Comparison of Inconsistency Measures, Model Order Reduction Methods and Interpolation/Regression Methods for Parametric Model Order Reduction by Matrix Interpolation

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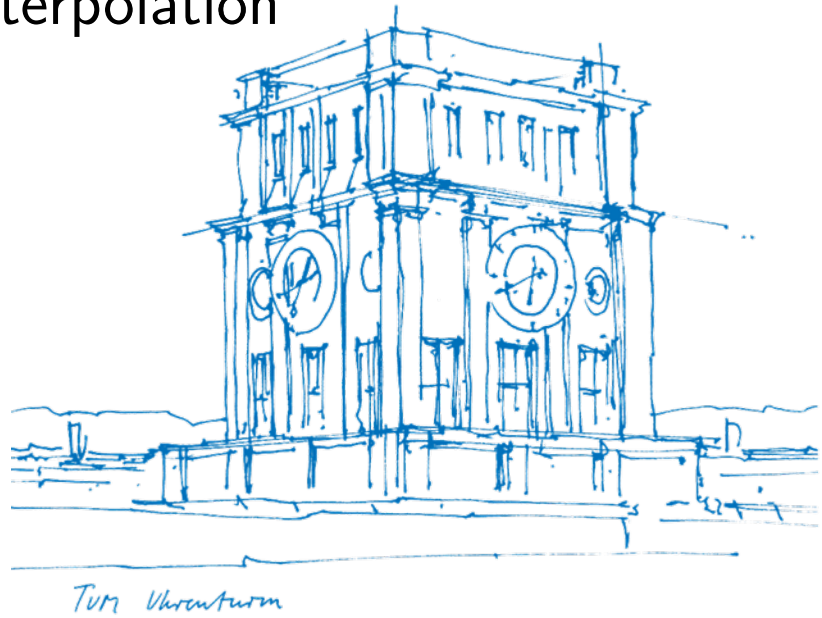
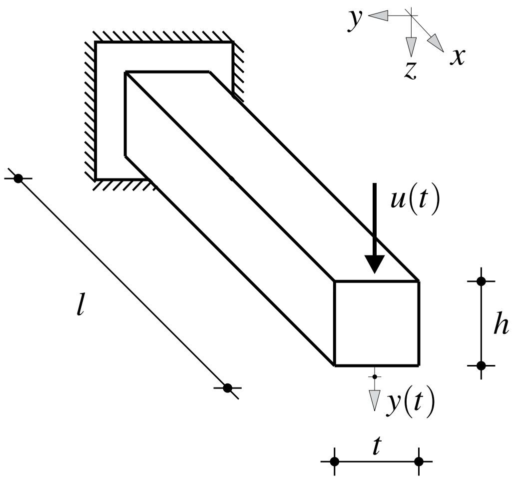
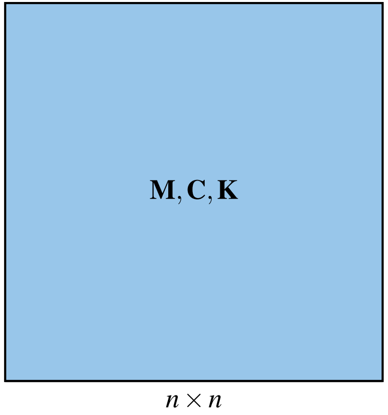


Motivation





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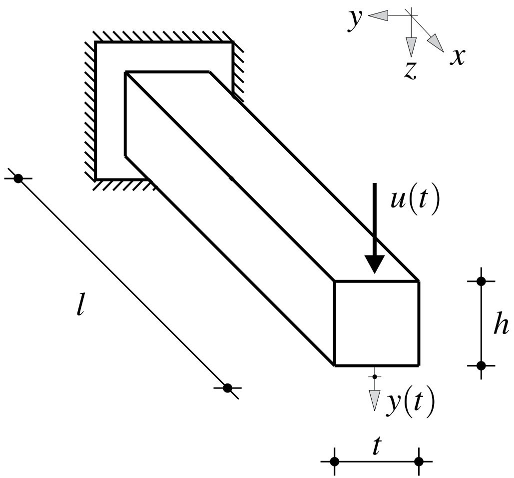

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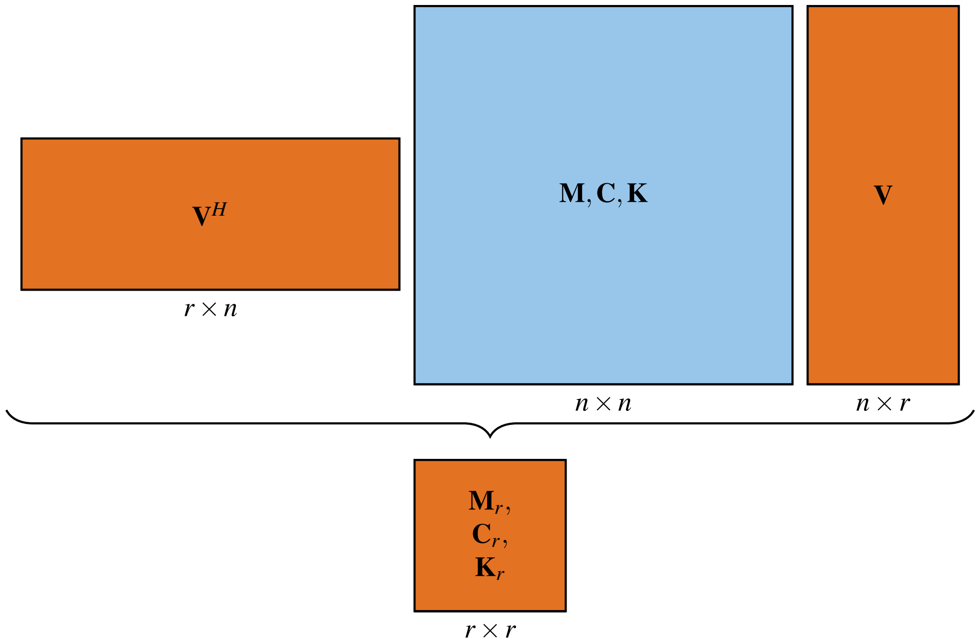

$\mathbf{V}^H$
$r \times n$
$\mathbf{M}, \mathbf{C}, \mathbf{K}$
$n \times n$
$\mathbf{V}$
$n \times r$
$\mathbf{M}_r$, $\mathbf{C}_r$, $\mathbf{K}_r$
$r \times r$

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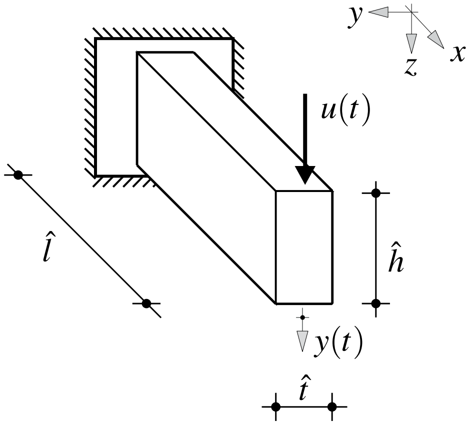

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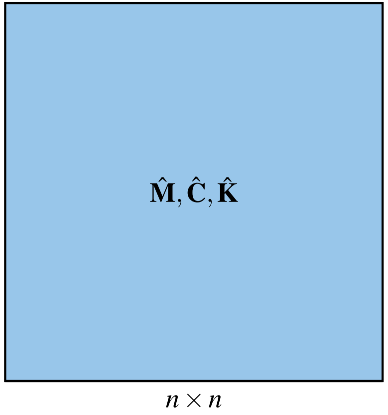

$\hat{\mathbf{M}}, \hat{\mathbf{C}}, \hat{\mathbf{K}}$
$n \times n$

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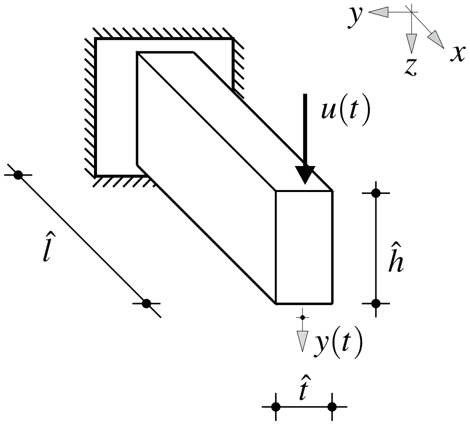

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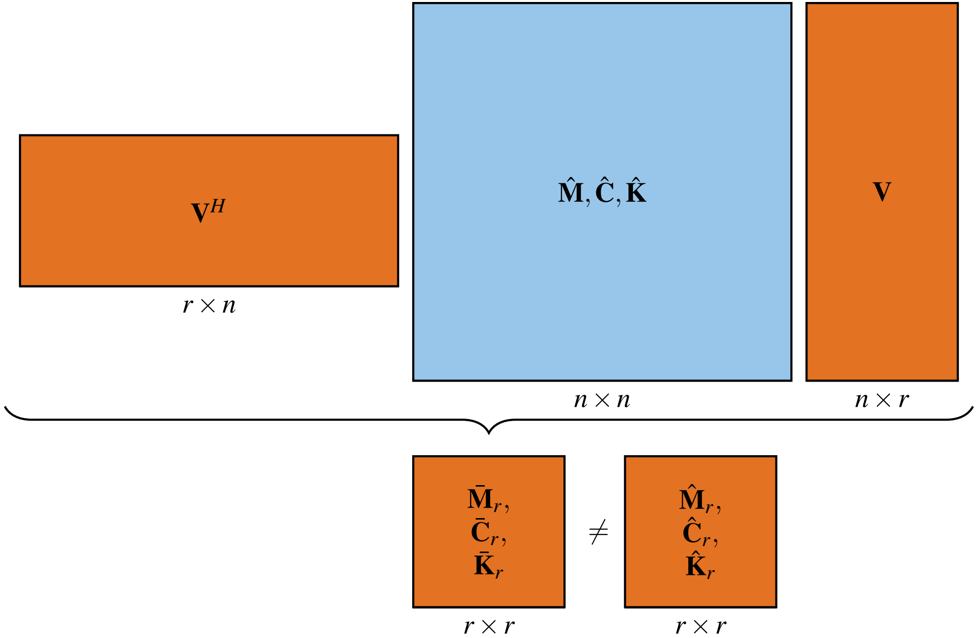

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$\neq$
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$r \times r$

Outline

Introduction

Mathematical System Description – Second-Order Systems

Linear-time invariant dynamical systems with single input and single output (SISO) in second-order form are regarded:

$$\Sigma : \begin{cases} \mathbf{M}\ddot{\mathbf{x}}(t) + \mathbf{C}\dot{\mathbf{x}}(t) + \mathbf{K}\mathbf{x}(t) &= \mathbf{f}u(t), \\ y(t) &= \mathbf{g}\mathbf{x}(t), \end{cases} \tag{1}$$

with mass, damping and stiffness matrix $\mathbf{M}, \mathbf{C}, \mathbf{K} \in \mathbb{R}^{n\times n}$ degrees of freedom $\ddot{\mathbf{x}}(t), \dot{\mathbf{x}}(t), \mathbf{x}(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$ and $\mathbf{f} \in \mathbb{R}^n$ and output $y(t) \in \mathbb{R}$ and $\mathbf{g} \in \mathbb{R}^{1\times n}$.

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After performing a Laplace transformation the transfer function of the system can be computed as

$$H(s) = \mathbf{g}\left(s^2\mathbf{M} + s\mathbf{C} + \mathbf{K}\right)^{-1}\mathbf{f}, \tag{2}$$

with the complex frequency $s \in \mathbb{C}$.

