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Axel Bücher, Miriam Jaser, and Aleksey Min*

Detecting departures from meta-ellipticity for multivariate stationary time series

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Abstract: A test for detecting departures from meta-ellipticity for multivariate stationary time series is proposed. The large sample behavior of the test statistic is shown to depend in a complicated way on the underlying copula as well as on the serial dependence. Valid asymptotic critical values are obtained by a bootstrap device based on subsampling. The finite-sample performance of the test is investigated in a large-scale simulation study, and the theoretical results are illustrated by a case study involving financial log returns.

Keywords: elliptical copula, empirical process, financial log returns, goodness-of-fit test, subsampling bootstrap

MSC: 62H15, 62M10

1 Introduction

In the recent decades, copula models have been successfully used in a wide range of applications, including finance, hydrology or risk management, see [18, 30, 31]. In the bivariate case, any of the most commonly applied copula families, including the Gaussian, Clayton, Gumbel, Frank or t-copula, can be identified as a member of one of the following large (nonparametric) subclasses: the class of Archimedean copulas, the class of extreme-value copulas or the class of elliptical copulas. The latter class also provides flexible parametric families in the higher-dimensional case, while more work is needed to define flexible models involving the former two classes. Multivariate extreme-value copulas typically arise from max-stable process models [12], while flexible copulas involving Archimedean building blocks may be defined based on certain hierarchical constructions [34]. Next to these approaches, vine copulas provide a versatile concept to connect mostly arbitrary bivariate building blocks into flexible multivariate models [1]. More recent approaches in the multivariate case comprise Archimax copulas [10] or non-central squared copulas [32].

While testing the goodness-of-fit of a certain parametric class of copulas has attracted a lot of attention [17, 21, 29], much less work has been devoted to testing whether a copula belongs to any of the large subclasses mentioned above. We refer to [7] for the case of Archimedean copulas, to [5] for tests for extreme-value copulas, while tests for the simplifying assumption in vine copula models can be found in [14]. Within this paper, we are interested in testing for the null hypothesis that a copula is elliptical; a question that is of particular interest in the context of financial risk management [35, 43], but see also [19] for applications in hydrology. Note that ellipticity of a copula is also referred to as meta-ellipticity of the underlying multivariate distribution, see [15] and [2]. For the case of observing i.i.d. data, respective tests have recently been investigated in [26] and [37], both of which exploit the fact that all bivariate margins of a d-variate elliptical copula exhibit equal values for Kendall's tau and Blomqvist's beta. While the former authors work under the unrealistic assumption that marginal distributions are known, the latter author considers suitable rank-based test statistics (see also

Axel Bücher: Heinrich-Heine-University Düsseldorf, E-mail: axel.buecher@hhu.de
Miriam Jaser: Technical University of Munich, E-mail: miriam.jaser@tum.de
***Corresponding Author: Aleksey Min:** Technical University of Munich, E-mail: min@tum.de

[25], where similar tests were worked out independently). Critical values in [37] are then obtained by a certain multiplier bootstrap procedure.

The present paper is motivated by the fact that available observations are often serially dependent time series (in particular in the important context of financial risk management), such that the tests mentioned in the previous paragraph are not valid anymore. We revisit the large sample theory for the respective test statistics, show that the asymptotic distribution is typically different than in the i.i.d. case, and propose a suitable bootstrap approach to calculate valid critical values. The bootstrap scheme relies on subsampling [36], and heavily exploits recent theoretical results in [28] on subsampling empirical copulas. It is important to mention that we believe to also close important gaps in the theoretical results in [37]: while we believe that his results regarding bootstrap validity are correct and provable, the given proofs lack mathematical rigorousness (for instance, in his Appendix A.4, weak limit fields are treated as if they were defined on the same probability space as the original data; moreover, they are partly considered non-random).

The remaining parts of this paper are organized as follows: some mathematical preliminaries on copulas, elliptical distributions and bivariate association measures are collected in Section 2. The test for meta-ellipticity is defined in Section 3, with respective large-sample theory and bootstrap results collected in Section 3.1 and 3.2, respectively. Results from a large-scale Monte Carlo simulation study are presented in Section 4. A case study on financial log returns is worked out in Section 5, while Section 6 briefly concludes. Finally, all proofs are postponed to Appendix A and B.

2 Mathematical preliminaries

Let $\boldsymbol{X} = (X_1, \dots, X_d) \in \mathbb{R}^d$ be a d-dimensional random vector with cumulative distribution function (c.d.f.) F and continuous univariate marginal c.d.f.s $F_1, \dots, F_d$. According to Sklar's theorem [42], there exists a unique copula $C : [0, 1]^d \mapsto [0, 1]$ such that, for all $\boldsymbol{x} \in \mathbb{R}^d$,

$$F(\boldsymbol{x}) = C(F_1(x_1), \dots, F_d(x_d)) .$$

The unique copula C may be written as

$$C(\boldsymbol{u}) = F(F_1^-(u_1), \dots, F_d^-(u_d)), \quad \boldsymbol{u} \in [0, 1]^d ,$$

where F_k^- denotes the generalized inverse of F_k, $k \in \{1, \dots, d\}$.

A copula C is called *elliptical* if it is the copula of some elliptical distribution that is absolutely continuous with respect to the Lebesgue measure. Recall that a random vector $\boldsymbol{Z} \in \mathbb{R}^d$ is said to have an *elliptical distribution* if it admits, for some $\boldsymbol{\mu} \in \mathbb{R}^d$, some $A \in \mathbb{R}^{d\times m}$ with $m \in \mathbb{N}$, and some non-negative random variable $\mathcal{R}$, the decomposition

$$\boldsymbol{Z} = \boldsymbol{\mu} + \mathcal{R}\boldsymbol{A}\mathcal{V} ,$$

where $\mathcal{V}$ is a random vector that is independent of $\mathcal{R}$ and uniformly distributed on the unit sphere in $\mathbb{R}^m$. Note that the distribution of $\boldsymbol{Z}$ is absolutely continuous with respect to the Lebesgue measure iff $\mathcal{R}$ has a Lebesgue density and if $\boldsymbol{\Sigma} = \boldsymbol{A}\boldsymbol{A}^\top$ is positive definite (Theorem 2.9 and the discussion on page 46 in [16]); the corresponding Lebesgue density of $\boldsymbol{Z}$ is then given

$$f_{\boldsymbol{Z}}(\boldsymbol{z}) = |\boldsymbol{\Sigma}|^{-1/2} g\big((\boldsymbol{z} - \boldsymbol{\mu})\boldsymbol{\Sigma}^{-1}(\boldsymbol{z} - \boldsymbol{\mu})\big), \quad \boldsymbol{z} \in \mathbb{R}^d,$$

for some function g that is in one-to-one correspondence with the density of $\mathcal{R}$. As suggested by the above construction, elliptical copulas are typically not available in closed form, two prime examples being the Gaussian and t-copula. Following [15], a distribution on $\mathbb{R}^d$ with continuous marginal c.d.f.s is called *meta-elliptical* if its associated copula is elliptical.

As explained in the next paragraph, elliptical copulas exhibit a remarkable relationship between two well-known pairwise association measures: Kendall's tau and Blomqvist's beta [4, 27]. For $k, \ell \in \{1, \dots, d\}$

distinct and C non necessarily elliptical, the latter are defined as

$$\begin{aligned}\tau_{k\ell} &:= \mathbb{E}[\operatorname{sgn}(X_{k1} - X_{k2})\operatorname{sgn}(X_{\ell 1} - X_{\ell 2})] \\ &= \mathbb{P}((X_{k1} - X_{k2})(X_{\ell 1} - X_{\ell 2}) > 0) - \mathbb{P}((X_{k1} - X_{k2})(X_{\ell 1} - X_{\ell 2}) < 0)\end{aligned}$$

and

$$\begin{aligned}\beta_{k\ell} &:= \mathbb{E}[\operatorname{sgn}(X_k - \tilde{x}_k)\operatorname{sgn}(X_\ell - \tilde{x}_\ell)] \\ &= \mathbb{P}((X_k - \tilde{x}_k)(X_\ell - \tilde{x}_\ell) > 0) - \mathbb{P}((X_k - \tilde{x}_k)(X_\ell - \tilde{x}_\ell) < 0)\,,\end{aligned}$$

where $(X_{k1}, X_{\ell 1})$ and $(X_{k2}, X_{\ell 2})$ are independent copies of (X_k, X_ℓ), where sgn denotes the signum function and where $\tilde{x}_k$ and $\tilde{x}_\ell$ denote the population medians of X_k and X_ℓ, respectively. It is well-known that the two coefficients are completely determined by the (unique) bivariate copula $C_{k\ell}$ of (X_k, X_ℓ), i.e.:

$$\tau_{k\ell} = \tau_{C_{k\ell}} = 4\int_0^1\int_0^1 C_{k\ell}(u_k, u_\ell)\, dC_{k\ell}(u_k, u_\ell) - 1$$

and

$$\beta_{k\ell} = \beta_{C_{k\ell}} = 4C_{k\ell}(0.5, 0.5) - 1\,. \tag{1}$$

Note that $C_{k\ell}$ can be retrieved from C, as $C_{k\ell}(u_k, u_\ell) = C(\boldsymbol{u}^{(k\ell)})$, where, for $\boldsymbol{u} = (u_1, \dots, u_d) \in [0, 1]^d$ and $A \subset \{1, \dots, d\}$, the vector $\boldsymbol{u}^{(A)} \in \mathbb{R}^d$ denotes the vector where all components of $\boldsymbol{u}$ except the components of the index set A are replaced by 1.

As a direct consequence of the definition of an elliptical distribution, all bivariate margins of an elliptical distribution are elliptical as well. As a consequence, the same is true for elliptical copulas. It then follows from Theorem 3.1 in [15] and Proposition 8 in [40] that, for all $k, \ell \in \{1, \dots, d\}$ with $k < \ell$,

$$\tau_{k\ell} = \frac{2}{\pi}\arcsin(\rho_{k\ell}) = \beta_{k\ell}, \tag{2}$$

where $\rho_{k\ell} = \sigma_{k\ell}/\sqrt{\sigma_{kk}\sigma_{\ell\ell}}$. As in [25, 26, 37], the latter will be the basis for the test for ellipticity. It is important to note that there exist non-elliptical copulas for which (2) is met:

Example 2.1. *(i) Consider the bivariate checkerboard copula C with Lebesgue-density*

$$c(u, v) = 4 \cdot (\mathbf{1}_A + \mathbf{1}_B + \mathbf{1}_C + \mathbf{1}_D)(u, v)$$

where $A = [0, 1/4]^2, B = [1/4, 1/2] \times [3/4, 1], C = [1/2, 3/4]^2, D = [3/4, 1] \times [1/4, 1/2]$. *A straightforward calculation shows that* $\tau = \beta = 0$. *The same is true for the copula whose induced law is the uniform distribution on* $\{(u, u) : u \in [0, 1]\} \cup \{(u, 1 - u) : u \in [0, 1]\}$.

