Unbounded Wiener–Hopf Operators and Isomorphic Singular Integral Operators

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Abstract

Some basics of a theory of unbounded Wiener–Hopf operators (WH) are developed. The alternative is shown that the domain of a WH is either zero or dense. The symbols for non-trivial WH are determined explicitly by an integrability property. WH are characterized by shift invariance. We study in detail WH with rational symbols showing that they are densely defined, closed and have finite dimensional kernels and deficiency spaces. The latter spaces as well as the domains, ranges, spectral and Fredholm points are explicitly determined. Another topic concerns semibounded WH. There is a canonical representation of a semibounded WH using a product of a closable operator and its adjoint. The Friedrichs extension is obtained replacing the operator by its closure. The polar decomposition gives rise to a Hilbert space isomorphism relating a semibounded WH to a singular integral operator of Hilbert transformation type. This remarkable relationship, which allows to transfer results and methods reciprocally, is new also in the thoroughly studied case of bounded WH.

Communicated by Pierre Portal.

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1 Introduction

There is an increasing interest in unbounded Toeplitz and Toeplitz-like operators (see 2.2), which will concern also the related Wiener–Hopf operators (WH). So far, results on WH W_κ with unbounded symbol κ are scarce and probably in the literature there exists no introduction to this subject. So Sect. 2 deals with preliminaries and basics regarding unbounded WH. In particular we are concerned with conditions on the symbol κ ensuring that the domain of W_κ is either the whole space or dense or trivial, and prove that dom W_κ is either trivial or dense. The symbols with non-trivial WH, which are called proper, are determined by a useful integrability property. A classical result on the eigenvalues of a WH is shown to remain valid in the unbounded case. It implies that non-trivial symmetric WH have no eigenvalues. A further result characterizes WH by their invariance under unilateral shifts. In Sect. 3 WH with rational symbols are studied. They constitute a welcome source of densely defined closed operators with finite index. An explicit description of the domains, ranges, kernels, deficiency spaces, spectral and Fredholm points is given. The remainder of this article deals in Sects. 4 and 5 with densely defined semibounded WH. A semibounded operator W_κ can be expressed by a product of a closable operator A and its adjoint. Replacing A by its closure one obtains quite naturally a self-adjoint extension $\tilde{W}_\kappa$. It is proven to coincide with the Friedrichs extension. Inverting the order of the factors one obtains a singular integral operator L_ϕ of Hilbert transformation type. For the operators of that type to be non-trivial there is a necessary condition analogous to that for WH. The self-adjoint extensions $\tilde{L}_\phi$ and $\tilde{W}_\kappa$ are isometric, which follows from the polar decomposition of $\overline{A}$. Actually $\tilde{W}_\kappa$ is Hilbert space isomorphic to the reduction of $\tilde{L}_\phi$ on $\ker(\tilde{L}_\phi)^\perp$, and the spectral representations of $\tilde{L}_\phi$ and $\tilde{W}_\kappa$ can be achieved in an explicit manner from each other. It is worth noting that this relationship is new also for bounded WH thus contributing to the well-developed theory of the latter. To conclude, this method is illustrated by a non-trivial example diagonalizing Lalescu's operator and the isometrically related singular integral operator. In [24, Sect. 3.3] the spectral representations of $W_{1_{[-1,1]}}$ and the finite Hilbert transformation are related to each other by this method.

Notations Let $\mathcal{F}$ denote the Fourier transformation on $L^2(\mathbb{R})$. For measurable $E \subset \mathbb{R}$ introduce the projection $P_E : L^2(\mathbb{R}) \to L^2(E), (P_E f)(x) := f(x)$, i.e., $P_E f = f|_E$. (For convenience define $L^2(E) = \{0\}$, $P_E = 0$ if $E = \emptyset$.) Its adjoint $P_E^* : L^2(E) \to L^2(\mathbb{R})$ is the injection $(P_E^* f)(x) = f(x)$ for $x \in E$ and $= 0$ otherwise. Note

$$P_E P_E^* = I_{L^2(E)}, \quad P_E^* P_E = M(1_E)$$

with $M(1_E)$ the multiplication by the indicator function 1_E for E. Put $\mathbb{R}_+ :=]0, \infty[$ and $P_+ := P_{\mathbb{R}_+}$. Analogously define P_-.—We call E **proper** if neither E nor the complement $\mathbb{R} \setminus E$ is a null set.

Throughout let $\kappa : \mathbb{R} \to \mathbb{C}$ denote a measurable function and $M(\kappa)$ the multiplication operator by κ in $L^2(\mathbb{R})$ with dense domain $\{f \in L^2(\mathbb{R}) : \kappa f \in L^2(\mathbb{R})\}$. Recall that $M(\kappa)$ is normal satisfying $M(\kappa)^* = M(\overline{\kappa})$ and $M(\overline{\kappa})M(\kappa) = M(|\kappa|^2)$.

Moreover, $M(\kappa)$ is self-adjoint if and only if κ is almost real, and $M(\kappa)$ is nonnegative if and only if $\kappa \geq 0$ a.e.

Definition 1.1 The operator in $L^2(\mathbb{R}_+)$

$$W_\kappa := P_+ \mathcal{F} M(\kappa) \mathcal{F}^{-1} P_+^*$$

is called the Wiener–Hopf operator (WH) with symbol κ. Occasionally we write $W(\kappa)$ instead of W_κ. Clearly, $W_\kappa = W_{\kappa'}$ if $\kappa = \kappa'$ a.e. Often we shall refer to this tacitly. The symbol κ is called **proper** if dom $W_\kappa \neq \{0\}$.

The theory of WH with bounded symbol is well developed. We content ourselves to refer here to the book [1, Chapter 9] and to mention the origins [2]. Obviously in case of a bounded symbol the operators W_κ are bounded with dom $W_\kappa = L^2(\mathbb{R}_+)$ and adjoint $W_\kappa^* = W_{\overline{\kappa}}$. W_κ is the convolution on the real half line with kernel k, i.e.,

$$(W_\kappa g)(x) = \int_0^\infty k(x-y)g(x)\,\mathrm{d}y$$

if $\kappa \in L^\infty \cap L^2$ and $k := (2\pi)^{-1/2}\mathcal{F}\kappa$, or if $\kappa = \int e^{i(\cdot)y} k(y)\,\mathrm{d}y$ for $k \in L^1(\mathbb{R})$. For the case of integrable kernel there is the rather exhaustive theory by M. G. Krein [3]. Generally, the tempered distribution $k := (2\pi)^{-1/2}\mathcal{F}\kappa$, where $\kappa \in L^\infty(\mathbb{R})$ is considered as a regular tempered distribution, satisfies $\mathcal{F} M(\kappa)\mathcal{F}^{-1}u = k \star u$ for every Schwartz function u in the distributional sense (e.g. [4, Theorem IX.4]). For instance the kernel for $W_{-\,\mathrm{sgn}}$ is the tempered distribution $k(x) = -\frac{1}{x}$ or that for $W_{-\,\tanh}$ equals $k(x) = \left(2\,\mathrm{i}\sinh(\pi x/2)\right)^{-1}$. In the literature the generalizations of WH stay mostly within the realm of bounded operators. One deals with the traces (compressions) of bounded bijective operators in a Banach space on a closed subspace [5]. The results concern the solvability of the associated Wiener–Hopf equations.

2 Unbounded Wiener–Hopf Operators

As put it by [6] results on unbounded WH are practically inexistent. Indeed they are scarce. See [7, 1.3] for some notes. An important result is due to M. Rosenblum [8], [9], obtained for Toeplitz operators and hence valid for the Hilbert space isomorphic WH W_κ (see 2.2). So in the case that the symbol κ is real bounded below not almost constant and $(1 + x^2)^{-1}\kappa$ is integrable, [9] furnishes the spectral representation of the extension $\tilde{W}_\kappa$, which is shown to be the Friedrichs extension 4.3, 4.4 and which by [9] is absolutely continuous. —There are investigations on unbounded general WH dealing with conditions for their invertibility [10]. —In [6] real bounded below Wiener–Hopf quadratic forms from distributional kernels k are considered, and it is shown that such a form determines a WH if and only if the form is closable or, equivalently, if and only if $\sqrt{2\pi}k$ is the Fourier transform of a locally integrable bounded below function κ with integrable $(1 + x^2)^{-n}\kappa$ for some $n \in \mathbb{N}$. Clearly κ is the symbol and dom $W_\kappa \supset C_c^\infty(\mathbb{R}_+)$ holds. See further [7]. —Furthermore, the

methods applied for the study of unbounded analytic Toeplitz operators [11] can also produce results on unbounded WH, see 2.10, 4.2, 4.3(c). One deals with $M_+(\kappa)$ (2.2), which is canonically Hilbert space isomorphic to W_κ. The former is the trace on the Hardy space H_+ of the multiplication operator by κ. Rather recently, as indicated to us by the reviewer, in [12] one finds a detailed study of the kernel of $M_+(\kappa)$.

Preliminaries

Starting the preliminary remarks and anticipating briefly some results on WH, note first that dom $W_\kappa = \text{dom } M(\kappa)\mathcal{F}^{-1}P_+^*$. So κ is proper if and only if κh is square-integrable for some Hardy function 2.1 $h \in \text{ran}(\mathcal{F}^{-1}P_+^*)$. Clearly dom $W_\kappa \subset \text{dom } W_{\kappa'}$ if $|\kappa'| \leq |\kappa|$. Moreover, dom $W_\kappa = \text{dom } W_{\overline{\kappa}} = \text{dom } W_{|\kappa|} = \text{dom } W(1 + |\kappa|)$. Note $\langle g', W_\kappa g \rangle = \langle W_{\overline{\kappa}} g', g \rangle$ for $g, g' \in \text{dom } W_\kappa$. Hence, if W_κ is densely defined then $W_\kappa \subset W_{\overline{\kappa}}^*$, whence W_κ is closable and dom $W_\kappa \subset \text{dom } W_\kappa^*$ holds. But unbounded WH may and may not be closed 3.2, 3.5.

   If the symbol κ is unbounded, then dom $W_\kappa \neq L^2(\mathbb{R}_+)$ 2.3. The alternative holds that either dom W_κ is trivial or dom W_κ is dense. In other words, as shown in 2.10, if κ is proper, then W_κ is densely defined. In 2.6 and 2.8 explicit characterizations of proper symbols are given. There is also the useful criterion in 2.4 for κ to be proper. So proper symbols may have polynomial growth and countably many singularities with integrable logarithm like as $\exp |x|^\alpha$, $-1 < \alpha < 0$. It is easy to give examples of non-proper symbols 2.9.

   If W_κ is densely defined then W_κ is symmetric, i.e., $W_\kappa \subset W_\kappa^*$, if and only if κ is almost real 2.14. If W_κ is densely defined symmetric, then W_κ is bounded below if and only if κ is essentially bounded below 2.15. Recall that the numerical range $\{\langle g, W_\kappa g \rangle : g \in \text{dom } W_\kappa, \; ||g|| = 1\}$ of W_κ is convex. (Indeed, the numerical range of every operator in Hilbert space is convex [13].) It is determined in 2.15, 2.16.

   Densely defined symmetric WH, which are not a multiple of I, have no eigenvalues 2.11. As already mentioned, if W_κ is bounded below not a multiple of I with integrable $(1 + x^2)^{-1}\kappa$, it follows from [9] that the Friedrichs extension $\tilde{W}_\kappa$ 4.4 of W_κ is even absolutely continuous. Section 4 is concerned with the general case of densely defined symmetric semibounded WH and their Friedrichs extensions.

   If κ is proper real and even (i.e. $\kappa(-x) = \kappa(x)$) then W_κ is densely defined symmetric and has a self-adjoint extension. This holds true since $L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$, $g \mapsto \overline{g}$ is a conjugation, which leaves dom W_κ invariant and satisfies $W_\kappa \overline{g} = \overline{W_\kappa g}$ (see [4, Theorem X.3]). If κ is odd instead of even then in general W_κ has no self-adjoint extension. Examples are furnished by real rational symbols as e.g. $\kappa(x) = x$. In 3.4 an explicit description of the deficiency spaces of W_κ for real rational κ are given yielding further examples of densely defined symmetric WH with self-adjoint extensions.

Hardy spaces 2.1 Recall the Hardy spaces $H_\pm := \text{ran}(\mathcal{F}^{-1}P_\pm^*)$. Obviously $L^2(\mathbb{R}) = H_+ \oplus H_-$ and $h \in H_+ \Leftrightarrow \overline{h} \in H_-$ as well as $h \in H_+ \Leftrightarrow \check{h} \in H_-$ with $\check{h}(x) := h(-x)$. We tacitly refer to the well-known Paley–Wiener Theorem characterizing the Fourier transforms of L^2-functions vanishing on a half-axis, see e.g. [14, Theorem 95]. In particular $h \in H_+$ if and only if there is a ϕ holomorphic on the upper half-plane such

that its partial maps $\phi_y(x) := \phi(x + i\,y)$ for $y > 0$ satisfy $\phi_y \in L^2(\mathbb{R})$, $\{\|\phi_y\|: y > 0\}$ bounded, and $\phi_y \to h$ for $y \to 0$ in the mean and pointwise a.e. Actually ϕ converges to h non-tangentially a.e., and $\|\phi_y\|\uparrow\|h\|$ for $y \downarrow 0$ (see e.g. [15, III 3.3, II 2.6]). Moreover, every $h \in H_{\pm}\backslash\{0\}$ vanishes only on a null set. Indeed, according to a Luzin-Privalov Theorem [16, IV 2.5] a meromorphic function on the upper or lower half-plane which takes non-tangential boundary value zero on a set of positive Lebesgue measure is zero. The former property is also an immediate consequence of the following result on the modulus of a Hardy function. Let $f \in L^2(\mathbb{R}) \setminus \{0\}$:

$$There\ is\ h \in H_+ satisfying\ |h| = |f|\ if\ and\ only\ if\ \frac{\ln|f|}{1 + x^2} \in L^1(\mathbb{R}) \quad (2.1)$$

One proves (2.1) using the outer function with prescribed modulus on the torus ([17, Chap. 3, Def. 1.1 and Prop. 3.2]) and the Hilbert space isomorphism Γ in 2.2. Recall also [18, Theorem XII] that $\ln|h|/(1 + x^2) \in L^1$ if $h \in H_+$.

Let $M_+(\kappa)$ denote the trace of $M(\kappa)$ on H_+, i.e.

$$M_+(\kappa) = P_{H_+} M(\kappa) P_{H_+}^* \quad (2.2)$$

with $P_{H_+} : L^2(\mathbb{R}) \to H_+$ the orthogonal projection. Note that $P_{H_+}\mathcal{F}^{-1}P_+^* : L^2(\mathbb{R}_+) \to H_+$ is a Hilbert space isomorphism with its inverse $P_+\mathcal{F}P_{H_+}^*$, by which W_κ is Hilbert space isomorphic to $M_+(\kappa)$. Often it is convenient to deal with $M_+(\kappa)$ in place of W_κ. Note that $M_+(\kappa)$ extends the multiplication operator in H_+

$$M(\kappa, H_+)h := \kappa\,h\ with\ \operatorname{dom} M(\kappa, H_+) = \{h \in H_+ : \kappa h \in H_+\}$$

The latter is Hilbert space isomorphic to its counterpart the so-called analytic Toeplitz operator with symbol $\omega := \kappa \circ C^{-1}$, see 2.2. $M_+(\kappa)$ and $M(\kappa, H_+)$ coincide for rational symbols κ holomorphic in the upper half-plane 3.1(c), or more general, if κ is the nontangential limit of an outer function on the upper half-plane. Indeed, in this case $\frac{\ln|\kappa|}{1+x^2}$ is integrable, whence κ is proper by 2.8 since $\ln|1 + |\kappa|| \leq \ln 2 + \ln|\kappa|$, and [12, 3.3] applies. We do not put forward this topic.

Finally H_+^∞ is the set of all measurable bounded $\alpha : \mathbb{R} \to \mathbb{C}$ such that there is a bounded holomorphic A on the upper half-plane with the partial maps $A_y \to \alpha$ for $y \to 0$ pointwise a.e. Actually A converges to α non-tangentially a.e. (see e.g. [15, III 3.3, II 2.6]). Clearly, $H_+^\infty H_+ \subset H_+$.