Mathematical System Description – First-Order Systems

One possibility to reformulate a second-order system into a first-order system is as follows:

$$\Sigma_I : \begin{cases} \underbrace{\begin{bmatrix} \mathbf{J} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} \end{bmatrix}}_{\mathbf{E}_I} \underbrace{\begin{bmatrix} \dot{\mathbf{x}}(t) \\ \ddot{\mathbf{x}}(t) \end{bmatrix}}_{\dot{\mathbf{x}}_I(t)} &= \underbrace{\begin{bmatrix} \mathbf{0} & \mathbf{J} \\ -\mathbf{K} & -\mathbf{C} \end{bmatrix}}_{\mathbf{A}_I} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}}_{\mathbf{x}_I(t)} + \underbrace{\begin{bmatrix} \mathbf{0} \\ \mathbf{f} \end{bmatrix}}_{\mathbf{B}_I} \mathbf{u}(t), \\ \mathbf{y}(t) &= \underbrace{\begin{bmatrix} \mathbf{g} & \mathbf{0} \end{bmatrix}}_{\mathbf{C}_I} \underbrace{\begin{bmatrix} \mathbf{x}(t) \\ \dot{\mathbf{x}}(t) \end{bmatrix}}_{\mathbf{x}_I(t)}. \end{cases} \tag{3}$$

where $\mathbf{E}_I, \mathbf{A}_I \in \mathbb{R}^{2n\times 2n}$, $\mathbf{B}_I \in \mathbb{R}^{2n}$ and $\mathbf{C}_I \in \mathbb{R}^{1\times 2n}$. $\mathbf{J} \in \mathbb{R}^{2n\times 2n}$ is an arbitrary invertible matrix, for example the identity.
An application of the Laplace transformation leads to the transfer function

$$\mathbf{H}(s) = \mathbf{C}_I \left(s\mathbf{E}_I - \mathbf{A}_I\right)^{-1} \mathbf{B}_I, \tag{4}$$

with the complex frequency $s \in \mathbb{C}$.

Parametric Model Order Reduction by Matrix Interpolation

Parametric Dynamic Systems

The system matrices and the degrees of freedom depend on d parameters $\mathbf{p} = [p_1, p_2, \dots, p_d]$.

$$H(s,\mathbf{p}) = \mathbf{g}(\mathbf{p}) \left(s^2\mathbf{M}(\mathbf{p}) + s\mathbf{C}(\mathbf{p}) + \mathbf{K}(\mathbf{p})\right)^{-1} \mathbf{f}(\mathbf{p}), \tag{5}$$

with parameter-dependent mass, damping and stiffness matrix $\mathbf{M}(\mathbf{p})$, $\mathbf{C}(\mathbf{p})$, $\mathbf{K}(\mathbf{p}) \in \mathbb{R}^{n\times n}$ and input and output vector $\mathbf{f}(\mathbf{p}) \in \mathbb{R}^n$ and $\mathbf{g}(\mathbf{p}) \in \mathbb{R}^{1\times n}$.

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Furthermore, it is assumed that it is **not** possible to efficiently compute an affine representation of the parametric dependency of the following form (exemplarily for the stiffness matrix):

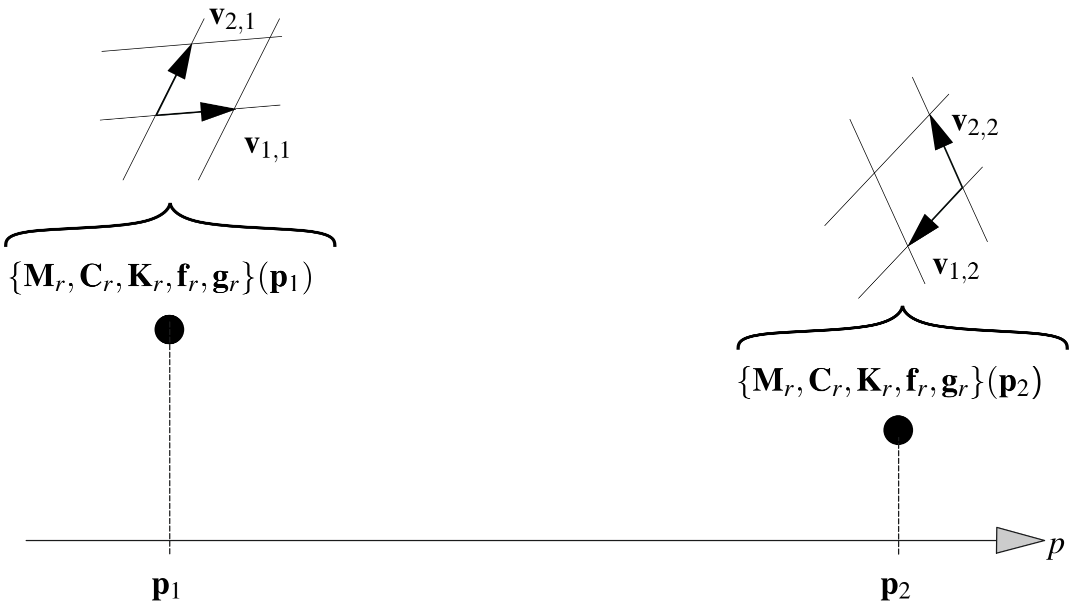
$$\mathbf{K}(\mathbf{p}) = \mathbf{K}_0 + \sum_{i=1}^{M} f_i(\mathbf{p}) \mathbf{K}_i, \qquad i = 1, \ldots, M, \tag{6}$$

where $f_i(\mathbf{p})$ are scalar functions. [BGW15]

Parametric Model Order Reduction by Matrix Interpolation

To handle non-affine parametric dependencies, the following workflow was proposed by [PMEL10]:

1. Sampling of local reduced systems



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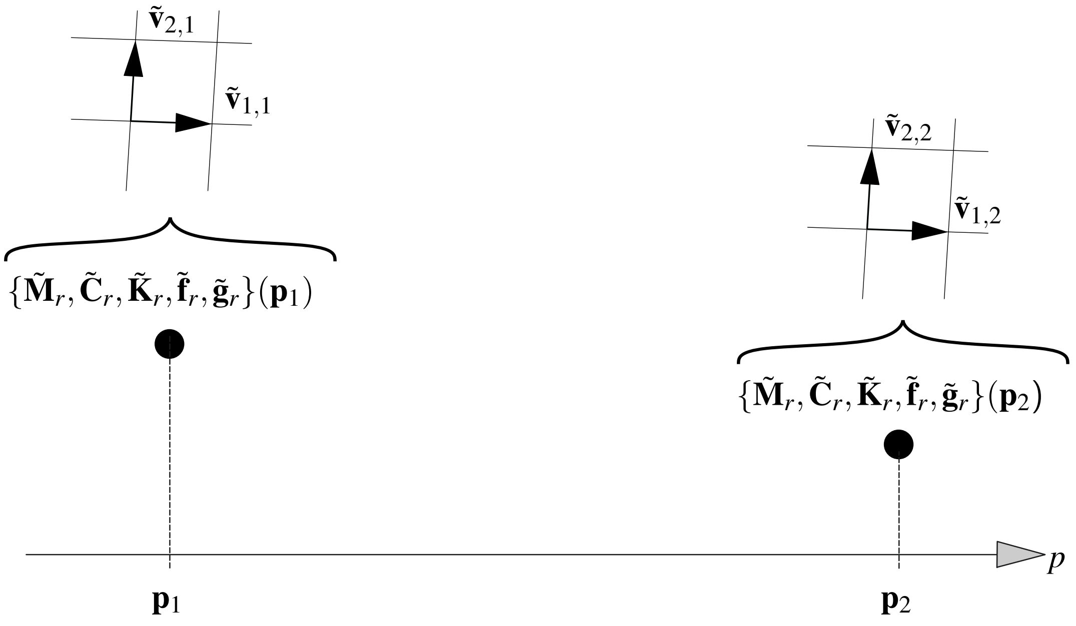
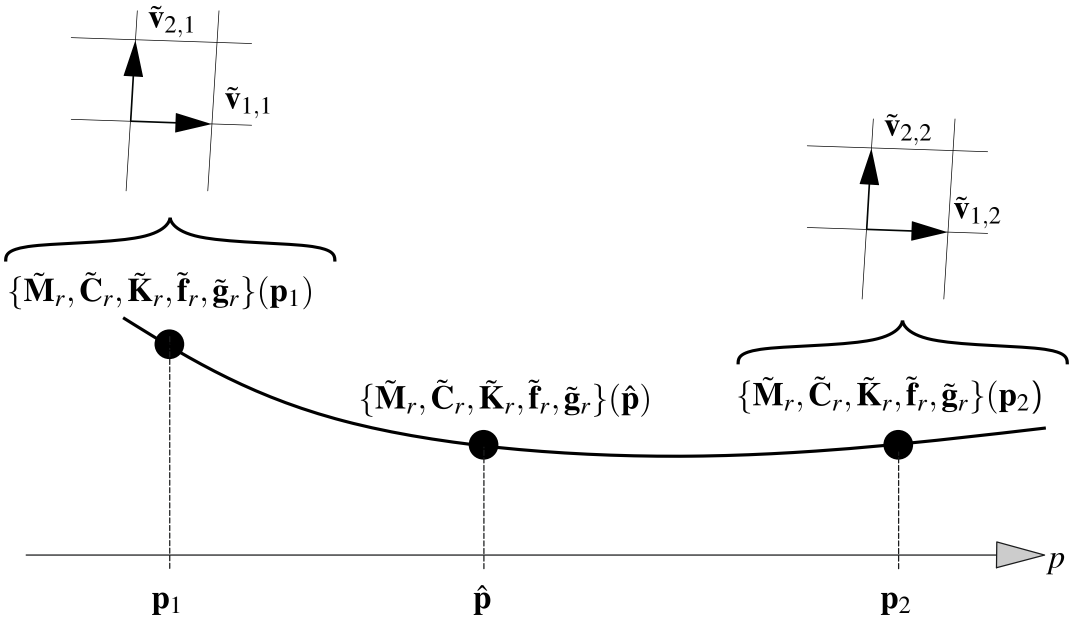
2. Transformation to generalized coordinate system



Parametric Model Order Reduction by Matrix Interpolation

To handle non-affine parametric dependencies, the following workflow was proposed by [PMEL10]:

1. Sampling of local reduced systems
2. Transformation to generalized coordinate system
3. Interpolation/ regression of reduced operators



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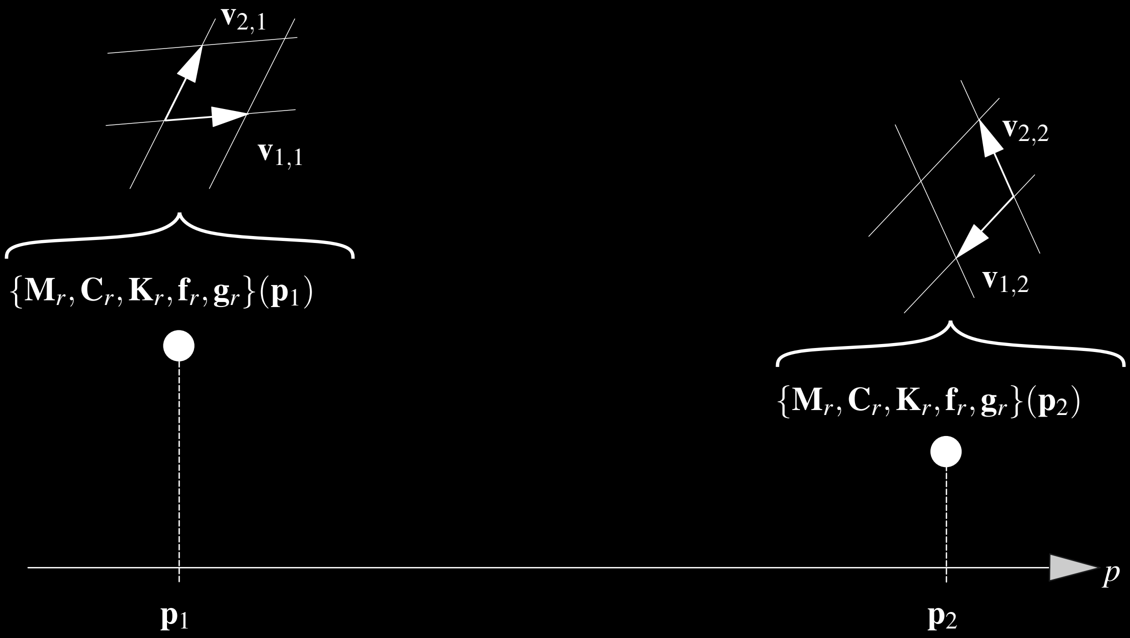

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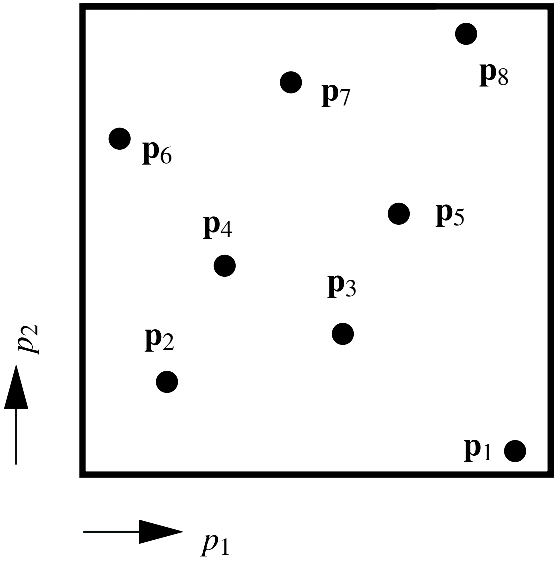


$$\{\mathbf{M}(\mathbf{p}_k), \mathbf{C}(\mathbf{p}_k), \mathbf{K}(\mathbf{p}_k), \mathbf{f}(\mathbf{p}_k), \mathbf{g}(\mathbf{p}_k)\}$$

