Fakultät für Mathematik
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Two discretizations of Koenigs nets and their connection

Zwei Diskretisierungen von Königsnetzen und ihr Zusammenhang

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I hereby confirm that this is my own work, and that I used only the cited sources and materials.

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Abstract

We investigate the relation between two discretizations of Koenigs nets: The classical discretization of Bobenko and Suris, which defines discrete two-dimensional Koenigs nets as nets where the intersection points of diagonals build a net of planar quadrilaterals, and Doliwa's discretization, where a Koenigs lattice is defined as net which has six of its Laplace transforms on a conic at each quadrilateral. We prove that the nets defined by intersection points of diagonals of a classical discretized two-dimensional Koenigs net are exactly Doliwas lattices. We describe how a classical Koenigs net can be constructed on a Doliwa lattice. Also we introduce an 8-point configuration – a slight generalization of the Menelaus' configuration in $n = 4$ – which can be used to characterize both discretizations similarly.

Zusammenfassung

In dieser Arbeit untersuchen wir die Beziehung zwischen zwei Diskretisierungen von Königsnetzen: Der klassischen Diskretisierung von Bobenko und Suris, die ein zweidimensionales Königsnetz als Netz definiert, dessen Diagonalenschnittpunkte ein neues Netz mit planaren Vierecken bildet und Doliwas Königsgitter, ein Netz bei dem an jedem Viereck sechs Laplacetransformationen auf einem Kegelschnitt liegen. Es wird bewiesen, dass die Netze aus Diagonalenschnittpunkten eines klassischen diskreten zweidimensionalen Königsnetzes genau die Königsgitter von Doliwa sind. Es wird gezeigt, wie man ein solches klassisches diskretes Königsnetz zu einem Königsgitter von Doliwa konstruiert. Außerdem wird eine Beschreibung beider Diskretisierungen mithilfe einer 8-Punkt Konfiguration gegeben, die eine Verallgemeinerung der Menelaus Konfiguration in $n = 4$ darstellt.

Contents

Chapter 1

Introduction

In this thesis we investigate (discrete) Koenigs nets, which are a part of (discrete) differential geometry.
Differential geometry deals with objects like curves or surfaces – or more general: nets – in space. These are described by parametrizations $f : \mathbb{R}^m \to \mathbb{R}^N$, which are assumed to be sufficiently smooth. One defines notions like normals or curvature on the nets. A big part of differential geometry is to investigate special classes of nets. The most basic special class is the class of conjugate nets, which fulfill $\partial_i \partial_j f \in \operatorname{span}(\partial_i f, \partial_j f)$ for all $u \in \mathbb{R}^m$ and all $1 \leq i \neq j \leq m$ (∂_i marks the i-th partial derivative). In Chapter 2 we will take a short look at the class of Koenigs nets, which is a subclass of two-dimensional (i.e. $m = 2$) conjugate nets. They have the additional property of equal Laplace invariants (see Chapter 2), and can also be characterized by the existence of a so called dual.

We will mainly be interested in discrete differential geometry, which deals with

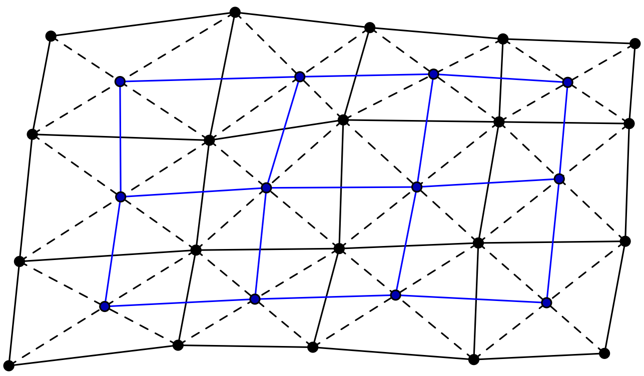

Figure 1.1: A classical discrete Koenigs net with its diagonals (black) and its D-net (blue), which will turn out to be a Doliwas Koenigs lattice.

modeling the notions of differential geometry on a discrete domain. Parametrization of discrete nets are maps $f : \mathbb{Z}^m \to \mathbb{R}^N$. Every four points $f(u)$, $f(u+e_i)$, $f(u+e_j)$ and $f(u+e_i+e_j)$ ($u \in \mathbb{Z}^m$, $1 \leq i \neq j \leq m$) define a quadrilateral of such a net. Again notions like normals or curvature can be defined with similar properties as in the smooth case. Interestingly discrete differential geometry is richer than smooth geometry in the following sense: The smooth objects can always be obtained as a limit of the discrete objects. There is an obstruction in discretizing smooth notions: There can be more than one suitable discretization. This thesis deals with such a case. We are concerned with two different discretizations of Koenigs nets. To explain their definitions we first need the notion of a Q-net. Q-nets are the discretized version of conjugate nets. They are defined by the fact that every quadrilateral of the net is planar.

In Chapter 3 we are concerned with the two discretizations of Koenigs nets. One is given by Bobenko and Suris[BS09]. They discretize the concept of duality. As in the smooth case, a discrete Koenigs net is then a Q-net which admits a dual. We will call this a *classical discrete Koenigs net*, since this discretization is the one which is used the most in todays research. For $m = 2$ there is also a geometric characterization: A Q-net is a classical discrete Koenigs net if and only if the intersection points of diagonals form another Q-net. We will call the nets, which one gets by intersecting the diagonals of a Koenigs nets, D-nets.
The other discretization of Koenigs nets is given by Doliwa[Dol03]. He calls a two-dimensional Q-net a Koenigs lattice if and only if the six Laplace transforms at a quadrilateral (introduced in Section 3.3) share a conic. We will call his discretization *Doliwas Koenigs lattice* (The words net and lattice can be exchanged in most of discrete differential geometry. We will use the word lattice only in the context of Doliwas Koenigs lattice to avoid confusion).
We will define an 8-point configuration as a slightly generalized configuration of the $n = 4$ case of Menelaus' theorem (Section 3.1). It can be used to characterize both discretizations in a similar manner.

In Chapter 4 we prove the main theorem of this paper. It states that the notions of D-net and Doliwas Koenigs lattice are equivalent. Bobenko and Suris already mention this in their paper, but they don't provide a poof.
We show how a classical discrete Koenigs net can be constructed on a Doliwas Koenigs lattice, such that the points of the lattice are the intersection points of diagonals of the Koenigs net. We will find a $(2N + 2)$-parameter freedom in this construction.

Chapter 2

Smooth Koenigs nets

Before looking at the discrete nets, we want to give some basic definitions for smooth nets, as they give motivation for the corresponding discrete definitions. This chapter is taken from [BS]. Nets are maps $f : \mathbb{R}^m \to \mathbb{R}^N$. To keep it simple, we will assume that every such map is regular and smooth enough (all partial derivatives that are used exist and are as linear independent as possible). For $m = 2$ we call the nets surfaces. It is the only case where smooth Koenigs nets can be defined.
First we will introduce a very basic class of smooth nets, the class of conjugate nets.

Definition 2.1 (Conjugate net) A map $f : \mathbb{R}^m \to \mathbb{R}^N$ is called a conjugate net if for all pairs of indices $1 \leq i \neq j \leq m$ we have $\partial_i \partial_j f \in \operatorname{span}(\partial_i f, \partial_j f)$.

Often functions $c_{ij} : \mathbb{R}^m \to \mathbb{R}$ are used to describe the linear dependency:

$$\partial_i \partial_j f = c_{ji} \partial_i f + c_{ij} \partial_j f, \; i \neq j \tag{2.1}$$

This is known as the Laplace equation of the net. Note that these equations need to fulfill some compatibility condition $\partial_i(\partial_j \partial_k f) = \partial_j(\partial_i \partial_k f)$. It can be expressed as the system

$$\partial_i c_{jk} = c_{ij} c_{jk} + c_{ji} c_{ik} - c_{jk} c_{ik}, i \neq j \neq k \neq i. \tag{2.2}$$

The existence of such functions which fulfill the Laplace equation is equivalent to f being a conjugate net. Many interesting classes of nets are subclasses of conjugate nets. One such class is the class of Koenigs nets. It is only defined for $m = 2$.

Definition 2.2 (Koenigs net) A map $f : \mathbb{R}^2 \to \mathbb{R}^N$ is called a Koenigs net if there exists a scalar function $\nu : \mathbb{R}^2 \to \mathbb{R}_+$, such that

$$\partial_1 \partial_2 f = (\partial_2 \log \nu) \partial_1 f + (\partial_1 \log \nu) \partial_2 f \tag{2.3}$$

Equivalently, a Koenigs net is a conjugate net with coefficients c_{12}, c_{21}, such that the coefficients fulfill $\partial_1 c_{21} = \partial_2 c_{12}$. This property is known as equality of the Laplace-invariants of f. There is a second characterization, which is important to us:

Theorem 2.3 (Christoffel dual) *A conjugate net $f : \mathbb{R}^2 \to \mathbb{R}^N$ is a Koenigs net if and only if there exists a scalar function $\nu : \mathbb{R}^2 \to \mathbb{R}_+$ such that the differential one-form df^* defined by*

$$\partial_1 f^* = \frac{\partial_1 f}{\nu^2}, \quad \partial_2 f^* = -\frac{\partial_2 f}{\nu^2} \tag{2.4}$$

is closed. In this case the map $f^* : \mathbb{R}^2 \to \mathbb{R}^N$, defined (up to translation) by the integration of this one-form, is also a Koenigs net, called Christoffel dual to f.

We can reformulate this condition to

$$\begin{gathered} \partial_1 f^* \parallel \partial_1 f, \;\; \partial_2 f^* \parallel \partial_2 f, \\ (\partial_1 + \partial_2) f^* \parallel (\partial_1 - \partial_2) f, \;\; (\partial_1 - \partial_2) f^* \parallel (\partial_1 + \partial_2) f. \end{gathered} \tag{2.5}$$

We will see that this equation can be discretized to define a notion of duality in the discrete domain.

