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Optimal power-constrained control of distributed systems with information constraints

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Abstract

This paper is concerned with a special case of stochastic distributed optimal control, where the objective is to design a structurally constrained controller for a system subject to state and input power constraints. The structural constraints are induced by the directed communication between local controllers over a strongly connected graph. Based on the information structure present, that is, who knows what and when, we provide a control synthesis with the optimal control law consisting of two parts: one that is based on the common information between the subsystems and one that uses more localized information. The developed method is applicable to an arbitrary number of physically interconnected subsystems.

KEYWORDS
distributed systems, networked control, optimal control, constrained control, communication delay, large-scale systems

1 | INTRODUCTION

Distributed control problems arise naturally in large-scale networked systems such as smart grids, transportation networks, and communication networks. In recent years, particular attention has been directed to understanding and controlling interconnected systems. In such networked control systems, communication is essential since different systems are aiming to minimize both local and global cost metrics. In order to meet the requirements of such cyber-physical applications it is essential to reduce time delay and packet loss probability within control loops. This goes in line with in-network control [1], where the control task is distributed to a potentially large number of in-network devices (routers, switches), reducing the communication distance. Rüth et al. [2] demonstrate the applicability of in-network processing for control, a previously unexplored area, by offloading small but critical control tasks into network elements managed and organized through remote environments.

In the controller design process of large-scale networked systems, it is immediately evident that some control actions must be based solely on locally available information. At the same time, the globally available information, which is strictly monotonically increasing, must also be taken into consideration in order to minimize global metrics. We effortlessly identify three challenges: (i) can the globally available information, which is strictly monotonically increasing with time, be summarized by some constant size information? In other words, can we identify sufficient statistics for the globally available information? (ii) can we prove the existence of optimal distributed policies and compute them efficiently? (iii) does the solution to distributed optimal control problem scale to an arbitrary

Abbreviations: DM, decision maker; NCS, networked control system.

number of subsystems? The work presented in this paper addresses these three questions for large-scale networked systems of arbitrary dimension subject to state and input power constraints, under the assumption that the communication between decision makers occurs as fast as the state interconnections propagate through the plant.

2 | RELATED WORK AND CONTRIBUTIONS

The design of optimal distributed control laws is difficult in general because the "information structure" (i.e., "who knows what and when") decides on the convexity of the problem [3-5]. Intuitively, this difficulty arises because of additional sparsity constraints on the feedback control that model the availability of information to local decision makers [6]. In Witsenhausen, [4] this is demonstrated by providing an example of information structure which results in optimal nonlinear policy even for a linear system and quadratic cost function. The linearity of optimal policies with respect to the associated information is proven for so-called partially nested information structures in Ho and Chu [7]. A more general result is given in Rotkowitz and Lall [8], where the quadratically invariant information structures are characterized which allow for the convex formulation of the problem. Lastly, the design of quadratically invariant information structures for distributed systems with intermittent observations is presented in Abara et al. [9]. The latter two results concern the computational tractability of the optimal information-constrained control problem. On the other hand, a lot of focus has been given to the design of optimal control laws for fixed information structures that have the property of being partially nested. The authors in Nayyar et al. [10] argue that the information hierarchy between the decision makers (DMs) can be exploited to obtain the optimal control law. In other studies [11–13], based on such hierarchy, the structure of the controller is given for linear quadratic Gaussian team problems with partially nested information structure. An extension of the latter results towards output feedback and correlated process and measurement noises is given in [14]. However, all of those works [11–14] consider either a two-player team or three-player chain, and it is not straightforward to see how those methodologies can be extended to an arbitrary number of DMs. To the best of our knowledge, this is partially due to the fact that no existing paper in the literature identifies sufficient statistics for general partially nested team over graphs. In Mahajan and Nayyar, [15] sufficient statistics are derived for linear control strategies and systems with partial information sharing. However, the approach does not provide the computation of the optimal control law, but rather its structure, and it is not easily seen if their approach simplifies for partially nested information structures. Interestingly, Wang et al. [16] address the design of optimal decentralized output feedback control, given a one-step delayed information pattern, over a connected digraph. However, while individual states are assumed to be communicated with one-step delay between neighboring decision makers, the authors assume that each DM has access to global control input vector at current time. This assumption seems unrealistic as communication delay should affect not only transmission of individual states but also control inputs. However, the optimal control design for an arbitrary number of physically coupled subsystems under partially nested information structure induced by one-step communication delays between neighboring DMs and state/input constraints is largely open.

The main contribution of this paper is a methodology for the design of optimal control laws for large-scale information-constrained system, given state and/or input power constraints in the system. Our approach differs from existing related works in the field of large-scale distributed optimal control subject to delayed information sharing in the following lines:

- We assume no process noise histories available to DMs at each time instant—contrary to the approach in Wang et al. [17]. Our approach is motivated by limited memory of in-network devices and therefore builds on the derivation of constant-size sufficient statistics for the part of control based on common information between DMs. This sufficient statistics (as later proven) can be computed efficiently using a recursive form. We then prove independence between derived sufficient statistics and locally available information, assuming that neighboring controllers communicate with one-step delay, over a connected directed graph.
- Our problem formulation permits definition of safety/actuation constraints - formulated as state/input power constraints. These can be defined locally (for the power of individual subsystems) or globally (for the overall power). Unlike model-predictive control (MPC) methodology that handles constrained problems, in our approach the effect of state/input constraints is incorporated and can be directly seen as part of the computed optimal control gains.
- Based on the control hierarchy induced by the derived optimal control policy we provide an interpretation of such hierarchy in terms of its implementation on in-network elements. This enables both optimal control actions and scalable orchestration through cloud environments.

The problem is reformulated in its dual form, and according to the information constraints, the structure of the controller as well as computation of controller gains is given.

The remainder of the paper is outlined as follows. We start with problem setting in Section 3. The method to decouple problem into several sub-problems via covariance decomposition is presented in Section 4. In Section 5, we provide structural characterization of the solution to the problem, and in Section 6, we illustrate the developed methodology via simulations. Finally, conclusions are given in Section 7.

2.1 | Notation

In this paper, for appropriate matrices C_i, the matrix $D = \text{blkdiag}\,(C_1, C_2, \ldots, C_n)$ is the block-diagonal matrix such that $D_{ii} = C_i$ and $D_{ij} = 0$ for $i \neq j$. Given a matrix A, we denote by $[A]_{ij}$ its element with position $(i.j)$ and furthermore we denote by $\mathbf{1}$ a matrix of all ones. For a time-varying vector $x(k)$, we denote by $x(k_1 : k_2)$ stacked vector $x(k_1 : k_2)^\top = [x^\top(k_1), x^\top(k_1 - 1), \ldots, x^\top(k_2)]$. where $k_1 > k_2$. By $\mathbb{S}^n$, we denote a space of symmetric matrices of dimension $n \times n$. The symbol $\circ$ denotes the element-wise multiplication of matrices. Finally, we define the left-multiplication operator $\prod$ as

$$\prod_{z=a}^{b} P(z) = P(b)P(b-1) \ldots P(a+1)P(a)$$

for $a \leq b$ and some square matrices $P(\cdot)$.