$\Downarrow$ Project into $\mathbf{V}_k \in \mathbb{C}^{n\times r}$, $(\mathbf{x}(\mathbf{p}_k) \approx \mathbf{V}_k \mathbf{x}_r(\mathbf{p}_k))$

$$\{\mathbf{M}_r(\mathbf{p}_k), \mathbf{C}_r(\mathbf{p}_k), \mathbf{K}_r(\mathbf{p}_k), \mathbf{f}_r(\mathbf{p}_k), \mathbf{g}_r(\mathbf{p}_k)\}$$

with

$$\mathbf{M}_r(\mathbf{p}_k) = \mathbf{V}_k^{\mathsf{H}} \mathbf{M}(\mathbf{p}_k)\mathbf{V}_k, \qquad \mathbf{f}_r(\mathbf{p}_k) = \mathbf{V}_k^{\mathsf{H}} \mathbf{f}(\mathbf{p}_k),$$
$$\mathbf{C}_r(\mathbf{p}_k) = \mathbf{V}_k^{\mathsf{H}} \mathbf{C}(\mathbf{p}_k)\mathbf{V}_k, \qquad \mathbf{g}_r(\mathbf{p}_k) = \mathbf{g}(\mathbf{p}_k)\mathbf{V}_k,$$
$$\mathbf{K}_r(\mathbf{p}_k) = \mathbf{V}_k^{\mathsf{H}} \mathbf{K}(\mathbf{p}_k)\mathbf{V}_k$$

1. Sampling of Local Reduced Systems – Modal Truncation (MT)

In modal truncation (MT), selected eigenmodes of a proportionally damped structure build the reduced basis $\mathbf{V}$. For this, the eigenvectors Φ of the undamped system are computed:

$$(\omega^2\mathbf{M}+\mathbf{K})\Phi = \mathbf{0}. \tag{7}$$

To build the reduced basis, the r eigenmodes with the largest dominancy according to the following index are selected: [BKTT15]

$$\frac{\|\mathbf{g}\phi_i\phi_i^{\mathsf{T}}\mathbf{f}\|_2}{\mathrm{Re}(\omega_{d+,i})\,\mathrm{Re}(\omega_{d-,i})}, \tag{8}$$

with the damped eigenfrequency

$$\omega_{d\pm,i} = -\omega_i\xi_i \pm \omega_i\sqrt{\xi_i - 1}, \tag{9}$$

and

$$\Phi^{\mathsf{T}}\mathbf{C}\Phi = \Xi = \mathrm{diag}(2\omega_1\xi_1,\ldots,2\omega_n\xi_n). \tag{10}$$

1. Sampling of Local Reduced Systems – Proper Orthogonal Decomposition (POD)

For Proper Orthogonal Decomposition (POD), snapshots of the state are computed for various frequencies s_i, $i = 1, \dots r$:

$$\mathbf{X} = [\mathbf{x}(s_1), \mathbf{x}(s_2), \dots, \mathbf{x}(s_r)] \,. \tag{11}$$

Afterwards, a singular value decomposition (SVD) of the snapshots is performed:

$$\mathbf{X} = \mathbf{V}\boldsymbol{\Sigma}\mathbf{S}^{\mathsf{H}}, \tag{12}$$

where $\mathbf{V} \in \mathbb{C}^{n \times n}$ and $\mathbf{S} \in \mathbb{C}^{r \times r}$ are the left and right singular vectors. $\boldsymbol{\Sigma} \in \mathbb{R}^{n \times r}$ is a diagonal matrix with the non-negative singular values σ_i, $i = 1, \dots, r$ on the diagonal in a descending order. [GHV21]

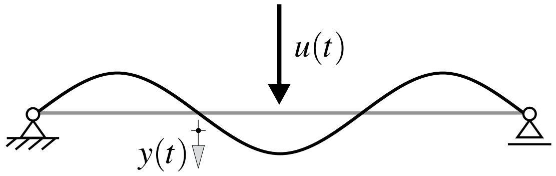
1. Sampling of Local Reduced Systems – Second-Order Iterative Rational Krylov Algorithm (SO-IRKA)

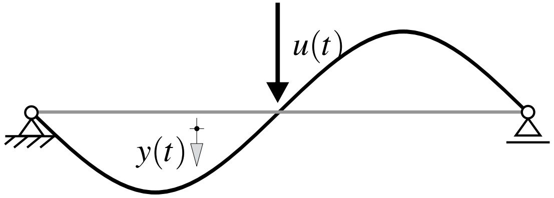
In the iterative rational Krylov algorithm, expansion frequencies are found iteratively in the following steps: [GAB08], [Wya12]

1. Choose an initial set of r expansion frequencies s_i with $i = 1, \ldots, r$ closed under complex conjugation.
2. Compute reduced basis:
$$\mathbf{V} = \left[\left(s_1^2 \mathbf{M} + s_1 \mathbf{C} + \mathbf{K} \right)^{-1} \mathbf{f}, \ldots, \left(s_r^2 \mathbf{M} + s_r \mathbf{C} + \mathbf{K} \right)^{-1} \mathbf{f} \right]. \tag{13}$$
2. Compute reduced order model:
$$\mathbf{M}_r = \mathbf{V}^{\mathsf{H}} \mathbf{M}(\mathbf{p}_i) \mathbf{V}, \, \mathbf{C}_r = \mathbf{V}^{\mathsf{H}} \mathbf{C} \mathbf{V}, \, \mathbf{K}_r = \mathbf{V}^{\mathsf{H}} \mathbf{K} \mathbf{V}. \tag{14}$$
3. Solve quadratic eigenvalue problem $\left(\lambda^2 \mathbf{M}_r + \lambda \mathbf{C}_r + \mathbf{K}_r \right) \mathbf{x} = 0$.
4. Select r eigenvalues from the set of $2r$ eigenvalues as new expansion frequencies
5. Repeat steps 2. to 4. until convergence

1. Sampling of Local Reduced Systems – Balanced Truncation (BT)

Balanced Truncation (BT) is based on the concepts of controllability and observability:





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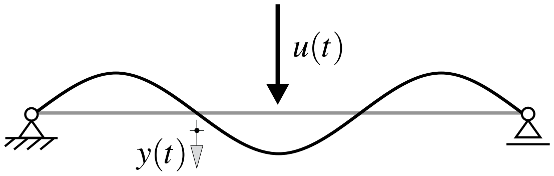
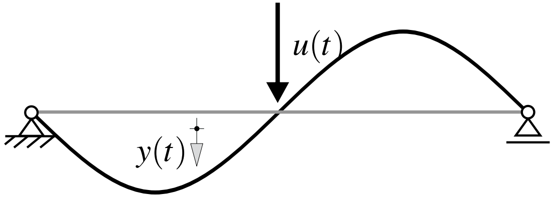




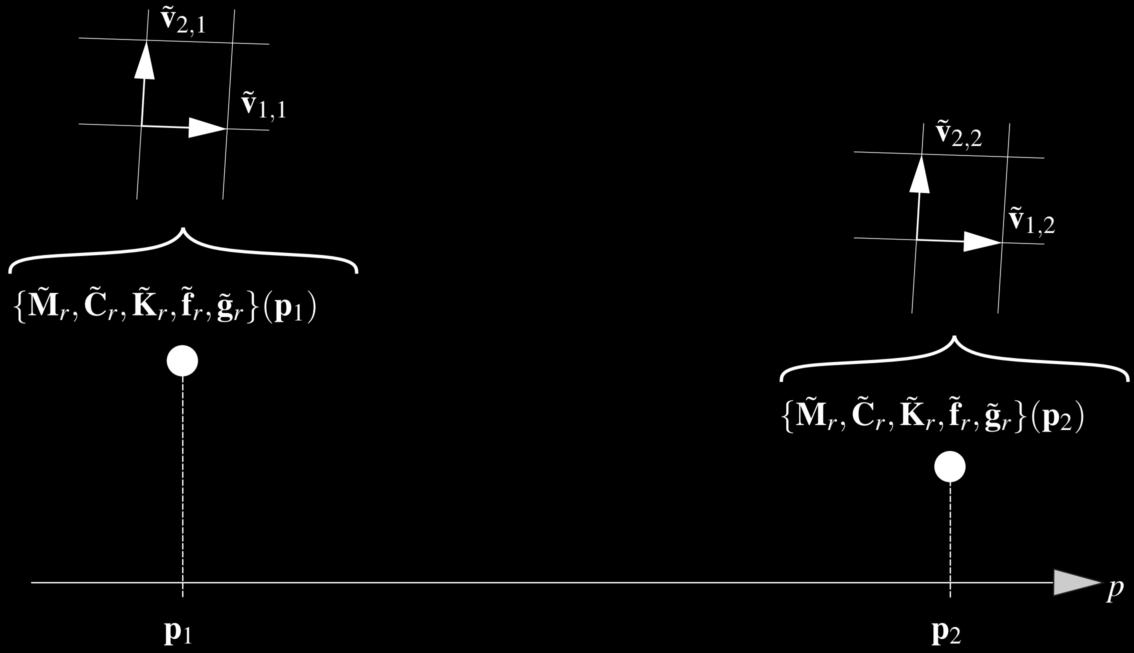
To find the most controllable and observable states, the controllability and observability Gramians $\mathbf{P}$ and $\mathbf{Q}$ have to be computed by solving the following Lyapunov equations:

$$\mathbf{E}_I \mathbf{P} \mathbf{A}_I^\mathsf{T} + \mathbf{A}_I \mathbf{P} \mathbf{E}_I^\mathsf{T} = -\mathbf{B}_I \mathbf{B}_I^\mathsf{T}, \tag{15}$$

$$\mathbf{E}_I \mathbf{Q} \mathbf{A}_I^\mathsf{T} + \mathbf{A}_I \mathbf{Q} \mathbf{E}_I^\mathsf{T} = -\mathbf{C}_I^\mathsf{T} \mathbf{C}_I. \tag{16}$$

The reduced basis is then obtained from SVDs of $\mathbf{P}$ and $\mathbf{Q}$ [MS96].

2. Transformation to Generalized Coordinate System



2. Transformation to Generalized Coordinate System

To make the interpolation meaningful, the reduced operators should be in the same coordinate system. To achieve this, the following approach was suggested in [PMEL10]:

1. Find a generalized coordinate system. For this purpose, find the most significant basis vectors by concatenating all N sampled bases and then performing an SVD:

$$[\mathbf{V}_1, \mathbf{V}_2, \ldots, \mathbf{V}_N] = \mathbf{U\Sigma Y}, \qquad \mathbf{V}_k \in \mathbb{C}^{n \times r}, \quad k = 1, \ldots, N \tag{17}$$

