Annales de l'Institut Henri Poincaré - Probabilités et Statistiques
2018, Vol. 54, No. 3, 1165–1203
https://doi.org/10.1214/17-AIHP836

ANNALES DE L'INSTITUT HENRI POINCARÉ PROBABILITÉS ET STATISTIQUES
www.imstat.org/aihp

The velocity of 1d Mott variable-range hopping with external field

Alessandra Faggionato[a], Nina Gantert[b] and Michele Salvi[c]

[a]*Dipartimento di Matematica, Università di Roma "La Sapienza", P.le Aldo Moro 2, 00185 Roma, Italy. E-mail: faggiona@mat.uniroma1.it*
[b]*Fakultät für Mathematik, Technische Universität München, 85748 Garching, Germany. E-mail: gantert@ma.tum.de*
[c]*Université Paris-Dauphine, PSL Research University, CNRS, [UMR 7534], CEREMADE, 75016 Paris, France. E-mail: salvi@ceremade.dauphine.fr*

Received 11 May 2016; revised 6 March 2017; accepted 3 April 2017

Abstract. Mott variable-range hopping is a fundamental mechanism for low-temperature electron conduction in disordered solids in the regime of Anderson localization. In a mean field approximation, it reduces to a random walk (shortly, Mott random walk) on a random marked point process with possible long-range jumps.

We consider here the one-dimensional Mott random walk and we add an external field (or a bias to the right). We show that the bias makes the walk transient, and investigate its linear speed. Our main results are conditions for ballisticity (positive linear speed) and for sub-ballisticity (zero linear speed), and the existence in the ballistic regime of an invariant distribution for the environment viewed from the walker, which is mutually absolutely continuous with respect to the original law of the environment. If the point process is a renewal process, the aforementioned conditions result in a sharp criterion for ballisticity. Interestingly, the speed is not always continuous as a function of the bias.

Résumé. Le « Mott variable-range hopping » est un mécanisme décrivant la conduction des electrons dans des solides désordonnés dans le régime de localisation d'Anderson. Sous l'approximation de champ moyen, le modèle se réduit à une marche aléatoire (marche aléatoire de Mott) sur un processus ponctuel. Cette marche peut sauter d'un point du processus ponctuel à n'importe quel autre, les sauts ne sont donc pas limités en taille.

Nous considerons une marche aléatoire de Mott unidimensionelle soumis à un champ extérieur (équivalent à un biais à droite). Nous montrons que la marche biaisée est transiente, et nous étudions sa vitesse linéaire. Nos résultats principaux sont des conditions pour la ballisticité (vitesse strictement positif) et la sous-ballisticité (vitesse nulle). Dans le regime ballistique, nous montrons l'existence d'une mesure invariante pour l'environment vu par la particule, absolument continue par rapport à la mesure originale. Si le processus ponctuel est un processus de renouvellement, nos conditions deviennent une condition nécessaire et suffisante pour la ballisticité. Nous montrons ainsi que la vitesse de la marche n'est pas, en général, une fonction continue du biais.

MSC: 60K37; 82D30; 60G50; 60G55

Keywords: Random walk in random environment; Disordered media; Ballisticity; Environment viewed from the walker; Electron transport in disordered solids

1. Introduction

Mott variable-range hopping is a fundamental mechanism at the basis of low-temperature electron conduction in disordered solids (e.g. doped semiconductors) in the regime of Anderson localization (see [2,16–18,20]). By localization, and using a mean-field approximation, Mott variable-range hopping can be described by a suitable random walk $(\mathbb{Y}_t)_{t\geq 0}$ in a random environment ω. The environment ω is given by a marked simple point process $\{(x_i, E_i)\}_{i\in\mathbb{Z}}$ with law $\mathbb{P}$. The sites $x_i \in \mathbb{R}^d$ correspond to the points in the disordered solid around which the conduction electrons are localized, and $E_i \in [-A, A]$ is the ground energy of the associated localized wave function. The random walk $\mathbb{Y}_t$ has state space $\{x_i\}$ and can jump from a site x_i to any other site $x_k \neq x_i$ with probability rate

$$r_{x_i,x_k}(\omega) := \exp\bigl\{-|x_i - x_k| - \beta\bigl(|E_i| + |E_k| + |E_i - E_k|\bigr)\bigr\},$$

β being the inverse temperature.

We refer to [5–7,11,12] for rigorous results on the random walk $\mathbb{Y}_t$, including the stretched exponential decay of the diffusion matrix as $\beta \to \infty$ in accordance with the physical Mott law for $d \geq 2$. Here we focus on the one-dimensional case, i.e. $\{x_i\}_{i\in\mathbb{Z}} \subset \mathbb{R}$ (we order the sites x_i's in increasing order, with $x_0 = 0$), and study the effect of applying an external field. This corresponds to modifying the above jump rates $r_{x_i,x_k}(\omega)$ by a factor $e^{\lambda(x_k - x_i)}$, where $\lambda \in (0, 1)$ has to be interpreted as the intensity of the external field. Moreover, we generalize the form of the jump rates, finally taking

$$r^{\lambda}_{x_i,x_k}(\omega) := \exp\bigl\{-|x_i - x_k| + \lambda(x_k - x_i) + u(E_i, E_k)\bigr\},$$

with u a symmetric bounded function. For simplicity, we keep the same notation $\mathbb{Y}_t$ for the resulting random walk starting at the origin.

Under rather weak assumptions on the environment, we will show that $\mathbb{Y}_t$ is a.s. transient for almost every environment ω (cf. Theorem 1(i)). In the rest of Theorem 1 we give two conditions in terms of the exponential moments of the inter-point distances, both assuring that the asymptotic velocity $v_{\mathbb{Y}}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{Y}_t}{t}$ is well defined and almost surely constant, that is, it does not depend on the realization of ω. Call $\mathbb{E}$ the expectation with respect to $\mathbb{P}$. The first condition, namely $\mathbb{E}[e^{(1-\lambda)(x_1-x_0)}] < \infty$ and u continuous, implies ballisticity, i.e. $v_{\mathbb{Y}}(\lambda) > 0$. The second condition, namely $\mathbb{E}[e^{(1-\lambda)(x_1-x_0)-(1+\lambda)(x_0-x_{-1})}] = \infty$, implies sub-ballisticity, i.e. $v_{\mathbb{Y}}(\lambda) = 0$. In particular, if the points $\{x_i\}_{i\in\mathbb{Z}}$ are given by a renewal process, our two conditions give a sharp dichotomy (when u is continuous). We point out that there are cases in which $v_{\mathbb{Y}}(\lambda)$ is not continuous in λ (see Example 2 in Section 2.2).

Under the condition leading to ballisticity we also show that the Markov process given by the environment viewed from the walker admits a stationary ergodic distribution $\mathbb{Q}^\infty$, which is mutually absolutely continuous to the original law $\mathbb{P}$ of the environment. Moreover, we give an upper bound for the Radon–Nikodym derivative $\frac{d\mathbb{Q}^\infty}{d\mathbb{P}}$ in terms of an explicit function in $L^1(\mathbb{P})$ and we give a lower bound in terms of a positive constant. We also characterize the asymptotic velocity as the expectation of the local drift with respect to the measure $\mathbb{Q}^\infty$ (cf. Theorem 2).

The study of ballisticity for the Mott random walk is the first fundamental step towards proving the Einstein Relation, which states the proportionality of diffusivity and mobility of the process (see e.g. [14]). Among other important applications, the Einstein Relation would allow to conclude the proof of the physical Mott law, which was originally stated for the mobility of the process and has only been proved for its diffusivity (see [5,11] and [12]). The Einstein Relation will be addressed in future work (some remarks in the paper will stress the behavior of some crucial bounds in the limit $\lambda \to 0$).

The techniques used to prove ballisticity and sub-ballisticity are different. In order to comment them it is convenient to refer to the discrete-time random walk[1] $(X_n)_{n\in\mathbb{N}}$ on $\mathbb{Z}$ such that $X_n = i$ if after n jumps the random walk $\mathbb{Y}_t$ is at site x_i. Due to our assumptions on the environment, the ballistic/sub-ballistic behavior of $(\mathbb{Y}_t)_{t\geq 0}$ is indeed the same as that of $(X_n)_{n\in\mathbb{N}}$, and therefore we focus on the latter.

We first comment the ballistic regime. Considering first a generic random walk on $\mathbb{Z}$ starting at the origin and a.s. transient to the right, ballisticity is usually derived by proving a law of large numbers (LLN) for the hitting times $(T_n)_{n\geq 1}$, where T_n is the first time the random walk reaches the half-line $[n, +\infty)$. In the case of nearest neighbor random walks, T_n is simply the hitting time of n, and considering an ergodic environment one can derive the LLN for $(T_n)_{n\geq 1}$ by showing that the sequence $(T_{n+1} - T_n)_{n\geq 1}$ is stationary and mixing for the annealed law as in [1,21]. This technique cannot be applied in the present case, since our random walk has infinite range and much information about the environment to the right is known, when a site in $[n, +\infty)$ is visited for the first time. A very useful tool is the method developed in [8] where the authors have studied ballisticity for a class of random walks on $\mathbb{Z}$ with arbitrarily long jumps. Their strategy is as follows. First one introduces for any positive integer ρ a truncated random walk obtained from the original one by forbidding all jumps of length larger than ρ. The ergodicity of the environment

[1]We use the convention $\mathbb{N}_+ := \{1, 2, \dots\}$ and $\mathbb{N} := \{0, 1, 2, \dots\}$.

and the finite range of the jumps allow to introduce a regenerative structure related to the times $T_{\rho n}$, and to analyze the asymptotic behavior of the ρ-truncated random walk. In particular, one proves that the environment viewed from the ρ-truncated random walk admits a stationary ergodic distribution $\mathbb{Q}^\rho$ which is mutually absolutely continuous to the original law of the environment. A basic ingredient here is the theory of cycle-stationarity and cycle-ergodicity (cf. [22, Chapter 8] and [9] for an example in a simplified setting). Finally, one proves that the sequence $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$ converges weakly to a probability distribution $\mathbb{Q}^\infty$, which is indeed a stationary and ergodic distribution for the environment viewed from the random walker $(X_n)_{n\in\mathbb{N}}$ and is also mutually absolutely continuous to the law of the environment $\mathbb{P}$. Since, as usual, the random walk can be written as an additive functional of the environment viewed from the random walker, one can apply Birkhoff's ergodic theorem and use the ergodicity of $\mathbb{Q}^\infty$ to get the strong LLN for the random walk (hence its asymptotic velocity) for $\mathbb{Q}^\infty$-a.e. environment. Using the fact that $\mathbb{P} \ll \mathbb{Q}^\infty$, the above strong LLN holds for $\mathbb{P}$-a.e. environment, too. Finally, since the velocities of the ρ-truncated walks are uniformly bounded from below by a strictly positive constant and since they converge to the velocity of $(X_n)_{n\in\mathbb{N}}$ when $\rho \to \infty$, we obtain a ballistic behavior.

To analyze ballisticity we have used the same method as in [8], although one cannot apply [8, Theorems 2.3, 2.4] directly to the present case, since some hypotheses are not satisfied in our context. In particular, in [8] three conditions (called E, C, D) are assumed, and only condition C is satisfied by our model. By means of estimates based on electrical networks, we are able to extend the method developed in [8] to the present case. We point out that a crucial tool in the study of effective conductances is given by a comparison with the nearest-neighbor conductance model. Indeed, a posteriori, the ballistic/subballistic behavior of Mott random walk appears very similar to the one of the modified version with only nearest-neighbor jumps.

We now move to sub-ballisticity (the regime of zero velocity is not covered in [8] and our method could be in principle applied to random walks on $\mathbb{Z}$ with arbitrarily long jumps). We define a coupling between the random walk $(X_n)_{n\geq 0}$, a sequence of suitable $\mathbb{N}_+$-valued i.i.d. random variables $\xi_1, \xi_2, \dots$ with finite mean, and an ergodic sequence of random variables $S_1, S_2, \dots$ with the following properties: Fix ω and call now T_{k+1} the first time the random walk overjumps the point $\xi_1 + \cdots + \xi_k$. S_k is a geometric random variable of parameter $s_k = s(\tau_{\xi_1+\cdots+\xi_k}\omega)$, where $\tau_\cdot$ is the usual shift and s a deterministic function. The coupling guarantees that $X_{T_{k+1}}$ does not exceed $\xi_1 + \cdots + \xi_k + \xi_{k+1}$ and also ensures that the time $T_{k+1} - T_k$ is larger than S_k. Notice that

$$\frac{X_n}{n} \leq \frac{X_{T_{k+1}}}{T_k} \leq \frac{\xi_1 + \cdots + \xi_{k+1}}{S_1 + S_2 + \cdots + S_k} \quad \text{if } T_k \leq n < T_{k+1}, \tag{1}$$

and therefore the sub-ballisticity of $(X_n)_{n\geq 0}$ follows from the LLN for $(\xi_k)_{k\in\mathbb{N}_+}$ and the LLN for $(S_k)_{k\in\mathbb{N}_+}$, since our assumption $\mathbb{E}[\mathrm{e}^{(1-\lambda)(x_1-x_0)-(1+\lambda)(x_0-x_{-1})}] = \infty$ implies that $\mathbb{E}[1/s(\omega)] = +\infty$.

1.1. *Outline*

In Section 2 we rigorously introduce the (perturbed) Mott random walk in its continuous and discrete-time versions. Theorem 1 states the transience to the right and gives conditions implying ballisticity or subballisticity. Theorem 2 deals with the Radon–Nikodym derivative of the invariant measure for the environment viewed from the walker with respect to the original law $\mathbb{P}$ of the environment and gives a characterization of the limiting speed of the walk. Section 2.1 comments the assumptions we made for the Theorems, while two important (counter-)examples can be found in Section 2.2.

In Section 3 we collect results on the ρ-truncated walks. Estimates of the effective conductances induced by these walks and of the time they spend on a given interval are carried out in Sections 3.1 and 3.2, respectively. In Section 3.3 we show that the probability for them to hit a specific site to the right is uniformly bounded from below.

