1-\phi)$ and $\frac{n_{11}}{M}(1-\phi)$ multiplied, respectively, with the excess degrees $E_0^{(0,1)}(1)$ and $1 + E_1^{(1,1)}(1)$. The full contribution to n_{11} is then

$$\frac{d}{dt}n_{11} = 2 \left[\underbrace{-\frac{n_{11}}{M}\phi}_{\text{rew. 11}} + \underbrace{\frac{n_{00}}{M}(1-\phi)}_{\text{b.r. 00-edge}} E_0^{(0,1)}(1) - \underbrace{\frac{n_{11}}{M}(1-\phi)}_{\text{b.r. 11-edge}} \left(1 + E_1^{(1,1)}(1)\right) \right]. \quad (7)$$

Pairwise approximation of the excess degrees, $E_0^{(0,1)}(1) \approx \frac{n_{01}}{n_0}$ and $E_1^{(1,1)}(1) \approx \frac{n_{11}}{n_1}$, see Appendix B 1, gives

$$\frac{d}{dt}n_{11} = 2 \left[-\frac{n_{11}}{M} + \frac{n_{00}}{M}(1-\phi) \frac{n_{01}}{n_0} - \frac{n_{11}}{M}(1-\phi) \frac{n_{11}}{n_1} \right]. \quad (8)$$

B. Adaptive three-player game

Three-player interactions take place on simplices. This leads to modifications of the existing terms of the two-player model and additional multiplayer terms. Most importantly, the probability to perform an action on simplices plays a crucial role. For xy -edges,

it is given by the probability P_{any}^{xy} that the edge is part of any simplex multiplied with the probability ρ to take such a simplex action, ρP_{any}^{xy} .

The number of cooperating nodes changes due to best response on edges and on simplices. Best response on 00-edges decreases n_0 by one, while best response on 11-edges increases n_0 by one. The corresponding probabilities are modified in comparison with the two-player game by a factor $(1 - \rho P_{\text{any}}^{00})$ and $(1 - \rho P_{\text{any}}^{11})$, respectively. For details, see Appendix A. Best response on simplices decreases the number of cooperators n_0 when the initial node cooperates, and there is at least one other cooperator. Similarly, best response on simplices increases the number of cooperators if the action takes place within a fully defecting simplex of type S_3 . This can be expressed in terms of probabilities $P_{S_j}^{xy}$ for an xy -type edge to be part of a simplex of type j , conditioned it is part of any simplex; for a full derivation, see Appendix A. Altogether, the number n_0 of cooperators changes according to

$$\begin{aligned} \frac{d}{dt}n_0 = & - \underbrace{\frac{n_{00}}{M} \left(1 - \rho P_{\text{any}}^{00}\right) (1 - \phi)}_{\text{b.r. on 00 and 11-edges}} + \underbrace{\frac{n_{11}}{M} \left(1 - \rho P_{\text{any}}^{11}\right) (1 - \phi)}_{\text{b.r. on 00 and 11-edges}} \\ & - \underbrace{\frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left(P_{S_0}^{00} + P_{S_1}^{00}\right)}_{\text{b.r. 00-edge in } S_0 \text{ and } S_1} - \underbrace{\frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}}_{\text{b.r. 01-edge in } S_1} + \underbrace{\frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11}}_{\text{b.r. 11-edge in } S_3}. \quad (9) \end{aligned}$$

The number of 00-edges changes due to rewiring and best response. The contribution from rewiring of edges is similar to the two-player game with an additional factor $(1 - \rho P_{\text{any}}^{01})$ for not taking a simplex action. The contribution from best response on 00- and 11-edges is again given by the direct contribution (for the 00-edge) and the corresponding excess factors, $1 + E_0^{(0,0)}(1)$ and $E_1^{(1,0)}(1)$; see Sec. III A and Appendix B 1. Similarly, best response on simplices contributes directly to n_{00} for S_0 and S_1 and indirectly from adjacent nodes outside of the considered simplex. For this, the simplex-excess degree $E_{S_x}^j(z)$ is defined as the expected number of z -type neighbors of an x -type node, which is part of a simplex of type j , i.e., which contains j defecting nodes. Adding all contributions leads to the following moment equation for n_{00} , for a detailed derivation, see Appendix A:

$$\begin{aligned} \frac{d}{dt}n_{00} = & 2 \left[\underbrace{\frac{n_{01}}{M} \left(1 - \rho P_{\text{any}}^{01}\right) \phi}_{\text{rew. 01-edges}} - \underbrace{\frac{n_{00}}{M} \left(1 - \rho P_{\text{any}}^{00}\right) (1 - \phi) \left(1 + E_1^{(0,0)}(0)\right)}_{\text{b.r. 00-edges}} + \underbrace{\frac{n_{11}}{M} \left(1 - \rho P_{\text{any}}^{11}\right) (1 - \phi) E_1^{(1,1)}(0)}_{\text{b.r. 11-edges}} - \underbrace{\frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left(2P_{S_0}^{00} + P_{S_1}^{00}\right)}_{\text{b.r. 00-edge in } S_0 \text{ and } S_1 \text{ (direct)}} \right. \\ & - \underbrace{\frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}}_{\text{b.r. 01-edge in } S_1 \text{ (direct)}} - \underbrace{\frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left[P_{S_0}^{00} E_{S_0}^0(0) + P_{S_1}^{00} E_{S_1}^0(0)\right]}_{\text{b.r. 00-edge in } S_0 \text{ and } S_1 \text{ (indirect)}} - \underbrace{\frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01} E_{S_1}^0(0)}_{\text{b.r. 01-edge in } S_1 \text{ (indirect)}} + \underbrace{\frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11} E_{S_3}^1(0)}_{\text{b.r. 11-edge in } S_3 \text{ (indirect)}} \left. \right]. \quad (10) \end{aligned}$$

Finally, the number of 11-nodes also changes according to rewiring and best response with a full derivation in Appendix A. Rewiring and best response on edges gives similar contributions as in the two-player game; see Sec. III A. Similarly to n_{00} , best response on simplices contributes directly within simplices of type S_1 and S_3 , and indirectly from adjacent nodes outside of the simplex, expressed in terms of simplex-excess degrees $E_{S_x}^j(z)$. This leads to

$$\begin{aligned} \frac{d}{dt} n_{11} = & 2 \left[\underbrace{-\frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) \phi}_{\text{rew. 11-edges}} + \underbrace{\frac{n_{00}}{M} (1 - \rho P_{\text{any}}^{00}) (1 - \phi) E_1^{(0,0)}(1)}_{\text{b.r. 00-edges}} - \underbrace{\frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) (1 - \phi) (1 + E_1^{(1,1)}(1))}_{\text{b.r. 11-edges}} \right. \\ & + \underbrace{\frac{n_{00}}{M} P_{\text{any}}^{00} \rho P_{S_1}^{00}}_{\text{b.r. 00-edge in } S_1 \text{ (direct)}} + \underbrace{\frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}}_{\text{b.r. 01-edge in } S_1 \text{ (direct)}} - \underbrace{2 \frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11}}_{\text{b.r. 11-edge in } S_3 \text{ (direct)}} \\ & \left. + \underbrace{\frac{n_{00}}{M} P_{\text{any}}^{00} \rho [P_{S_0}^{00} E_{S_0}^0(1) + P_{S_1}^{00} E_{S_0}^1(1)]}_{\text{b.r. 00-edge in } S_1 \text{ (indirect)}} + \underbrace{\frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01} E_{S_0}^1(1)}_{\text{b.r. 01-edge in } S_1 \text{ (indirect)}} - \underbrace{\frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11} E_{S_1}^3(1)}_{\text{b.r. 11-edge in } S_1 \text{ (indirect)}} \right]. \end{aligned} \tag{11}$$

C. Closed moment system

We consider pairwise approximation to close Eqs. (9)–(11). Such closure methods are well established in epidemiology and dynamics on two-player networks.^{41–44} Therefore, we approximate the excess degrees, i.e., the expected numbers of neighbors of type z for x -type nodes with $j = 1$ neighbor of type y as $E_1^{(x,y)}(z) \approx \frac{n_{xyz}}{n_{xy}}$ $\approx \frac{n_{xy} n_{yz}}{n_y n_{xy}} \approx \frac{n_{yz}}{n_y}$; see detailed derivation in Appendix B. The excess

degrees for nodes within simplices are approximated with $E_{S_0}^0(1) \approx E_1^{(0,0)}(1)$, $E_{S_0}^1(1) \approx E_1^{(0,1)}(1)$, and $E_{S_1}^3(1) \approx E_1^{(1,1)}(1)$; see Appendix B. Detailed approximations for the probabilities $P_{S_z}^{xy}$ and $P_{S_z}^{yy}$ are also given in Appendix B. Furthermore, by definition, we have $P_{S_0}^{00} + P_{S_1}^{00} = 1$ since 00-type edges, which are part of any simplex, must be either within S_0 or S_1 . Applying these approximations to the moment equations leads to the following closed system of equations for the lowest moments:

$$\frac{d}{dt} n_0 = -\frac{n_{00}}{M} \left[(1 - \rho P_{\text{any}}^{00}) (1 - \phi) + \rho P_{\text{any}}^{00} \right] + \frac{n_{11}}{M} \left[(1 - \rho P_{\text{any}}^{11}) (1 - \phi) + P_{\text{any}}^{11} \rho P_{S_3}^{11} \right] - \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}, \tag{12}$$

$$\begin{aligned} \frac{d}{dt} n_{00} = & 2 \left[\frac{n_{01}}{M} (1 - \rho P_{\text{any}}^{01}) \phi - \frac{n_{00}}{M} (1 - \rho P_{\text{any}}^{00}) (1 - \phi) \left(1 + \frac{n_{00}}{n_0} \right) + \frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) (1 - \phi) \frac{n_{01}}{n_1} \right. \\ & \left. - \frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left(1 + P_{S_0}^{00} + \frac{n_{00}}{n_0} \right) - \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01} \left(1 + \frac{n_{00}}{n_0} \right) + \frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11} \frac{n_{01}}{n_1} \right], \end{aligned} \tag{13}$$

$$\begin{aligned} \frac{d}{dt} n_{11} = & 2 \left[-\frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) + \frac{n_{00}}{M} (1 - \rho P_{\text{any}}^{00}) (1 - \phi) \frac{n_{01}}{n_0} - \frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) (1 - \phi) \frac{n_{11}}{n_1} \right. \\ & \left. + \frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left(1 - P_{S_0}^{00} + \frac{n_{01}}{n_0} \right) + \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01} \left(1 + \frac{n_{01}}{n_0} \right) - \frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11} \left(2 + \frac{n_{11}}{n_1} \right) \right]. \end{aligned} \tag{14}$$

