A Simple Test Case for Error Reduction of Black-Box Coupling Schemes

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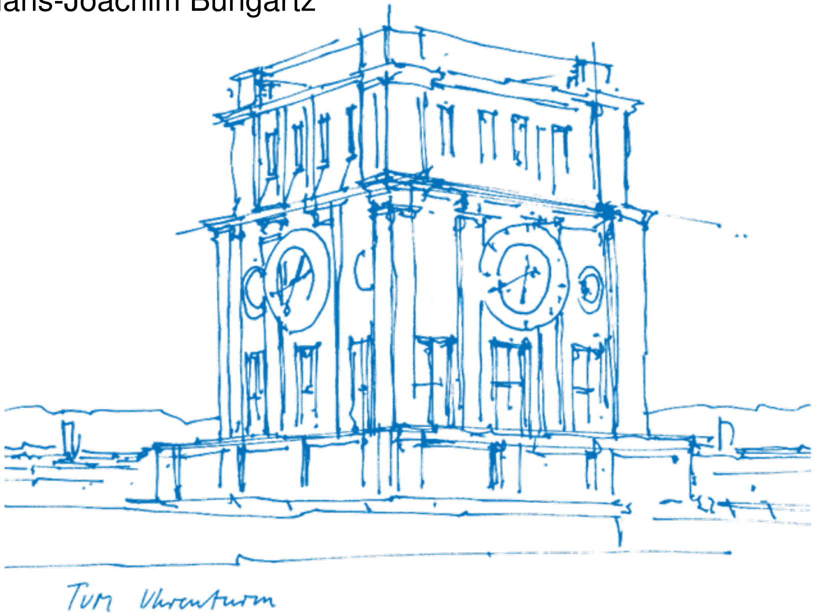
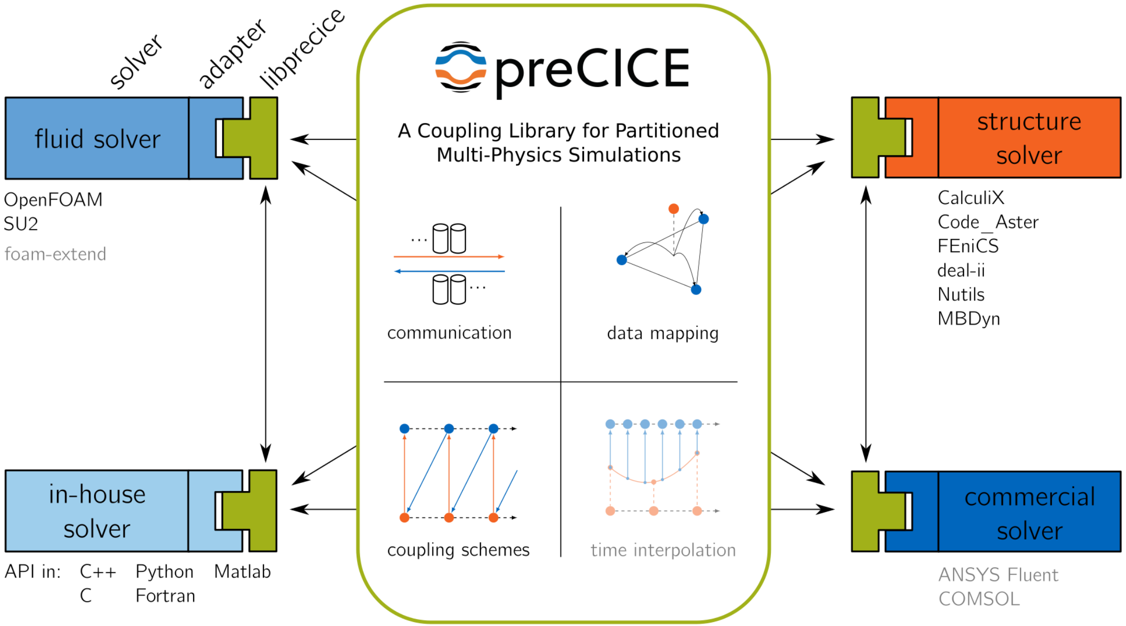
9th GACM Colloquium on Computational Mechanics

Essen, September 22, 2022



Why black-box coupling?



Why study the numerics of coupling schemes?