Section 4 introduces the regenerative structure for the ρ-truncated random walks and in Section 4.1 we give estimates on the regeneration times. The existence and positivity of the limiting speed for the truncated walks is proven in Section 4.2.

In Section 5 we characterize the density of the invariant measure for the process viewed from the ρ-truncated walker with respect to the original law of the environment. In Section 5.1 we bound the Radon–Nikodym derivatives from above by an L^1 function, while in Section 5.2 we give a uniform lower bound. In Section 5.3 we finally pass to the limit $\rho \to \infty$ and obtain an invariant measure for the environment viewed from the non-truncated walker and show that it is also absolutely continuous with respect to $\mathbb{P}$ (see Lemma 5.9).

To conclude, in Sections 6, 7 and 8 we prove, respectively, parts (i), (ii) and (iii) of Theorem 1. Some technical results are collected in the Appendixes A, B and C.

2. Mott random walk and main results

One-dimensional Mott random walk is a particular random walk in a random environment. The environment ω is given by a double-sided sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$, with $Z_k \in (0, +\infty)$ and $E_k \in \mathbb{R}$ for all $k \in \mathbb{Z}$. We denote by $\Omega = ((0, +\infty) \times \mathbb{R})^{\mathbb{Z}}$ the set of all environments. Let $\mathbb{P}$ be a probability on Ω, standing for the law of the environment. We denote by $\mathbb{E}$ the associated expectation. Given $\ell \in \mathbb{Z}$, we define the shifted environment $\tau_\ell\omega$ as $\tau_\ell\omega := (Z_{k+\ell}, E_{k+\ell})_{k\in\mathbb{Z}}$. From now on, with slight abuse of notation, we will denote by Z_k, E_k also the random variables on $(\Omega, \mathbb{P})$ such that $(Z_k(\omega), E_k(\omega))$ is the kth projection of the environment ω.

Our main assumptions on the environment are the following:

(A1) The random sequence $(Z_k, E_k)_{k\in\mathbb{Z}}$ is stationary and ergodic with respect to shifts;
(A2) $\mathbb{E}[Z_0]$ is finite;
(A3) $\mathbb{P}(\omega = \tau_\ell\omega) = 0$ for all $\ell \in \mathbb{Z}$;
(A4) There exists $d > 0$ satisfying $\mathbb{P}(Z_0 \geq d) = 1$.

We postpone to Section 2.1 some comments on the above assumptions.

It is convenient to introduce the sequence $(x_k)_{k\in\mathbb{Z}}$ of points on the real line, where $x_0 = 0$ and $x_{k+1} = x_k + Z_k$. Then the environment ω can be thought also as the marked simple point process $(x_k, E_k)_{k\in\mathbb{Z}}$, which will be denoted again by ω (with some abuse of notation). In this case, the ℓ-shift reads $\tau_\ell\omega = (x_{k+\ell} - x_\ell, E_{k+\ell})_{k\in\mathbb{Z}}$. For physical reasons, E_k is called the energy mark associated to point x_k, while Z_k is the interpoint distance (between x_{k-1} and x_k).

Fix now a symmetric and bounded (from below by $u_{\min}$ and from above by $u_{\max}$) measurable function $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Given an environment ω, the Mott random walk $(\mathbb{Y}_t)_{t\geq 0}$ is the continuous-time random walk on $\{x_k\}_{k\in\mathbb{Z}}$ with probability rate

$$r_{x_i,x_k}(\omega) := \exp\bigl\{-|x_i - x_k| + u(E_i, E_k)\bigr\} \tag{2}$$

for a jump from x_i to $x_k \neq x_i$. For convenience, we set $r_{x,x}(\omega) \equiv 0$. Note that the Mott walk is well defined for $\mathbb{P}$-a.a. ω. Indeed, since the interpoint distance is a.s. at least d and the function u is uniformly bounded, the holding time parameter $r_x(\omega) := \sum_y r_{x,y}(\omega)$ can be bounded from above by a constant $C > 0$ uniformly in $x \in \{x_k\}_{k\in\mathbb{Z}}$, hence explosion does not take place.

We now introduce a bias λ which corresponds to the intensity of the external field. For a fixed $\lambda \in [0, 1)$, the biased Mott random walk $(\mathbb{Y}_t)_{t\geq 0}$ with environment ω is the continuous-time random walk on $\{x_k\}_{k\in\mathbb{Z}}$ with probability rates

$$r^\lambda_{x,y}(\omega) = \mathrm{e}^{\lambda(y-x)} r_{x,y}(\omega) \tag{3}$$

for a jump from x to $y \neq x$. For convenience, we set $r^\lambda_{x,x}(\omega) \equiv 0$ and denote the holding time parameter by $r^\lambda_x(\omega) := \sum_y r^\lambda_{x,y}(\omega)$. When $\lambda = 0$, one recovers the original Mott random walk. Since for $\lambda \in (0, 1)$ we have, for $\mathbb{P}$-a.a. ω, $r^\lambda_x(\omega) \leq \sum_{k\in\mathbb{Z}} \mathrm{e}^{-(1-\lambda)|k|d+u_{\max}} < \infty$, the biased Mott random walk with environment ω is well defined for $\mathbb{P}$-a.a. ω.

We can consider also the jump chain $(Y_n)_{n\geq 0}$ associated to the biased Mott random walk (we call it the *discrete-time Mott random walk*). Given the environment ω, $(Y_n)_{n\geq 0}$ is the discrete-time random walk on $\{x_k\}_{k\in\mathbb{Z}}$ with jump probabilities

$$p^\lambda_{x,y}(\omega) := \frac{r^\lambda_{x,y}(\omega)}{r^\lambda_x(\omega)}, \quad x \neq y. \tag{4}$$

A similar definition holds for the unbiased case ($\lambda = 0$).

The following result concerns transience, sub-ballisticity and ballisticity:

Theorem 1. *Fix $\lambda \in (0, 1)$. Then, for $\mathbb{P}$-a.a. ω, the continuous-time Mott random walk $(\mathbb{Y}_t)_{t\geq 0}$ with environment ω, bias λ and starting at the origin satisfies the following properties*:

(i) *Transience to the right*: $\lim_{t\to\infty} \mathbb{Y}_t = +\infty$ *a.s.*

(ii) *Ballistic regime*: *If* $\mathbb{E}[e^{(1-\lambda)Z_0}] < +\infty$ *and* $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *is continuous*, *then the asymptotic velocity*

$$v_{\mathbb{Y}}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{Y}_t}{t}$$

exists a.s., *it is deterministic* (*and does not depend on* ω), *finite and strictly positive* (*an integral representation of* $v_{\mathbb{Y}}$ *is given in Section* 7, *see* (85) *and* (95)).

(iii) *Sub-ballistic regime*: *If*

$$\mathbb{E}\big[e^{(1-\lambda)Z_0-(1+\lambda)Z_{-1}}\big] = \infty, \tag{5}$$

then

$$v_{\mathbb{Y}}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{Y}_t}{t} = 0 \quad a.s. \tag{6}$$

In particular, if $\mathbb{E}[Z_{-1}|Z_0] \le C$ *for some constant* C *which does not depend on* ω *and* $\mathbb{E}[e^{(1-\lambda)Z_0}] = \infty$, *then condition* (5) *is satisfied and* $v_{\mathbb{Y}}(\lambda) = 0$.

In addition, *for* $\mathbb{P}$*-a.a.* ω *the above properties remain valid* (*restricting to integer times* $n \ge 0$) *for the discrete-time Mott random walk* $(Y_n)_{n\ge 0}$ *with environment* ω, *bias* λ *and starting at the origin*, *and its velocity* $v_Y(\lambda) := \lim_{n\to\infty} \frac{Y_n}{n}$.

Remark 2.1. In the case $\lambda = 0$ the Mott random walks $\mathbb{Y}_t$ and Y_n are recurrent and have a.s. zero velocity. Recurrence follows from [6, Theorem 1.2(iii)] and the recurrence of the spatially homogeneous discrete-time random walk on $\mathbb{Z}$ with probability to jump from x to y proportional to $e^{-|x-y|}$. To see that the velocity is zero, set $\mathbb{Q}(d\omega) = \frac{r_0(\omega)}{\mathbb{E}[r_0]}\mathbb{P}(d\omega)$. $\mathbb{Q}$ is a reversible and ergodic distribution for the environment viewed from the discrete-time Mott random walker Y_n (see [5]). By writing Y_n as an additive function of the process "environment viewed from the walker" and using the ergodicity of $\mathbb{Q}$, one gets that $v_Y(\lambda = 0)$ is zero a.s., for $\mathbb{Q}$-a.a. ω and therefore for $\mathbb{P}$-a.a. ω. Similarly, $v_{\mathbb{Y}}(\lambda = 0) = 0$ a.s., for $\mathbb{P}$-a.a. ω (use that $\mathbb{P}$ is reversible and ergodic for the environment viewed from $\mathbb{Y}_t$, see [12]).

Remark 2.2. If the random variables Z_k are i.i.d. (or even when only Z_k, Z_{k+1} are independent for every k) and u is continuous, the above theorem implies the following dichotomy: $v_{\mathbb{Y}}(\lambda) > 0$ if and only if $\mathbb{E}[e^{(1-\lambda)Z_0}] < +\infty$, otherwise $v_{\mathbb{Y}}(\lambda) = 0$. The same holds for $v_Y(\lambda)$. We point out that, if the Z_k's are correlated, $\mathbb{E}[e^{(1-\lambda)Z_0}] = +\infty$ does not imply in general zero velocity (see Example 1 in Section 2.2).

Remark 2.3. Theorem 1 shows that there are cases in which the limiting speed $v_{\mathbb{Y}}(\lambda)$ is not continuous in λ. See Example 2 in Section 2.2.

Remark 2.4. When considering the continuous-time nearest neighbor random walk on $\{x_k\}_{k\in\mathbb{Z}}$ with probability rate for a jump from x to a neighboring site y given by (3), the random walk is ballistic if and only if

$$\sum_{i=1}^{\infty} \mathbb{E}\big[\exp\big\{(1-\lambda)Z_0 - (1+\lambda)Z_{-i} - 2\lambda(Z_{-i+1} + \cdots + Z_{-1})\big\}\big] < \infty \tag{7}$$

(the same criterion holds for the discrete-time version of the random walk). A derivation of this fact from classical results on random walks in random environment (cf. [24]) is given in Appendix C. Note that, if the Z_k's are i.i.d., the above condition (7) reduces to the bound $\mathbb{E}[e^{(1-\lambda)Z_0}] < \infty$, thus leading to the same dichotomy as pointed out in Remark 2.2. Formula (7) would suggest that, in order to ensure ballisticity for Mott random walk, the condition $\mathbb{E}[e^{(1-\lambda)Z_0}] < +\infty$ introduced in Theorem 1(ii) could be weakened. As outlined in Remark 3.13, this can indeed be achieved at the cost of dealing with rather ugly formulas having some analogy with (7).

One of the most interesting technical results we use in the proof of Theorem 1, Part (ii), concerns the invariant measure for the environment seen from the point of view of the walker:

Theorem 2. *Fix $\lambda \in (0,1)$. Suppose that $\mathbb{E}[e^{(1-\lambda)Z_0}] < +\infty$ and $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then the following holds*:

(i) *The environment viewed from the discrete-time Mott random walker $(Y_n)_{n\ge 0}$, i.e. the process $(\tau_{\phi(Y_n)}\omega)_{n\ge 0}$ where $\phi(x_i) = i$, admits an invariant and ergodic distribution $\mathbb{Q}^\infty$ absolutely continuous to $\mathbb{P}$ such that*

$$0 < \gamma \le \frac{d\mathbb{Q}^\infty}{d\mathbb{P}} \le F, \quad \mathbb{P}\text{-a.s.} \tag{8}$$

for a suitable universal constant γ and an explicit function $F \in L^1(\mathbb{P})$ (defined in (65)).

(ii) *By writing $\mathbb{E}^\infty$ for the expectation with respect to $\mathbb{Q}^\infty$, the velocities $v_{\mathbb{Y}}(\lambda)$, $v_Y(\lambda)$ can be expressed as*

$$v_{\mathbb{Y}}(\lambda) = v_Y(\lambda)/\mathbb{E}^\infty\big[1/r_0^\lambda(\omega)\big], \tag{9}$$

$$v_Y(\lambda) = \mathbb{E}[Z_0]\mathbb{E}^\infty\Big[\sum_{k\in\mathbb{Z}} k p_{0,x_k}^\lambda(\omega)\Big], \tag{10}$$

and the expectations in (9), (10) are finite and positive (recall that $r_0^\lambda(\omega) = \sum_k r_{0,k}^\lambda(\omega)$).

Proof. Theorem 2(i) is part of Proposition 5.3 given at the end of Section 5. The proof of Theorem 2(ii) is part of Section 7, more precisely (9) and (10) are an immediate consequence of (85), (94) and the observation just after (88). □

In the rest of the paper, if not stated otherwise, λ will be thought of as a fixed constant in $(0,1)$.

2.1. *Comments on assumptions* (A1)–(A4)

By Assumption (A1) both random sequences $(Z_k)_{k\in\mathbb{Z}}$ and $(E_k)_{k\in\mathbb{Z}}$ are stationary and ergodic with respect to shifts. The physically interesting case is given by two independent random sequences $(Z_k)_{k\in\mathbb{Z}}$ and $(E_k)_{k\in\mathbb{Z}}$, the former stationary and ergodic, while the latter given by i.i.d. random variables. In this case assumption (A1) is satisfied (see Lemma B.4 in Appendix B).

Assumption (A3) ensures that a.s. the environment is not periodic. If the energy marks E_k are i.i.d. and non-constant, as in the physically interesting case, then (A3) is automatically fulfilled. Note that the sequence $(Z_k)_{k\in\mathbb{Z}}$ could be periodic, without violating our assumptions (e.g. take $Z_k = 1$ for all $k \in \mathbb{Z}$).