(ii) Among the most common bivariate copulas that are non-elliptical are the members from the Gumbel–Hougaard, the Clayton and the Frank copula family (except for some cases at the boundary of the parameter space). In Figure 1, we depict the absolute difference $|\beta - \tau|$ *as a function of* $\tau \in [0, 1]$ *within the respective families. It can be seen that the difference is largest for the Frank copula. Quite remarkably, we have* $\tau = \beta$ *for some non-trivial members from the Clayton and Gumbel family.*

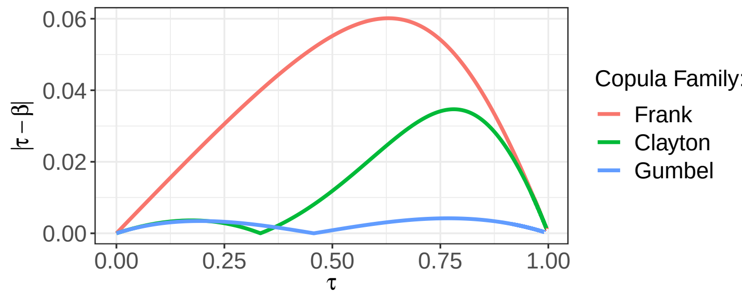

Figure 1: Absolute difference $|\tau - \beta|$ as a function of τ for the Frank, Clayton and Gumbel–Hougaard family.

3 Testing meta-ellipticity

Throughout this section, let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ with $\boldsymbol{X}_i = (X_{1i}, \dots, X_{di}) \in \mathbb{R}^d$ be a stretch of a strictly stationary time series $(\boldsymbol{X}_i)_{i \in \mathbb{Z}}$ of d-dimensional random vectors. The common c.d.f. of $\boldsymbol{X}_i$ is F, which is assumed to have continuous univariate c.d.f.s $F_1, \dots, F_d$, and its copula is denoted by C. We are going to test for the hypotheses

$$H_0 : C \in \mathcal{C}_{\text{elliptical}} \qquad \text{vs.} \qquad H_1 : C \notin \mathcal{C}_{\text{elliptical}},$$

where $\mathcal{C}_{\text{elliptical}}$ denotes the set of all elliptical copulas. A respective test statistic will be defined in Section 3.1. For carrying out the test, we rely on suitable bootstrap approximations, which will be investigated in Section 3.2.

3.1 The test statistic and its asymptotic behavior

By (2), the null hypothesis is equivalent to the fact that $\tau_{k\ell} = \beta_{k\ell}$ for all $k, \ell \in \{1, \dots, d\}$ with $k < \ell$. For detecting departures from ellipticity, it hence makes sense to investigate the difference between empirical counterparts of the two coefficients. It is important to note that, by construction, the test's ability to detect departures from meta-ellipticity is limited by (1) the fact that it is completely based on investigating bivariate margins, and by (2) the fact that the difference between Kendall's tau and Blomquist's beta may be small even for non-elliptical copulas (see Example 2.1).

The classical sample version of Kendall's tau is defined as

$$\widehat{\tau}_{k\ell,n} = \frac{2}{n(n-1)} \sum_{1 \le i < j \le n} \operatorname{sgn}(X_{ki} - X_{kj}) \operatorname{sgn}(X_{\ell i} - X_{\ell j})\,.$$

Obviously, $\widehat{\tau}_{k\ell,n}$ is unbiased in case the underlying sample is serially independent. Under this assumption, large sample theory dates back to [24] and can be found in classical monographs such as [45], Section 12. In the case of serial dependence, large-sample theory may for instance be deduced from simple multivariate extensions of the results in [13], see also Proposition 2.3 in [6].

Next, a suitable sample version of Blomqvist's beta motivated by (1) is given by

$$\widehat{\beta}_{k\ell,n} = 4\widehat{C}_{k\ell,n}(\tfrac{1}{2}, \tfrac{1}{2}) - 1 = \frac{4}{n} \sum_{i=1}^{n} \mathbf{1}(\widehat{U}_{ki} \le \tfrac{1}{2}, \widehat{U}_{\ell i} \le \tfrac{1}{2}) - 1,$$

where $\widehat{U}_{ki} = (n+1)^{-1} \operatorname{rank}(X_{ki} \text{ among } X_{k1}, \dots, X_{kn})$ and where, for $u_k, u_\ell \in [0,1]^2$,

$$\widehat{C}_{k\ell,n}(u_k, u_\ell) = \frac{1}{n} \sum_{i=1}^{n} (\widehat{U}_{ki} \le u_k, \widehat{U}_{k\ell} \le u_\ell) \tag{3}$$

denotes the empirical copula. Note that, in the case of serial independence, $\widehat{\beta}_{k\ell,n}$ is in fact an asymptotically equivalent version of the estimator initially proposed in [4], see [23]. Large sample theory in the case of serial dependence is an immediate consequence of the results in [9].

For the definition of suitable test statistics for H_0, let $\widehat{\boldsymbol{\beta}}_n = (\widehat{\beta}_{12,n}, \widehat{\beta}_{13,n}, \ldots, \widehat{\beta}_{d-1\,d,n})^\top$ and $\widehat{\boldsymbol{\tau}}_n = (\widehat{\tau}_{12,n}, \widehat{\tau}_{13,n}, \ldots, \widehat{\tau}_{d-1\,d,n})^\top$ denote vectors in $\mathbb{R}^{d(d-1)/2}$ obtained by concatenating all pairwise estimators. Moreover, let

$$\widehat{\boldsymbol{D}}_n = \widehat{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\tau}}_n\,. \tag{4}$$

We will next introduce three suitable conditions that will be sufficient to deduce asymptotic normality of $\widehat{\boldsymbol{D}}_n$ under H_0.

The first condition concerns the serial dependence of the time series, and is taken from [9]. Define unobservable observations $U_{ki} = F_k(X_{ki})$ for $k \in \{1, \ldots, d\}$ and $i \in \{1, \ldots, n\}$ and let

$$C_n(\boldsymbol{u}) = \frac{1}{n}\sum_{i=1}^{n} \mathbf{1}\{U_{1i} \le u_1, \ldots, U_{di} \le u_d\}, \qquad \boldsymbol{u} = (u_1, \ldots, u_d)^\top \in [0,1]^d\,. \tag{5}$$

Moreover, let $\ell^\infty([0,1]^d)$ denote the set of all bounded, real-valued functions on $[0,1]^d$ and let $\mathcal{C}([0,1]^d)$ denote the subset of continuous functions, both equipped with the supremum metric. Weak convergence in $\ell^\infty([0,1]^d)$ is to be understood in the sense of [44] and denoted by '$\rightsquigarrow$'.

Condition 3.1. *The empirical process $\alpha_n = \sqrt{n}(C_n - C)$ converges weakly towards a tight, centered Gaussian field $\mathbb{B}_C$ concentrated on $\mathbb{D}_0$, that is*

$$\alpha_n = \sqrt{n}(C_n - C) \rightsquigarrow \mathbb{B}_C \quad \textit{in } \ell^\infty([0,1]^d)\,,$$

where $\mathbb{D}_0$ is given by

$$\mathbb{D}_0 = \left\{\alpha \in C([0,1]^d) \mid \alpha(1, \ldots, 1) = 0 \textit{ and } \alpha(\boldsymbol{u}) = 0 \textit{ if some of the components of } \boldsymbol{u} \textit{ are equal to } 0\right\}.$$

The condition is trivially satisfied in the i.i.d. case, in which case the limit is a standard C-brownian bridge on $[0,1]^d$. As outlined in [9], it is also met for the majority of the most common stationary time series models like ARMA and GARCH processes or, more generally, for strongly mixing processes with mixing coefficients

$$\alpha(h) := \sup\left\{|P(A \cap B) - P(A)P(B)| : A \in \sigma(\ldots, \boldsymbol{X}_{-1}, \boldsymbol{X}_0), B \in \sigma(\boldsymbol{X}_h, \boldsymbol{X}_{h+1}, \ldots)\right\} \tag{6}$$

of the order $\alpha(h) = O(h^{-a})$ for some $a > 1$. The covariance kernel of $\mathbb{B}_C$ is then given by

$$\mathrm{Cov}(\mathbb{B}_C(\boldsymbol{u}), \mathbb{B}_C(\boldsymbol{v})) = \sum_{h \in \mathbb{Z}} \mathrm{Cov}(\mathbf{1}(\boldsymbol{U}_0 \le \boldsymbol{u}), \mathbf{1}(\boldsymbol{U}_h \le \boldsymbol{v})).$$

Finally, note that there exists an abundance of tests for hypotheses like stationarity, serial independence, or the goodness-of-fit of a specific time series model; all of which may provide empirical evidence for the circumstance that Condition 3.1 is met.

The second condition is essentially a further condition on the serial dependence, as it is trivially met for i.i.d. data. It is, however, not met in general for time series, even for continuous stationary c.d.f.s: consider for instance a random repetition process, where, at time t, the previous observation is repeated with positive probability p or a new observation is generated independently with probability $1 - p$.

Condition 3.2. *For any $k \in \{1, \ldots, d\}$, the kth component sample $X_{k1}, \ldots, X_{kn}$ does not contain any ties with probability one.*

The third condition concerns the regularity of C, and is taken from [41]. It is non-restrictive in the sense that it is necessary for weak convergence of the empirical copula process with respect to the supremum distance to a limit with a.s. continuous sample paths.

Condition 3.3. *For any $k \in \{1, \ldots, d\}$, the first-order partial derivatives $\partial_k C(\boldsymbol{u})$ exist and are continuous on the set $U_k = \{\boldsymbol{u} \in [0,1]^d : u_k \in (0,1)\}$.*

It is worthwhile to mention that, if one were only interested in weak convergence of $\widehat{\boldsymbol{\beta}}_n$, then it would be sufficient to assume existence and continuity of the partial derivatives in a neighbourhood of the points $\{\boldsymbol{u} \in [0,1]^d : \sum_k \mathbf{1}(u_k = 1/2) = 2, \sum_k \mathbf{1}(u_k = 1) = d-2\}$ only (this follows from a straightforward modification of the arguments in [9], see also Lemma B.1 for the case of ellipticity). However, under such an assumption only, proving our bootstrap consistency results in Section 3.2 would require substantial additional effort.