- **order degradation**: the coupling scheme can decrease the achievable convergence order in time
- additional impacts on stability, **energy conservation**

$\Rightarrow$ isolate and study these phenomena systematically (this talk)

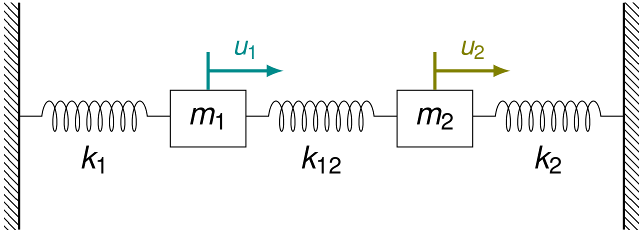
$\Rightarrow$ support more sophisticated black-box coupling schemes in preCICE ($\rightarrow$ Outlook)

The oscillator example

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

or, in shorthand notation,

$$M\ddot{\mathbf{u}} + K\mathbf{u} = 0$$



some properties of this system:

- analytical solution simple to compute
- energy conservation

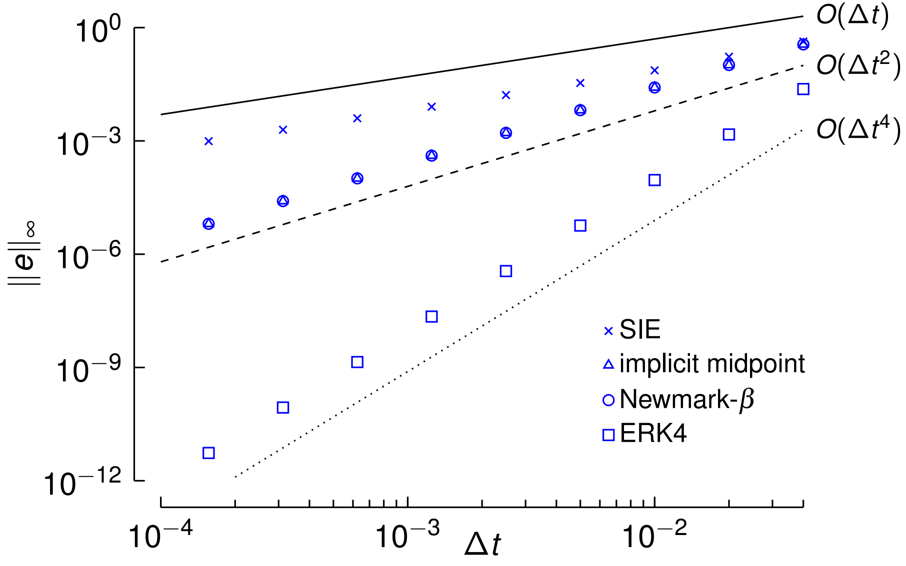
The oscillator example: Convergence study

4 time integration methods:

- $O(\Delta t)$: semi-implicit Euler (SIE)
- $O(\Delta t^2)$: implicit midpoint
- $O(\Delta t^2)$: Newmark-β
- $O(\Delta t^4)$: classical Runge-Kutta (ERK4)

decreasing time step sizes Δt

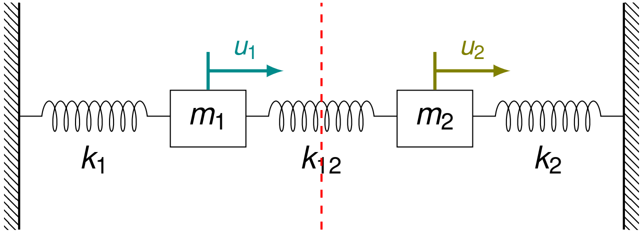
measure error w.r.t. analytical solution $||e||_\infty$



The oscillator example: Partitioned system

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{pmatrix} \ddot{u}_1 \\ \ddot{u}_2 \end{pmatrix} + \begin{bmatrix} k_1 + k_{12} & -k_{12} \\ -k_{12} & k_2 + k_{12} \end{bmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = 0$$