Each term represents one of the considered actions in the adaptive simplicial Snowdrift game, as specified in Eqs. (9)–(11), with its corresponding implications on the numbers of labeled nodes and edges. However, even in their closed form, a complete analytical treatment is not likely to work. Instead, the expected time evolution of the lowest moments from Eqs. (12)–(14) is evaluated numerically. In the following sections, we compare this to full direct simulations of the adaptive Snowdrift game on simplicial complexes. Note that setting $\rho = 0$ leads to the closed moment equations of the adaptive two-player game; see Sec. III A.

D. Numerical results

The closed moment equations and the approximations of subgraphs, excess degrees, and probabilities used therein from Appendix B are compared in this section with numerical simulations

on simplicial complexes. For the simulations, we consider random Erdős–Rényi graphs^{45,46} as initial networks, on top of which we add a specific number of simplices in order to get a simplicial complex. Note that this is just one exemplary ensemble, even though the system could be applied to any simplicial complex. We find similar results for the uniform ensemble $G(N, \tilde{M})$ and the binomial ensemble $G(N, p)$.

Graphs in $G(N, \tilde{M})$ consist of N nodes and exactly \tilde{M} undirected edges (such that the total number of edges is $|\mathcal{E}_2| = M = 2\tilde{M}$). Here, we choose $\tilde{M} = \mu N$ such that the average degree of each node is $k_{\text{deg}} = 2\mu$. Graphs in $G(N, p)$ consist of N nodes, and each of the possible $N(N - 1)/2$ undirected edges is independently drawn with probability p . Here, we obtain the same total number of edges $M = k_{\text{deg}} N$ on average by choosing $p = \frac{k_{\text{deg}}}{N-1}$. In order to create a simplicial complex, we assign a fraction σ of all

existing triangles as simplices in \mathcal{E}_3 . The expected number of triangles in $G(N, p)$ is given by $\mathbb{E}[|T|] = \binom{N}{3} p^3$. This asymptotically also holds for $G(N, \tilde{M})$ with $N \rightarrow \infty$ and $\frac{\tilde{M}}{N(N-1)} = p$ such that we expect $\mathbb{E}[|T|] = \frac{N(N-1)(N-2)}{6} \frac{k_{\text{deg}}^3}{(N-1)^3} = \frac{1}{6} k_{\text{deg}}^3 \frac{N^2 - 2N}{N^2 - 2N + 1} \approx \frac{1}{6} k_{\text{deg}}^3$.

In the following, numerical results are presented for the uniform ensemble only, and equivalent results are obtained for the binomial ensemble. We choose $N = 1000$ nodes, $M = 38\,000$ edges (such that the average degree is $k_{\text{deg}} = 38$), and $\sigma = 0.5$. For the chosen parameters, we get approximately 9000 triangles and 4500 simplices (each counted six times). The probability that a random edge is part of at least one simplex is approximately $P \approx 1 - (1 - \frac{6}{M})^{\sigma \mathbb{E}[|T|]}$, which gives $P \approx 0.5$ for the chosen parameters. Furthermore, we consider the probabilities for action on a simplex and for rewiring of edges as $\rho = 0.5$ and $\phi = 0.5$, respectively.

The results for the lowest moments n_x and n_{xy} are illustrated in Fig. 2 for five different realizations of the initial network (blue lines). The simulation is compared to the evolution with the closed moment equations (black lines). We observe that within $t = 10\,000$ time steps, the closed moment equations converge to a steady state. The simulation shows oscillations that are in the range of the steady state of the closed moment equations. However, there are larger deviations for the number n_{00} and n_{11} of 00-edges and 11-edges.

Nevertheless, the closed moment equations are a good approximation to the time evolution of the number of nodes and edges within the adaptive simplicial Snowdrift model. Both simulation and closed system show, in general, the same trends, and the oscillations of the number of nodes and edges in the simulation are close to the non-trivial steady state of the closed moment equations. We account the observed differences to the strong simplifications made in the closure approximations; see Appendix B.

Note that the number of simplices remains constant within each of the realizations (not shown) because due to $\sigma = 0.5$, there are initially enough triangles left to compensate a potential destruction of simplices by rewiring. For different realizations, however, this number can be different due to the probabilistic number of initial triangles and the proportional dependence of the corresponding simplices. For the corresponding time evolution of the different number of simplex types, see Fig. 9 in Appendix B 2 b.

Moreover, similar good agreement between numerical simulation and moment equations has been checked with additional simulations for different numbers of nodes, edges, and simplices, probabilities for actions on nodes or simplices, and initial binomial random graph $G(N, p)$.

In Fig. 3, we illustrate how the adaptive simplicial Snowdrift model depends on the probability ρ to choose an action on simplices. For this, the time dependence of the lowest moments n_0 , n_{00} , and n_{11} is illustrated for different values of ρ as specified in the figure (blue colored lines). This is compared to the corresponding solution of the closed moment equations (gray colored lines). We observe that for all ρ , the lowest moments quickly approach an equilibrium, around which there are fluctuations in the simulations. The observed steady states for the moment equations suggest monotone dependence on the parameter ρ . For example, the fraction of cooperators in the equilibrium $f_0 = n_0/N$ decreases from $f_0^* = 1/2$ (with zero multiplayer interactions) to

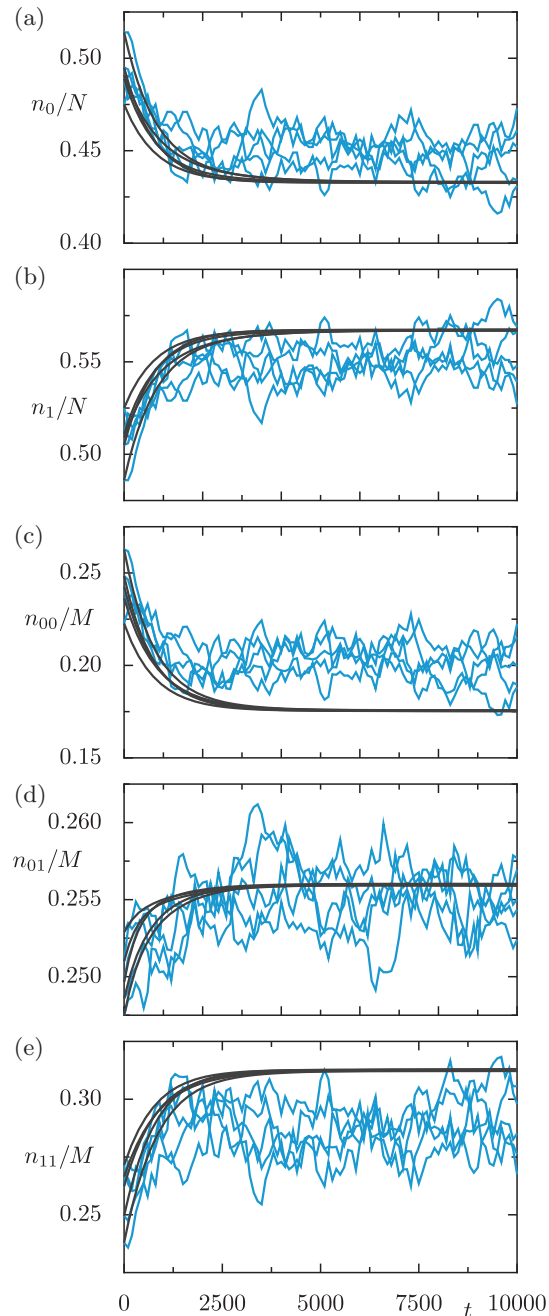


FIG. 2. Numerical simulations of the adaptive simplicial Snowdrift model compared to closed moment equations [Eqs. (12)–(14)] on random uniform networks with $N = 1000$ and $\mu = 19$. A fraction $\sigma = 0.5$ of triangles are initially considered simplices. Illustrated is the time evolution of the fractions of (a) cooperators n_0/N , (b) defectors n_1/N , (c) 00-edges n_{00}/M , (d) 01-edges n_{01}/M , and (e) 11-edges n_{11}/M for the simulation (blue lines). Different realizations imply a different number of initial simplices, leading to different corresponding solutions of the closed moment equations (black lines). Other parameters are $\phi = 0.5$ and $\rho = 0.5$.