3 | PROBLEM SETTING

Consider a large-scale physically interconnected system composed of N linear subsystems, where the interconnections are described through a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, called the physical interconnection graph. Each node $i \in \mathcal{V}$ corresponds to one of the subsystems $i \in \{1, \ldots, N\}$. An edge $(j, i) \in \mathcal{E}$ if the dynamics of node i is directly affected by node j. We consider a directed graph $\mathcal{G}$ and assume it to be strongly connected. The set of direct neighbors of subsystem i is defined as $\mathcal{N}_i = \{j \,|\, (j, i) \in \mathcal{E}, j \neq i\}, i = 1, \ldots, N$. The dynamics of the ith subsystem is given by a first order stochastic difference equation

$$x_i(k+1) = A_i x_i(k) + B_i u_i(k) + \sum_{j \in \mathcal{N}_i} A_{ij} x_j(k) + w_i(k), \quad (1)$$

where $x_i(k) \in \mathbb{R}^{n_i}$ is the state, $u_i(k) \in \mathbb{R}^{m_i}$ is the control signal, $A_i \in \mathbb{R}^{n_i \times n_i}$, $A_{ij} \in \mathbb{R}^{n_i \times n_j}$ and $B_i \in \mathbb{R}^{n_i \times m_i}$. The noise process $w_i(k) \in \mathbb{R}^{n_i}$ is zero mean i.i.d. Gaussian with covariance matrix Σ_{w_i}. The initial state $x_i(0)$ is a random variable with zero mean and finite covariance Σ_{x_i}. Moreover, $x_i(0)$ and $w_i(k)$ are assumed to be pair-wise independent at each time instant k and every i. Equation 1 can be written as

$$x(k+1) = Ax(k) + Bu(k) + w(k) \quad (2)$$

where the stacked vectors are defined as $x^\top(k) = (x_1^\top(k), \ldots, x_N^\top(k)) \in \mathbb{R}^n$, $w^\top(k) = (w_1^\top(k), \ldots, w_N^\top(k)) \in \mathbb{R}^n$ and $u^\top(k) = (u_1^\top(k), \ldots, u_N^\top(k)) \in \mathbb{R}^m$. The corresponding dimensions are $n = \sum_{i=1}^{N} n_i$, $m = \sum_{i=1}^{N} m_i$. Additionally, we define $\Sigma_w = \text{blkdiag}(\Sigma_{w_1}, \ldots, \Sigma_{w_N})$ and $\Sigma_x = \text{blkdiag}(\Sigma_{x_1}, \ldots, \Sigma_{x_N})$. For each DM i, the admissible control policies at time instant k are measurable functions of the available information $\mathcal{I}_k^i$

$$u_i(k) = \gamma_k^i(\mathcal{I}_k^i) \quad (3)$$

where we define $\mathcal{I}_k^i$ as

$$\mathcal{I}_k^i = \{\mathcal{I}_{k-1}^i, x_k^i\} \bigcup_{j \in \mathcal{N}_i} \{\mathcal{I}_{k-1}^j\}, \quad (4)$$

that is, the information set of each DM is updated by the current state value and the one-step delayed information from the direct neighbors $\mathcal{N}_i$. At $k = 0$, it holds $\mathcal{I}_0^i = \{x_0^i\}$. Thus, we assume here that DMj communicates its state to DM i with one-step delay iff $j \in \mathcal{N}_i$.

Remark 1. In practice, the validity of the assumption that the controllers communicate with a maximum delay of one sampling interval [18] depends on the communication technology and sampling rates. In case of wired communication between subsystems, communication delay is typically low and can be assumed within the range of one sampling interval for systems with low and high sampling rates. Wireless communication typically induces larger delays, but technology developments such as 5G aim to reduce this delay.

Remark 2. Notice that (4) represents the information history that is in principle available to each decision maker i and that increases with time. However, for the ease of computation and memory optimization for each DM, we will later introduce sufficient statistics for control policy (3). This is of particular relevance as we assume that local controllers have limited memory, computation, and access (as they represent in-network elements).

The objective is to minimize the global cost

$$J_C = \mathbb{E}\left[\sum_{k=0}^{T-1} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top \Lambda \begin{bmatrix} x(k) \\ u(k) \end{bmatrix} + x(T)^\top Q_T x(T)\right], \quad (5)$$

where $\Lambda = \text{blkdiag}(Q, R)$. The matrix R is assumed to be positive-definite, while matrices Q, Q_T are assumed to be positive semi-definite. The cost (5) is to be minimized under state/input power constraints

$$\mathbb{E}\left[\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top W_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\right] \le p_k^i, \ \forall i = 1, \dots, M \quad (6)$$

where p_k^i is the maximum average power at every time instant $k = 1, \dots, T-1$, and W_i, $i = 1, \dots, M$, is a symmetric weighting matrix. By appropriate choice of W_i, the set of constraints in (6) captures limited power of actuators and/or safety constraints imposed on the state variables. Also, as shown in Shannon, [19] Gaussian channel capacity limitation can be modeled via power constraints.

Remark 3. Note that constraints (6) are defined in expectation, that is, we require satisfaction of those constraints on average. This (later proven), together with assumptions on one-step delay between neigboring decision makers, implies the optimality of linear control policies for the information-constrained problem addressed here. An interesting example [20] that illustrates the role of state power constraints is a vehicle platoon that suddenly increases its velocity due to an increased speed limit on the road. As the target speed of the platoon increases, it is crucial to limit the deviation of distances from desired ones (as the failure of one vehicle can have bigger consequences when the platoon is moving at higher velocity).

Ultimately, the problem is formally stated as

$$\min_{\gamma_{0:T-1}} J_C \ \text{ s.t. } (2), (3), (6), \quad (7)$$

where $\gamma_k = [\gamma_k^1, \dots, \gamma_k^N]$ contains all DMs control policies.

Example 1. In Figure 2 a physical interconnection graph and at the same time information topology that satisfy the assumptions are given. The nodes $\mathcal{S}_i (i = 1, \dots, 4)$ represent either physical subsystems $\mathcal{P}_i$ or corresponding control units $\mathcal{C}_i$. The links denote the physical coupling between $\mathcal{P}_i$ and also the communication delay between $\mathcal{C}_i$. Indeed, one can see that dynamics of subsystem $\mathcal{P}_1$ affects the dynamics of subsystem $\mathcal{P}_4$ with delay of three steps, which defines the communication delay between control units $\mathcal{C}_1$ and $\mathcal{C}_4$.

4 | LOCAL CONTROL ACTIONS AND COMMON CONTROL

In this section, we give the structure of the optimal control law, for a fixed partially nested information pattern defined in (4) and given constraints on control input and state introduced in (6). To this end, we derive sufficient statistics for the part of the control law that is based on common information between DMs and establish its independency from locally available information.

4.1 | Information decomposition

In this subsection, the information history defined in (4) is split into two sets: common history known by all DMs and more localized information. To this end, we recall the definitions of path and diameter of a graph.

Definition 1 (Path). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed graph. A path $p(a, b)$ between any two vertices $v_a, v_b \in \mathcal{V}$ is defined as an ordered tuple $p(a, b) = (v_{i_1}, \dots, v_{i_n})$ where $(a, b) = (i_1, i_n)$, $(v_{i_r}, v_{i_{r+1}}) \in \mathcal{E}$ and $r \in [1, n-1]$. Let $l_p(a, b)$ denote the number of vertices in p. Then length of a path is $L_p(a, b) = l_p(a, b) - 1$.

Definition 2 (Diameter of a graph). Consider a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$. The diameter of a graph $\mathcal{G}$ is defined as

$$\mathcal{D} = \mathcal{D}(\mathcal{G}) = \max_{a,b \in \mathcal{V}} \min_{p} L_p(a, b)$$

that is, it is the length of the longest shortest path in $\mathcal{G}$.