Idea: cut through the connecting spring k_{12}

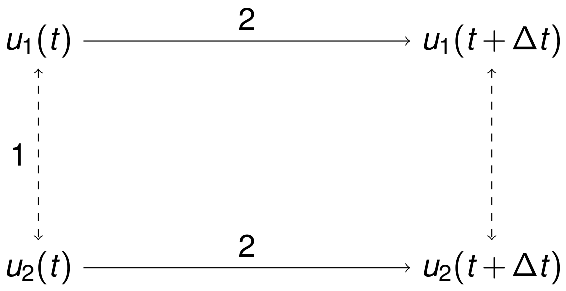


$\Rightarrow$ interface forces and two decoupled initial value problems:

$$m_1 \ddot{u}_1 = -(k_1 + k_{12}) u_1 + F_2(t)$$
$$m_2 \ddot{u}_2 = -(k_2 + k_{12}) u_2 + F_1(t)$$

with $F_i(t) = k_{12} u_i(t)$

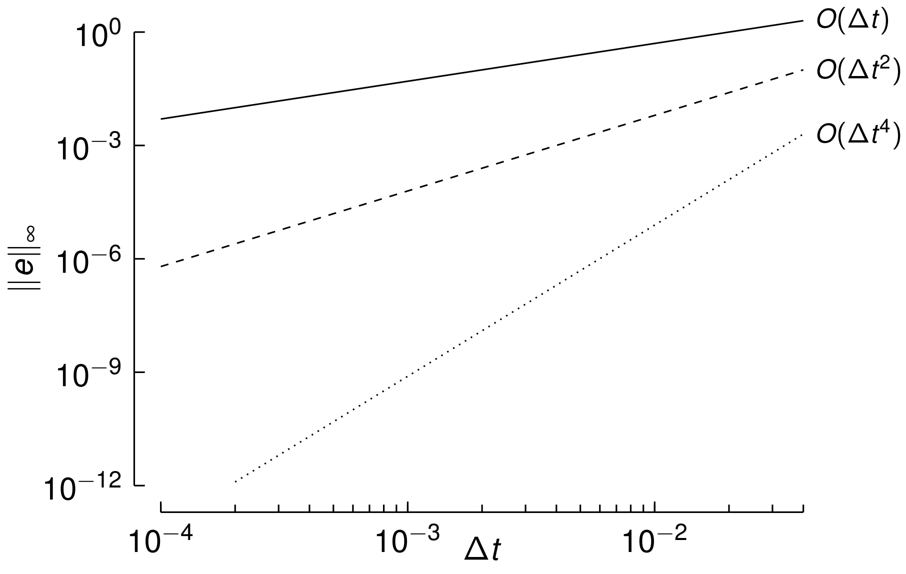
Naive coupling: A parallel scheme



conventional parallel staggered (CPS) scheme

(Farhat and Lesoinne 1998)

achievable convergence order: $O(\Delta t)$



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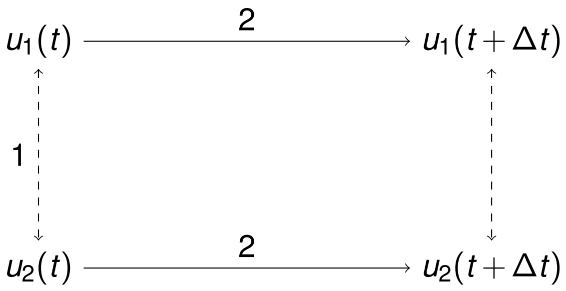