Chapter 3

Discretizations of Koenigs nets

3.1 Ratios of directed lengths and Menelaus' theorem

We introduce ratios of directed lengths, as they are an important tool of Euclidean geometry. They are used in many proofs regarding discrete Koenigs nets. First, we need the notion of a directed length: Let A_1, A_2 and P be three distinct points on a line and e_1 be any unit vector along the line. Then we find unique γ and δ, such that:

$$A_1 - P = \gamma e_1, \tag{3.1}$$
$$A_2 - P = \delta e_1 \tag{3.2}$$

We call γ and δ the directed lenghts of the respective line segments. They are just the distances up to sign:

$$|\gamma| = |PA_1|, \quad |\delta| = |PA_2| \tag{3.3}$$

The lengths are not invariant under the choice of the unit vector e_1. However, their ratio is:

Definition 3.1 (Ratio of directed lengths) Let A_1, A_2 and P be three distinct points on a line and let γ and δ be defined as above. Then we call

$$q(A_1, P, A_2) := \frac{\delta}{\gamma} \tag{3.4}$$

the ratio of directed lengths.

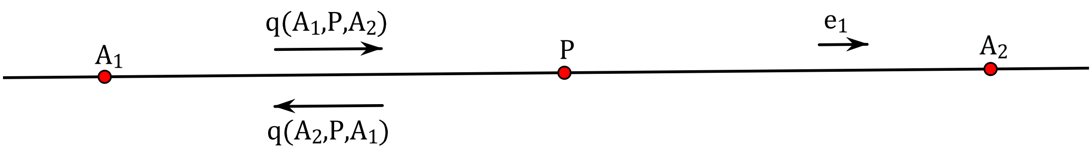

Figure 3.1: In pictures we will mark the direction of a ratio of directed lengths with an arrow pointing from A_1 to A_2 or vice versa.

It is well-defined (i.e. invariant under choice of e_1), since the absolute value is well-defined anyway and for the sign the following holds: $q(A_1, P, A_2)$ is negative if and only if A_1 and A_2 "lie" on different sides of P.
Note that $q(A_1, P, A_2)$ can, by definition, never take the values 0, 1 and ∞, since the points are all distinct from each other. We can invert the *direction* of the ratio:

$$q(A_1, P, A_2) = \frac{1}{q(A_2, P, A_1)}. \tag{3.5}$$

If we know the ratio and two of the points, we can calculate the third point:

$$A_2 = P + q(A_1, P, A_2)(A_1 - P), \tag{3.6}$$
$$A_1 = P + q(A_2, P, A_1)(A_2 - P), \tag{3.7}$$
$$\begin{aligned} P &= \frac{1}{1 - q(A_1, P, A_2)} A_2 - \frac{q(A_1, P, A_2)}{1 - q(A_1, P, A_2)} A_1 \\ &= \frac{1}{1 - q(A_1, P, A_2)} A_2 + \frac{1}{1 - q(A_2, P, A_1)} A_1 \end{aligned} \tag{3.8}$$

The most famous theorem using these ratios is Menelaus' theorem.

Theorem 3.2 (Menelaus' theorem) *Let A_1, A_2, A_3 be three points, which define a triangle in some plane. Let P_{12}, P_{23}, P_{31} be points on the lines (A_1A_2), (A_2A_3), (A_3A_1), respectively, such that none of them coincides with any A_i. Then:*

$$q(A_1, P_{12}, A_2) \cdot q(A_2, P_{23}, A_3) \cdot q(A_3, P_{31}, A_1) = -1 \tag{3.9}$$

is equivalent to P_{12}, P_{23}, P_{31} being collinear.

This theorem can be generalized to arbitrary dimensions:

Theorem 3.3 (Generalized Menelaus' theorem) *Let A_1, A_2,... A_n be n points in general position, i.e. they span a $(n-1)$-dimensional affine space. For $1 \leq i \leq n$ let $P_{i,i+1}$ be n points on the lines (A_iA_{i+1}), respectively, such that none of them coincides with any A_i (indices are taken modulo n). Then:*

$$\prod_{i=1}^{n} q(A_i, P_{i,i+1}, A_{i+1}) = (-1)^n \tag{3.10}$$

is equivalent to $P_{i,i+1}$, $1 \leq i \leq n$ spanning a $(n-2)$-dimensional affine space.

Proof is given in [BS]. Menelaus' theorem is the $n = 3$ case.
In this paper we will use a configuration of points and lines, which is similar to the configuration in the $n = 4$ case of the generalized Menelaus' theorem. We call it *8-point configuration.*

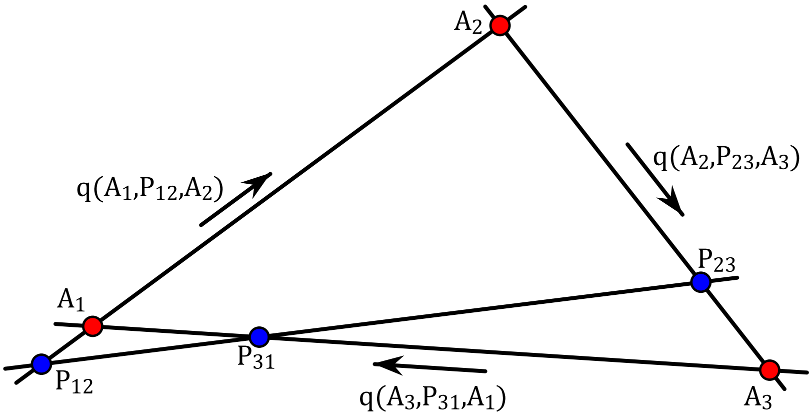

Figure 3.2: Menelaus' theorem

Definition 3.4 (8-point configuration) Let A_1, A_2, A_3, A_4 be four points, which define a quadrilateral. Let P_{12}, P_{23} ,P_{34}, P_{41} be points on the lines (A_1A_2), (A_2A_3), (A_3A_4), (A_4A_1), respectively, such that none of them coincides with any A_i. We call this configuration an 8-point configuration if

$$q(A_1, P_{12}, A_2) \cdot q(A_2, P_{23}, A_3) \cdot q(A_3, P_{34}, A_4) \cdot q(A_4, P_{41}, A_1) = 1 \tag{3.11}$$

The only difference to a configuration of Menelaus' theorem is that we don't require $A_1, ..., A_4$ to be in general position. In this case general position means that the four points are distinct from each other and are not planar. If $A_1, ..., A_4$ still appear to be in general position (i.e. not planar) this definition is equivalent to P_{12},..., P_{41} being planar, by the generalized Menelaus' theorem. However, if $A_1, ..., A_4$ share a plane we can't apply the theorem and we need another geometric characterization. In this case the quadrilateral P_{12},..., P_{41} is planar, even if it doesn't lie in 8-point configuration. We can find a geometric characterization which holds in both cases.

Theorem 3.5 (Geometric characterization of the 8-point configuration) *Let A_1, A_2, A_3, A_4 be four points, which define a quadrilateral. Let P_{12}, P_{23}, P_{34}, P_{41} be points on the lines (A_1A_2), (A_2A_3), (A_3A_4), (A_4A_1), respectively, such that none of them coincides with any A_i. This defines an 8-point configuration if and only if*

$$(P_{12}P_{23}), (A_1A_3) \textit{ and } (P_{34}P_{41}) \textit{ are concurrent} \tag{3.12}$$

or, equivalently,

$$(P_{23}P_{34}), (A_2A_4) \textit{ and } (P_{41}P_{12}) \textit{ are concurrent} \tag{3.13}$$

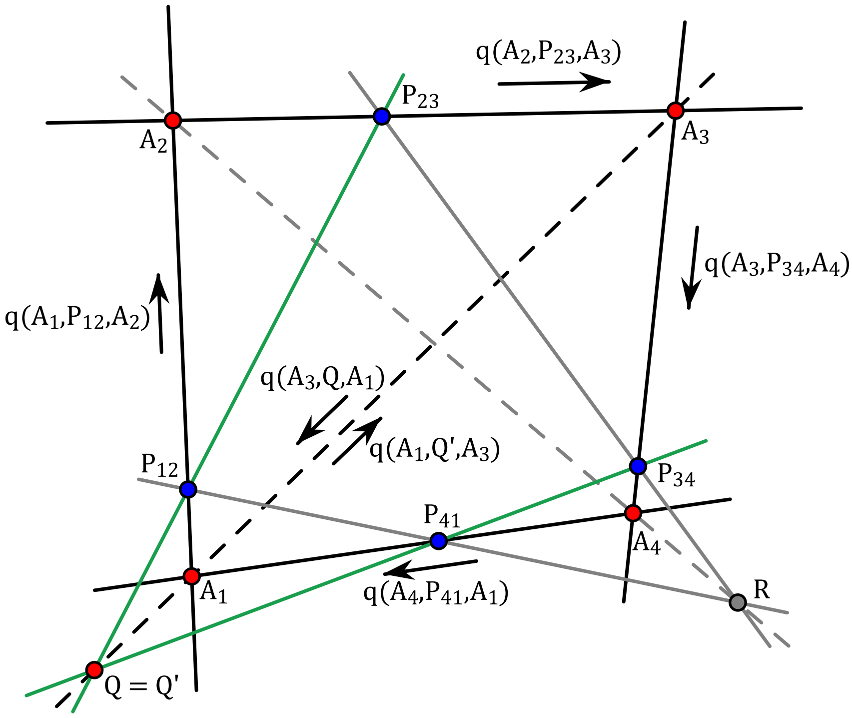

Figure 3.3: 8-point configuration (Together with the points $Q = Q'$ and R this is exactly Desargues' theorem)

Proof. (A_1A_3) and $(P_{12}P_{23})$ intersect in a point Q, since both lines share the plane $(A_1A_2A_3)$. By the n = 3 case of Menelaus' theorem, we have

$$q(A_1, P_{12}, A_2)q(A_2, P_{23}, A_3)q(A_3, Q, A_1) = -1. \tag{3.14}$$

The same argument can be applied for the intersection point Q' of (A_1A_3) and $(P_{34}P_{41})$ to get

$$q(A_3, P_{34}, A_4)q(A_4, P_{41}, A_1)q(A_1, Q', A_3) = -1. \tag{3.15}$$

Multiplication yields

$$\begin{aligned} 1 &= q(A_1, P_{12}, A_2)q(A_2, P_{23}, A_3)q(A_3, P_{34}, A_4)q(A_4, P_{41}, A_1)q(A_3, Q, A_1)q(A_1, Q', A_3) \\ &= q(A_1, P_{12}, A_2)q(A_2, P_{23}, A_3)q(A_3, P_{34}, A_4)q(A_4, P_{41}, A_1)\frac{q(A_3, Q, A_1)}{q(A_3, Q', A_1)}. \end{aligned} \tag{3.16}$$

The points lie in an 8-point configuration if and only if $q(A_3, Q, A_1) = q(A_3, Q', A_1)$, which is equivalent to $Q = Q'$, since the ratio of directed lengths defines the third point uniquely. This is equivalent to $(P_{12}P_{23})$, (A_1A_3) and $(P_{34}P_{41})$ being concurrent with intersection point $Q = Q'$.

