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Optimal inventory control with cyclic fixed order costs

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Abstract

We consider a periodic review single-item inventory model under stochastic demand. Every m periods, in the regular order period, fixed order costs are K. In the periods in-between, the intraperiods, higher fixed order costs of $L > K$ apply. The literature on optimal inventory policies under fixed order costs does not account for these time-dependent fixed order costs. By generalizing existing proofs for optimal inventory policies, we close this gap in inventory theory. The optimal inventory policy is complex in the regular order period and in the intraperiods, a period-dependent (s, S) policy is optimal. We describe and prove this optimal policy based on the notion of K-convexity and the optimal ordering behavior in the presence of non–K-convex cost functions. In a numerical study, we find that a major driver of the optimal policy is a forward-buying effect that shifts the probability of ordering from the intraperiods to the regular order period. The cost differences between the optimal and a pure period-dependent (s, S) policy are, however, small.

KEYWORDS

inventory control, K-convexity, proof of optimality, varying fixed cost

1 | INTRODUCTION

In the context of stochastic inventory control, the optimality of (s, S) policies under rather general assumptions is widely known. Although generalizations in many directions (e.g., random discounts, Markovian demands) have been established for more applications, one crucial assumption of existing results is that fixed costs do not increase over time (or at least not in their expectation) (see Iglehart, 1963; Scarf, 1959; Sethi & Cheng, 1997; Veinott & Wagner, 1965). This assumption is, however, violated in several practical problems.

To reduce the complexity of changes in delivery times in the retail industry, stores or product groups are frequently assigned to a fixed delivery schedule, for example, one delivery per week. This fixed delivery schedule is set to exploit economies of scale concerning the fixed costs of order handling in the warehouse and the transport costs to all stores (e.g., Gaur & Fisher, 2004). Within a store, however, goods might run out between two scheduled deliveries and have to be reordered. As these orders will not be optimally synchronized with the orders of other stores, higher fixed costs per order occur. As a generalization of the retail problem above, consider the *joint replenishment problem* where order schedules of several products are synchronized to purchase them jointly, resulting in a low share of fixed order costs per individual product. Consequently, a cyclic cost structure is present whenever managers are allowed to place single orders in between those schedules. Another example is the spare parts replenishment process of an original equipment manufacturer to its dealers. To minimize fixed costs, dealers are asked to order in bulk only once per week on a fixed day. However, due to the critical nature of spare parts, small "express" orders are regularly ordered in between. Naturally, those small orders incur larger fixed costs per order as neither picking processes in the warehouse nor transportation tours are highly utilized. This setting includes order systems with deterministic discount opportunities, where fixed costs are known to drop and increase cyclically, for example, when a vendor offers reduced fixed order costs for end-of-season sales.

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By explicitly considering that fixed costs might increase over time, we arrive at a generalization of existing research regarding the (s, S) policy (e.g., of Sethi & Cheng, 1997). We assume cyclical fixed order costs, where a cycle consists of multiple discrete periods, for example, 1 week. In that cycle, there is one regular order period, where low fixed order costs of K apply. All other periods in that cycle (so-called intraperiods) have higher fixed order costs of L. We show that the optimal policy is not necessarily an (s, S) policy, but of a more complex type in the order period with low fixed costs. The policy is of (s_j, S_j) type in all other periods (with j being an index for the intraperiods).

In particular, we answer the following research questions:

(RQ1) How does the optimal policy look like in the presence of cyclical varying fixed costs?
(RQ2) What drives the complex policy structure in the regular order period?
(RQ3) What is the benefit of having the option to order in the intraperiods rather than only in the regular order period?
(RQ4) What is the (cost) impact of the complex order structure in the regular order period in comparison to an (s, S) type of policy?

The remainder of this paper is structured as follows. We give an overview of the relevant literature in Section 2. In Section 3, we state the problem formally. We describe and prove the optimal order policy in Section 4 and extend the findings in Section 5. In Section 6, we provide numerical insights on the behavior of the optimal policy and Section 7 concludes the paper.

2 | LITERATURE REVIEW

After Arrow et al. (1951) introduced the stochastic inventory control problem and the (s, S) policy, many publications dealt with proving optimality for similar control settings. The basic inventory problem considers a single-stage and single-item problem where fixed order costs prevail. The seminal paper of Scarf (1959), directly extending Arrow et al. (1951) and introducing K-convexity, and Veinott and Wagner (1965), Song and Zipkin (1993), or Sethi and Cheng (1997) generalize these findings (e.g., for Markovian demands, positive lead times). For a definition of K-convexity, see, for example, Porteus (2002), and for a survey of extensions of optimal (s, S) policies in the discrete-time setting, see Perera and Sethi (2023b), and in the continuous-time setting, Perera and Sethi (2023a). However, existing results require fixed costs to be either constant or at least nonincreasing in their values over time (or in their expected values of state-dependent fixed order costs for the Markovian case). Perera et al. (2018) show that optimality of the (s, S) policy also holds for very different cost functions, such as multiple setup costs, piecewise concave costs, and so forth, but only with the same cost structure over time and in a continuous review setting, where demands occur unit-sized and inventory levels are changing in unit-sized steps. However, in our periodic review setting, inventory levels might change in larger steps between periods, and fixed costs change over time. Hence, their results cannot be applied and the proof of optimality for the (s, S) policy under more general cost functions remains nontrivial.