Assumption (A4), corresponding to interpoint distances which are uniformly bounded from below, is not restrictive from a physical viewpoint and d can be taken of the angstrom order. On the other hand, (A4) plays a crucial role in our proofs.

2.2. *Examples*

In this section we give two examples highlighting the importance of the assumptions in Theorem 1 and showing some of its consequences.

Example 1. $\mathbb{E}[e^{(1-\lambda)Z_0}] = \infty$ *does in general not imply that* $v_{\mathbb{Y}}(\lambda) = 0$, $v_Y(\lambda) = 0$.

We set $u(\cdot,\cdot) \equiv 0$ and take $p \in (0, 1/2)$. We choose $(Z_k)_{k\in\mathbb{Z}}$ as the reversible Markov chain with values in $\{\gamma, 2\gamma, 3\gamma, \dots\}$ for some $\gamma \ge 1$ and with transition probabilities defined as follows:

$$\begin{cases} P(k\gamma, (k+1)\gamma) = p & \text{for } k \ge 1, \\ P(k\gamma, (k-1)\gamma) = 1-p & \text{for } k \ge 2, \\ P(\gamma,\gamma) = 1-p. \end{cases}$$

The equilibrium distribution is given by $\pi(k\gamma) = c(p/(1-p))^k$ for $k \ge 1$, c being the normalizing constant. Hence, $\mathbb{P}(Z_0 = k\gamma) = \pi(k\gamma)$, for each $k \ge 1$. Notice that $\mathbb{E}[e^{(1-\lambda)Z_0}] = c\sum_{k\ge 1} e^{(1-\lambda)k\gamma}(p/(1-p))^k$ is infinite if and only if

$$\lambda \leq 1 - \frac{1}{\gamma}\log\frac{1-p}{p}. \tag{11}$$

We now show that we can choose the parameters such that $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}] = \infty$ and $\sum_k k r^\lambda_{0,x_k}(\omega) > 0$ for each ω, the latter implying that $v_{\mathbb{Y}}(\lambda), v_Y(\lambda) > 0$ due to Theorem 2(ii) and the definition of $p^\lambda_{0,x_k}(\omega)$ in (4).

If $Z_0 = j\gamma$, for some $j \geq 1$, the local drift $\sum_k k r^\lambda_{0,x_k}(\omega)$ can be bounded from below by the drift of the configuration with longer and longer interpoint distances to the right and shorter and shorter interpoint distances to the left: $Z_k = (j+k)\gamma$ for all $k \geq -j+1$ and $Z_k = \gamma$ for all $k \leq -j$. Note that in this case

$$\begin{cases} x_k = \gamma[kj + \frac{k(k-1)}{2}] & \text{if } k \geq 1, \\ x_{-k} = -\gamma[kj - \frac{k(k+1)}{2}] & \text{if } 1 \leq k \leq j-1, \\ x_{-k} = -\gamma[\frac{j(j-1)}{2} + k - j + 1] & \text{if } k \geq j. \end{cases}$$

Hence we can write

$$\sum_k k r^\lambda_{0,x_k}(\omega) \geq A(\lambda,\gamma,j) - B(\lambda,\gamma,j) - C(\lambda,\gamma,j),$$

where $A(\lambda,\gamma,j) = \sum_{k\geq 1} k\exp\{-(1-\lambda)\gamma(kj + k(k-1)/2)\}$, $B(\lambda,\gamma,j) = \sum_{1\leq k\leq j-1} k\exp\{-(1+\lambda)\gamma(kj - k(k+1)/2)\}$ and $C(\lambda,\gamma,j) = \sum_{k\geq j} k\exp\{-(1+\lambda)\gamma(j(j-1)/2 + k - j + 1)\}$. We bound $A(\lambda,\gamma,j)$ from below with its first summand $\exp\{-(1-\lambda)\gamma j\}$ and prove that

$$\lim_{\gamma\to\infty}\sup_{j\in\mathbb{N}}\big[B(\lambda,\gamma,j) + C(\lambda,\gamma,j)\big]\exp\big\{(1-\lambda)\gamma j\big\} < 1, \tag{12}$$

since this will imply the positivity of the local drift $\sum_k k r^\lambda_{0,x_k}(\omega)$ for any possible ω, for γ big enough. Using that $Z_{-1} = \gamma(j-1)$ we bound $B(\lambda,\gamma,j) \leq j^2\exp\{-(1+\lambda)\gamma(j-1)\}$ and, using that $j(j-1)/2 + 1 \geq j/2$, we bound

$$C(\lambda,\gamma,j) \leq \mathrm{e}^{-\frac{(1+\lambda)}{2}\gamma j}\left(\sum_{k\geq j}(k-j)\mathrm{e}^{-(1+\lambda)\gamma(k-j)} + j\sum_{k\geq 0}\mathrm{e}^{-(1+\lambda)\gamma k}\right) \leq jK\mathrm{e}^{-\frac{(1+\lambda)}{2}\gamma j},$$

for some constant $K > 0$ independent of λ and γ (recall that $\gamma \geq 1$). With these bounds we see that (12) holds as soon as $\lambda > 1/3$. On the other hand, by (11) we can choose p close enough to $1/2$ so that $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}]$ is infinite.

Example 2. *The velocities $v_{\mathbb{Y}}(\lambda)$, $v_Y(\lambda)$ are not continuous in general.*

Take $u \equiv 0$. Let the Z_k be i.i.d. random variables larger than 1 such that e^{Z_0} has probability density $f(x) := \frac{c}{x^\gamma(\ln x)^2}\mathbb{1}_{[e,+\infty)}(x)$, with $\gamma \in (1,2)$ and c is the normalizing constant. Since, for $x \geq e$, $\frac{1}{x^\gamma(\ln x)^2} \leq \frac{1}{x(\ln x)^2} = -\frac{d}{dx}(1/\ln x)$, the constant c is well defined and $E[\mathrm{e}^{(1-\lambda)Z_0}] = \int_{\mathrm{e}}^\infty \frac{cx^{1-\lambda}}{x^\gamma(\ln x)^2}\,dx < \infty$ if and only if $\lambda \geq 2-\gamma$. Note that $\lambda_c := 2-\gamma \in (0,1)$. Then the above observations and Theorem 1 imply that $v_{\mathbb{Y}}(\lambda), v_Y(\lambda)$ are zero for $\lambda \in (0,\lambda_c)$ and are strictly positive for $\lambda \in [\lambda_c, 1)$.

3. A class of random walks on $\mathbb{Z}$ with jumps of size at most $\rho \in [1,+\infty]$

Given $i, j \in \mathbb{Z}$ we replace, with a slight abuse of notation, $r^\lambda_{i,j}(\omega) := r^\lambda_{x_i,x_j}(\omega)$ and the associated conductance $c_{i,j}(\omega) := \mathrm{e}^{2\lambda x_i} r^\lambda_{i,j}(\omega)$ (note that in $c_{i,j}(\omega)$ the dependence on λ has been omitted). Hence we have $c_{i,i} \equiv 0$ and

$$c_{i,j}(\omega) = \mathrm{e}^{\lambda(x_i+x_j) - |x_j - x_i| + u(E_i,E_j)} = c_{j,i}(\omega), \quad i \neq j \text{ in } \mathbb{Z}. \tag{13}$$

Given $\rho \in \mathbb{N}_+ \cup \{+\infty\}$ we introduce the discrete-time random walk $(X_n^\rho)_{n\geq 0}$ with environment ω as the Markov chain on $\mathbb{Z}$ such that the ω-dependent probability to jump from i to j in one step is given by

$$\begin{cases} c_{i,j}(\omega)/\sum_{k\in\mathbb{Z}} c_{i,k}(\omega), & \text{if } 0<|i-j|\leq \rho, \\ 0 & \text{if } |i-j|>\rho, \\ 1-\sum_{j:|j-i|\leq\rho} c_{i,j}(\omega)/\sum_{k\in\mathbb{Z}} c_{i,k}(\omega) & \text{if } i=j. \end{cases} \tag{14}$$

Warning 3.1. When the Markov chain $(X_n^\rho)_{n\geq 0}$ starts at $i\in\mathbb{Z}$, we write $P_i^{\omega,\rho}$ for its law and $E_i^{\omega,\rho}$ for the associated expectation. In order to make the notation lighter, inside $P_i^{\omega,\rho}(\cdot)$ and $E_i^{\omega,\rho}[\cdot]$ we will usually write X_n instead of X_n^ρ.

It is convenient to introduce the random bijection $\psi:\mathbb{Z}\to\{x_k\}_{k\in\mathbb{Z}}$ defined as $\psi(i)=x_i$, and also the continuous-time random walk $(\mathbb{X}_t^\infty)_{t\geq 0}$ on $\mathbb{Z}$ with probability rate $r_{i,j}^\lambda(\omega)$ for a jump from i to j. Since

$$\frac{c_{i,j}(\omega)}{\sum_{k\in\mathbb{Z}} c_{i,k}(\omega)} = \frac{r_{i,j}^\lambda(\omega)}{\sum_{k\in\mathbb{Z}} r_{i,k}^\lambda(\omega)},$$

we conclude that realizations of $\mathbb{Y}$ and Y can be obtained as

$$\mathbb{Y}_t = \psi\big(\mathbb{X}_t^\infty\big), \qquad Y_n = \psi\big(X_n^\infty\big). \tag{15}$$

In particular, when the denominators are nonzero, we can write

$$\frac{\mathbb{Y}_t}{t} = \frac{\psi(\mathbb{X}_t^\infty)}{\mathbb{X}_t^\infty}\frac{\mathbb{X}_t^\infty}{t}, \qquad \frac{Y_n}{n} = \frac{\psi(X_n^\infty)}{X_n^\infty}\frac{X_n^\infty}{n}. \tag{16}$$

By Assumptions (A1) and (A2), $\lim_{i\to\infty}\psi(i)/i = \mathbb{E}[Z_0]<\infty$, $\mathbb{P}$-a.s. By this limit, together with (15) and (16), we will see in Sections 7, 8 that in order to prove Theorem 1 it is enough to show the same properties for $\mathbb{X}^\infty$, X^∞ instead of $\mathbb{Y}$, Y.

In what follows, we write $v_{X^\rho}(\lambda)=v$ if, for $\mathbb{P}$-a.a. ω, $\lim_{n\to\infty}\frac{X_n^\rho}{n}=v$ $P_0^{\omega,\rho}$-a.s. A similar meaning is assigned to $v_{\mathbb{X}^\rho}(\lambda)$.

Remark 3.2. In the rest of this section we will present propositions, lemmas and corollaries containing several bounds with positive constants. These will be denoted by the letter K possibly with some subindex, which are uniform in ρ and ω. As the reader can easily check, these constants can be taken independent also from λ when λ varies e.g. in $[0,1/2)$. The same is true also for the positive constant ε appearing in Lemma 3.16. This observation could be very relevant in the derivation of the Einsten relation.

3.1. *Estimates on effective conductances*

We take again $\rho\in\mathbb{N}_+\cup\{+\infty\}$. For A, B disjoint subsets of $\mathbb{Z}$, we introduce the effective conductance between A and B as

$$C_{\mathrm{eff}}^\rho(A\leftrightarrow B) := \min\Biggl\{\sum_{i<j\in\mathbb{Z}:|i-j|\leq\rho} c_{i,j}\big(f(j)-f(i)\big)^2 : f|_A=0,\ f|_B=1\Biggr\}. \tag{17}$$

We also set

$$\pi^\rho(i) := \sum_{j\in\mathbb{Z}:|j-i|\leq\rho} c_{i,j}, \quad i\in\mathbb{Z}, \tag{18}$$

and define $p_{\mathrm{esc}}^\rho(i)$ as the escape probability of X_n^ρ from $i\in\mathbb{Z}$, i.e.

$$p_{\mathrm{esc}}^\rho(i) := P_i^{\omega,\rho}(X_n\neq i \text{ for all } n\geq 1). \tag{19}$$

It is known (see the discussion before Theorem 2.3 in [15, Section 2.2], formula (2.4) and exercise 2.13 therein) that

$$p^{\rho}_{\mathrm{esc}}(i) := \lim_{N\to\infty} \frac{C^{\rho}_{\mathrm{eff}}(i \leftrightarrow (-\infty, i-N] \cup [i+N, \infty))}{\pi^{\infty}(i)} \tag{20}$$

(recall that the probability for X^{ρ}_n to jump from i to j (for $0 < |i-j| \le \rho$) is given by $c_{i,j}/\pi^{\infty}(i)$, cf. (14)). We will see (cf. Corollary 3.5) that the escape probability of each ρ-random walk can be uniformly bounded from below and above by the escape probability of the nearest neighbor walk times constants.

Warning 3.3. Note that $C^{\rho}_{\mathrm{eff}}(A \leftrightarrow B)$, $\pi^{\rho}(i)$ and $p^{\rho}_{\mathrm{esc}}(i)$ depend on the environment ω, although we have omitted ω from the notation.