The equivalence to $(P_{23}P_{34})$, (A_2A_4) and $(P_{41}P_{12})$ being concurrent can be shown exactly the same way. One can also prove the equivalence of both concurrent statements with Desargues' theorem. □

3.2 Classical discretization of Koenigs nets

The classical definition of discrete Koenigs nets is given by Bobenko and Suris. This chapter is based on their work in [BS09]. They discretize the notion of the existence of a dual net, which was a characterization for smooth Koenigs nets. First we need a discrete version of duality.

3.2.1 Duality of quadrilaterals

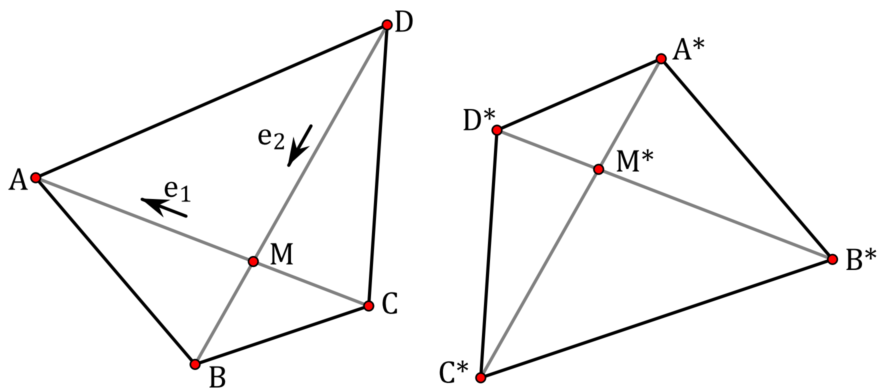

Figure 3.4: Dual Quadrilaterals

Definition 3.6 (Dual quadrilateral) Two quadrilaterals (A, B, C, D) and (A^*, B^*, C^*, D^*) in a plane are called dual if their corresponding sides are parallel:

$$(AB) \parallel (A^*B^*), \quad (BC) \parallel (B^*C^*), \quad (CD) \parallel (C^*D^*), \quad (DA) \parallel (D^*A^*), \tag{3.17}$$

and their non-corresponding diagonals are parallel:

$$(AC) \parallel (B^*D^*), \quad (BD) \parallel (A^*C^*). \tag{3.18}$$

Lemma 3.7 (Existence and uniqueness of a dual quadrilateral) *For any planar quadrilateral (A, B, C, D) a dual one exists and is unique up to scaling and translation.*

The proof gives some geometric inside into the concept of dual quadrilaterals.

Proof. We show this by constructing a dual quadrilateral (A^*, B^*, C^*, D^*) and keeping track of the freedom we have in the construction. This will show existence and uniqueness. Let e_1 and e_2 be some unit vectors along the diagonals (AC) and (BD) respectively. Let $M = (AC) \cap (BD)$ be the intersection point of the diagonals. M exists, since the quad is planar. Also there exist unique coefficients α, β, γ, δ, such that

$$A = M + \alpha e_1, \quad B = M + \beta e_2, \quad C = M + \gamma e_1, \quad D = M + \delta e_2. \tag{3.19}$$

We arbitrarily choose M^*, which is the intersection point of diagonals in the dual quad. This corresponds to the freedom of translation. Now we know the diagonals of the dual quad: The line through M^* which is parallel to e_2 and the line through M^* which is parallel to e_1. Next, we choose A^* on the corresponding diagonal. Since we know the diagonal, we only have a real number λ as freedom. We choose it, such that

$$A^* = M^* + \frac{\lambda}{\alpha} e_2. \tag{3.20}$$

Choosing λ corresponds to the freedom of scaling. The rest of the construction is uniquely determined:
B^* is the unique point on the diagonal of the dual quad parallel to e_1 which fulfills $(A^*B^*) \parallel (AB)$. In fact we find B^* to be

$$B^* = M^* + \frac{\lambda}{\beta} e_1, \tag{3.21}$$

since this point is on the diagonal and also fulfills $(A^*B^*) \parallel (AB)$:

$$B^* - A^* = \frac{\lambda}{\beta} e_1 - \frac{\lambda}{\alpha} e_2 = \frac{-\lambda}{\alpha\beta}(\beta e_2 - \alpha e_1) = \frac{-\lambda}{\alpha\beta}(B - A). \tag{3.22}$$

The same argument can be used to find C^* uniquely as

$$C^* = M^* + \frac{\lambda}{\gamma} e_2, \tag{3.23}$$

since this point lies on the diagonal parallel to e_2 and fulfills the condition $(B^*C^*) \parallel (BC)$:

$$C^* - B^* = \frac{\lambda}{\gamma} e_2 - \frac{\lambda}{\beta} e_1 = \frac{-\lambda}{\beta\gamma}(\gamma e_1 - \beta e_2) = \frac{-\lambda}{\beta\gamma}(C - B) \tag{3.24}$$

Last we find D^* uniquely as

$$D^* = M^* + \frac{\lambda}{\delta} e_1, \tag{3.25}$$

since this point lies on the diagonal parallel to e_1 and fulfills the condition $(C^*D^*) \parallel (CD)$:

$$D^* - C^* = \frac{\lambda}{\delta} e_1 - \frac{\lambda}{\gamma} e_2 = \frac{-\lambda}{\gamma\delta}(\delta e_2 - \gamma e_1) = \frac{-\lambda}{\gamma\delta}(D - C). \tag{3.26}$$

Now every parallelity except $(D^*A^*) \parallel (DA)$ is fulfilled by construction. We can simply check this last condition:

$$A^* - D^* = \frac{\lambda}{\alpha} e_2 - \frac{\lambda}{\delta} e_1 = \frac{-\lambda}{\delta\alpha}(\alpha e_1 - \delta e_2) = \frac{-\lambda}{\delta\alpha}(A - D). \tag{3.27}$$

This implies that the constructed quadrilateral is indeed a dual.
We have proven that a dual quadrilateral can always be constructed and one has freedom of translation and scaling in the construction. □

3.2.2 Discrete Koenigs nets as nets admitting a dual

A discrete m-dimensional net is a map $f : \mathbb{Z}^m \to \mathbb{R}^N$. It is practicable to use the so called *shift notation*, when working with local properties of such maps: We write f, f_i, f_j and f_{ij} instead of $f(u)$, $f(u+e_i)$, $f(u+e_j)$ and $f(u+e_i+e_j)$ for some $u \in \mathbb{Z}^m$ and lattice directions $1 \le i, j \le m$ (e_i denotes the i-th vector of the standard basis). These four points define a quadrilateral of the net. We will always assume the net to be regular, i.e. no two points of a quadrilateral coincide.
For $m = 2$ the net is a discrete surface. We will mostly be concerned with this case.

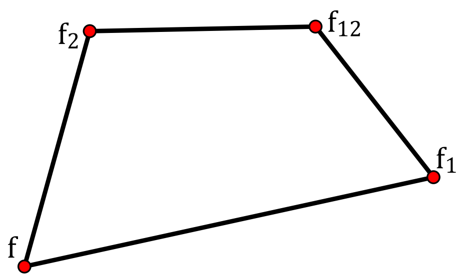

Figure 3.5: Shift notation for a quadrilateral of a 2-dimensional net

Definition 3.8 (Q-net) A Q-net is a map $f : \mathbb{Z}^m \to \mathbb{R}^N$, such that every quadrilateral is planar, i.e. for all $1 \le i, j \le m$ the points f, f_i, f_j and f_{ij} lie in a common plane.

This is the discretization of conjugate nets. Most discrete nets (for example Koenigs nets) are Q-nets.

Bobenko's and Suris' characterization of Koenigs nets is given by discretizing the property of a net having a dual:

Definition 3.9 (Discrete Koenigs net) A Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is called a discrete Koenigs net if it admits a dual, i.e. a Q-net $f^* : \mathbb{Z}^m \to \mathbb{R}^N$, such that all elementary quadrilaterals of the net f^* are dual to the corresponding quadrilaterals of f:

$$f_i - f \parallel f_i^* - f^*, \quad f_j - f \parallel f_j^* - f^*, \tag{3.28}$$
$$f_{ij} - f \parallel f_j^* - f_i^*, \quad f_j - f_i \parallel f_{ij}^* - f^* \tag{3.29}$$
$$\text{for all } 1 \leq i, j \leq m. \tag{3.30}$$

It might seem as if this definition is trivial, since we have proven that any quadrilateral admits a dual. However, in general we can not find dual quadrilaterals for a net, such that these fit together and build a new net. This is why the definition is not trivial.

Since we are mostly concerned with discrete Koenigs nets instead of smooth Koenigs nets, we will omit the word "discrete".