Finally, recall the empirical copula $\widehat{C}_n$ defined in (3) and let $\mathbb{C}_n = \sqrt{n}(\widehat{C}_n - C)$ denote the empirical copula process. As shown in [9] we have, under the previous conditions,

$$\mathbb{C}_n = \sqrt{n}(\widehat{C}_n - C) \rightsquigarrow \mathbb{G}_C$$

in $\ell^\infty([0,1]^d)$, where the limiting Gaussian field $\mathbb{G}_C$ is defined, for all $\boldsymbol{u} \in [0,1]^d$, by

$$\mathbb{G}_C(\boldsymbol{u}) = \mathbb{B}_C(\boldsymbol{u}) - \sum_{k=1}^{d} \partial_k C(\boldsymbol{u}) \mathbb{B}_C(\boldsymbol{u}^{(k)})$$

with $\boldsymbol{u}^{(k)} = (1, \dots, 1, u_k, 1, \dots, 1)$. The following theorem is one of the main theoretical results of this paper.

Theorem 3.4. *Let $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ be a stretch of a strictly stationary time series $(\boldsymbol{X}_i)_{i\in\mathbb{Z}}$ of d-dimensional random vectors with common c.d.f F, continuous univariate marginal c.d.f.s $F_1, \dots, F_d$ and copula C. If Conditions 3.1, 3.2 and 3.3 are met, then, for all $(k,\ell) \in B_{d,2} = \{(k,\ell) \in \{1,\dots,d\} : k < \ell\}$ and as $n \to \infty$,*

$$\sqrt{n}(\widehat{\beta}_{k\ell,n} - \beta_{k\ell}) = 4 \cdot \mathbb{C}_{k\ell,n}(1/2, 1/2), \tag{7}$$

$$\sqrt{n}(\widehat{\tau}_{k\ell,n} - \tau_{k\ell}) = 8 \int \mathbb{C}_{k\ell,n}(u,v)\, dC_{k\ell}(u,v) + o_P(1), \tag{8}$$

where $\mathbb{C}_{k\ell,n}(u,v) = \mathbb{C}(1,\dots,1,u,1\dots,1,v,1,\dots,1)$ with u and v at the kth and ℓth position, respectively. As a consequence, under the null hypothesis of ellipticity, we have

$$\sqrt{n}\,\widehat{\boldsymbol{D}}_n \rightsquigarrow \boldsymbol{Z} \sim N_{d(d-1)/2}(0, \boldsymbol{\Sigma}), \tag{9}$$

where $\boldsymbol{Z} = (Z_{k\ell})_{(k,\ell)\in B_{d,2}}$ with

$$Z_{k\ell} := \Psi_{k\ell}(\mathbb{G}_C) := 4 \cdot \mathbb{G}_{k\ell,C}(1/2, 1/2) - 8 \int \mathbb{G}_{k\ell,C}(u,v)\, dC_{k\ell}(u,v). \tag{10}$$

and where $\boldsymbol{\Sigma} = (\Sigma_{(k\ell),(k'\ell')})_{(k,\ell),(k',\ell')\in B_{d,2}}$ with $\Sigma_{(k\ell),(k'\ell')} = \mathrm{Cov}(Z_{k\ell}, Z_{k'\ell'})$.

Remark 3.5. *Under slightly more restrictive mixing conditions (see, e.g., [13]) and less restrictive conditions on C, it can be shown that the limiting covariance may alternatively be written as*

$$\Sigma_{(k\ell),(k'\ell')} = \sum_{h\in\mathbb{Z}} \mathrm{Cov}\left\{(h^{(\beta)} - h^{(\tau)}_{k\ell})(U_{kh}, U_{\ell h}), (h^{(\beta)} - h^{(\tau)}_{k\ell})(U_{k'h}, U_{\ell' h})\right\}, \tag{11}$$

where, for $u, v \in [0,1]$,

$$h^{(\beta)}(u,v) = 4 \cdot \mathbf{1}(u \le \tfrac{1}{2}, v \le \tfrac{1}{2}) - 2 \cdot \mathbf{1}(u \le \tfrac{1}{2}) - 2 \cdot \mathbf{1}(v \le \tfrac{1}{2}) \tag{12}$$

$$h^{(\tau)}_{k\ell}(u,v) = 8 C_{k\ell}(u,v) - 4u - 4v + 2. \tag{13}$$

A sketch-proof relying on U-statistic theory for strongly mixing observations is given in Section B.

For testing meta-ellipticity, one may use various real-valued functionals of $\widehat{\boldsymbol{D}}_n$ defined in (4). Throughout this paper, we opt for the L_2-type test statistic

$$\widehat{T}_n := n \cdot \widehat{\boldsymbol{D}}_n^\top \widehat{\boldsymbol{D}}_n\,. \tag{14}$$

In the i.i.d. case, related Wald-type statistics have been found to provide worse accuracy, see [26] and [37]. The latter may be explained by the fact that Wald-type statistics involve an estimator for an inverse covariance matrix of possibly small signals. Likewise L_1- or L_∞-type test statistics have been found to be of comparable quality to the L_2-statistic, see [37].

Now, Theorem 3.4 and the Continuous Mapping Theorem (see Theorem 1.3.6 in [44]) immediately yield

$$\widehat{T}_n \rightsquigarrow T := \boldsymbol{Z}^\top \boldsymbol{Z}\,.$$

The limiting variable can be written as a weighted sum of independent chi-square variables with one degree of freedom, where the weights depend in a complicated, statistically intractable way on the copula C and the serial dependence of the time series. For that purpose, we will introduce a suitable bootstrap scheme in the next section.

Remark 3.6. *The proposed tests can straightforwardly be adapted to the situation where one is only interested in testing whether some of the bivariate margins are elliptical.*

3.2 A subsampling procedure

Among the abundance of bootstrap procedures, the subsampling approach [36] has recently attracted attention when working with empirical copulas for a number of practical reasons, see [28]. First of all, in comparison to bootstrap schemes that are based on resampling with replacement, the approach does not artificially introduce ties into the bootstrap samples, thereby avoiding what might be called a 'tie-bias'. Next, in comparison to various versions of the multiplier bootstrap [6, 38], subsampling does not require expensive case-by-case implementation of the bootstrap approximation (see also [37] for a multiplier bootstrap for testing ellipticity). Finally, the subsampling approach may easily be modified in such a way that it is valid for time series data.

Following [28], we define two different subsampling schemes. The first one is only valid in the i.i.d. case, while the latter may be applied to a general stationary time series (including the i.i.d. case). In the former case, let $N_{b,n}^{(\mathrm{iid})} = \binom{n}{b}$ denote the number of subsamples of size b that may be taken from $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ and denote the subsamples by

$$\mathcal{X}_b^{[m]} = (\boldsymbol{X}_1^{[m]}, \dots, \boldsymbol{X}_b^{[m]}), \quad m \in \{1, \dots, N_{b,n}^{(\mathrm{iid})}\}.$$

Under the plain assumption of observing a strictly stationary time series, let $N_{b,n}^{(\mathrm{ts})} = n - b + 1$ denote the number of possible subsamples that consist of b successive observations, and denote them by

$$\mathcal{X}_b^{[m]} = (\boldsymbol{X}_1^{[m]}, \dots, \boldsymbol{X}_b^{[m]}) = (\boldsymbol{X}_m, \dots, \boldsymbol{X}_{m+b-1}), \quad m \in \{1, \dots, N_{b,n}^{(\mathrm{ts})}\}.$$

Algorithm 3.7. *For a given sample $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ of size n:*

1. *Compute the statistic $\widehat{T}_n$ from* (14).
2. *Choose a number $S \in \mathbb{N}$ of bootstrap replicates and a subsampling size $b \in \{1, \dots, n\}$ such that $S \le N_{b,n}$, where $N_{b,n} = N_{b,n}^{(\mathrm{iid})}$ if the sample is (believed to be) i.i.d. and $N_{b,n} = N_{b,n}^{(\mathrm{ts})}$ if the sample is (believed to be) a stationary time series that is not i.i.d.*
3. *For $s \in \{1, \dots, S\}$:*
   - *Randomly select a subsample $\mathcal{X}_b^{[I_{s,n}]} = (\boldsymbol{X}_1^{[I_{s,n}]}, \dots, \boldsymbol{X}_b^{[I_{s,n}]})$ of size b by drawing $I_{s,n}$ randomly from $\{1, \dots, N_{b,n}\}$.*
   - *Compute the statistic $\widehat{\boldsymbol{\beta}}_{b,n}^{[I_{s,n}]}$ and $\widehat{\boldsymbol{\tau}}_{b,n}^{[I_{s,n}]}$ from the subsample.*
   - *Compute the bootstrap statistic*

   $$\widehat{T}_{b,n}^{[I_{s,n}]} = (1 - b/n)^{-1} b (\widehat{\boldsymbol{\beta}}_{b,n}^{[I_{s,n}]} - \widehat{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\tau}}_{b,n}^{[I_{s,n}]} - \widehat{\boldsymbol{\tau}}_n)^\top (\widehat{\boldsymbol{\beta}}_{b,n}^{[I_{s,n}]} - \widehat{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\tau}}_b^{[I_{s,n}]} - \widehat{\boldsymbol{\tau}}_n).$$

4. *An approximate p-value for the test based on $\widehat{T}_n$ is then given by*

   $$\widehat{p}_{S,b,n} = \frac{1}{S} \sum_{s=1}^{S} I\{\widehat{T}_{b,n}^{[I_{s,n}]} > \widehat{T}_n\}\,.$$

The following result concerning the validity of the subsampling procedure is the second main theorem.

Theorem 3.8. *Suppose that $\boldsymbol{X}_1, \dots, \boldsymbol{X}_n$ is either i.i.d. or an excerpt from a strongly mixing stationary time series with mixing coefficient $\alpha(h) = O(h^{-a})$ for some $a > 0$, as $h \to \infty$ (see (6) for the definition of $\alpha(h)$ and note that, as a consequence, Condition 3.1 is met). Further, assume that Conditions 3.2 and 3.3 are met. If $b = b_n \to \infty$, $b = o(n)$ and $S = S_n \to \infty$ as $n \to \infty$, then*

$$\widehat{p}_{S,b,n} \xrightarrow{d} \begin{cases} \text{Uniform}([0,1]) & , \text{ if } \beta_{k\ell} = \tau_{k\ell} \text{ for all } (k,\ell) \in B_{d,2}, \\ 0 & , \text{ if } \beta_{k\ell} \neq \tau_{k\ell} \text{ for some } (k,\ell) \in B_{d,2} \end{cases}$$

as $n \to \infty$. In particular, for $\alpha \in (0,1)$, the test $\varphi_{S,b,n} = \mathbf{1}(\widehat{p}_{S,b,n} \le \alpha)$ is an asymptotic level α for H_0 which is consistent against all alternatives with $\beta_{k\ell} \neq \tau_{k\ell}$ for some $(k,\ell) \in B_{d,2}$.

It is important to note that, in view of Example 2.1, test $\varphi_{S,b,n}$ is not consistent against any non-elliptical copula. In practice, it is therefore advisable to complement the above test by suitable nonparametric tests involving other important qualitative features of bivariate elliptical copulas, such as symmetry or radial symmetry (see [22] and [20], respectively, for the i.i.d. case).