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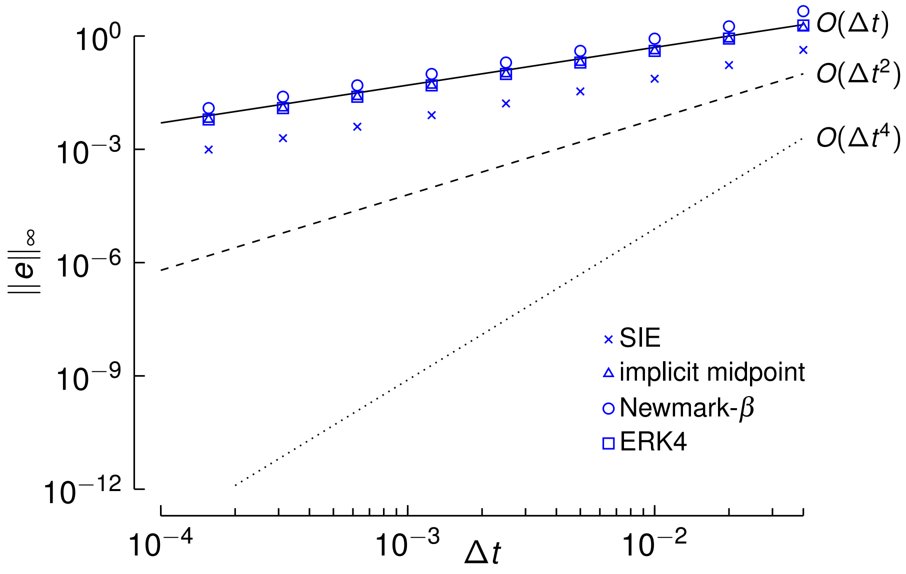
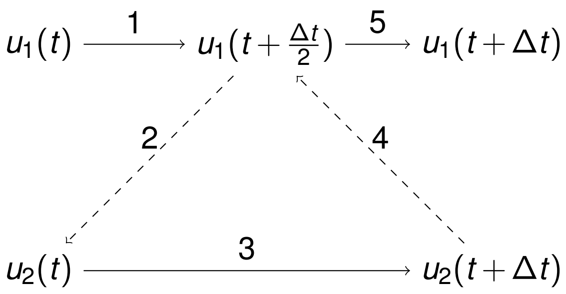
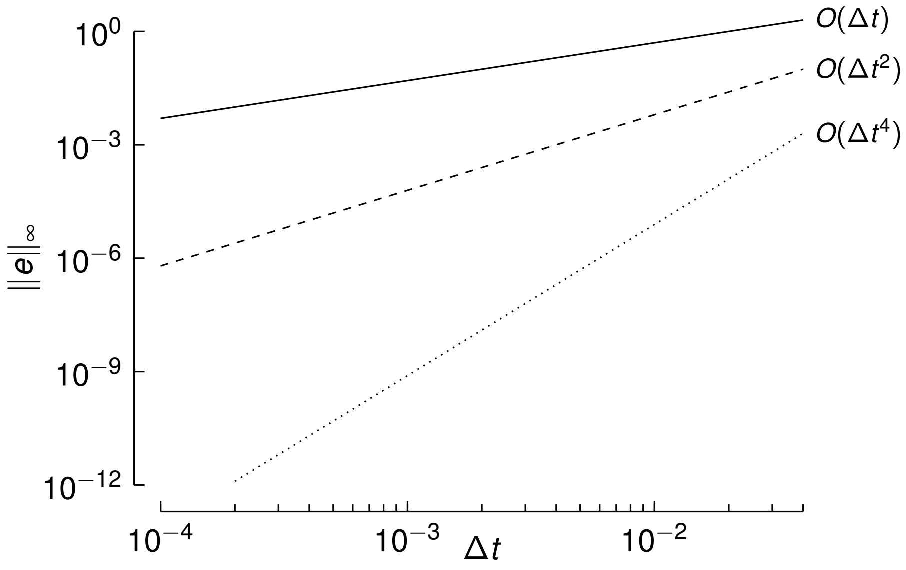


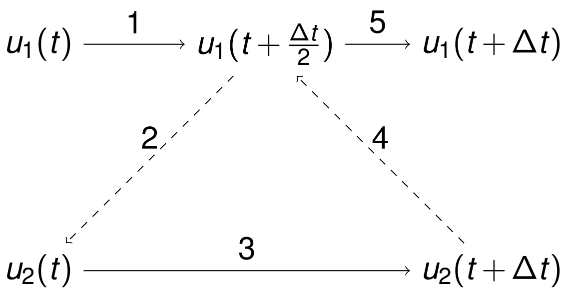
Strang splitting



achievable convergence order: $O(\Delta t^2)$ (Strang 1968)