In the particular example in Figure 1, the diameter is $\mathcal{D} = 3$. Referring to (3), (4) the diameter $\mathcal{D} = \mathcal{D}(\mathcal{G})$ determines a common information history for all N DMs at time k

$$\mathcal{I}_k^C = \{x(0 : k - \mathcal{D}(\mathcal{G}))\}. \quad (8)$$

Indeed, the complete history until time k is divided into two intervals: $[0 : k - \mathcal{D}]$ that defines the information set in (8) and $[k - \mathcal{D} + 1 : k]$ which defines more localized information. Introducing a vector $\bar{x}(k) = x(k : k - \mathcal{D} + 1)$, and $\bar{w}^\top(k) = (w^\top(k), 0^\top, \dots, 0^\top)$, the augmented system representing the evolution of the localized information is written as

$$\bar{x}(k+1) = \bar{A}\bar{x}(k) + \bar{B}u(k) + \bar{w}(k) \quad (9)$$

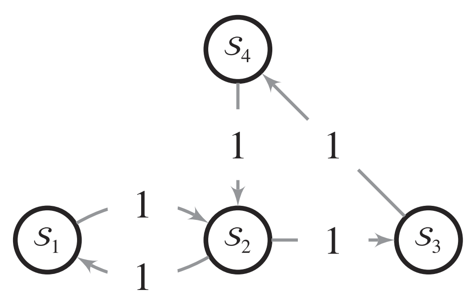

FIGURE 1 Example of information (and physical interconnection) graph

where the matrices $\bar{A}$, $\bar{B}$ are defined as

$$\bar{A} = \left[\begin{array}{c|ccccc} A & 0 & 0 & \dots & 0 \\ I & 0 & 0 & \dots & 0 \\ 0 & I & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \dots & \vdots \\ 0 & 0 & \dots & I & 0 \end{array}\right], \ \bar{B} = \begin{bmatrix} B \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

4.2 | Structure of the optimal control law

In this section, in order to derive the structure of the optimal control law, we prove an important property of the information structure (4), that is, partial nestedness. To this end, we first give the definition of partial nestedness [21].

Definition 3. The information structure $\mathcal{I}_k = \left\{\mathcal{I}_k^1, \dots, \mathcal{I}_k^N\right\}$ and system (2) are partially nested if, for every admissible policy (3), whenever $u_i(\tau)$ affects $\mathcal{I}_k^j$, then $\mathcal{I}_\tau^i \subset \mathcal{I}_k^j$.

Lemma 1 (Partial nestedness). *The information structure defined by (4) and system (2) are partially nested.*

Proof. Let d_{ji} be the length of the shortest path $j \to i$ in the physical interconnection graph. Considering (4), the information set $\mathcal{I}_k^i$ contains the measurement $x_j(k-d_{ji})$ which is affected by $u_j(k-d_{ji}-1)$. Thus, to check if information structure (4) is partially nested, one should verify the condition: $\mathcal{I}_{k-d_{ji}-1}^j \subset \mathcal{I}_k^i$. Recalling the assumption that graph $\mathcal{G}$ is connected we have

$$\mathcal{I}_k^i = \bigcup_{n=1,\dots,N} \{x_n(0:k-d_{ni})\},$$
$$\mathcal{I}_{k-d_{ji}-1}^j = \bigcup_{n=1,\dots,N} \left\{x_n(0:k-d_{nj}-d_{ji}-1)\right\},$$

which reduces the partial nestedness condition to $d_{nj}+d_{ji}+1 \geq d_{ni}$. Since d_{ni} is the length of the shortest path between nodes n and i in $\mathcal{G}$, one can write $d_{ni} \leq d_{nj}+d_{ji} < d_{nj}+d_{ji}+1$ which concludes the proof.

Remark 4. Even though Lemma 1 verifies partial nestedness of information structure in (4), this does not guarantee that optimal control inputs for the problem (7) are linear in the associated information, due to the presence of constraints (6). A complete proof for linearity of optimal control policies is given in the next corollary.

Proposition 1. *[Linearity of optimal control policies] Considering problem (7), the optimal control policies (3) are linear in the associated information, that is, of the form*

$$u_i(k) = \gamma_k^i(\mathcal{I}_k^i), \ k = 0, \dots, T-1, i = 1, \dots, N,$$

where γ_k^i is a linear admissible map.

Proof. Let us define $l_i(k)$ as

$$l_i(k) = \mathbb{E}\left[\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top W_i \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\right]. \tag{10}$$

There exists a dual multiplier $\lambda_i^\star(k)$ [22] such that minimizing (5) under power constraints (6) is equivalent to minimizing

$$J_\mathcal{P} = J_C + \sum_{k=0}^{T-1}\sum_{i=1}^{M} \lambda_i^\star(k) l_i(k)$$

Since $J_\mathcal{P}$ is quadratic and the information structure, (4) is partially nested, the optimal policies (3) to the problem (7) are linear in the associated information [7]. This concludes the proof.

From Lemma 2, the optimal control input $u(k)$ is a linear function of the complete history

$$u(k) = \Gamma(k)x(k:0) = [\Gamma_2(k)\,\Gamma_1(k)]\left[\frac{\bar{x}(k)}{x(k-\mathcal{D}:0)}\right] \tag{11}$$

where $\Gamma_1(k), \Gamma_2(k)$ are control gains of appropriate dimensions. The dimension of the vector $x(k-\mathcal{D}:0)$ increases in time, and therefore, we derive a sufficient statistics for it in order to express the control law (11) as a function of a finite-dimensional recursive estimate. The control part

$$u_c(k) = \Gamma_1(k)x(k-\mathcal{D}:0) \tag{12}$$

is referred to as the *coordinator*'s control actions. The coordinator observes the common history in (8) by updating its measurement set at each time instant k with $y_c(k) = x(k-\mathcal{D})$. Formally, combining Equations 9, (11) and (12) the coordinator is represented as the following dynamical system

$$\bar{x}(k+1) = A_c\bar{x}(k) + \bar{B}u_c(k) + \bar{w}(k) \tag{13}$$

$$y_c(k) = \tilde{C}\bar{x}(k-1) = x(k-\mathcal{D}) \tag{14}$$

where $\tilde{C} = [0 \dots 0\, I]$ and $A_c(k) = \bar{A} + \bar{B}\Gamma_2(k)$.

Remark 5. Note that the system above and in particular matrix $A_c(k)$ are derived assuming that Γ_2 is known to the coordinator. As we will demonstrate later, this gain can be computed offline, which justifies the assumption.

Using the identity $x(k) = F\bar{x}(k), F = [I\,0\,\dots\,0]$ and Equations 11, (12) the cost function in (5) can be written as

$$J_C = \mathbb{E}\left[\sum_{k=0}^{T-1}\begin{bmatrix}\bar{x}(k)\\ u_c(k)\end{bmatrix}^\top \Lambda_c \begin{bmatrix}\bar{x}(k)\\ u_c(k)\end{bmatrix} + \bar{x}(T)^\top Q_{T,c}\bar{x}(T)\right],$$

where

$$\Lambda_c = \begin{bmatrix}\Gamma_2^\top R\Gamma_2 + F^\top QF & \Gamma_2^T R\\ R\Gamma_2 & R\end{bmatrix}, Q_{T,c} = F^\top Q_T F.$$

To conclude, due to the quadratic nature of the cost above and Equations (13) and (14), the coordinator is a centralized, partially observable LQG system whose state estimate is computed by standard results in linear stochastic control [23]

$$\begin{aligned}\hat{x}(k+1) = \mathbb{E}\left[\bar{x}(k+1)|\mathcal{I}_{k+1}^C\right] = A_c\hat{x}(k)\\ + \bar{B}u_c(k) + K_c(k)(y_c(k+1) - \tilde{C}\hat{x}(k))\end{aligned} \tag{15}$$

with corresponding Kalman gain K_c computed as

$$K_c(k) = A_c(k)V_{ee}(k)\tilde{C}^T(\tilde{C}V_{ee}(k)\tilde{C}^\top)^{-1} \tag{16}$$

where $V_{ee}(k)$ is the variance of estimation error.