   The most significant basis vectors are the first r columns in $\mathbf{U}$ and denoted with $\mathbf{R}$:

$$\mathbf{R} = \mathbf{U}(:, 1 : r). \tag{18}$$

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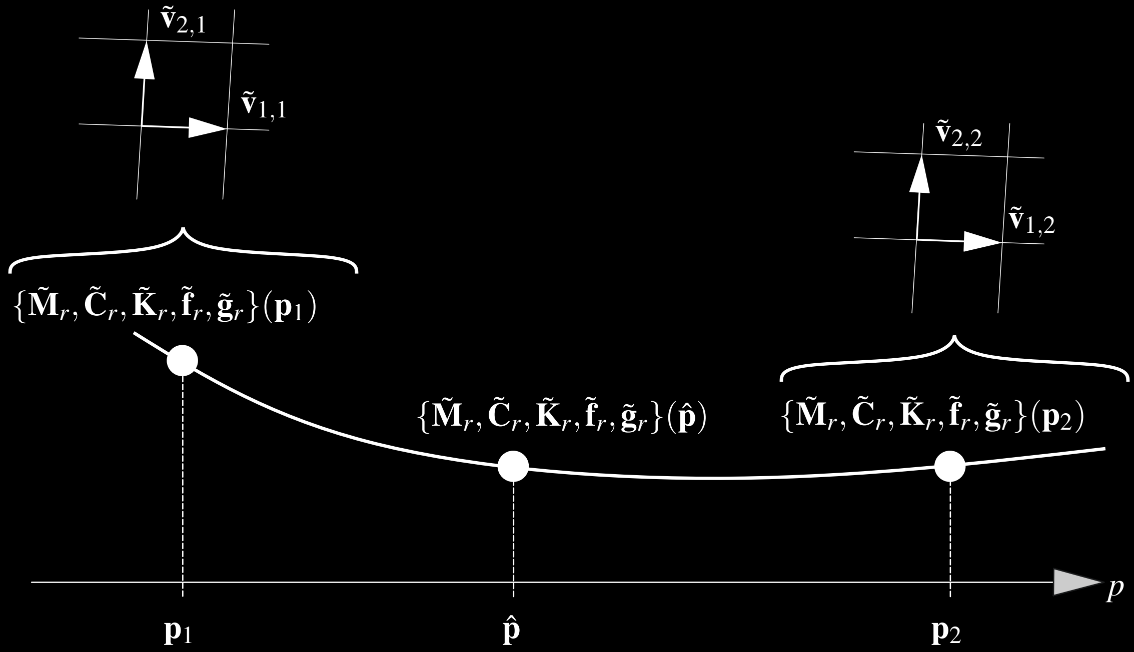
2. Transform the individual reduced operators from their individual bases $\mathbf{V}_k$ to the generalized coordinate system $\mathbf{R}$:

$$\tilde{\mathbf{K}}_r(\mathbf{p}_k) = \mathbf{T}_k^{\mathsf{T}}\mathbf{K}_r(\mathbf{p}_k)\mathbf{T}_k, \quad \tilde{\mathbf{C}}_r(\mathbf{p}_k) = \mathbf{T}_k^{\mathsf{T}}\mathbf{C}_r(\mathbf{p}_k)\mathbf{T}_k, \quad \tilde{\mathbf{M}}_r(\mathbf{p}_k) = \mathbf{T}_k^{\mathsf{T}}\mathbf{M}_r(\mathbf{p}_k)\mathbf{T}_k, \quad \tilde{\mathbf{f}}_r(\mathbf{p}_k) = \mathbf{T}_k^{\mathsf{T}}\mathbf{f}_r(\mathbf{p}_k), \quad \tilde{\mathbf{g}}_r(\mathbf{p}_k) = \mathbf{g}_r(\mathbf{p}_k)\mathbf{T}_k, \tag{19}$$

   with

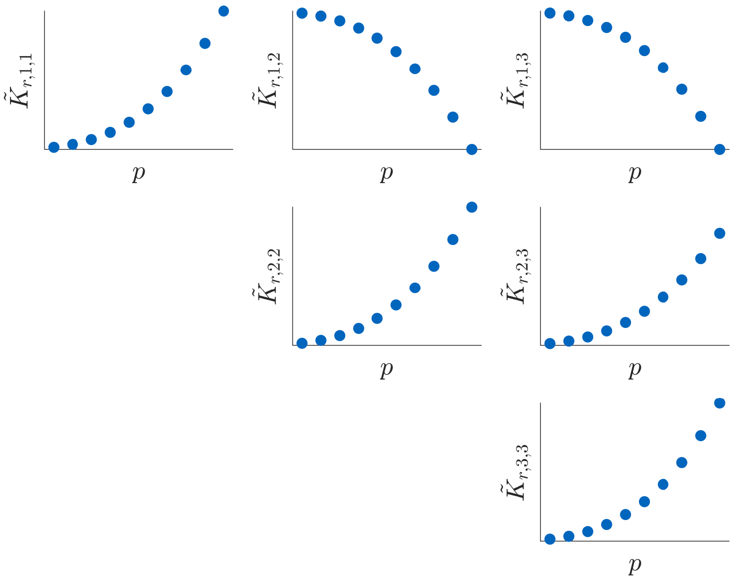
$$\mathbf{T}_k = (\mathbf{R}^T\mathbf{V}_k)^{-1}. \tag{20}$$

3. Interpolation/Regression of Reduced Operators



3. Interpolation/Regression of reduced operators

When all local reduced systems are described in a similar coordinate system, an interpolation/regression of the reduced operators is meaningful. Any interpolation/regression method can be used to learn the reduced operators entry-wise.



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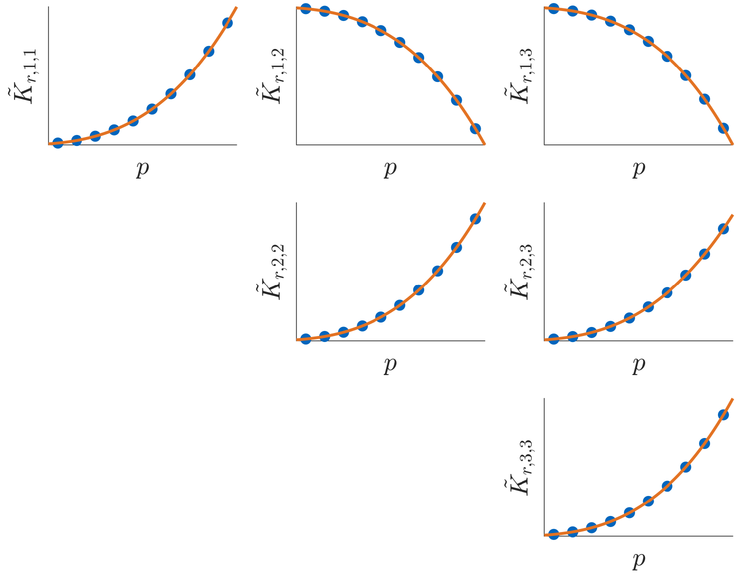

Possible methods for the interpolation/regression are

- Polynomial Regression
- Radial Basis Function
- Kriging

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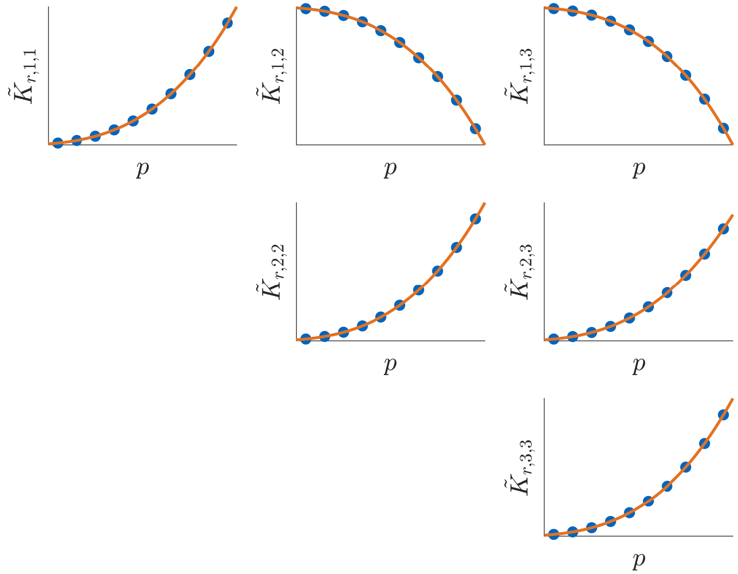

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To ensure positive definiteness of the predicted system matrices, the Cholesky decomposition of the transformed system matrices [XHD21]

$$\tilde{\mathbf{K}} = \mathbf{L}_{\mathbf{K}}^{\intercal}\mathbf{L}_{\mathbf{K}}, \tag{21}$$

or the logarithmic mapping of the transformed system matrices [AF11]

$$\mathbf{\Gamma}_{\mathbf{K}} = \log\left(\mathbf{I}^{-1/2}\tilde{\mathbf{K}}\mathbf{I}^{-1/2}\right) \tag{22}$$

can be learned instead of the transformed system matrices.

Transformation to Generalized Coordinate System – Inconsistency Measures

However, it is not guaranteed that all local reduced systems can be transformed to the generalized coordinate system. Possible measures to judge whether this is/was possible are:

(a) [ATF15] proposed for a different pMOR approach to compute the subspace angles between the reduced bases obtained in the sampling. The subspace angles between the subspaces spanned by the two bases $\mathbf{V}_i$ and $\mathbf{V}_j$, which both have to be orthonormal, are computed by first performing an SVD on the following product:

$$\mathbf{V}_i^{\mathrm{H}}\mathbf{V}_j = \mathbf{U}\boldsymbol{\Sigma}\mathbf{Y}^{\mathsf{T}}, \qquad i,j = 1,\dots,N \tag{23}$$

The subspace angles can then be found as

$$\varphi_l = \arccos(\sigma_l), \qquad l = 1,\dots,r. \tag{24}$$

In [ATF15] it is stated that in case any angle $\varphi_l \geq \frac{\pi}{4}$, consistency between the subspaces cannot be achieved.

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(a) Subspace Angles φ

(b) In pMOR by Matrix Interpolation, the basis of the local reduced systems after the transformation can be computed as

$$\tilde{\mathbf{V}}_k = \mathbf{V}_k \mathbf{T}_k. \tag{25}$$

Consistency can then be judged by computing the angle between the lth transformed basis vectors of samples i and j:

$$\psi_l = \arccos\left(\frac{< \tilde{\mathbf{v}}_{l,i}, \tilde{\mathbf{v}}_{l,j} >}{\|\tilde{\mathbf{v}}_{l,i}\|_2 \cdot \|\tilde{\mathbf{v}}_{l,j}\|_2}\right), \qquad l = 1, \ldots, r, \quad i, j = 1, \ldots, N, \tag{26}$$

where $< \cdot, \cdot >$ denotes the scalar product.

Results

Error measures

The following error measures are used for the investigated SISO systems:

- Relative error per frequency point:

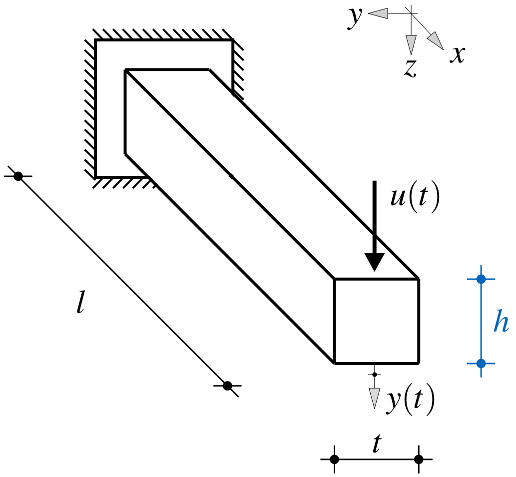
$$\varepsilon(s;\hat{\mathbf{p}}) = \frac{|y(s;\hat{\mathbf{p}}) - y_r(s;\hat{\mathbf{p}})|}{|y(s;\hat{\mathbf{p}})|} \tag{27}$$

- Relative $\mathscr{H}_\infty$ error:

$$\|\varepsilon(\cdot;\hat{\mathbf{p}})\|_{\mathscr{H}_\infty} = \sup_{s\in\mathbb{C}} |\varepsilon(s;\hat{\mathbf{p}})| \tag{28}$$

Results – Timoshenko Beam – Beam Height h

A 3D cantilevered beam discretized with Timoshenko beam elements is investigated. The beam is excited at the tip with a harmonic force of varying frequency ($[0, 1000]$ Hz). Rayleigh damping is used: $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$.