The above-mentioned publications under periodic review use K-convexity in their proofs. If K-convexity is not given, other notions have been used in the inventory control literature. In the area of capacity expansion/reduction and cash balancing problems, (K_1, K_2)-convexity (Ye & Duenyas, 2007) and *weak*-(K_1, K_2)-convexity (Semple, 2007) have been introduced. These problems have in common that there are two different fixed costs for buying and selling. The optimal policies obtained are to some degree similar to the optimal policy found in our case, in that they include multiple buying and selling areas. However, those notions do not apply to problems with cyclic fixed order costs as the decision causing the fixed costs is taken in the same period for both, buying and selling. Also, fixed costs are assumed to be nonincreasing over time. Chen and Simchi-Levi (2004) introduce *sym*-K-convexity to characterize the optimality of an (s, S, p) policy for integrated inventory control and price setting with additive random demand, again assuming nonincreasing fixed costs. Simchi-Levi et al. (2014) extend the symmetric K-convexity along with properties on symmetric $\max(Q, K)$-convex functions. Similar to this symmetric $\max(Q, K)$-convexity, we prove that in our setting general L-convexity prevails even if K-convexity is not given (under $L > K$). Porteus (1971) introduced $quasi-K$-convexity for concave increasing order costs. This can be applied to problems with multiple suppliers and different fixed and variable order costs. As opposed to our setting, all supply options are available in every period, which leads to the same order cost structure for all periods. Gallego and Sethi (2005) extend K-convexity to order decisions placed for multiple products. In their approach, joint replenishment effects can be obtained when multiple products are ordered together and thus K-convexity is expanded to the $\mathbb{R}^n$ space. An approach for finding optimal order policies, in the case where convexity-based properties are not given, especially in highly dimensional inventory control systems (such as perishable inventory or dual sourcing systems with long lead times), is asymptotic analysis. See, for example, Goldberg et al. (2021) for a survey on the application of asymptotic analysis for (among others) dual sourcing problems and Zhang et al. (2020) for an application in perishable inventory under fixed order costs, resulting in near-optimal simple heuristics.

In the dual sourcing literature, Johansen and Thorstenson (2014) include different fixed costs for the regular and emergency order mode and propose an (s, Q) policy for the regular and an (s, S) policy for the emergency order mode. They use a policy iteration algorithm to obtain optimal parameters, yet do not provide analytical proofs for an optimal policy. Chiang (2003) integrates fixed costs as being zero either for both or at least for the regular supply mode. The resulting policy has two order-up-to levels (one for the regular and one for the emergency supply) and a reorder point for both modes. Moinzadeh and Nahmias (1988) assume different fixed costs for the two supply modes in a continuous review setting with lost sales. They suggest a heuristic with separate reorder points and lot sizes for both modes in a (Q_1, Q_2, R_1, R_2) policy, which achieves good results in case fixed costs are high and a stock-out penalty is low. Jain et al. (2011) apply varying fixed costs for both supply modes that differ in their lead time and characterize the complex optimal order policy. They indicate that this complex policy reduces to an (s, S) policy when the difference in fixed order costs of choosing one order mode over the other is negligible. They numerically show that the employment of a pure (s, S) strategy yields good approximations. The difference to our problem is that both order modes are available in all periods and that it is allowed to split orders between both modes, whereas in our setting, only one order mode is available at any given point in time.

Another related research area is the single-item inventory problem with random discount opportunities in a continuous review setting. In this scenario, it is assumed that possibilities with reduced costs for ordering (fixed and/or variable costs) occur randomly. Federgruen et al. (1984) assume that combined orders of multiple products occur randomly based on an exponential distribution and thus, for a single item, order costs are also reduced randomly due to economies of scale. Simple, yet not necessarily optimal, policies are of an (s, c, S) type, where s and S are used as reorder point and order-up-to level. Parameter c states a can-order point that is considered when a random discount opportunity is at hand. Hurter and Kaminsky (1968) introduce an inventory problem with random discount opportunities and provide an algorithm to determine the optimal parameters for an (s, c, S) policy. The optimal policies when variable and fixed order costs are changing for discount orders are of (r, R, d, D) type with reorder points (order-up-to levels) r (R) for the regular and d (D) for the discount orders (see Feng & Sun, 2001). However, all papers in the area of random discount opportunities have in common that demand is (compound) Poisson and the discount might happen at any time, thus they do not have a cyclic cost structure. We analyze a case where discounts appear deterministically and only apply to the fixed costs. Also, our setting is not limited to Poisson demand. This is why the above papers cannot be directly extended to our setting. In fact, we show that an (r, R, d, D) is not optimal in case of general demand structures and a deterministic discount opportunity.