4 Simulation study

The finite-sample performance of the proposed test for meta-ellipticity was investigated in a large-scale Monte Carlo simulation study. The study was designed to primarily illustrate the test's level and power properties for varying (1) sample size, (2) block length parameter, (3) dimension, (4) strength of the serial dependence, and (5) strength of the cross-sectional dependence. We also illustrate that an application of the related test from [37], which is designed for i.i.d. data, can fail in case of serial dependence.

4.1 Setup

The aforementioned goals were tackled by considering four different copula families (Gaussian and t_5 for H_0, Clayton and Frank for H_1), three different dimensions $d \in \{2, 3, 6\}$, five different levels of serial dependence, and five different levels of cross-sectional dependence. With respect to the cross-sectional dependence, the respective copula parameters were chosen in such a way that all bivariate margins exhibit the same Kendall's τ, taken from the set $\{0.1, 0.25, 0.5, 0.75, 0.9\}$.

With respect to the serial dependence, we opted for the following transformation of a classical Gaussian AR(1)-model. First, starting from d independent AR(1)-models

$$Y_{k,i} = \varphi Y_{k,i-1} + \varepsilon_{k,i}, \qquad (k \in \{1,\dots,d\}, i \in \{1,\dots,n\}),$$

with $\varepsilon_{k,i}$ i.i.d. $N(0,1)$ and $Y_{k,0}$ i.i.d. $N\big(0, 1/(1-\varphi^2)\big)$, whose stationary distribution is well-known to be $N\big(0, 1/(1-\varphi^2)\big)$, we may construct random vectors $\boldsymbol{V}_i = (V_{1,i}, \dots, V_{d,i})$ with independent standard uniformly distributed coordinates by setting $V_{k,i} = \Phi\big((1-\varphi^2)^{1/2} Y_{k,i}\big)$; Φ the c.d.f. of the standard normal distribution. Next, for some given copula C as specified above, the vectors $\boldsymbol{V}_i$ may be transformed to (serially dependent) observations $\boldsymbol{U}_i$ from C by applying the inverse Rosenblatt transformation [39]. Overall, the serial dependence is controlled by a single parameter φ, which was chosen in such a way that the lag 1 auto-correlation version of Kendall's tau of $(Y_{k,i})_i$ varies in the set $\{0, 0.2, 0.4, 0.6, 0.8\}$.

Finally, the sample size n was chosen to vary in $\{100, 250, 500, 1000\}$, while the block length parameter was chosen to vary in $\{0.05n, 0.1n, \dots, 0.6n\}$. The number of Monte replications was set to N=1000, the number of subsampling replications to S=300, and all tests were performed at a significance level of $\alpha = 0.05$.

Since the plain subsampling approach described in Algorithm 3.7 suffers from the fact that observations at the start and at the end of the observation period have a reduced chance of appearing in a randomly selected block of size b, we applied the following slight modification: instead of drawing (in step 3) from the blocks starting at observation $\boldsymbol{X}_i$ with $i \in \{1, \dots, n-b+1\}$ only, we also allow to subsample a block starting at $\boldsymbol{X}_{n-i}$ with $i \in \{0, \dots, b-2\}$, with the respective block being defined as $(\boldsymbol{X}_{n-i}, \dots, \boldsymbol{X}_n, \boldsymbol{X}_1, \dots, \boldsymbol{X}_{b-i+1})$ (which is similar in spirit to the circular bootstrap). Since $b = o(n)$, this modification does not make a difference asymptotically, but we observed increased accuracy for finite samples. Finally, for $S = 300 \geq n$, there are only n blocks to draw from, whence we did draw each block exactly once, instead of S times with replacement.

In terms of computing time, we remark that the subsampling approach with a single fixed subsampling size b is advantageous over the multiplier bootstrap from [37], as calculating each bootstrap statistic relies on only $b = o(n)$ observations, compared to n observations for the latter. Within a small experiment with $b = \lfloor n^{0.95}/4 \rfloor$, we found that the relative computing time 'multiplier/subsampling' ranges from 2.44 ($d = 2, n = 100$) up to 61 ($d = 6, n = 1000$). As a consequence, even evaluating the subsampling approach for various block sizes from a grid does not necessarily make it computationally heavier than the multiplier method.

4.2 Empirical level and empirical power results

In this section, we partially report the results from the simulation study, after thoroughly weighing completeness against brevity.

First of all, Figure 2 shows empirical rejection probabilities for samples from the Gaussian model and the Frank model in dimension d=3 for all chosen sample sizes, block sizes, serial dependencies and cross-sectional dependencies as described in the previous section. Little dots at the left-hand side of each plot refer to the empirical rejection probability of the test from [37] (which is designed for the iid case only). The triangles at the right-hand side will be explained below.

In terms of level approximation (upper panel), we see that our test does not show a huge dependence on the choice of the block length in most cases. Moreover, it is slightly conservative in many cases, in particular for small sample sizes and large block sizes. For high levels of serial dependence, the test becomes liberal for small block sizes. In comparison, Quessy's test does not hold its level for moderate to high levels of serial dependence. Similar results were obtained for dimensions $d \in \{2, 6\}$ and for the t-copula.

In terms of power (lower panel), we observe very little power for sample size $n = 100$, but a great increase with increasing sample size. Moreover, large block sizes tend to reduce the test's ability to detect the alternative. The latter may be explained by continuity reasons, observing that in the extreme case, $n = b$, we have $N_{b,n}^{(\mathrm{ts})} = 1$, which amounts to no power at all. Qualitatively similar results were obtained for dimensions $d \in \{2, 6\}$ and for the Clayton copula.

Based on the results summarized in Figure 2, as well as on those that were not reported for the sake of brevity, we propose to use $b = \lfloor n^{0.95}/4 \rfloor$ as a general formula for choosing the subsample size (which is in accordance with the assumptions from Theorem 3.8). The empirical rejection probabilities for this block size are displayed as triangles on the right-hand side of each plot in Figure 2. It can be seen that the choice guarantees that the level is not exceeded, while at the same time reaching decent power properties.

Next, Tables 1, 2, and 3 report empirical rejection probabilities of our test for fixed subsample size $b = \lfloor n^{0.95}/4 \rfloor$, and for the test from [37] based on a multiplier bootstrap (given in parentheses) for dimension $d = 2, 3$, and 6, respectively. For the sake of brevity, we excluded the case $\tau = \tau(\text{cross}) \in \{0.1, 0.9\}$ (for which the results were qualitatively the same) and the sample size $n = 100$ (for which little to no power was visible). The general findings are similar as for Figure 2: while our test keeps its level, Quessy's test fails to do so for moderate to high level of serial dependence. There are almost no differences between the t-copula and the Gaussian copula models. In terms of detecting alternatives, the Frank copula exhibits much larger rejection probabilities. Moreover, the rejection probabilities are largest for high levels of cross-sectional dependence and small levels of temporal dependence. Finally, note that the Clayton copula may more easily be detected to be non-elliptical by applying a suitable test for radial symmetry (adapted to the time series case).

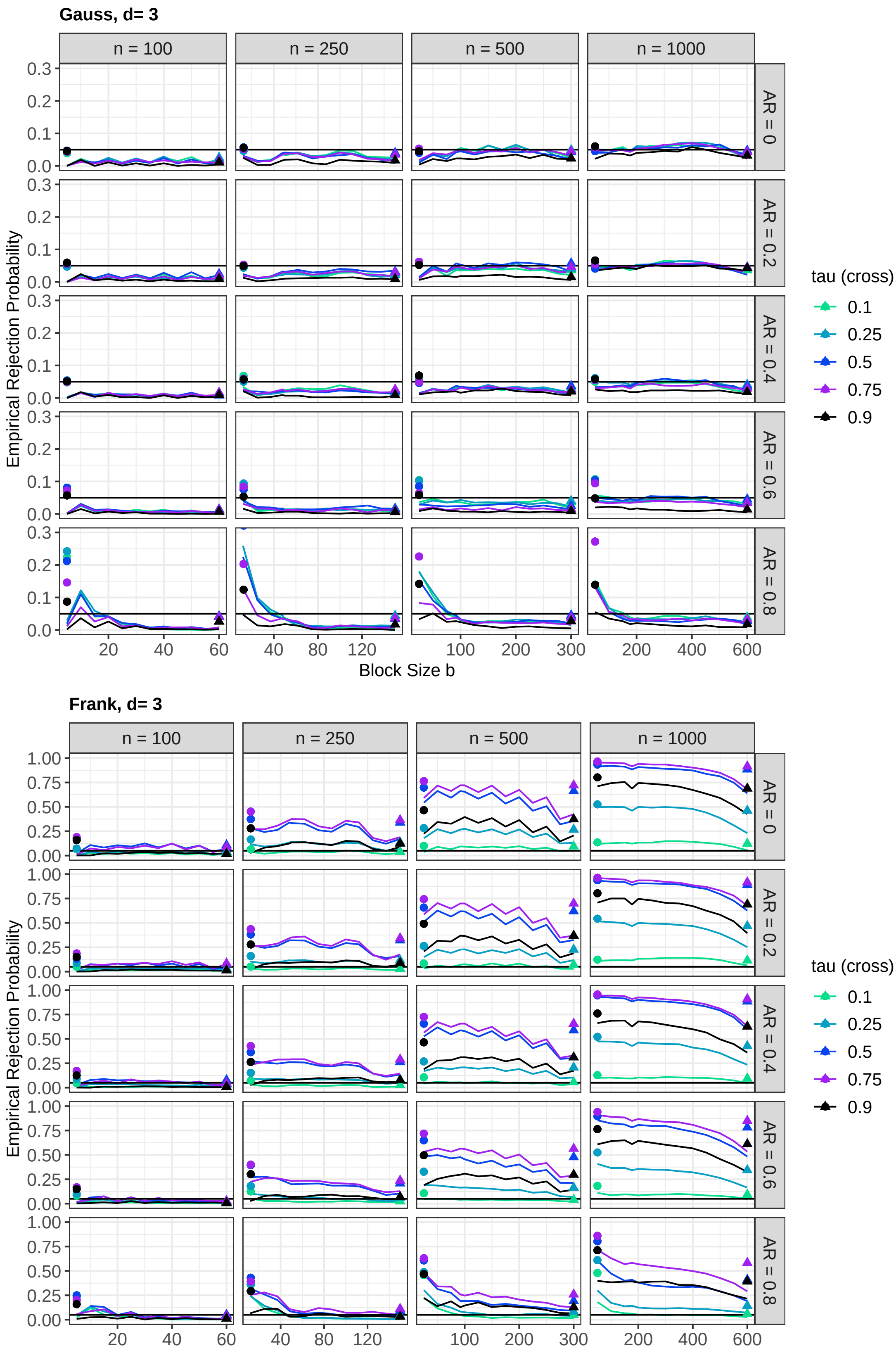

Figure 2: Empirical rejection probabilities for the Gaussian model (upper part) and Frank model (lower part) in dimension d=3 against the block size b for $n \in \{100, 250, 500, 1000\}$. AR $=$ c refers to the fact that the marginal AR models were chosen in such a way that the lag 1 auto-Kendall-rankcorrelation equals $c \in \{0, .2, .4, .6, .8\}$. Likewise, τ(cross) $\in \{0.1, 0.25, 0.5, 0.75, 0.9\}$ specifies the (pairwise) cross sectional dependencies in terms of Kendall's tau. Empirical rejection probabilities for the test from [37] and for our test with fixed block size $b = \lfloor n^{0.95}/4 \rfloor$ are depicted as points on the left side and triangles on the right side, respectively.