3.2.3 Algebraic characterization of Koenigs nets

There is an algebraic characterization of Koenigs nets, which uses ratios of directed lengths on the diagonals of two bipartite parts of the net.
We can split the lattice $\mathbb{Z}^m$ into two bipartite parts: For any $u \in \mathbb{Z}^m$, consider the sum of its entries $|u| = u_1 + u_2 + ... + u_m \in \mathbb{Z}$. It is either even or odd. We introduce the two parts as

$$\mathbb{Z}^m_{even} = \{u \in \mathbb{Z}^m |\ |u| \text{ is even}\} \tag{3.31}$$
$$\mathbb{Z}^m_{odd} = \{u \in \mathbb{Z}^m |\ |u| \text{ is odd}\} \tag{3.32}$$

Note that every edge connects a point of $\mathbb{Z}^m_{even}$ with a point of $\mathbb{Z}^m_{odd}$, whereas a diagonal either connects a point of $\mathbb{Z}^m_{even}$ with a point of $\mathbb{Z}^m_{even}$ or connects a point of $\mathbb{Z}^m_{odd}$ with a point of $\mathbb{Z}^m_{odd}$.

A bipartite part together with the connecting diagonals forms a lattice. In the case $m = 2$ this lattice has the same combinatorics as $\mathbb{Z}^2$. If we have a discrete net $f : \mathbb{Z}^m \to \mathbb{R}^N$, we get two *subnets* by restriction of f to one of the bipartite parts. We interprete the corresponding diagonals of f as the edges of the subnet.

On a planar quadrilateral (A, B, C, D) we have an intersection point of diagonals $M = (AC) \cap (BD)$. We will interpret the diagonal (AC) of the quad as the combination

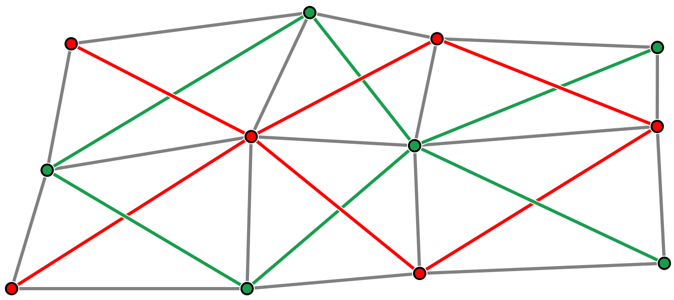

Figure 3.6: Six quadrilaterals of a discrete two-dimensional net(grey) and its two bipartite parts (green and red), which form discrete nets themselves

of the two oriented diagonals $\overrightarrow{AC}$ and $\overrightarrow{CA}$, and equivalently (BD) as the combination of $\overrightarrow{BD}$ and $\overrightarrow{DB}$. Then we can define the ratio of diagonal segments by

$$q(\overrightarrow{AC}) = q(A, M, C), \quad q(\overrightarrow{CA}) = q(C, M, A), \tag{3.33}$$

$$q(\overrightarrow{BD}) = q(B, M, D), \quad q(\overrightarrow{DB}) = q(D, M, B). \tag{3.34}$$

We can extend this to a be a map on the oriented diagonals of a Q-net or equivalently on the oriented edges of the two subnets:

Definition 3.10 (Quantity q) Let $f : \mathbb{Z}^m \to \mathbb{R}^N$ be a Q-net and $x \in \{\text{even}, \text{odd}\}$. On every oriented edge of the subnet $f|_{\mathbb{Z}^m_x}$ the quantity q can be defined as the ratio of the corresponding diagonal segments of f.
q is called closed if for any circle of directed edges the corresponding q's multiply to one.

This quantity q can be used to characterize a Koenigs net:

Theorem 3.11 (Algebraic characterization) *A Q-net $f : \mathbb{Z}^m \to \mathbb{R}^N$ is a Koenigs net if and only if the quantity q is closed on both $\mathbb{Z}^m_{even}$ and $\mathbb{Z}^m_{odd}$.*

Proof. We will only prove this for $m = 2$. Other dimensions can be proven similarly. We will show three equalities:

$$\begin{aligned} & \text{A dual net } f^* \text{ can be constructed} && (3.35)\\ \Leftrightarrow\ & \text{Duals can be constructed for any four quads sharing a point} && (3.36)\\ \Leftrightarrow\ & \text{q multiplies to one around any quadrilateral of a subnet} && (3.37)\\ \Leftrightarrow\ & \text{q is closed on both } \mathbb{Z}^m_{even} \text{ and } \mathbb{Z}^m_{odd}. && (3.38) \end{aligned}$$

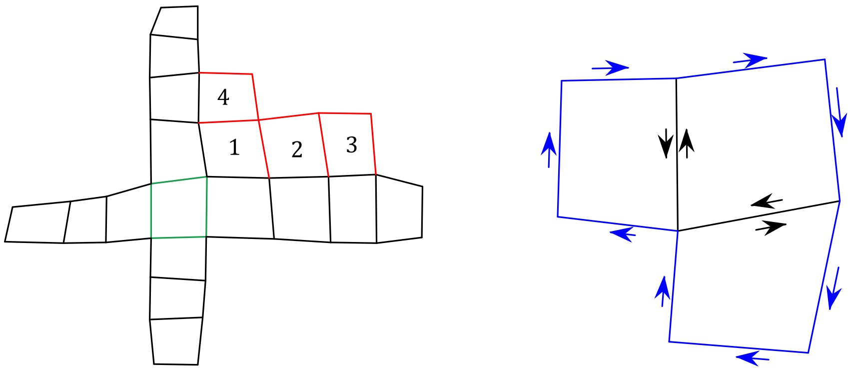

Figure 3.7: Left: Construction of the dual. Start with the green quad, which is unique up to scaling and translation. The rest is unique. The red quadrilaterals can be constructed by assumption. Right: A circle of directed edges (blue) can be split into elementary circles. Every black edge appears once in both directions.

For the first equality we only have to show that we can build the hole dual net f^* if we can build fitting duals for four quadrilaterals sharing a point. For a single quadrilateral we can always find a dual, which is unique up to scaling and translation. If we have a second quad which shares an edge with the first one, then we can find a fitting dual quad by using the translation and scaling freedom in the choice of the second dual quad: We need to scale and translate the second dual quad, such that the edges which are dual to the common edge of the two original quads match again. This means that the second dual quad is uniquely determined. We can iterate this procedure in both lattice directions. Then we can use the assumption to construct the rest of the dual net (see Figure 3.7).
Next we will prove the third equality, which says that q multiplying to 1 around any circle of a subnet is equivalent to q multiplying to 1 around any quadrilateral of a subnet. Any circle of directed edges can be split into elementary circles (quadrilaterals with an orientation on the edges). See Figure 3.7. The product of the q's on the circle is the same as the product of the q's of all of the elementary circles, since all "inside" q's cancel: All "inside" edges appear once in both directions and therefore their product is 1 (see Equation 3.5). This already proves the equivalence.
We will proof the second equality by prescribing four quadrilaterals sharing a point, and proving that the condition that duals can be constructed is equivalent to q multiplying to 1 around the subnet quadrilateral contained in this construction (see Figure 3.8). In the Proof of Lemma 3.7 we described a quadrilateral and its dual by quantities α,..., δ and a scaling factor λ. Similarly, we can introduce the quantities α_i,..., δ_i and a scaling factor λ_i $(i = 1, ..., 4)$ on all four quadrilaterals.

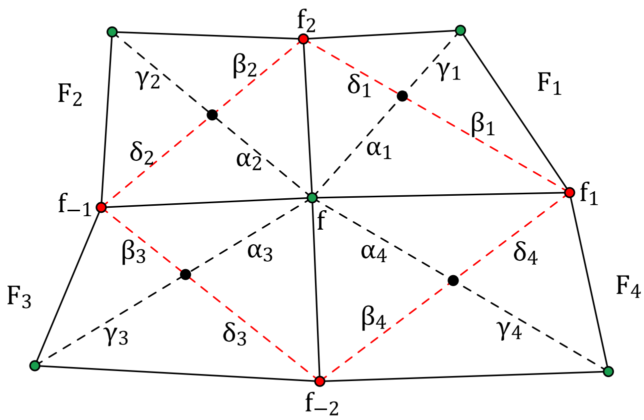

Figure 3.8: Four quadrilaterals of a Q-net, which share the point f. $(f_1 f_2 f_{-1} f_{-2})$ is an elementary subnet quadrilateral (red).

We can choose the scaling factors freely, while the rest is determined by our given Q-net quadrilaterals (and the choice of unit vectors, which is not so important here). By the translation freedom we can make sure that all four dual quadrilaterals share the point f^*, which is dual to f. Now we need to make sure that the edges shared by two of the quadrilaterals can be dualized: The edge shared by F_1 and F_2 is $(f f_2)$. Its dual is well-defined if the dual quads of F_1 and F_2 both scale this edge by the same number. In the proof of Lemma 3.7 we found the scaling factors of the edges to be

$$\frac{-\lambda_1}{\alpha_1\delta_1} \text{ and } \frac{-\lambda_2}{\alpha_2\beta_2}. \tag{3.39}$$

The dual of the edge is well-defined if these scaling factors are equal, i.e.

$$\frac{\lambda_1}{\lambda_2} = \frac{\alpha_1\delta_1}{\alpha_2\beta_2}. \tag{3.40}$$

On the other edges, we get similar conditions:

$$\frac{\lambda_2}{\lambda_3} = \frac{\alpha_2\delta_2}{\alpha_3\beta_3}, \quad \frac{\lambda_3}{\lambda_4} = \frac{\alpha_3\delta_3}{\alpha_4\beta_4}, \quad \frac{\lambda_4}{\lambda_1} = \frac{\alpha_4\delta_4}{\alpha_1\beta_1}. \tag{3.41}$$

The condition to find λ_i's that fulfill all four equations turns out to be

$$1 = \frac{\alpha_1\delta_1}{\alpha_2\beta_2}\frac{\alpha_2\delta_2}{\alpha_3\beta_3}\frac{\alpha_3\delta_3}{\alpha_4\beta_4}\frac{\alpha_4\delta_4}{\alpha_1\beta_1} = \frac{\delta_1}{\beta_1}\frac{\delta_2}{\beta_2}\frac{\delta_3}{\beta_3}\frac{\delta_4}{\beta_4}. \tag{3.42}$$