Zipkin (1989) investigates optimal order policies under periodically changing variable costs, which results in a period-dependent base-stock policy. Our paper extends this direction by focusing on cyclical changing fixed order costs. In another continuous review setting with varying fixed costs for two supply modes with differing lead times, Jain et al. (2010) find the optimal parameters for a (Q, r) policy. A split of orders between the two supply modes is only preferable if the additional fixed costs from the second order are outweighed by savings of otherwise higher holding and variable costs of a single order. We observe a similar trade-off of expected holding against expected fixed order costs when we split orders over time in our setting. In Jain et al. (2010), the decision on either supply mode can be done simultaneously (i.e., in the same state of information). In our setting, the second supply mode is only available at a different point in time, given a different state of information.

3 | PROBLEM FORMULATION

We consider a finite planning horizon with $n = 1, \ldots, N$ discrete order cycles, each covering m periods. Later we will generalize to the infinite horizon problem. Orders are placed at the beginning of each period and arrive instantaneously, followed by random demand Ξ with realization ξ and probability density ϕ. Lead time is assumed to be zero (we relax this assumption in Section 5.1). In the following analyses and proofs, we describe the cost functions and Bellman equations that assume continuous distributions, but the same holds for discrete distributions (similar to Kalymon, 1971). There are two different fixed order costs. K applies in the regular order period every m periods and fixed order costs L ($K < L$) occur in the intraperiods $j \in \{1, \ldots, m-1\}$. Let the period-dependent fixed order costs $K_j = L$ for all $j = 1, \ldots, m-1$, $K_m = K$ and, for the convenience of writing, $K_{m+1} = K_1 = L$. Variable costs c are incurred for any item ordered. They do not differ between the two order modes. Positive inventory at the end of the period incurs holding costs of c_H per unit and period and any demand that cannot be fulfilled is backlogged at a cost c_P per unit and period. $0 < \alpha \leq 1$ is the one-period discount factor.

Let x be the inventory on hand before and y the inventory on hand after ordering. The expected single-period holding and penalty cost function $\mathcal{L}(y)$ is

$$\mathcal{L}(y) = \int_0^y c_H(y - \xi)\phi(\xi)d\xi + \int_y^\infty c_P(\xi - y)\phi(\xi)d\xi. \quad (1)$$

For any given cycle n, period j, and inventory state before ordering x, the optimal inventory after ordering y minimizes variable and fixed order costs, expected single period holding and penalty cost as well as (discounted) expected costs of acting optimally in all future periods, starting with the next period in state $y - \xi$ (see Porteus, 2002, for the case with constant fixed order costs). We define this optimal expected costs $f_{n,j}$ as

$$f_{n,j}(x) = \min_{y \geq x} \Big\{ c(y - x) + \delta(y - x)K_j + \mathcal{L}(y) + \alpha \int_0^\infty f_{\theta(n,j)}(y - \xi)\phi(\xi)d\xi \Big\} \quad \forall j = 1, \ldots, m,$$

where $\delta(a) = 1$ if $a > 0$, and 0 otherwise and $\theta(n, j) = \begin{cases} (n, j+1) & \text{for } j \leq m-1 \\ (n+1, 1) & \text{for } j = m \end{cases}$ is a one-period time shift operator that either points to the next period in a cycle or the first period of the next cycle. We furthermore rewrite the optimal expected costs $f_{n,j}(x)$ as

$$f_{n,j}(x) = -cx + \min\left\{ g_{n,j}(x), \min_{y>x} \left\{ K_j + g_{n,j}(y) \right\} \right\} \quad \forall j = 1, \dots, m, \qquad \text{where} \tag{2}$$

$$g_{n,j}(y) = cy + \mathcal{L}(y) + \alpha \int_0^\infty f_{\Theta(n,j)}(y-\xi)\phi(\xi)d\xi \quad \forall j = 1, \dots, m. \tag{3}$$

The optimal decision for inventory state x in cycle n and period j is found by determining the minimum of $g_{n,j}(x)$ and the minimizing value of $g_{n,j}(y)$ to the right of x, that is, $y > x$, plus the fixed order costs. Hence, we can limit our investigations regarding the optimal policy by investigating functions $g_{n,j}$ as in Scarf (1959) and Porteus (2002). As we consider a finite horizon problem, let $f_{N,m+1} = -cx$ be a terminal value function that is charged at the end of period m in cycle N. This terminal value function represents the charge that is incurred for any leftover inventory or backlogged demand at the end of the finite time horizon, that is, all positive inventory is salvaged and all backlog has to be produced at the unit cost c.

4 | OPTIMAL POLICY

To fully describe the optimal policy, we define terminology and critical points along a continuous function $g(x)$. Note that we characterize the general behavior of a cost function and thus refrain from period-specific indices at this point. For a visual representation of these points, see Figure 1.

Let a set of order areas I describe an order policy for the defined problem. An order area $i \in I$ is defined by a reorder point $\underline{a}_i$, a stay-put point $\bar{a}_i$, and an order-up-to level a_i^*. If the inventory level drops below the reorder point, an order is placed to fill inventory to the order-up-to level, but only in case the inventory level does not drop to a level below the stay-put point and above the reorder point of the next order area.