Table 1: Dimension $d = 2$: Empirical level (Panel A) and empirical power (Panel B) of our test for ellipticity based on the subsampling procedure with block size $b = \lfloor n^{0.95}/4 \rfloor$ and the test from [37] based on a multiplier bootstrap (in parentheses) with significance level $\alpha = 0.05$. The model parameters are as in Figure 2.

AR	τ	$n = 250$	$n = 500$	$n = 1000$	$n = 250$	$n = 500$	$n = 1000$
Panel A:			*Gaussian*			t_5	
0	0.25	4.5 (5.6)	5.0 (5.1)	7.1 (5.5)	4.0 (5.5)	5.8 (5.8)	6.4 (5.5)
	0.50	4.0 (6.2)	5.2 (5.0)	5.8 (6.0)	3.7 (5.7)	4.9 (4.3)	5.9 (4.4)
	0.75	2.4 (4.8)	4.6 (4.8)	4.9 (5.2)	3.4 (4.5)	6.0 (6.2)	5.8 (6.1)
0.2	0.25	4.4 (5.1)	5.5 (6.3)	6.2 (5.5)	3.4 (5.7)	6.3 (6.3)	5.0 (5.0)
	0.50	3.9 (6.7)	5.5 (6.1)	4.3 (4.1)	4.7 (6.2)	4.6 (4.6)	4.5 (4.6)
	0.75	2.5 (4.5)	4.1 (5.7)	4.8 (5.3)	2.1 (5.8)	4.6 (5.8)	5.1 (5.5)
0.4	0.25	3.5 (6.2)	4.8 (6.7)	5.2 (6.5)	2.4 (4.2)	3.0 (5.1)	4.3 (5.3)
	0.50	3.4 (5.2)	4.3 (5.9)	3.9 (4.8)	2.6 (5.4)	3.8 (6.2)	4.1 (5.3)
	0.75	1.5 (5.4)	2.6 (5.8)	3.6 (5.0)	1.6 (6.2)	3.5 (5.6)	5.5 (7.3)
0.6	0.25	1.8 (7.8)	3.6 (7.8)	3.9 (6.9)	1.7 (7.8)	3.4 (8.8)	4.8 (8.4)
	0.50	1.6 (6.3)	3.3 (9.3)	3.7 (7.8)	1.9 (8.3)	4.0 (8.6)	4.8 (8.9)
	0.75	1.5 (9.4)	2.6 (8.0)	3.5 (7.1)	1.4 (6.9)	3.0 (6.7)	4.3 (8.0)
0.8	0.25	2.5 (22.3)	2.7 (23.8)	4.0 (26.5)	3.0 (20.0)	4.5 (25.1)	4.7 (27.7)
	0.50	3.3 (23.1)	4.9 (24.3)	4.5 (25.6)	3.1 (18.5)	2.6 (23.0)	4.4 (23.7)
	0.75	2.6 (15.7)	4.3 (20.0)	3.6 (19.7)	3.3 (17.0)	3.0 (18.5)	2.8 (19.7)
Panel B:			*Frank*			*Clayton*	
0	0.25	9.4 (11.0)	18.9 (18.5)	28.9 (29.9)	4.6 (5.3)	5.5 (5.4)	6.8 (4.6)
	0.50	25.4 (28.5)	43.0 (44.8)	71.9 (78.4)	4.6 (5.7)	7.0 (7.1)	9.9 (8.0)
	0.75	25.0 (33.2)	54.2 (56.9)	80.0 (86.3)	10.5 (14.9)	25.1 (25.6)	45.9 (46.2)
0.2	0.25	8.5 (12.7)	18.9 (18.7)	26.1 (29.1)	3.5 (5.0)	4.9 (5.2)	5.5 (5.1)
	0.50	19.7 (25.0)	46.0 (47.1)	67.7 (72.9)	4.3 (6.0)	6.0 (5.4)	8.9 (7.5)
	0.75	24.6 (32.9)	55.0 (57.5)	77.0 (84.1)	11.0 (17.9)	23.7 (27.4)	43.2 (45.0)
0.4	0.25	6.4 (10.9)	14.7 (18.6)	27.8 (31.2)	3.2 (5.1)	4.5 (5.7)	5.6 (6.4)
	0.50	20.3 (24.9)	39.8 (45.2)	65.8 (73.0)	3.3 (5.5)	5.2 (5.5)	9.7 (8.8)
	0.75	20.8 (34.9)	51.2 (59.1)	78.5 (86.3)	7.6 (13.8)	19.8 (23.1)	44.4 (45.8)
0.6	0.25	4.8 (14.1)	12.0 (20.9)	21.3 (30.1)	3.1 (8.3)	3.1 (7.6)	4.5 (7.1)
	0.50	12.4 (23.8)	33.4 (44.9)	58.1 (70.2)	2.1 (7.4)	4.9 (9.5)	8.9 (11.0)
	0.75	15.4 (31.1)	42.9 (56.2)	69.3 (81.0)	4.6 (15.3)	15.7 (25.7)	35.9 (45.1)
0.8	0.25	4.7 (24.8)	6.5 (29.3)	13.4 (39.7)	3.8 (22.8)	3.8 (21.5)	3.1 (25.3)
	0.50	5.8 (28.2)	13.1 (40.8)	31.7 (61.7)	3.1 (21.0)	3.5 (22.2)	5.1 (21.5)
	0.75	7.4 (32.0)	18.8 (45.2)	44.7 (71.6)	3.6 (22.1)	7.7 (29.3)	17.0 (39.9)

5 Case Study

Elliptical copulas are popular for financial data due to their tractability, their flexibility and in particular their ability to capture the dependence of extreme events. However, recent studies also report an observed non-(meta-)ellipticity of stock returns, see, e.g., [11], [26]. Within this section, we illustrate our method with a case study that allows to further enlighten this topic.

More precisely, we reconsider the case study from Section 5 in [26], consisting of a three dimensional data set of 2663 daily log returns of the DAX, the Dow Jones Industrial Average and the Euro Stoxx 50 indices for 11 years in the period January 1, 2006 till December 31, 2016. Unlike in the empirical analysis in that paper, we do not necessarily need to apply ARMA-GARCH-filters to obtain approximately independent observations,

Table 2: Analogue of Table 1 in dimension $d = 3$.

AR	τ	$n = 250$	$n = 500$	$n = 1000$	$n = 250$	$n = 500$	$n = 1000$
Panel A:			*Gaussian*			t_5	
0	0.25	3.6 (4.6)	4.7 (4.7)	4.5 (5.6)	2.4 (3.6)	5.0 (5.3)	4.0 (3.6)
	0.50	4.0 (5.7)	4.2 (4.0)	4.7 (4.5)	3.6 (4.8)	5.4 (5.9)	5.0 (5.3)
	0.75	3.5 (5.2)	4.5 (5.3)	4.3 (5.7)	2.9 (4.9)	4.9 (6.3)	4.6 (4.7)
0.2	0.25	2.1 (4.4)	4.1 (5.6)	4.4 (5.6)	2.0 (4.2)	4.3 (5.6)	3.7 (4.9)
	0.50	3.2 (4.9)	5.7 (6.2)	4.5 (4.1)	2.5 (5.0)	3.2 (4.2)	4.3 (5.4)
	0.75	2.9 (5.3)	4.7 (6.1)	4.4 (5.4)	2.0 (4.7)	2.8 (4.7)	3.9 (4.0)
0.4	0.25	1.7 (5.0)	2.8 (5.7)	4.1 (6.1)	2.4 (5.5)	4.3 (6.1)	3.5 (5.3)
	0.50	2.1 (5.6)	3.7 (4.6)	3.6 (5.5)	2.6 (6.3)	2.8 (5.1)	3.4 (6.2)
	0.75	2.6 (5.9)	2.5 (4.9)	2.9 (5.6)	1.7 (4.3)	3.0 (6.2)	2.4 (5.1)
0.6	0.25	1.7 (9.4)	4.0 (10.2)	3.9 (10.2)	1.8 (9.1)	2.8 (10.1)	3.7 (11.1)
	0.50	1.5 (7.5)	2.4 (8.5)	4.5 (10.4)	2.1 (9.4)	1.6 (8.0)	3.5 (9.2)
	0.75	0.9 (8.4)	1.5 (6.4)	3.4 (9.4)	1.4 (6.4)	2.2 (7.2)	3.2 (7.3)
0.8	0.25	4.4 (36.3)	4.3 (39.9)	2.7 (44.5)	3.2 (33.0)	3.4 (41.0)	3.9 (43.2)
	0.50	3.4 (32.1)	4.5 (35.5)	2.8 (37.2)	4.2 (29.9)	3.0 (36.4)	4.3 (38.9)
	0.75	3.6 (20.3)	3.8 (22.6)	3.3 (27.2)	3.9 (23.5)	3.7 (24.6)	2.9 (26.1)
Panel B:			*Frank*			*Clayton*	
0	0.25	13.4 (16.6)	26.7 (28.3)	46.1 (52.6)	3.2 (4.1)	5.6 (5.0)	5.0 (5.3)
	0.50	33.9 (37.4)	66.2 (69.8)	88.4 (93.3)	4.5 (5.1)	8.8 (7.8)	11.0 (9.0)
	0.75	36.4 (45.2)	72.1 (76.4)	91.5 (96.4)	14.6 (17.1)	33.9 (33.2)	60.7 (62.0)
0.2	0.25	11.6 (16.0)	22.6 (26.3)	46.7 (54.3)	2.3 (3.8)	4.6 (5.1)	5.7 (6.0)
	0.50	32.1 (38.1)	61.9 (65.7)	88.9 (93.7)	5.0 (6.0)	6.8 (6.6)	10.3 (9.8)
	0.75	34.2 (43.6)	69.9 (74.3)	91.6 (96.2)	12.0 (16.1)	33.9 (32.1)	58.4 (61.0)
0.4	0.25	8.1 (15.2)	20.7 (26.9)	42.7 (52.1)	3.0 (6.7)	4.1 (6.0)	4.8 (6.3)
	0.50	26.2 (36.4)	58.8 (65.6)	88.4 (94.6)	1.9 (5.2)	4.6 (5.8)	9.7 (11.0)
	0.75	28.9 (42.7)	65.6 (72.5)	90.9 (95.5)	8.2 (17.3)	28.9 (33.3)	57.4 (61.5)
0.6	0.25	6.1 (18.1)	16.4 (32.7)	34.4 (52.5)	1.8 (9.2)	2.2 (8.9)	3.9 (9.1)
	0.50	20.8 (39.5)	47.7 (65.0)	78.2 (89.8)	1.7 (7.1)	4.1 (10.0)	9.1 (14.2)
	0.75	23.7 (40.0)	56.4 (71.9)	84.9 (94.0)	5.6 (16.7)	20.8 (32.2)	49.0 (58.6)
0.8	0.25	4.5 (36.3)	8.3 (48.9)	14.3 (61.0)	3.7 (36.9)	4.1 (37.5)	4.7 (42.4)
	0.50	9.1 (43.1)	19.2 (60.8)	40.9 (80.4)	4.0 (31.1)	4.0 (34.1)	5.3 (37.0)
	0.75	11.1 (39.2)	25.9 (63.0)	58.3 (86.0)	4.1 (25.3)	8.1 (39.3)	27.8 (57.3)

but may also investigate the raw data for ellipticity. For the sake of completeness, we chose to analyse both, the raw data and filtered observations.