Proposition 3.4. *There exists a constant $K > 0$ not depending on ω, ρ, A and B such that*

$$C^{1}_{\mathrm{eff}}(A \leftrightarrow B) \le C^{\rho}_{\mathrm{eff}}(A \leftrightarrow B) \le K C^{1}_{\mathrm{eff}}(A \leftrightarrow B).$$

Proof. Since $C^{\rho}_{\mathrm{eff}}(A \leftrightarrow B)$ is increasing in ρ, it is enough to show the second inequality for $\rho = \infty$. To this aim take any valid $f : \mathbb{Z} \to \mathbb{R}$ and note that

$$\begin{aligned}
\sum_{i<j\in\mathbb{Z}} c_{i,j}\big(f(j)-f(i)\big)^2 &= \sum_{i<j\in\mathbb{Z}} c_{i,j}\left(\sum_{z=i}^{j-1} f(z+1)-f(z)\right)^2 \\
&\le \sum_{i<j\in\mathbb{Z}} c_{i,j}\cdot(j-i)\sum_{z=i}^{j-1}\big(f(z+1)-f(z)\big)^2 \\
&= \sum_{z\in\mathbb{Z}}\big(f(z+1)-f(z)\big)^2\left[\sum_{i\le z}\sum_{j\ge z+1} c_{i,j}\cdot(j-i)\right],
\end{aligned} \tag{21}$$

where we have used the Cauchy–Schwarz inequality for the second step. Define the new conductances

$$\bar{c}_{z,z+1} = \sum_{i\le z}\sum_{j\ge z+1} c_{i,j}\cdot(j-i).$$

Now we are left to show that $\bar{c}_{z,z+1} \le K c_{z,z+1}$ for some K and this will conclude the proof. Using the fact that $\forall k > k'$ we have $x_k - x_{k'} > d(k-k')$, we have

$$\begin{aligned}
\bar{c}_{z,z+1} &= \sum_{i\le z}\sum_{j\ge z+1} \mathrm{e}^{\lambda(x_i+x_j)-(x_j-x_i)+u(E_i,E_j)}(j-i) \\
&\le \mathrm{e}^{\lambda(x_z+x_{z+1})-(x_{z+1}-x_z)+u_{\max}}\sum_{i\le z}\sum_{j\ge z+1} \mathrm{e}^{-(x_j-x_{z+1})(1-\lambda)}\mathrm{e}^{-(x_z-x_i)(1+\lambda)}(j-i) \\
&\le c_{z,z+1}\mathrm{e}^{u_{\max}-u_{\min}}\sum_{i\le z}\sum_{j\ge z+1} \mathrm{e}^{-d(j-z-1)(1-\lambda)}\mathrm{e}^{-d(z-i)(1+\lambda)}(j-i) \\
&\le c_{z,z+1}\mathrm{e}^{u_{\max}-u_{\min}}\sum_{l=0}^{\infty}\sum_{h=0}^{\infty}\mathrm{e}^{-d(1-\lambda)h}\mathrm{e}^{-d(1+\lambda)l}(h+l+1) =: c_{z,z+1}K.
\end{aligned}$$

Since the last double sum is bounded for each $\lambda \in [0,1)$, we obtain the claim. □

As a byproduct of (20) and Proposition 3.4 we get:

Corollary 3.5. *There exist constants $K_1, K_2 > 0$ which do not depend on ω, ρ such that*

$$K_1 p^1_{\text{esc}}(i) \le p^\rho_{\text{esc}}(i) \le K_2 p^1_{\text{esc}}(i), \quad \forall i \in \mathbb{Z}.$$

Lemma 3.6. *There exists a constant $K > 0$ which does not depend on ω, ρ such that*

$$\pi^1(k) \le \pi^\rho(k) \le K\pi^1(k), \quad \forall k \in \mathbb{Z}.$$

Proof. Since $\pi^\rho(k)$ is increasing in ρ it is enough to prove that $\pi^\infty(k) \le K\pi^1(k)$ for all $k \in \mathbb{Z}$. We easily see that

$$\begin{aligned}\sum_{j>k} c_{k,j} &= \mathrm{e}^{\lambda(x_{k+1}+x_k)-(x_{k+1}-x_k)} \sum_{j>k} \mathrm{e}^{-(1-\lambda)(x_j-x_{k+1})+u(E_j,E_k)} \\ &\le c_{k,k+1}\mathrm{e}^{u_{\max}-u_{\min}} \sum_{j>k} \mathrm{e}^{-d(j-k-1)(1-\lambda)} =: c_{k,k+1}K_1. \end{aligned} \tag{22}$$

Analogously,

$$\begin{aligned}\sum_{j<k} c_{k,j} &= \mathrm{e}^{\lambda(x_{k-1}+x_k)-(x_k-x_{k-1})} \sum_{j<k} \mathrm{e}^{-(x_{k-1}-x_j)(1-\lambda)+u(E_j,E_k)} \\ &\le c_{k-1,k}\mathrm{e}^{u_{\max}-u_{\min}} \sum_{j>k} \mathrm{e}^{-d(k-1-j)(1-\lambda)} =: c_{k-1,k}K_2. \end{aligned} \tag{23}$$

□

The following lemma is well known and corresponds to formula (2.1.4) in [24]:

Lemma 3.7. *Let $\{\bar{c}_{k,k+1}\}_{k\in\mathbb{Z}}$ be any system of strictly positive conductances on the nearest neighbor bonds of $\mathbb{Z}$. Let H_A be the first hitting time of the set $A \subset \mathbb{Z}$ for the associated discrete-time nearest-neighbor random walk among the conductances $\{\bar{c}_{k,k+1}\}_{k\in\mathbb{Z}}$, which jumps from k to $k \pm 1$ with probability $\bar{c}_{k,k\pm1}/(\bar{c}_{k,k-1} + \bar{c}_{k,k+1})$. Take $-\infty < M < x < N < \infty$, with $M, x, N \in \mathbb{Z}$ and write H_M, H_N for $H_{\{M\}}$, $H_{\{N\}}$. Then*

$$P^{\text{n.n.}}_x(H_M < H_N) = \frac{C^{\text{n.n.}}_{\text{eff}}(x \leftrightarrow (-\infty, M])}{C^{\text{n.n.}}_{\text{eff}}(x \leftrightarrow (-\infty, M] \cup [N, \infty))},$$

where $P^{\text{n.n.}}_x$ is the probability for the nearest-neighbor random walk starting at x, and $C^{\text{n.n.}}_{\text{eff}}(A \leftrightarrow B)$ is the effective conductance of the nearest-neighbor walk between A and B.

We state another technical lemma which will be frequently used when dealing with conductances:

Lemma 3.8. $\mathbb{P}(\sum_{j=0}^\infty \frac{1}{c_{j,j+1}} < +\infty) = 1.$

Proof. By assumption (A1), $(x_{j+1} - x_j)_{j\in\mathbb{Z}}$ is a stationary ergodic sequence. By writing $x_j = \sum_{k=0}^{j-1}(x_{k+1} - x_k)$, the ergodic theorem implies that $\lim_{j\to\infty} \frac{x_j}{j} = \mathbb{E}[x_1]$, $\mathbb{P}$-a.s. As a consequence we get that

$$\lim_{j\to\infty} \frac{-\lambda(x_j + x_{j+1}) + (x_{j+1} - x_j)}{j} = -2\lambda\mathbb{E}[x_1] < 0, \quad \mathbb{P}\text{-a.s.}$$

Since $\sum_{j=0}^\infty \frac{1}{c_{j,j+1}} = \sum_{j=0}^\infty \mathrm{e}^{-\lambda(x_j+x_{j+1})+(x_{j+1}-x_j)}$, the claim follows. □

We conclude this section with a simple estimate leading to an exponential decay of the transition probabilities:

Lemma 3.9. *There exists a constant K which does not depend on ω, ρ, such that $\mathbb{P}$-a.s.*

$$P^{\omega,\rho}_i\big(|X_1 - i| > s\big) \le \sum_{j:|j-i|>s} \frac{c_{i,j}}{\sum_{k\in\mathbb{Z}} c_{i,k}} \le K\mathrm{e}^{-ds(1-\lambda)} \quad \forall s, \rho \in \mathbb{N}_+ \cup \{+\infty\}, \forall i \in \mathbb{Z}. \tag{24}$$

Proof. The first inequality follows from the definitions. To prove the second one, we can estimate

$$\begin{aligned}\sum_{j>i+s} c_{i,j} &= \mathrm{e}^{\lambda(x_{i+1}+x_i)-(x_{i+1}-x_i)} \sum_{j>i+s} \mathrm{e}^{-(x_j-x_{i+1})(1-\lambda)+u(E_j,E_i)}\\ &\le c_{i,i+1}\mathrm{e}^{u_{\max}-u_{\min}} \sum_{j>i+s} \mathrm{e}^{-d(j-i-1)(1-\lambda)}\\ &= c_{i,i+1}\mathrm{e}^{-ds(1-\lambda)} \frac{\mathrm{e}^{u_{\max}-u_{\min}}}{1-\mathrm{e}^{-d(1-\lambda)}} =: c_{i,i+1}\mathrm{e}^{-ds(1-\lambda)} K_1, \qquad (25)\\ \sum_{j<i-s} c_{i,j} &= \mathrm{e}^{\lambda(x_{i-1}+x_i)-(x_i-x_{i-1})} \sum_{j<i-s} \mathrm{e}^{-(x_{i-1}-x_j)(1-\lambda)+u(E_j,E_i)}\\ &\le c_{i-1,i}\mathrm{e}^{u_{\max}-u_{\min}} \sum_{j<i-s} \mathrm{e}^{-d(i-1-j)(1-\lambda)} = c_{i-1,i}\mathrm{e}^{-ds(1-\lambda)} K_1. \qquad (26)\end{aligned}$$

The second bound in (24) now follows from (25), (26) and Lemma 3.6. □

3.2. *Expected number of visits*

We fix some notations which will be frequently used below. For $I \subseteq \{0, 1, 2, \dots\}$ and $A \subset \mathbb{Z}$, we denote by $N_I^\rho(A)$ the time spent by the random walk X_n^ρ in the set A during the time interval I:

$$N_I^\rho(A) := \sum_{k\in I} \mathbb{1}_{X_k^\rho \in A}.$$

If $I := \{0, 1, 2, \dots\}$ we simply write $N_\infty^\rho(A)$ and if $A = \{x\}$ we write $N_\infty^\rho(x)$.

Warning 3.10. When appearing inside $\mathbb{P}^{\omega,\rho}(\cdot)$ or $\mathbb{E}^{\omega,\rho}(\cdot)$, $N_I(A)$, $N_\infty(A)$ will usually replace $N_I^\rho(A)$, $N_\infty^\rho(A)$.

We can state our main result on the expected number of visits to a site k for a given environment:

Proposition 3.11. *There exists a constant K_0, not depending on ρ, ω, such that the function $g_\omega : \{0, 1, \dots\} \to \mathbb{R}_+$, defined as*

$$g_\omega(n) := K_0\pi^1(-n)\sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_j+(1-\lambda)(x_{j+1}-x_j)}, \quad n \ge 0, \tag{27}$$

satisfies

$$E_0^{\omega,\rho}\big[N_\infty(k)\big] \le g_\omega\big(|k|\big), \quad \forall k \le 0. \tag{28}$$

We recall that $\pi^1(k) = c_{k-1,k} + c_{k,k+1}$ for all $k \in \mathbb{Z}$. Moreover, we point out that $g_\omega(n)$ can be rewritten as $K_0\pi^1(-n)\sum_{j=0}^{\infty} \frac{1}{c_{j,j+1}}$, therefore it is finite $\mathbb{P}$-a.s. by Lemma 3.8. We remark that estimate (28) is not uniform in the environment ω, and in general one cannot expect a uniform bound. This technical fact represents a major difference with the setting of [8], where the existence of an ω-independent upper bound of the expected number of visits is required (cf. Condition D therein).

Proof. During the proof K will denote a generic positive constant, not depending on ρ, ω, whose value may change from line to line.

Fix $k < 0$. Starting from 0, the random variable $N_\infty^\rho(k)$ is equal to

$$N_\infty^\rho(k) = \begin{cases} 0 & \text{with probability } 1 - P_0^{\omega,\rho}(X_\cdot \text{ eventually reaches } k),\\ Y(k) & \text{with probability } P_0^{\omega,\rho}(X_\cdot \text{ eventually reaches } k),\end{cases}$$

where $Y(k)$ is a geometric random variable whose parameter is the escape probability $p^{\rho}_{\text{esc}}(k)$ from k (recall Warning 3.3). Therefore

$$E^{\omega,\rho}_0\big[N_\infty(k)\big] = \frac{1}{p^{\rho}_{\text{esc}}(k)} P^{\omega,\rho}_0(X.\text{ eventually reaches } k). \tag{29}$$

Let us start by giving an upper bound for the probability of reaching k in finite time:

$$\begin{aligned} P^{\omega,\rho}_0(X.\text{ eventually reaches } k) &\le P^{\omega,\rho}_0\big(X.\text{ eventually reaches } A := (-\infty,k]\big) \\ &= \lim_{N\to\infty} P^{\omega,\rho}_0(H_{B_N} > H_A), \end{aligned} \tag{30}$$

where $B_N := [N,\infty)$ and the H's are the hitting times of the respective sets. By a well-known formula (see [3, Proof of Fact 2], [15, Exercise 2.36])

$$P^{\omega,\rho}_0(H_{B_N} > H_A) \le \frac{C^{\rho}_{\text{eff}}(0 \leftrightarrow A)}{C^{\rho}_{\text{eff}}(0 \leftrightarrow A \cup B_N)}. \tag{31}$$

Using now Proposition 3.4 we have that there exists a K such that

$$\begin{aligned} P^{\omega,\rho}_0(X.\text{ eventually reaches } k) &\le \lim_{N\to\infty} K \frac{C^{1}_{\text{eff}}(0 \leftrightarrow A)}{C^{1}_{\text{eff}}(0 \leftrightarrow A \cup B_N)} \\ &= K \frac{C^{1}_{\text{eff}}(0 \leftrightarrow A)}{C^{1}_{\text{eff}}(0 \leftrightarrow A \cup B_\infty)}, \end{aligned} \tag{32}$$

where $C^1_{\text{eff}}(0 \leftrightarrow A \cup B_\infty) := \lim_{N\to\infty} C^1_{\text{eff}}(0 \leftrightarrow A \cup B_N)$.

Call $C_N := (-\infty, -N+k] \cup [N+k,\infty)$. By Corollary 3.5 and equation (20), we know that

$$\begin{aligned} p^{\rho}_{\text{esc}}(k) &\ge \frac{1}{K} \lim_{N\to\infty} \frac{C^1_{\text{eff}}(k \leftrightarrow C_N)}{\pi^1(k)} \\ &= \frac{1}{K} \frac{C^1_{\text{eff}}(k \leftrightarrow C_\infty)}{\pi^1(k)}, \end{aligned} \tag{33}$$

where $C^1_{\text{eff}}(k \leftrightarrow C_\infty) := \lim_{N\to\infty} C^1_{\text{eff}}(k \leftrightarrow C_N)$.