Remark on Toeplitz operators 2.2 Let the torus $\mathbb{T}$ be endowed with the normalized Lebesgue measure. The Hardy space $H^2(\mathbb{T})$ is the subspace of $L^2(\mathbb{T})$ with orthonormal basis $e_n(w) := w^n$, $n \in \mathbb{N}_0$. Given a measurable $\omega : \mathbb{T} \to \mathbb{C}$ then, quite analogous to (2.2), the Toeplitz operator T_ω with symbol ω is defined by

$$T_\omega := P_{H^2(\mathbb{T})} M(\omega)\, P_{H^2(\mathbb{T})}^*$$

Let $\Gamma : H^2(\mathbb{T}) \to H_+$, $(\Gamma u)(x) := \frac{i}{\sqrt{\pi}(x+i)} u\big(C(x)\big)$ be the Hilbert space isomorphism based on the Cayley transformation $C(x) := \frac{x-i}{x+i}$. One has

$$T_\omega = \Gamma^{-1} M_+(\omega \circ C)\,\Gamma \tag{2.3}$$

Obviously by this well-known relationship (see e.g. [1, 9.5(e)], [24, 3.3.2 (13)]) results and methods regarding Toeplitz operators may be transferred for the study of WH and vice versa. For unbounded Toeplitz and Toeplitz-like operators see: [8], [9] for a spectral theory of Toeplitz operators with bounded below integrable symbols, [7] for closable quadratic forms for semibounded Toeplitz operators, [11] for analytic Toeplitz operators, [19] for Toeplitz-like operators with rational symbols.

Results

As to 2.3 one recalls that in general an everywhere defined linear operator in a Hilbert space need not be bounded.

Proposition 2.3 dom $W_\kappa = L^2(\mathbb{R}_+) \Rightarrow \kappa$ is bounded $\Rightarrow$ dom $W_\kappa = L^2(\mathbb{R}_+)$ and W_κ is bounded.

Proof The second implication is obvious. As to the first one, assume that κ is not bounded. Then there exists $f \in L^2(\mathbb{R})$ with $\kappa f \notin L^2(\mathbb{R})$. Write $f = h_+ + h_-$ with $h_\pm \in H_\pm$. As dom $M_+(\kappa) = H_+$ one has $\kappa h_+ \in L^2(\mathbb{R})$, and hence $\kappa h_- \notin L^2(\mathbb{R})$. Similarly, $\kappa \overline{f} \notin L^2(\mathbb{R})$, $\overline{f} = \overline{h}_- + \overline{h}_+$ with $\overline{h}_\mp \in H_\pm$, and $\kappa \overline{h}_- \in L^2(\mathbb{R})$. This contradicts $\kappa h_- \notin L^2(\mathbb{R})$.   □

There follows a first result on the domain of a WH which applies for instance to rational symbols.

Lemma 2.4 *If $\kappa\, qs \in L^2(\mathbb{R})$, where q is a polynomial and s is the inverse Fourier transform of a Schwartz function with support in $[0, \infty[$, then $qs \in$ dom W_κ and κ is proper. —Suppose that $\kappa\, qs \in L^2(\mathbb{R})$ for a polynomial q with only real zeros and for every Schwartz function s. Then dom $W_\kappa \supset \{q(i \frac{d}{dx})\phi : \phi \in C_c^\infty(\mathbb{R}_+)\}$.*

Proof As to the first claim note that s is a Schwartz function in H_+, whence $qs \in H_+$ by 3.1(c).

Now let D denote the differential operator $i \frac{d}{dx}$ and let u be any Schwartz function with support in $\mathbb{R}_+$. Then $q(D)u$ is still such a function, and $\mathcal{F}^{-1}u$, $\mathcal{F}^{-1}\big(q(D)u\big) = q\mathcal{F}^{-1}u$ are Schwartz functions in H_+. Hence it remains to show that $\{q(D)\phi : \phi \in \mathcal{D}\}$ is dense in $L^2(\mathbb{R}_+)$, where $\mathcal{D}$ denotes the space of test functions $C_c^\infty(\mathbb{R}_+)$. Assume $g \in L^2(\mathbb{R}_+)$ with $g \perp q(D)\phi$, i.e., $\int_{\mathbb{R}_+} \overline{g}\, q(D)\phi\, dx = 0$. The claim is $g = 0$.

Indeed, regarding $\overline{g}$ as a regular distribution in $\mathcal{D}'$, one has $\overline{g}\big(q(D)\phi\big) = q(-D)\overline{g}\,(\phi) = 0$ for all test functions ϕ, whence $q(-D)\overline{g} = 0$. Thus $\overline{g}$ is a solution of the differential equation $q(-D)F = 0$ for $F \in \mathcal{D}'$. As known all its solutions are regular. Hence $\overline{g} \in T$, where T denotes the space of linear combinations of functions on $\mathbb{R}_+$ of the kind $x \to x^k e^{i\lambda x}$, $k \in \mathbb{N}_0$ and $\lambda \in \mathbb{R}$. Then $G \in T$ for $G(x) := \int_0^x |g(t)|^2\, dt$.

One has $G(x) \to \|g\|^2 < \infty$ for $x \to \infty$. Write $G = \sum_{k=0}^{n} x^k A_k$ with A_k a linear combination of periodic functions $e^{i\lambda x}$. Then $A_n(x) \to 0$ for $x \to \infty$. Since A_n is almost periodic this implies $A_n = 0$. The result follows. □

The following invariance H_+^{∞} dom $M_+(\kappa) \subset$ dom $M_+(\kappa)$ is essential for the density of dom W_κ in 2.10.

Lemma 2.5 *(a)* $W_\alpha(\text{dom } W_\kappa) \subset \text{dom } W_\kappa \ \forall \ \alpha \in H_+^{\infty}$, *equivalently* $\alpha h \in \text{dom } M_+(\kappa)$
$\forall \ \alpha \in H_+^{\infty}, h \in \text{dom } M_+(\kappa)$.
(b) $\forall \ \alpha \in H_+^{\infty}$ *one has* (1) $W_\kappa W_\alpha = W_{\alpha\kappa}$ *and* (2) $W_\alpha^* W_\kappa \subset W_{\overline{\alpha}\kappa}$.
(c) For $b \geq 0$ let $S_b : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ be the unilateral translation $S_b g(x) := g(x - b)$ if $x > b$ and $:= 0$ otherwise. One has the translational invariance $W_\kappa = S_b^* W_\kappa S_b$.

Proof (a) Let $h \in \text{dom } M_+(\kappa)$. The claim is $M_+(\alpha)h \in \text{dom } M_+(\kappa)$. Now $M_+(\alpha)h = \alpha h$ since obviously $\alpha h \in H_+$. Moreover $\kappa\alpha h \in L^2(\mathbb{R})$, whence the claim.
(b) For (1) note $h \in \text{dom}\big(M_+(\kappa)M_+(\alpha)\big) \Leftrightarrow h \in H_+, P_{H_+}(\alpha h) \in \text{dom } M_+(\kappa)$. Since $\alpha h \in H_+$, the latter is equivalent to $\alpha h \in \text{dom } M_+(\kappa) \Leftrightarrow \kappa\alpha h \in L^2(\mathbb{R}) \Leftrightarrow h \in \text{dom } M_+(\alpha\kappa)$. —As to (2), $M_+^*(\alpha) = M_+(\overline{\alpha})$, dom $M_+(\kappa) = \text{dom}\big(M_+^*(\alpha)M_+(\kappa)\big)$, and $P_{H_+}^* P_{H_+} = I - P_{H_-}^* P_{H_-}$ for $P_{H_-} := I - P_{H_+}$. So for $h \in \text{dom } M_+(\kappa)$ one has $M_+^*(\alpha)M_+(\kappa)h = M_+(\overline{\alpha})P_{H_+}(\kappa h) = P_{H_+}M(\overline{\alpha})P_{H_+}^* P_{H_+}(\kappa h) = P_{H_+}(\overline{\alpha}\kappa h) - P_{H_+}\big(\overline{\alpha}P_{H_-}^* P_{H_-}(\kappa h)\big) = M_+(\overline{\alpha}\kappa)h - 0$, since $\overline{\alpha}P_{H_-}^* P_{H_-}(\kappa h) \in H_-$.
(c) Check $S_b = W(e^{ib(\cdot)})$, dom $W(e^{-ib(\cdot)}\kappa) = \text{dom } W_\kappa$ and apply the foregoing results. □

We turn to characterizations of proper symbols κ, as in particular by the condition of the integrability of $\ln(1 + |\kappa|)/(1 + x^2)$ in 2.8.

Proposition 2.6 *Let $p \in]0, \infty]$. Let j denote a real-valued function such that $\frac{j}{1+x^2}$ is integrable. Then*

$$\kappa \text{ proper } \Leftrightarrow \kappa\, e^j \in L^p(\mathbb{R}) \text{ for some } j$$

Proof Here we prove the case $p = 2$ and the implication $\Leftarrow$ for $p = \infty$ and $p = 1$. The remainder is shown in the proof of 2.7.
κ being proper there is $h \in \text{dom } M_+(\kappa) \setminus \{0\}$. Then $\kappa h \in L^2(\mathbb{R})$, whence $\kappa\, e^{\ln|h|} \in L^2(\mathbb{R})$. By (2.1), $\ln|h|/(1 + x^2)$ is integrable. For the converse implication put $j' := 1_{\{j \leq 0\}}j - |x|^{1/2}$. Then $j'/(1 + x^2)$ is integrable, $e^{j'}$ is square-integrable. By (2.1) there is $h \in H_+$ with $|h| = e^{j'}$. Moreover, $\kappa\, e^{j'}$ is square-integrable since $j' \leq j$. So $h \in \text{dom } M_+(\kappa) \setminus \{0\}$, whence κ is proper. —Next turn to $\Leftarrow$ for $p = \infty$. For j' from above $|\kappa|\, e^{j'} \leq |\kappa|\, e^j\, e^{-|x|^{1/2}}$ is square-integrable. Hence κ is proper by the case $p = 2$ just shown. —Now consider $\Leftarrow$ for $p = 1$. Due to the assumption $|\kappa|\, e^j(1 + x^2)^{-1}$ is integrable, whence $j'(1 + x^2)^{-1}$ is integrable for $j' := \ln(1 + |\kappa|\, e^j)$. Therefore $(1 + |\kappa|\, e^j)\, e^{-j'} = 1$ implying $|\kappa|\, e^{j-j'} \leq 1$, whence the result by $\Leftarrow$ for $p = \infty$. □

Lemma 2.7 κ *is proper if and only if* κ^2 *is proper. More generally, let* $r > 0$ *and let* κ_1, κ_2 *be two symbols satisfying* $|\kappa_1| = |\kappa_2|^r$. *Then* κ_1 *is proper if and only if* κ_2 *is proper. Finally, if* κ_1 *and* κ_2 *are proper symbols then so are* $\kappa_1\kappa_2$ *and* $\kappa_1 + \kappa_2$.

Proof Let κ^2 be proper. Then $1+|\kappa|^2$ is proper and $|\kappa| \leq 1+|\kappa|^2$, whence κ is proper. —Now let κ be proper. Then by 2.6 $(p = 2)$, $\kappa\,e^j \in L^2(\mathbb{R})$ for some real-valued j with integrable $j/(1 + x^2)$. Since $\kappa' := |\kappa|^2 e^{2j}$ is integrable, κ' is proper by 2.6 ($\Leftarrow$ for $p = 1$). Then 2.6 $(p = 2)$ yields $\kappa' e^{j'} \in L^2(\mathbb{R})$ for some real-valued j' with integrable $j'/(1+x^2)$. Hence $|\kappa|^2 e^{2j+j'} \in L^2(\mathbb{R})$, whence the claim by 2.6 $(p = 2)$.

The general case is easily reduced to the claim that, for $r > 1$ and $\kappa \geq 0$, κ is proper if and only if κ^r is proper. So let κ be proper. Let $n \in \mathbb{N}$ satisfy $r \leq 2^n$. By the foregoing result κ^{2^n} is proper. Then $1 + \kappa^{2^n}$ is proper and $\kappa^r \leq 1 + \kappa^{2^n}$. Hence κ^r is proper. Conversely, if κ^r is proper, then $1 + \kappa^r$ is proper and $\kappa \leq 1 + \kappa^r$, whence κ is proper.

Now we complete the proof of 2.6. Consider first the case $p \in]0, \infty[$. Let $\kappa\,e^j \in L^p(\mathbb{R})$ for some j. Then $|\kappa|^{p/2} e^{\frac{p}{2}j} \in L^2(\mathbb{R})$. Hence $|\kappa|^{p/2}$ is proper by 2.6 $(p = 2)$. The foregoing result applies, whence κ is proper. The converse follows quite similarly. —Now let $p = \infty$ and let κ be proper. As just shown, κe^j is integrable for some real-valued j with integrable $j(1 + x^2)^{-1}$. Then $|\kappa|\,e^{j''} \leq 1$ for $j'' := j - \ln(1 + |\kappa|\,e^j)$ as shown in the proof of 2.6 ($\Leftarrow$ for $p = 1$).

Finally, let κ_1, κ_2 be proper. Apply 2.6 for $p = \infty$. So $|\kappa_i| \leq e^{j_i}$, $i = 1, 2$. Then $|\kappa_1\kappa_2| \leq e^{j_1+j_2}$. Assume without restriction $j_i \geq 0$, $i = 1, 2$. Then $|\kappa_1 + \kappa_2| \leq 2\,e^{j_1+j_2}$. $\qquad\square$

Theorem 2.8 κ *is proper if and only if* $\frac{\ln(1+|\kappa|)}{1+x^2}$ *is integrable.*

Proof Let κ be proper. Then $1+|\kappa|$ is proper, and by 2.6 $(p = \infty)$ one has $1+|\kappa| \leq e^j$ for some real-valued j with integrable $\frac{j}{1+x^2}$. Then $\ln(1 + |\kappa|) \leq j$, whence the claim. Conversely let $\frac{j}{1+x^2}$ be integrable for $j := \ln(1 + |\kappa|)$. Then $1 + |\kappa| = e^j$, whence $|\kappa| \leq e^j$ and κ is proper by 2.6 $(p = \infty)$. $\qquad\square$

Corollary 2.9 κ *is not proper if* $|\kappa(x)|$ *increases not less than exponentially for* $x \to \infty$ *or* $x \to -\infty$, *i.e., if there are positive constants* a, δ, λ *such that* $|\kappa(x)| \geq \delta\,e^{\lambda|x|}$ *for* $x \geq a$ *or* $x \leq -a$.

Proof $\ln(1 + |\kappa(x)|) \geq \ln|\kappa(x)| \geq \ln(\delta) + \lambda|x|$ for all $x \geq a$ or $x \leq -a$. Hence $\ln(1 + |\kappa|)/(1 + x^2)$ is not integrable. Apply 2.8. $\qquad\square$

Now the alternative is shown that a WH is either trivial or densely defined.

Theorem 2.10 *If* κ *proper, then* $\mathrm{dom}\,W_\kappa$ *is dense. More precisely, if* $h \in \mathrm{dom}\,M_+(\kappa) \setminus \{0\}$, *then there is* $h' \in H_+$ *with* $|h| = |h'|$ *such that* $H_+^\infty \frac{1}{x+\mathrm{i}} h'$ *is dense and contained in* $\mathrm{dom}\,M_+(\kappa)$.

Proof Let $h \in \mathrm{dom}\,M_+(\kappa) \setminus \{0\}$. Then $\kappa h \in L^2(\mathbb{R})$, whence $\kappa\,e^j \in L^2(\mathbb{R})$ for $j := \ln|h|$. By (2.1), $j/(1 + x^2)$ is integrable.

Recall 2.2. Put $c := C^{-1}$. Check that $j \circ c \geq 0$ is integrable on the torus $\mathbb{T}$. Hence $U(w) := \exp\left(\frac{1}{2\pi} \int_0^{2\pi} \frac{e^{\mathrm{i}t} + w}{e^{\mathrm{i}t} - w} j \circ c(e^{\mathrm{i}t})\,\mathrm{d}t\right)$ is an outer function on the disc $\mathbb{D}$ (see e.g.

[17, Chapter 3]). So U converges non-tangentially a.e. to a function u on $\mathbb{T}$ satisfying $|u| = e^{j \circ c} \in L^2(\mathbb{T})$, whence $U \in H^2(\mathbb{D})$. Therefore, as known (see e.g. [21, Sect. 3]), $U H^\infty(\mathbb{D})$ is dense in $H^2(\mathbb{D})$. This implies that $u H^\infty(\mathbb{T})$ is dense in $H^2(\mathbb{T})$.