Starting with the former, we may even investigate the six-dimensional data set $(x_t, y_t, z_t, x_{t+1}, y_{t+1}, z_{t+1})$ for $t \in \{1, \dots, 2662\}$, where x_t, y_t and z_t is the log return at day t of the DAX, Dow Jones and EURO Stoxx index, respectively. Note that $n = 2662$. Eight hypotheses are of interest:

- **Multivariate.** Meta-ellipticity of the six-dimensional data set (denoted '1:6') and of the three-dimensional cross-sectional dependence of $(x_t, y_t, z_t)_{t=1,,\dots,2662}$ (denoted '1:3').
- **Temporal.** Meta-ellipticity of the three marginal temporal dependencies of (x_t, x_{t+1}) (DAX, denoted '(1,4)'), of (y_t, y_{t+1}) (Dow, denoted '(2,5)'), and of (z_t, z_{t+1}) (Eurostoxx, denoted '(3,6)').
- **Crosssectional Pairs.** Meta-ellipticity of the three pairwise dependencies of (x_t, y_t) (DAX-Dow, denoted '(1,2)'), of (x_t, y_t) (DAX-Eurostoxx, denoted '(1,3)'), and of (y_t, z_t) (Dow-Eurostoxx, denoted '(2,3)').

Note that, for consistency, we restrict attention to $t \in \{1, \dots, 2662\}$ for the cross-sectional dependencies, despite the fact that we might include the observations for day 2663. Our test has been applied to each of

Table 3: Analogue of Table 1 in dimension $d = 6$.

AR	τ	$n = 250$	$n = 500$	$n = 1000$	$n = 250$	$n = 500$	$n = 1000$
Panel A:			*Gaussian*			t_5	
0	0.25	2.1 (3.4)	3.6 (3.8)	3.9 (5.4)	2.2 (3.4)	3.5 (5.3)	2.5 (3.4)
	0.50	2.2 (3.5)	3.5 (4.3)	3.9 (5.1)	2.6 (3.6)	4.0 (4.1)	3.9 (5.5)
	0.75	2.1 (4.1)	4.8 (4.7)	3.5 (4.4)	2.1 (4.0)	3.0 (4.8)	4.5 (5.6)
0.2	0.25	2.0 (4.9)	1.6 (3.9)	2.7 (4.0)	1.1 (3.3)	3.2 (5.2)	2.3 (4.9)
	0.50	2.9 (4.4)	2.9 (4.3)	2.4 (3.9)	1.2 (2.8)	3.3 (4.4)	2.3 (4.0)
	0.75	2.3 (4.5)	2.2 (4.7)	2.7 (3.7)	1.9 (4.5)	3.0 (4.8)	3.1 (4.5)
0.4	0.25	1.1 (4.2)	1.7 (5.0)	2.5 (6.5)	1.1 (5.2)	1.5 (5.4)	2.4 (4.9)
	0.50	1.2 (4.7)	3.1 (6.8)	2.4 (5.2)	1.4 (5.7)	2.2 (6.1)	2.4 (6.0)
	0.75	1.7 (4.8)	2.0 (4.8)	3.0 (4.7)	1.1 (4.5)	2.1 (4.5)	2.8 (5.5)
0.6	0.25	0.5 (12.2)	1.1 (15.4)	1.6 (14.1)	1.1 (11.5)	2.0 (14.3)	2.4 (15.2)
	0.50	1.7 (10.5)	2.7 (11.7)	2.5 (12.1)	1.1 (8.8)	2.0 (12.2)	3.3 (12.1)
	0.75	1.4 (6.7)	2.2 (8.0)	3.3 (7.8)	1.1 (7.0)	2.3 (9.1)	2.2 (8.6)
0.8	0.25	3.7 (67.6)	2.5 (79.5)	2.7 (87.1)	3.5 (65.7)	3.8 (79.6)	3.3 (84.5)
	0.50	4.8 (55.2)	2.8 (66.9)	4.0 (75.2)	4.2 (54.1)	3.8 (66.1)	3.6 (69.7)
	0.75	3.5 (30.5)	3.9 (37.4)	2.8 (37.8)	4.0 (29.1)	2.6 (34.9)	2.3 (38.5)
Panel B:			*Frank*			*Clayton*	
0	0.25	20.8 (28.1)	49.3 (56.2)	78.5 (89.2)	1.9 (2.4)	3.2 (4.1)	4.8 (4.7)
	0.50	56.7 (62.6)	91.1 (93.5)	99.6 (100.0)	4.0 (4.4)	8.9 (6.8)	13.6 (11.7)
	0.75	52.4 (61.3)	86.5 (90.8)	98.3 (99.6)	20.9 (21.8)	51.6 (47.3)	82.6 (82.8)
0.2	0.25	17.6 (26.8)	45.1 (54.0)	78.2 (88.5)	2.1 (3.9)	2.8 (3.4)	2.7 (3.6)
	0.50	53.7 (62.8)	90.2 (93.8)	99.1 (100.0)	3.4 (4.4)	6.7 (7.0)	14.4 (13.6)
	0.75	52.1 (60.9)	85.6 (90.1)	99.3 (99.8)	18.8 (20.6)	48.0 (45.4)	79.5 (81.9)
0.4	0.25	12.2 (25.5)	38.5 (52.5)	78.5 (88.7)	1.1 (4.0)	3.4 (6.4)	3.4 (6.6)
	0.50	50.0 (62.4)	86.6 (92.2)	99.7 (100.0)	2.5 (4.4)	6.3 (7.7)	10.2 (12.5)
	0.75	45.4 (57.7)	84.2 (90.4)	98.2 (99.8)	14.9 (20.4)	45.7 (47.8)	75.7 (81.3)
0.6	0.25	8.7 (30.4)	26.4 (57.3)	65.0 (89.3)	1.0 (10.8)	1.7 (11.5)	2.5 (15.9)
	0.50	32.1 (56.3)	75.1 (90.5)	95.7 (99.6)	1.1 (8.5)	3.9 (12.4)	10.7 (21.1)
	0.75	35.8 (53.8)	75.7 (87.4)	95.3 (99.5)	10.6 (23.7)	32.6 (45.1)	67.3 (77.6)
0.8	0.25	6.9 (73.5)	10.2 (87.0)	27.3 (95.0)	4.4 (66.3)	2.6 (78.5)	2.6 (85.4)
	0.50	15.1 (65.6)	32.3 (88.4)	64.6 (97.5)	4.1 (49.4)	3.1 (61.0)	7.3 (69.7)
	0.75	14.7 (52.8)	36.7 (75.8)	67.5 (95.5)	4.2 (31.0)	11.5 (48.8)	37.9 (75.0)

the aforementioned situations with subsample sizes $b \in \{\lfloor cn \rfloor : c \in 0.1, 0.11, \dots, 0.39, 0.4\}$ and with $b = b_{\text{opt}} = \lfloor n^{0.95}/4 \rfloor = 449$. The results for $b = \lfloor cn \rfloor$ are summarized in Figure 3 and in the lower row of Table 4, where we state the proportion of significant p-values (≤ 0.5). The p-value for $b = b_{\text{opt}}$ can be found in the upper row of Table 4.

The results can be summarized as follows: p-values for the six-dimensional data set, as well as for two of the temporal dependencies (DAX and Eurostoxx) are clearly significant at the five-percent level, even after a Bonferroni correction. On the other hand, the test did not find any evidence against meta-ellipticity for the temporal dependence of the Dow Jones index; this difference to the European indices may possibly be explained by differences of financial market regulations in the European Union and the US.

Furthermore, the cross-sectional dependency of (x_t, y_t, z_t) is found to be weakly significant, despite the fact that none of the respective pairs is significant when considered on its own. The latter is hence an instance of the circumstance that comparably mediocre (pairwise) signals may add up to a strong overall signal. For simplicity ignoring that the data is serially dependent, the findings may further be supported by standard

Table 4: Upper row: p-values for the test with $b = b_{\text{opt}} = \lfloor n^{0.95}/4 \rfloor = 449$. Lower row: proportion of p-values smaller than 0.05 among all tests with $b \in \{\lfloor cn \rfloor : c \in 0.10, 0.11, \ldots, 0.39, 0.40\}$.

Dependency	1:6	1:3	(1,4)	(2,5)	(3,6)	(1,2)	(1,3)	(2,3)
P-value(b_{opt})	0.000	0.055	0.009	0.989	0.000	0.062	0.264	0.082
Prop. of Rej.	1.000	0.452	0.871	0.000	1.000	0.097	0.000	0.065

model selection procedures. More precisely, for all three bivariate cross-sectional dependencies, the family of t-copulas has been selected among 37 bivariate candidate models based on AIC and BIC model selection. However, the estimated degrees of freedom are equal to 2.808 (standard error 0.236), 3.733 (0.398) and 2.900 (0.247), respectively, which are incompatible with a three-variate t-copula. Moreover, among the candidate three-variate models, a (non-elliptical) d-vine model (with t-pair copulas) has been selected over the family of t-copulas based on AIC and BIC.

Overall, the obtained results yield additional evidence for the non-(meta-)ellipticity of stock returns. However, we would like to stress once again that our empirical findings and interpretations should be treated with caution when the null hypothesis cannot be rejected: we only test for the equality between Kendall's tau and Blomqvist's beta, which is not a characterizing property of elliptical copulas (see Example 2.1).