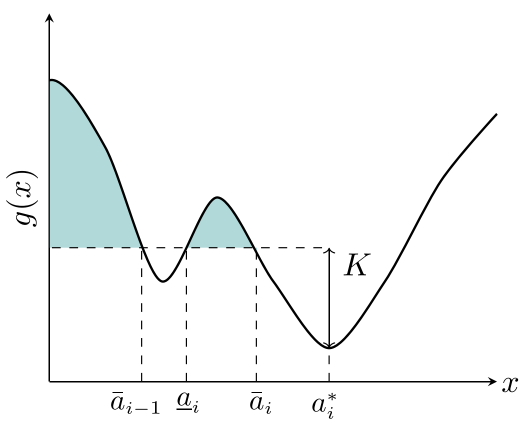

FIGURE 1 Illustration of critical points on function $g(x)$ to define order area i. [Color figure can be viewed at wileyonlinelibrary.com]

In Figure 1, we see that a_i^* is the minimizer of the function on the shown interval from 0 to a_i^*. When observing the interval $(-\infty, \infty)$, the minimizer a_i^* is equal to the global minimum M, where $M = \min_{x\in(-\infty,\infty)} g(x)$. Note that $a_i^* = M$ for all $i \in I | \bar{a}_i \leq M$, that is, the global minimum M represents the order-up-to level for all order areas to the left of M. To be more precise on a_i^*, define it as the minimizer of any interval to the right of reorder point $\underline{a}_i$ with $a_i^* := arg\,min_{x\in[\underline{a}_i,\infty]}\, g(x)$. This definition is necessary, as there might be an interval on $g(x)$ where a_i^* is not the global minimum, which is the case if an order area appears to the right of M.

Focusing on the interval $[\underline{a}_i, a_i^*]$ in Figure 1, we observe the known behavior of order policies under K-convex functions. An order is beneficial if $g(x) \geq g(a_i^*) + K$, indicated by the shaded areas. $\bar{a}_i$ is the first point left of a_i^* where this is true, indicating the reorder point. Define $\bar{a}_i := arg\,max\{\underline{a}_i \leq x \leq a_i^* : g(y) \geq g(a_i^*) + K$ for all $\underline{a}_i \leq y \leq x\}$. Now, going further to the left in the interval $]\bar{a}_{i-1}, \underline{a}_i[$, it is not optimal to order as $g(x) < g(a_i^*) + K$. Hence $\underline{a}_i$ is the last point, where it is still beneficial to order, indicating the stay-put level. We define $\underline{a}_i := arg\,min\{\bar{a}_{i-1} < x \leq \infty : g(x) \geq g(y) + K, y = arg\,min_{z\in[x,\infty]}\, g(z)\}$. The stay-put level is different from the traditional (s, S) order policies, where it does not exist. We furthermore introduce $\underline{A}$ as the level below which it is always optimal to order, defined as $\underline{A} = \min_{i\in I}\{\bar{a}_i\}$ and $\bar{A}$ as the level above which it is never optimal to order, defined by $\bar{A} = \max_{i\in I}\{\bar{a}_i\}$. For technical purposes, let $\underline{a}_0 := -\infty$.

Definition 1. Critical Points

(i) $M = \min_{x\in[-\infty,\infty]} g(x)$.
(ii) $\underline{a}_i := arg\,min\{\bar{a}_{i-1} < x \leq \infty : g(x) \geq g(y) + K, y = arg\,min_{z\in[x,\infty]}\, g(z)\}$.
(iii) $a_i^* := arg\,min_{x\in[\underline{a}_i,\infty]}\, g(x)$.
(iv) $\bar{a}_i := arg\,max\{\underline{a}_i \leq x \leq a_i^* : g(y) \geq g(a_i^*) + K$ for all $\underline{a}_i \leq y \leq x\}$.
(v) $\underline{a}_0 := -\infty$.
(vi) $\underline{A} = \min_{i\in I}\{\bar{a}_i\}$.
(vii) $\bar{A} = \max_{i\in I}\{\bar{a}_i\}$.

To characterize the optimal policy, we use Definition 1 to show that a function, which might not be K-convex, still is L-convex (for a formal definition of K-convexity, see Supporting Information EC.1).

Proposition 1. *Let L and K be nonnegative, with $L \geq K$, and $g : R \to R$ be an L-convex continuous function such that* $\lim_{x\to+\infty} g(x) = +\infty$. *Let* $g^* = \min_{x\in(-\infty,\infty)} g(x) > -\infty$.