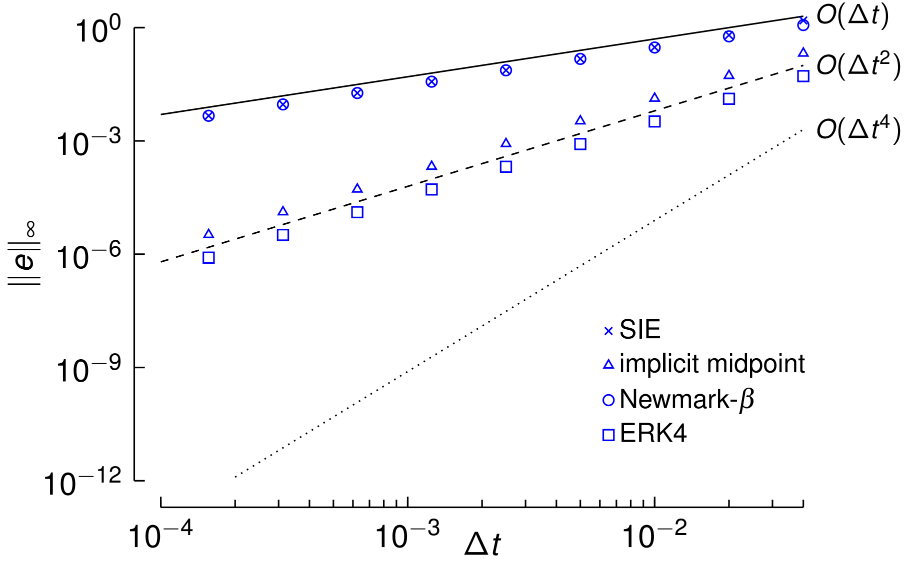


Strang splitting



achievable convergence order: $O(\Delta t^2)$ (Strang 1968)

$\Rightarrow$ **What is surprising?**

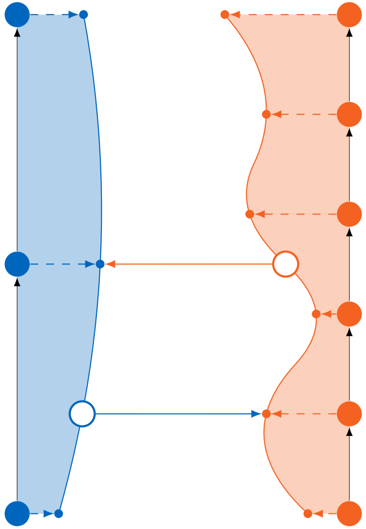


Waveform iterations

$$m_1 \ddot{u}_1 = -(k_1 + k_{12})u_1 + \mathbf{F_2}(\mathbf{t})$$
$$m_2 \ddot{u}_2 = -(k_2 + k_{12})u_2 + \mathbf{F_1}(\mathbf{t})$$

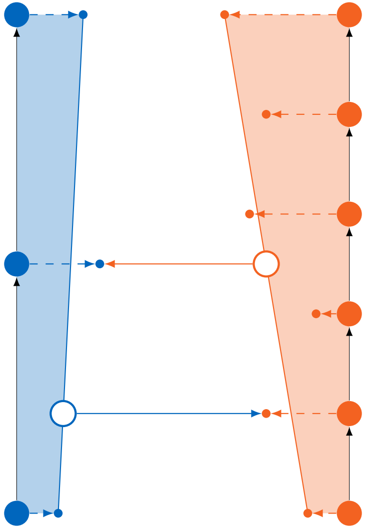
new approach: (Rüth et al. 2018, 2021)

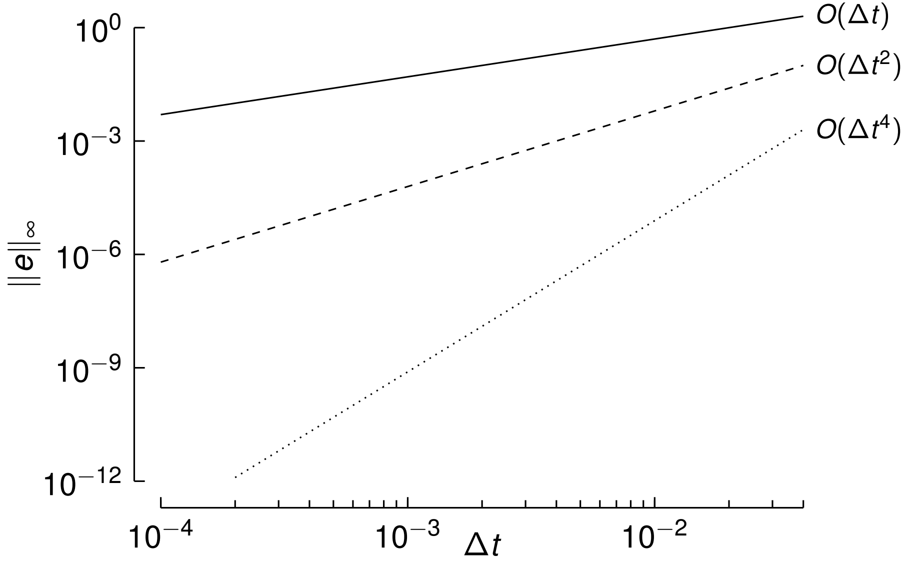
- exchange **polynomial interpolants** of boundary terms ($\mathbf{F_1}, \mathbf{F_2}$) instead of single values
- initial guess: constant interpolation
- *iterate* until convergence