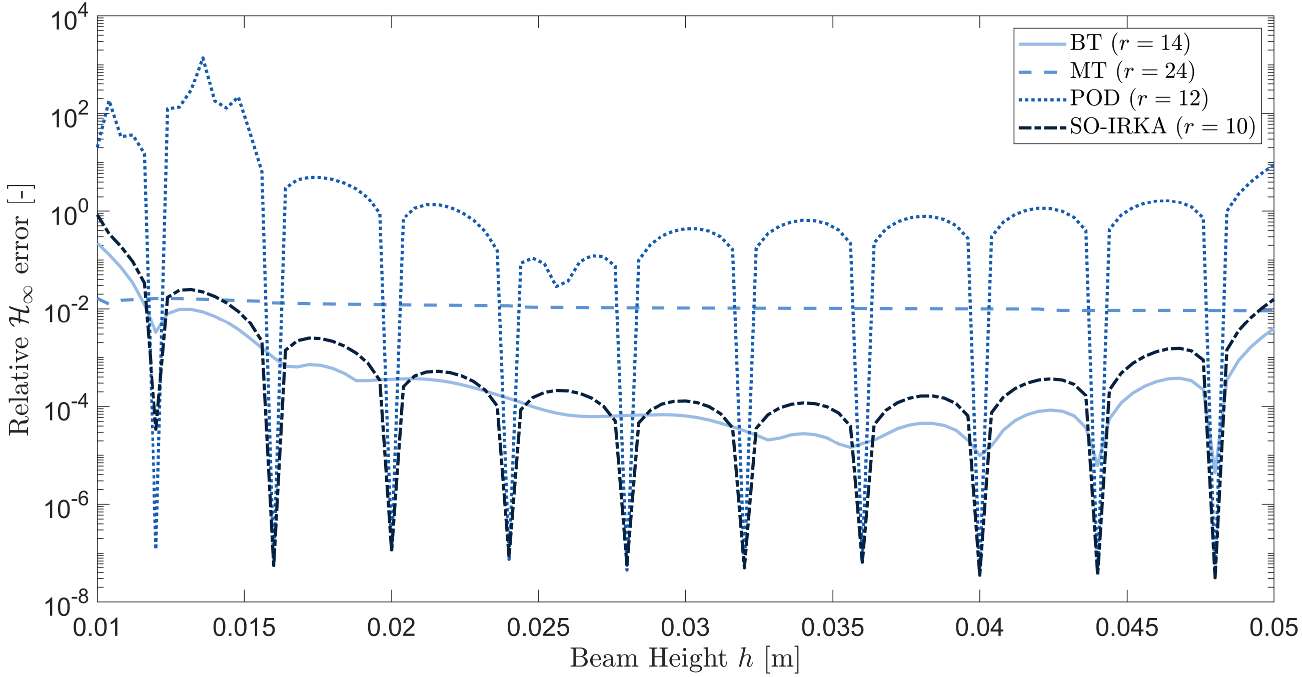


Parameter	Range/Value	Unit
Height h	$[0.01, 0.05]$	m
Thickness t	0.01	m
Length l	1.0	m
Young's modulus E	$2.1 \cdot 10^{11}$	N/m^2
Poisson's ratio ν	0.3	-
Density ρ	7860	kg/m^3
Rayleigh damping α	$8 \cdot 10^{-6}$	1/s
Rayleigh damping β	8	s

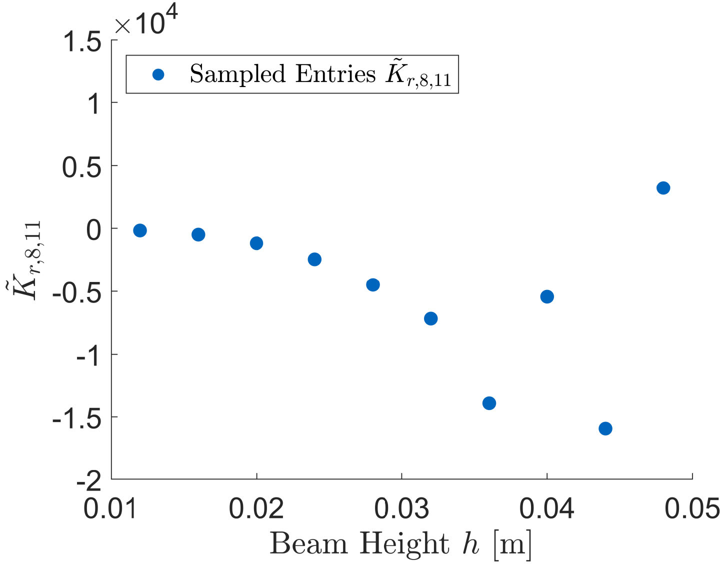
Table: Geometry and material parameters of the 3D cantilevered beam.

Training samples: 10 equally distanced samples within $[0.012, 0.048]$ m.
Test samples: 101 equally distanced samples within $[0.01, 0.05]$ m.

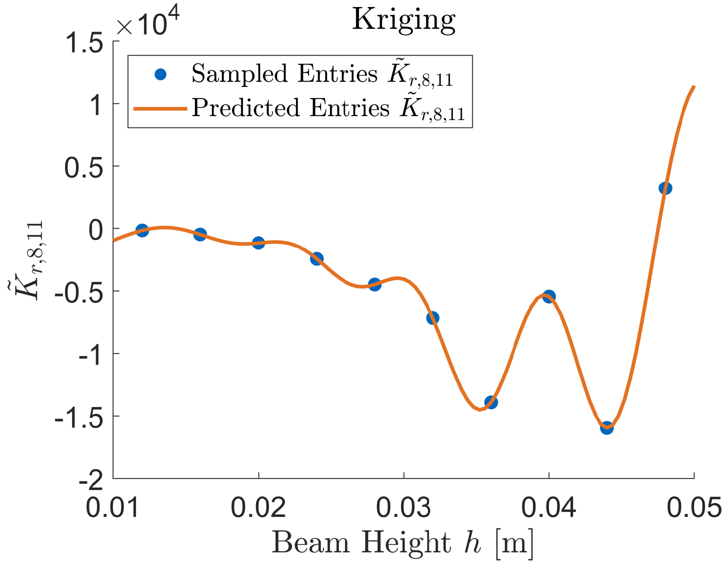
Timoshenko Beam – Kriging



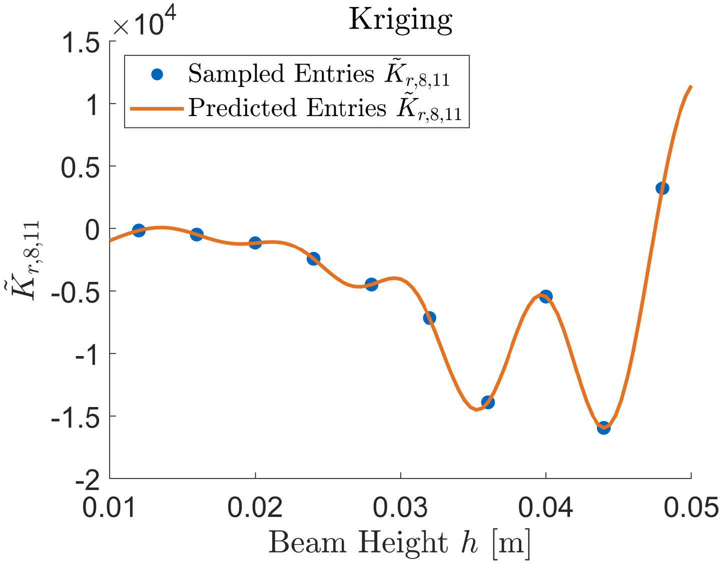
Timoshenko Beam – Proper Orthogonal Decomposition

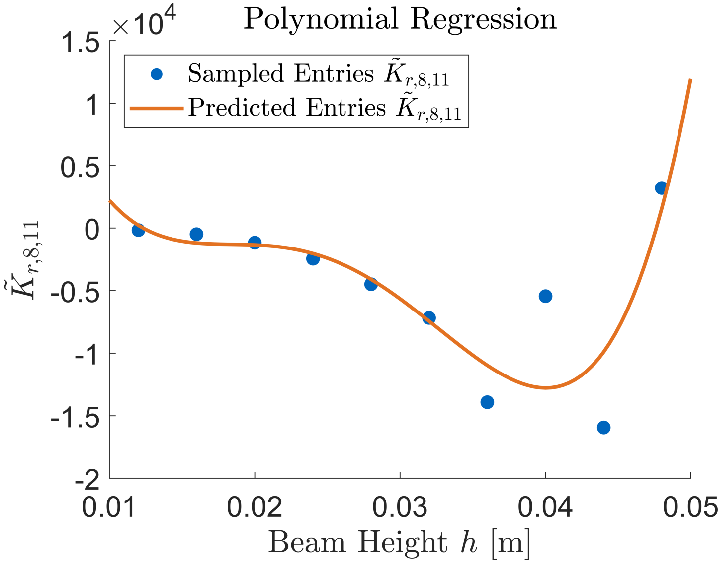


Timoshenko Beam – Proper Orthogonal Decomposition

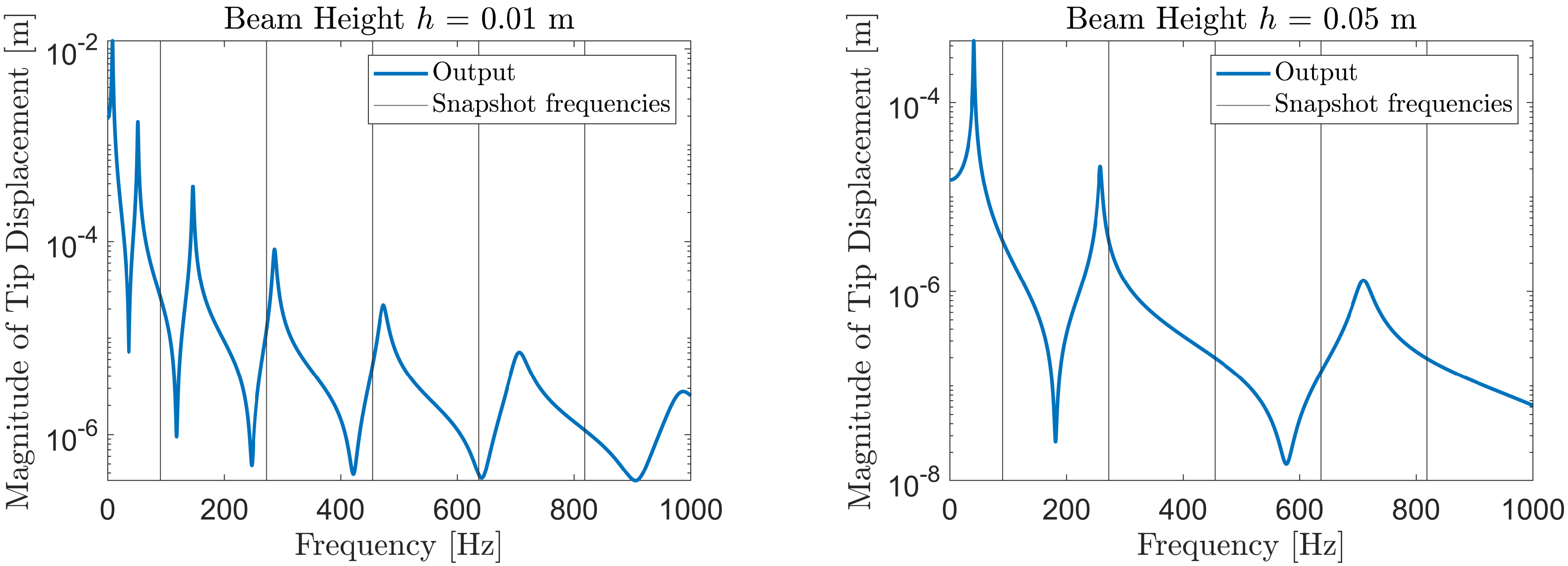


Timoshenko Beam – Proper Orthogonal Decomposition

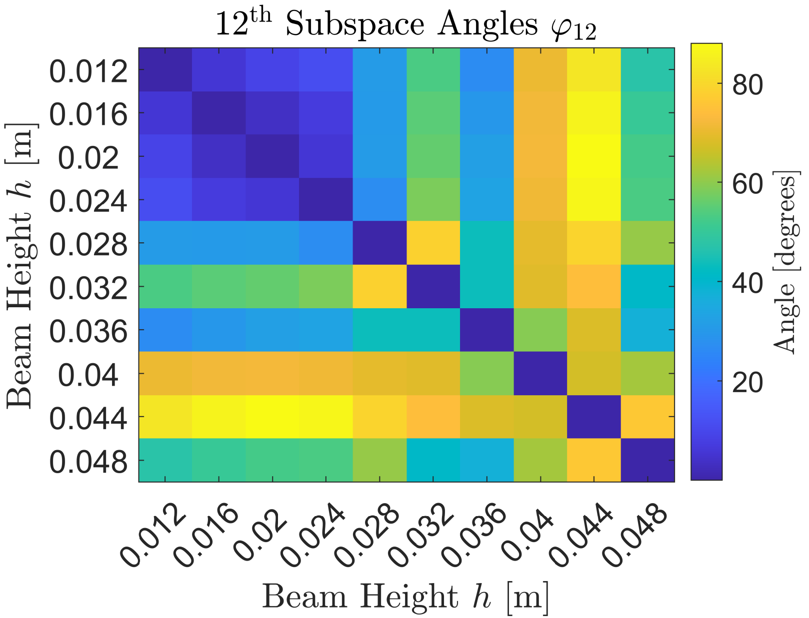




Timoshenko Beam – Proper Orthogonal Decomposition



Timoshenko Beam – POD – Inconsistency Measures



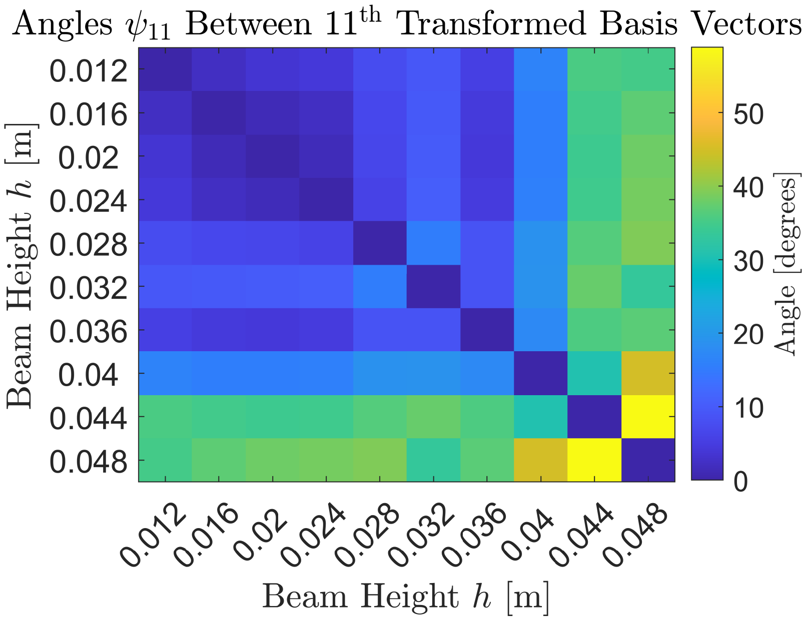
Reminder:
The lth subspace angle φ_l between the reduced bases $\mathbf{V}_i$ and $\mathbf{V}_j$ of samples i and j is computed as