Since we have conductances in series, we can write

$$C^1_{\text{eff}}(k \leftrightarrow C_\infty) = \left(\sum_{j=-\infty}^{k-1} \frac{1}{c_{j,j+1}}\right)^{-1} + \left(\sum_{j=k}^{\infty} \frac{1}{c_{j,j+1}}\right)^{-1}. \tag{34}$$

We claim that

$$\begin{aligned} &\sum_{j=-\infty}^{k-1} \frac{1}{c_{j,j+1}} = +\infty, \\ &\sum_{j=k}^{\infty} \frac{1}{c_{j,j+1}} < +\infty \quad \mathbb{P}\text{-a.s.} \end{aligned} \tag{35}$$

Indeed, the first series diverges a.s. since, for $j \le -1$, $1/c_{j,j+1} \ge K e^{-\lambda(x_j+x_{j+1})+(x_{j+1}-x_j)} \ge K$ (note that $x_j, x_{j+1} \le 0$). The second series is finite a.s. due to Lemma 3.8.

Due to (29), (32), (33), (34) and (35) we can write

$$
\begin{aligned}
E_0^{\omega,\rho}\big[N_\infty(k)\big] &\le K\frac{\pi^1(k)}{C^1_{\mathrm{eff}}(k\leftrightarrow C_\infty)}\cdot\frac{C^1_{\mathrm{eff}}(0\leftrightarrow A)}{C^1_{\mathrm{eff}}(0\leftrightarrow A\cup B_\infty)}\\
&= K\frac{\pi^1(k)}{(\sum_{j=-\infty}^{k-1}\frac{1}{c_{j,j+1}})^{-1}+(\sum_{j=k}^{\infty}\frac{1}{c_{j,j+1}})^{-1}}\cdot\frac{(\sum_{j=k}^{-1}\frac{1}{c_{j,j+1}})^{-1}}{(\sum_{j=k}^{-1}\frac{1}{c_{j,j+1}})^{-1}+(\sum_{j=0}^{\infty}\frac{1}{c_{j,j+1}})^{-1}}\\
&= K\pi^1(k)\Bigg(\sum_{j=0}^{\infty}\frac{1}{c_{j,j+1}}\Bigg)\le K_0\pi^1(k)\sum_{j=0}^{\infty}\mathrm{e}^{-\lambda(x_j+x_{j+1})+(x_{j+1}-x_j)}\le g_\omega\big(|k|\big).
\end{aligned}\tag{36}
$$

We now consider the case $k=0$. By (29), (33), (34) and (35) we have

$$
\begin{aligned}
E_0^{\omega,\rho}\big[N_\infty(0)\big] &= \frac{1}{p^{\rho}_{\mathrm{esc}}(0)}\\
&\le K\frac{\pi^1(0)}{C^1_{\mathrm{eff}}(0\leftrightarrow C_\infty)}=K\pi^1(0)\sum_{j=0}^{\infty}\frac{1}{c_{j,j+1}}
\end{aligned}
$$

and we can conclude as in (36). □

We now collect some properties of the function g_ω:

Lemma 3.12. *There exist constants $K_*>0$ which do not depend on ρ, ω, such that*

$$\pi^1(k)\le K_*\mathrm{e}^{2\lambda dk},\quad \forall k\le 0,\tag{37}$$

$$\mathbb{E}\big[g_\omega(k)\big]\le K_*\frac{\mathrm{e}^{-2\lambda dk}}{1-\mathrm{e}^{-2\lambda d}}\mathbb{E}\big[\mathrm{e}^{(1-\lambda)x_1}\big],\quad \forall k\ge 0,\tag{38}$$

$$g_\omega(k)\ge g_{\tau_\ell\omega}(k+\ell),\quad \forall k,\ell\ge 0,\tag{39}$$

$$\mathbb{E}E_k^{\omega,\rho}\big[N_\infty(\mathbb{Z}_-)\big]\le K_*\bigg(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2}+\frac{|k|}{1-\mathrm{e}^{-2\lambda d}}\bigg)\mathbb{E}\big[\mathrm{e}^{(1-\lambda)x_1}\big],\quad \forall k\le 0.\tag{40}$$

Trivially, the second and fourth estimates are effective when $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}]<\infty$.

Proof. We first prove (37). Recall $\pi^1(k)=c_{k-1,k}+c_{k,k+1}$. Given $i\le 0$ we have $x_i\le id$, implying $c_{i-1,i}\le \mathrm{e}^{u_{\max}}\mathrm{e}^{\lambda(x_{i-1}+x_i)-(x_i-x_{i-1})}\le K\mathrm{e}^{2\lambda di}$. By the same argument, for $i<0$ one gets $c_{i,i+1}\le K\mathrm{e}^{2\lambda di}$ and, for $i=0$, $c_{0,1}=\mathrm{e}^{\lambda x_1-x_1+u(E_0,E_1)}\le K$.

(38) is obtained noting that, by (37), $\mathbb{E}[g_\omega(k)]\le K_*\mathrm{e}^{-2\lambda dk}\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda jd}\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}]$.

To get (39) we first observe that $x_{i-\ell}(\tau_\ell\omega)=x_i(\omega)-x_\ell(\omega)$ and $E_{i-\ell}(\tau_\ell\omega)=E_i(\omega)$ for all $i\in\mathbb{Z}$. As a consequence, we get $\pi^1(-k-\ell)[\tau_\ell\omega]=\mathrm{e}^{-2\lambda x_\ell}\pi^1(-k)$ (the r.h.s. refers to the environment ω). Therefore, using also that $x_j(\tau_\ell\omega)=x_{j+\ell}(\omega)-x_\ell(\omega)$ and that $x_{j+1}(\tau_\ell\omega)-x_j(\tau_\ell\omega)=x_{j+1+\ell}(\omega)-x_{j+\ell}(\omega)$, we have

$$
\begin{aligned}
g_{\tau_\ell\omega}(k+\ell) &= K_0\pi^1(-k)\mathrm{e}^{-2\lambda x_\ell(\omega)}\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda x_j(\tau_\ell\omega)+(1-\lambda)(x_{j+1}(\tau_\ell\omega)-x_j(\tau_\ell\omega))}\\
&= K_0\pi^1(-k)\sum_{j=0}^{\infty}\mathrm{e}^{-2\lambda x_{j+\ell}+(1-\lambda)(x_{j+1+\ell}-x_{j+\ell})}\le g_\omega(k),
\end{aligned}\tag{41}
$$

thus completing the proof of (39).

Finally, for (40), we write, thanks to Proposition 3.11,

$$\begin{aligned}\mathbb{E}E_k^{\omega,\rho}\big[N_\infty(\mathbb{Z}_-)\big] &= \sum_{z\le k}\mathbb{E}E_k^{\omega,\rho}\big[N_\infty(z)\big] + \sum_{k<z\le 0}\mathbb{E}E_k^{\omega,\rho}\big[N_\infty(z)\big]\\ &\le \sum_{z\le k}\mathbb{E}E_k^{\omega,\rho}\big[N_\infty(z)\big] + \sum_{k<z\le 0}\mathbb{E}E_z^{\omega,\rho}\big[N_\infty(z)\big] \quad \text{(Markov Property)}\\ &\le \sum_{i\ge 0}\mathbb{E}\big[g_{\tau_k\omega}(i)\big] + |k|\mathbb{E}\big[g_\omega(0)\big],\end{aligned}$$

and the claim then follows from (38). □

Remark 3.13. In the spirit of Remark 2.4, we point out that we could consider weaker conditions than $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}] < +\infty$, at the cost of dealing with rather involved formulas. Take for simplicity $u \equiv 0$. In our case, $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}] < +\infty$ guarantees, by Lemma 3.12, that $\mathbb{E}[g_\omega(k)]$ is finite and summable over $k \ge 0$. But what is actually required is that $g_\omega(k)$ bounds from above the quantity $\alpha_\omega(k) := K\pi^1(-k)\sum_{j\ge 0}\frac{1}{c_{j,j+1}}$ (see the proof of Proposition 3.11). By stationarity, one has

$$\mathbb{E}[c_{k,k+1}/c_{k+i,k+i+1}] = \mathbb{E}\big[\mathrm{e}^{-(1+\lambda)Z_0 - 2\lambda(Z_1+\cdots+Z_{i-1}) + (1-\lambda)Z_i}\big].$$

This identity allows to provide conditions for $\sum_{k\ge 0}\mathbb{E}[\alpha_\omega(k)]$ to be finite, which are weaker than $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}] < +\infty$. One could go on in weakening conditions, also inside Proposition 5.4, and still get the ballisticity of the Mott random walks $\mathbb{Y}_t$ and Y_n.

Corollary 3.14. *There exist constants $K_1, K_2 > 0$ which do not depend on ρ, ω such that*

$$E_0^{\omega,\rho}\big[N_\infty(k)\big] \le K_1 E_0^{\omega,1}\big[N_\infty(k)\big] \le K_1 g_\omega\big(|k|\big) \quad \forall k \le 0, \tag{42}$$

$$E_0^{\omega,\rho}\big[N_\infty(k)\big] \le K_2 E_0^{\omega,1}\big[N_\infty(k)\big] \quad \forall k > 0. \tag{43}$$

Proof. First we consider (42). Its second inequality is a restatement of Proposition 3.11. For the first inequality we distinguish the cases $k < 0$ and $k = 0$. When $k < 0$ note that (32) and Lemma 3.7 imply that

$$P_0^{\omega,\rho}(X_\cdot \text{ eventually reaches } k) \le K P_0^{\omega,1}(X_\cdot \text{ eventually reaches } k). \tag{44}$$

Then put together equation (29) (and its analogous version for $\rho = 1$), equation (44) and Corollary 3.5. For $k = 0$ use that $E_0^{\omega,\rho}[N_\infty(0)] = \frac{1}{p_{\mathrm{esc}}^\rho(0)}$ (also in the case $\rho = 1$) and use Corollary 3.5.

Let us now consider equation (43). Start with (29). Due to Corollary 3.5 and the fact that $P_0^{\omega,1}(X_\cdot$ eventually reaches $k) = 1$ for each $k > 0$ (cf. Lemma 3.15 below) it is simple to conclude. □

3.3. *Probability to hit a site on the right*

Following [8], given $k, z \in \mathbb{Z}$, we set

$$T_z^\rho := \inf\big\{n \ge 0 : X_n^\rho \ge z\big\}, \qquad T^\rho := T_0^\rho, \qquad r_k^\rho(z) := P_k^{\omega,\rho}(X_{T_z} = z).$$

Note that the dependence of ω has been omitted. Again (see Warnings 3.1 and 3.10), we simply write T_z, $r_k(z)$ inside $P_k^{\omega,\rho}(\cdot)$, $E_k^{\omega,\rho}(\cdot)$.

Lemma 3.15. *For $\mathbb{P}$-a.a. ω and for each $\rho \in \mathbb{N}_+ \cup \{\infty\}$ it holds that*

$$P_k^{\omega,\rho}(T_z < \infty) = 1 \quad \forall k < z \text{ in } \mathbb{Z}.$$

Proof. Without loss of generality we take $k < 0 =: z$ and prove that $P_k^{\omega,\rho}(T_0 = \infty) = 0$. As in (31), setting $C_N := (-\infty, -N]$ and $D = [0, \infty)$, we can bound

$$P_k^{\omega,\rho}(T_0 = \infty) = \lim_{N\to\infty} \mathbb{P}_k^{\omega,\rho}(\tau_{C_N} < \tau_D) \le \liminf_{N\to\infty} \frac{C_{\text{eff}}^{\rho}(k \leftrightarrow C_N)}{C_{\text{eff}}^{\rho}(k \leftrightarrow C_N \cup D)}.$$

We observe that $C_{\text{eff}}^1(k \leftrightarrow C_N \cup D) = C_{\text{eff}}^1(k \leftrightarrow C_N) + C_{\text{eff}}^1(k \leftrightarrow D)$, while (recall (35))

$$\lim_{N\to\infty} C_{\text{eff}}^1(k \leftrightarrow C_N) = \left(\sum_{j=-\infty}^{k-1} \frac{1}{c_{j,j+1}}\right)^{-1} = 0, \qquad C_{\text{eff}}^1(k \leftrightarrow D) = \left(\sum_{j=k}^{-1} \frac{1}{c_{j,j+1}}\right)^{-1} > 0.$$

Together with Proposition 3.4, this allows to conclude that $P_k^{\omega,\rho}(T_0 = \infty) = 0$. □

Our next result, Lemma 3.16, is the analog of Lemma 3.1 in [8]. Our proof follows a different strategy in order to avoid to deal with Conditions D, E of [8], which are not satisfied in our context.

Lemma 3.16. *There exists $\varepsilon > 0$ which does not depend on ρ, ω such that, $\mathbb{P}$-a.s., $r_k^{\rho}(0) \ge 2\varepsilon$ for all $k < 0$ and for all $\rho \in \mathbb{N}_+ \cup \{\infty\}$.*

Proof. We just make a pathwise analysis. By the Markov property we get

$$\begin{aligned} r_k^{\rho}(0) &= \sum_{-\rho \le j < 0} \sum_{n=1}^{\infty} P_k^{\omega,\rho}(X_n = 0, X_{n-1} = j, X_0, X_1, \dots, X_{n-2} < 0) \\ &= \sum_{-\rho \le j < 0} \sum_{n=1}^{\infty} P_j^{\omega,\rho}(X_1 = 0) P_k^{\omega,\rho}(X_{n-1} = j, X_0, X_1, \dots, X_{n-2} < 0). \end{aligned} \tag{45}$$

We claim that there exists $\varepsilon > 0$ such that, for all j and ω,

$$P_j^{\omega,\rho}(X_1 = 0) \ge 2\varepsilon P_j^{\omega,\rho}(X_1 \ge 0).$$

Indeed, given j with $-\rho \le j < 0$, we can write

$$\begin{aligned} \frac{P_j^{\omega,\rho}(X_1 = 0)}{P_j^{\omega,\rho}(X_1 \ge 0)} &\ge \frac{c_{j,0}}{\sum_{l=0}^{\infty} c_{j,l}} \ge K \frac{e^{\lambda x_j + x_j}}{\sum_{l=0}^{\infty} e^{\lambda(x_l + x_j) - (x_l - x_j)}} = K \frac{1}{\sum_{l=0}^{\infty} e^{-(1-\lambda)x_l}} \\ &\ge K \frac{1}{\sum_{l=0}^{\infty} e^{-(1-\lambda)dl}} =: 2\varepsilon. \end{aligned}$$

Coming back to (45), using the Markov property and the fact that $T_0 < \infty$ a.s., we get

$$\begin{aligned} r_k^{\rho}(0) &\ge 2\varepsilon \sum_{-\rho \le j < 0} \sum_{n=1}^{\infty} P_j^{\omega,\rho}(X_1 \ge 0) P_k^{\omega,\rho}(X_{n-1} = j, X_0, X_1, \dots, X_{n-2} < 0) \\ &= 2\varepsilon \sum_{-\rho \le j < 0} \sum_{n=1}^{\infty} P_k^{\omega,\rho}(X_n \ge 0, X_{n-1} = j, X_0, X_1, \dots, X_{n-2} < 0) \\ &= 2\varepsilon P_k^{\omega,\rho}(X_{T_0} \ge 0) = 2\varepsilon. \end{aligned}$$

□

4. Regenerative structure for the ρ-truncated random walk with $\rho < \infty$

In this section we take $\rho < \infty$. We recall the regenerative structure of [8] for the ρ-truncated random walk with ρ finite.