Also from the definition of ratios of directed lenghts we find

$$q(\overrightarrow{f_1 f_2}) = \frac{\delta_1}{\beta_1}, \; q(\overrightarrow{f_2 f_{-1}}) = \frac{\delta_2}{\beta_2}, \; q(\overrightarrow{f_{-1} f_{-2}}) = \frac{\delta_3}{\beta_3}, \; q(\overrightarrow{f_{-2} f_1}) = \frac{\delta_4}{\beta_4}. \tag{3.43}$$

Now the condition reads

$$1 = q(\overrightarrow{f_1 f_2})q(\overrightarrow{f_2 f_{-1}})q(\overrightarrow{f_{-1} f_{-2}})q(\overrightarrow{f_{-2} f_1}), \tag{3.44}$$

which is exactly the condition that q multiplies to 1 around the quadrilateral $(f_1 f_2 f_{-1} f_{-2})$. □

3.2.4 Geometric characterization of two-dimensional Koenigs nets

We will extend the use of shift notation and write f_{-1} and f_{-2} instead of $f(u - e_1)$ and $f(u - e_2)$ respectively.
We now introduce the important geometric characterization of two-dimensional Koenigs nets:

Theorem 3.12 (Geometric characterization) *Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be a Q-net such that for every point $f = f(u)$ its four neighbours $f_{\pm 1}$, $f_{\pm 2}$ are not coplanar. Then f is a discrete Koenigs net if and only if for every point $f = f(u)$ the intersection points of diagonals of the four quadrilaterals adjacent to f are coplanar, that is, if the intersection points of diagonals build a Q-net.*

Proof. This follows directly from Equation 3.44 and the generalized Menelaus' theorem. □

The extra condition that $f_{\pm 1}$, $f_{\pm 2}$ are not coplanar is necessary for the generalized Menelaus' theorem to apply. However, we introduced the notion of an 8-point configuration, which can be used to eliminate this extra condition.

Theorem 3.13 (Geometric characterization via 8-point configuration) *Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be a Q-net. Then f is a discrete Koenigs net if and only if for every point $f = f(u)$ the intersection points of diagonals of the four quadrilaterals adjacent to f lie in 8-point configuration on the quadrilateral $(f_1 f_2 f_{-1} f_{-2})$.*

Proof. This follows directly from Equation 3.44 and Theorem 3.5. □

It appears that the condition to be a Koenigs net can be formulated in terms of the intersection points of diagonals. These points form a net as well.

Definition 3.14 (D-net) A map $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is called a D-net if there exists a Koenigs net g, s.t. f is the net defined by intersecting the diagonals of g.

It is obvious from the geometric characterizations that any D-net is already a Q-net. However, we will see that not every Q-net is a D-net. We are interested in finding a more direct characterization of D-nets, which allows to directly compute whether a net is a D-net or not. We will do this by introducing another discretization of Koenigs nets and showing its equivalence to the notion of D-nets.

Note that one has to be careful with combinatorics of the nets, since every point of a D-net corresponds to a quadrilateral of a Koenigs net. From this perspective it makes more sense to define a D-net on $(\mathbb{Z}^2)^*$, the space of quadrilaterals of $\mathbb{Z}^2$. For simpler notation, we will keep $\mathbb{Z}^2$ as domain, while assigning a point (k,l) of a D-net to the quadrilateral of a Koenigs net which contains the points (k,l) ,$(k+1,l)$, $(k+1,l+1)$ and $(k,l+1)$. The D-net is then characterized by the existence of a Koenigs net, s.t. every point of the D-net is an intersection point of diagonals of the corresponding quadrilateral of the Koenigs net.

3.3 Doliwa's discretization of Koenigs nets

This section is based on the discretization of Koenigs nets given in [Dol03]. Doliwa's definitions only work for the case $m=2$, i.e. for nets defined on $\mathbb{R}^2$ in the smooth case and $\mathbb{Z}^2$ in the discrete case.
We will use the homogeneous coordinates of the projective space $\mathbb{RP}^N$. This is the set of equivalences classes of $\mathbb{R}^{N+1}\setminus\{0\}$ together with the equivalence relation

$$x \sim y \Leftrightarrow x = \lambda y, \quad x,y \in \mathbb{R}^{N+1}\setminus\{0\}, \quad \lambda \in \mathbb{R}\setminus\{0\}. \tag{3.45}$$

Every element of $\mathbb{RP}^N$ is a 1-dimensional subspace of $\mathbb{R}^{N+1}$ without 0. We identify $\mathbb{R}^N$ (which is the Euclidean space we are working with) with the projective space $\mathbb{RP}^N$ by the bijection

$$x \in \mathbb{R}^N \;\; \leftrightarrow \;\; [(x,1)] \in \mathbb{RP}^N, \tag{3.46}$$

where $[v]$ denotes the equivalence class of $v \in \mathbb{R}^{N+1}\setminus\{0\}$.
Note the following facts about homogeneous coordinates:

- In homogeneous coordinates, every affine n-dimensional Euclidean space forms a $(n+1)$-dimensional linear space without 0 (in $\mathbb{R}^{N+1}\setminus\{0\}$).
- A point p is in the affine space spanned by other points p_1,..., p_k if and only if p can be written as a linear combination of p_1,..., p_k
- In a (projective) plane, we can define a conic as the solution set of the equation $p^\top Ap = 0$. Five points in general position define such a conic uniquely. Four points define a pencil (1-dimensional linear system) of conics. Two lines which should be tangent to a conic in given points also define a pencil of conics.

Next, we will give a characterization for smooth Koenigs nets in terms of homogeneous coordinates. It is equivalent to the notion of Koenigs nets defined in Chapter 2. The condition for a map $f : \mathbb{R}^2 \to \mathbb{RP}^N$ to be a conjugate net is the Laplace equation written in homogeneous coordinates:

$$\partial_{12} f = a\partial_1 f + b\partial_2 f + cf, \tag{3.47}$$

where a, b, c are real functions of $\mathbb{R}^2$ (i.e. $\mathbb{R}^2 \to \mathbb{R}$). To be able to continue working with a, b, c one must fix representatives for the net f, because otherwise a, b, c are not well-defined: Scaling f with some non-zero function results in a scaling of a, b, c in the Laplace equation as well. Assume fixed representatives from now on.
One defines the Laplace transforms of a conjugate net f to be

$$l^1 = \partial_2 f - af, \quad l^2 = \partial_1 f - bf. \tag{3.48}$$

These are also conjugate nets $\mathbb{R}^2 \to \mathbb{RP}^N$, which fulfill the following condition: The x-tangent line of f coincides with the y-tangent line of l^2 and the y-tangent line of f coincides with the x-tangent line of l^1 (see Figure 3.9). This also implies that the

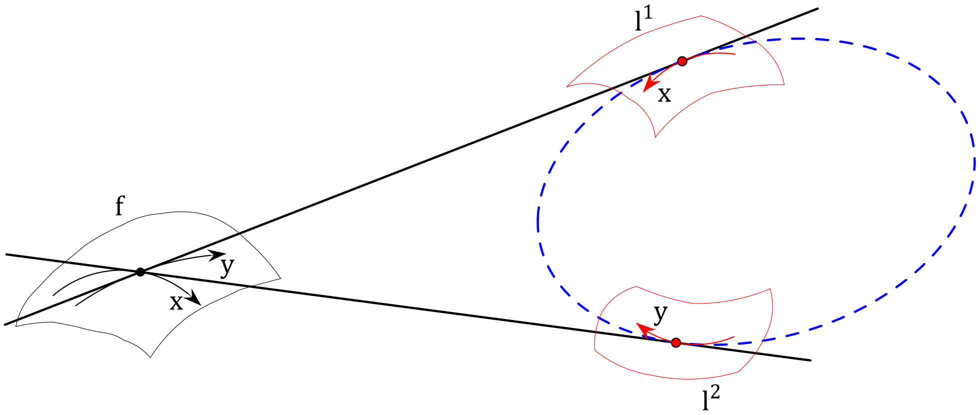

Figure 3.9: The Koenigs net f with its x- and y-coordinate line and the corresponding tangents (black). The Laplace transforms l^1 and l^2 with x- and y-coordinate line respectively (red).

points $f(x, y)$, $l^1(x, y)$ and $l^2(x, y)$ share a plane with these tangent lines for all (x, y). In this plane we can use the notion of conics. There exists a pencil of conics in the plane, such that all conics are tangent to both tangent lines. This means that the conics have first order contact with the x-coordinate line of l^1 and the y-coordinate line of l^2. The condition for f to be a smooth Koenigs net is now: There exists a conic in the pencil which has second order contact with both coordinate lines.
A useful property of the Laplace equation of Koenigs nets in homogeneous coordinates

is that it can – by clever choice of representatives – be written as

$$\partial_{12} f = \alpha f, \tag{3.49}$$

where α is a real function of $\mathbb{R}^2$.

Doliwa discretizes this property for his definition of discrete Koenigs nets. The condition of a map $f : \mathbb{Z}^2 \to \mathbb{RP}^N$ to be a Q-net can be rewritten in homogeneous coordinates:

$$f_{12} = A_1 f_1 + B_2 f_2 + Cf, \tag{3.50}$$

where A, B, C are real functions of $\mathbb{Z}^2$. The Laplace transforms discretize analogously:

Definition 3.15 (Discrete Laplace transforms) Let $f : \mathbb{Z}^2 \to \mathbb{RP}^N$ be a Q-net, such that

$$f_{12} = A_1 f_1 + B_2 f_2 + Cf. \tag{3.51}$$

Then we call the discrete nets defined by

$$L^1 = f_2 - Af, \quad L^2 = f_1 - Bf \tag{3.52}$$

the Laplace transforms of f.

In the discrete case the Laplace transformations are just the intersection points of opposite edges of the quadrilaterals. To see this one needs to check the collinearities.