$$\varphi_l = \arccos(\sigma_l), \qquad l = 1, \dots, r, \tag{29}$$

with $i, j = 1, \dots, N$ and

$$\mathbf{V}_i^{\mathrm{H}} \mathbf{V}_j = \mathbf{U} \boldsymbol{\Sigma} \mathbf{Y}^{\mathsf{T}}, \qquad i, j = 1, \dots, N. \tag{30}$$

Timoshenko Beam – POD – Inconsistency Measures



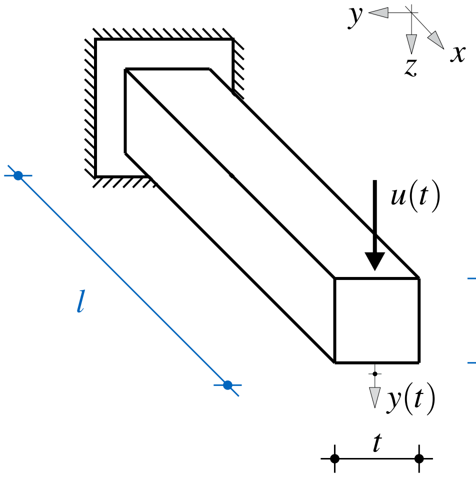
Reminder:
The angle ψ_l between the lth transformed basis vectors $\tilde{\mathbf{v}}_{l,i}$ and $\tilde{\mathbf{v}}_{l,j}$ of samples i and j is computed as

$$\psi_l = \arccos\left(\frac{<\tilde{\mathbf{v}}_{l,i}, \tilde{\mathbf{v}}_{l,j}>}{\|\tilde{\mathbf{v}}_{l,i}\|_2 \cdot \|\tilde{\mathbf{v}}_{l,j}\|_2}\right), \qquad l = 1, \ldots, r, \qquad (31)$$

where $< \cdot, \cdot >$ denotes the scalar product and $i, j = 1, \ldots, N$.

Results – Timoshenko Beam – Beam Length l and Beam Height h

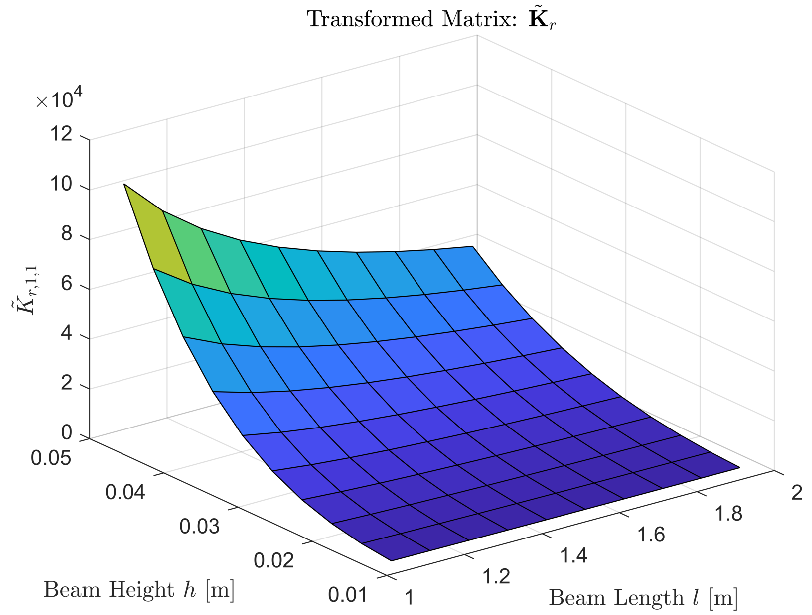
A 3D cantilevered beam discretized with Timoshenko beam elements is investigated. The beam is excited at the tip with a harmonic force of varying frequency ($[0, 1000]$ Hz). Rayleigh damping is used: $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$. The full order model is reduced using SO-IRKA with $r = 10$.

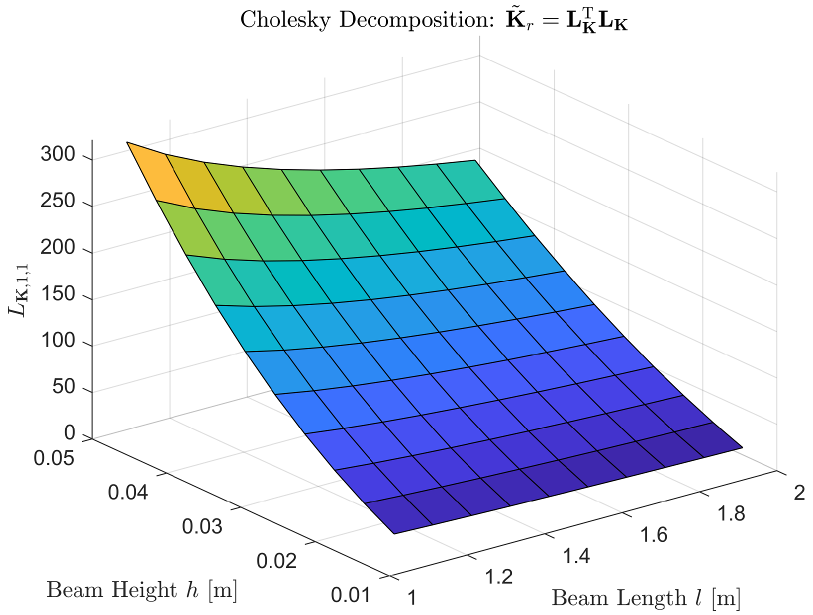


Parameter	Range/Value	Unit
Height h	$[0.01, 0.05]$	m
Thickness t	0.01	m
Length l	$[1.0, 2.0]$	m
Young's modulus E	$2.1 \cdot 10^{11}$	N/m^2
Poisson's ratio ν	0.3	-
Density ρ	7860	kg/m^3
Rayleigh damping α	$8 \cdot 10^{-6}$	$1/s$
Rayleigh damping β	8	s

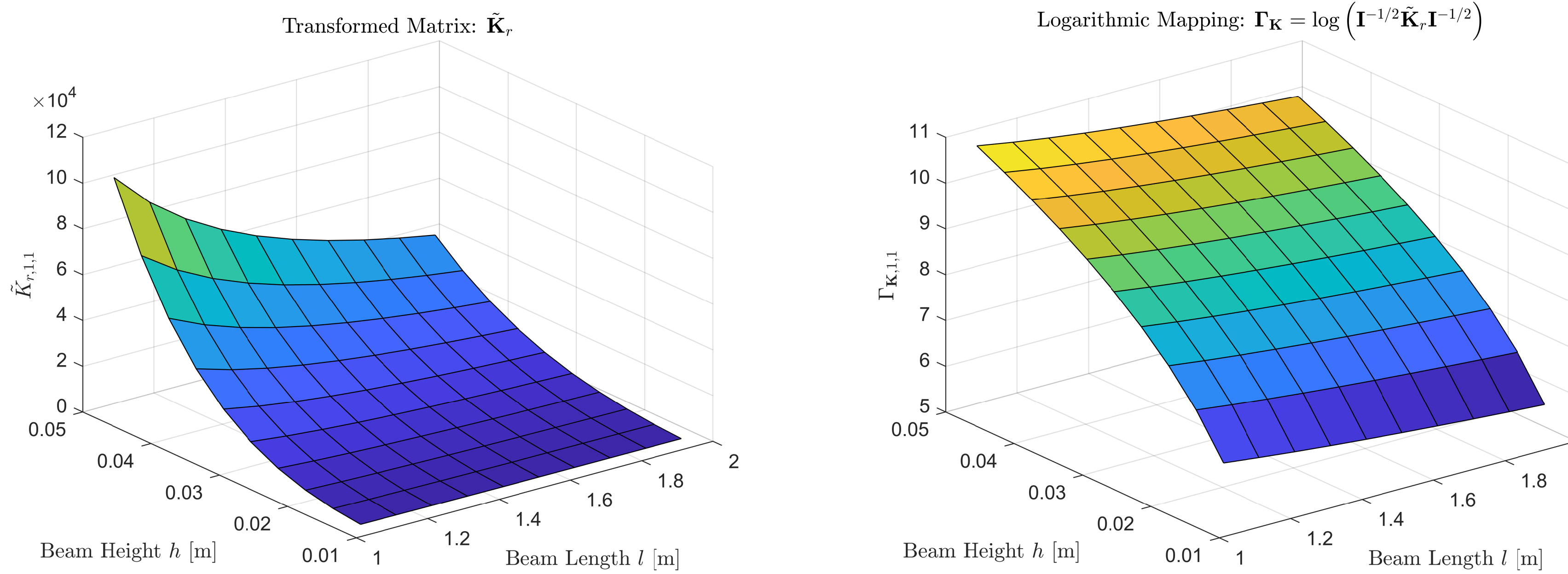
Table: Geometry and material parameters of the 3D cantilevered beam.

Timsohenko Beam – Cholesky Decomposition





Timsohenko Beam – Exponential Map



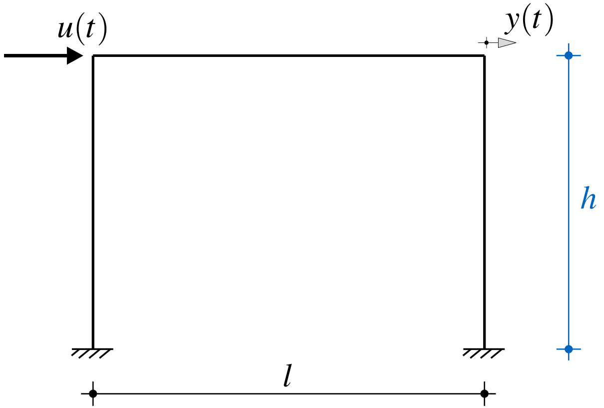
Conclusion and Future Work

Conclusions

- Regarding MOR, Balanced Truncation, Modal Truncation and the Iterative Rational Krylov Algorithm proved to be suited for pMOR by Matrix Interpolation.
- The subspace angles φ and the angles ψ between the transformed basis vectors seem to be indicators for inconsistency of the sampled subspaces.