Then, $h(x)$ is the optimal cost function after ordering in the regular order period and $k(x)$ in the intraperiods, where

(i) $h(x) \equiv \min_{y \geq x, -\infty < y < \infty} [K\delta(y-x) + g(y)] = \begin{cases} K + g(M) & x < \underline{A} \\ K + g(a_i^*) & i \in I, x \in (\underline{a}_i, \bar{a}_i) \\ g(x) & i \in I, x \in [\bar{a}_{i-1}, \underline{a}_i] \\ g(x) & \bar{A} \leq x \end{cases}$ *and* $h : R \rightarrow R$ *is continuous;*

(ii) *h is L-convex on* $(-\infty, \infty)$, *following the general logic of Simchi-Levi et al. (2014);*

(iii) $k(x) \equiv \min_{y \geq x, -\infty < y < \infty} [L\delta(y-x) + g(y)] = \begin{cases} L + g(S) & x < s \\ g(x) & s \leq x < \infty \end{cases}$ *and* $k : R \rightarrow R$ *is continuous;*

(iv) *k is L-convex on* $(-\infty, \infty)$.

Proof. See Supporting Information EC.2.1. □

When we apply Proposition 1 to cost functions (3) and (2), (i) introduces the optimal ordering decisions in the regular order period, and (ii) proves that L-convexity is preserved under this policy. Based on the preserved L-convexity, the standard arguments as in Scarf (1959) and Sethi and Cheng (1997) to show that an (s, S) policy is optimal in the intraperiods apply.

Theorem 1.

(i) *In the regular order period m of order cycle n, there exist critical points such that, for a given inventory state x, the optimal inventory after ordering is*

$$y_{n,m} = \begin{cases} M_n & x \in (-\infty, \underline{A}_n) \\ a_{n,i}^* & i \in I_n, x \in (\underline{a}_{n,i}, \bar{a}_{n,i}) \\ x & i \in I_n, x \in [\bar{a}_{n,i-1}, \underline{a}_{n,i}] \\ x & x \in [\bar{A}_n, \infty). \end{cases}$$

(ii) *For all intraperiods $j = 1, \ldots, m-1$ of order cycle n, there exists an optimal $(s_{n,j}, S_{n,j})$ policy, such that, for a given inventory state x, the optimal inventory after ordering is*

$$y_{n,j} = \begin{cases} S_{n,j} & x \in [-\infty, s_{n,j}) \\ x & x \in [s_{n,j}, \infty). \end{cases}$$

Proof. See Supporting Information EC.2.2. □

The proof for the intraperiod policy uses Proposition 1 and the basic properties of K-convexity of Scarf (1959) and Sethi and Cheng (1997). It proves by induction that L-convexity is given in every cycle and intraperiod and thus an (s, S) policy is optimal. Note that we need parameters for all order cycles. Hence, we add index n to the parameters defined in Definition 1.

5 | EXTENSIONS

5.1 | Positive constant lead times

For positive constant lead times $\lambda \geq 1$, we follow the argumentation in Scarf (1959) and define x_k as the stock to be delivered in period $k = 1, 2 \ldots, \lambda - 1$. Furthermore, $\hat{x} = x + \sum_{k=1}^{\lambda-1} x_k$ is the inventory position before ordering, z is the amount ordered in the current period, and $\hat{y} = \hat{x} + z$ is the inventory position after ordering. Define

$$\hat{\mathcal{L}}(\hat{y}) = \alpha^{\lambda} \int_0^{\infty} \ldots \int_0^{\infty} \mathcal{L}(\hat{y} - \sum_{i=1}^{\lambda} \xi_i) \phi(\xi_1) \ldots \phi(\xi_\lambda) d\xi_1 \ldots d\xi_\lambda.$$

We can now replace $\hat{x}$ for x, $\hat{y}$ for y, and $\hat{\mathcal{L}}$ for $\mathcal{L}$ in the formulation of the optimal expected costs $f_{n,j}$ in (2) and (3). As $\hat{y}$ and $\hat{x}$ have no impact on the properties of the cost function and $\hat{\mathcal{L}}$ is convex (since $\mathcal{L}$ is convex), the same argumentation as in Section 4 applies to arrive at the optimality of the same policy structure as described in Theorem 1, dependent on $\hat{x}$.

5.2 | Generic fixed costs changes

For a generic change of fixed costs $K_j \geq 0 \ \forall j \in 1, \ldots, m$, intraperiods' fixed costs are not bound to be equal within a cycle and might even fall below the regular order costs. This covers cyclical and nondecreasing costs within one cycle. For any period j with the largest fixed costs $\max_j\{K_j\} = L$, we can follow Proposition 1 to show L-convexity for $f_{n,j}$. In any other period, we know that $K_j \leq L$, and hence we can use Proposition 1.(ii) to show that again L-convexity is given, however obviously not K_j-convexity. Using the backward induction logic as in Supporting Information EC.2.2, L-convexity is preserved and thus for all periods with $\arg\max_j\{K_j\} = L$, the (s, S) policy of Theorem 1.(ii) is optimal, while for all other periods (also respective intraperiods), the more complex order policy of Theorem 1.(i) applies.