Warning 4.1. In order to avoid heavy notation, in this section ρ is fixed once and for all in $\mathbb{N}_+$ and we write P_x^ω, T_k, $r_k(z)$, $X_n, \dots$ instead of $P_x^{\omega,\rho}$, T_k^ρ, $r_k^\rho(z)$, $X_n^\rho, \dots$. The whole section refers to the ρ-truncated random walk. Only in Section 4.2, in which we collect the main conclusions, we will indicate ρ explicitly according to the usual notation.

Consider a sequence of i.i.d. Bernoulli r.v.'s $\zeta_1, \zeta_2, \dots$ with parameter $P(\zeta_1 = 1) = \varepsilon$ (the same ε as in Lemma 3.16) which does not depend on the environment ω. P and E denote the probability law and the expectation of the ζ's. We couple the sequence $\zeta = (\zeta_1, \zeta_2, \dots)$ with the random walk X_n in such a way that $\zeta_j = 1$ implies $X_{T_{j\rho}} = j\rho$.

To this aim we construct the quenched measure $P_0^{\omega,\zeta}$ of the random walk starting at 0 once both ω and ζ are fixed. Recall Lemma 3.16. First, the law of $(X_n)_{n \le T_\rho}$ is defined by

$$\mathbb{1}_{\{\zeta_1=1\}} P_0^\omega(\cdot|X_{T_\rho} = \rho) + \mathbb{1}_{\{\zeta_1=0\}}\left[\frac{r_0(\rho)-\varepsilon}{1-\varepsilon} P_0^\omega(\cdot|X_{T_\rho} = \rho) + \frac{1-r_0(\rho)}{1-\varepsilon} P_0^\omega(\cdot|X_{T_\rho} > \rho)\right]. \tag{46}$$

Then, given $j \ge 1$ and $X_{T_{j\rho}} = y \in [j\rho, (j+1)\rho)$, the law of $(X_{T_{j\rho}+n})_{n\in[0,T_{(j+1)\rho}-T_{j\rho}]}$ is

$$\begin{aligned}&\mathbb{1}_{\{\zeta_{j+1}=1\}} P_y^\omega\big(\cdot|X_{T_{(j+1)\rho}} = (j+1)\rho\big) + \mathbb{1}_{\{\zeta_{j+1}=0\}}\left[\frac{r_y((j+1)\rho)-\varepsilon}{1-\varepsilon} P_y^\omega\big(\cdot|X_{T_{(j+1)\rho}} = (j+1)\rho\big)\right.\\ &\quad\left. + \frac{1-r_y((j+1)\rho)}{1-\varepsilon} P_y^\omega\big(\cdot|X_{T_{(j+1)\rho}} > (j+1)\rho\big)\right].\end{aligned} \tag{47}$$

One can check that, by averaging $P_0^{\omega,\zeta}$ over ζ, one obtains the law P_0^ω of the original random walk $(X_n)_{n\ge0}$.

We introduce by iteration the sequence $(\ell_k)_{k\ge0}$ as follows:

$$\ell_0 := 0, \qquad \ell_{k+1} := \min\{j > \ell_k : \zeta_j = 1\}, \quad k \ge 0.$$

Note that by construction we have $X_{T_{\ell_k\rho}} = \ell_k\rho$.

Given $k \ge 0$ let $\mathcal{C}_k := (\tau_{X_{j+T_{\ell_k\rho}}}\omega : 0 \le j < T_{\ell_{k+1}\rho} - T_{\ell_k\rho})$. As in [8] one can prove the following result (cf. [8, Lemma 3.2] and the corresponding proof):

Lemma 4.2. *Let $\rho < \infty$. Then the sequence of random pieces $(\mathcal{C}_k)_{k\ge0}$ is stationary and ergodic under the measure $P \otimes \mathbb{P} \otimes P_0^{\omega,\zeta}$. In particular, $\tau_{\ell_k\rho}\omega$ has the same law $\mathbb{P}$ as ω for all $k = 1, 2, \dots$.*

As in [22], the fact that $(\mathcal{C}_k)_{k\ge1}$ is stationary and ergodic can be restated as follows: under $P \otimes \mathbb{P} \otimes P_0^{\omega,\zeta}$ the random path $(X_n)_{n\ge0}$ with time points $0 < T_{\ell_1\rho} < T_{\ell_2\rho} < \cdots$ is *cycle-stationary and ergodic*. This is the regenerative structure pointed out in [8].

In what follows, we will consider also the random walk $(X_n)_{n\ge0}$ starting at x and with law $P_x^{\omega,\zeta}$. This random walk is built as follows. Fix a such that $x \in [a\rho, (a+1)\rho)$. Then, the law of $(X_n)_{n\le T_{(a+1)\rho}}$ is defined by (4) with j replaced by a and y replaced by x. Note that $T_{a\rho} = 0$. Given $j \ge a+1$ and $X_{T_{j\rho}} = y \in [j\rho, (j+1)\rho)$, the law of $(X_n)_{n\in[T_{j\rho}+1,T_{(j+1)\rho}]}$ is then given by (4). Again, the average over ζ of $P_x^{\omega,\zeta}$ gives P_x^ω.

4.1. *Estimates on the regeneration times*

As in [8] we set

$$\mathbb{P}' := P \otimes \mathbb{P}, \qquad \mathbb{E}'[\cdot] = E\big[\mathbb{E}[\cdot]\big].$$

In what follows we assume that $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}] < \infty$.

Lemma 4.3. *Let $\rho < \infty$. There exist constants $K_1, K_2 > 0$ not depending on ω, ρ such that*

$$E\big[E_0^{\omega,\zeta}[T_{\ell_1\rho}]\big] \geq K_1\rho, \tag{48}$$

$$\mathbb{E}'\big[E_0^{\omega,\zeta}[T_{\ell_1\rho}]\big] \leq K_2\left(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2}+\frac{\rho}{1-\mathrm{e}^{-2\lambda d}}\right)\mathbb{E}\big[\mathrm{e}^{(1-\lambda)x_1}\big]. \tag{49}$$

Proof. The proof of (48) is very similar to the derivation of the first inequality of (19) in [8]. We make some comments. One first gets that

$$E\big[E_0^{\omega,\zeta}[T_{\ell_1\rho}]\big] \geq E\big[E_0^{\omega,\zeta}[T_\rho]\big] \geq \varepsilon E_0^{\omega}[T_\rho], \tag{50}$$

since, arguing as in [8], one derives from Lemma 3.16 that $E_0^{\omega,\zeta}[T_\rho] \geq \frac{\varepsilon}{1-\varepsilon}E_0^{\omega}[T_\rho]$ on the event $\{\zeta_1 = 0\}$. Now take a sequence $Y_1, Y_2, \ldots$ of i.i.d. positive random variables with $P(Y_i \geq s) = (K\mathrm{e}^{-ds(1-\lambda)}) \wedge 1$ for $s \geq 1$ integer, K being the constant appearing in Lemma 3.9. Due to this lemma, under P_0^{ω}, X_k is stochastically dominated by $Y_1 + \cdots + Y_k$ for any $k \geq 0$. This domination allows to bound $E_0^{\omega}[T_\rho]$ from below as in [8].

We concentrate on (49). Exactly like on page 731, formulas (21) and (22) of [8], we also have that for any ζ and for all $j \geq 0$

$$E_y^{\omega,\zeta}[T_{(j+1)\rho}] \leq \frac{1}{\varepsilon(1-\varepsilon)}E_y^{\omega}[T_{(j+1)\rho}] \tag{51}$$

for all $y \in [j\rho, (j+1)\rho - 1)$.

When $\ell_1 = k$ we can write

$$T_{\ell_1\rho} = T_\rho + (T_{2\rho} - T_\rho) + \cdots + (T_{k\rho} - T_{(k-1)\rho}).$$

Now for each $j \geq 1$ we have

$$\begin{aligned} E_0^{\omega,\zeta}[T_{(j+1)\rho} - T_{j\rho}] &= \sum_{y\in[j\rho,(j+1)\rho)} E_y^{\omega,\zeta}[T_{(j+1)\rho}]P_0^{\omega,\zeta}(X_{T_{j\rho}} = y) \\ &\leq \frac{1}{\varepsilon(1-\varepsilon)}\sum_{y\in[j\rho,(j+1)\rho)} E_y^{\omega}[T_{(j+1)\rho}]P_0^{\omega,\zeta}(X_{T_{j\rho}} = y), \end{aligned}$$

where we have used (51). Now we see that, for any $y \in [j\rho, (j+1)\rho)$,

$$\begin{aligned} E_y^{\omega}[T_{(j+1)\rho}] &\leq E_y^{\omega}\big[N_\infty\big((-\infty,(j+1)\rho]\big)\big] \\ &\leq K E_y^{\omega,1}\big[N_\infty\big((-\infty,(j+1)\rho]\big)\big] \\ &\leq K E_{j\rho}^{\omega,1}\big[N_\infty\big((-\infty,(j+1)\rho]\big)\big], \end{aligned}$$

where the second inequality is due to Corollary 3.14.

Hence

$$\begin{aligned} E_0^{\omega,\zeta}[T_{(j+1)\rho} - T_{j\rho}] &\leq K\frac{1}{\varepsilon(1-\varepsilon)}E_{j\rho}^{\omega,1}\big[N_\infty\big((-\infty,(j+1)\rho]\big)\big] \\ &\leq K\frac{1}{\varepsilon(1-\varepsilon)}\left(\sum_{z\leq j\rho}E_{j\rho}^{\omega,1}\big[N_\infty(z)\big] + \sum_{j\rho<z\leq(j+1)\rho}E_{j\rho}^{\omega,1}\big[N_\infty(z)\big]\right) \\ &= K\frac{1}{\varepsilon(1-\varepsilon)}\left(\sum_{z\leq j\rho}E_{j\rho}^{\omega,1}\big[N_\infty(z)\big] + \sum_{j\rho<z\leq(j+1)\rho}E_{z}^{\omega,1}\big[N_\infty(z)\big]\right). \end{aligned}$$

Using the results of Lemma 3.12 and Corollary 3.14 we obtain for every $j \geq 1$

$$\mathbb{E}E_0^{\omega,\zeta}[T_{(j+1)\rho} - T_{j\rho}] \leq K \frac{1}{\varepsilon(1-\varepsilon)} \biggl(\sum_{k \geq 0} \mathbb{E}\bigl[g_{\tau_{j\rho}\omega}(k)\bigr] + \sum_{j\rho < z \leq (j+1)\rho} \mathbb{E}\bigl[g_{\tau_z\omega}(0)\bigr] \biggr)$$

$$\leq K \frac{1}{\varepsilon(1-\varepsilon)} \biggl(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2} + \frac{\rho}{1-\mathrm{e}^{-2\lambda d}} \biggr) \mathbb{E}\bigl[\mathrm{e}^{(1-\lambda)x_1}\bigr]$$

and hence

$$\mathbb{E}E_0^{\omega,\zeta}[T_{k\rho}] \leq K \frac{k}{\varepsilon(1-\varepsilon)} \biggl(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2} + \frac{\rho}{1-\mathrm{e}^{-2\lambda d}} \biggr) \mathbb{E}\bigl[\mathrm{e}^{(1-\lambda)x_1}\bigr].$$

Since $P(\ell_1 = k) = \varepsilon(1-\varepsilon)^{k-1}$, we obtain

$$\mathbb{E}'\bigl[E_0^{\omega,\zeta}[T_{\ell_1\rho}]\bigr] \leq K \biggl(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2} + \frac{\rho}{1-\mathrm{e}^{-2\lambda d}} \biggr) \mathbb{E}\bigl[\mathrm{e}^{(1-\lambda)x_1}\bigr] \sum_{k=1}^{\infty} k(1-\varepsilon)^{k-2}$$

$$= \bar{K} \biggl(\frac{1}{(1-\mathrm{e}^{-2\lambda d})^2} + \frac{\rho}{1-\mathrm{e}^{-2\lambda d}} \biggr) \mathbb{E}\bigl[\mathrm{e}^{(1-\lambda)x_1}\bigr]. \tag{52}$$

□

Recall the definition of the function g_ω given in Proposition 3.11.