Remark 6. Note that $\bar{x}(0) = x(0 : -\mathcal{D}+1)$ and $V_{ee}(0) = \text{blkdiag}(\Sigma_x, \Sigma_{x_{-1}}, \dots, \Sigma_{x_{-\mathcal{D}+1}})$ that is, states $x(k-i)(i = 1, \dots, \mathcal{D}-1)$ for $k = 0$ are considered as mutually independent zero-mean Gaussian variables with finite covariance $\Sigma_{x_{k-i}} \geqslant 0$.

To conclude, the estimator in (15) represents sufficient statistics [15] for (12), and the optimal control law (11) becomes

$$u(k) = K_1(k)\hat{x}(k) + \Gamma_2(k)\bar{x}(k) \tag{17}$$

where $K_1(k)$ represents the gain with constant dimension compared to Γ_1 defined in (11) due to constant dimension of $\hat{x}(k)$. Differently than $K_1(k)$, the gain $\Gamma_2(k)$ needs to satisfy sparsity constraints coming from the imposed information structure (3). In fact, partitioning Γ_2 according to u and $\bar{x}$, it is seen that Γ_2 has $N \times N\mathcal{D}$ blocks, and therefore, the sparsity constraints can be defined as

$$\Gamma_2^{i,j+N\tau}(k) = 0 \iff x_j(k-\tau) \notin \mathcal{I}_k^i \tag{18}$$

where $\Gamma_2^{i,j+N\tau}$ indicates the block matrix in position $(i,j+N\tau)$, for $i\in[1:N]$, $j\in[1:N]$ and $\tau \in [0 : \mathcal{D}-1]$. Hence, we define the subspace S as

$$S = \left\{\Gamma_2 \in \mathbb{R}^{m\times n\mathcal{D}} | \Gamma_2 \text{ satisfies (18)}\right\} \tag{19}$$

and the structural identity of S is given as

$$[I_S]_{ij} = \begin{cases}1, \text{ if } S_{ij} \text{ can take arbitrary value}\\ 0, \text{ if } S_{ij} = 0\end{cases} \tag{20}$$

with structural identity of the complementary subspace S^c defined as $I_S^c = \mathbf{1} - I_S$.

Remark 7. As seen in (15), the estimator $\hat{x}$ is conditioned on the information that is common for all DMs. In terms of in-network control implementation this means that its value will be computed by each in-network node.

4.3 | Orthogonality of control decomposition

In this subsection, we write the optimal control law (17) in a form that will allow us to decompose the cost in (7) into independent cost functions, exploiting the information decomposition already introduced. Introducing the estimation error

$$e(k) = \bar{x}(k) - \hat{x}(k) \tag{21}$$

the control law in (17) is written as

$$u(k) = \phi_1(k) + \phi_2(k), \tag{22}$$

where

$$\phi_1(k) = (K_1(k) + \Gamma_2(k))\hat{x}(k), \ \phi_2(k) = \Gamma_2(k)e(k). \tag{23}$$

From (13) and (15), the evolution of error e is written as

$$\begin{aligned}e(k+1) =& \bar{x}(k+1) - \hat{x}(k+1) = \left(A_c(k) - K_c(k)\tilde{C}\right)e(k)\\ &+ \bar{w}(k) = \Phi(k+1,0)e_0 + \sum_{l=0}^{k-1}\Phi(k+1,l+1)\\ &\bar{w}(l) + \bar{w}(k)\end{aligned} \tag{24}$$

where the transition matrix $\Phi(k+1,l)$ for $l = 0, \dots, k$ is

$$\Phi(k+1,l) = \begin{cases}A_c(k) - K_c(k)\tilde{C} & l = k\\ \left(A_c(k) - K_c(k)\tilde{C}\right)\Phi(k,l) & l < k\end{cases}$$

Referring to (24) and (15), it follows

$$\mathbb{E}\left[e(k+1)\hat{x}(k+1)^\top\right] = \mathbb{E}\left[e(k+1)\right]\hat{x}(k+1)^\top$$

since the coordinator at time instant $(k+1)$ knows its previous estimate $\hat{x}(k)$, the previously computed control $u_c(k)$ and its partial measurement of the overall state vector $y_c(k+1)$. Finally, referring to (24), it follows $\mathbb{E}\left[e(k+1)\right] = 0\ \forall k$ due to assumption on zero mean Gaussian noise $w(k)$ and proper initialization of Kalman filter $\hat{x}(0) = \mathbb{E}[\bar{x}(0)]$, since $\mathbb{E}[e(0)] = \mathbb{E}[\bar{x}(0) - \hat{x}(0)] = 0$ and $\mathbb{E}[\bar{w}(k)] = 0$ for every k. Thus, it follows that $e(k)$ and $\hat{x}(k)$ are orthogonal, which also implies independency between ϕ_1 and ϕ_2 as defined in (23).

Remark 8. Notice that the optimal control policy (22) is a superposition of two components. The component $\phi_1(k)$ is proportional to the estimator $\hat{x}(k)$ of the state $\bar{x}(k)$, conditioned on the common information between all decision makers (in-network elements). This common information is not the full information available at the in-network elements. Indeed, each in-network element, at time instant k, also measures only locally available state value as well as state that was communicated to it with a number of steps less than the diameter of the physical interconnection graph. Thus, local corrections $\phi_2(k)$ have to be applied to compensate for the discrepancy of estimator $\hat{x}(k)$ and actual state $x(k)$, due to the process noise. In other words, additional available local information has to be considered as the part of the optimal control action of each in-network element.

5 | COMPUTATION OF CONTROL GAINS

In this section, we write the problem (7) in terms of variance of the state and input components defined in (21), (22), and split it into two subproblems: computation of the coordinator's control actions $\phi_1(k)$ and computation of local control actions $\phi_2(k)$.

5.1 | Cost function decomposition

The cost function in (7) is decoupled introducing the vector $\bar{z}$ as a decision variable

$$\bar{z}(k) = \left[\hat{x}(k)^\top \ \phi_1(k)^\top | e(k)^\top \ \phi_2(k)^\top\right]^\top, \tag{25}$$

which was proven to be block-wise independent in previous section. Due to the quadratic cost J_C and constraints in (6), we introduce the covariance matrix

$$\bar{V}(k) = \mathbb{E}\left[\bar{z}(k)\bar{z}(k)^\top\right] = \mathrm{blkdiag}(V_1(k), V_2(k)) \tag{26}$$

where the covariance matrices of the individual blocks of $\bar{z}(k)$ are

$$V_1(k) = \mathbb{E}\left[\begin{bmatrix}\hat{x}(k)\\ \phi_1(k)\end{bmatrix}\begin{bmatrix}\hat{x}(k)\\ \phi_1(k)\end{bmatrix}^\top\right] = \begin{bmatrix}V_{\hat{x}\hat{x}}(k) & V_{\hat{x}\phi_1}(k)\\ V_{\phi_1\hat{x}}(k) & V_{\phi_1\phi_1}(k)\end{bmatrix}, V_2(k) = \mathbb{E}\left[\begin{bmatrix}e(k)\\ \phi_2(k)\end{bmatrix}\begin{bmatrix}e(k)\\ \phi_2(k)\end{bmatrix}^\top\right] = \begin{bmatrix}V_{ee}(k) & V_{e\phi_2}(k)\\ V_{\phi_2 e}(k) & V_{\phi_2\phi_2}(k)\end{bmatrix}.$$