Lemma 3.16 *The four points L^1_1, f, f_2 and L^1 are collinear and the four points L^2_2, f, f_1, L^2 are collinear.*

Proof. We can simply use the discrete Laplace equation to find

$$\begin{aligned} L^1_1 &= f_{12} - A_1 f_1 = B_2 f_2 + Cf = B_2 L^1 + (B_2 A + C)f, \\ L^2_2 &= f_{12} - B_2 f_2 = A_1 f_1 + Cf = A_1 L^2 + (A_1 B + C)f. \end{aligned}$$ □

That means we can calculate the Laplace transforms purely with incidence geometry:

$$L^1_1 = (f f_2) \cap (f_1 f_{12}), \quad L^2_2 = (f f_1) \cap (f_2 f_{12}), \tag{3.53}$$

where $\cap$ denotes the intersection of the two lines.

As in the smooth case, we will now look at conics in the tangent plane of f (In the discrete case, the tangent plane is just the plane of the quadrilateral). Note that the Laplace transforms L^1, L^1_1, L^2 and L^2_2 all lie in the tangent plane. We can check that

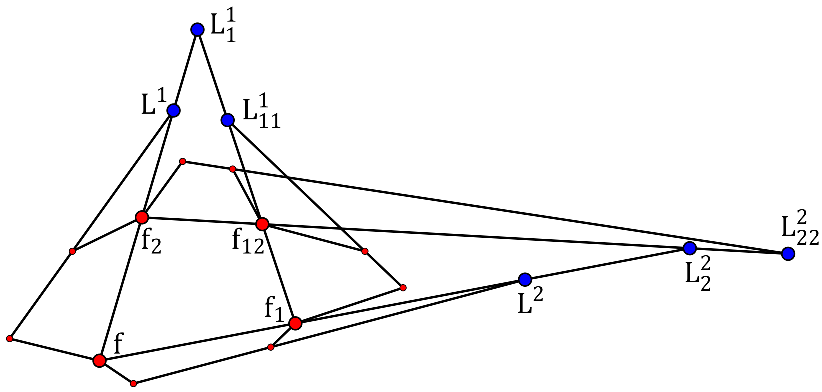

Figure 3.10: Laplace transforms

L_{11}^1 and L_{22}^2 lie in the tangent plane aswell: Shift the equations from the proof of the above lemma once in the corresponding directions to obtain:

$$L_{11}^1 = (B_{12}A_1 + C_1)f_1 + B_{12}B_2 f_2 + B_{12}Cf; \tag{3.54}$$

$$L_{22}^2 = (A_{12}B_2 + C_2)f_2 + A_{12}A_1 f_1 + A_{12}Cf; \tag{3.55}$$

We know there exists a pencil of conics through the four points L^1, L_1^1, L^2 and L_2^2. We discretize the notion of second order contact by the condition that the conic passes through one more of the Laplace transforms in each lattice direction:

Definition 3.17 (Doliwas Koenigs lattice) Let $f : \mathbb{Z}^2 \to \mathbb{RP}^N$ be a Q-net. It is called a Koenigs lattice if for every point of the net there exists a conic passing through the six points L^1, L_1^1, L_{11}^1, L^2, L_2^2, L_{22}^2.

We saw that a smooth Koenigs net had a simpler Laplace equation than a normal conjugate net. This holds for discrete Koenigs lattices as well:

Theorem 3.18 *The Laplace equation of a Koenigs lattice $f : \mathbb{Z}^2 \to \mathbb{RP}^N$ can be gauged into the canonical form*

$$f_{12} + f = \alpha_1 f_1 + \alpha_2 f_2, \tag{3.56}$$

with some scalar function $\alpha : \mathbb{Z}^2 \to \mathbb{R}$.

This result indicates that the discretization makes sense. Proof can be found in [Dol03].
There is also a geometrically simpler way to describe the Koenigs lattice, which uses only incidence:

Theorem 3.19 *Let $f : \mathbb{Z}^2 \to \mathbb{RP}^N$ be a Q-net. f is a Koenigs lattice if and only if at every point the three lines $(L^2 L^1_{11})$, $(f f_{12})$ and $(L^1 L^2_{22})$ are concurrent.*

Proof. Pascals' theorem tells us that six points P_1, P_2, P_3, P_4, P_5 and P_6 lie on a conic, if and only if $X = (P_1P_5) \cap (P_2P_4)$, $Y = (P_3P_4) \cap (P_1P_6)$ and $Z = (P_2P_6) \cap (P_3P_5)$ are collinear.

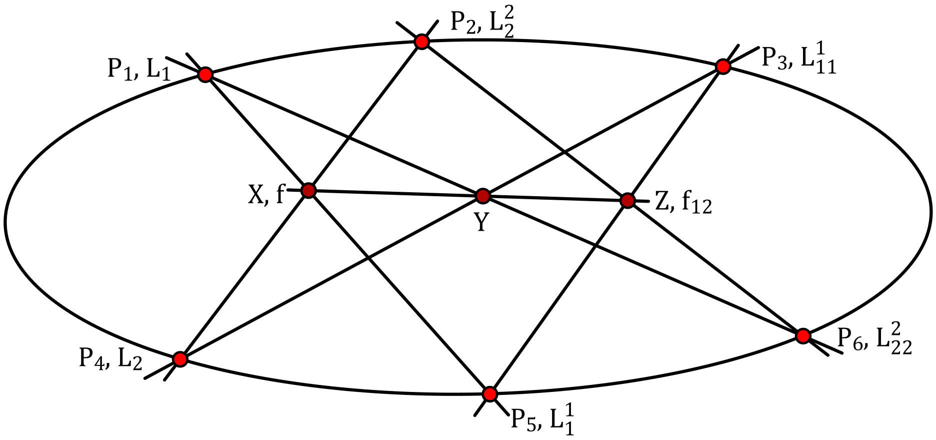

Figure 3.11: Pascals' theorem

Applying this to the points L^1, L^2_2, L^1_{11}, L^2, L^1_1 and L^2_{22} yields

$$
\begin{aligned}
& L^1, L^2_2, L^1_{11}, L^2, L^1_1 \text{ and } L^2_{22} \text{ lie on a conic} && (3.57)\\
\Leftrightarrow\ & f = (L_1 L^1_1) \cap (L^2_2 L^2), f_{12} = (L^2_2 L^2_{22}) \cap (L^1_{11} L^1_1) \text{ and} && (3.58)\\
& Y = (L^1_{11} L^2) \cap (L^1 L^2_{22}) \text{ are collinear} && \\
\Leftrightarrow\ & (f f_{12}), (L^1_{11} L^2) \text{ and } (L^1 L^2_{22}) \text{ intersect in a point.} && (3.59)
\end{aligned}
$$

□

Since incidence is equivalent in $\mathbb{R}^N$ and $\mathbb{RP}^N$, we can omit the projectivization and work with Euclidean coordinates again. In Euclidean space we know this configuration to be an 8-point configuration.

Corollary 3.20 *Let $f : \mathbb{Z}^2 \to \mathbb{R}^N$ be a Q-net. f is a Koenigs lattice if and only if every quadrilateral f, f_1, f_2 and f_{12} has its Laplace transforms L^1, L^2, L^1_{11} and L^2_{22} in 8-point configuration on the quadrilateral.*

Proof. Follows from Theorem 3.19 and Theorem 3.5. □

Chapter 4

Connection of both discretizations

4.1 Main theorem

Since classical discrete two-dimensional Koenigs nets and Doliwas Koenigs lattices are both discretizations of smooth Koenigs nets, it seems natural that these notions have some connection. We introduced the notion of D-nets as nets made of intersection points of diagonals of classical discrete Koenigs nets. It is the main goal of this chapter to show that the notion of a D-net is already equivalent to the notion of Doliwas Koenigs lattice.

Theorem 4.1 (Equivalence of Doliwas Koenigs lattices and D-nets) *A map $f : \mathbb{Z}^2 \to \mathbb{R}^N$ is a D-net if and only if it is a Koenigs lattice of Doliwa.*

We already know that for every Koenigs net we find a corresponding D-net. In the proof of the theorem we will take a look at the construction of a Koenigs net upon a D-net. We will find out that we have a $(2N+2)$-degree of freedom in this construction. This means that we can biject between D-nets (Doliwa lattices) and $(2N+2)$-parameter families of Koenigs nets. Therefore, there are "more" classical Koenigs nets than Doliwa lattices.

4.2 Proof of the theorem

We will prove the theorem by finding an algebraic condition for a Q-net to be a D-net and proving that this condition is equivalent to the Q-net being a Koenigs lattice. In this section $f : \mathbb{Z}^2 \to \mathbb{R}^N$ will be a prescribed Q-net and $x : \mathbb{Z}^2 \to \mathbb{R}^N$ will be the Koenigs net we try to construct. We will find out that the condition on f for this to work out is equivalent to the condition on f to be a Doliwa Koenigs lattice.

We will now take a look at the construction. Since the diagonal intersection points are given, we will construct the diagonals and points of the Koenigs net. We need to find two diagonals of the Koenigs net through each Q-net point, such that each four of the edges meet in a point (the Koenigs-net point). The edges of Q-net and Koenigs net don't play a role in this construction.

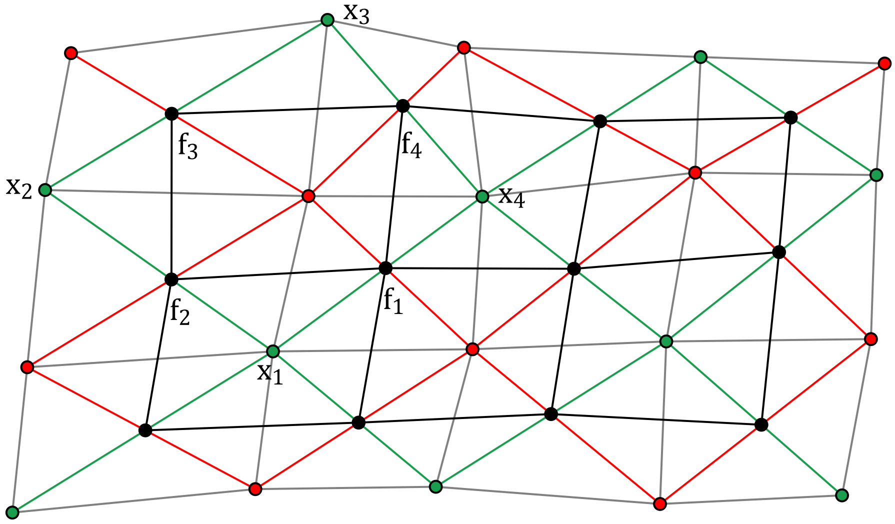

Figure 4.1: Construction of a Koenigs net f (grey edges) on a given Q-net x (black). The Koenigs net can be split into two subnets (green and red).