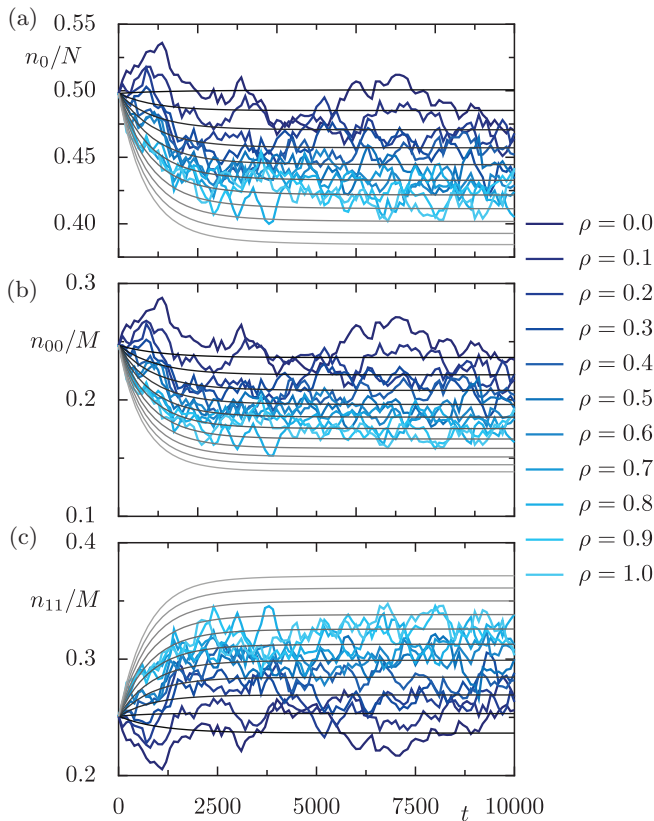


FIG. 3. Dependence of an adaptive simplicial Snowdrift model on the probability of choosing an action on the simplex ρ . Shown are numerical simulations (colored lines) compared to closed moment equations [Eqs. (12)–(14)] (gray lines) with same parameters as in Fig. 2 and with ρ as specified. Illustrated is the time evolution of the fractions of (a) cooperators n_0/N , (b) 00-edges n_{00}/M , and (c) 11-edges n_{11}/M .

approximately $f_0^* = 1/3$ (with many multiplayer interactions). This means that increasing the influence of these multiplayer interactions by increasing ρ does not lead to a change in stability or a bifurcation. In contrast, the observed non-trivial steady states seem to be stable for the simplicial Snowdrift game, even when certain irrational perturbations are considered, as discussed in Sec. IV.

IV. STABILITY ANALYSIS

The closed moment equations that describe the evolution of the adaptive simplicial Snowdrift game cannot be examined completely analytically. However, it is possible to analyze steady states and their local stability for the different operations, rewiring, best response on edge, and best response in simplices, separately. For this purpose, we consider the fractions of labeled nodes and edges $f_0 = \frac{n_0}{N}$, $f_1 = \frac{n_1}{N} = 1 - f_0$, $f_{00} = \frac{n_{00}}{M}$, $f_{11} = \frac{n_{11}}{M}$, and $f_{01} = \frac{n_{01}}{M} = \frac{1}{2}(1 - f_{00} - f_{11})$ and rewrite the closed moment equations accordingly.

A. Rewiring

The dynamics of rewiring can be extracted from the closed moment equations (12)–(14) by choosing $\phi = 1, \rho = 0$ or $\phi = 1, S = 0$. Therefore, the corresponding dynamics of the adaptive network in relative variables is given by

$$\begin{aligned} \frac{d}{dt}f_0 &= 0, \\ \frac{d}{dt}f_{00} &= \frac{1 - f_{00} - f_{11}}{M}, \\ \frac{d}{dt}f_{11} &= -2\frac{f_{11}}{M}. \end{aligned} \tag{15}$$

This system has a line of steady states $\{(f_0^*, f_{00}^*, f_{11}^*) = (f_0^*, 1, 0) \mid f_0^* \in (0, 1]\}$. Here, $f_0^* = 0$ is excluded because without 0-nodes, there are no 00-edges, which contradicts $f_{00}^* = 1$ in the steady state. Note that in the derivation of the moment equations, Appendix A, it is assumed that there always are enough 0-nodes, to which the selected edge can be rewired. In networks with a small fraction of 0-nodes but with high node degrees μ , this assumption will be violated and Eq. (15) cannot be applied.

The second and third equations in Eq. (15) are independent of f_0 , which allows one to reduce the system to the f_{00} and f_{11} variables. The Jacobian of the right-hand side of the reduced system is given by

$$J(f_{00}, f_{11}) = \frac{1}{M} \begin{pmatrix} -1 & -1 \\ 0 & -2 \end{pmatrix}, \tag{16}$$

which has two negative real eigenvalues $\lambda_1 = -\frac{1}{M}$ and $\lambda_2 = -\frac{2}{M}$. Therefore, the steady state is locally asymptotically stable in this reduced system. Since the fraction of cooperators f_0 is always constant in the system, the line of steady states defined above is stable in the system with only rewiring.

B. Best response on edges

For the parameter choices $\phi = 0, \rho = 0$ or $\phi = 0, S = 0$ in the closed moment equations (12)–(14), only best response on edges occurs in the adaptive network. In terms of $f_0, f_{00}, f_{11} \in [0, 1]$, this is given by

$$\begin{aligned} \frac{d}{dt}f_0 &= -\frac{f_{00}}{N} + \frac{f_{11}}{N}, \\ \frac{d}{dt}f_{00} &= -2\frac{f_{00}}{M} - 2\frac{f_{00}^2}{Nf_0} + \frac{f_{11}(1 - f_{00} - f_{11})}{N(1 - f_0)}, \\ \frac{d}{dt}f_{11} &= -2\frac{f_{11}}{M} + \frac{f_{00}(1 - f_{00} - f_{11})}{Nf_0} - \frac{2f_{11}^2}{N(1 - f_0)}. \end{aligned} \tag{17}$$

For $N, M \neq 0$, this system has a line of trivial steady states $\{(f_0^*, f_{00}^*, f_{11}^*) = (f_0^*, 0, 0) \mid f_0^* \in (0, 1)\}$. The cases $f_0^* = 0$ and $f_0^* = 1$ are excluded since this contradicts $f_{00} = 0$ and $f_{11} = 0$. Note that depending on f_0^* , there might not be enough possible 01-edges in simulations on some finite network. The Jacobian in one of the

steady state points $(f_0^*, f_{00}^*, f_{11}^*) = (f_0^*, 0, 0)$ is given by

$$J(f_0^*) = \begin{pmatrix} 0 & -\frac{1}{N} & \frac{1}{N} \\ 0 & -\frac{2}{M} & \frac{1}{N(1-f_0^*)} \\ 0 & \frac{1}{Nf_0^*} & -\frac{2}{M} \end{pmatrix}, \tag{18}$$

with eigenvalues $\lambda_0 = 0$ and $\lambda_{2,3} = -\frac{2}{M} \pm \frac{1}{N\sqrt{f_0^*(1-f_0^*)}}$. The second eigenvalue becomes positive for all f_0^* if $\frac{M}{N} > \sqrt{2}$, i.e., if the average degree $\mu > 1/\sqrt{2}$. This is satisfied for realistic networks. The Hartman–Grobman theorem implies that under this condition, each of the trivial steady states is unstable.

Furthermore, the system Eq. (17) has a non-trivial steady state $(f_0^*, f_{00}^*, f_{11}^*) = (\frac{1}{2}, \frac{1}{4}[1 - \frac{N}{M}], \frac{1}{4}[1 + \frac{N}{M}])$. Therefore, if the number of edges M is sufficiently high relative to the number of nodes N , this steady state corresponds to the state in which all node labels and edges are uniformly distributed, i.e., $(f_0^*, f_{00}^*, f_{11}^*) \rightarrow (\frac{1}{2}, \frac{1}{4}, \frac{1}{4})$

for $N/M \rightarrow 0$. Calculating the eigenvalues and applying the Hartman–Grobman theorem reveals that this steady state is asymptotically stable for $\mu = \frac{M}{N} > 1$, which is fulfilled for all reasonable networks. For large M , this can be seen as the realization of the only mixed Nash equilibrium of the two-player Snowdrift game globally in the network, where both players select either of the two actions C and D with the same probability $1/2$.