Put $h' := u \circ C$. Note $|h'| = e^j = |h|$. The above result is transferred to H_+ by Γ in 2.2. Accordingly, $\frac{1}{x+\mathrm{i}} h' \in H_+$ and $\frac{1}{x+\mathrm{i}} h' H_+^\infty$ is dense in H_+. The latter is contained in dom $M_+(\kappa)$ by 2.5 (a), since $\kappa \frac{1}{x+\mathrm{i}} h'$ is square-integrable and hence $\frac{1}{x+\mathrm{i}} h' \in$ dom $M_+(\kappa)$. Finally $h' \in H_+$ by 3.1(c). □

Two results on the eigenvalues of WH follow. The proof of 2.11 uses an alternative argument with respect to [22, 2.8] for the case of bounded κ.

Theorem 2.11 *Let κ be not almost constant. If $\lambda \in \mathbb{C}$ is an eigenvalue of W_κ, then $\bar{\lambda}$ is not an eigenvalue of $W_{\bar{\kappa}}$. If κ is almost real, then W_κ has no eigenvalues and in particular W_κ is injective.*

Proof Keep 2.1 in mind. Since $W_\kappa - \lambda I = W(\kappa - \lambda 1_\mathbb{R})$ assume without restriction $\lambda = 0$. Suppose $W_\kappa u = 0$, $W_{\bar{\kappa}} v = 0$. Set $h_+ := \mathcal{F}^{-1} P_+^* u$. Then $h_+ \in H_+$ and $h_- := \kappa h_+ \in H_-$. Set $k_- := \overline{\mathcal{F}^{-1} P_+^* v}$. Then $k_- \in H_-$ and $\bar{\kappa} \bar{k}_- \in H_-$, whence $k_+ := \kappa k_- \in H_+$.

Note that $j := h_- k_- = \kappa h_+ k_- = h_+ k_+$. Hence there is a holomorphic $\chi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}$ such that its partial maps satisfy $\chi_y \in L^1(\mathbb{R})$, $K := \sup\{\|\chi_y\|_1 : y \neq 0\} < \infty$, $\chi_y \to j$ pointwise a.e. and $\|\chi_y - j\|_1 \to 0$ for $y \to 0$. By a standard argument (see also [23, Theorem II]) χ extends to an entire function still called χ with $\chi|_\mathbb{R} = j$ a.e. Fix $z \in \mathbb{C}$, $|z| > 1$. We use the representation $\pi \chi(z) = \int_\mathbb{D} \chi(z+w) \, \mathrm{d}^2 w$. Then $\pi |\chi(z)| \leq \int_{-1}^1 \int_{-\infty}^\infty |\chi(x+u+\mathrm{i}(y+v))| \, \mathrm{d} u \, \mathrm{d} v \leq 2K$ so that χ is constant equal to 0, whence $h_- k_- = 0$, $h_+ k_+ = 0$. ([22, 2.8] obtains this result using alternatively [14, Theorem 76], by which $(2\pi)^{1/2} \mathcal{F}_{L^1}(h_\pm k_\pm) = \mathcal{F} h_\pm \star \mathcal{F} k_\pm$. Accordingly, $\mathcal{F} h_- \star \mathcal{F} k_- = \mathcal{F} h_+ \star \mathcal{F} k_+ = 0$, since $\mathcal{F} h_-$, $\mathcal{F} k_-$ vanish on $[0, \infty[$ and $\mathcal{F} h_+$, $\mathcal{F} k_+$ vanish on $]-\infty, 0]$.) The cases $k_- = 0$ or $h_+ = 0$ are trivial. Otherwise $h_+ \neq 0$ and $\kappa h_+ = h_- = 0$, whence $\kappa = 0$ a.e.

Now let κ be real, let $\lambda \in \mathbb{C}$, and let $g \in$ dom W_κ satisfy $W_\kappa g = \lambda g$. Then $\lambda \langle g, g \rangle = \langle g, W_\kappa g \rangle = \langle W_\kappa g, g \rangle = \bar{\lambda} \langle g, g \rangle$. Hence $g = 0$, since otherwise $\lambda \in \mathbb{R}$ would contradict the foregoing result. □

Proposition 2.12 *Let W_κ be not a multiple of I. If $\lambda \in \mathbb{C}$ is an eigenvalue of W_κ, then $\kappa \neq \lambda$ a.e.*

Proof Put $E := \kappa^{-1}(\{\lambda\})$. $\mathbb{R} \setminus E$ is not a null set. Let $h \in$ dom $M_+(\kappa) \setminus \{0\}$ satisfy $M_+(\kappa) h = \lambda h$. Then $M_+(\kappa') h = 0$ for $\kappa' := \kappa - \lambda$. Hence $h_- := \kappa' h \in H_-$. If E were not a null set, then $h_- = 0$ as vanishing on E, whence h vanishes on $\mathbb{R} \setminus E$ implying the contradiction $h = 0$. □

The following result from [12, 3.2] obviously is stronger than 2.12: *Let λ be an eigenvalue of $W_\kappa \neq \lambda I$. Then $\frac{\ln |\kappa - \lambda|}{1 + x^2}$ is integrable.* The converse still does not hold as the case of real-valued symbols shows by 2.11.

By 2.13 the symbol of a WH is uniquely determined.

Lemma 2.13 *Let W_{κ_1} and W_{κ_2} coincide on a dense set of $L^2(\mathbb{R}_+)$. Then $\kappa_1 = \kappa_2$ a.e. In particular, if κ is proper and $W_\kappa \subset 0$, then $\kappa = 0$ a.e. and $W_\kappa = 0$.*

Proof Put $\beta := \kappa_1 - \kappa_2$. Then $W_\beta|_D = 0$ for some dense $D \subset L^2(\mathbb{R}_+)$. Let $k \in$ dom W_β^*. Then for all $g \in D$ one has $0 = \langle k, W_\beta g \rangle = \langle W_\beta^* k, g \rangle$, whence $W_\beta^* k = 0$. Since $W_{\overline{\beta}} \subset W_\beta^*$, it follows $W_{\overline{\beta}}|_D = 0$. So β is almost constant by 2.11, whence $\beta = 0$ a.e. □

We turn to symmetric WH.

Corollary 2.14 *Let W_κ be densely defined. Then W_κ is symmetric if and only if κ is almost real.*

Proof Let $W_\kappa \subset W_\kappa^*$. Since generally $W_{\overline{\kappa}} \subset W_\kappa^*$ and dom $W_\kappa =$ dom $W_{\overline{\kappa}}$, it follows $W_\kappa = W_{\overline{\kappa}}$, whence $\kappa = \overline{\kappa}$ a.e. by 2.13. The converse is obvious. □

Hence due to 2.11 densely defined symmetric WH have no eigenvalues. —Now the numerical range of densely defined semibounded WH is determined.

Theorem 2.15 *Let W_κ be densely defined symmetric. Then W_κ is bounded below if and only if κ is real essentially bounded below, and the maximal lower bound of W_κ equals the maximal essential lower bound of κ. If W_κ is bounded below and not bounded, then the numerical range $\{\langle g, W_\kappa g \rangle : g \in$ dom $W_\kappa, \ ||g|| = 1\}$ equals $]\alpha, \infty[$ with α the maximal lower bound.*

Proof By 2.14 let κ be real. First suppose that κ is bounded below with maximal lower bound a. Put $\kappa' := \kappa - a \geq 0$. Then for all $h \in$ dom $M_+(\kappa') =$ dom $M_+(\kappa)$ one has $\langle h, M_+(\kappa')h \rangle = \int \kappa'|h|^2 \, dx \geq 0$, whence $\langle h, M_+(\kappa)h \rangle \geq a \, \|h\|^2$. So a is a lower bound for $M_+(\kappa)$.

Now let $M_+(\kappa)$ be bounded below with maximal lower bound α. Put $\kappa' := \kappa - \alpha$. Then $\langle h, M_+(\kappa')h \rangle \geq 0 \ \forall \ h \in$ dom $M_+(\kappa')$. We show that $B := \{x \in \mathbb{R} : \kappa'(x) < 0\}$ is a null set. Assume the contrary. Let $h_0 \in$ dom $M_+(\kappa') \setminus \{0\}$. Then $c := \int_B \kappa'|h_0|^2 \, dx < 0$, since h_0 does not vanish on a set of positive measure. Put $A := \mathbb{R} \setminus B$ and let $\lambda \in]0, 1[$. Then $f := \lambda 1_A|h_0| + 1_B|h_0| \leq |h_0|$. Hence $f \in L^2(\mathbb{R}) \setminus \{0\}, |\ln f| \leq |\ln |h_0||$, and $\ln f/(1+x^2)$ is integrable by (2.1). Therefore by (2.1) there is $h \in H_+$ with $|h| = |f|$. Note that $h \in$ dom $M_+(\kappa')$, since $|\kappa' h|^2 \leq |\kappa' h_0|^2$ is integrable. Hence the contradiction $0 \leq \langle h, M_+(\kappa')h \rangle = \int \kappa'|h|^2 \, dx = \lambda^2 \int_A \kappa'|h_0|^2 \, dx + c < 0$ follows for λ sufficiently small. Therefore B is a null set, whence $\kappa' \geq 0$ a.e. and α is an essential lower bound of κ. —This proves $a = \alpha$.

Since $M_+(\kappa)$ is not bounded above by assumption, the numerical range R of $M_+(\kappa)$ is not bounded above. As R is convex one infers $]\alpha, \infty[\subset R \subset [\alpha, \infty[$. It remains to show $\alpha \notin R$ or, equivalently, that $\langle h, M_+(\kappa')h \rangle = 0$ implies $h = 0$, where $\kappa' \geq 0$ is not almost zero. Indeed, $0 = \langle h, M_+(\kappa')h \rangle = \int \kappa'|h|^2 \, dx \Rightarrow h$ vanishes on a non-null set $\Rightarrow h = 0$. □

For bounded WH, 2.15 implies the following well-known result.

Corollary 2.16 *Let W_κ be bounded symmetric and not a multiple of I. Then the numerical range equals $]a, b[$ with a and b the minimum and maximum, respectively, of the essential range of κ.*

Concluding this section we deal with the unilateral translation invariance of WH. We are inspired by [6, Sect. 2.3)] which treats the bounded case 2.18. Observe the easily verifiable relation

$$T_b^* P_+^* W_\kappa P_+ T_b = M(1_{]-b,\infty[}) \mathcal{F} M(\kappa) \mathcal{F}^{-1} M(1_{]-b,\infty[}) \tag{2.4}$$

where T_b, $b \in \mathbb{R}$ denotes the unitary one-parameter group of translations $T_b f(x) := f(x - b)$ on $L^2(\mathbb{R})$. It shows again the invariance 2.5(c)

$$W_\kappa = S_b^* W_\kappa S_b \tag{2.5}$$

under the unilateral translations S_b, $b \geq 0$, since $S_b = P_+ T_b P_+^*$. Recall $S_b = W(e^{ib(\cdot)})$. Moreover it implies that $T_b^* P_+^* W_\kappa P_+ T_b f$ converges as $b \to \infty$ if $f \in T_a^* P_+^*(\text{dom } W_\kappa)$ for some $a \geq 0$, yielding the limit $\mathcal{F} M(\kappa) \mathcal{F}^{-1} f$. In particular $\{\|W_\kappa S_b g\|: b \geq 0\}$ is bounded for every $g \in \text{dom } W_\kappa$. If W_κ is densely defined then also $\{\|W_\kappa^* S_b g\|: b \geq 0\}$ is bounded as $W_\kappa^*|_{\text{dom } W_\kappa} = W_{\overline{\kappa}}$.

Theorem 2.17 *Let A be a densely defined operator in $L^2(\mathbb{R}_+)$ satisfying*

$$A \subset S_b^* A S_b \quad \forall\, b \geq 0 \tag{2.6}$$

If $\text{dom } A \subset \text{dom } A^*$ *and* $\{\|A S_b g\|: b \geq 0\}$ *and* $\{\|A^* S_b g\|: b \geq 0\}$ *are bounded for every* $g \in \text{dom } A$ *then there is a WH W_κ extending A.*

Proof (i) For $a \geq 0$ put $D_a := T_a^* P_+^*(\text{dom } A)$ and $D := \bigcup_{a \geq 0} D_a$. Then D is a dense translation invariant subspace of $L^2(\mathbb{R})$.

Indeed, let $f \in L^2(\mathbb{R})$ and $\varepsilon > 0$. Since $f_a := 1_{]-a,\infty[} f \to f$ for $a \to \infty$ in the mean, one has $\|f - f_a\| \leq \varepsilon/2$ for some $a \geq 0$. As $T_a f_a = P_+^* P_+ T_a f$, there is $g \in \text{dom } A$ with $\|P_+ T_a f_a - g\| \leq \varepsilon/2$. Hence $\|f_a - T_a^* P_+^* g\| \leq \varepsilon/2$. It follows that D is dense.

Note that $S_b(\text{dom } A) \subset \text{dom } A$ for $b \geq 0$ due to $A \subset S_b^* A S_b$. Then $D_a \subset D_b$ for $a \leq b$. Indeed, put $c := b - a \geq 0$. Then check $T_a^* P_+^* = T_b^* P_+^* S_c$, whence $D_a = T_b^* P_+^* S_c(\text{dom } A) \subset T_b^* P_+^*(\text{dom } A) = D_b$. It follows that D is a subspace of $L^2(\mathbb{R})$.

For the translation invariance of D it suffices to show $T_c^* P_+^* g \in D$ for $c \in \mathbb{R}$, $g \in \text{dom } A$. If $c \geq 0$ this is obvious. Let $c < 0$. Then $T_c^* P_+^* g = T_{-c} P_+^* g = P_+^* P_+ T_{-c} P_+^* g = P_+^* S_{-c} g \in D$ as $S_{-c} g \in \text{dom } A$.

(ii) Let $f \in D$. Then $f = T_a^* P_+^* g$ for some $a \geq 0$, $g \in \text{dom } A$, whence $f_b := T_b^* P_+^* A P_+ T_b f = T_b^* P_+^* A S_{b-a} g$ is well-defined for all $b \geq a$. We are going to show that $\lim_{b \to \infty} f_b$ exists.

Let $c \geq b \geq a$: $\| f_c - f_b \|^2 = \| f_c \|^2 + \| f_b \|^2 - \langle f_c, f_b \rangle - \langle f_b, f_c \rangle$, with $\|f_c\|^2 = \|A P_+ T_c f\|^2$, $\| f_b \|^2 = \| A P_+ T_b f \|^2$, $\langle f_c, f_b \rangle = \langle S_{c-b}^* A P_+ T_{c-b} T_b f, A P_+ T_b f \rangle = \langle S_{c-b}^* A P_+ T_{c-b}(P_+^* P_+ T_b f), A P_+ T_b f \rangle = \langle S_{c-b}^* A S_{c-b} P_+ T_b f, A P_+ T_b f \rangle$, which equals $\|A P_+ T_b f\|^2$ by (2.6) since $P_+ T_b f \in \text{dom } A$. So $\| f_c - f_b \|^2 = \|A P_+ T_c f\|^2 - \|A P_+ T_b f\|^2$. This implies

that $b \mapsto \|A P_+ T_b f\|^2$ is increasing. Being bounded by the assumption it follows $\|f_c - f_b\|^2 \to 0$ for $b, c \to \infty$ so that $\lim_{b \to \infty} f_b$ exists.

(iii) Thus $C_1 f := \lim_{b \to \infty} f_b$ defines an operator C_1 on D. Since one has $T_c^* C_1 T_c f = \lim_{b \to \infty} T_c^* T_b^* P_+^* A P_+ T_b T_c f = \lim_{b \to \infty} T_{c+b}^* P_+^* A P_+ T_{c+b} f = \lim_{b' \to \infty} T_{b'}^* P_+^* A P_+ T_{b'} f = C_1 f$, it is translation invariant.