The sparsity of $\bar{V}$ is due to the following facts: (1) $\hat{x}(k)$ and $e(k)$ are ortoghonal; (2) both process noise and initial state are zero-mean. The power constraints are written as

$$\mathrm{E}\left[\begin{bmatrix}x(k)\\ u(k)\end{bmatrix}^\top W_i \begin{bmatrix}x(k)\\ u(k)\end{bmatrix}\right] = \mathrm{E}\left[\begin{bmatrix}F\bar{x}(k)\\ u(k)\end{bmatrix}^\top W_i \begin{bmatrix}F\bar{x}(k)\\ u(k)\end{bmatrix}\right] = tr(\tilde{W}_i V_1(k)) + tr(\tilde{W}_i V_2(k)) \le p_k^i \tag{27}$$

where $\tilde{W}_i = (\mathrm{blkdiag}(F, I))^\top W_i \,\mathrm{blkdiag}(F, I)$. Analogously, the quadratic cost is written as

$$J_C = tr(\bar{Q}_T V_{\hat{x}\hat{x}}(T)) + \sum_{k=0}^{T-1} tr(\bar{\Lambda}(k)V_1(k)) + \sum_{k=0}^{T-1} tr(\bar{\Lambda}(k)V_2(k)) + tr(\bar{Q}_T V_{ee}(T)) \tag{28}$$

where

$$\bar{\Lambda}(k) = (\mathrm{blkdiag}(F, I))^\top \Lambda(k)\,\mathrm{blkdiag}(F, I), \ \bar{Q}_T = F^\top Q_T F.$$

Proposition 2. *Suppose that $\Gamma_2(0:T-1)$ in Equation (11) are given such that the information constraints in (18) are satisfied. Problem (7) is reduced to*

$$\begin{aligned}\min_{V_1(0:T-1)\succcurlyeq 0} \ & tr(\bar{Q}_T M V_1(T) M^\top) + \sum_{k=0}^{T-1} tr(\bar{\Lambda}(k)V_1(k))\\ s.t.\ & MV_1(0)M^\top = 0\\ & MV_1(k+1)M^\top = \left[\bar{A}\ \bar{B}\right] V_1(k) \left[\bar{A}\ \bar{B}\right]^\top + (K_c\tilde{C})MV_2(k)M^\top (K_c\tilde{C})^\top\\ & tr(\tilde{W}_i V_1(k)) + tr(\tilde{W}_i V_2(k)) \le p_k^i\end{aligned} \tag{29}$$

where M is such that $V_{\hat{x}\hat{x}}(k) = MV_1(k)M^\top$.

Proof. Given the gains $\Gamma_2(0:T-1)$ that satisfy (18), in the cost function (28) only the first two terms are dependent on $V_1(0:T-1)$. Such cost is to be minimized with respect to $V_1(0:T-1)$ with constraints on the evolution of the dynamics of the estimator given by Equation (15), and subject to power constraints in (27). Indeed, calculating the variance of (15) yields the second constraint in the Proposition 1. The initial condition $V_{\hat{x}\hat{x}}(0) = MV_1(0)M^\top = 0$ holds since $\hat{x}(0) = E[\bar{x}(0)]$ i.e. it follows $V_{\hat{x}\hat{x}}(0) = \mathrm{E}\left[\hat{x}(0)\hat{x}(0)^\top\right] = \mathrm{E}[\bar{x}(0)]\,\mathrm{E}[\bar{x}(0)^\top] = 0$. Finally, the last constraint is from (27).

5.2 | Computation of dual variables

In this section, we exploit Langrangian duality to solve the problem in (29). We introduce dual variables $S(k) \in \mathbb{S}^n (k = 0, \ldots, T)$ to account for constraints on the evolution of $V_1(k)$ and its initial value, and dual variables $\tau_i(k) \in \mathbb{R}^+ (k = 0, \ldots, T-1)$ to account for the power constraints in (27). The Lagrangian function $\mathcal{L}(S_{0:T}, \tau_{i,0:T-1}, V_1(0:T))$ for problem (29) is

$$
\begin{aligned}
\mathcal{L} = {} & \mathcal{H}(T) + \sum_{k=0}^{T-1}\{\mathcal{H}(k) + tr\{S(k+1)\Sigma_{V_2}\}\} \\
& + \sum_{k=0}^{T-1}\sum_{i=1}^{M}\tau_i(k)(tr(\tilde{W}_i V_2(k)) - p_k^i), \\
\Sigma_{V_2} = {} & (K_c\tilde{C})MV_2(k)M^\top(K_c\tilde{C})^\top,
\end{aligned}
\tag{30}
$$

where the Hamiltonian function $\mathcal{H}(k), (k = 0, \dots, T)$ is

$$
\begin{aligned}
\mathcal{H}(T) &= tr\{M^T\left(\bar{Q}_T - S(T)\right)MV_1(T)\}, \\
\mathcal{H}(k) &= tr\{(\bar{\Lambda} + \begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix}^\top S(k+1)\begin{bmatrix}\bar{A} & \bar{B}\end{bmatrix} - M^\top S(k)M \\
&\quad + \sum_{i=1}^{M}\tau_i(k)\tilde{W}_i)V_1(k)\} \triangleq tr\{\xi(k)V_1(k)\}.
\end{aligned}
$$

The value of $S(T)$ can be computed by imposing the boundary condition on the Hamiltonian, that is, $\mathcal{H}(T) = 0$ which implies $S(T) = \bar{Q}_T$. The dual function is

$$
g(S_{(0:T)}, \tau_{i,(0:T-1)}) = \min_{V_1(0:T)} \mathcal{L}
$$

and since it is linear in V_1 the dual problem is feasible iff $\xi(k) \succcurlyeq 0$. In fact, since the cost function as well as constraints in (29) are linear, Slater's condition [22] reduces to feasibility of either primal or dual problem, and hence, optimal duality gap is zero. Thus, the dual problem is

$$
\begin{aligned}
\max_{S(0:T),\tau_i(0:T-1)} & \sum_{k=0}^{T} tr\{S(k)\Sigma_{V_2}\} + r_T \\
\text{s.t. } & \xi(k) \succcurlyeq 0, \ S(T+1) = 0.
\end{aligned}
\tag{31}
$$

It can be proven analogously to Gattami [24] that, with fixed values of τ_i, the previous equation is maximized with

$$
\begin{aligned}
S(k) &= \bar{A}^\top S(k+1)\bar{A} + \tilde{\Lambda}_{xx}(k) - L(k)^\top(\bar{B}^\top S(k+1)\bar{B} \\
&\quad + \tilde{\Lambda}_{uu}(k))L(k) \succ 0 \\
L(k) &= (\bar{B}^\top S(k+1)\bar{B} + \tilde{\Lambda}_{uu}(k))^{-1}(\bar{B}^\top S(k+1)\bar{A} + \tilde{\Lambda}_{xu}^\top(k))
\end{aligned}
\tag{32}
$$

where

$$
\tilde{\Lambda}(k) = \begin{bmatrix}\tilde{\Lambda}_{xx}(k) & \tilde{\Lambda}_{xu}(k) \\ \tilde{\Lambda}_{ux}(k) & \tilde{\Lambda}_{uu}(k)\end{bmatrix} = \begin{cases}\bar{\Lambda}(k) + \sum_{i=1}^{M}\tau_i(k)\tilde{W}_i, & k < T \\ \text{blkdiag}(\bar{Q}_T, 0), & k = T\end{cases}
$$

where $\tilde{\Lambda}_{uu}(k) \in \mathbb{R}^{m\times m}$. The variables τ_i are computed from cost in (31) accounting for (32). Interestingly, as a conclusion, the dual variables can be computed offline.