Lemma 4.4. *Let $\rho < \infty$. Given $k \leq 0$ it holds*

$$E_0^{\omega,\zeta}\bigl[N_\infty(k)\bigr] \leq \frac{1}{\varepsilon(1-\varepsilon)} \sum_{j=0}^{\infty} g_{\tau_{j\rho}\omega}\bigl(|k| + j\rho\bigr). \tag{53}$$

Proof. As for the derivation of (33) in [8] one can prove that, if $y \in [j\rho, (j+1)\rho)$, then

$$E_y^{\omega,\zeta}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr] \leq \frac{1}{\varepsilon(1-\varepsilon)} E_y^{\omega}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr]. \tag{54}$$

On the other hand, by applying Proposition 3.11 , we get

$$E_y^{\omega}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr] \leq E_y^{\omega}\bigl[N_\infty(k)\bigr] = E_0^{\tau_y\omega}\bigl[N_\infty(k-y)\bigr] \leq g_{\tau_y\omega}\bigl(|k| + y\bigr). \tag{55}$$

At this point we write y as $y = j\rho + \ell$ and set $\omega' := \tau_{j\rho}\omega$. Then, by applying (39) in Lemma 3.12, we get

$$g_{\tau_y\omega}\bigl(|k| + y\bigr) = g_{\tau_\ell\omega'}\bigl(|k| + j\rho + \ell\bigr) \leq g_{\omega'}\bigl(|k| + j\rho\bigr) = g_{\tau_{j\rho}\omega}\bigl(|k| + j\rho\bigr). \tag{56}$$

As a byproduct of (54), (55) and (56) we conclude that

$$E_y^{\omega,\zeta}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr] \leq \frac{1}{\varepsilon(1-\varepsilon)} g_{\tau_{j\rho}\omega}\bigl(|k| + j\rho\bigr). \tag{57}$$

The above bound and the strong Markov property applied at time $T_{j\rho}$ (which holds by construction of $P_0^{\omega,\zeta}$) imply that

$$E_0^{\omega,\zeta}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr] = E_0^{\omega,\zeta}\bigl[E_{X_{T_{j\rho}}}^{\omega,\zeta}\bigl[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\bigr]\bigr] \leq \frac{1}{\varepsilon(1-\varepsilon)} g_{\tau_{j\rho}\omega}\bigl(|k| + j\rho\bigr). \tag{58}$$

Since $N_\infty(k) = \sum_{j=0}^{\infty} N_{[T_{j\rho},T_{(j+1)\rho})}(k)$, the above bound (58) implies (53). □

4.2. *Speed for the truncated process*

Recall that $\rho < \infty$ is fixed and recall Warning 4.1. Here we follow the usual notation, indicating explicitly ρ, and we also write $P_0^{\omega,\zeta,\rho}$ instead of $P_0^{\omega,\zeta}$ to stress the dependence on ρ.

Proposition 4.5. *Fix $\rho < +\infty$. For $\mathbb{P}$-a.a. $\omega \in \Omega$ it holds*

$$v_{X^\rho}(\lambda) := \lim_{n\to\infty} \frac{X_n^\rho}{n} = \frac{\rho E[\ell_1]}{\mathbb{E}'[E_0^{\omega,\zeta}[T_{\ell_1\rho}^\rho]]} = \frac{\rho}{\varepsilon\mathbb{E}'[E_0^{\omega,\zeta}[T_{\ell_1\rho}^\rho]]} P_0^{\omega,\rho}\text{-a.s.}, \tag{59}$$

where ε is the same as in Lemma 3.16. Moreover, $v_{X^\rho}(\lambda)$ does not depend on ω and

$$v_{X^\rho}(\lambda) \in (c_1, c_2) \tag{60}$$

for strictly positive constants c_1, c_2, which do neither depend on ω nor on ρ.

Proof. We work on the probability space $(\Theta, P \otimes \mathbb{P} \otimes P_0^{\omega,\zeta,\rho})$ where $\Theta := \{0,1\}^{\mathbb{N}_+} \times \Omega \times \mathbb{Z}^{\mathbb{N}}$. For $n \in [T_{\ell_k\rho}^\rho, T_{\ell_{k+1}\rho}^\rho)$ we have $\ell_{k+1}\rho - (T_{\ell_{k+1}\rho}^\rho - T_{\ell_k\rho}^\rho)\rho < X_n^\rho < \ell_{k+1}\rho$ (note in particular that X_n^ρ has to be thought as a function on Θ). It then follows

$$\frac{\ell_{k+1}\rho - (T_{\ell_{k+1}}^\rho - T_{\ell_k\rho}^\rho)\rho}{T_{\ell_{k+1}\rho}^\rho} < \frac{X_n^\rho}{n} < \frac{\ell_{k+1}\rho}{T_{\ell_k\rho}^\rho}. \tag{61}$$

Due to the cycle stationarity and ergodicity stated in Lemma 4.2, we let $n \to \infty$ in (61) and obtain that the limit in (59) holds $P \otimes \mathbb{P} \otimes P_0^{\omega,\zeta,\rho}$-a.s. This can be restated as $\mathbb{E}[E[P_0^{\omega,\zeta,\rho}(X_n^\rho/n \to \rho/\varepsilon\mathbb{E}'[E_0^{\omega,\zeta}[T_{\ell_1\rho}^\rho]])]] = 1.$ To conclude the proof of (59), it is enough to recall that, by averaging $P_0^{\omega,\zeta,\rho}$ over ζ, one obtains the law $P_0^{\omega,\rho}$ of the original random walk $(X_n^\rho)_{n\geq 0}$, in particular we get $\mathbb{E}[P_0^{\omega,\rho}(X_n^\rho/n \to \rho/\varepsilon\mathbb{E}'[E_0^{\omega,\zeta}[T_{\ell_1\rho}^\rho]])] = 1$.

Finally, we observe that $v_{X^\rho}(\lambda)$ does not depend on ω since the last term in (59) doesn't, and that $v_{X^\rho}(\lambda) \in (c_1, c_2)$ due to (48) and (49). □

5. Stationary distribution $\mathbb{Q}^\rho$ of the environment viewed from the ρ-walker

In this section we assume that $\mathbb{E}[e^{(1-\lambda)x_1}] < \infty$ and we fix $\rho < \infty$. We consider the process *environment viewed from the ρ-walker*, which is the Markov chain $(\tau_{X_n^\rho}\omega)_{n\in\mathbb{N}}$ on the space of environments Ω with transition mechanism induced by $P_0^{\omega,\rho}$. When starting with initial distribution Q, we denote by $\mathcal{P}_Q^\rho$ its law as probability distribution on $\Omega^{\mathbb{N}}$.

Lemma 4.2 and bound (49) in Lemma 4.3 guarantee (cf. [22, Section 4, Chapter 8]) the existence of a stationary distribution $\mathbb{Q}^\rho$ of the process *environment viewed from the ρ-walker*, such that $\mathbb{Q}^\rho$ is absolutely continuous with respect to $\mathbb{P}$.

From [22, Chapter 8, Equation (4.14°)], $\mathbb{Q}^\rho$ can be characterized by its expectation:

$$\mathbb{E}^\rho[f(\omega)] = \frac{1}{\mathbb{E}'[E_0^{\omega,\zeta,\rho}[T_{\ell_1\rho}]]}\mathbb{E}'\left[E_0^{\omega,\zeta,\rho}\left[\sum_{k=1}^{T_{\ell_1\rho}} f(\tau_{X_k}\omega)\right]\right]. \tag{62}$$

As in [8, Proposition 3.4] one can prove that $\mathbb{Q}^\rho$ is absolutely continuous to $\mathbb{P}$ with Radon–Nikodym derivative

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) = \frac{1}{\mathbb{E}'[E_0^{\omega,\zeta,\rho}[T_{\ell_1\rho}]]}\sum_{k\in\mathbb{Z}} E E_0^{\tau_{-k}\omega,\zeta,\rho}\left[N_{T_{\ell_1\rho}}(k)\right]. \tag{63}$$

Note that the denominator in the r.h.s. is finite due to (49) and the numerator is positive. As a consequence, $\mathbb{P}$ is also absolutely continuous to $\mathbb{Q}^\rho$.

Lemma 5.1. *Fix $\rho \in \mathbb{N}_+$. Then $\mathbb{Q}^\rho$ is ergodic with respect to shifts for the environment seen from the ρ-walker.*

Remark 5.2. The above ergodicity means that any Borel subset of the path space $\Omega^{\mathbb{N}}$, which is left invariant by shifts, has $\mathcal{P}^\rho_{\mathbb{Q}^\rho}$-probability equal to 0 or 1.

Due to Theorem 6.9 in [23] (cf. also [19, Chapter IV]), the above ergodicity is equivalent to the following fact: $\mathbb{Q}^\rho(A) \in \{0,1\}$ whenever $A \subset \Omega$ is an invariant Borel set, in the sense that "$\tau_{X_n^\rho}\omega \in A$ for any $n \in \mathbb{N}$" holds $\mathbb{Q}^\rho \otimes P_0^{\omega,\rho}$-a.s. on $\{\omega \in A\}$ and "$\tau_{X_n^\rho}\omega \in A^c$ for any $n \in \mathbb{N}$" holds $\mathbb{Q}^\rho \otimes P_0^{\omega,\rho}$-a.s. on $\{\omega \in A^c\}$. As usual, $\mathbb{Q}^\rho \otimes P_0^{\omega,\rho}$ is the probability measure on $\Omega \times \mathbb{Z}^{\mathbb{N}}$ such that the expectation of a function f is given by $\int \mathbb{Q}^\rho(d\omega) E_0^{\omega,\rho}[f(\omega,(X_n)_{n\geq 0})]$.

Proof of Lemma 5.1. The proof can be obtained as in [8, pages 735–736]. The only difference is that in [8] the authors use their formula (29), which is not satisfied in our case. More precisely, they use their formula (29) to argue that $0 < \mathbb{P}(A) < 1$ for any $\mathbb{Q}^\rho$-nontrivial set A. On the other hand, this claim follows simply from the absolute continuity of $\mathbb{Q}^\rho$ to $\mathbb{P}$. □

The rest of this section is devoted to the proof of Lemma 5.9 which leads to the following result:

Proposition 5.3. *Suppose $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}] < \infty$ and that $u : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous. Then the sequence $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$ converges weakly to a unique measure $\mathbb{Q}^\infty$ as $\rho \to \infty$. $\mathbb{Q}^\infty$ is absolutely continuous to $\mathbb{P}$ and, $\mathbb{P}$-a.s., $0 < \gamma \leq \frac{d\mathbb{Q}^\infty}{d\mathbb{P}} \leq F$ (cf. (65)). Furthermore, $\mathbb{Q}^\infty$ is invariant and ergodic for the dynamics from the point of view of the ∞-walker.*[2]

Having Lemma 3.9 and Lemma 5.9 below, Proposition 5.3 can be proved by the same arguments used in [8, p. 735], with some slight modifications. For completeness, we give the proof in Appendix A.

5.1. *Upper bound for the Radon–Nikodym derivative $d\mathbb{Q}^\rho/d\mathbb{P}$*

Proposition 5.4. *Suppose $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}] < \infty$. Then, uniformly in $\rho \in \mathbb{N}_+$,*

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) \leq F(\omega) \quad \mathbb{P}\text{-}a.s., \tag{64}$$

where

$$F(\omega) := K\pi^1(0)\sum_{j=0}^{\infty}(j+2)^2\mathrm{e}^{-2\lambda x_j+(1-\lambda)(x_{j+1}-x_j)}, \tag{65}$$

for some constant $K > 0$. Moreover, $\mathbb{E}[F] < \infty$.

Before proving Proposition 5.4 we state a technical result:

Lemma 5.5. *Let $F_*(\omega) := K_0\sum_{i=0}^{\infty}(i+1)\mathrm{e}^{-2\lambda x_i+(1-\lambda)(x_{i+1}-x_i)}$, with K_0 as in Proposition 3.11. Then*

$$\sum_{j=0}^{\infty} g_{\tau_{j\rho}\omega}\big(|k|+j\rho\big) \leq \sum_{r=0}^{\infty} g_{\tau_r\omega}\big(|k|+r\big) \leq \pi^1\big(-|k|\big)F_*(\omega). \tag{66}$$

[2] Ergodicity means that the law $\mathcal{P}^\infty_{\mathbb{Q}^\infty}$ on the path space $\Omega^{\mathbb{N}}$ is ergodic with respect to shifts (cf. Remark 5.2).