As in Section 3.2.3 we can split the Koenigs net into two bipartite parts, such that each diagonal connects two points of the same part. The parts form nets themselves, which we will again call the *subnets* of the Koenigs net.
Assume we can prescribe some starting point x_1 in the construction. Then the four adjacent diagonals are set. If we pick two points x_2 and x_4 on these corresponding diagonals, the subnet quadrilateral might not close, because we don't know whether the lines $(x_2 f_3)$ and $(x_4 f_4)$ intersect in a point x_3. If it exists we still need to make sure that the Q-net points are in 8-point configuration on the constructed subnet quad (see Theorem 3.13). We deal with this first obstruction in the next subsection. If you construct further Koenigs net points from x_1, notice that they will always be points from the same subnet. There are no dependencies to the other subnet. This means we can construct both subnets completely independent of each other. Therefore we only consider one subnet in the remains of the proof. In the following subsection we first construct one subnet quadrilateral, then find a condition to be able to construct four subnet quadrilaterals sharing a point, and then prove that the condition being fulfilled everywhere is enough to construct the hole subnet.

4.2.1 Construction of a subnet quadrilateral

Constructing a quadrilateral of the subnet is already nontrivial. For the construction we need a quadrilateral of the Q-net f and we need to construct the subnet quad "around" the Q-net quad, which means that every point of the Q-net quad lies on one edge of the subnet quad. Also the Q-net points must be in 8-point configuration on the subnet quad. We will assume that any arbitrarily chosen point is distinct from all other known points. We have $N + 1$ degree of freedom in this construction:

Lemma 4.2 *For a given Q-net quadrilateral F we can construct a surrounding subnet quad X uniquely if we prescribe one of the subnet edges, such that the edge contains the corresponding Q-net point.*

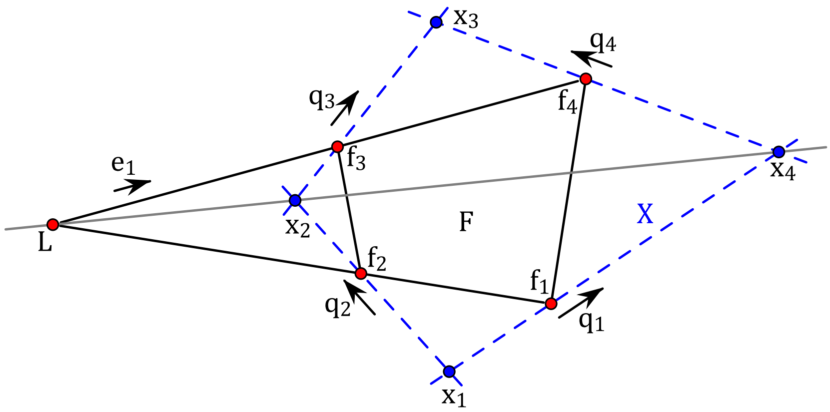

Figure 4.2: A Q-net quadrilateral F with one of its Laplace transforms L and a subnet quadrilateral X, which has been constructed "around" F in 8-point configuration.

Proof. Let $F = (f_1 f_2 f_3 f_4)$ be the prescribed Q-net quadrilateral. One point in $\mathbb{R}^N$ (w.l.o.g. x_1) can be chosen arbitrarily, then one neighbouring point (w.l.o.g. x_2) can be chosen freely on the line $(x_1 f_2)$. Now the rest is uniquely determined: x_4 must be on the line $(L x_2)$ for F to be in 8-point configuration on X (see Figure 4.2). Therefore x_4 is the intersection point of $(L x_2)$ and $(x_1 f_1)$. It exists, since both lines share the plane $(f_1 f_2 x_1)$. In the plane $(x_2 f_3 f_4)$ we find x_3 uniquely as intersection point of the lines $(x_2 f_3)$ and $(x_4 f_4)$. $\square$

This construction is geometrically straight forward, but makes further constructions complicated. We will make use of the quantity q on the Q-net points to get an algebraic formular. We will prescribe ratios q_1, q_2, q_3 and q_4, which should fulfill (after construction of the subnet quad X)

$$q_1 = q(x_1, f_1, x_4), \quad q_2 = q(x_1, f_2, x_2), \quad q_3 = q(x_2, f_3, x_3), \quad q_4 = q(x_4, f_4, x_3). \tag{4.1}$$

We will now check under which condition we can construct a quad X fitting these ratios. After choosing x_1 we can calculate x_2 and x_4:

$$x_2 = f_2 + q_2(x_1 - f_2), \tag{4.2}$$
$$x_4 = f_1 + q_1(x_1 - f_1). \tag{4.3}$$

Now we have two possibilities to calculate x_3:

$$x_3^1 = f_3 + q_3(x_2 - f_3), \tag{4.4}$$
$$x_3^2 = f_4 + q_4(x_4 - f_4). \tag{4.5}$$

The construction with the given q's works if and only if $x_3^1 = x_3^2$. Plugging in the above equations yields

$$(1 - q_3)f_3 + q_3x_2 = (1 - q_4)f_4 + q_4x_4, \tag{4.6}$$

and even further:

$$(1 - q_3)f_3 + q_3(1 - q_2)f_2 + q_3q_2x_1 = (1 - q_4)f_4 + q_4(1 - q_1)f_1 + q_4q_1x_1, \tag{4.7}$$

which is the condition that the quadrilateral closes. However if we want X to be a subnet quad of a Koenigs net we can't prescribe arbitrary ratios q_i. In fact X is a suitable subnet quad if and only if the hole construction is an 8-point configuration, i.e. if

$$q(x_4, f_1, x_1)q(x_1, f_2, x_2)q(x_2, f_3, x_3)q(x_3, f_4, x_4) = 1. \tag{4.8}$$

Comparing this with our choice of the q's, we find that this is equivalent to

$$q_3q_2 = q_4q_1. \tag{4.9}$$

For f to be a D-net quadrilateral it is therefore necessary to prescribe q's, which fulfill that condition. If we add this as an assumption, the closing conditions becomes

$$(1 - q_3)f_3 + q_3(1 - q_2)f_2 = (1 - q_4)f_4 + q_4(1 - q_1)f_1, \tag{4.10}$$

which does not depend on any x_i. The independence of x_i means that for these given q's the subnet quad X will close for any starting point x_1. Comparing this to Lemma 4.2 yields: We can choose x_1 arbitrarily. Choosing x_2 on the line (x_1f_2) is equivalent to choosing q_2. From the Lemma we know that the rest is unique. This means we can choose one q and calculate the rest from the closing condition:

Lemma 4.3 *Given a planar quadrilateral F and one ratio q_i, we can uniquely solve the closing condition*

$$(1 - q_3)f_3 + q_3(1 - q_2)f_2 = (1 - q_4)f_4 + q_4(1 - q_1)f_1, \tag{4.11}$$

such that $q_3q_2 = q_4q_1$ is fulfilled.

It is difficult to explicitly solve this equation. In $\mathbb{R}^3$ one can use Cramer's rule to do that. In arbitrary dimensions it appears to be useful to write one f_i as linear combination of the other three before solving. This is always possible, since the four points share a plane. We will use a similar trick to solve the equation.

Remark 4.4 Note that our closing condition is not very symmetric in the sense that it includes the q_i's and f_i's in different manner. The reason is that the direction of the q_i's is not chosen symmetrically. We can formulate a more symmetric closing condition if we choose the direction of the q_i's cyclically:

$$q_1 = q(x_4 f_1 x_1), \quad q_2 = q(x_1 f_2 x_2), \quad q_3 = q(x_2 f_3 x_3), \quad q_4 = q(x_3 f_4 x_4). \tag{4.12}$$

Then the closing condition becomes

$$(1 - q_1)f_1 + q_1(1 - q_4)f_4 + q_1 q_4(1 - q_3)f_3 + q_1 q_4 q_3(1 - q_2)f_2 = 0. \tag{4.13}$$

It doesn't look more symmetric at first glance, however by multiplying q_2, then q_3, then q_4, and making use of $q_1 q_2 q_3 q_4 = 1$ we get the equivalent equations

$$(1 - q_2)f_2 + q_2(1 - q_1)f_1 + q_2 q_1(1 - q_4)f_4 + q_2 q_1 q_4(1 - q_3)f_3 = 0, \tag{4.14}$$
$$(1 - q_3)f_3 + q_3(1 - q_2)f_2 + q_3 q_2(1 - q_1)f_1 + q_3 q_2 q_1(1 - q_4)f_4 = 0, \tag{4.15}$$
$$(1 - q_4)f_4 + q_4(1 - q_3)f_3 + q_4 q_3(1 - q_2)f_2 + q_4 q_3 q_2(1 - q_1)f_1 = 0. \tag{4.16}$$

This shows that every q_i and f_i play the same role in the condition.