C. Best response in simplices

Choosing $\rho = 1$ and a large number S of simplices implies that every edge is part of at least one simplex in Eqs. (12)–(14). Thus, only best response in simplices is executed in the adaptive network. In particular, here, we assume that $P_{any}^{xy} = 1$ for all xy -edges and apply pair approximation for the probabilities of xy -edges to be part of specific simplex types $P_{S_i}^{xy}$; see Appendix B 2. This leads to the following system in terms of $f_0, f_{00}, f_{11} \in [0, 1]$:

$$\begin{aligned} \frac{d}{dt}f_0 &= -\frac{f_{00}}{N} - \frac{f_{01}}{N} \frac{f_{00}f_1}{f_{00}f_1 + f_{11}f_0} + \frac{f_{11}}{N} \frac{f_{11}^2f_0}{3f_{01}^2f_1 + f_{11}^2f_0}, \\ \frac{d}{dt}f_{00} &= -2\frac{f_{00}}{M} \left(\frac{f_{00}^2f_1}{f_{00}^2f_1 + 3f_{01}^2f_0} + 1 \right) - \frac{2f_{00}^2}{Nf_0} - 2 \left(\frac{f_{01}}{M} + \frac{f_{00}f_{01}}{Nf_0} \right) \frac{f_{00}f_1}{f_{00}f_1 + f_{11}f_0} + 2\frac{f_{01}f_{11}}{Nf_1} \frac{f_{11}^2f_0}{3f_{01}^2f_1 + f_{11}^2f_0}, \\ \frac{d}{dt}f_{11} &= +2\frac{f_{00}}{M} \frac{3f_{01}^2f_0}{f_{00}^2f_1 + 3f_{01}^2f_0} + 2\frac{f_{00}f_{01}}{Nf_0} + 2 \left(\frac{f_{01}}{M} + \frac{f_{01}^2}{Nf_0} \right) \frac{f_{00}f_1}{f_{00}f_1 + f_{11}f_0} - 2 \left(\frac{2f_{11}}{M} + \frac{f_{11}^2}{Nf_1} \right) \frac{f_{11}^2f_0}{3f_{01}^2f_1 + f_{11}^2f_0}. \end{aligned} \tag{19}$$

Again, we find a line of trivial steady states for best response in simplices, $\{(f_0^*, f_{00}^*, f_{11}^*) = (f_0^*, 0, 0) | f_0^* \in (0, 1)\}$, which are the same as for best response on edges. Further steady states can only be computed implicitly by the usage of mathematical software. In order to analyze the dynamics of best response in simplices besides the trivial steady states, we assume in the following that $P_{S_i}^{xy} = \frac{1}{2}$ for all possible edge and simplex types. This is equivalent to the assumption that the simplex types are uniformly distributed within all simplices; i.e., their total amount is the same $|\mathcal{S}_0| = |\mathcal{S}_1| = |\mathcal{S}_2| = |\mathcal{S}_3|$. The resulting system in terms of $f_0, f_{00}, f_{11} \in (0, 1)$ is given by

$$\begin{aligned} \frac{d}{dt}f_0 &= -\frac{f_{00}}{N} - \frac{1}{2} \frac{f_{01}}{N} + \frac{1}{2} \frac{f_{11}}{N}, \\ \frac{d}{dt}f_{00} &= -3\frac{f_{00}}{M} - \frac{2f_{00}^2}{Nf_0} - \frac{f_{01}}{M} - \frac{f_{00}f_{01}}{Nf_0} + \frac{f_{01}f_{11}}{Nf_1}, \\ \frac{d}{dt}f_{11} &= \frac{f_{00}}{M} + 2\frac{f_{00}f_{01}}{Nf_0} + \frac{f_{01}}{M} + \frac{f_{01}^2}{Nf_0} - 2\frac{f_{11}}{M} - \frac{f_{11}^2}{Nf_1}. \end{aligned} \tag{20}$$

For this simplified system, we find the nontrivial steady state $(f_0^*, f_{00}^*, f_{11}^*) = \left(\frac{1}{3}, \frac{2\mu - 2 + \sqrt{4\mu^2 + 2\mu + 4}}{9\mu} - \frac{1}{3}, \frac{2\mu - 2 + \sqrt{4\mu^2 + 2\mu + 4}}{9\mu} \right)$ with $\mu = \frac{M}{N}$. Again, $f_0^* = \frac{1}{3}$ corresponds to the fraction of cooperators C in the Nash equilibrium of a three-player Snowdrift game, which is

consistent with the assumption of an initial uniform distribution of labels, edges, and simplices.

D. Numerical analysis with MatCont

We perform a numerical stability and bifurcation analysis of the adaptive simplicial Snowdrift model with the help of the software tool MatCont⁴⁷ as described in Appendix C. The numerical analysis confirms the analytical results of the equations for rewiring, best response on edges, and best response in simplices separately. Furthermore, the analysis in MatCont indicates that the closed moment equations, Eqs. (12)–(14), of the full adaptive simplex Snowdrift model have a non-trivial steady state, which is asymptotically stable for all reasonable parameter choices $\phi \in [0, 1], \rho \in [0, 1], N > 3, \mu > 1$, and $\sigma \in [0, 1]$.

An exemplary parameter diagram is shown in Fig. 4 as a function of ρ and for different parameters ϕ . The fractions of cooperators and 00-edges in the equilibrium are shown in panels (a) and (b), while the real parts of the corresponding eigenvalues are shown in (c). Apparently, all eigenvalues are below zero such that the steady states are stable for all parameters. Yet, we observe that the eigenvalues are relatively close to zero so that the stability is relatively weak, which does allow after a perturbation for an extended duration of phases away from the steady state.

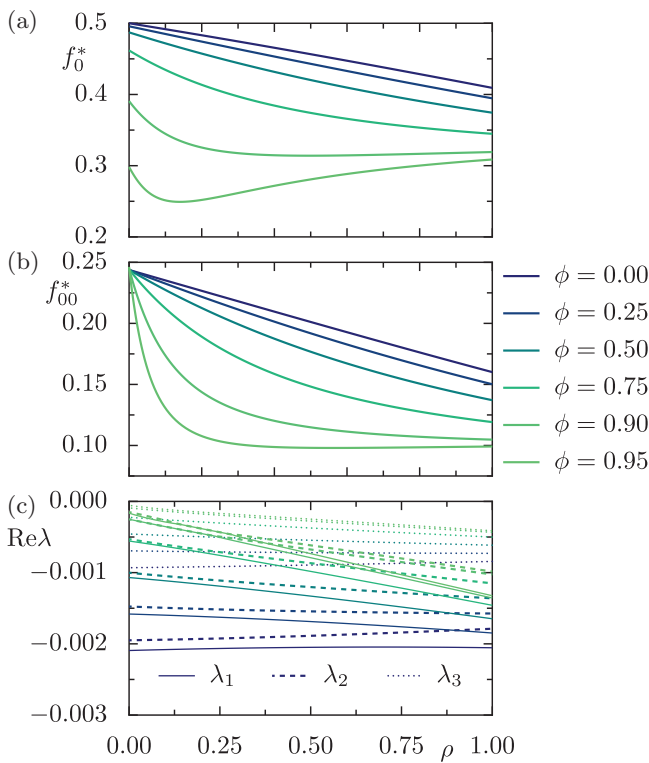


FIG. 4. Stability analysis with MatCont. Shown are (a) the fraction of cooperators f_0^* and (b) the fraction of 00-edges f_{00}^* as a function of ρ in the equilibrium of the moment system with $N = 1000$, $\mu = 20$, and $\sigma = 0.5$ for different values of ϕ . In (c), the corresponding, numerically obtained real parts of the eigenvalues λ_i are illustrated, which are smaller than zero for all ρ .

E. Robustness

Furthermore, we are interested how robust the presented adaptive simplicial Snowdrift model is against the introduction of irrational actions of players. An action is called irrational if the payoff of the selected player is decreasing (or at least not increasing). For example, rewiring from cooperating C to defecting D-nodes leads to smaller payoff. Here, we consider irrational replacements for the *best-response simplex* action in order to understand the effect of multiplayer interactions on the stability of the system. For this, we define the probability $\alpha \in [0, 1]$ to replace the simplex action that is replaced with one of the following irrational strategies. In particular, for $\alpha = 0$, the original adaptive simplicial Snowdrift model is obtained, and for $\alpha = 1$, all best-response actions on simplices are replaced.

For this, we consider (i) the all-defecting strategy, (ii) the worst response strategy, where the opposite of best response in simplices is considered, (iii) the all-cooperating strategy, (iv) the adaption to the majority action within the simplex, (v) copying the neighbors’ action, (vi) always changing the action, (vii)–(ix) increasing the number of 00-, 01-, or 11-edges, and (x)–(xii) decreasing the number of 00-, 01-, or 11-edges, respectively. Each of these extensions

leads to additional terms in the moment equations, which are proportional to α and depend on the considered action (i)–(xii). We consider the resulting closed moment equations using the approximations of Appendix B. The numerical bifurcation analysis of these extended models is performed in MatCont, see Appendix C, for parameters $N \in [3, 10^6]$, $M = \mu N$ with $\mu \in (1, N(N - 1))$, $\phi \in (0, 1)$, $\rho \in (0, 1)$, $\sigma \in (0, 1)$, and $\alpha \in (0, 1)$. The results (not shown) suggest that the non-trivial steady state remains stable for all reasonable parameter choices and is only shifted under variation of α . Yet, the same transient excursions due to weakly stable eigenvalues can be possible as discussed above. Hence, we conclude that the dynamics of the adaptive simplicial Snowdrift model is stable, even if the multiplayer interactions are perturbed.

V. SUMMARY AND OUTLOOK

We propose a multiplayer Snowdrift model on adaptive simplicial complexes. This model contains rewiring rules, changing the topology, and best-response rules, changing the state of the system. For the lowest moments, the number of cooperators n_0 and the number of edges between cooperators n_{00} and between defectors n_{11} , we derive moment equations based on the dynamical rules of adaption. Based on the structure of simplicial complexes, we derive closure relations with pairwise approximation, relying on the numbers of labeled nodes and edges only. We confirm by numerical comparison that the corresponding closed system leads to a good approximation of the system, even though only the lowest moments are considered. Since the closed system is analytically not fully solvable, we show for specific limiting subsystems, where only one of the considered actions takes place, that the non-trivial steady state is stable for reasonable hypergraphs as long as (sufficient) rationality governs the dynamics. These results are further supported with numerical stability analysis of the full system, including direct numerical simulations of the full hypergraph dynamics as well as numerical continuation calculations for the reduced moment equations.

In total, we observe quite surprising robustness of non-trivial steady states of adaptive multiplayer games, which we attribute to sufficient rational behavior for each player. Furthermore, even considering some irrational, multiplayer update rules does not destabilize the non-trivial steady state. We expect this stability to prevail if rational higher-order interactions of more than three players are considered and also if multiple agents are updated with a best response in each step.