(iv) Also $C_2 f := \lim_{b \to \infty} T_b^* P_+^* A^* P_+ T_b f$ exists for $f \in D$ thus defining an operator C_2 on D. This result follows replacing A in (ii) by $A^\# := A^*|_{\operatorname{dom} A}$. It remains to verify $A^\# \subset S_b^* A^\# S_b$, $b \geq 0$. Indeed, for $g, g' \in \operatorname{dom} A$ one has $\langle g, S_b^* A^\# S_b g' \rangle = \langle S_b g, A^* S_b g' \rangle = \langle A S_b g, S_b g' \rangle$ since $S_b g \in \operatorname{dom} A \subset \operatorname{dom} \overline{A} = \operatorname{dom} A^{**}$. So $\langle g, S_b^* A^\# S_b g' \rangle = \langle S_b^* A S_b g, g' \rangle = \langle A g, g' \rangle = \langle g, A^\# g' \rangle$ by (2.6) and $g, g' \in \operatorname{dom} A$. This implies $S_b^* A^\# S_b g' = A^\# g'$, whence the claim.

(v) Obviously $C_2 \subset C_1^*$. Hence C_1^* is densely defined and the closure $C := \overline{C}_1$ exists. Clearly translation invariance $C = T_b^* C T_b$ holds. Equivalently $\mathcal{F}^{-1} C \mathcal{F}$ commutes with $M(e^{i b(\cdot)}) = \mathcal{F}^{-1} T_b \mathcal{F}$ for all $b \in \mathbb{R}$. Thus $\mathcal{F}^{-1} C \mathcal{F} = M(\kappa)$ for some measurable function κ. Hence $P_+ C P_+^* = W_\kappa$. Finally, for $g \in \operatorname{dom} A$ one has $P_+^* g \in D$, $W_\kappa g = P_+ C_1 P_+^* g = \lim_{b \to \infty} P_+ T_b^* P_+^* A P_+ T_b P_+^* g = \lim_{b \to \infty} S_b^* A S_b g = \lim_{b \to \infty} A g = A g$. $\qquad\square$

Corollary 2.18 *Let A be a bounded operator on $L^2(\mathbb{R}_+)$. Then A is a WH if and only if*

$$A = S_b^* A S_b \quad \forall\, b \geq 0$$

Proof It remains to observe that $A \subset W_\kappa$ by 2.17 implies $A = W_\kappa$. $\qquad\square$

For 2.18 see also [6, (2.10)], where the existence of $\lim_{b \to \infty} T_b^* P_+^* A P_+ T_b f$ (see (ii) of the proof of 2.17) is not proven.

3 Rational Symbols

WH for rational symbols $\kappa = \frac{P}{Q}\big|_{\mathbb{R}}$ with polynomials $P \neq 0$, $Q \neq 0$ permit some more general analysis. By 2.4 they are densely defined. In 3.2 we show that they are closed and we determine their domains, ranges, and kernels and deficiency spaces, which are finite dimensional, and their spectral and Fredholm points. In particular, in the symmetric case, i.e., for a real rational symbol the deficiency spaces and indices are explicitly available 3.4.

Recall that a densely defined closed operator between Banach spaces with finite dimensional kernel and cokernel is called a Fredholm operator if its range is closed (cf. [25]).

Mostly we will omit $|_{\mathbb{R}}$ indicating the restriction on $\mathbb{R}$. A polynomial with a negative degree is the null function. For convenience we will deal with $M_+(\kappa)$ (2.2) in place of W_κ. We are indebted to the reviewer for a significantly simplified proof of (b) and (c) of the following lemma 3.1.

Lemma 3.1 *Let $P \neq 0$ and $Q \neq 0$ be polynomials.*

(a) *Let P and Q have no common zeros. Then $\frac{P}{Q} \in H_+$ ($\in H_-$) if and only if $\deg P < \deg Q$ and all zeros of Q are in the lower (upper) half-plane.*

(b) *Let $h \in H_+ \setminus \{0\}$ such that $\frac{P}{Q}h \in H_-$. Then there is a polynomial R with $\deg R < \min\{\deg P, \deg Q\}$ such that all zeros of P in the closed upper half-plane as well as all zeros of Q in the closed lower half-plane are zeros of R with at least the same multiplicities and such that $h = \frac{R}{P}$. Conversely, it is obvious that $h_+ := \frac{R}{P}$ and $h_- := \frac{R}{Q}$ satisfy $h_\pm \in H_\pm$ and $\frac{P}{Q}h_+ = h_-$.*

(c) *Let Q have no zeros in the upper half-plane. If $h \in H_+$ and $\frac{P}{Q}h \in L^2(\mathbb{R})$, then $\frac{P}{Q}h \in H_+$.*

(d) *Let P and Q have no common zeros. Suppose that $h \in H_+$ and $\frac{P}{Q}h \in H_+$. Then $h/Q \in H_+$.*

Proof (a) The result is known and we omit the proof.

(b) Put $n := \max\{\deg P, \deg Q\}$, $k := \frac{P}{Q}h$, $\alpha := (x+\mathrm{i})^{-n}P$, $\beta := (x-\mathrm{i})^{-n}Q$. Then $\alpha h \in H_+$ and $\beta k \in H_-$, since $\alpha \in H_+^\infty$, $\beta \in H_-^\infty$, and

$$\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^n \alpha h = \beta k$$

Hence $\alpha h \in \ker M_+\left(\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^n\right)$, whence $\alpha h = (x+\mathrm{i})^{-n}R$ for some polynomial R with $\deg R < n$ by [22, Theorem 3.1]. Therefore $h = R/P$ and $k = R/Q$. Now, $h \in H_+$ implies by (a) that $\deg R < \deg P$ and all zeros of P in the closed upper half-plane are zeros of R with at least the same multiplicities. The analogous result regarding Q follows from $k \in H_-$.

(c) For $g := \frac{P}{Q}h \in L^2(\mathbb{R})$ write $g = g_+ + g_-$ with $g_+ \in H_+$, $g_- \in H_-$. Then, using n, α, β from the proof of (b) and $\gamma := (x+\mathrm{i})^{-n}Q$, it follows

$$\left(\frac{x+\mathrm{i}}{x-\mathrm{i}}\right)^n (\alpha h - \gamma g_+) = \beta g_-$$

whence analogously $Ph - Qg_+ = Qg_- = R$ for some polynomial R. Then $g_- = R/Q$ implies $R = 0$ by (a), since Q has no zeros in the upper half-plane by assumption.

(d) Assume first that Q has no real zeros. Let ϕ and ψ be the holomorphic functions on the upper half-plane related to h and $\frac{P}{Q}h$, respectively. Then $\frac{P}{Q}\phi$ is meromorphic on the upper half-plane and converges non-tangentially to $\frac{P}{Q}h$ a.e. Since ψ does the same, according to [16, IV 2.5], $\psi = \frac{P}{Q}\phi$ holds. Hence ϕ/Q is holomorphic on the upper half-plane with $(\phi/Q)_y \to h/Q$ for $0 < y \to 0$ a.e. Let $0 < \delta < c$ such that $C := [-c, c] \times \mathrm{i}[\delta, c]$ is a neighborhood of the zeros of Q in the upper half-plane. Then $|1/Q|$ is bounded by some constant L on $\{z : \operatorname{Im} z \geq 0\} \setminus C$, and $|\phi/Q|$ is bounded on C by some M. Recall that $\|\phi_y\|_2^2$ is bounded for $y > 0$ by some K. Then $\int |(\phi/Q)(x+\mathrm{i}y)|^2\,\mathrm{d}x \leq L^2 \int |\phi(x+\mathrm{i}y)|^2\,\mathrm{d}x + \int_{-c}^{c} M^2\,\mathrm{d}x \leq L^2 K + 2cM^2$

for all $y > 0$. Finally, for $0 < y < \delta$, $\int |(\phi/Q)(x + \mathrm{i}\,y) - (h/Q)(x)|^2\,\mathrm{d}\,x \leq L^2 \int |\psi(x + \mathrm{i}\,y) - h)(x)|^2\,\mathrm{d}\,x \to 0$ for $y \to 0$, whence $h/Q \in H_+$.

Now we turn to the general case. Note $h/Q \in L^2(\mathbb{R})$, cf. (v). Write $Q = q\,Q_0$, where q has only real zeros and Q_0 has no real zeros. Let $q_\epsilon(z) := q(z + \mathrm{i}\,\epsilon)$ for $\epsilon > 0$ and put $Q_\epsilon := q_\epsilon\,Q_0$. Note $|q(z)/q_\epsilon(z)| < 1$ on the upper half-plane. Therefore $\frac{P}{Q_\epsilon}h = \frac{q}{q_\epsilon}\frac{P}{Q}h \in H_+$, where Q_ϵ has no real zeros. Moreover, for $\epsilon > 0$ small enough, P and Q_ϵ have no common zeros. Hence the foregoing result applies so that $h/Q_\epsilon \in H_+$. Now $h/Q_\epsilon = \frac{q}{q_\epsilon}h/Q \to h/Q$ for $\epsilon \to 0$ in the mean implying $h/Q \in H_+$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

Let $\kappa = \frac{P}{Q}$ be a rational function, where the polynomials P and Q have no common zeros. Then let q be the polynomial, the zeros of which are the real zeros of Q with the same multiplicities. Put $\varsigma := \max\{\deg q, \deg P - \deg Q + \deg q\}$. Moreover put $P = P_<P_\geq$, where the zeros of $P_<(P_>)$ and $P_\geq$ are exactly the zeros of P in the lower(upper) half-plane and in the closed upper half-plane, respectively. $\tilde{P}$ denotes the polynomial whose coefficients are the complex conjugates of P. Analogous notations concern Q.

Theorem 3.2 *Let $\kappa = \frac{P}{Q}$ be a rational function, where the polynomials P and Q have no common zeros. Then $M_+(\kappa)$ is densely defined and closed and*

(a) $\operatorname{dom} M_+(\kappa) = \frac{q}{(x+\mathrm{i})^\varsigma}H_+$ *and* $\operatorname{ran} M_+(\kappa) = P_{H_+}\left(\frac{P}{Q_> Q_<(x+\mathrm{i})^\varsigma}H_+\right)$

(b) $\ker M_+(\kappa) = \left\{(Q_\leq/P_<)r \; : \quad r \text{ polynomial with } \deg r < \min\{\deg P_< - \deg Q_\leq,\ \deg Q_> - \deg P_\geq\}\right\}$

(c) $\left(\operatorname{ran} M_+(\kappa)\right)^\perp = \left\{(\tilde{Q}_</\tilde{P}_<)r : r \text{ polynomial with } \deg r < \deg P_> - \deg Q_>\right\}$

(d) *the following statements are equivalent:*

    (1) $M_+(\kappa)$ *is a Fredholm Operator*

    (2) $\operatorname{ran} M_+(\kappa)$ *closed*

    (3) $\deg Q \leq \deg P$ *and* P *without real zeros*

    (4) $0 \notin \overline{\kappa(\mathbb{R})}$

Proof For the closeness of $M_+(\kappa)$ write $\frac{P}{Q}$ in the form $\frac{P}{Q} = \frac{P_0}{Q_0} + \frac{p}{q}$ with polynomials P_0, Q_0, p, q such that $Q = Q_0q$, Q_0 has no real zeros, q has only real zeros, $\deg P_0 < \deg Q_0$, and p and q have no common zeros and satisfy $\varsigma = \max\{\deg p, \deg q\}$.

Since $\kappa_0 := \frac{P_0}{Q_0}\big|_\mathbb{R}$ is bounded, $M_+(\kappa_0)$ is bounded. It follows $M_+(\kappa) = M_+(\kappa_0) + M_+(\frac{p}{q})$ and it remains to show that $M_+(\frac{p}{q})$ is closed. Let $h_n \in \operatorname{dom} M_+(\frac{p}{q})$ such that (h_n) converges to some $h \in H_+$ and $(M_+(\frac{p}{q})h_n)$ converges to some $k \in H_+$. By 3.1(c), $\frac{p}{q}h_n \in H_+$. Hence one has $h_n \to h$ and $\frac{p}{q}h_n \to k$ in $L^2(\mathbb{R})$. Since $M(\frac{p}{q})$ is closed, $h \in \operatorname{dom} M_+(\kappa)$ and $k = M_+(\frac{p}{q})h$ follows.

(a) $\operatorname{dom} M_+(\kappa)$ is dense by 2.4. Arguing as above it remains to show $\operatorname{dom} M_+(\frac{p}{q}) = \frac{q}{(x+\mathrm{i})^\varsigma}H_+$. Let $h \in H_+$. Then, by 3.1(c), $\frac{q}{(x+\mathrm{i})^\varsigma}h \in H_+$ and $\frac{p}{q}\frac{q}{(x+\mathrm{i})^\varsigma}h \in H_+$ implying $\frac{q}{(x+\mathrm{i})^\varsigma}H_+ \subset \operatorname{dom} M_+(\frac{p}{q})$. For the converse inclusion argue $g \in \operatorname{dom} M_+(\frac{p}{q})$

$\Rightarrow g \in H_+$, $\frac{p}{q}g \in L^2 \Rightarrow h := \frac{p}{q}g \in H_+$ by 3.1(c). Hence $g = \frac{q}{p}h$, whence $\frac{1}{p}h \in H_+$ by 3.1(d). Since $\varsigma = \max\{\deg p, \deg q\}$ one infers $k := \frac{(x+i)^\varsigma}{p}h \in L^2$, whence $k \in H_+$ applying 3.1(c) to $\frac{1}{p}h \in H_+$. This shows $g = \frac{q}{(x+i)^\varsigma}k \in \frac{q}{(x+i)^\varsigma}H_+$. —
Now the claim about ran $M_+(\kappa)$ is obvious.

(b) Check: $h_0 \in \ker M_+(\kappa) \Leftrightarrow h_0 \in \operatorname{dom} M_+(\kappa)$, $M_+(\kappa)h_0 = 0 \Leftrightarrow h_0 \in H_+$, $\kappa h_0 \in L^2(\mathbb{R})$, and $\kappa h_0 \in H_-$. According to 3.1(b) this means $h_0 = R/P$, where R is a polynomial with $\deg R < \min\{\deg P, \deg Q\}$ and $R = P_{\geq}Q_{<}r$, whence the claim.

(c) Using (a) one has $h_0 \in (\operatorname{ran} M_+(\kappa))^{\perp} \Leftrightarrow 0 = \langle h_0, \frac{P}{Q_{>}Q_{<}(x+i)^\varsigma}h \rangle = \langle \frac{\tilde{P}}{(Q_{>})^{\sim}(Q_{<})^{\sim}(x-i)^\varsigma}h_0, h \rangle \forall h \in H_+ \Leftrightarrow \frac{\tilde{P}}{(Q_{>})^{\sim}(Q_{<})^{\sim}(x-i)^\varsigma}h_0 \in H_-$. By 3.1(b) this means $h_0 = R/\tilde{P}$, where R is a polynomial such that $\deg R < \min\{\deg \tilde{P}, \deg \tilde{Q} - \deg q + \varsigma\} = \deg P$ and $R = \tilde{P}_{\geq}(Q_{>})^{\sim}r = \tilde{P}_{\geq}\tilde{Q}_{<}r$, whence the claim.

(d) By the forgoing results $(1)\Leftrightarrow(2)$ holds. Moreover $(3)\Leftrightarrow(4)$ is quite obvious. So we turn to $(2)\Leftrightarrow(3)$.

Show first $(3)\Rightarrow(2)$. Put $R := \frac{P}{(Q_{>})^{\sim}Q_{<}(x+i)^\varsigma}$. Since by the assumptions nominator and denominator of R have equal degree and have no real zeros, R and $\frac{1}{R}$ are bounded on $\mathbb{R}$. Hence $M(R)$ is a homeomorphism on $L^2(\mathbb{R})$, whence RH_+ is closed. Since $(Q_{>})^{\sim} = \tilde{Q}_{<}$ it follows by 3.1(c) that $RH_+ \subset H_+$. Hence $RH_+ = \operatorname{ran} M_+(\kappa')$ for $\kappa' := \frac{P}{Q'}$ with $Q' := \tilde{Q}_{<}Q_{<}q$, whence $\dim(RH_+)^{\perp} < \infty$ by (c). So it suffices to show that $RH_+ \subset \operatorname{ran} M_+(\kappa)$. By 3.1(c), $\frac{Q_{>}}{Q_{<}}H_+ \subset H_+$. So by (a), $\operatorname{ran} M_+(\kappa) \supset P_{H_+}\left(\frac{P}{Q_{>}Q_{<}(x+i)^\varsigma}\frac{Q_{>}}{(Q_{>})^{\sim}}H_+\right) = P_{H_+}(RH_+) = RH_+$.