Now, assume that fixed costs are increasing from one cycle to the next, for example, $K_{n,j} < K_{n+1,k} \ \forall n \in 1, \ldots, N-1, j, k \in 1, \ldots, m$. In that case, the largest fixed costs over the whole time horizon will appear in the last cycle N, that is, $L = \max_j\{K_{N,j}\}$. While we can show that L-convexity is still preserved in all periods and cycles, only the last cycle N will order under the fixed costs of L and an optimal (s, S) policy. Hence, in all other cycles, we observe the more complex order policy of Theorem 1.(i).

In case of nonincreasing fixed cost values over all periods and cycles (more precisely, if $K_{n,j} \geq \alpha K_{\theta(n,j)} \ \forall j \in 1, \ldots, m, n \in 1, \ldots, N$), the conditions of existing optimality proofs of period-dependent (s, S) policies of, for example,

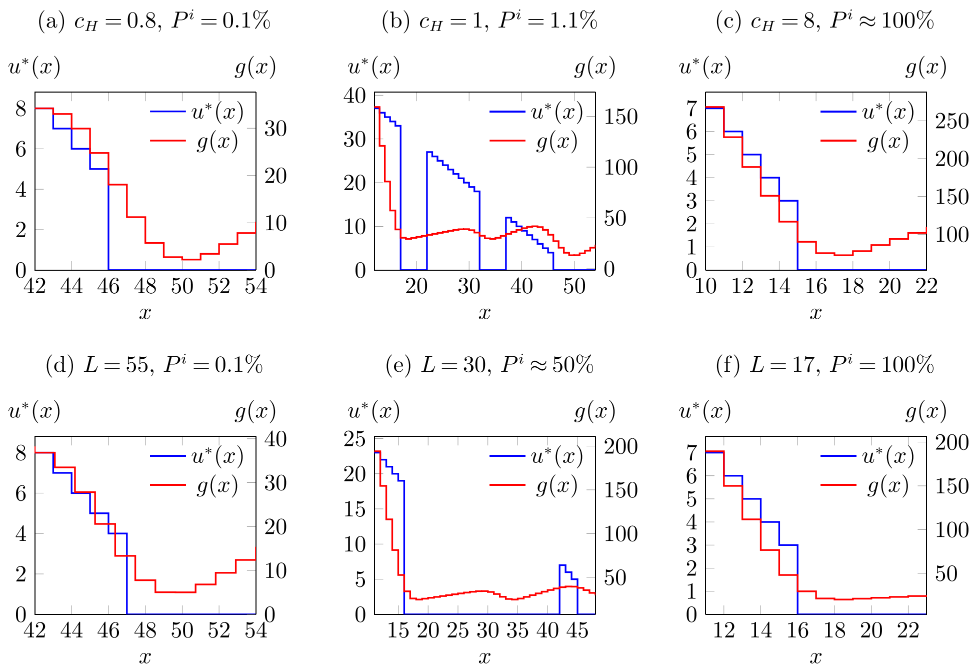

FIGURE 2 Illustration of the forward-buying effect (Observation 1). [Color figure can be viewed at wileyonlinelibrary.com]

Sethi and Cheng (1997) hold. Hence period-dependent (s, S) policies for all periods and cycles are optimal.

In Supporting Information EC.4, we show that the results of Section 4 further hold for the infinite horizon case in Supporting Information EC.4.1 and for Markov-modulated demands and Markov-modulated fixed costs in Supporting Information EC.4.2.

6 | NUMERICAL STUDY

We numerically investigate the optimal order policies to answer research questions (RQ2)–(RQ4). We focus on highlighting the effects of forward-buying, the value of the intraperiod option, and the impact of the multiple order areas. Assuming $\alpha = 1$, we hence optimize under the average cost criterion. Uncertain demands are assumed to be either binomial or negative binomial and are selected by and fitted to the first two moments as in Adan et al. (1995). To capture the usage of the intraperiods' order option, we define P^i as the steady-state probability of ordering under the condition that the system is currently in an intraperiod. For our numerical investigations, we introduce the $(Opt|m)$ policy as a benchmark, which always only orders in the regular order period, that is, every m periods. For details on the algorithms, see Supporting Information EC.3.

Observation 1. The multiple order areas in the regular order period are driven by a *forward-buying effect*. This effect describes the ordering of more units in the regular order period to reduce the probability of having to order in the intraperiods.

Figures 2a–c illustrate this ordering behavior with changes in holding costs $c_H \in \{0.8, 1, 8\}$ and Figures 2d–f with decreases in the fixed order costs of the intraperiods $L \in \{55, 30, 17\}$. The average demand is $\mu = 15$ and the coefficient of variation is $CV = 0.1$. The cycle length $m = 3$, $c = 0$, and $c_P = 40$. In Figures 2a–c, we set $K = 20$ and $L = 50$ and in Figures 2d–f, $c_H = 1$ and $K = 10$.

In Figure 2a with $c_H = 0.8$, we observe one order area which represents an (s, S) policy, with $s = 45$ and $S = 50$. The probability of ordering when the system is in an intraperiod is 0.01%. Hence, the policy will order as much as needed at the regular order period to (nearly) never order in the intraperiods. The other extreme can be found in Figure 2c, where we again see a classic (s, S) policy, with $s = 14$ and $S = 17$. In an intraperiod, the probability of ordering is nearly 100%. Hence, the regular order period does not engage in forward-buying at all, due to higher holding costs. Figure 2b shows the multiple order areas for the case with $c_H = 1$. The probability of ordering in an intraperiod is 1.1%. Hence, the order in the regular order period covers the demand of the intraperiods in some, but not all, cases.