Finally, following [26], we have also applied our test to the three-dimensional data set of sample size $n = 2663$ obtained from independently fitting ARMA-GARCH-models to the margins of (x_t, y_t, z_t) and calculating respective standardized residuals (see the last-named paper for precise model specifications). For simplicity, we only report the result for the null hypothesis of three-dimensional cross-sectional meta-ellipticity, which gets rejected at the 5% level with a p-value of 0.011. This is in line with the findings of [26] who obtained a p-value of 0.030.

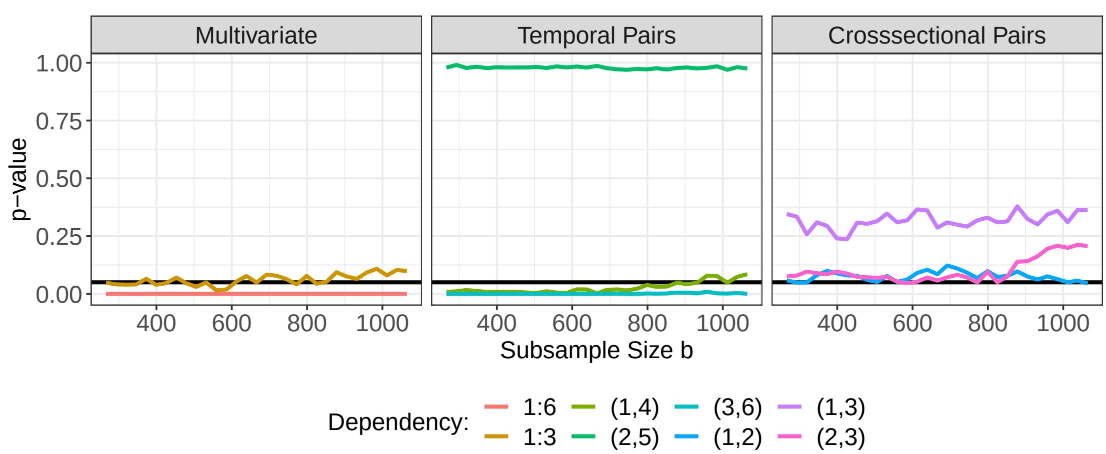

Figure 3: P-values as a function of the block size b for the eight hypotheses described in Section 5.

6 Conclusion

A test for detecting departures from meta-ellipticity for multivariate stationary time series has been proposed. Carrying out the test requires (approximate) critical values of a complex asymptotic distribution, which were obtained using the subsampling bootstrap. Large-sample validity was proven. The test was found to perform well for moderate sample sizes in a simulation study. An application to financial log returns provided evidence for their non-(meta-)ellipticity.

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A Proofs of the main results

Proof of Theorem 3.4. The assertion in (7) is obvious. For the proof of (8), we simplify the notation by occasionally omitting the index k, ℓ; i.e., we write $\widehat{\tau}_n = \widehat{\tau}_{k\ell,n}$, $\widehat{C}_n = \widehat{C}_{k\ell,n}$ etc. Then,

$$\begin{aligned}\frac{1}{4}\sqrt{n}(\widehat{\tau}_n - \tau) &\overset{(a)}{=} \sqrt{n}\Big(\int \widehat{C}_n d\widehat{C}_n - \int C dC\Big) + O_p(n^{-1/2})\\ &= \int \mathbb{C}_n d\widehat{C}_n + \sqrt{n}\Big(\int C d\widehat{C}_n - \int C dC\Big) + O_p(n^{-1/2})\\ &\overset{(b)}{=} \int \mathbb{C}_n d\widehat{C}_n + \int \mathbb{C}_n dC + O_p(n^{-1/2})\\ &\overset{(c)}{=} 2\int \mathbb{C}_n dC + o_P(1),\end{aligned} \tag{15}$$

which is (8). Explanations:

(a) Note that $\widehat{\tau}_n = 2U_n - 1$, where

$$\begin{aligned}U_n &= \frac{2}{n(n-1)} \sum_{1\le i<j\le n} \mathbf{1}\{(X_{ki} - X_{kj})(X_{\ell i} - X_{\ell j}) > 0\}\\ &= \frac{2}{n(n-1)} \sum_{1\le i\ne j\le n} \mathbf{1}(X_{ki} > X_{kj}, X_{\ell i} > X_{\ell j})\\ &= \frac{2}{n^2}\sum_{i,j=1}^{n} \mathbf{1}(X_{ki} \ge X_{kj}, X_{\ell i} \ge X_{\ell j}) + O_p(n^{-1}).\end{aligned}$$

Further,

$$\frac{2}{n^2}\sum_{i,j=1}^{n} \mathbf{1}(X_{ki} \ge X_{kj}, X_{\ell i} \ge X_{\ell j}) = \frac{2}{n^2}\sum_{i,j=1}^{n} \mathbf{1}(\widehat{U}_{ki} \ge \widehat{U}_{kj}, \widehat{U}_{\ell i} \ge \widehat{U}_{\ell j}) = 2\int \widehat{C}_n d\widehat{C}_n.$$

(b) It is sufficent to show that $\int C d\widehat{C}_n = \int \widehat{C}_n dC$, which is related to the arguments given in the proof of Theorem 5.1.1 in [33]. For the ease of reading, we give a self-contained proof. Conditional on $(\boldsymbol{X}_1, \dots, \boldsymbol{X}_n)$ consider independent random vectors $(U, V) \sim C$ and $(U_n, V_n) \sim \widehat{C}_n$. We may then write

$$\int \widehat{C}_n dC = \Pr(U_n \le U, V_n \le V)$$

and

$$\int C d\widehat{C}_n = \Pr(U_n \ge U, V_n \ge V).$$

Furthermore, by Condition 3.2, the distribution of U_n is uniform on $\{\frac{1}{n+1}, \dots, \frac{n}{n+1}\}$, whence

$$\Pr(U \le U_n) = \int_{[0,1]}\int_{[0,u_n]} dF_U(u) dF_{U_n}(u_n) = \int_{[0,1]} u_n dF_{U_n}(u_n) = \sum_{i=1}^{n} \frac{i}{n+1}\frac{1}{n} = \frac{1}{2}.$$

This implies

$$\begin{aligned}\Pr(U_n \le U, V_n \le V) &= 1 - \Pr(U_n > U) - \Pr(V_n > V) + \Pr(U_n > U, V_n > V)\\ &= 1 - \Pr(U_n \ge U) - \Pr(V_n \ge V) + \Pr(U_n \ge U, V_n \ge V)\\ &= \Pr(U_n \ge U, V_n \ge V),\end{aligned}$$

and hence (b).

(c) We have to show that $\int \mathbb{C}_n d(\widehat{C}_n - C) = o_P(1)$. This is direct consequence of the continuous mapping theorem and Lemma C.8 in [3].

The convergence result in (9) is a direct consequence of (7),(8), the continuous mapping theorem and the fact that, under ellipticity,

$$\sqrt{n}(\widehat{\beta}_{k\ell,n} - \widehat{\tau}_{k\ell,n}) = \sqrt{n}(\widehat{\beta}_{k\ell,n} - \beta_{k\ell}) - \sqrt{n}(\widehat{\tau}_{k\ell,n} - \tau_{k\ell}).$$

Finally, normality of the limit in (9) follows from the fact that $\mathbb{G}_C$ is Gaussian and $\Psi_{k\ell}$ defined in (10) is linear. □

Proof of Theorem 3.8. We only need to prove the weak convergence result for $\widehat{p}_{S,b,n}$. For simplicity, we only consider the strong mixing case, the proof for the i.i.d. case is essentially the same. For the weak convergence result under the null hypothesis, it is sufficient to show that

$$(\widehat{T}_n, \widehat{T}_{b,n}^{[I_{1,n}]}, \widehat{T}_{b,n}^{[I_{2,n}]}) \xrightarrow{d} (T, T^{[1]}, T^{[2]}) \tag{16}$$

where $T^{[1]}, T^{[2]}$ are i.i.d. copies of T, the weak limit of $\widehat{T}_n$. Indeed, the assertion then follows from Corollary 4.3 in [8], observing that T has a continuous c.d.f.

For the proof of (16), recall that, by Theorem 3.4, $\widehat{T}_n = \{\Psi(\mathbb{C}_n) + o_P(1)\}^\top \{\Psi(\mathbb{C}_n) + o_P(1)\}$ with

$$\Psi(\mathbb{C}_n) = \begin{pmatrix} 4 \cdot \mathbb{C}_{12,n}(\tfrac{1}{2}, \tfrac{1}{2}) - 8 \cdot \int_{[0,1]^2} \mathbb{C}_{12,n}(u_1, u_2) dC_{12}(u_1, u_2) \\ 4 \cdot \mathbb{C}_{13,n}(\tfrac{1}{2}, \tfrac{1}{2}) - 8 \cdot \int_{[0,1]^2} \mathbb{C}_{13,n}(u_1, u_3) dC_{13}(u_1, u_3) \\ \vdots \\ 4 \cdot \mathbb{C}_{d-1,d,n}(\tfrac{1}{2}, \tfrac{1}{2}) - 8 \cdot \int_{[0,1]^2} \mathbb{C}_{d-1,d,n}(u_{d-1}, u_d) dC_{d-1,d}(u_{d-1}, u_d) \end{pmatrix}.$$

Suppose we have shown that

$$\widehat{T}_{b,n}^{[I_{s,n}]} = \{\Psi(\widehat{\mathbb{C}}_b^{[I_{s,n}]}) + o_P(1)\}^\top \{\Psi(\widehat{\mathbb{C}}_b^{[I_{s,n}]}) + o_P(1)\}, \tag{17}$$

where $\widehat{\mathbb{C}}_b^{[I_{s,n}]} = \sqrt{b}(\widehat{C}_b^{[I_{s,n}]} - \widehat{C}_n)$ with $\widehat{C}_b^{[I_{s,n}]}$ the empirical copula based on the subsample $\mathcal{X}_b^{[I_{s,n}]}$. The assertion in (16) then follows from the continuous mapping theorem and

$$(\mathbb{C}_n, \widehat{\mathbb{C}}_b^{[I_{1,n}]}, \widehat{\mathbb{C}}_b^{[I_{2,n}]}) \rightsquigarrow (\mathbb{C}_C, \mathbb{C}_C^{[1]}, \mathbb{C}_C^{[2]}) \quad \text{in } \{\ell^\infty([0,1]^d)\}^3; \tag{18}$$

the latter convergence being a consequence of Theorem 3.3 in [28].