Now turn to $(2)\Rightarrow(3)$. Consider first the case $\deg P < \deg Q$. Put $R := Q_{>}Q_{<}(x+i)^\varsigma$. Note $\deg R = \deg Q$ and $\operatorname{ran} M_+(\kappa) = P_{H_+}\left(\frac{P}{R}H_+\right)$ by (a). By 3.1(c) and since $M\left(\frac{R_{>}}{R_{<}}\right)$ is unitary on $L^2(\mathbb{R})$, $\frac{R_{>}}{R_{<}}H_+$ is a closed subspace of H_+. By (c) one has $\left(\frac{R_{>}}{R_{<}}H_+\right)^{\perp} = \left\{\frac{r}{\tilde{R}_{<}} : \deg r < \deg R_{>}\right\}$ and hence

$$\frac{P}{R}H_+ = \left\{\frac{Pr}{R\tilde{R}_{<}} : \deg r < \deg R_{>}\right\} + \frac{P}{R_{<}\tilde{R}_{<}}H_+$$

Applying (c) for $\kappa = \frac{P}{R_{<}\tilde{R}_{<}}$ and $\kappa = \frac{P_{\geq}}{\tilde{P}_{<}}$ check $\left(\frac{P}{R_{<}\tilde{R}_{<}}H_+\right)^{\perp} = \left(\frac{P_{\geq}}{\tilde{P}_{<}}H_+\right)^{\perp} = \left\{\frac{r}{\tilde{P}_{<}} : \deg r < \deg P_{>}\right\}$. Hence $\frac{P}{R_{<}\tilde{R}_{<}}H_+$ is dense in $\frac{P_{\geq}}{\tilde{P}_{<}}H_+$, which is closed in H_+. Note that $P_{H_+}\left\{\frac{Pr}{R\tilde{R}_{<}} : \deg r < \deg R_{>}\right\} \subset V := \left\{\frac{B}{R_{<}\tilde{R}_{<}} : \deg B < \deg R\right\}$.

Now assume that $P_{H_+}\left(\frac{P}{R}H_+\right)$ is closed. Then the above considerations imply that $\frac{P_{\geq}}{\tilde{P}_{<}}H_+ \subset V + \frac{P}{R_{<}\tilde{R}_{<}}H_+$. Thus given $h_0 \in H_+$ there is $h \in H_+$ and some polynomial B with $\deg B < \deg R$ such that $h = \frac{R_{<}\tilde{R}_{<}}{P_{<}\tilde{P}_{<}}h_0 + \frac{B}{P}$. Hence $|h| \geq \left|\left|\frac{R_{<}\tilde{R}_{<}}{P_{<}\tilde{P}_{<}}\right||h_0| - \left|\frac{B}{P}\right|\right|$ with $m := \deg R - \deg P_{<} - \deg P_{>} \geq 1$ and $l := \deg B - \deg P \leq m - 1$. Choose $h_0 \in H_+$ satisfying $|h_0| = (1 + |x|)^{-3/4}$. (h_0 exists by (2.1) since square-integrable and $\ln(1 + |x|) \leq \sqrt{2|x|}$.) Because of $m - \frac{3}{4} - l > 0$ the right side tends to ∞ like $|x|^{m-3/4}$ for $|x| \to \infty$ contradicting $h \in L^2(\mathbb{R})$.

To complete the proof of the implication (2)⇒(3) it remains to treat the case that $\deg P \geq \deg Q$ and P has a real zero. Proceeding as in the forgoing case, here one has $\deg R = \deg P$. Hence, assuming that $P_{H_+}\left(\frac{P}{R}H_+\right)$ is closed, one has $m \geq 1$ and $l \leq -1$ and the same contradiction follows.   □

From 3.2 one immediately obtains

Corollary 3.3 *For $\lambda \in \mathbb{C}$ put $P^\lambda := P + \lambda Q$. Then referring to $M_+(\kappa)$, λ is*

- *a Fredholm point (i.e. $M_+(\kappa) - \lambda I$ is a Fredholm operator) iff $\lambda \notin \overline{\kappa(\mathbb{R})}$; if λ is a Fredholm point, then $\dim \ker(M_+(\kappa) - \lambda I) = \max\{0, \deg Q_> - \deg P_>^\lambda\}$, $\dim \operatorname{ran}(M_+(\kappa) - \lambda I)^\perp = \max\{0, \deg P_>^\lambda - \deg Q_>\}$, and further $\operatorname{ind}(M_+(\kappa) - \lambda I) = \deg Q_> - \deg P_>^\lambda$*
- *a regular value (i.e. $M_+(\kappa) - \lambda I$ is continuously invertible) iff $\lambda \notin \overline{\kappa(\mathbb{R})}$ and $\deg Q_> \leq \deg P_>^\lambda$*
- *in the resolvent set iff $\lambda \notin \overline{\kappa(\mathbb{R})}$ and $\deg Q_> = \deg P_>^\lambda$*
- *a spectral value iff $\lambda \in \overline{\kappa(\mathbb{R})}$ or $\deg Q_> \neq \deg P_>^\lambda$*
- *in the point spectrum iff $\deg Q_\leq < \deg P_<^\lambda$ and $\deg P_>^\lambda < \deg Q_>$*
- *in the continuous spectrum (i.e. $M_+(\kappa) - \lambda I$ is injective with dense not closed range) iff $\lambda \in \overline{\kappa(\mathbb{R})}$, $\deg P_>^\lambda \leq \deg Q_>$ and either $\deg P_<^\lambda \leq \deg Q_\leq$ or $\deg Q_> \leq \deg P_\geq^\lambda$*
- *in the residual spectrum (i.e. $M_+(\kappa) - \lambda I$ is injective with not dense range) iff $\deg Q_> < \deg P_>^\lambda$*

The characterization of the Fredholm points of $M_+(\kappa)$ and the fact that at a Fredholm point either the kernel or the deficiency space is trivial, are familiar from Krein's theory [3] for the case of integrable kernel. $\lambda \mapsto \deg P_>^\lambda$ is locally non-decreasing due to the continuity of the roots of a polynomial on its coefficients [26]. On $\mathbb{C} \setminus \overline{\kappa(\mathbb{R})}$ it is even locally constant, since there $P_>^\lambda = P_\geq^\lambda$. Hence, besides $\operatorname{ind}(M_+(\kappa) - \lambda I)$, also $\dim \ker(M_+(\kappa) - \lambda I)$ and $\dim \operatorname{ran}(M_+(\kappa) - \lambda I)^\perp$ are constant on the components of $\mathbb{C} \setminus \overline{\kappa(\mathbb{R})}$.

Theorem 3.4 *Let κ be a real rational function. Let p, q be real polynomials without common zeros such that $\kappa = \frac{p}{q}$ and put*

$$p - \mathrm{i}q = Q_+ Q_-$$

where the zeros of the polynomials Q_+ and Q_- are all in the upper and lower half-plane, respectively. Then $M_+(\kappa)$ is closed symmetric with $\operatorname{dom} M_+(\kappa) = \frac{q'}{(x+\mathrm{i})^\varsigma} H_+$. By definition the zeros of q' are the real zeros of q and $\varsigma := \max\{\deg q', \deg p - \deg q + \deg q'\}$. The deficiency indices are

$$n_\pm := \dim \operatorname{ran}\left(M_+(\kappa) \mp \mathrm{i}\, I\right)^\perp = \deg Q_\pm - \deg q_<$$

and the deficiency spaces are

$$\operatorname{ran}\left(M_+(\kappa) - \mathrm{i}\,I\right)^\perp = \{\frac{q_<}{Q_+}r : r \text{ polynomial with } \deg r < \deg Q_+ - \deg q_<\}$$

$$\operatorname{ran}\left(M_+(\kappa) + \mathrm{i}\,I\right)^\perp = \{\frac{q_<}{Q_-}r : r \text{ polynomial with } \deg r < \deg Q_- - \deg q_<\}$$

Proof For dom $M_+(\kappa)$ see 3.2(a). Adopting the notation of 3.2 one has $M_+(\kappa) - \mathrm{i}\,I = M_+\left(\frac{P}{Q}\right)$ with $P := p - \mathrm{i}\,q$ and $Q = q$. Obviously P and Q have no common zeros. In view of 3.2(c) note $\tilde{Q}_< = q_<$, $\tilde{P}_< = \tilde{Q}_+$, $\deg P_> = \deg Q_+$, $\deg Q_> = \deg q_> = \deg q_<$. Hence the formula for $\operatorname{ran}\left(M_+(\kappa) - \mathrm{i}\,I\right)^\perp$ holds, which implies $n_+ = \max\{0, \deg Q_+ - \deg q_<\}$. It remains to show $\deg q_< \le \deg Q_+$. The assertions about the deficiency space for $-\mathrm{i}$ follow similarly.

Let m_t denote the number of zeros in the upper half-plane of $p - \mathrm{i}\,tq$ for $t > 0$. $p - \mathrm{i}\,tq$ has no real zeros. So by the continuity of the roots of a polynomial, m_t is locally constant and hence constant $= \deg Q_+$. Obviously m_t is also the number of zeros in the upper half-plane of $\frac{1}{t}p - \mathrm{i}\,q$. Then by the same continuity the zeros of q in the upper half-plane stay there for all t large enough, whence $m_t \ge \deg q_>$. This yields the result. $\qquad\square$

W_κ for κ in 3.4 has unequal deficiency indices and hence no self-adjoint extension if $\max\{\deg p, \deg q\}$ is odd. The deficiency indices (n_+, n_-) of $W(x^l), l \in \mathbb{Z}$, are $(\frac{|l|}{2}, \frac{|l|}{2})$ if l is even and $(\frac{|l|+1|}{2}, \frac{|l|-1|}{2})$ otherwise. Other interesting examples are $W\left(\frac{x^2+1}{x}\right)$ and $W\left(\frac{x^2-1}{x}\right)$ with deficiency indices $(1, 1)$ and $(2, 0)$, respectively. Compare the later with $W\left(\frac{x^2-1}{x-2}\right)$ having deficiency indices $(1, 1)$. —This section is concluded by a much needed

Example 3.5 There are WH, even essentially self-adjoint semibounded ones, which are not closed, as for instance $W_{|x|}$ (or $W_{1/|x|}$).

As to the proof, by 3.2(a), dom $M_+(|x|) = \frac{1}{x+\mathrm{i}}H_+$ and hence $R_\pm := \operatorname{ran}(M_+(|x|)\mp \mathrm{i}\,I) = \frac{|x|\mp \mathrm{i}}{x+\mathrm{i}}H_+$. It suffices to show that R_+ is dense and $\ne H_+$.

The latter is easily inferred. Assume the contrary. Then there is $h \in H_+$ satisfying $\frac{|x|-\mathrm{i}}{x+\mathrm{i}}h = \frac{1}{x+\mathrm{i}}$. Hence $h = \frac{1}{|x|-\mathrm{i}} \in H_+$ and $h = \check{h} \in H_-$, whence the contradiction $h = 0$.

As to the former claim, let $h_\pm \in R_\pm^\perp$. Then $\langle h_\pm, \frac{|x|\mp \mathrm{i}}{x+\mathrm{i}}h\rangle = 0$ for all $h \in H_+$, whence $k_\pm := \frac{|x|\pm \mathrm{i}}{x-\mathrm{i}}h_\pm \in H_-$. Let $\phi_\pm$ be holomorphic on the upper half-plane with $\phi_{\pm,y} \to h_\pm$ for $0 < y \to 0$ pointwise a.e. and in the mean (cf. 2.1). Similarly let $\psi_\pm$ be holomorphic in the lower half-plane with $\psi_{\pm,y} \to k_\pm$ for $0 > y \to 0$. Consider $\phi := (z + \mathrm{i})\phi_-\phi_+$ and $\psi := (z - \mathrm{i})\psi_-\psi_+$ holomorphic on the upper and lower half-plane, respectively. Check that ϕ_y for $0 < y \to 0$ and ψ_y for $0 > y \to 0$ converge pointwise a.e. and in L^1_{loc} to $f := (x + \mathrm{i})h_-h_+$. By [23, Theorem II] there is an entire function χ extending ϕ and ψ. Let $\chi_\pm$ be equal to $\phi_\pm$ and $\psi_\pm$ on the upper and lower half-plane, respectively. Let $z = x + \mathrm{i}\,y$, $|z| > 1$. Then $\pi|\chi(z)| \le \int_{|w|\le 1}|\chi(z + w)|\,\mathrm{d}^2 w \le (|z| + 2)\left(\int_{|w|\le 1}|\chi_+(z + w)|^2\,\mathrm{d}^2 w \int_{|w|\le 1}|\chi_-(z + w)|^2\,\mathrm{d}^2 w\right)^{1/2}$. Now

$\int_{|w|\le 1} |\chi_\pm(z+w)|^2 \, d^2 w \le \int_{-1}^1 \int_{-\infty}^\infty |\chi_\pm(u+i(y+v))|^2 \, du \, dv \le \int_{-1}^1 C \, dv = 2C$
for some finite constant C. Hence $|\chi(z)| \le C'|z| \; \forall \, |z| > 1$. So χ is a polynomial
$a + bz$. For $y > 0$, $x \mapsto \phi_{-,y}(x)\phi_{+,y}(x) = \frac{a+bz}{z+i}$ is integrable on $\mathbb{R}$. This implies
$a = b = 0$ and hence $\chi = 0$. Therefore either $h_+ = 0$ or $h_- = 0$. This means that
one and hence both deficiency spaces are $\{0\}$. So R_+ is dense.

4 Semibounded Wiener–Hopf operators

In 4.3, 4.4 a densely defined semibounded WH W_κ is expressed in a canonical way
by the product of a closable operator and its adjoint. Replacing the operator by its
closure one obtains a self-adjoint extension $\tilde{W}_\kappa$ of W_κ, which is semibounded by the
same bound. The bound is not an eigenvalue of the extension. $\tilde{W}_\kappa$ is shown to be the
Friedrichs extension of W_κ. We start with a preparatory lemma. The condition $\gamma \ne 0$
not a.e. means that $\gamma^{-1}(\{0\})$ is not a null set.

Lemma 4.1 *Let $\gamma : \mathbb{R} \to \mathbb{C}$ be measurable. Put $E := \gamma^{-1}(\mathbb{C} \setminus \{0\})$ and let $A :=$
$P_+ \mathcal{F} M(\gamma) P_E^*$. Then A is densely defined and $A^* = P_E M(\overline{\gamma}) \mathcal{F}^{-1} P_+^*$ holds, and
$\mathrm{dom}\, A^* = \mathrm{dom}\, W_{\overline{\gamma}}$ is either $\{0\}$ or dense. If γ is not almost zero, then A^* is injective.
If $\gamma \ne 0$ not a.e., then A is injective. If $\gamma \ne 0$ a.e., then $\ker A = \{f \in L^2(\mathbb{R}) :
\gamma f \in H_-\}$, which equals $\{0\}$ if and only if $\frac{1}{\gamma}$ is not proper. Finally $\ker A^* A = \ker A$,
$W(|\gamma|^2) = A A^*$, and*

$$\gamma \text{ proper } \Leftrightarrow \mathrm{dom}\, W(|\gamma|^2) \ne \{0\} \Leftrightarrow \mathrm{dom}\, A^* \ne \{0\} \Leftrightarrow A \text{ closable} \qquad (\star)$$

Proof Note $M(\gamma) P_E^* = P_E^* M(\gamma|_E)$ and $P_E M(\gamma) = M(\gamma|_E) P_E$ with $M(\gamma|_E)$ the
multiplication operator in $L^2(E)$. Hence $A = P_+ \mathcal{F} P_E^* M(\gamma|_E)$ is densely defined, and
$A^* = P_E M(\overline{\gamma}) \mathcal{F}^{-1} P_+^*$ by [27, 13.2(2)]. Note that $\mathrm{dom}\, A^* = \mathrm{dom}\, M(\overline{\gamma}) \mathcal{F}^{-1} P_+^* =
\mathrm{dom}\, W_{\overline{\gamma}}$, whence the claim on $\mathrm{dom}\, A^*$ by 2.10.
   First let E be proper. Then $A^* g = M(\overline{\gamma}|_E)(P_E \mathcal{F}^{-1} P_+^* g) = 0$ implies
$P_E \mathcal{F}^{-1} P_+^* g = 0$, whence $g = 0$. Similarly, $Ak = P_+ \mathcal{F}(P_E^* M(\gamma|_E)k) = 0$
means that $f_- := P_E^* M(\gamma|_E)k \in H_-$ and vanishes on $\mathbb{R} \setminus E$, whence $f_- = 0$
and hence $k = 0$. —Now assume at once $E = \mathbb{R}$. Then $A^* = M(\overline{\gamma}) \mathcal{F}^{-1} P_+^*$ is
injective as $M(\overline{\gamma})$ is injective. Furthermore, $f \in \ker A \Leftrightarrow f, \gamma f \in L^2(\mathbb{R})$ with
$P_+ \mathcal{F}(\gamma f) = 0 \Leftrightarrow f \in L^2(\mathbb{R}), \gamma f \in H_- \Leftrightarrow f \in L^2(\mathbb{R}), \overline{f} \in \frac{1}{\overline{\gamma}} H_+$, and recall that
$\frac{1}{\gamma}$ is proper $\Leftrightarrow \frac{1}{\gamma} h \in L^2(\mathbb{R})$ for some $h \in H_+ \setminus \{0\}$.
   $\ker A^* A = \ker A$ is obvious since either $\gamma = 0$ a.e. or A^* is injective.
$M(\gamma) P_E^* P_E M(\overline{\gamma}) = M(\gamma) M(1_E) M(\overline{\gamma}) = M(|\gamma|^2)$, whence $A A^* = W(|\gamma|^2)$.
   Turn to the final claim $(\star)$. Recall that $\mathrm{dom}\, A^* = \mathrm{dom}\, W_{\overline{\gamma}}$ is either trivial or dense.
So the last equivalence is standard and the remaining equivalences hold by 2.7.   □

   In view of 4.1$(\star)$ recall the results on the domain of a WH in Sect. 2. If A is closable
it need not be closed, even if $A A^*$ is closed. $\big($Indeed, $W_{x^2} = A A^*$ is closed and
$\ne \tilde{W}_{x^2} = \overline{A} A^*$ by 3.4, 4.3.$\big)$ Recall that $\mathrm{ran}\, A^*$ is not dense if A is not injective. A is
not injective for $\gamma \ne 0$ a.e. if for instance $\frac{q}{\gamma} \mathcal{F}^{-1} s \in L^2(\mathbb{R})$ with q a polynomial and s

a Schwartz function with support in $]-\infty, 0]$. (Indeed, $q\mathcal{F}^{-1}s$ is a Schwartz function in H_-, whence $\frac{q}{\gamma}\mathcal{F}^{-1}s \in \ker A$.) Recall that A is injective if and only if $\gamma \neq 0$ not a.e. or $\gamma \neq 0$ a.e. and $\frac{1}{\gamma}$ not proper.