Similar behavior is exhibited for decreasing intraperiod order costs. In Figure 2d, where L is substantial, orders are primarily placed during the regular order period to exploit the cheaper option. However, in Figure 2e, where L is reduced, multiple order areas are observed, as forward-buying only covers the likelihood of ordering during an intraperiod to a certain extent. In contrast, as shown in Figure 2f, the cost premium of L decreases to a level where forward-buying to avoid an intraperiod order is not beneficial anymore.

Observation 2. The intraperiod order option provides the largest cost benefits when the order cycle length is not

TABLE 1 Cost increase of $(Opt|m)$ to the optimal policy (Observation 2).

TBO	$L/K = 1$	$L/K = 1.1$	$L/K = 1.2$
$0.25m$	165.79%	149.56%	135.20%
$0.5m$	29.66%	25.34%	22.17%
$0.75m$	5.31%	1.35%	0.30%
$1m$	0.38%	0.05%	0.02%
$1.25m$	0.16%	0.03%	0.01%
$1.5m$	1.40%	0.01%	0.00%
$1.75m$	3.57%	0.00%	0.00%
$2m$	5.49%	0.02%	0.00%

well aligned with the optimal time between orders ($TBO = \sqrt{2K/(c_H\mu)}$) based on the fixed costs of the regular order option. Furthermore, the benefit decreases linearly if fixed costs increase, that is, the decrease is not accelerating with higher cost premiums.

We observe the value of having additional order options to the regular order period in Table 1. Here, we set $c_H = 1$ and alter K and L. We set K to represent different values of the TBO, with $TBO \in \{0.25m, 0.5m, 0.75m, 1m, 1.25m, 1.5m, 1.75m, 2m\}$ and set $L \in \{K, 1.1K, 1.2K\}$. As before, $m = 3$, $\mu = 15$, $CV = 0.1$, $c = 0$, and $c_P = 40$.

The largest cost differences between $(Opt|m)$ and the optimal policy can be found in Table 1 when TBO is much smaller than the order cycle. In these cases, even a 10% or 20% cost premium for ordering in the intraperiod will diminish the savings rather linearly and not with an accelerating decrease at higher cost premiums. If $TBO > m$, a positive value of using the optimal policy can only be found when L equals K, that is, there is no cost premium for ordering in the intraperiod. This shows that the additional order options in the intraperiod might counteract a misaligned order cycle length and provide significant cost savings when fixed costs point to ordering more often than the cycle length.

Observation 3. Multiple order areas occur most likely when demand uncertainty is low and the cycle length is large. The benefit of multiple order areas in the regular order period against an (s_j, S_j) policy is negligible.

We conduct a full factorial design with a variation in $\mu \in \{5, 15\}$, $CV \in \{0.1, 0.2, 0.3\}$, $c_H \in \{0.5, 1, 1.5\}$, $c_P = 40$, $m \in \{2, 3, 4\}$, $K = 20$, and $L \in \{30, 50, 70, 90\}$ and compare the optimal policy with the best (s_j, S_j) policy. We select these parameters to focus on the effects of multiple order areas for illustration purposes. In our numerical study of 216 instances, 19 demonstrated multiple order areas in the optimal order policy during regular order periods (i.e., $|I| > 1$; see Table 2).

The multiple order areas were most prominent with high μ and cycle length m and low CV. Increasing stochasticity (i.e., higher CV) increases the costs of holding (safety) inventory

TABLE 2 Number of instances with multiple order areas $|I|$ (Observation 3).

		μ		m			c_H			CV			L/K			
		5	**15**	**2**	**3**	**4**	**0.5**	**1**	**1.5**	**0.1**	**0.2**	**0.3**	**1.5**	**2.5**	**3.5**	**4.5**
	1	101	96	69	66	62	68	64	65	59	66	72	46	46	52	53
$\|I\|$	**2**	6	10	3	4	9	4	5	7	10	6		7	7	1	1
	3	1	2		2	1		3		3			1	1	1	

to cover demand in intraperiods, reducing the attractiveness of forward-buying and the possible appearance of multiple order areas. Additionally, the number of order areas never exceeds the number of periods in the order cycle ($|I| \leq m$), indicating that the presence of multiple order areas is linked to the forward-buying effect of fully satisfying the demand of additional intraperiods. Therefore, with an increase in m, more order areas may appear.

The impact of higher μ can be explained by the trade-off between fixed order costs and holding costs as L/K rises. If $L/K = 1.5$, multiple order areas occur more frequently for lower μ, while for larger $L/K \in \{2.5, 3.5, 4.5\}$, they occur more frequently when μ is large. This suggests that multiple order areas arise only if the fixed costs differences between the order modes are counterbalanced by holding costs required to meet future demand, as corroborated by Observation 1 in Figures 2d–f.