5.3 | Optimal coordinator action

In this section, we show how to compute optimal coordinator control action $\phi_1(k)$. Consider the primal optimization problem in (29) and let $S(k)$ and $\tau_i(k)$ be the optimal values of the dual variables computed from (31) and (32). Similarly to Causevic et al. [11], the dual problem to (29) is written as

$$
\min_{V_1(0:T-1)\succcurlyeq 0} \sum_{k=0}^{T-1} tr(Z(k)V_1(k)) + tr(S(k)\Sigma_{V_2}) - \sum_{k=0}^{T-1}\sum_{i=1}^{M}\tau_i(k)p_k^i
\tag{33}
$$

where $Z(k)$ is given by

$$
Z(k) = \begin{bmatrix} X(k)Y^{-1}(k)X^\top(k) & X(k) \\ X^\top(k) & Y(k)\end{bmatrix}
$$

where the values of $X(k), Y(k)$ are computed as

$$
\begin{aligned}
X(k) &= \bar{A}^\top S(k+1)\bar{B} + \tilde{\Lambda}_{xu}, \\
Y(k) &= \bar{B}^\top S(k+1)\bar{B} + \tilde{\Lambda}_{uu}
\end{aligned}
\tag{34}
$$

Proposition 3. *The optimal covariances $V_1(k)$ in (29) and the corresponding control action $\phi_1(k)$ are*

$$
V_1 = \begin{bmatrix} V_{\hat{x}\hat{x}} & -V_{\hat{x}\hat{x}}^\top K^\top \\ -KV_{\hat{x}\hat{x}} & KV_{\hat{x}\hat{x}}^\top K^\top \end{bmatrix}, \phi_1(k) = -K(k)\hat{x}(k) \tag{35}
$$

where the gain $K(k) = Y^{-1}(k)X^\top(k)$, with $Y(k), X(k)$ defined in (34).

Proof. To prove (35), note that dual problem in (33) is an unconstrained minimization problem. In order to compute the optimal covariances $V_1(k)$, it is sufficient to verify if the condition $tr(Z(k)V_1(k)) = 0$ is satisfied for a certain choice of the covariance matrix V_1. Recalling the definition of V_1 we get

$$
tr(ZV_1) = tr\begin{bmatrix} XY^{-1}X^\top V_{\hat{x}\hat{x}} + XV_{\phi_1\hat{x}} & * \\ * & X^\top V_{\hat{x}\phi_1} + YV_{\phi_1\phi_1}\end{bmatrix} \tag{36}
$$

Due to the linearity of the problem, it is sufficient to find $V_1(k)$ such that the block diagonal elements in (36) are zero. Additionally, from the assumption on positive-definiteness (and thus invertibility) of $\tilde{\Lambda}_{uu}$ we obtain

$$
\begin{aligned}
V_{\phi_1\hat{x}} &= -Y^{-1}X^\top V_{\hat{x}\hat{x}} \\
V_{\phi_1\phi_1} &= -Y^{-1}X^\top V_{\hat{x}\phi_1} = Y^{-1}X^\top V_{\hat{x}\hat{x}}(Y^{-1}X^\top)^\top
\end{aligned}
$$

From Equation (23) and latter expression for $V_{\phi_1\phi_1}$ it follows that $K_1(k) + \Gamma_2(k) = -Y^{-1}X^\top \triangleq K(k)$.

5.4 | Local control updates

Finally, we now provide the main result of the paper, which gives the computation of optimal local control actions $\phi_2(k)$ or equivalently control gains $\Gamma_2(k)$. The computation is given in the form of an iterative algorithm, namely, Algorithm 1.

Theorem 1. *Given the problem in (7) and control policy in (11) the gains $\Gamma_2(k), k = 0, \ldots, T-1$ computed via Algorithm 1 are optimal.*

Algorithm 1 Computation of local gains

procedure
  $G^0(k) \leftarrow 0, \forall k, c^0 \leftarrow 0, V_{ee}(0) \leftarrow \Sigma_x$
  for $h = 0, 1, 2, \ldots$ **do**
    Solve with respect to $\Gamma_2(0 : T-1)$

$$
\begin{aligned}
&P(k)\Gamma_2(k) + c^h \left(\Gamma_2(k) \circ I_S^c\right) V_{ee}^{-1}(k) = -N(k), \forall k \\
&P(k) = \bar{B}^\top R(k+1)\bar{B} + 2\tilde{\Lambda}_{uu} \\
&N(k) = \tilde{\Lambda}_{xu}^\top + \tilde{\Lambda}_{ux} + \bar{B}^\top R(k+1)(\bar{A} - K_c(k)\tilde{C}) + M(k) \\
&M(k) = (G^h(k) + c^h \sum_{j=k+1}^{T-1} \Gamma_2(k) \circ I_S^c) V_{ee}^{-1}(k) \\
&V_{ee}^+ = (A_c(k) - K_c(k)\tilde{C})V_{ee}(A_c(k) - K_c(k)\tilde{C})^\top + \Sigma_{\bar{w}}.
\end{aligned}
\tag{37}
$$

  where $R(T) = \bar{Q}_T$ and for $k < T$ we have

$$
R(k) = \Upsilon^\top(T)\bar{Q}_T\Upsilon(T) + 2\sum_{j=k}^{T-1} \Upsilon(j)^\top(\tilde{\Lambda}(j) + \tilde{S}(j+1))\Upsilon(j)
$$

$$
\Upsilon(j) = \prod_{z=k+1}^{j-1} (\bar{A} + \bar{B}\Gamma_2(z) - K_c(z)\tilde{C}),
$$

    Update $G^{h+1}(k) = G^h(k) + c^h(\Gamma_2^h(k) \circ I_S^c)$
    Update $c^{h+1} = \gamma c^h, \gamma > 1$
    Stop the algorithm if $\max_k ||\Gamma_2^h(k) \circ I_S^c|| < \epsilon$
  end for
end procedure

Proof. As the optimal cost for problem (29) sets Hamiltonian in (30) to zero, from (28) the cost for computing $\Gamma_2(0 : T-1)$ becomes

$$
\begin{aligned}
J_2(V_2(0 : T-1)) &= tr(\bar{Q}_T V_{ee}(T)) + \sum_{k=0}^{T-1} tr(\tilde{\Lambda}(k)V_2(k)) \\
&\quad + \sum_{k=0}^{T} tr(S(k+1)\Sigma_{V_2}) \\
&= tr(\bar{Q}_T V_{ee}(T)) + \sum_{k=0}^{T-1} tr(\tilde{\Lambda}(k)V_2(k)) \\
&\quad + \sum_{k=0}^{T-1} tr(\tilde{S}(k+1)V_{ee}(k))
\end{aligned}
$$

where the second row is obtained by taking into account the expression for Σ_{V_2} as defined in (30) and $\tilde{S}(k+1) = (K_c(k)\tilde{C})^\top S(k+1)(K_c(k)\tilde{C})$. The latter cost function is to be minimized with respect to $V_2(0 : T-1)$ such that the constraints on the evolution of $V_{ee}(k)$ due to Equation (24) are satisfied, that is, the following equation has to be satisfied at every time instant k:

$$
V_{ee}(k+1) = (A_c(k) - K_c(k)\tilde{C})V_{ee}(k)(A_c(k) - K_c(k)\tilde{C})^\top + \Sigma_{\bar{w}}.
\tag{38}
$$

Additionally, structural constraints on Γ_2 as defined in (18) have to be preserved. As we want to minimize J_2 with respect to $\Gamma_2(0 : T-1)$ we first express elements of V_2 as a function of it. Indeed, rewriting the elements of V_2 explicitly we get

$$
\begin{aligned}
V_{e\phi_2}(k) &= V_{\phi_2 e}^\top(k) = V_{ee}(k)\Gamma_2^\top(k) \\
V_{\phi_2\phi_2}(k) &= \Gamma_2(k)V_{ee}(k)\Gamma_2^\top(k)
\end{aligned}
$$