Lemma 4.2 *Let γ be proper. Suppose that $\gamma \neq 0$ not a.e. or that $\gamma \neq 0$ a.e. and $\frac{1}{\gamma}$ is not proper. Then A is closable and $\overline{A}$ is injective.*

Proof A is closable by 4.1($\star$). According to 2.6 ($p = \infty$) one has $|\gamma|\,e^j \leq 1$ for some j. Put $j' := 1_{\{j \leq 0\}}\, j - |x|^{1/2}$. Then $|\gamma|\,e^{j'} \leq 1$, and $e^{j'}$ and $|\gamma|\,e^{j'}$ are square integrable. By (2.1) there is $h \in H_+$ with $|h| = e^{j'}$. Hence $h \in \operatorname{dom} M_+(\overline{\gamma}) \setminus \{0\}$. Then $hD \subset \operatorname{dom} M_+(\overline{\gamma})$ for $D := \frac{1}{x+\mathrm{i}}H_+^\infty$. D is dense in H_+ since $D = \Gamma(H^\infty(\mathbb{T}))$ (see 2.2). —Now let $f \in \ker \overline{A} = (\operatorname{ran} A^*)^\perp$. For $d \in D$ one has $0 = \langle f, P_E M(\overline{\gamma}) P_{H_+}^*(hd)\rangle = \langle P_E^* f, \overline{\gamma}hd\rangle = \langle \gamma \overline{h} P_E^* f, d\rangle$, whence $\gamma \overline{h} P_E^* f \in H_-$. If $\gamma \neq 0$ not a.e., then it follows $\gamma \overline{h} P_E^* f = 0$, whence $f = 0$. If $\gamma \neq 0$ a.e., then $h\overline{f} = \frac{1}{\gamma}h'$ for some $h' \in H_+$. Assume $f \neq 0$. Then $h' \neq 0$ and $|\gamma|^{-1}||h'|$ is integrable. Hence $|\gamma|^{-1}$ is proper by (2.1) and 2.6 ($p = 1$), which however is excluded by the premise. □

The foregoing lemma is needed only in Sect. 5. The main result of this section follows.

Theorem 4.3 *Let $\kappa \geq 0$. Put $E := \kappa^{-1}(\mathbb{R} \setminus \{0\})$ and $A := P_+\mathcal{F}M(\sqrt{\kappa})P_E^*$. Then $W_\kappa = AA^*$. —Now let κ be proper not almost zero. Put $\tilde{W}_\kappa := \overline{A}A^*$. Then*

(a) W_κ is densely defined symmetric nonnegative and $\tilde{W}_\kappa$ is an injective nonnegative self-adjoint extension of W_κ.
(b) $\operatorname{dom} W_\kappa$ is a core of A^* and $\operatorname{dom} A^* \cap \operatorname{ran}(I + W_\kappa)^\perp = \{0\}$ holds.
(c) $\tilde{W}_\kappa$ is the Friedrichs extension of W_κ.

Proof (a) Apply 4.1 for $\gamma := \sqrt{\kappa}$. Accordingly, $W_\kappa = AA^*$ and, if κ is proper, W_κ is densely defined and symmetric nonnegative by 2.14, 2.15, and A^* is densely defined. Then $\overline{A} = A^{**}$ and by [27, 13.13(a)] $\overline{A}A^*$ is self-adjoint. Clearly $\overline{A}A^*$ is nonnegative. Check that $\overline{A}A^*$ is injective as A^* is injective by 4.1 for κ not almost zero.

(b), (c) According to [4, Theorem X.23], $\tilde{W}_\kappa$ is the Friedrichs extension only if $\operatorname{dom} \tilde{W}_\kappa \subset H_{W_\kappa}$, where H_{W_κ} is the completion of $\operatorname{dom} W_\kappa$ with respect to the sesquilinear form $\langle g, g'\rangle_{W_\kappa} := \langle g, g'\rangle + \langle g, W_\kappa g'\rangle$.

Endow $\operatorname{dom} A^*$ with the inner product $\langle g, g'\rangle_{A^*} := \langle g, g'\rangle + \langle A^*g, A^*g'\rangle$, by which $\operatorname{dom} A^*$ becomes a Hilbert space $\mathcal{K}$ since A^* is closed. Then the subspace $\operatorname{dom} \overline{A}A^*$ is dense in $\mathcal{K}$ since $\operatorname{dom} \overline{A}A^*$ is a core for A^*, see [27, 13.13(b)]. One easily checks that H_{W_κ} is the closure of $\operatorname{dom} W_\kappa$ in $\mathcal{K}$. Therefore $\operatorname{dom} \overline{A}A^* \subset H_{W_\kappa}$ if and only if $H_{W_\kappa} = \mathcal{K}$, which means that $\operatorname{dom} W_\kappa$ is a core of A^*. A short computation shows also that $H_{W_\kappa} = \mathcal{K}$ is equivalent to $\operatorname{dom} A^* \cap \operatorname{ran}(I + W_\kappa)^\perp = \{0\}$.

Hence it remains to show $\operatorname{dom} A^* \cap \operatorname{ran}(I + W_\kappa)^\perp = \{0\}$. Explicitly this means that $h_0 = 0$ if

$$h_0 \in H_+,\ \sqrt{\kappa}h_0 \in L^2(\mathbb{R}),\ \langle h_0, (1 + \kappa)h\rangle = 0\ \forall\, h \in H_+ \text{ with } \kappa h \in L^2(\mathbb{R}) \quad (\star)$$

To this end a sequence $(\alpha_n)_n$ in H_+^∞ will be constructed with $|\alpha_n| = e^{j_n}$ a.e., where $j_n(x) := 0$ if $1 + \kappa(x) < n$ and $j_n(x) := -\frac{1}{2}\ln\big(1 + \kappa(x)\big)$ otherwise, and

satisfying $\alpha_{n_k} \to 1$ pointwise a.e. for some subsequence (n_k). Provided $(\alpha_n)_n$ set $h_n := \alpha_n h_0$. Then $h_n \in H_+$ and almost everywhere $(1 + \kappa(x))|h_n(x)|$ is less than $n|h_0(x)|$ if $1 + \kappa(x) \le n$ and equals $\sqrt{1 + \kappa(x)}|h_0(x)|$ otherwise, which proves $(1 + \kappa)h_n \in L^2(\mathbb{R})$. Moreover $\sqrt{1 + \kappa}|h_n| \le \sqrt{1 + \kappa}|h_0|$ since $|\alpha_n| \le 1$ and $\sqrt{1 + \kappa}h_{n_k} \to \sqrt{1 + \kappa}h_0$ pointwise a.e., whence $\sqrt{1 + \kappa}h_{n_k} \to \sqrt{1 + \kappa}h_0$ in $L^2(\mathbb{R})$ by dominated convergence. Thus $(\star)$ holds for $h = h_{n_k}$, whence $0 = \langle h_0, (1 + \kappa)h_{n_k}\rangle = \langle \sqrt{1 + \kappa}h_0, \sqrt{1 + \kappa}h_{n_k}\rangle \to \langle \sqrt{1 + \kappa}h_0, \sqrt{1 + \kappa}h_0\rangle = ||\sqrt{1 + \kappa}h_0||^2$ implying $h_0 = 0$.

We turn to the construction of $(\alpha_n)_n$. By 2.8, $\frac{\ln(1 + \kappa)}{1 + x^2}$ and hence all $\frac{j_n}{1 + x^2}$ are integrable. For convenience we pass from $\mathbb{R}$ to the torus $\mathbb{T}$ by means of the Cayley transformation C (see 2.2). So let $\tilde{j}_n := j_n \circ C^{-1}$, which is integrable on $\mathbb{T}$. Put

$$F_n(w) := \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + w}{e^{it} - w} \tilde{j}_n(e^{it}) \, dt$$

for $w \in \mathbb{D}$. Then $\exp \circ F_n$ is an outer function. Let $\tilde{\alpha}_n$ denote its non-tangential limit a.e. on $\mathbb{T}$. It satisfies $|\tilde{\alpha}_n| = e^{\tilde{j}_n}$ a.e. Hence $\tilde{\alpha}_n \in H^\infty(\mathbb{T})$. It remains to show the existence of a subsequence (n_k) satisfying $\tilde{\alpha}_{n_k} \to 1$ a.e. The formula $Tf(z) := \lim_{r \uparrow 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + rz}{e^{it} - rz} f(e^{it}) \, dt$, $z \in \mathbb{T}$ defines a bounded operator on $L^1(\mathbb{T})$ into weak-$L^1(\mathbb{T})$, whence $|\{z \in \mathbb{T} : |Tf| > \delta\}| \le C \, \| f \|_1 \, / \delta$ for all $\delta > 0$ and $f \in L^1(\mathbb{T})$. Therefore, if $f_n \to 0$ in $L^1(\mathbb{T})$, then $Tf_n \to 0$ in probability, which implies $Tf_{n_k} \to 0$ a.e. for some subsequence (n_k). This applies to $(\tilde{j}_n)_n$ yielding $\tilde{\alpha}_{n_k} = \exp \circ T \tilde{j}_{n_k} \to 1$ a.e. $\qquad \square$

For $\kappa \ge 0$ and W_κ densely defined recall that the deficiency subspace $\mathrm{ran}(I + W_\kappa)^\perp$ of W_κ at -1 is trivial if and only if W_κ is essentially self-adjoint.

Semibounded symbol 4.4 *Let κ be real semibounded. Then there are $\alpha > 0$ and $\eta \in \{1, -1\}$ such that $\kappa' := \alpha 1 + \eta\kappa \ge 0$. Clearly $W_{\kappa'} = \alpha I + \eta W_\kappa$. Let W_κ be densely defined. Then so is $W_{\kappa'}$, and according to 4.3 there is the injective nonnegative self-adjoint extension $\tilde{W}_{\kappa'}$ of $W_{\kappa'}$. So*

$$\tilde{W}_\kappa := -\eta\alpha I + \eta\tilde{W}_{\kappa'} \tag{4.1}$$

is a semibounded self-adjoint extension of W_κ with bound $-\eta\alpha$, which is not an eigenvalue of $\tilde{W}_\kappa$. It is the Friedrichs extension of W_κ.

5 Isomorphic Singular Integral Operators

This section is concerned with the symmetric singular integral operator in $L^2(E)$ for proper E or $E = \mathbb{R}$

$$\left(L_\phi f\right)(x) := \frac{1}{2}\phi(x)f(x) + \frac{1}{2\pi \, i} \int_E \frac{\sqrt{\phi(x)}\sqrt{\phi(y)}}{y - x} f(y) \, dy \tag{5.1}$$

(in the sense of the principal value at x) where $\phi : E \to \mathbb{R}$ is measurable positive. L_ϕ will turn out to be closely related to W_κ, where κ extends ϕ on $\mathbb{R}$ by zero.

L_ϕ is rather general. Indeed, it belongs to the studied class of singular integral operators in $L^2(E)$ of Hilbert transformation type

$$(L(a, b)f)(x) := a(x)f(x) + \frac{1}{i\pi} \int_E b(x)\overline{b(y)}(y - x)^{-1} f(y) \, d y \qquad (5.2)$$

where a, b are measurable functions on E with a real and $b \neq 0$ a.e. There is the obvious unitary equivalence

$$U L(a, b) U^{-1} = L_\phi + M(\alpha) \qquad (5.3)$$

for $\phi = 2|b|^2$ and $\alpha = a - |b|^2$, where U is the multiplication operator by $\overline{b}/|b|$ and $M(\alpha)$ the multiplication operator by α in $L^2(E)$. So we are concerned with the case $a = |b|^2$.

The operator $L(a, b)$ for bounded b and bounded below a is treated by Rosenblum in [28]. It is shown to be self-adjoint on dom $L(a, b) = $ dom $M(a)$, and its diagonalization is achieved. See also [29] and the literature cited in [28,29]. The really unbounded case however is there when b is unbounded. [30] is concerned with this case replacing $L(a, b)$ by the limit of truncated $L(a_n, b_n)$ which are bounded. Our analysis of L_ϕ will show 5.3 that $L(a, b) \subset M(\alpha)$ if the extension of b on $\mathbb{R}$ by zero is not proper. Hence for $L(a, b)$ in (5.2) being not trivial it is necessary that the extension of b is proper. In this case L_ϕ in (5.3) has a self-adjoint extension 5.5.

The Hilbert transformation H on $L^2(\mathbb{R})$ is defined by the singular integral

$$Hf(x) = \frac{1}{i\pi} \int_{-\infty}^{\infty} \frac{f(y)}{y - x} \, d y \qquad (5.4)$$

Recall its representation

$$H = \mathcal{F}^{-1} M(\text{sgn})\mathcal{F} = -\mathcal{F}M(\text{sgn})\mathcal{F}^{-1} \qquad (5.5)$$

on $L^2(\mathbb{R})$ with sgn the signum function on $\mathbb{R}$ (see e.g. [14, Theorems 91, 95] or [22, Lemma 1.35] or [15, Chapter II 4.3], and for more details [31, Teorema 1.1.1]). Let us introduce

$$H_E := P_E H P_E^* \qquad (5.6)$$

the trace on $L^2(E)$ of the Hilbert transformation H. Its spectrum is determined in [32]. For $E = [a, b]$, $-\infty \leq a < b \leq \infty$, H_E is called finite and semi-finite Hilbert transformation if E is bounded and semi-bounded, respectively. Its spectral representation is achieved in [33].