Future Work – Frame

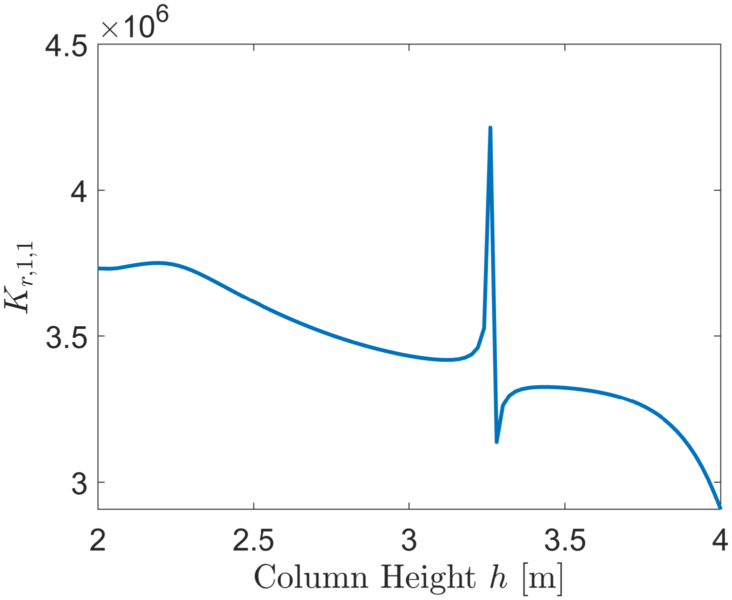
A frame structure discretized with Timoshenko beam elements is investigated. The frame is excited at the top left corner with a harmonic force of varying frequency ($[0, 100]$ Hz), the output is the displacement at the top right corner. Rayleigh damping is used: $\mathbf{C} = \alpha\mathbf{K} + \beta\mathbf{M}$. The full order model is reduced using SO-IRKA with $r = 10$.



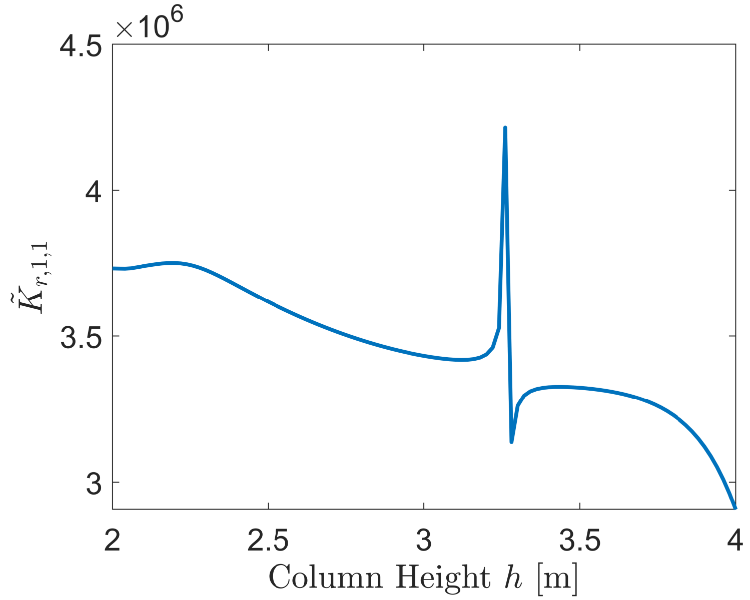
Parameter	Range/Value	Unit
Height h	$[2.0, 4.0]$	m
Length l	5.0	m
Young's modulus E	$2.1 \cdot 10^{11}$	$\mathrm{N/m^2}$
Poisson's ratio ν	0.3	-
Density ρ	7860	$\mathrm{kg/m^3}$
Rayleigh damping α	$8 \cdot 10^{-6}$	$1/\mathrm{s}$
Rayleigh damping β	8	s

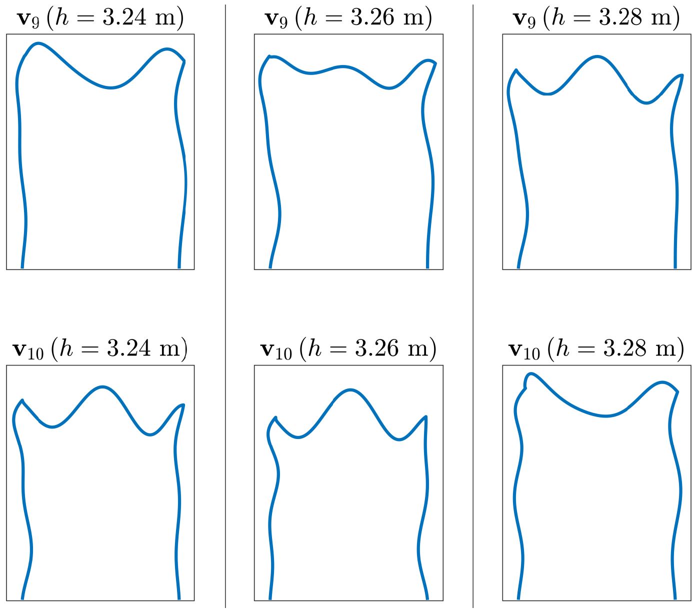
Table: Geometry and material parameters of the frame.

Results – Frame

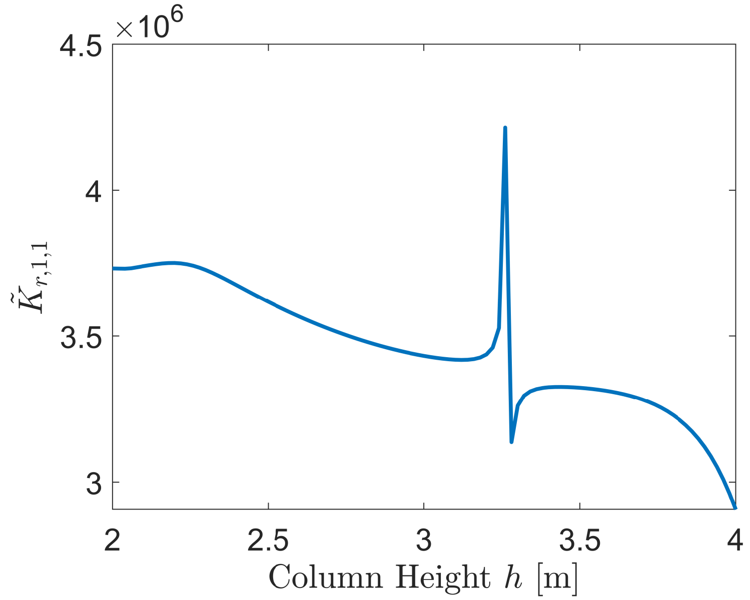


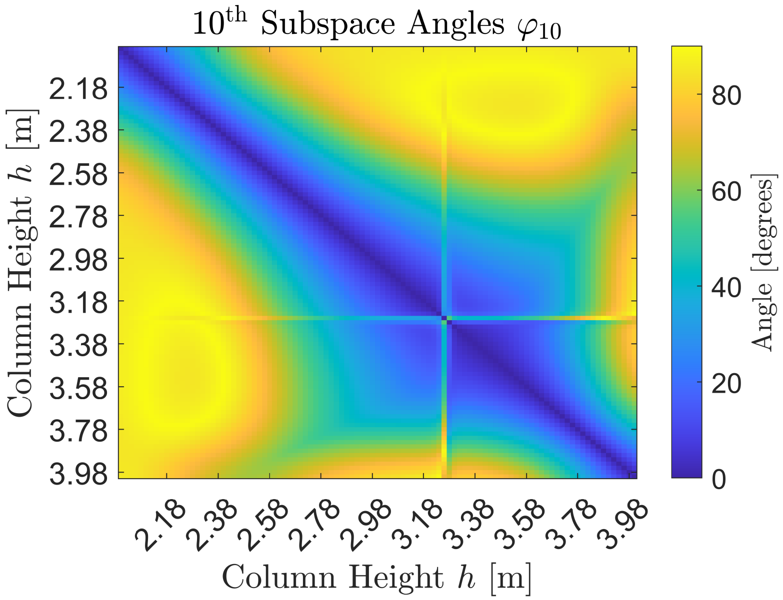
Results – Frame



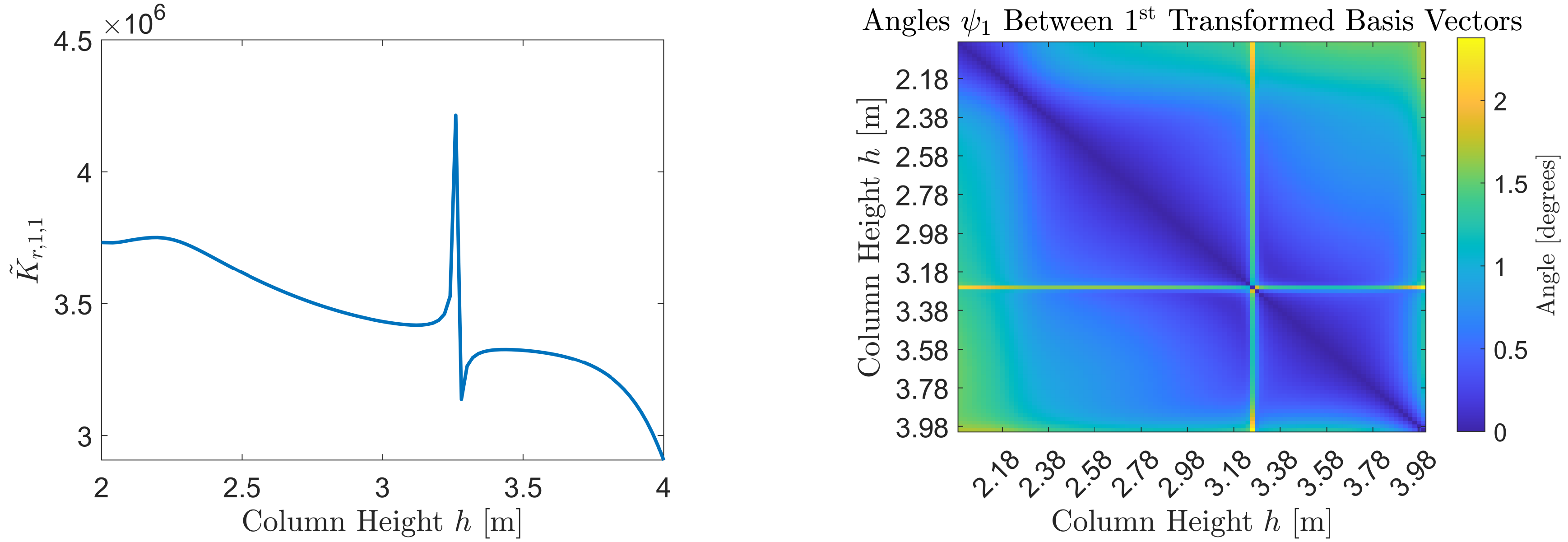


Frame – Subspace Angles φ

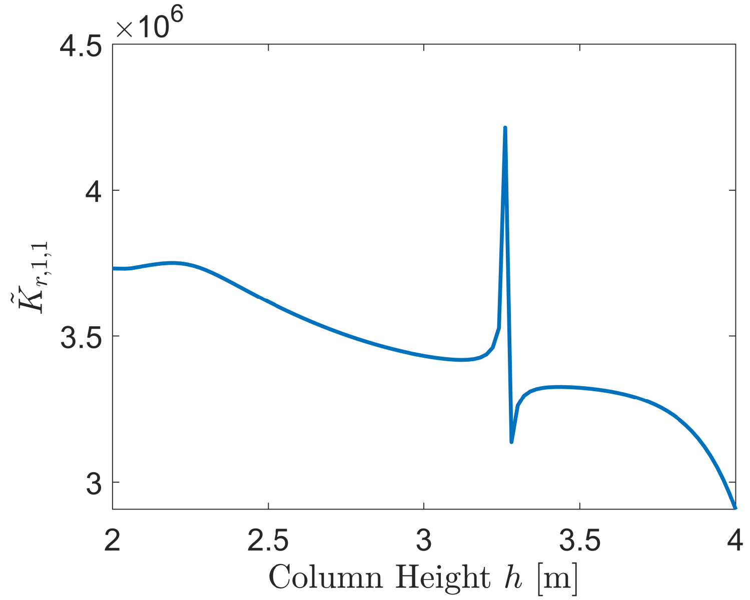


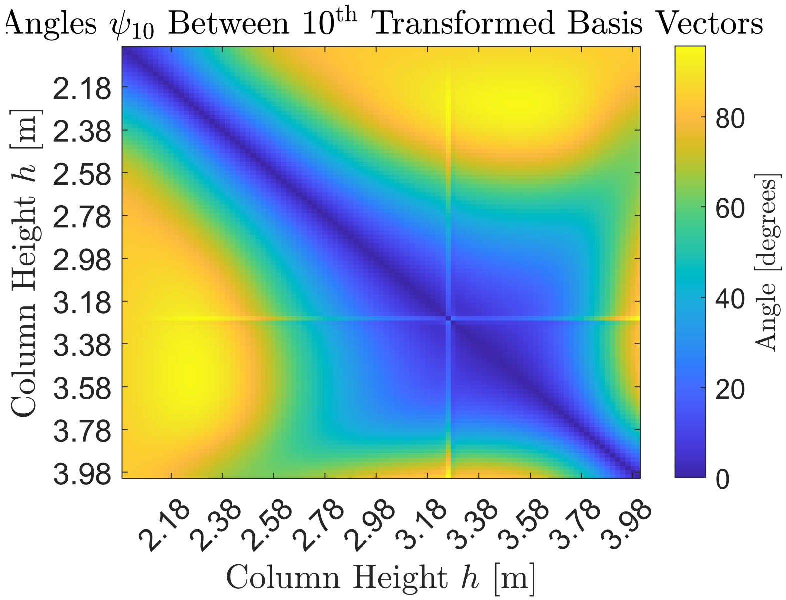


Frame – Angles ψ Between Transformed Basis Vectors



Frame – Angles ψ Between Transformed Basis Vectors





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1. Sampling of local reduced systems – Balanced Truncation (BT)