The occurrence of multiple order areas (i.e., $|I| > 1$) does not exhibit a clear pattern when varying c_H and L/K. For $c_H \in \{0.5, 1, 1.5\}$, the number of instances with $|I| > 1$ increases from 4 to 8, but then drops to 7. For $L/K \in \{1.5, 2.5\}$, the number of instances with $|I| > 1$ remains constant at 8, indicating that L/K has no discernible effect. However, for $L/K \in \{3.5, 4.5\}$, the number of instances with $|I| > 1$ decreases to 2 and 1, respectively. These findings show that the occurrence of multiple order areas is driven by the trade-off and thus the combination of holding costs and fixed cost differences, hence altering a single parameter does not produce a straightforward impact.

However, in the instances where multiple order areas were observed, the deviations from the optimal (s_j, S_j) policy were negligible, with the maximum cost difference being only 0.03%. This is evident when examining the optimal policies and steady-state probabilities during regular order periods, as the multiple order areas occurred in states with negligible steady-state probability, making them almost transient and having minimal impact on the long-term average cost of the infinite horizon problem. Similar performance results were reported in Jain et al. (2011), where the cost differences between an (s, S) policy and the optimal policy were also small, averaging 0.029%.

Observation 4. Optimal values for s_j and S_j in the intraperiods do not monotonically decrease or increase within an order cycle.

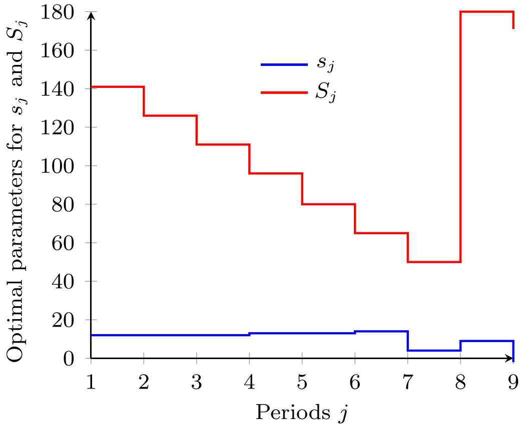

FIGURE 3 Example values for s_j, S_j for each intraperiod of the order cycle (Observation 4), with $m = 10$, $c_H = 1$, $c_P = 40$, $K = 750$, $L = 1275$ (i.e., $TBO = 1$), $\mu = 15$, $CV = 0.1$. [Color figure can be viewed at wileyonlinelibrary.com]

An illustration of the aforementioned effect can be found in Figure 3.

The value of S progressively decreases until it experiences a sharp increase. The reduction in S during the initial periods is a result of covering the demand until the subsequent regular order period. This, in turn, enhances the likelihood of placing an order during that period rather than during an intraperiod, thereby exploiting the fixed costs difference. However, in intraperiods 8 and 9 in Figure 3, it is advantageous to maintain inventory levels sufficient to satisfy the entire demand for the subsequent order cycle to avoid placing two orders within the same cycle.

7 | CONCLUSION

We investigated inventory control policies where, at regular order periods, orders can be placed at lower fixed costs than in the intraperiods in between. We summarize our findings to the research questions (RQ1)–(RQ4).

(RQ1) How does the optimal policy look like in the presence of cyclical varying fixed costs?

We proved the optimality of an (s_j, S_j) inventory policy for the intraperiods. The optimal policy in the regular order period, although complex, can be described by multiple order areas. Comparing it to a policy that potentially requires a different order logic for every inventory state, we showed that the order policy contains a certain structure, which allows rule-based decision making for managers (see Theorem 1).

(RQ2) What drives the policy structure in the regular order period?

The cost differences between benchmarks and the optimal policy reveal a forward-buying effect, which redirects orders from the intra- to the regular order period. This effect drives the policy structure in the regular order period toward (s, S) structures or the more complex policy with multiple order areas (see Observation 1).

(RQ3) What is the benefit of having the option to order in the intraperiods rather than only in the regular order period?

The option of ordering in an intraperiod is especially valuable when the length of the order cycle is not well aligned with the optimal TBO. As long as the TBO corresponds to the order cycle or is larger, the benefits of the intraperiod order option are small. However, benefits can be large if the TBO is much smaller than the order cycle. While a cost premium for the intraperiod order option reduces the benefits, it will only have a linear effect and not an accelerating negative impact on the total cost, when cost premiums are rising (see Observation 2).

(RQ4) What is the (cost) impact of the complex order structure in the regular order period in comparison to an (s, S) type of policy?

Cost differences between the complex order structure to an (s, S) type of policy are negligible if the parameters of the (s, S) policy are period-dependent (see Observation 3).

For further research, it would be interesting to investigate the case of an expedited lead time for the more expensive order option, as in, for example, Jain et al. (2011) and investigate the situation where the more expensive sourcing option has a base capacity, which can be exceeded at a cost premium, as in Gijsbrechts et al. (2022).

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