Waveform iterations: Convergence study

here: implementation with linear interpolant





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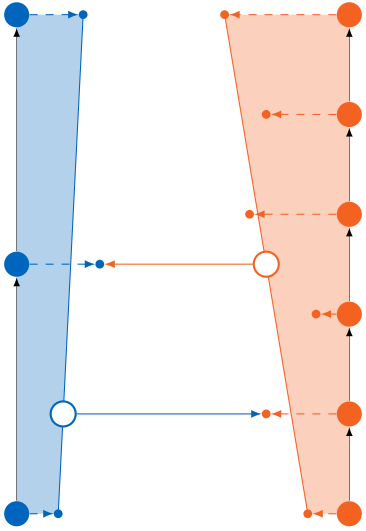
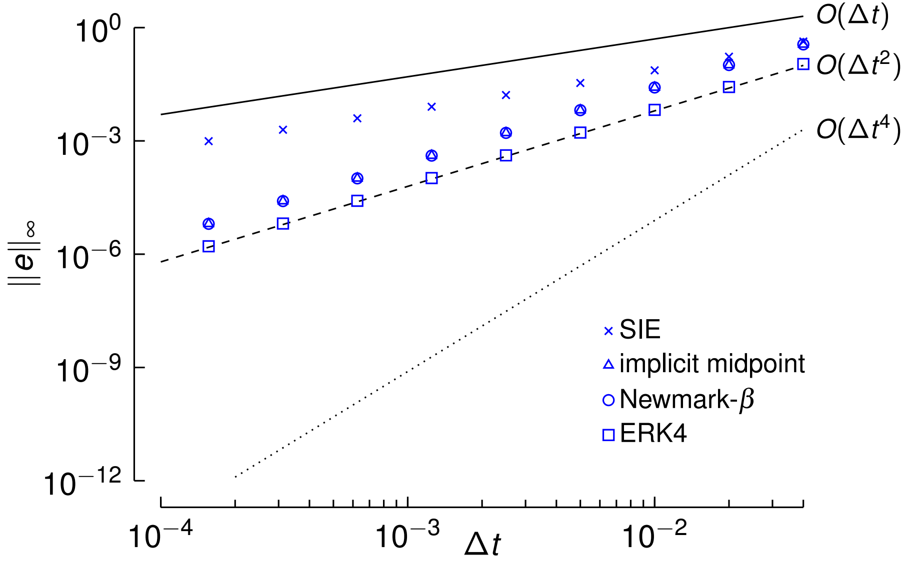