5.1 Isometry relating $\tilde{L}_\phi$ to $\tilde{W}_\kappa$

In what follows we use the polar decomposition $C = S|C|$ of a closed densely defined operator C from a Hilbert space $\mathcal{H}$ into another $\mathcal{H}'$ (see e.g. [4, VIII.9]). $|C|$ denotes the square root of the self-adjoint nonnegative operator C^*C in $\mathcal{H}$. One has dom $|C| =$ dom C and dom C^*C is a core for C. S is a partial isometry from $\mathcal{H}$ into $\mathcal{H}'$. Its initial space $(\ker S)^\perp$ equals $\overline{\operatorname{ran} C^*} = \overline{\operatorname{ran} |C|} = \overline{\operatorname{ran} |C|^2}$. Similarly, its final space $\operatorname{ran} S$ equals $\overline{\operatorname{ran} C} = \overline{\operatorname{ran} |C^*|} = \overline{\operatorname{ran} |C^*|^2}$. The partial isometry S^* satisfies $(\ker S^*)^\perp =$ ran S and ran $S^* = (\ker S)^\perp$. Note the important relation

$$CC^* = SC^*CS^* \tag{5.7}$$

which means that the reductions of CC^* and C^*C on the orthogonal complements of their respective null spaces are Hilbert space isomorphic by the restriction of the partial isometry S to its initial state $(\ker S)^\perp$ and its final state ran S.

Throughout this section $\kappa \geq 0$ and $\phi > 0$ are related to each other by

$$\phi = \kappa|_E \ \text{for} \ E = \kappa^{-1}(\mathbb{R}_+)$$

Lemma 5.1 *Let* $\kappa \geq 0$, $A = P_+ \mathcal{F} M(\sqrt{\kappa}) P_E^*$, $A^* = P_E M(\sqrt{\kappa}) \mathcal{F}^{-1} P_+^*$. *Then*

(a) $W_\kappa = AA^*$
(b) $L_\phi = \frac{1}{2} M(\phi) + \frac{1}{2} M(\sqrt{\phi}) H_E M(\sqrt{\phi}) = P_E M(\sqrt{\kappa}) P_{H_+}^* P_{H_+} M(\sqrt{\kappa}) P_E^* = A^*A$

Now suppose that κ *is proper. Let* $\overline{A} = T|\overline{A}|$ *be the polar decomposition of* $\overline{A}$. *Then the partial isometry* $T : L^2(E) \to L^2(\mathbb{R}_+)$ *is surjective, its adjoint* T^* *is injective, and*

(c) $\tilde{W}_\kappa := \overline{A} A^*$ *is a self-adjoint extension of* W_κ.
(d) $\tilde{L}_\phi := A^* \overline{A}$ *is a self-adjoint extension of* L_ϕ.
(e) $\ker \tilde{L}_\phi = \ker \overline{A}$, $\left(\ker T \right)^\perp = \left(\ker \tilde{L}_\phi \right)^\perp = \overline{\operatorname{ran} A^*}$.

Proof (a) See 4.3. (b) Check $A^*A = P_E M(\sqrt{\kappa}) \mathcal{F}^{-1} P_+^* P_+ \mathcal{F} M(\sqrt{\kappa}) P_E^* = P_E M(\sqrt{\kappa}) P_{H_+}^* P_{H_+} M(\sqrt{\kappa}) P_E^* = P_E M(\sqrt{\kappa}) \frac{1}{2} (I + H) M(\sqrt{\kappa}) P_E^*$ by (5.5). So $A^*A = L_\phi$ as $P_E M(\sqrt{\kappa}) = M(\sqrt{\phi}) P_E$, $M(\sqrt{\kappa}) P_E^* = P_E^* M(\sqrt{\phi})$. Recall (5.6). — (c) and (d) are obvious. —Finally, T is surjective as ran $T = (\operatorname{ran} \overline{A})^- = (\ker A^*)^\perp = L^2(\mathbb{R}_+)$ by 4.1, and (e) follows directly from polar decomposition.   □

As a first result we note that L_ϕ in (5.1) can be trivial:

Corollary 5.2 *Let* $\kappa \geq 0$ *be not proper. Then* $L_\phi \subset 0$. *Moreover,* dom $L_\phi = \{0\}$ *if and only if either* $\kappa \neq 0$ *not a.e. or otherwise* $\frac{1}{\kappa}$ *is not proper. If* $\kappa \neq 0$ *a.e. and* $\frac{1}{\kappa}$ *is proper, then* dom $L_\phi = \frac{1}{\sqrt{\phi}} H_- \cap L^2(\mathbb{R})$.

Proof dom $A^* = \{0\}$ by 4.1($\star$), whence $L_\phi \subset 0$ and dom $L_\phi = \ker A$ because of 5.1(b), 4.1. Recall 2.7 by which $1/\kappa$ is not proper if and only if $1/\sqrt{\kappa}$ is not proper. The remainder is shown in 4.1.   □

For example $\kappa = e^x$ is positive and, by 2.9, κ and $1/\kappa$ are not proper. Hence dom $L_{e^x} = \{0\}$.

Now 5.2 and (5.3) imply that for $L(a, b)$ in (5.2) being not trivial it is necessary that the extension of b on $\mathbb{R}$ by zero be proper:

Theorem 5.3 *Let the extension of b on $\mathbb{R}$ by zero be not proper. Then $L(a, b) \subset M(a - |b|^2)$ with*

(a) dom $L(a, b) = \{f \in L^2(\mathbb{R}) : |b| f \in H_-, (a - |b|^2) f \in L^2(\mathbb{R})\}$ *if $E = \mathbb{R}$*

(b) dom $L(a, b) = \{0\}$ *if and only if E is proper, or $E = \mathbb{R}$ and $\frac{1}{b}$ is not proper, or $E = \mathbb{R}$ and $\frac{1}{b}$ is proper and $\frac{a - |b|^2}{b}$ is not proper.*

Corollary 5.4 *Let $E = \mathbb{R}$ and suppose that $\frac{1}{\phi}$ is bounded.. Then L_ϕ is densely defined and*

$$\mathrm{dom}\, L_\phi = \{\frac{h + k}{\sqrt{\phi}} : h \in H_+, \sqrt{\phi}h \in L^2(\mathbb{R}),\ k \in H_-\},\ L_\phi\big(\frac{h + k}{\sqrt{\phi}}\big) = \sqrt{\phi}h \quad (\star)$$

with ker $L_\phi = \frac{1}{\sqrt{\phi}} H_-$ *and* (ker $L_\phi)^\perp = \{\sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+\}$. *—If ϕ is proper, then* ker $\tilde{L}_\phi = \overline{\mathrm{ker}\, L_\phi} \neq L^2(\mathbb{R})$. *If ϕ is not proper, then $\overline{L}_\phi = 0$.*

Proof Note $\kappa = \phi$, $L_\phi = M(\sqrt{\phi}) P_{H_+}^* P_{H_+} M(\sqrt{\phi})$ by 5.1(b). So $(\star)$ is easily verified and ker $L_\phi = \frac{1}{\sqrt{\phi}} H_-$ follows. Check $f \in L^2(\mathbb{R})$, $f \perp \frac{1}{\sqrt{\phi}} H_- \Leftrightarrow f \in L^2(\mathbb{R})$, $0 = \langle f, \frac{1}{\sqrt{\phi}} k \rangle = \langle \frac{1}{\sqrt{\phi}} f, k \rangle \forall k \in H_- \Leftrightarrow f \in L^2(\mathbb{R})$, $\frac{1}{\sqrt{\phi}} f \in H_+$, proving the claim on (ker $L_\phi)^\perp$.

Note $D := \mathrm{dom}\, A^* = \{h \in H_+ : \sqrt{\phi}h \in L^2(\mathbb{R})\}$, whence ran $A^* = \{\sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+\}$ which equals (ker $L_\phi)^\perp$.

Now let ϕ be proper. Then D is dense by 4.1 and ran $A^* \neq \{0\}$. Hence by 5.1(e) on has ker $\tilde{L}_\phi = (\mathrm{ran}\, A^*)^\perp = \overline{\mathrm{ker}\, L_\phi} \neq L^2(\mathbb{R})$. Moreover, dom L_ϕ is dense, since $f \in (\mathrm{dom}\, L_\phi)^\perp \Rightarrow 0 = \langle f, (h + k)/\sqrt{\phi} \rangle = \langle f/\sqrt{\phi}, (h + k) \rangle \forall h \in D, k \in H_- \Rightarrow f/\sqrt{\phi} \in H_+, \langle f/\sqrt{\phi}, h \rangle \forall h \in D \Rightarrow f/\sqrt{\phi} = 0 \Rightarrow f = 0$. —If ϕ is not proper then $D = \{0\}$ by 4.1, whence dom $L_\phi = \frac{1}{\sqrt{\phi}} H_-$. The latter equals ker L_ϕ. Finally, (ker $L_\phi)^\perp = \{\sqrt{\phi}h \in L^2(\mathbb{R}) : h \in H_+\} = \{0\}$ since $D = \{0\}$. □

For example, $L_{e^{|x|}} \subset 0$ with dense dom $L_{e^{|x|}} = e^{-|x|/2} H_-$. The main outcome of this section is the following:

Theorem 5.5 *Let $\kappa \geq 0$ be proper. Then*

$$\tilde{L}_\phi = T^* \tilde{W}_\kappa T \tag{5.8}$$

Let $\tilde{L}'_\phi$ and T' denote the reduction of $\tilde{L}_\phi$ and the restriction of T on the orthogonal complement of the null space of $\tilde{L}_\phi$, respectively. Then T' is a Hilbert space isomorphism onto $L^2(\mathbb{R}_+)$ satisfying $\tilde{L}'_\phi = T'^{-1} \tilde{W}_\kappa T'$.

Proof (5.8) holds by polar decomposition $\overline{A} = T|\overline{A}|$ (cf. (5.7)). By 5.1 T is surjective. So T' is a Hilbert space isomorphism by 5.1(e). Recall (5.8). □

Of course one has $\tilde{W}_\kappa = T \tilde{L}_\phi T^*$ as well. It allows to study $\tilde{W}_\kappa$ starting from $\tilde{L}_\phi$. For the still remarkable bounded case see (5.9). By 4.4 it is easy to extend 5.5 to semibounded symbols.

Corollary 5.6 *Let $\frac{\kappa}{1+x^2}$ be integrable. Then $\tilde{L}'_\phi$ is absolutely continuous.*

Proof Recall that $\tilde{T}_{\kappa\circ\gamma}$ and hence $\tilde{W}_\kappa$ (see 2.2) is absolutely continuous by [9]. Apply 5.5. □

The special case of 5.6 that ϕ is bounded is treated in [28, Sect. 3] and [29, Theorem]. —We remind that for $\tilde{L}_\phi$ being injective it is necessary that E is proper or that $1/\kappa$ is not proper.

Corollary 5.7 *Let $\kappa \geq 0$ be proper. Suppose that E is proper or $1/\kappa$ is not proper. Then $\tilde{L}_\phi$ is injective and $T : L^2(E) \to L^2(\mathbb{R}_+)$ is a Hilbert space isomorphism with*

$$\tilde{L}_\phi = T^{-1} \tilde{W}_\kappa T$$

Proof $\overline{A}$ is injective by 4.2. Hence $\{0\} = \ker \tilde{L}_\phi = \ker T$ and T is an isomorphism. Apply 5.5. □

So it is worth noting that $\tilde{L}_\phi$ is absolutely continuous, if $\frac{\kappa}{1+x^2}$ is integrable and if E is proper or $1/\kappa$ is not proper.

Let κ be bounded. Then L_ϕ, W_κ are bounded, $\tilde{L}_\phi = L_\phi$, $\tilde{W}_\kappa = W_\kappa$, and

$$L_\phi = T^* W_\kappa T, \quad W_\kappa = T L_\phi T^* \tag{5.9}$$

Moreover, if E is proper or $1/\kappa$ is not proper, then even $L'_\phi = L_\phi$ by 5.7, whence $L_\phi = T^{-1} W_\kappa T$. An example for the latter case is $L_{e^{-|x|}}$, which is injective. Generally, if $\kappa > 0$ does not decrease too rapidly (so that $1/\kappa$ is proper) the kernel of L_κ is not trivial. For an instructive example see Sect. 5.2. The trivial example here is $\kappa = 1_\mathbb{R}$. Note that $W(1_\mathbb{R}) = I_{L^2(\mathbb{R}_+)}$ and $L_{1_\mathbb{R}}$ is the orthogonal projection on H_+, and $T' : H_+ \to L^2(\mathbb{R}_+)$, $T'h := P_+ \mathcal{F}h$. Examples for proper E are the isomorphic pairs $W(1_E) \simeq \frac{1}{2}(I + H_E)$ with H_E in (5.6), which we like to write as

$$H_E = T^{-1} W_{21_E - 1} T \tag{5.10}$$

The case of the finite Hilbert transformation $H_{[-1,1]}$ is studied in detail in [24, Sect. 3.3], [24, (3.20)].

In conclusion we remark on the spectral representations of $\tilde{W}_\kappa$ and $\tilde{L}_\phi$ in 5.5. By the spectral theorem in the multiplication operator version, $\tilde{W}_\kappa$ is Hilbert space isomorphic to the multiplication operator $M(\varphi)$ on $L^2_\mu(\mathbb{R})$ for some Borel-measurable positive $\varphi : \mathbb{R} \to \mathbb{R}$ and finite Borel measure μ. Let $V : L^2_\mu(\mathbb{R}) \to L^2(\mathbb{R}_+)$ be an isomorphism satisfying

$$\tilde{W}_\kappa = V M(\varphi) V^{-1} \tag{5.11}$$

The spectral measure for $\tilde{W}_\kappa$ is given by $E_{\tilde{W}_\kappa}(\Delta) = VM(1_{\varphi^{-1}(\Delta)})V^{-1}$ for measurable $\Delta \subset \mathbb{R}$.

Corollary 5.8 *Suppose (5.11). Then $W_0 := A^*VM(\frac{1}{\sqrt{\varphi}})$ with A^* in 5.1 is closable, its closure W is the Hilbert space isometry T^*V with $\mathrm{ran}\, W = (\ker \tilde{L}_\phi)^\perp$, by which $\tilde{L}_\phi = WM(\varphi)W^*$.*

Proof By (5.11) and 5.1(a), $W_0 = A^*VM(\frac{1}{\sqrt{\varphi}})V^{-1}V = A^*\tilde{W}_\kappa^{-1/2}V = A^*|A^*|^{-1}V = T^*|_{\mathrm{dom}(|A^*|^{-1})}V$. For the last equality recall $A^* = T^*|A^*|$. Further note that $\mathrm{dom}(|A^*|^{-1}) = \mathrm{ran}(|A^*|)$ is dense as $|A^*| = \tilde{W}_\kappa^{1/2}$ is self-adjoint injective. Since T^* is bounded, W_0 is closable and its closure W equals T^*V. The remainder is obvious. $\qquad\square$

5.2 Example: Lalescu's Operator

Supposing $\kappa > 0$, $\kappa \in L^2$, and kernel $k \in L^1$, in [34] a spectral theory of W_κ is proposed by a reduction to a previously developed theory for singular integral operators. Its application in [34] to Lalescu's operator W_λ with symbol

$$\lambda(x) := \frac{2}{1+x^2} \tag{5.12}$$

however does not produce the right normalization (5.14) of the generalized eigenfunctions (5.13). In establishing (5.14) we get also the result 5.10 on orthogonal polynomials.

The diagonalization of Lalescu's operator W_λ is achieved in Sect. 5.2.1 and that of the associated singular integral operator L_λ is derived in Sect. 5.2.2. The generalized eigenfunctions (5.18) of the latter are no longer regular distributions.

5.2.1 Spectral Representation of W_λ

Clearly, the spectrum of W_λ lies in [0, 2]. The integral kernel for W_λ is $e^{-|x|}$. The obvious ansatz $a\,e^{\alpha x} + b\,e^{\beta x}$ for $u(x)$, $x > 0$ in $s\,u(x) - \int_0^\infty e^{-|x-y|}u(y)\,dy = 0$ yields for every $s \in\,]0, 2[$ the generalized eigenfunction (cf. [2])

$$q_s(x) = n(s)\left((\tau - i)\,e^{i\tau x} + (\tau + i)\,e^{-i\tau x}\right), \quad \tau := (\frac{2}{s} - 1)^{1/2} \tag{5.13}$$

The claim is that there is the unique positive normalization constant

$$n(s) = (4\pi s\tau)^{-1/2} \tag{5.14}$$

such that $q(s, x) := q_s(x)$ is the kernel for a Hilbert space isomorphism

$$V : L^2(0, 2) \to L^2(\mathbb{R}_+), \quad Vh = \mathrm{l.\,i.\,m.} \int_{\downarrow 0}^{\uparrow 2} q(s, \cdot)h(s)\,ds \tag{5.15}$$

Indeed, the assertion follows immediately from the following Lemma 5.9 as $V = U^*\Gamma$ holds for the Hilbert space isomorphism

$$\Gamma : L^2(0, 2) \to L^2(\mathbb{R}_+), \quad \Gamma h(y) := \frac{2\sqrt{y}}{1+y^2} h\left(\frac{2}{1+y^2}\right) \tag{5.16}$$

due to the change of variable $\gamma(y) := \frac{2}{1+y^2}$.