This suggests two interesting conclusions: (a) Complex socio-economic systems with many agents and higher-order multiplayer interactions might be quite robust, at least if each player/agent has access to enough information to make sufficiently rational decisions and perturbations from the steady states are sufficiently small. (b) Although several examples of de-stabilization of network dynamical systems by such higher-order interactions have been found,^{26,29} one should not overlook the possibility that the stability of systems with multiplayer interactions could often reduce to more classical systems with pairwise interactions. Both conclusions are somewhat re-assuring as there are (i) indeed long periods of socio-economic stability in data and (ii) we can still hope to rely on results for more classical pairwise models for certain modeling situations.

Of course, we emphasize that our study is only a step toward a deeper understanding of coevolutionary adaptivity in models with higher-order, multiplayer interactions. Many open questions remain, and we mention a few natural ones. From a structural perspective, it would be important to generalize our results from simplicial complexes to more general hypergraphs. A similar remark applies to the derivation of more detailed sequences of moment equations going beyond second-order moments. Unfortunately, both generalizations become quite involved as the combinatorial complexity increases quickly when higher-order moments or non-homogeneous degree distributions are considered.^{41–44} Thus, developing an efficient mathematical scheme to algorithmically generate moment equations for hypergraphs would be very desirable.

From the viewpoint of applications, one could investigate other multiplayer games. In fact, we have also cross-validated our results in variants of the prisoner's dilemma⁴⁸ with similar results and conclusions. Yet, there could be many other multiplayer games arising in various fields, where adaptivity might have a different impact. In this regard, it would also be interesting to develop a more precise quantifier as to how adaptivity and the amount of rational agent behavior influence stability or can induce bifurcations in behavioral socio-economic systems. One possible direction is the generalization to Fermi-update rules considering individual payoffs instead of rational best response.^{15,18,39} In order to understand the limitations of the considered closures, one could generalize evolutionary multiplayer games with multiple strategies,²⁴ where pairwise interaction fails to describe the dynamics, to adaptive simplicial complexes or hypergraphs.

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AUTHOR DECLARATIONS

Conflict of Interest

The authors have no conflicts to disclose.

DATA AVAILABILITY

The data that support the findings of this study are available from the corresponding author upon reasonable request.

APPENDIX A: DERIVATION OF MOMENT EQUATIONS

In this section, we derive moment equations for the adaptive simplicial Snowdrift model. We are particularly interested in the lowest moments, i.e., the expected numbers n_0 and n_1 of C and D-nodes, as well as the expected numbers of edges n_{00} , n_{01} , and n_{11} . Recall that we consider a fixed total number of nodes $N = n_0 + n_1$ and a fixed number of edges $M = n_{00} + 2n_{01} + n_{11}$. Therefore, it is sufficient to derive moment equations for n_0 , n_{00} , and n_{11} .

1. Evolution of n_0

The number of cooperating nodes n_0 changes due to best response on edges and best response on simplices. Let P_{any}^{xy} be the probability that a randomly chosen edge $(i, j) \in \mathcal{E}$ of type xy , i.e., $X(i) = x, X(j) = y$, is part of at least one simplex and P_{no}^{xy} be the probability that it is not part of any simplex.

a. Best response on edges. The number of C nodes increases by one for best response on 11-edges and decreases by one for best response on 00-edges. Based on the proposed dynamics, the probability to take an action on edges is given by the probability to be in any simplex P_{any}^{xy} plus the probability to be in any simplex P_{no}^{xy} times the probability of not taking a simplex action $(1 - \rho)$, which gives $P_{\text{no}}^{xy} + (1 - \rho)P_{\text{any}}^{xy} = 1 - \rho P_{\text{any}}^{xy}$, where $P_{\text{no}}^{xy} + P_{\text{any}}^{xy} = 1$ has been used. This leads to the following contributions to $\frac{d}{dt}n_0$:

$$\begin{aligned} a_0^- &= \frac{n_{00}}{M} (1 - \rho P_{\text{any}}^{00}) (1 - \phi), \\ a_0^+ &= \frac{n_{11}}{M} (1 - \rho P_{\text{any}}^{11}) (1 - \phi), \end{aligned} \quad (\text{A1})$$

where n_{00}/M and n_{11}/M are the probabilities that the chosen edge is of type 00 and 11, respectively. The other factors account for the probability to perform best response on edges.

b. Best response on simplices. For best response in simplices, we define $P_{S_d}^{xy}$ the probability that an edge of type xy is part of a simplex with d defecting nodes, conditioned that it is part of at least one simplex. Best response on simplices decreases the number of cooperating nodes if the selected node is in state C and at least one other node is cooperating. Similarly, it increases the number of C nodes if all nodes are defecting. This gives the following contributions to $\frac{d}{dt}n_0$:

$$\begin{aligned} b_0^- &= \frac{n_{00}}{M} P_{\text{any}}^{00} \rho (P_{S_0}^{00} + P_{S_1}^{00}) + \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}, \\ b_0^+ &= \frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11}. \end{aligned} \quad (\text{A2})$$

Note that n_{01}/M is the probability to select an edge, which goes from a 0-type edge to a 1-type edge. Since we consider undirected graphs, both types of edges, 01 and 10, exist in \mathcal{E}_2 for each such pair of connected nodes. Since we perform the best response on the first node of the selected edge, it is necessary to distinguish 01 from 10 edges. However, since $n_{01} = n_{10}$, the probabilities to select either are the same.

c. Resulting equation for n_0 . Altogether, the evolution of the number of cooperating 0-nodes in the adaptive simplex Snowdrift model is governed by the differential equation $\frac{d}{dt}n_0 = -a_0^- + a_0^+ - b_0^- + b_0^+$, which is equivalent to Eq. (9).

2. Evolution of n_{00}

The number of 00-edges n_{00} changes directly due to rewiring, as well as directly and indirectly due to best response on edges and simplices.

a. Rewiring. The number of 00-edges increases if a 01-edge is rewired to a 00-edge. We do not consider the opposite direction since this would decrease the payoff. Respecting the probability to choose rewiring of edges (either in or outside of any simplex), we obtain the following contribution to $\frac{d}{dt}n_{00}$:

$$a_{00}^+ = \frac{n_{01}}{M} \left(1 - \rho P_{\text{any}}^{01}\right) \phi, \tag{A3}$$

where again, it is respected that only 01-edges rewire to 00 edges, but not 10-edges, since in the latter case, the first node that performs the action is already connected to a 0-type node. Note for each rewired edge, the number of both 01 and 10-edges decreases by one and the number of 00-edges increases by 2 such that the total contribution is $2a_{00}^+$.

b. Best response on edges. The change in the number of cooperating nodes n_0 also influences the number of 00-edges. In particular, if a cooperating node changes from C to D, all its adjacent 00-edges to other C-nodes become 01-edges. The contribution to n_{00} from the switching node comes from its other neighbors of type 0. For this, let $E_n^{(x,y)}(z)$ be the expected number of z -type neighbors for a node i with type $X(i) = x$, which is already connected to n nodes j with type $X(j) = y$. Together with the probability of best response on edges, this gives the total number of added or removed n_{00} connections. Altogether, this leads to

$$b_{00}^- = -a_0^- \left(1 + E_1^{(0,0)}(0)\right), \quad b_{00}^+ = a_0^+ E_1^{(1,1)}(0), \tag{A4}$$

accounting for both direct and indirect contributions.

c. Best response in simplices. Best response in simplices also changes the number of 00-edges directly and indirectly. If the simplex is S_0 , two 00-edges are destroyed due to best response. For S_1 simplices, changing the node state from 0 to 1 destroys one 00-edge in each case. Note that within the simplex, best response does not create new 00-edges. This implies the contribution

$$c_{00}^- = \frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left(2P_{S_0}^{00} + P_{S_1}^{00}\right) + \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01}. \tag{A5}$$

The indirect influence comes from the changes of edge types outside of the simplex due to the change of the node state within the simplex. For this, the simplex-excess degree $E_{S_n}^x(y)$ is defined²³ as the expected number of additional neighbors j with type $X(j) = y$ of a node with type $X(i) = x$ given that it is part of a simplex of type n . In particular, the x -node is within the simplex already connected to $n - \delta_{x1}$ neighbors of type 1, where δ_{ij} is the Kronecker delta and $3 - n + \delta_{x1}$ neighbors of type 0. With this, the indirect contributions to the change of 00-type edges can easily be written in terms of the probabilities for best response in the specific simplices multiplied with the expected number of additional 0-type neighbors. We end with

$$d_{00}^- = \frac{n_{00}}{M} P_{\text{any}}^{00} \rho \left[P_{S_0}^{00} E_{S_0}^0(0) + P_{S_1}^{00} E_{S_1}^0(0) \right] \tag{A6}$$

$$+ \frac{n_{01}}{M} P_{\text{any}}^{01} \rho P_{S_1}^{01} E_{S_1}^0(0), \tag{A7}$$

$$d_{00}^+ = \frac{n_{11}}{M} P_{\text{any}}^{11} \rho P_{S_3}^{11} E_{S_3}^1(0). \tag{A8}$$

d. Resulting equation for n_{00} . Altogether, the evolution of the number of 00-nodes in the adaptive simplex Snow-drift model is governed by the differential equation $\frac{d}{dt}n_{00} = 2(a_{00}^+ - b_{00}^- + b_{00}^+ - c_{00}^- - d_{00}^- + d_{00}^+)$, where the factor of 2 reflects the double counting of undirected edges. Altogether, the change in the number of 00-edges in the adaptive simplicial Snow-drift model is given by Eq. (10).