Computation of gains $\Gamma_2(k), k = 0, \ldots, T-1$ is done using the principle of optimality. Indeed, lets first compute $\Gamma_2(T-1)$. We construct a Lagrangian:

$$
\mathcal{L} = tr(\bar{Q}_T V_{ee}(T)) + tr(\tilde{\Lambda}(T-1)V_2(T-1)) + tr(G^\top(T-1)(\Gamma_2(T-1) \circ I_S^c))
$$

where the last term with multiplier $G(T-1)$ is introduced to account for sparsity on $\Gamma_2(T-1)$ according to (18). In what follows a quadratic term is introduced to penalize infeasible points, yielding the augmented Lagrangian

$$
\mathcal{L}_c = \mathcal{L} + \frac{c}{2}\|\Gamma_2(T-1) \circ I_S^c\|^2
\tag{39}
$$

where the penalty weight c is a positive scalar and $\|\cdot\|$ is the Frobenius norm. Starting with initial estimate of the Lagrange multiplier $G(T-1) = 0$, the augmented Lagrangian method [25] iterates between minimizing $\mathcal{L}_c(G(T-1), \Gamma_2(T-1))$ with respect to unstructured $\Gamma_2(T-1)$ and updating:

$$
G(T-1)^{h+1} = G(T-1)^h + c^h(\Gamma_2(T-1)^h \circ I_S^c)
\tag{40}
$$

where $\Gamma_2(T-1)^h$ is the minimizer of $\mathcal{L}_c(G(T-1)^h, \Gamma_2(T-1))$. The latter update rule for $G(T-1)$ guarantees that $G(T-1)^h$ belongs to the subspace S^c. Since $\mathcal{L}_c(G(T-1), \Gamma_2(T-1))$ is convex it can be shown [26] that sequence $\{G(T-1)^h\}$ converges to the optimal value $G(T-1)^*$, and consequently, local control gain $\Gamma_2(T-1)$ converges to the optimal structured gain $\Gamma_2(T-1)^*$. To this end, replacing $V_{ee}(T)$ as a function of $\Gamma_2(T-1)$ using (38) and deriving $\mathcal{L}_c$ with respect to $\Gamma_2(T-1)$ yields

$$
\begin{aligned}
&2\bar{B}^\top \bar{Q}_T(\bar{A} + \bar{B}\Gamma_2(T-1) - K_c(T-1)\tilde{C})V_{ee}(T-1) \\
&\quad + G(T-1) + (\tilde{\Lambda}_{xu}^\top + \tilde{\Lambda}_{ux})V_{ee}(T-1) \\
&\quad + 2\tilde{\Lambda}_{uu}\Gamma_2(T-1)V_{ee}(T-1) + c(\Gamma_2(T-1) \circ I_S^c) = 0
\end{aligned}
$$

which proofs the Equation (37) for $k = T-1$. Then, for time $k, k < T-1$ the cost-to-go is written as

$$\mathcal{L} = tr(\bar{Q}_T V_{ee}(T)) + tr(\tilde{\Lambda}(k)V_2(k)) + \sum_{j=k+1}^{T-1} tr(\tilde{\Lambda}(j)V_2(j)) + \sum_{j=k+1}^{T-1} tr(\tilde{S}(j+1)V_{ee}(j)) + \sum_{j=k}^{T-1} tr(G^\top(j)(\Gamma_2(j)\circ I_S^c)) + \sum_{j=k}^{T-1}\frac{c}{2}\|\Gamma_2(j)\circ I_S^c\|^2 \tag{41}$$

The latter function should be minimized accounting for dependency of $V_{ee}(j)(j = k+1, \dots, T-1)$ on $\Gamma_2(k)$ and assuming that $\Gamma_2(k+1:T-1)$ are known. Thus, it is important to express $V_{ee}(j)(j > k)$ as function of $\Gamma_2(k)$

$$V_{ee}(j) = (\prod_{l=k}^{j-1}(A_c(l) - K_c(l)\tilde{C}))V_{ee}(k)(\prod_{l=k}^{j-1}(A_c(l) - K_c(l)\tilde{C}))^\top + \sum_{l=k+1}^{j-1}(\prod_{z=l}^{j-1}(A_c(z) - K_c(z)\tilde{C}))\Sigma_{\bar{w}}\prod_{z=k+1}^{l}(A_c(z) - K_z(l)\tilde{C}))^\top + \Sigma_{\bar{w}} \tag{42}$$

For compactness, we denote by $\Upsilon(j) = \prod_{l=k+1}^{j-1}((A_c(l) - K_c(z)\tilde{C}))$. Replacing (42) into (41) and deriving with respect to $\Gamma_2(k)$, assuming that $\Gamma_2(k+1:T-1)$ are known, we get

$$2\bar{B}^\top\Upsilon^\top(T)\bar{Q}_T\Upsilon(T)(\bar{A} - K_c(k)\tilde{C} + \bar{B}\Gamma_2(k))V_{ee}(k) + G(k) + (\tilde{\Lambda}_{xu}^\top + \tilde{\Lambda}_{ux} + 2\tilde{\Lambda}_{uu}\Gamma_2(k))V_{ee}(k) + \sum_{j=k+1}^{T-1} 2\bar{B}^\top\Upsilon^\top(j)\theta(j)\Upsilon(j)(A_c(k) - K_c(k)\tilde{C} + \bar{B}\Gamma_2(k))V_{ee}(k) + \sum_{j=k}^{T-1} c(\Gamma_2(j)\circ I_S^c)$$

where

$$\theta(j) = \tilde{\Lambda}_{xx} + \tilde{\Lambda}_{xu}^\top\Gamma_2(j) + \Gamma_2^\top(j)(\tilde{\Lambda}_{ux} + \tilde{\Lambda}_{uu}\Gamma_2(j)) + \tilde{S}(j+1)$$

Setting the derived expression to zero, and solving explicitly for $\Gamma_2(k)$ the proof is concluded.

As a conclusion, since both gains $K(k)$ and $\Gamma_2(k)$ can be computed offline, in terms of in-network control implementation, they can be computed beforehand and communicated to each in-network device on time. Moreover, from (23), it follows $K_1(k) = K(k) - \Gamma_2(k)$, which finalizes the computation of gains in the control law (17).

5.5 | Control structure interpretation

5.5.1 | Computational nodes

Consider a network with N subsystems where neighboring DMs communicate with one-step delay. In addition to process $\mathcal{P}_i (i = 1, \dots, N)$, each subsystem $\mathcal{S}_i$ is collocated with a local computing unit C_i. The control units C_i are assumed to be of limited computing power and memory, or the access to their resources might be restricted (e.g., routers and switches). Additionally, the network contains a computationally and memory-wise powerful device, denoted by C.

5.5.2 | Offline computation

In order to account for memory and computation constraints of in-network computational units C_i, a task decomposition is introduced. Indeed, primarily C computes and stores the sequences of dual variables $S(0:T)$, $\tau_i(0:T-1)$ using the Equations (31) and (32). Then it computes the gains $K(0:T-1), \Gamma_2(0:T-1)$ according to (35) and Alghoritm 1, respectively.

5.5.3 | Online computation

At each time k, each C_i computes the estimate of $\bar{x}(k)$ according to (15) as it is based on common information defined in (8). This means that C needs to send to each C_i, respectively ith row of gain $K_1(k)$ and ith row of gain $\Gamma_2(k)$, one step in advance. Finally, local computational units C_i can calculate individual control input using (17). Indeed, what C_i computes is computationally inexpensive as it is given in an explicit form and based on operations of summing and multiplying matrices.