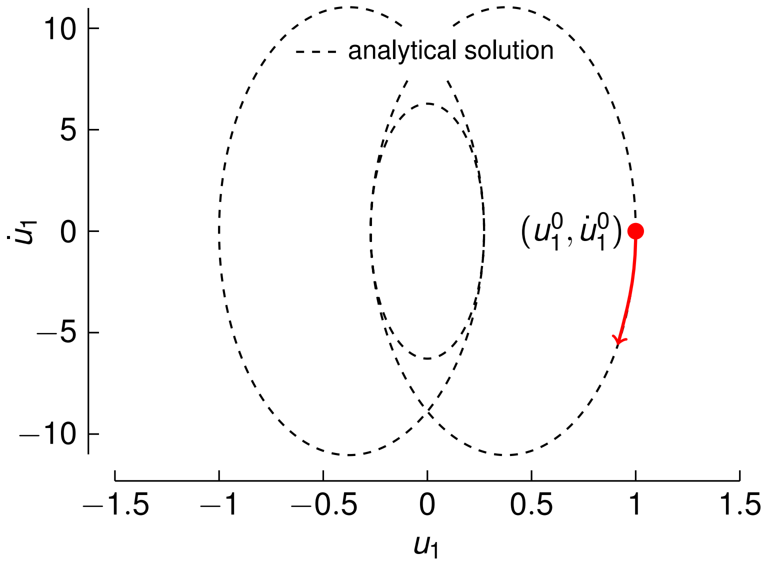


Energy Conservation

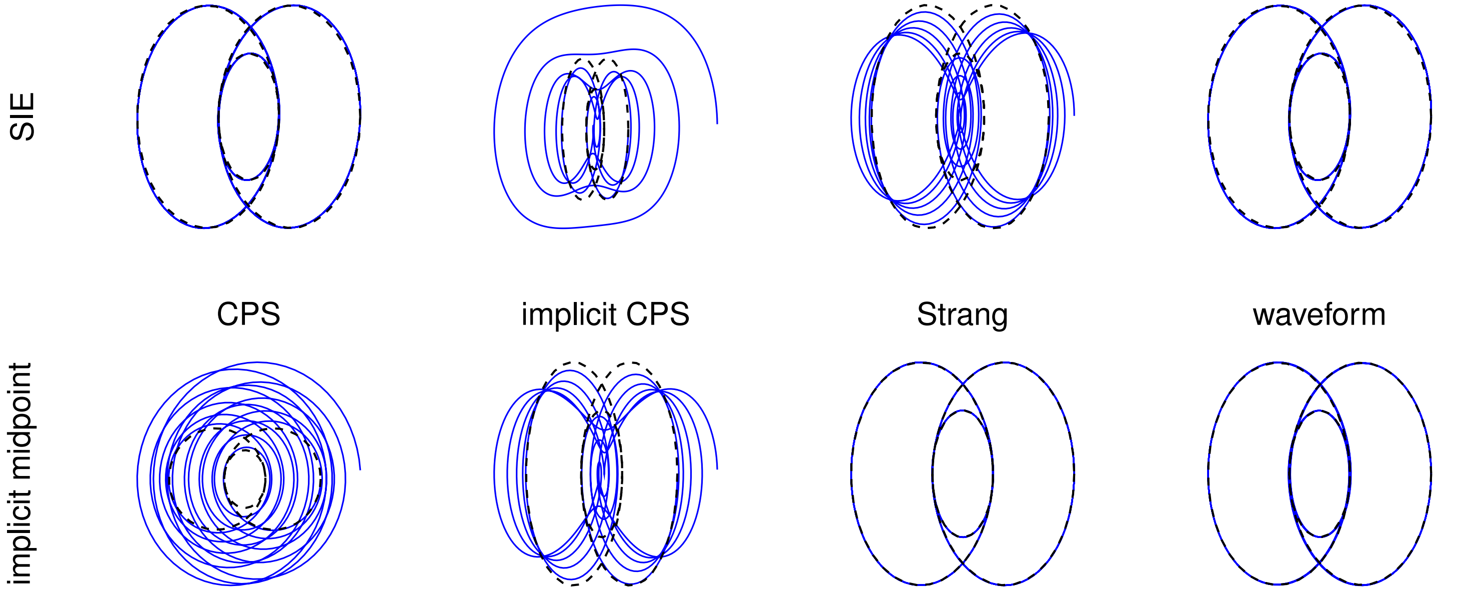
$$E = \frac{1}{2}\dot{\mathbf{u}}^T M \dot{\mathbf{u}} + \frac{1}{2}\mathbf{u}^T K \mathbf{u}$$

→ visualize the solution in phase space

→ use methods with good energy conservation properties (semi-implicit Euler, implicit midpoint)



Energy Conservation



Summary & Outlook

- coupling schemes and time integration methods *interact*
- simple coupling schemes lead to order degradation and worse behavior regarding energy conservation *most of the time*
- Strang splitting is a cheap way to get a better solution in both aspects *most of the time*
- waveform iterations are an expensive way to get a better solution *reliably*
  - more potential for applications with multiscale characteristics (cf. slide 17)

Summary & Outlook

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- waveform iterations are an expensive way to get a better solution *reliably*
  - more potential for applications with multiscale characteristics (cf. slide 17)

$\Rightarrow$ experimental linear waveform iterations are already part of preCICE v2.4

$\Rightarrow$ quadratic, cubic,... waveform iterations will be part of preCICE v3 ($\sim$ start of 2023)

more info + tutorials: `https://precice.org/`, or talk to me!

Full paper

"A Simple Test Case for Convergence Order in Time and Energy Conservation of Black-Box Coupling Schemes."
Schüller et al. 2022, DOI: 10.23967/wccm-apcom.2022.038.

Thank you!

Questions?