In Balanced Truncation (BT), states that are equally observable and controllable are used as reduced basis $\mathbf{V}$. For this, the controllability and observability Gramians $\mathbf{P}$ and $\mathbf{Q}$ need to be computed by solving the following Lyapunov equations:

$$\mathbf{E}_I \mathbf{P} \mathbf{A}_I^\top + \mathbf{A}_I \mathbf{P} \mathbf{E}_I^\top = -\mathbf{B}_I \mathbf{B}_I^\top, \tag{32}$$

$$\mathbf{E}_I \mathbf{Q} \mathbf{A}_I^\top + \mathbf{A}_I \mathbf{Q} \mathbf{E}_I^\top = -\mathbf{C}_I^\top \mathbf{C}_I. \tag{33}$$

The reduced basis $\mathbf{V}$ is then computed as [MS96]

$$\mathbf{V} = \mathbf{R}_p \mathbf{S}_1 \mathbf{\Sigma}^{-\frac{1}{2}}, \tag{34}$$

with

$$\begin{bmatrix} \mathbf{U}_1 & \mathbf{U}_2 \end{bmatrix} \begin{bmatrix} \mathbf{\Sigma}_1 & \mathbf{0} \\ \mathbf{0} & \mathbf{\Sigma}_2 \end{bmatrix} \begin{bmatrix} \mathbf{S}_1 \\ \mathbf{S}_2 \end{bmatrix} = \mathbf{L}_p^\top \mathbf{R}_p \quad \text{and} \quad \mathbf{P} = \begin{bmatrix} \mathbf{R}_p \\ \mathbf{R}_v \end{bmatrix} \begin{bmatrix} \mathbf{R}_p \\ \mathbf{R}_v \end{bmatrix}^\top, \quad \mathbf{Q} = \begin{bmatrix} \mathbf{L}_p \\ \mathbf{L}_v \end{bmatrix} \begin{bmatrix} \mathbf{L}_p \\ \mathbf{L}_v \end{bmatrix}^\top \tag{35}$$

3. Interpolation/regression of reduced operators

When all local reduced systems are described in a similar coordinate system, an interpolation/regression of the reduced operators is meaningful. Any interpolation/regression method can be used to learn the reduced operators entry-wise:

$$\theta(\hat{\mathbf{p}}) \to \tilde{\mathbf{K}}_r(\mathbf{p}_k), \tilde{\mathbf{D}}_r(\mathbf{p}_k), \tilde{\mathbf{M}}_r(\mathbf{p}_k), \tilde{\mathbf{F}}_r(\mathbf{p}_k), \tilde{\mathbf{G}}_r(\mathbf{p}_k) \quad (36)$$

However, this way it is not guaranteed that important properties of the reduced operators such as positive-definiteness of the mass, damping and stiffness matrix are preserved. Two different approaches can be used to ensure this:

3. Interpolation/regression of reduced operators

When all local reduced systems are described in a similar coordinate system, an interpolation/regression of the reduced operators is meaningful. Any interpolation/regression method can be used to learn the reduced operators entry-wise:

$$\theta(\hat{\mathbf{p}}) \rightarrow \tilde{\mathbf{K}}_r(\mathbf{p}_k), \tilde{\mathbf{D}}_r(\mathbf{p}_k), \tilde{\mathbf{M}}_r(\mathbf{p}_k), \tilde{\mathbf{F}}_r(\mathbf{p}_k), \tilde{\mathbf{G}}_r(\mathbf{p}_k) \tag{36}$$

However, this way it is not guaranteed that important properties of the reduced operators such as positive-definiteness of the mass, damping and stiffness matrix are preserved. Two different approaches can be used to ensure this:

a) Train interpolation/regression model with the Cholesky decomposition of the stiffness, damping and mass matrix [Quelle!]:

$$\tilde{\mathbf{K}}_r(\mathbf{p}_k) = \mathbf{L}_{\mathbf{K}}(\mathbf{p}_k)^\intercal \mathbf{L}_{\mathbf{K}}(\mathbf{p}_k), \quad \tilde{\mathbf{C}}_r(\mathbf{p}_k) = \mathbf{L}_{\mathbf{C}}(\mathbf{p}_k)^\intercal \mathbf{L}_{\mathbf{C}}(\mathbf{p}_k), \quad \tilde{\mathbf{M}}_r(\mathbf{p}_k) = \mathbf{L}_{\mathbf{M}}(\mathbf{p}_k)^\intercal \mathbf{L}_{\mathbf{M}}(\mathbf{p}_k) \tag{37}$$

$$\theta(\hat{\mathbf{p}}) \rightarrow \mathbf{L}_{\mathbf{K}}(\hat{\mathbf{p}}), \mathbf{L}_{\mathbf{C}}(\hat{\mathbf{p}}), \mathbf{L}_{\mathbf{M}}(\hat{\mathbf{p}}), \tilde{\mathbf{F}}_r(\hat{\mathbf{p}}), \tilde{\mathbf{G}}_r(\hat{\mathbf{p}}) \tag{38}$$

3. Interpolation/regression of reduced operators

When all local reduced systems are described in a similar coordinate system, an interpolation/regression of the reduced operators is meaningful. Any interpolation/regression method can be used to learn the reduced operators entry-wise:

$$\theta(\hat{\mathbf{p}}) \rightarrow \tilde{\mathbf{K}}_r(\mathbf{p}_k), \tilde{\mathbf{D}}_r(\mathbf{p}_k), \tilde{\mathbf{M}}_r(\mathbf{p}_k), \tilde{\mathbf{F}}_r(\mathbf{p}_k), \tilde{\mathbf{G}}_r(\mathbf{p}_k) \tag{36}$$

However, this way it is not guaranteed that important properties of the reduced operators such as positive-definiteness of the mass, damping and stiffness matrix are preserved. Two different approaches can be used to ensure this:

a) Cholesky decomposition
b) Train interpolation/regression model with the exponential map of the reduced operators [AF11]:

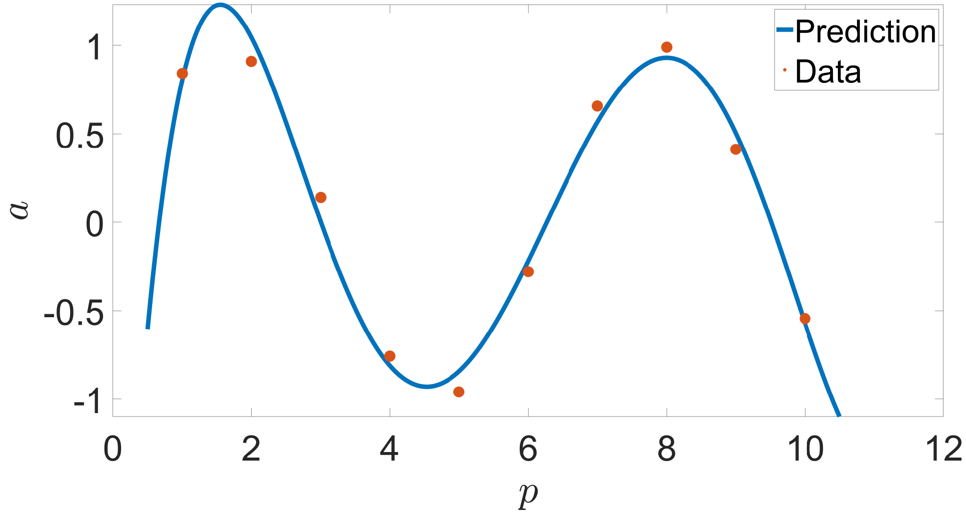
$$\mathbf{\Gamma} = \mathrm{Log}_{\mathbf{X}}(\mathbf{Y}) = \log\left(\mathbf{X}^{-1/2}\mathbf{Y}\mathbf{X}^{-1/2}\right), \quad \mathbf{Y} = \mathrm{Exp}_{\mathbf{X}}(\mathbf{\Gamma}) = \mathbf{X}^{1/2}\exp(\mathbf{\Gamma})\mathbf{X}^{1/2} \tag{37}$$

$$\mathbf{\Theta} = \mathrm{Log}_{\mathbf{X}}(\mathbf{Y}) = \mathbf{Y} - \mathbf{X}, \qquad \mathbf{Y} = \mathrm{Exp}_{\mathbf{X}}(\mathbf{\Theta}) = \mathbf{X} + \mathbf{\Theta} \tag{38}$$

$$\theta(\hat{\mathbf{p}}) \rightarrow \mathbf{\Gamma}_{\mathbf{K}}(\hat{\mathbf{p}}), \mathbf{\Gamma}_{\mathbf{C}}(\hat{\mathbf{p}}), \mathbf{\Gamma}_{\mathbf{M}}(\hat{\mathbf{p}}), \mathbf{\Theta}_{\mathbf{F}}(\hat{\mathbf{p}}), \mathbf{\Theta}_{\mathbf{G}}(\hat{\mathbf{p}}) \tag{39}$$

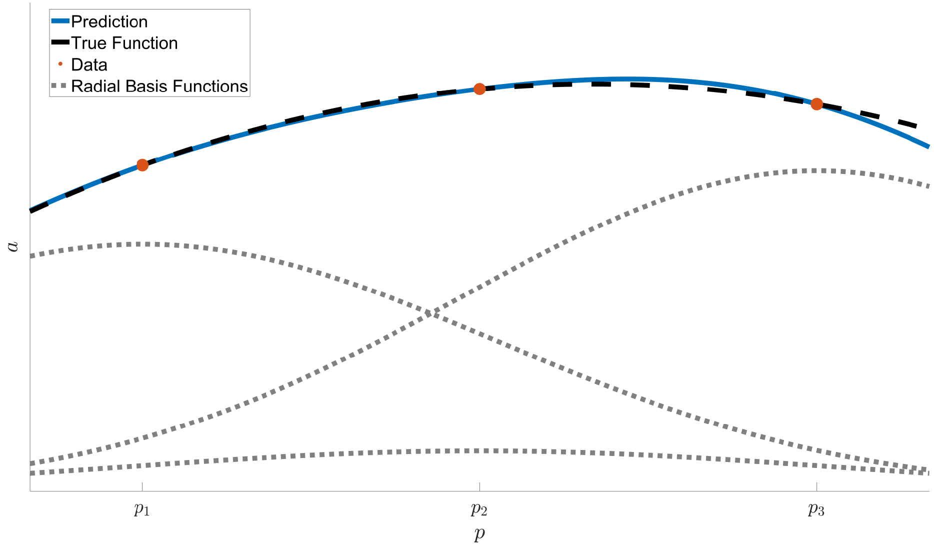
3. Interpolation/regression of reduced operators – Polynomial Regression

$$\hat{a}(\mathbf{p}) = \alpha_0 + \sum_{j_1=1}^{d} \alpha_{j_1} p_{j_1} + \sum_{j_1=1}^{d} \sum_{j_2=j_1}^{d} \alpha_{j_1 j_2} p_{j_1} p_{j_2} + \dots \tag{40}$$



3. Interpolation of reduced operators – Radial Basis Function

$$\hat{a}(\mathbf{p}) = \sum_{k=1}^{K} c_k \varphi\left(\|\mathbf{p} - \mathbf{p}_k\|\right) \tag{41}$$



3. Interpolation/regression of reduced operators – Kriging

$$\hat{a}(\mathbf{p}) = \mathbf{f}_{\text{reg}}^{\mathsf{T}}(\mathbf{p})\hat{\boldsymbol{\beta}} + \mathbf{r}^{\mathsf{T}}(\mathbf{p})\mathbf{R}_{\text{corr}}^{-1}\left(\mathbf{a}_s - \mathbf{F}_{\text{reg}}\hat{\boldsymbol{\beta}}\right) \quad (42)$$