It remains to show (17), which follows from

$$\sqrt{b}(\widehat{\beta}_{k\ell,b}^{[I_{s,n}]} - \widehat{\beta}_{k\ell,n}) = 4 \cdot \widehat{\mathbb{C}}_b^{[I_{s,n}]}(\tfrac{1}{2}, \tfrac{1}{2})$$

and

$$\begin{aligned}\frac{1}{4}\sqrt{b}(\widehat{\tau}_{k\ell,b}^{[I_{s,n}]} - \widehat{\tau}_{k\ell,n}) &\overset{(a)}{=} \sqrt{b}\Big(\int \widehat{C}_{k\ell,b}^{[I_{s,n}]} d\widehat{C}_{k\ell,b}^{[I_{s,n}]} - \int \widehat{C}_{k\ell,n} d\widehat{C}_{k\ell,n}\Big) + O_P(b^{-1/2})\\ &= \int \widehat{\mathbb{C}}_{k\ell,b}^{[I_{s,n}]} d\widehat{C}_{k\ell,b}^{[I_{s,n}]} + \sqrt{b}\Big(\int \widehat{C}_{k\ell,n} d\widehat{C}_{k\ell,b}^{[I_{s,n}]} - \int \widehat{C}_{k\ell,n} d\widehat{C}_{k\ell,n}\Big) + O_P(b^{-1/2})\\ &\overset{(b)}{=} \int \widehat{\mathbb{C}}_{k\ell,b}^{[I_{s,n}]} d\widehat{C}_{k\ell,b}^{[I_{s,n}]} + \int \widehat{\mathbb{C}}_{k\ell,b}^{[I_{s,n}]} d\widehat{C}_{k\ell,n} + O(b^{1/2}/n) + O_P(b^{-1/2})\\ &\overset{(c)}{=} 2\int \widehat{\mathbb{C}}_{k\ell,b}^{[I_{s,n}]} dC + o_P(1).\end{aligned}$$

Whence it remains to explain (a), (b) and (c) in the latter equation. For that purpose, as in the proof of Theorem 3.4, we will omit the index k, ℓ.

(a) This follows by the same arguments as for the proof of (a) in (15).
(b) Conditional on $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ and $I_{s,n}$, consider independent random vectors $(U_n, V_n) \sim \widehat{C}_n$ and $(U_b, V_b) \sim \widehat{C}_b^{[I_{s,n}]}$. We may then rewrite

$$\int \widehat{C}_n d\widehat{C}_b^{[I_{s,n}]} = \Pr(U_n \le U_b, V_n \le V_b), \qquad \int \widehat{C}_b^{[I_{s,n}]} d\widehat{C}_n = \Pr(U_n \ge U_b, V_n \ge V_b).$$

Under the no-ties condition in Condition 3.2, the subsample $\mathcal{X}_b^{[I_{s,n}]}$ does not contain ties either, whence

$$\begin{aligned}\Pr(V_n < V_b) = \Pr(U_n < U_b) &= \sum_{j=1}^{b} \Big(\sum_{i: \frac{i}{n+1} < \frac{j}{b+1}} \frac{1}{n} \Big) \frac{1}{b} \\ &= \frac{1}{nb} \sum_{j=1}^{b} \Big\{ \frac{j(n+1)}{b+1} + O(1) \Big\} = \frac{n+1}{2n} + O\Big(\frac{1}{n}\Big) = \frac{1}{2} + O\Big(\frac{1}{n}\Big)\end{aligned}$$

and

$$\Pr(V_n = V_b) = \Pr(U_n = U_b) = \frac{1}{b} \sum_{j=1}^{b} \Pr(U_n = \tfrac{j}{b+1}) \le \frac{1}{n}.$$

These two equations imply

$$\begin{aligned}\Pr(U_b \le U_n, V_b \le V_n) &= 1 - \Pr(U_b > U_n) - \Pr(V_b > V_n) + \Pr(U_b > U_n, V_b > V_n) \\ &= \Pr(U_b \ge U_n, V_b \ge V_n) + O(1/n),\end{aligned}$$

where the O-terms are not depending on $(\boldsymbol{X}_1, \ldots, \boldsymbol{X}_n)$ and $I_{s,n}$. This implies (b).
(c) We have to show that $\int \mathbb{C}_b^{[I_{s,n}]} d(\widehat{C}_b^{[I_{s,n}]} - C) = o_P(1)$ and $\int \mathbb{C}_b^{[I_{s,n}]} d(\widehat{C}_n - C) = o_P(1)$. This is a direct consequence of (18), the continuous mapping theorem and Lemma C.8 in [3].
Finally, consider the alternative. By the same arguments as under the null hypothesis, we have

$$(\widehat{T}_{b,n}^{[I_{1,n}]}, \widehat{T}_{b,n}^{[I_{2,n}]}) \xrightarrow{d} (T^{[1]}, T^{[2]}),$$

where $T^{[1]}$ and $T^{[2]}$ are as in (16). By Lemma 2.3 in [8], we have

$$\sup_{x \in \mathbb{R}} \Big| \frac{1}{S} \sum_{s=1}^{S} \mathbf{1}(\widehat{T}_{b,n}^{[I_{1,s}]} \le x) - F_T(x) \Big| = o_P(1),$$

where F_T denotes the c.d.f. of T. As a consequence,

$$\widehat{p}_{S,b,n} = 1 - F_T(\widehat{T}_n) + o_P(1) = o_P(1),$$

where the last equality follows from $\widehat{T}_n \to \infty$ in probability under the alternative.

□

B Sketch-proof of Remark 3.5

Lemma B.1. *Under the null hypothesis of ellipticity, and if each $C_{k\ell}$ has continuous partial derivatives in a neighbourhood of* $(1/2, 1/2)$*, we have*

$$\sup_{k \neq \ell} \Big| \sqrt{n}\{\widehat{\beta}_{k\ell,n} - \beta_{k\ell}\} - \mathbb{B}_{k\ell,n} \Big| = o_{\mathbb{P}}(1), \qquad n \to \infty,$$

where, for $k, \ell \in \{1 \ldots, d\}$ with $k \neq \ell$,

$$\mathbb{B}_{k\ell,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \Big\{ h^{(\beta)}(U_{ki}, U_{\ell i}) - \mathbb{E}[h^{(\beta)}(U_{ki}, U_{\ell i})] \Big\},$$

and where $h^{(\beta)}$ is defined in (12).

Proof. Fix $k, \ell \in \{1, \dots, d\}$ with $k \neq \ell$. By ellipticity of $C_{k\ell}$, we have $C_{k\ell}(u, v) = u + v - 1 + C_{k\ell}(1-u, 1-v)$, which implies $\partial_j C_{k\ell}(u, v) = 1 - \partial_j C_{k\ell}(1-u, 1-v)$ for $j \in \{1, 2\}$; in particular $\partial_j C_{k\ell}(1/2, 1/2) = 1/2$. A straightforward modification of Corollary 2.5 in [9] then implies

$$\begin{aligned}\sqrt{n}\{\widehat{C}_{k\ell,n}(\tfrac{1}{2}, \tfrac{1}{2}) - C_{k\ell}(\tfrac{1}{2}, \tfrac{1}{2})\} &= \alpha_{k\ell,n}(\tfrac{1}{2}, \tfrac{1}{2}) - \tfrac{1}{2}\{\alpha_{k\ell,n}(\tfrac{1}{2}, 1) + \alpha_{k\ell,n}(1, \tfrac{1}{2})\} + o_{\mathbb{P}}(1) \\ &= \tfrac{1}{4}\mathbb{B}_{k\ell,n} + o_{\mathbb{P}}(1),\end{aligned}$$

where $\alpha_n = \sqrt{n}(C_n - C)$ with C_n as defined in (5) and where the $o_{\mathbb{P}}(1)$-term is uniform in k, ℓ. Since $\sqrt{n}(\widehat{\beta}_{k\ell} - \beta_{k\ell}) = 4\sqrt{n}\{\widehat{C}_{k\ell,n}(\tfrac{1}{2}, \tfrac{1}{2}) - C_{k\ell}(\tfrac{1}{2}, \tfrac{1}{2})\}$, we obtain the assertion. □

Lemma B.2. *For $k, \ell \in \{1 \dots, d\}$ with $k \neq \ell$, let*

$$\mathbb{T}_{k\ell,n} = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left\{ h_{k\ell}^{(\tau)}(U_{ki}, U_{\ell i}) - \mathbb{E}[h_{k\ell}^{(\tau)}(U_{ki}, U_{\ell i})] \right\},$$

where $h_{k\ell}^{(\tau)}$ is defined in (13). *Then, under suitable mixing conditions (e.g., Theorem 2.1 in [13]),*

$$\sup_{k \neq \ell} \left| \sqrt{n}\{\widehat{\tau}_{k\ell,n} - \tau_{k\ell}\} - \mathbb{T}_{k\ell,n} \right| = o_{\mathbb{P}}(1), \qquad n \to \infty.$$

It is important to note that no regularity condition on C is needed (see also [45] for the i.i.d. case).

Proof. Note that $\widehat{\tau}_{k\ell,n}$ may be identified as a non-degenerate U-statistic with kernel

$$H_{kl}(\boldsymbol{u}, \boldsymbol{v}) = 2 \cdot I(\boldsymbol{u}^{(k,\ell)} < \boldsymbol{v}^{(k,\ell)}) + 2 \cdot I(\boldsymbol{v}^{(k,\ell)} < \boldsymbol{u}^{(k,\ell)}) - 1.$$

Further note that, for $\boldsymbol{V} \sim C$,

$$\boldsymbol{E}[H_{k\ell}(\boldsymbol{u}, \boldsymbol{V})] = 1 - 2P(V_k \le u_k) - 2P(V_\ell \le u_\ell) + 4C_{k\ell}(u_k, u_\ell) = \frac{1}{2} h_{k\ell}^{(\tau)}(u_k, u_\ell)$$

and, for an independent copy $\boldsymbol{U}$ of $\boldsymbol{V}$,

$$\boldsymbol{E}[H_{k\ell}(\boldsymbol{U}, \boldsymbol{V})] = -1 + 4\mathbb{E}[C_{k\ell}(U_k, U_l)] = \tau_{k\ell}.$$

The assertion then follows from a straightforward multivariate extension of Theorem 2.1 in [13]. □

Under the combined assumptions from the previous two lemmas, we obtain

$$\sup_{k \neq \ell} \left| \sqrt{n}(\widehat{\beta}_{k\ell,n} - \widehat{\tau}_{k\ell,n}) - n^{-1/2} \sum_{i=1}^{n} (h^{(\beta)} - h_{k\ell}^{(\tau)})(U_{ki}, U_{\ell i}) - \mathbb{E}[(h^{(\beta)} - h_{k\ell}^{(\tau)})(U_{ki}, U_{\ell i})] \right| = o_{\mathbb{P}}(1).$$

As a consequence,

$$\sqrt{n}(\widehat{\boldsymbol{\beta}}_n - \widehat{\boldsymbol{\tau}}_n) \xrightarrow{d} \mathcal{N}_{d(d-1)/2}(0, \boldsymbol{\Sigma}),$$

where the entries of $\boldsymbol{\Sigma}$ are given by (11). This is exactly the result claimed to be true in Remark 3.5.

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