3. Evolution of n_{11}

Similar as for n_{00} , the number of 11-edges changes due to rewiring and due to best response on edges and simplices, directly and indirectly. Therefore, the derivation follows analogously. Summing up direct and indirect contributions leads to the differential equation in Eq. (11).

We emphasize that these three equations for the lowest moments depend on higher-order moments, such as the distribution of triples through the excess degrees and the distribution of simplices of different types.

APPENDIX B: MOMENT-CLOSURE APPROXIMATIONS

In the following, the moment equations are closed at the level of triples such that the right-hand side of the ODEs depends on the lowest moments $n_0, n_1, n_{00}, n_{01}, n_{11}$, only. This means that the full information on the network structure and the simplex distribution is reduced to information on the pairs.^{11,41–44} Note that we focus on structural configurations of pairs, which is different from approaches where three-player interactions are modeled by effective two-player interactions. We consider a system with a fixed number of nodes $N = n_0 + n_1$, a fixed number of edges $M = n_{00} + 2n_{01} + n_{11}$, and also a fixed number of simplices S .

1. Approximation of excess degrees

a. Approximation of $E_1^{(a,b)}(c)$. By definition, the excess degree $E_1^{(a,b)}(c)$ is the expected number of c -type neighbors of an a -type node, which already has one b -type neighbor; see Fig. 5(a). Thus, it is equivalent to the expected number of cab -triplets containing the ab -edge. Let n_{abc} be the number of triplets with states $X(i) = a, X(j) = b$, and $X(k) = c$; i.e., $n_{abc} = \sum_{i,j,k=1}^N A_{kj} A_{ji} \delta_{X(i),a} \delta_{X(j),b} \delta_{X(k),c}$. Note that symmetry implies $n_{001} = n_{100}$ and $n_{011} = n_{110}$. The expected number of cab -triplets n_{cab} is approximately given by the number of ab -edges multiplied with the expected number of c -neighbors of a . Therefore, we have

$$E_a^{(b,c)}(1) \approx \frac{n_{cab}}{n_{ab}}. \tag{B1}$$

Furthermore, the numbers of triplets n_{cab} are approximated in terms of pairs and nodes as follows. The proportion of edges from a -type to b -type nodes is given by the number of ab -edges divided by the total number of edges starting at a , which is approximately

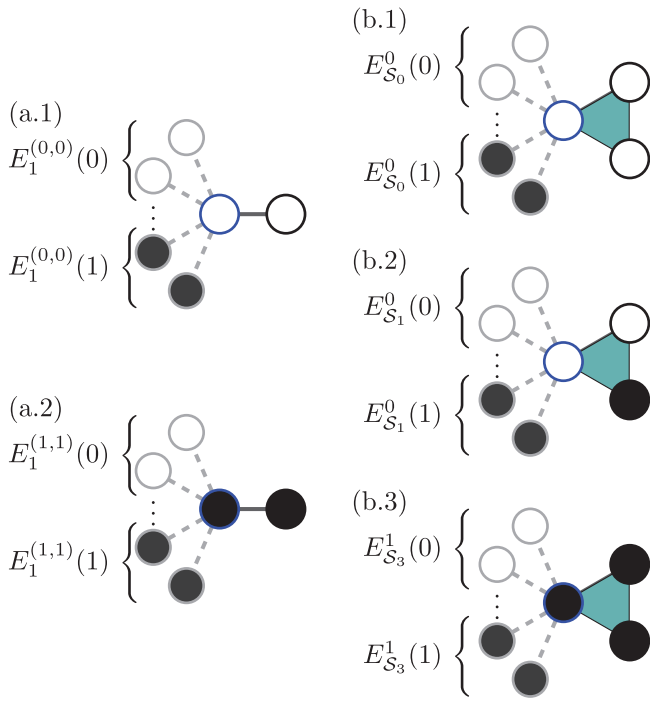


FIG. 5. Illustration of excess degrees (a) $E_x^{(y,z)}(j)$ and (b) $E_{S_x}^j(z)$.

$k_{\text{deg}} n_a$. Thus, we have $\frac{n_{ab}}{k_{\text{deg}} n_a}$. The probability for two neighbors of some a type node to be of type c and b , respectively, is thus approximately $\frac{n_{ca} n_{ab}}{k_{\text{deg}} (k_{\text{deg}} - 1) n_a^2}$. For the total number of cab -triplets, we have to multiply this with the possible number of ways to choose the neighbors, which is $k_{\text{deg}} (k_{\text{deg}} - 1)$, and with the total number of a -nodes, and we get

$$n_{cab} \approx \frac{n_{ca} n_{ab}}{n_a} \quad (\text{B2})$$

such that

$$\begin{aligned} E_1^{(0,0)}(0) &\approx \frac{n_{00}}{n_0}, E_1^{(1,1)}(0) \approx \frac{n_{01}}{n_1}, \\ E_1^{(0,0)}(1) &\approx \frac{n_{01}}{n_0}, E_1^{(1,1)}(1) \approx \frac{n_{11}}{n_1}. \end{aligned} \quad (\text{B3})$$

We emphasize that for this pair approximation of the triplets, the approximative excess degrees of a -type nodes are independent on the number of already connected b -nodes and, therefore, are simply given by the expected number of c -node neighbors.

Note that $n_0 = 0$ also implies $n_{00} = 0$ and $n_{01} = 0$, and the terms containing the excess degrees do not contribute. For illustration of the agreement of the closure with simulation, see Fig. 6.

b. Approximation of $E_{S_i}^a(c)$. The simplex-excess degree $E_{S_i}^a(c)$ is the expected number of c -type-neighbors of an a -node within a simplex of type i ; see Fig. 5(b). In order to apply pair approximation we

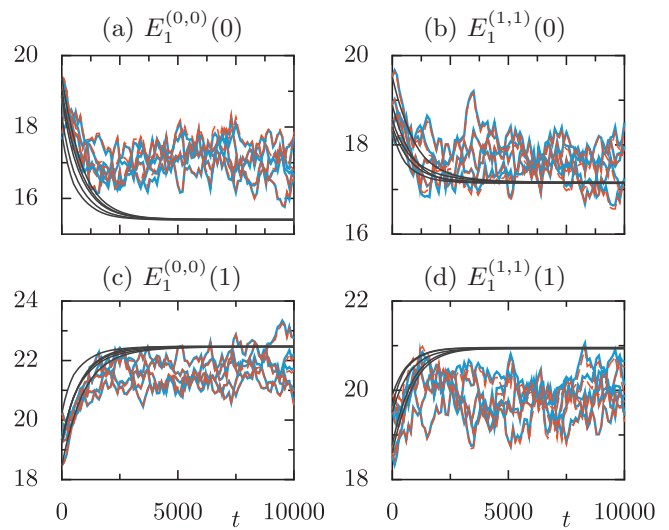


FIG. 6. Approximation of excess degrees $E_a^{(b,c)}(j)$ for the same simulation as in Fig. 2. Shown are values directly obtained from simulation (blue lines), compared to the approximations in Eq. (B3) using n_x and n_y from simulation (orange dashed lines) and from the closed moment equations (black lines). The agreement of the approximation in Eq. (B3) with the simulation is very good, in particular, if the current values of the lowest moments are used. Differences to the closed moment result are of similar kind as in Fig. 2. Interestingly, we find that $E_0^{(0,1)}(0) \approx E_1^{(1,1)}(0) - 1$ and $E_0^{(0,1)}(1) \approx E_1^{(1,1)}(1) + 1$. This means that the expected number of 0-type neighbors is similar for 0-type and 1-type nodes and similar for the number of 1-type neighbors. The proposed closure is, therefore, only justified for large average degrees k_{deg} when this difference of ± 1 becomes negligible.

notice that an a -type node within a simplex of type i is already connected to $i - \delta_{a,1}$ neighbors of type 1. For example, if $a = 0, i = 0$, we have $E_{S_0}^0(0) \approx E_2^{(0,0)}(0)$ and $E_{S_0}^0(1) \approx E_2^{(0,0)}(1)$. Similarly, $E_{S_3}^1(0) \approx E_2^{(1,1)}(0)$ and $E_{S_3}^1(1) \approx E_2^{(1,1)}(1)$. If the average degree k_{deg} in the network is large, these excess degrees are approximately $E_2^{(a,b)}(c) \approx E_1^{(a,b)}(c)$. If $a = 0$ and $i = 1$, the a -node is already connected to one neighbor of type 0 and one neighbor of type 1. Therefore, we can approximate $E_{S_1}^0(0) \approx E_1^{(0,0)}(0)$ but also $E_{S_1}^0(0) \approx E_1^{(0,1)}(0)$. In pair approximation, both quantities are the same. Similarly, we obtain $E_{S_1}^1(1) \approx E_1^{(0,0)}(1) \approx E_1^{(0,1)}(1)$.

Thus, for large node degrees k_{deg} , the approximations in Eq. (B3) from Subsection 1 a of Appendix B are applied. We illustrate the time dependence of the simplex-excess degrees in Fig. 7, comparing values from the simulation (colored lines) to the corresponding results from the moment-closure system (black lines).