References / Further Reading

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- Schüller, Valentina et al. (July 2022). "A Simple Test Case for Convergence Order in Time and Energy Conservation of Black-Box Coupling Schemes". In: *WCCM-APCOM 2022*. Yokohama, Japan. DOI: 10.23967/wccm-apcom.2022.038.
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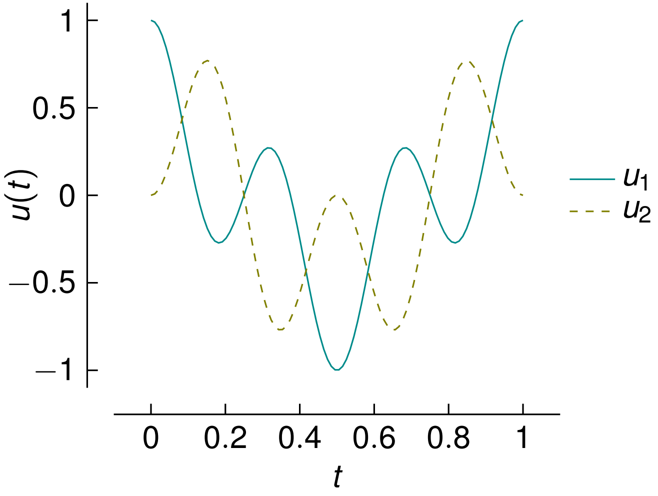
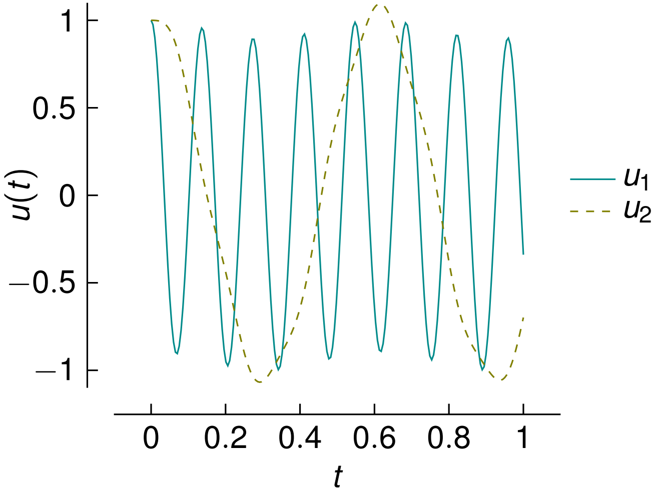
Appendix

The oscillator example: Analytical solutions



$k_1 = k_2 = 4\pi^2$, $k_{12} = 16\pi^2$ – studied here



$k_1 = 2000$, $k_2 = 10$, $k_{12} = 100$

Naive coupling: A parallel scheme with iterations

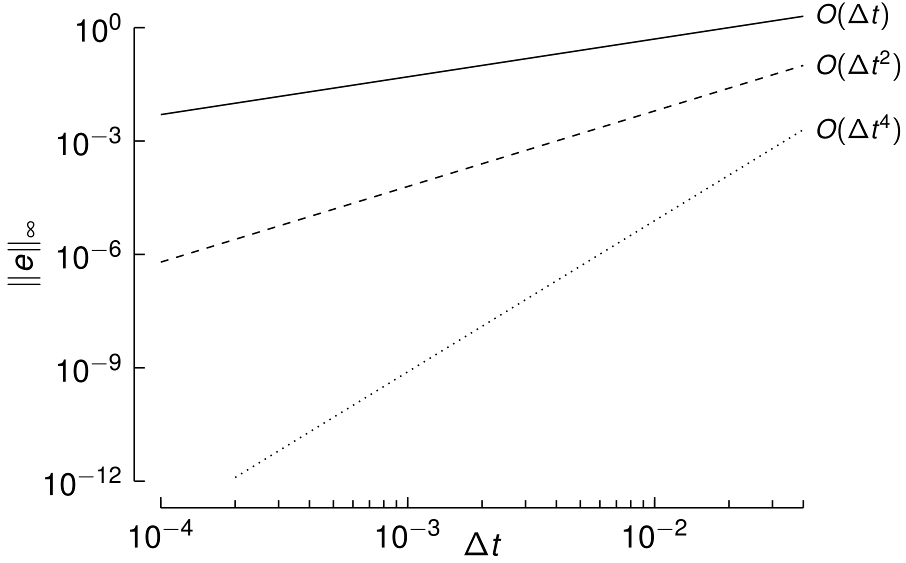
Fixed-point iterations over the CPS scheme:

$$\left\| \left(u_i^{n+1}\right)^{k+1} - \left(u_i^{n+1}\right)^{k} \right\| \leq \varepsilon.$$

"implicit CPS" (Farhat and Lesoinne 1998)

Achievable convergence order: $O(\Delta t)$

(but higher stability)



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⇒ **What happened here?**