Algorithm 2 In-network control implementation

```
 1: procedure OFFLINE COMPUTATION PERFORMED BY C
 2:     S(T) ← Q̄_T
 3:     compute S(k), τ_i(k) using (31), (32), k = 0, ..., T−1
 4:     compute K(k) using (35)
 5:     compute Γ_2(k) using Algorithm 1
 6:     compute K_1(k) = K(k) − Γ_2(k)
 7:     store the values K_1(k), Γ_2(k) at the unit C
 8: end procedure
 9: procedure ONLINE COMPUTATIONS
10:     for k = 0 to T − 1 do
11:         C sends i^th rows of K_1(k + 1), Γ_2(k + 1) to C_i
12:         compute x̂(k) at C_i using (15)
13:         compute inputs u_i(k) at C_i using (17)
14:         apply u_i(k) to the corresponding plant P_i
15:     end for
16: end procedure
```

6 | SIMULATIONS

We illustrate the the developed control methodology on the example of an interconnected plant consisting of four subsystems. The physical interconnections between subsystems are given via an undirected line graph (as

FIGURE 2 Physically-interconnected plant consisting of four subsystems interacting via line graph. Subsystems are denoted by $S_i (i = 1, 2, 3, 4)$

shown in Figure 2). An example of system whose dynamics is described via a line graph is a vehicle platoon [20]. Referring to (2), we define the following global dynamics matrices:

$$A = \begin{bmatrix} 1 & 10 & 0 & 0 \\ 1 & 0.1 & 1 & 0 \\ 0 & 0.1 & 1 & 1 \\ 0 & 0 & 0.1 & 1 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The initial state is drawn from zero-mean distribution with covariance $\Sigma_x = 10^{-2} \times \mathbb{I}^{4x4}$, and the system noise is drawn from the zero-mean distribution with covariance $25 \times 10^{-4} \times \mathbb{I}^{4x4}$, where $\mathbb{I}^{4x4}$ denotes identity matrix with corresponding dimensions. For the state and input penalty matrices we define $Q = R = \mathbb{I}^{4x4}$. Given a control horizon $T = 30$, for time instants $k = 1, \dots, 29$ we impose the following power constraints

$$\mathbb{E}\left[u^\top(k)u(k)\right] \leq \frac{1}{4}\mathbb{E}\left[x^\top(k)x(k)\right] \iff \mathbb{E}\left[\begin{bmatrix} x(k) \\ u(k) \end{bmatrix}^\top \begin{bmatrix} -\mathbb{I}^{4x4} & 0^{4x4} \\ 0^{4x4} & 4\mathbb{I}^{4x4} \end{bmatrix} \begin{bmatrix} x(k) \\ u(k) \end{bmatrix}\right] \leq 0$$

where the latter inequality is written in the form of constraint (6).

Considering the graph structure in Figure 2 which has a diameter of $D = 3$ the optimal control policy is of the form

$$u(k) = K_1(k)\hat{x}(k) + \Gamma_2(k)\bar{x}(k) \tag{43}$$

where $\bar{x}^\top(k) = [x^\top(k)x^\top(k-1)x^\top(k-2)]$ and the estimator $\hat{x}(k) = \mathbb{E}[\bar{x}(k)|x(0 : k-3)]$ is computed according to (15). The gains $K(k) \in \mathbb{R}^{4x12}, k = 1, \dots, 29$ while $\Gamma_2(k) \in \mathbb{R}^{4x12}, k = 0, \dots, 29$ satisfy the following sparsity constraints $\Gamma_2^{i,j+4\tau}(k) = 0$, for $i \in [1:4], j \in [1:4], \tau \in [0:2]$. By choosing $\gamma = 0.5$ the gains $\Gamma_2(k)$ are computed by applying Algorithm 1. Subsequently, the gains $K_1(k)$ are computed as $K_1(k) = K(k) - \Gamma_2(k)$ where $K(k)$ are computed via Riccati-like forms in (35). In Figure 3, one can see the evolution of costs for computed optimal control and given 500 different realizations of the process noise and the initial state. It can also be seen that the optimal expected cost with the imposed power constraints (6) (total cost $J^*_{\text{con}} = 13.96$) is always higher compared to optimal expected cost when power constraints neglected (total cost $J^* = 5.72$). This is not to surprise since power constraints

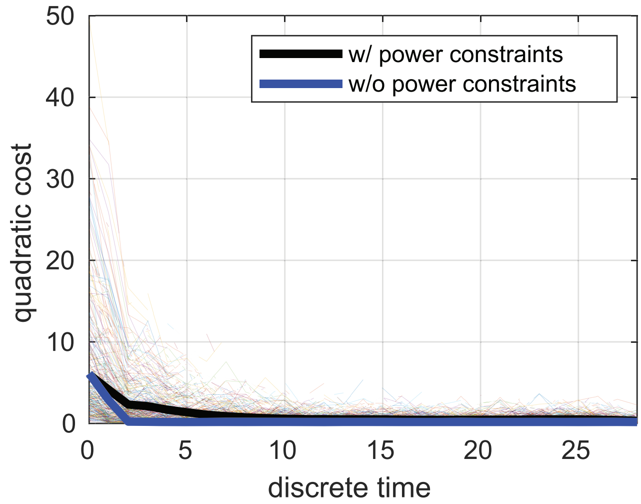

FIGURE 3 Monte Carlo simulation showing performance of computed optimal control for 500 different process noise realizations (thin lines), the optimal expected cost with power constraints (thick black line), and the optimal expected cost without power constraints (thick blue line)

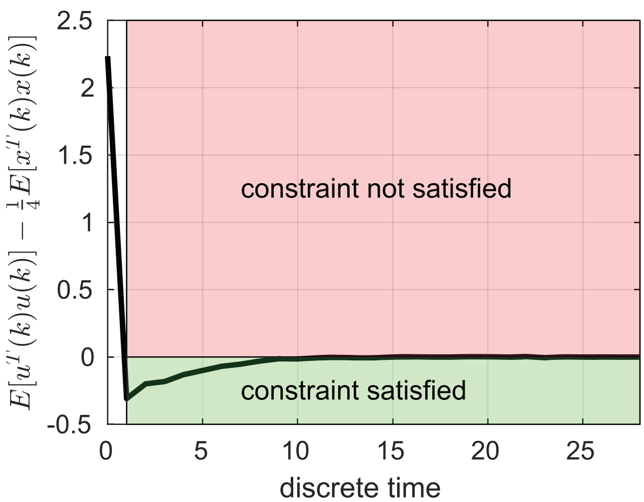

FIGURE 4 Power constraint function (solid line), the area of allowed values (shaded green area), and the area of constraint violation (shaded red area)

introduce additional limits on the range of control actions. The satisfaction of power constraint is validated in the Figure 4.

7 | CONCLUSIONS

In this paper, a framework for large-scale information-constrained optimal control with power constraints on input and state variables, is introduced. The optimal control law is linear and a superposition of two control components—one based on common information in the system and the other based on locally available information. The approach presented is applicable to an arbitrarily interconnected systems and an arbitrary number N of decision makers, under the assumption that communication between decision makers occurs as fast as information travels through the plant [27]. Finally, the implementation aspects of the resulting optimal control law on the in-network elements are provided.

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AUTHOR CONTRIBUTIONS

Vedad Causevic: Formal analysis, investigation, methodology. **Precious Ugo Abara:** Formal analysis. **Sandra Hirche:** Methodology, supervision.

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