2. Approximation of probabilities

a. Approximation of P_{any}^{xy} . First, we consider the probability that a 01-edge is not part of any simplex. Intuitively, only simplices in S_1 and S_2 contain 01-edges, and there are for each type exactly two such edges. Thus, a randomly drawn 01-edge $e_{01} \in \mathcal{E}$ is part of a specific

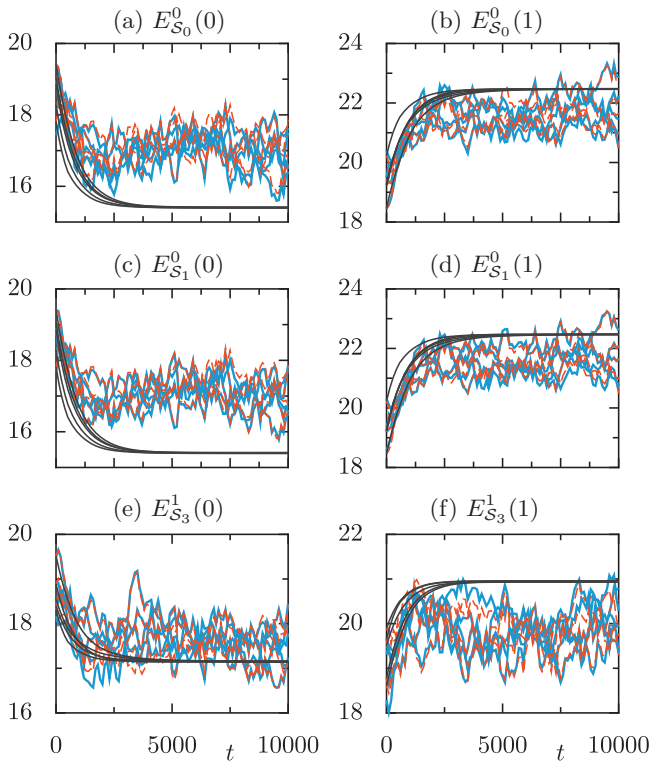


FIG. 7. Approximation of simplex-excess degrees $E_{S_i}^j(y)$ for the same simulation as in Fig. 2. Shown are values directly obtained from simulation (blue lines), compared to the approximations in Eq. (B3) using n_x and n_{xy} from simulation (orange dashed lines) and from the closed moment equations (black lines).

simplex $s^* \in \mathcal{S}_1 \cup \mathcal{S}_2$ with probability

$$P(e_{01} \in s^*) = \frac{2}{n_{01}} \tag{B4}$$

because s^* contains two out of all n_{01} edges of type 01. Furthermore, we note that if the 01-edge is part of $s^* = (i, j, k) \in \mathcal{S}$, then it is also part of any permutative simplex, e.g., $\tilde{s} = (j, i, k) \in \mathcal{S}$, which is of the same type. This means that each $3! = 6$ simplices are equivalent. Let $\tilde{\mathcal{S}}_i := \mathcal{S}_{i|\geq}$ be the set of all unique simplices defined by this equivalence condition; i.e., $|\tilde{\mathcal{S}}_i| = |\mathcal{S}_i|/6$. The probability that the 01-edge is not part of any simplex is thus given by

$$\begin{aligned} P_{no}^{01} &= P(e_{01} \notin s^* \forall s^* \in \mathcal{S}_1 \cup \mathcal{S}_2) \\ &\approx \prod_{s^* \in \tilde{\mathcal{S}}_1 \cup \tilde{\mathcal{S}}_2} P(e_{01} \notin s^*) \\ &= \left(1 - \frac{2}{n_{01}}\right)^{(|\tilde{\mathcal{S}}_1| + |\tilde{\mathcal{S}}_2|)/6}, \end{aligned} \tag{B5}$$

where the approximation assumes that the simplices in $\tilde{\mathcal{S}}_i$ are independent.

Second, edges of 00-type can only be part of simplices in \mathcal{S}_0 and \mathcal{S}_1 . Recall that $s \in \mathcal{S}_0$ contains three 0-nodes and three 00-edges counted twice, while $s \in \mathcal{S}_1$ contains exactly one 00 edge counted twice. Thus, a randomly drawn 00-edge $e_{00} \in \mathcal{E}$ is part of a simplex $s_0^* \in \mathcal{S}_0$ with probability

$$P(e_{00} \in s_0^*) = \frac{6}{n_{00}} \tag{B6}$$

and is part of a simplex $s_1^* \in \mathcal{S}_1$ with probability

$$P(e_{00} \in s_1^*) = \frac{2}{n_{00}}. \tag{B7}$$

The probability that e_{00} is not part of any simplex is thus

$$\begin{aligned} P_{no}^{00} &= P(e_{00} \notin s_0 \forall s_0 \in \mathcal{S}_0 \wedge e_{00} \notin s_1 \forall s_1 \in \mathcal{S}_1) \\ &\approx \prod_{s_0 \in \mathcal{S}_0} P(e_{00} \notin s_0) \cdot \prod_{s_1 \in \mathcal{S}_1} P(e_{00} \notin s_1) \\ &= \left(1 - \frac{6}{n_{00}}\right)^{|\mathcal{S}_0|/6} \left(1 - \frac{2}{n_{00}}\right)^{|\mathcal{S}_1|/6}. \end{aligned} \tag{B8}$$

Analogously, one derives the approximations

$$P_{no}^{11} \approx \left(1 - \frac{2}{n_{11}}\right)^{|\mathcal{S}_2|/6} \left(1 - \frac{6}{n_{11}}\right)^{|\mathcal{S}_3|/6}. \tag{B9}$$

Altogether, this approximates the probabilities of some edge e_{xy} of type xy to be part of any simplex,

$$P_{any}^{xy} = 1 - P_{no}^{xy}, \tag{B10}$$

with approximations for P_{no}^{xy} as given above. Suitable approximations for the numbers of simplices $|\mathcal{S}_i|$ are given at the end of this section.

For a comparison of these approximative probabilities for the three edge types with the numerical simulation, see Figs. 8(a)–8(c).

b. Approximation of $P_{S_i}^{xy}$. Recall that $P_{S_i}^{xy}$ is the probability that an xy -type edge is in a specific type $i \in \{0, 1, 2, 3\}$ of simplex, conditioned on the event that the edge is part of at least one simplex. Considering the possible type of simplices that a specific edge can be part of, these probabilities are given by

$$\begin{aligned} P_{S_0}^{00} &= \frac{|\mathcal{S}_0|}{|\mathcal{S}_0| + |\mathcal{S}_1|}, & P_{S_1}^{00} &= \frac{|\mathcal{S}_1|}{|\mathcal{S}_0| + |\mathcal{S}_1|}, \\ P_{S_1}^{01} &= \frac{|\mathcal{S}_1|}{|\mathcal{S}_1| + |\mathcal{S}_2|}, & P_{S_2}^{01} &= \frac{|\mathcal{S}_2|}{|\mathcal{S}_1| + |\mathcal{S}_2|}, \\ P_{S_2}^{11} &= \frac{|\mathcal{S}_2|}{|\mathcal{S}_2| + |\mathcal{S}_3|}, & P_{S_3}^{11} &= \frac{|\mathcal{S}_3|}{|\mathcal{S}_2| + |\mathcal{S}_3|}. \end{aligned} \tag{B11}$$

These approximated probabilities are compared to the values from numerical simulation in Figs. 8(d)–8(f), where the following approximations for the numbers of simplices are considered.

c. Approximation of $|\mathcal{S}_i|$. For a simplicial complex as considered in this paper, it is possible to approximate the number of simplices with the numbers of labeled edges in \mathcal{E}_2 . We assume that

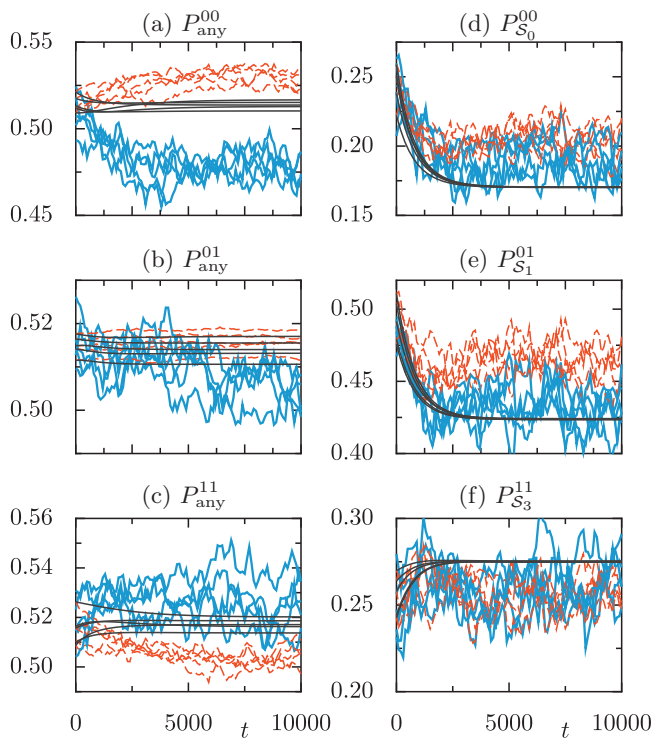


FIG. 8. Approximation of probabilities for the same simulation as in Fig. 2. Shown are values directly obtained from simulation (blue lines), compared to the approximations for P_{any}^{xy} in Eq. (B10) and for $P_{S_i}^{xy}$ in Eq. (B11) using n_x and n_y from simulation (orange dashed lines) and from the closed moment equations (black lines). The most significant deviations between approximation and numerical values are observed for P_{any}^{00} , for which the approximation is considerably larger. This is reasonable to expect since new 00-edges are created due to rewiring of 01-edges, without simultaneously increasing the number of adjacent simplices. For the corresponding numbers of simplices, see Fig. 9.

the simplices are distributed proportional to the triangles in the network,

$$|S_i| \approx S \cdot P(t \in T_i), \quad i \in \{0, 1, 2, 3\}, \quad (B12)$$

where S is the number of simplices, t some triangle, and T_i the set of all triangles of type i in the network. The number of simplices S is constant in our model. If a simplex is destroyed due to rewiring, a new simplex is added to the set of triangles. We want to express the probabilities that some triangle t is of type $i \in \{0, 1, 2, 3\}$,

$$P(t \in T_i) = \frac{|T_i|}{T}, \quad (B13)$$

in terms of the nodes and edges, where $T = \sum_i |T_i|$.