We will now return to the choice of q_i's as in Lemma 4.3 and explicitly solve the equation for q_2, q_3 and q_4 given q_1. To do this we need to write some of the points of F in coordinates relative to the others. We will use the Laplace transform L, which is the intersection point of $(f_1 f_2)$ and $(f_3 f_4)$. Let e_1 be a unit vector parallel to the line $(f_3 f_4)$. Then we find scalars γ_3, γ_4, such that

$$f_3 = L + \gamma_3 e_1, \tag{4.17}$$
$$f_4 = L + \gamma_4 e_1. \tag{4.18}$$

Also we can describe L on the line $(f_1 f_2)$ by a ratio of directed lengths $p_1 \in \mathbb{R}$:

$$p_1 = q(f_1, L, f_2). \tag{4.19}$$

L can be calculated (see equation 3.8) as

$$L = \frac{1}{1 - p_1} f_2 - \frac{p_1}{1 - p_1} f_1. \tag{4.20}$$

The quad closing condition becomes

$$\begin{aligned}&(1-q_3)(\frac{1}{1-p_1}f_2-\frac{p_1}{1-p_1}f_1+\gamma_3 e_1)+q_3(1-q_2)f_2\\=&(1-q_4)(\frac{1}{1-p_1}f_2-\frac{p_1}{1-p_1}f_1+\gamma_4 e_1)+q_4(1-q_1)f_1.\end{aligned}\tag{4.21}$$

Sorting this after e_1,f_1 and f_2 yields

$$\begin{aligned}((1-q_3)\gamma_3-(1-q_4)\gamma_4)e_1+\\(\frac{p_1}{1-p_1}((1-q_4)-(1-q_3))-q_4(1-q_1))f_1+\\(\frac{1}{1-p_1}((1-q_3)-(1-q_4))+q_3(1-q_2))f_2=0.\end{aligned}\tag{4.22}$$

Since we assume everything to be in sufficiently general position, the three vectors e_1, f_1 and f_2 are linearly independent, which means that this is equivalent to the three scalar equations

$$(1-q_3)\gamma_3-(1-q_4)\gamma_4=0,\tag{4.23}$$

$$\frac{p_1}{1-p_1}(q_3-q_4)-q_4(1-q_1)=0,\tag{4.24}$$

$$\frac{1}{1-p_1}(q_4-q_3)+q_3(1-q_2)=0.\tag{4.25}$$

Solving these for q_2, q_3, q_4 is some simple algebra, however it is a quite long computation, which is why we won't write down the hole process. It is important to note that the solution always exists and is unique as long as we require the ratios to be unequal to 0 or 1 (or infinity), which is justified, since these values correspond to two points coinciding, which we excluded anyway. Solving yields

$$q_2=\frac{p_1q_1}{1+q_1(p_1-1)},\tag{4.26}$$

$$q_3=\frac{(\gamma_3-\gamma_4)(1+(p_1-1)q_1)}{\gamma_3-\gamma_4p_1+\gamma_3q_1(p_1-1)},\tag{4.27}$$

$$q_4=\frac{(\gamma_3-\gamma_4)p_1}{\gamma_3-\gamma_4p_1+\gamma_3q_1(p_1-1)}.\tag{4.28}$$

We know that the solution has to fulfill $q_2q_3=q_1q_4$, which can now easily be checked.

From now on it suffices to construct the quantity q on the subnet, since the explicit choice of a starting point x_1 is not important. We need to find the quantity q, such that the closing condition is fulfilled on any quad of the subnet. This results in a hole N-parameter family of subnets, since we get a subnet for any arbitrary starting point x_1.

4.2.2 Condition for four subnet quadrilaterals

The next step is to construct four quadrilaterals of a subnet, which share a point. From the last chapter we know that it is enough to construct the quantity q on the four quadrilaterals. If we find q's, such that the quad closing condition is fufilled on all four quadrilaterals we are finished.

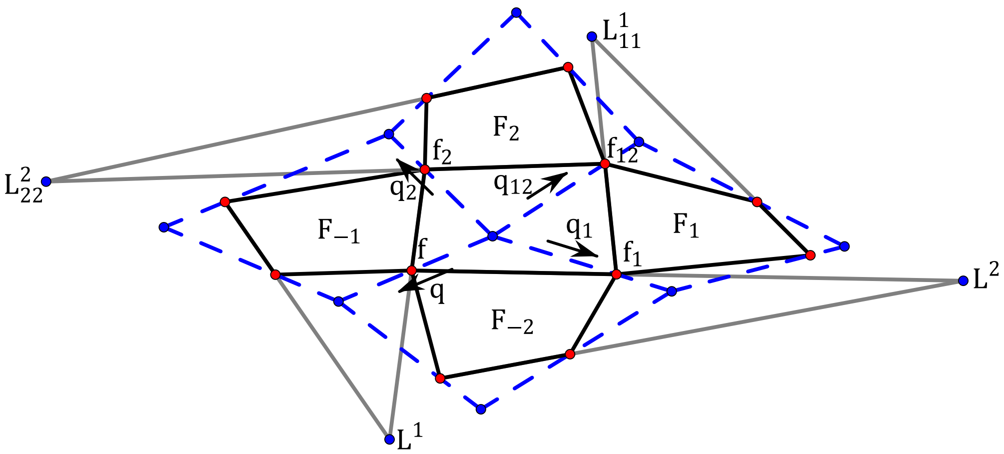

Figure 4.3: Construction of four subnet quadrilaterals (blue) "around" the four Q-net quads F_1, F_2, F_{-1} and F_{-2}.

We name the quantities q, q_1, q_{12} and q_2 as shown in Figure 4.3. Also p_{-2}, p_1, p_2 and p_{-1} should be defined by

$$p_{-2} = q(f, L^2, f_1),\ p_1 = q(f_1, L^1_{11}, f_{12}),\ p_2 = q(f_{12}, L^2_{22}, f_2),\ p_{-1} = q(f_2, L^1, f). \tag{4.29}$$

We can choose q arbitrarily. We know from last section that q_1 is then already unique if the closing condition on F_{-2} should be fulfilled. We can calculate it to be

$$q_1 = \frac{p_{-2}q}{1 + q(p_{-2} - 1)}. \tag{4.30}$$

We can use this q_1 to uniquely determine q_{12}, and from q_{12} we get q_2 uniquely. Now the condition for the construction to work out is that q_2 and q need to fulfill the closing condition on F_{-1}. The corresponding formulars are

$$q_{12} = \frac{p_1 q_1}{1 + q_1(p_1 - 1)}, \tag{4.31}$$

$$q_2 = \frac{p_2 q_{12}}{1 + q_{12}(p_2 - 1)}, \tag{4.32}$$

$$q = \frac{p_{-1} q_2}{1 + q_2(p_{-1} - 1)}. \tag{4.33}$$

It is important to note that the q's on the "outer" subnet edges exist and can be calculated, however they don't have any connection to the other three quadrilaterals, which is why we don't need to pay attention to them anymore. Plugging the formulars into each other yields

$$q_{12} = \frac{p_1 q_1}{1 + q_1(p_1 - 1)} = \frac{p_{-2} p_1 q}{1 + (p_{-2} p_1 - 1)q}, \tag{4.34}$$

$$q_2 = \frac{p_2 q_{12}}{1 + q_{12}(p_2 - 1)} = \frac{p_{-2} p_1 p_2 q}{1 + (p_{-2} p_1 p_2 - 1)q}, \tag{4.35}$$

$$q = \frac{p_{-1} q_2}{1 + q_2(p_{-1} - 1)} = \frac{p_{-2} p_1 p_2 p_{-1} q}{1 + (p_{-2} p_1 p_2 p_{-1} - 1)q}. \tag{4.36}$$

This last equation is the condition we are looking for:

$$p_{-2} p_1 p_2 p_{-1} q = (1 + (p_{-2} p_1 p_2 p_{-1} - 1)q)q \tag{4.37}$$

$$\Leftrightarrow \qquad 0 = q(1 - q)(p_{-2} p_1 p_2 p_{-1} - 1). \tag{4.38}$$

If we note that $q \notin \{0, 1\}$, this is equivalent to

$$1 = p_{-2} p_1 p_2 p_{-1} \tag{4.39}$$

$$\Leftrightarrow 1 = q(f, L^2, f_1) q(f_1, L^1_{11}, f_{12}) q(f_{12}, L^2_{22}, f_2) q(f_2, L^1, f), \tag{4.40}$$

which is finally the condition that the four subnet quads can be constructed. Note that the equation is independent of any q_i, which makes this a condition directly on the points of f. It is the condition that the Laplace transforms lie in 8-point configuration on the quadrilateral $(f f_1 f_{12} f_2)$. We now know that this equation needs to hold on any quadrilateral of the Q-net f, for f to be a D-net. We don't yet know why it is already enough. The condition is the same as the condition for f to be Koenigs lattice of Doliwa. It is exactly the condition described in Corollary 3.20. It is noteworthy that we found the equation by looking at conditions on the quads F_{-2}, F_1, F_2 and F_{-2}, while the equation we found is formulated on the quad $(f f_1 f_{12} f_2)$.

4.2.3 Global construction of a subnet

We have to prove the following equivalence:

Four subnet quads sharing a point can be constructed everywhere (4.41)

$\Leftrightarrow$ Both subnets can be constructed. (4.42)

The argument is exactly the same as in the proof of Theorem 3.11 (see Figure 3.7 on the left): Start at some subnet quadrilateral. Construct subnet quads in both lattice directions (The lattice directions of the subnet are different than those of the Q-net). Then construct the rest by making use of the assumption that a forth quadrilateral can always be constructed. With this procedure one can construct both subnets.
We have proven everything: f is a D-net if and only if both subnets can be constructed, which is equivalent to f being a Doliwas Koenigs lattice.

□

4.3 Conclusion

We have proven the main theorem. We also know how to construct a Koenigs net on a D-net. The freedom in the construction of a subnet was in the construction of the first quadrilateral: Choose one point and one q arbitrarily. If we consider that we have two subnets, we see that we have a $(2N+2)$-parameter freedom in the construction of the Koenigs net.
We have shown that both discretizations of Koenigs nets are closely related. We have also shown that both discretizations have a characterization via 8-point configurations. One can wonder how properties of the nets relate. For example: If two Koenigs nets are dual to each other, how do the corresponding Doliwa lattices relate? How do the Koenigs transformations of classical Koenigs nets and Doliwa lattices relate? These questions might be answered in future papers.

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Figures

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