Lemma 5.9 $Ug(x) := \text{l. i. m.} \int_{\downarrow 0}^{\uparrow\infty} u(x, y)g(y)\, d\, y$ with

$$u(x, y) = (2\pi)^{-1/2}(1 + x^2)^{-1/2}\left((x - \mathrm{i})\, \mathrm{e}^{\mathrm{i}xy} + (x + \mathrm{i})\, \mathrm{e}^{-\mathrm{i}xy}\right)$$

is unitary on $L^2(\mathbb{R}_+)$.

Proof Using the generating function $l_q(x) = (1 - q)^{-1} \exp\left(-\frac{1}{2}\frac{1+q}{1-q} x\right)$, $|q| < 1$ of the Laguerre functions l_n, $n \geq 0$ one finds $\lambda_n(x) := Ul_n(x) = \sqrt{\frac{2}{\pi}}(1 + x^2)^{-1/2}\left(\frac{x+\mathrm{i}}{1+2\mathrm{i}x}\gamma(x)^n + \frac{x-\mathrm{i}}{1-2\mathrm{i}x}\gamma(x)^{-n}\right)$ with $\gamma(x) := \frac{-1+2\mathrm{i}x}{1+2\mathrm{i}x}$. Note that $\int_{-\infty}^{\infty}(1 + x^2)^{-1}\left(\frac{x+\mathrm{i}}{1+2\mathrm{i}x}\right)^2\gamma(x)^k\, d\, x = 0$ for $k \in \mathbb{Z}$, since the integral is a parametrization of the complex integration of $\frac{\mathrm{i}}{4}\frac{3z-\mathrm{i}}{3-z}z^k$ along the unit circle. Using this it follows easily that λ_n, $n = 0, 1, 2, \ldots$ are orthonormal in $L^2(\mathbb{R}_+)$. It remains to show the completeness of $(\lambda_n)_n$.

Let $\varphi \in]0, \pi[$ satisfy $\mathrm{e}^{\mathrm{i}\varphi} = \gamma(x)$, $x > 0$. Put $\xi := \cos\varphi$. Then $\lambda_n(x) = \sqrt{\frac{18(1-\xi)(1-\xi^2)}{\pi(5-3\xi)}}\left(\frac{\sin(n+1)\varphi}{\sin\varphi} - \frac{1}{3}\frac{\sin n\varphi}{\sin\varphi}\right) = \sqrt{\frac{18(1-\xi)(1-\xi^2)}{\pi(5-3\xi)}}\left(U_n(\xi) - \frac{1}{3}U_{n-1}(\xi)\right)$, where U_n denotes the Chebyshev polynomial of the second kind of degree n for $n = 0, 1, 2, \ldots$ and $U_{-1} := 0$ [35, 10.11(2)]. Therefore the Hilbert space isomorphism $\Gamma : L^2(\mathbb{R}_+) \to L^2(-1, 1)$ due to the change of variable $\gamma(\xi) := \frac{1}{2}\sqrt{\frac{1+\xi}{1-\xi}}$, $\gamma'(\xi) = \frac{1}{2}\sqrt{\frac{1-\xi}{1+\xi}}(1 - \xi)^{-2}$ maps λ_n onto $\Gamma\lambda_n(\xi) = \sqrt{\frac{9}{\pi(5-3\xi)}}(1 - \xi^2)^{1/4}Q_n(\xi)$, where $Q_n := U_n - \frac{1}{3}U_{n-1}$ is a polynomial of degree n for $n = 0, 1, 2, \ldots$ Thus 5.10 is shown, which implies the assertion. $\qquad\square$

Corollary 5.10 Let $Q_n := U_n - \frac{1}{3}U_{n-1}$, $n = 0, 1, 2, \ldots$ with U_n the Chebyshev polynomial of the second kind of degree n and $U_{-1} := 0$. Then $(Q_n)_{n\geq0}$ is a sequence of orthonormal polynomials on $[-1, 1]$ with respect to the weight function $\xi \mapsto \frac{9}{\pi}\frac{\sqrt{1-\xi^2}}{(5-3\xi)}$. It obeys the recurrence $Q_{n+1} = 2\xi Q_n - Q_{n-1}$ for $Q_0 = 1$, $Q_1(\xi) = 2\xi - \frac{1}{3}$.

Theorem 5.11 The spectral representation $W_\lambda = V M(\mathrm{id}_{[0,2]})V^{-1}$ holds.

Proof First note $|q(s, x)| \leq \left(s(2-s)\right)^{-1/2}$. Let $h \in L^2(0, 2)$ with support in $[\delta, 2-\delta]$ for some $0 < \delta < 1$. Then $\left(V M(\mathrm{id}_{[0,2]})h\right)(x) = \int_\delta^{2-\delta} q(s, x)s\, h(s)\, d\, s$. Substitute $s\, q(s, x) = \int_0^\infty \mathrm{e}^{-|x-y|} q(s, y)\, d\, y$. The integrations can be interchanged by Fubini's theorem yielding $\int_0^\infty \mathrm{e}^{-|x-y|}\left(\int_\delta^{2-\delta} q(s, y)h(s)\, d\, s\right) d\, y = \int_0^\infty \mathrm{e}^{-|x-y|} Vh(y)\, d\, y = W_\lambda Vh(y)$. This implies the result. $\qquad\square$

5.2.2 Spectral Representation of L_λ

Let us first establish the kernel of L_λ. By 5.1(e) the latter equals that of $A = P_+\mathcal{F}M(\sqrt{\lambda})$, whence $\ker(L_\lambda) = \{f \in L^2(\mathbb{R}) : \sqrt{\lambda}f \in H_-\}$ by 4.1. Put $u(x) := \frac{x+\mathrm{i}}{\sqrt{1+x^2}}$, $x \in \mathbb{R}$. Note that the multiplication operator $M(u)$ is unitary. It follows that

$$\ker(L_\lambda) = M(u)H_-, \quad \ker(L_\lambda)^\perp = M(u)H_+ \qquad (5.17)$$

Indeed, as to the less trivial implication let $f \in \ker(L_\lambda)$. Then $h := \sqrt{\lambda}f \in H_-$, whence $(x - \mathrm{i})h \in L^2(\mathbb{R})$. Therefore $k := 2^{-1/2}(x - \mathrm{i})h \in H_-$ using 3.1(c), and $f = uk$ follows.

Hence the reduction L'_λ of L_λ on the orthogonal complement of its kernel is self-adjoint bounded on $M(u)H_+$ with spectrum $[0, 2]$. The following computations are valid in a distributional sense. So generalized eigenfunctions Q_s for L_λ are given by $A^*q_s = M(\sqrt{\lambda})\mathcal{F}^{-1}P_+^*q_s$ since $L_\lambda A^*q_s = A^*A\,A^*q_s = sA^*q_s$. Recall $\int_{-\infty}^{\infty} 1_{]0,\infty[}\,\mathrm{e}^{\mathrm{i}xy}\,\mathrm{d}\,y = \frac{\mathrm{i}}{x} + \pi\delta(x) =: \Delta(x)$, see e.g. [36, Table of Fourier Transforms 1.23]. So one yields the not regular distributions on $\mathbb{R}$

$$Q_s(x) = \frac{n(s)}{\sqrt{s}}\sqrt{\frac{2}{1+x^2}}\,\frac{1}{\sqrt{2\pi}}\big((\tau - \mathrm{i})\Delta(x + \tau) + (\tau + \mathrm{i})\Delta(x - \tau)\big) \quad (5.18)$$

The claim is that $\frac{n(s)}{\sqrt{s}}$ is the unique positive normalization constant such that $Q(s, x) := Q_s(x)$ is the kernel for an isometry W on $L^2(0, 2)$ in $L^2(\mathbb{R})$ satisfying

$$Wh = \int_0^2 Q(s, \cdot)h(s)\,\mathrm{d}\,s \qquad (5.19)$$

for test functions h. The additional factor $\frac{1}{\sqrt{s}}$ regarding the normalization constant of Q_s corresponds to the factor $\frac{1}{\sqrt{\varphi}}$ for W_0 in 5.8 and is suggested heuristically by $\langle A^*q_s, A^*q_s\rangle = \langle q_s, AA^*q_s\rangle = s\langle q_s, q_s\rangle$. For the proof recall the isomorphism Γ (5.16) and the Hilbert transformation H (5.5). For $g : \mathbb{R}_+ \to \mathbb{C}$ let g_{oe} denote the odd extension $g_{oe}(x) = -g(-x)$ for $x \le 0$ of g.

Theorem 5.12 *The integral operator (5.19) determines a Hilbert space isometry W : $L^2(0, 2) \to L^2(\mathbb{R})$ with $\operatorname{ran} W = M(u)H_+$. It satisfies $Wh = M(u)\frac{1}{2}(I + H)(\Gamma h)_{oe}$ for $h \in L^2(0, 2)$ and yields the representation*

$$L_\lambda = WM(\mathrm{id}_{[0,2]})W^*$$

Proof By the change of variable $t = 2/(1 + s^2)$ for the integration in (5.19) one obtains $\sqrt{1 + x^2}\,Wh(x) = \frac{1}{2\pi}\int_0^\infty \big((t - \mathrm{i})\Delta(x + t) + (t + \mathrm{i})\Delta(x - t)\big)\Gamma h(t)\,\mathrm{d}\,t$. The integral is easily done yielding $(\star)$ $Wh = M(u)\frac{1}{2}(I + H)(\Gamma h)_{oe}$ for test functions h. Check $\langle Hf, f\rangle = 0$ if $f \in L^2(\mathbb{R})$ is odd. Therefore $\|Wh\|^2 = \|\frac{1}{2}(I + H)(\Gamma h)_{oe}\|^2 =$

$\langle \frac{1}{2}(I + H)(\Gamma h)_{oe}, (\Gamma h)_{oe}\rangle = \frac{1}{2} \|(\Gamma h)_{oe}\|^2 = \|h\|^2$. So W is an isometry, and $(\star)$ extends to all $h \in L^2(0, 2)$. Obviously $Wh \in M(u)H_+$. Moreover, if $k \in H_+$ then $k = \frac{1}{2}(I + H)(k - \check{k})$ as $\check{k} \in H_-$. Since $k - \check{k}$ is odd, this implies ran $W = M(u)H_+$.

The spectral representation of L_λ follows from $WM(\mathrm{id}_{[0,2]})h = L_\lambda Wh$ for test functions h. The latter is shown along the lines of the proof of 5.11. We omit the technicalities. □

Declarations

Conflict of interest Not applicable.

Availability of data and material Not applicable.

Code availability Not applicable.

References

1. Böttcher, A., Silbermann, B.: Analysis of Toeplitz Operators. Springer, Berlin (1990)
2. Wiener, N., Hopf, E.: Über eine Klasse singulärer Integralgleichungen, Sitz. Berlin. Akad. Wiss. pp. 696–706 (1931). See also [18, Chap. IV] and [14, 11.17]
3. Krein, M.G.: Integral equations on a half-line with kernels depending upon the difference of the arguments. Amer. Math. Soc. Transl. **2**(22), 163–288 (1962)
4. Reed, M., Simon, B.: Methods of Modern Mathematical Physics. Academic Press, New York (1975)
5. Devinatz, A., Shinbrot, M.: General Wiener-Hopf Operators, Trans. AMS **145**, 467-494 (1969). F. O. Speck: General Wiener–Hopf factorization methods, Pitman, Boston (1985)
6. Yafaev, D.R.: On Semibounded Wiener-Hopf Operators, J. London Math. Soc. **95**, 742–762 (2017) or arXiv:1606.01361v1 [math.FA]
7. Yafaev, D.R.: Toeplitz versus Hankel: Semibounded Operators. Opuscula Math. **38**(4), 573–590 (2018). https://doi.org/10.7494/OpMath.2018.38.4.573
8. Rosenblum, M.: Self-adjoint Toeplitz operators and associated orthonormal functions. Proc. Am. Math. Soc. **13**, 590–595 (1962)
9. Rosenblum, M.: A concrete spectral theory for self-adjoint Toeplitz operators. Am. J. Math. **87**, 709–718 (1965)
10. Pellegrini, V.J.: Unbounded general Wiener–Hopf operators. Indiana Univ. Math. J. **21**, 85–90 (1971)
11. Sarason, D.: Unbounded Toeplitz operators. Integr. Equ. Oper. Theory **61**, 281–298 (2008)
12. Câmara, M.C., Malheiro, M.T., Partington, J.R.: Kernels of unbounded Toeplitz operators and factorization of symbols. Results Math. **76**(1), 10 (2021)
13. Gustafson, K.: The Toeplitz-Hausdorff Theorem for Linear Operators. In Proceedings of the American Mathematical Society, Shorter Notes. pp. 203–204 (1970)
14. Titchmarsh, C.: Introduction to the Theory of Fourier Integrals, 2nd edn. Clarendon Press, Oxford (1948)

15. Ricci, F.: Hardy Spaces in One Complex Variable, http://homepage.sns.it/fricci/papers/hardy.pdf (2005)

16. Privalov, I.I.: Randeigenschaften analytischer Funktionen. VEB Deutscher Verlag der Wissenschaften, Berlin (1965)

17. Aleman, A.: Introduction to Hardy spaces, www.ctr.maths.lu.sehardy-spaces-lecture-notes2 (2014)

18. Paley, R.E.A.C., Wiener, N.: Fourier transforms in the complex domain. Amer. Math. Soc. Coll. Pub. **19**, (1934)

19. Groenewald, G.J., ter Horst, S., Jaftha, J., Ran, A.C.M.: A Toeplitz-like operator with rational symbol having poles on the unit circle I: Fredholm properties, II: The Spectrum, III: The Adjoint, Operator Theory: Advances and Applications **271**, 239 - 268 (2018), Operator Theory: Advances and Applications **272**, 133 -154 (2019), Integr. Equ. Oper. Theory 91:43 (2019)

20. Hewitt, E., Stromberg, K.: Real and Abstract Analysis. Springer, New York (1969)

21. Vukotić, D.: Analytic Toeplitz operators on the hardy space H^p: a Survey. Bull. Belg. Math. Soc. **10**, 101-113 (2003)

22. Duduchava, R.: Integral Equations with Fixed Singularities. Teubner, Leipzig (1979)

23. Carleman, T.: L'integrale de Fourier et questions qui s'y rattachent. Almquist Wiksells, Uppsala (1944)

24. Castrigiano, D.P.L.: Spectral Representation of the Wiener-Hopf Operator for the sinc Kernel and some Related Operators, arXiv:2002.09235v1 [math.FA] (2020)

25. Schechter, M.: Basic theory of fredholm operators. Annali della Scuola Normale Superiore di Pisa Classe di Scienze 3e série tome **21**(2): 261–280 (1967)

26. Cucker, F., Corbalan, A.G.: An alternate proof of the continuity of the roots of a polynomial. Am. Math. Monthly **96**, 342–345 (1989)

27. Rudin, W.: Functional Analysis. TATA McGraw-Hill TMH Edition, New Delhi (1974)

28. Rosenblum, M.: A spectral theory for self-adjoint singular integral operators. Am. J. Math. **88**, 314–328 (1966)

29. Putnam, C.R.: Absolute continuity of singular integral operators. Am. J. Math. **91**, 453–462 (1969)

30. Pincus, J.D.: Symmetric singular integral operators. Indiana Univ. Math. J. **23**, 537–556 (1973)

31. Talenti, G.: Sulle equazioni integrali di Wiener Hopf. Boll. Un. Mat. Ital. (4) **7**(1), 18–118 (1973)

32. Widom, H.: Singular integral equations in L_p. Trans. Amer. Math. Soc. **97**, (1960)

33. Koppelman, W., Pincus, J.D.: Spectral representations for finite Hilbert transformations. Math. Zeitschrift **71**, 399–407 (1959)

34. Pincus, J.D.: The spectral theory of self-adjoint Wiener–Hopf operators. Bull. Am. Math. Soc. **72**, 882–887 (1966)

35. Erdelyi, A., et al.: Higher Transcendental Functions, vol. 2. McGraw-Hill, New York (1953)

36. Gel'fand, I.M., Shilov, G.E.: Generalized Functions, vol. I. Academic Press, New York (1964)