For this, we consider a triangle t_{abc} with three nodes of status a, b, c as an abc -subgraph, where the a and c node are connected. For example, the number of 000-triangles is approximated as the number n_{000} of 000-triples, multiplied with the probability that the outer 0-type nodes are connected, which approximately is n_{00}/n_0^2 . This approximation becomes better for large average node degrees k_{deg} .

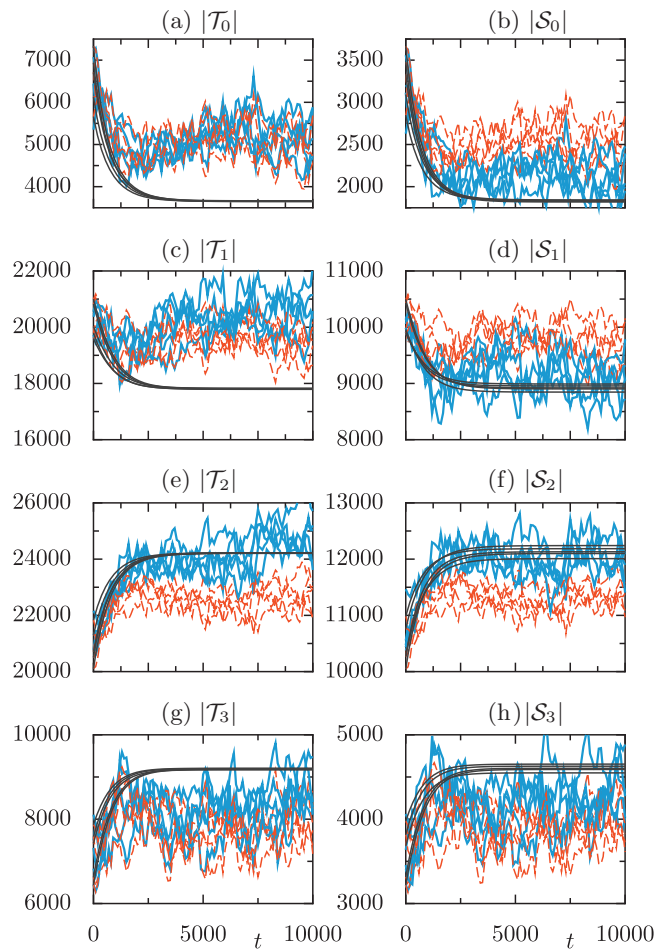


FIG. 9. Approximation of numbers of triangles and simplices of different types for the same simulation as in Fig. 2. Shown are values directly obtained from simulation (blue lines) compared to the approximations for $|T_i|$ in Eqs. (B14)–(B17) and for $|S_i|$ in Eq. (B19) using n_x and n_y from simulation (orange dashed lines) and from the closed moment equations (black lines). The most significant deviations between approximation and numerical values are observed for $|T_2|$, for which the approximation (orange line) is considerably smaller. This is reasonable to expect since simplices (and thus triangles) of type 2 are not destroyed by best response in simplices, which is not covered by uniformity assumptions and pair approximation.

Altogether, this leads to

$$|T_0| \approx n_{000} \frac{n_{00}}{n_0^2} \approx \frac{n_{00}^3}{n_0^3}. \quad (B14)$$

For triangles of type 1, we have to take into account the contributions from 001-triples, 010-triples, and 100-triples. The 001-triples contribute with n_{001} multiplied with the probability that the outer 0-node and 1-node are connected, which approximately is $n_{01}/(n_0 n_1)$ and the same contribution comes from 100-triples. The 010-triples contribute with n_{010} multiplied with the probability that the outer 0-type nodes are connected, which approximately

is n_{00}/n_0^2 . Altogether, we have

$$|\mathcal{T}_1| \approx 2n_{001} \frac{n_{01}}{n_0 n_1} + n_{010} \frac{n_{00}}{n_0^2} \approx 3 \frac{n_{00} n_{01}^2}{n_0^2 n_1}. \quad (B15)$$

A similar argumentation leads to the following approximations for the number of type-2 and type-3 triangles:

$$|\mathcal{T}_2| \approx 2n_{011} \frac{n_{01}}{n_0 n_1} + n_{101} \frac{n_{11}}{n_1^2} \approx 3 \frac{n_{11} n_{01}^2}{n_0 n_1^2}, \quad (B16)$$

$$|\mathcal{T}_3| \approx n_{111} \frac{n_{11}}{n_1^2} \approx \frac{n_{11}^3}{n_1^3}. \quad (B17)$$

These approximations are inserted into Eq. (B12), and we obtain with $|\mathcal{S}_i| \approx \frac{|\mathcal{T}_i|}{T}$ and

$$\begin{aligned} T &\approx \frac{n_{00}^3}{n_0^3} + 3 \frac{n_{00} n_{01}^2}{n_0^2 n_1} + 3 \frac{n_{11} n_{01}^2}{n_0 n_1^2} + \frac{n_{11}^3}{n_1^3} \\ &= \frac{n_{00}^3 n_1^3 + 3n_0 n_1^2 n_{00} n_{01}^2 + 3n_0^2 n_1 n_{11} n_{01}^2 + n_0^3 n_{11}^3}{n_0^3 n_1^3} \end{aligned} \quad (B18)$$

the following approximations for the different types of simplices:

$$\begin{aligned} |\mathcal{S}_0| &\approx \frac{S \cdot n_{00}^3 n_1^3}{n_{00}^3 n_1^3 + 3n_{00} n_{01}^2 n_0 n_1^2 + 3n_{11} n_{01}^2 n_0^2 n_1 + n_{11}^3 n_0^3}, \\ |\mathcal{S}_1| &\approx \frac{S \cdot 3n_{00} n_{01}^2 n_0 n_1^2}{n_{00}^3 n_1^3 + 3n_{00} n_{01}^2 n_0 n_1^2 + 3n_{11} n_{01}^2 n_0^2 n_1 + n_{11}^3 n_0^3}, \\ |\mathcal{S}_2| &\approx \frac{S \cdot 3n_{11} n_{01}^2 n_0^2 n_1}{n_{00}^3 n_1^3 + 3n_{00} n_{01}^2 n_0 n_1^2 + 3n_{11} n_{01}^2 n_0^2 n_1 + n_{11}^3 n_0^3}, \\ |\mathcal{S}_3| &\approx \frac{S \cdot n_{11}^3 n_0^3}{n_{00}^3 n_1^3 + 3n_{00} n_{01}^2 n_0 n_1^2 + 3n_{11} n_{01}^2 n_0^2 n_1 + n_{11}^3 n_0^3}. \end{aligned} \quad (B19)$$

Note that for these equations, we assume that $n_{0,1} \neq 0$ and $n_{00,01,11} \neq 0$. Let us emphasize that due to rewiring of edges, the actual number of triangles T and also its approximation are not constant in time. However, the considered approximations for the numbers of simplices satisfy $\sum_{i=0}^3 |\mathcal{S}_i| = S$ for all times. The approximations for the number of triangles and simplices are illustrated in Fig. 9. For the specific probabilities $P_{\mathcal{S}_i}^{xy}$ that are used in the closed moment equations, we get

$$\begin{aligned} P_{\mathcal{S}_0}^{00} &\approx \frac{n_{00}^2 n_1}{n_{00}^2 n_1 + 3n_{01}^2 n_0}, \\ P_{\mathcal{S}_1}^{01} &\approx \frac{n_{00} n_1}{n_{00} n_1 + n_{11} n_0}, \\ P_{\mathcal{S}_3}^{11} &= \frac{n_{11}^2 n_0}{3n_{01}^2 n_1 + n_{11}^2 n_0}. \end{aligned} \quad (B20)$$

APPENDIX C: STABILITY ANALYSIS WITH MATCONT

MatCont is a Matlab software project for the numerical continuation and bifurcation study of continuous and discrete parameterized dynamical systems.⁴⁷ In MatCont, the adaptive simplicial Snowdrift model is implemented via its closed moment equations in terms of relative variables. This is done for the system of closed

moment equations, Sec. III C, and also separately for the simplified parameter choices from Sec. IV.

For the simulations of the closed moment equations, initial values are selected on a grid like from the admissible range or in small neighborhoods around already known steady states. The analysis allows one to make statements about the local and also global stability. Furthermore, for the bifurcation analysis, we consider initial conditions at a steady state for a specific choice of parameters. This is followed by parameter variation, where one of the parameters changes while the others remain constant. The real parts of the eigenvalues of the Jacobian of the system reveal the stability of the steady state, and MatCont, therefore, detects possible bifurcations. This procedure is done for several steady states, choosing values of ϕ and ρ on a grid in $[0, 1]^2$. These parameters control the strength of influence of the three different adaptive operations. We also consider different parameters N, μ with $M = 2\mu N$ and σ with $S = \sigma E(T) \in \mathbb{N}_{>0}$, which determine the properties of the underlying network.

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