Proof. The first inequality in (66) is trivial. We prove the second one. By (41) we can write

$$\sum_{r=0}^{\infty} g_{\tau_r\omega}\big(|k|+r\big) \le K_0\pi^1\big(-|k|\big)\sum_{r=0}^{\infty}\sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_{j+r}+(1-\lambda)(x_{j+1+r}-x_{j+r})}$$

$$= K_0\pi^1\big(-|k|\big)\sum_{i=0}^{\infty}(i+1)\mathrm{e}^{-2\lambda x_i+(1-\lambda)(x_{i+1}-x_i)}.$$

□

We can now prove Proposition 5.4:

Proof of Proposition 5.4. Due to (48) and (63) we can bound

$$\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) \le \frac{H_+(\omega)+H_-(\omega)}{K_1\rho}, \tag{67}$$

where K_1 is the constant appearing in (48) and

$$H_+(\omega) := \sum_{k>0} E E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{T_{\ell_1\rho}}(k)\big], \qquad H_-(\omega) := \sum_{k\le 0} E E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{T_{\ell_1\rho}}(k)\big].$$

As a byproduct of Lemma 4.4 and Lemma 5.5 it holds (see the proof of (39) for the equality below)

$$H_-(\omega) \le \frac{1}{\varepsilon(1-\varepsilon)}\sum_{k\le 0}\pi^1(k)[\tau_{-k}\omega]F_*(\tau_{-k}\omega) = \frac{1}{\varepsilon(1-\varepsilon)}\pi^1(0)\sum_{k\le 0}\mathrm{e}^{-2\lambda x_{-k}}F_*(\tau_{-k}\omega). \tag{68}$$

Let us bound $H_+(\omega)$. We can write

$$\sum_{k=0}^{\infty} E E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{T_{\ell_1\rho}}(k)\big] = \sum_{m=0}^{\infty}\sum_{k\in[m\rho,(m+1)\rho)}\sum_{i=1}^{\infty} E\big[\mathbb{1}_{\ell_1=i}E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{T_{i\rho}}(k)\big]\big]$$

$$\le \sum_{m=0}^{\infty}\sum_{k\in[m\rho,(m+1)\rho)}\sum_{i=1}^{\infty}\sum_{j=0}^{i} E\big[\mathbb{1}_{\ell_1=i}E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\big]\big]. \tag{69}$$

Note that, given $m > j \ge 0$ and $k \in [m\rho,(m+1)\rho)$, it holds $N_{[T_{j\rho},T_{(j+1)\rho})}(k) = 0$, hence in the last expression of (69) we can restrict to $0 \le m \le j \le i$. Moreover note that (cf. (54))

$$E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\big] \le \frac{1}{\varepsilon(1-\varepsilon)}E_0^{\tau_{-k}\omega,\rho}\big[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\big]. \tag{70}$$

Consider then the case $k \in [m\rho,(m+1)\rho)$ with $0 \le m \le j \le i$. Note that $X^\rho_{T_{j\rho}} \in [j\rho,(j+1)\rho)$ due to the maximal length of the jump. Fix $y \in [j\rho,(j+1)\rho)$. Then, for any environment ω, we have

$$E_y^{\omega,\rho}\big[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\big] = E_y^{\omega,\rho}\big[N_{T_{(j+1)\rho}}(k)\big] \le \begin{cases} g_{\tau_k\omega}(0) & \text{if } j=m, \\ g_{\tau_{j\rho}\omega}(j\rho-k) & \text{if } j>m. \end{cases} \tag{71}$$

Indeed, consider first the case $j > m$. Then $k < y$ and by Proposition 3.11

$$E_y^{\omega,\rho}\big[N_{T_{(j+1)\rho}}(k)\big] \le E_y^{\omega,\rho}\big[N_{T_\infty}(k)\big] = E_0^{\tau_y\omega,\rho}\big[N_{T_\infty}(k-y)\big] \le g_{\tau_y\omega}(y-k), \tag{72}$$

Write $y = j\rho+\ell$ and $\omega' := \tau_{j\rho}\omega$. Then we have

$$g_{\tau_y\omega}(y-k) = g_{\tau_\ell\omega'}(j\rho-k+\ell) \le g_{\omega'}(j\rho-k) = g_{\tau_{j\rho}\omega}(j\rho-k)$$

(in the last step we have used (39)). This proves (71) for $j > m$. If $j = m$ we bound (by the Markov property at the first visit of k)

$$E_y^{\omega,\rho}\big[N_{T_{(j+1)\rho}}(k)\big] \le E_y^{\omega,\rho}\big[N_{T_\infty}(k)\big] \le E_k^{\omega,\rho}\big[N_{T_\infty}(k)\big] = E_0^{\tau_k\omega,\rho}\big[N_{T_\infty}(0)\big].$$

At this point (71) for $j = m$ follows from Proposition 3.11.

The above bound (71), the Markov property and (70) imply

$$E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{[T_{j\rho},T_{(j+1)\rho})}(k)\big] \le \frac{1}{\varepsilon(1-\varepsilon)} \cdot \begin{cases} g_\omega(0) & \text{if } j = m, \\ g_{\tau_{j\rho-k}\omega}(j\rho - k) & \text{if } j > m. \end{cases} \tag{73}$$

Coming back to (69) and due to the above observations we can bound

$$H_+(\omega) \le \sum_{k=0}^{\infty} E E_0^{\tau_{-k}\omega,\zeta,\rho}\big[N_{T_{\ell_1\rho}}(k)\big] \le A(\omega) + B(\omega), \tag{74}$$

where (distinguishing the cases $m = j$ and $m < j$)

$$\begin{aligned} A(\omega) &:= \sum_{i=1}^{\infty}\sum_{j=0}^{i}\sum_{k\in[j\rho,(j+1)\rho)} E[\mathbb{1}_{\ell_1=i}]g_\omega(0) = \rho\big(E(\ell_1)+1\big)g_\omega(0), \\ B(\omega) &:= \sum_{i=1}^{\infty}\sum_{j=1}^{i}\sum_{m=0}^{j-1}\sum_{k\in[m\rho,(m+1)\rho)} E[\mathbb{1}_{\ell_1=i}]g_{\tau_{j\rho-k}\omega}(j\rho-k). \end{aligned} \tag{75}$$

For what concerns $B(\omega)$ observe that $\sum_{i=j}^{\infty} E[\mathbb{1}_{\ell_1=i}] = (1-\varepsilon)^{j-1}$, hence

$$\begin{aligned} B(\omega) &\le \sum_{j=1}^{\infty}(1-\varepsilon)^{j-1}\sum_{m=0}^{j-1}\sum_{k\in[m\rho,(m+1)\rho)} g_{\tau_{j\rho-k}\omega}(j\rho-k) \\ &= \sum_{j=1}^{\infty}(1-\varepsilon)^{j-1}\sum_{k\in[0,j\rho)} g_{\tau_{j\rho-k}\omega}(j\rho-k) = \sum_{j=1}^{\infty}(1-\varepsilon)^{j-1}\sum_{h=1}^{j\rho} g_{\tau_h\omega}(h). \end{aligned} \tag{76}$$

Since, by (39), $g_{\tau_h\omega}(h) \le g_\omega(0)$, we get that $B(\omega) \le \sum_{j=1}^{\infty}(1-\varepsilon)^{j-1} j\rho g_\omega(0)$. Combining this estimate with (75) we conclude that $H_+(\omega) \le C\rho g_\omega(0)$, where the constant C depends only on ε. Coming back to (67) and (68), and observing that $g_\omega(0) \le \pi^1(0)F_*(\omega)$, we have

$$\begin{aligned} \frac{d\mathbb{Q}^\rho}{d\mathbb{P}}(\omega) &\le \frac{C'}{K_1}\left[\frac{\pi^1(0)\sum_{k\le 0}\mathrm{e}^{-2\lambda x_{-k}}F_*(\tau_{-k}\omega)}{\rho} + g_\omega(0)\right] \\ &\le \frac{2C'}{K_1}\pi^1(0)\sum_{k\le 0}\mathrm{e}^{-2\lambda x_{-k}}F_*(\tau_{-k}\omega), \end{aligned} \tag{77}$$

where C' depends only on ε. Since $x_a(\tau_{-k}\omega) = x_{a-k}(\omega) - x_{-k}(\omega)$, by definition of F_* (and setting $r = i - k$) we can write

$$\begin{aligned} \sum_{k\le 0}\mathrm{e}^{-2\lambda x_{-k}}F_*(\tau_{-k}\omega) &= K_0\sum_{k\le 0}\sum_{i\ge 0}(i+1)\mathrm{e}^{-2\lambda x_{i-k}+(1-\lambda)(x_{i-k+1}-x_{i-k})} \\ &= K_0\sum_{r\ge 0}\mathrm{e}^{-2\lambda x_r+(1-\lambda)(x_{r+1}-x_r)}\frac{(r+1)(r+2)}{2}. \end{aligned} \tag{78}$$

As byproduct of (77) and (78) we get (64).

Finally, by using (37) and that $x_j \geq jd$ for $j \geq 0$, we can bound $\mathbb{E}[F] \leq C \sum_{j\geq 0}(j+2)^2 e^{-2\lambda j}\mathbb{E}[e^{(1-\lambda)x_1}]$ for some positive constant C. Since by assumption $\mathbb{E}[e^{(1-\lambda)x_1}] < \infty$, we conclude that $\mathbb{E}[F] < \infty$. □

5.2. *Uniform lower bound for $d\mathbb{Q}^\rho/d\mathbb{P}$*

We remark that, following the proof of Proposition 3.4 in [8], we could easily obtain a lower bound on $d\mathbb{Q}^\rho/d\mathbb{P}$ which is independent of ρ, but which would in principle depend on the particular argument ω. Here we will do more: We will exhibit a lower bound that is uniform in both ρ and ω (see Corollary 5.8 below).

For fixed $\omega \in \Omega$, we denote by Q_n^ω the empirical measure at time n for the environment viewed from the ρ-walker. More precisely, Q_n^ω is a random probability measure on Ω defined as

$$Q_n^\omega := \frac{1}{n}\sum_{j=1}^{n} \delta_{\tau_{X_j^\rho}\omega}.$$

Averaging over the paths of the walk we obtain the probability $E_0^{\omega,\rho}[Q_n^\omega(\cdot)]$. For fixed $\omega \in \Omega$, we define another probability measure on Ω, given by

$$R_n^\omega := \frac{1}{m(n)}\sum_{j=1}^{m(n)} \delta_{\tau_j\omega},$$

where $m(n) := n \cdot v_{X^\rho}/2$ and v_{X^ρ} is the positive limiting speed of the truncated random walk given in (59) (we are omitting the dependence on λ; the $1/2$ could be replaced by any other constant smaller than 1).

We remark that R_n^ω and $E_0^{\omega,\rho}[Q_n^\omega(\cdot)]$ can be thought of as random variables on $(\Omega, \mathbb{P})$ with values in $\mathcal{P}(\Omega)$, the space of probability measures on Ω endowed with the weak topology. Note also that $\mathbb{P}, \mathbb{Q}^\rho \in \mathcal{P}(\Omega)$. Furthermore, Q_n^ω can be thought of as a random variable on the probability space $(\Omega \times \mathbb{Z}^{\mathbb{N}}, \mathbb{P} \otimes P_0^{\omega,\rho})$ with values in $\mathcal{P}(\Omega)$.

Proposition 5.6. *For $\mathbb{P}$-almost every $\omega \in \Omega$ we have that $R_n^\omega \to \mathbb{P}$ and $E_0^{\omega,\rho}[Q_n^\omega(\cdot)] \to \mathbb{Q}^\rho$ weakly in $\mathcal{P}(\Omega)$. Moreover, $\mathbb{P} \otimes P_0^{\omega,\rho}$-a.s., we have that $Q_n^\omega \to \mathbb{Q}^\rho$ weakly in $\mathcal{P}(\Omega)$.*

Proof. The a.s. convergence of R_n^ω to $\mathbb{P}$ comes directly from the ergodicity of $\mathbb{P}$ with respect to shifts.

We claim that $Q_n^\omega \to \mathbb{Q}^\rho$ weakly in $\mathcal{P}(\Omega)$, $\mathbb{Q}^\rho \otimes P_0^{\omega,\rho}$-a.s. This follows from Birkhoff's ergodic theorem applied to the Markov chain $\tau_{X_n^\rho}\omega$ starting from the ergodic distribution $\mathbb{Q}^\rho$ (cf. Lemma 5.1). As already observed after equation (63), $\mathbb{P}$ is absolutely continuous to $\mathbb{Q}^\rho$. Hence, due to the above claim, $Q_n^\omega \to \mathbb{Q}^\rho$ weakly in $\mathcal{P}(\Omega)$ also $\mathbb{P} \otimes P_0^{\omega,\rho}$-a.s.

Finally, the last a.s. convergence and the dominated convergence theorem imply that $E_0^{\omega,\rho}[Q_n^\omega(\cdot)] \to \mathbb{Q}^\rho$ weakly in $\mathcal{P}(\Omega)$, $\mathbb{P}$-a.s. □

Lemma 5.7. *There exists $\gamma > 0$, depending neither on ω nor on ρ, such that the following holds: For $\mathbb{P}$-almost every ω, there exists an $\bar{n}_\omega$ such that, $\forall n \geq \bar{n}_\omega$,*

$$\frac{E_0^{\omega,\rho}[Q_n^\omega(\{\tau_k\omega\})]}{R_n^\omega(\{\tau_k\omega\})} > \gamma, \quad \forall k : 1 \leq k \leq m(n).$$

Proof. For all $k = 1, \dots, m(n)$, we have

$$E_0^{\omega,\rho}\big[Q_n^\omega(\cdot)\big] \geq \frac{1}{n} P_0^{\omega,\rho}(\exists j \leq n : X_j = k)\delta_{\tau_k\omega}. \tag{79}$$

We claim that, for n big enough and $k = 1, \dots, m(n)$, it holds

$$P_0^{\omega,\rho}(\exists j \leq n : X_j = k) \geq \varepsilon, \tag{80}$$

where $\varepsilon > 0$ is the same as in Lemma 3.16. To prove our claim, we bound

$$\begin{aligned} P_0^{\omega,\rho}(\exists j \le n : X_j = k) &\ge P_0^{\omega,\rho}(X_{T_k} = k, T_k \le n) \\ &\ge P_0^{\omega,\rho}(X_{T_k} = k) - P_0^{\omega,\rho}(T_k > n) \\ &\ge 2\varepsilon - P_0^{\omega,\rho}(T_{m(n)} > n), \end{aligned}$$

where in the last line we have used Lemma 3.16. On the other hand, we also know, by the definition of the limiting speed, that for almost every $\omega \in \Omega$, there exists an $\bar{n}_\omega$ such that, $\forall n > \bar{n}_\omega$, $P_0^{\omega,\rho}(T_{m(n)} > n) \le P_0^{\omega,\rho}(X_n < m(n)) < \varepsilon$. This completes the proof of the claim.

Hence, putting together (79) and (80), for all $n \ge \bar{n}_\omega$ and $k = 1, \dots, m(n)$, we have

$$E_0^{\omega,\rho}\big[Q_n^\omega(\cdot)\big] \ge \frac{\varepsilon}{n}\delta_{\tau_k\omega}.$$

On the other hand, by definition, $R_n^\omega(\{\tau_k\omega\}) = \frac{1}{m(n)}$ for all $k = 1, \dots, m(n)$ and for $\mathbb{P}$-a.a. ω (since periodic environments have $\mathbb{P}$-measure zero by Assumption (A3)). It then follows that, for all $k = 1, \dots, m(n)$ and for $\mathbb{P}$-a.a. ω,

$$\frac{E_0^{\omega,\rho}[Q_n^\omega(\{\tau_k\omega\})]}{R_n^\omega(\{\tau_k\omega\})} \ge \frac{\frac{\varepsilon}{n}}{\frac{1}{m(n)}} = \frac{\varepsilon v_{X^\rho}}{2} \ge \frac{\varepsilon c_1}{2} =: \gamma > 0, \tag{81}$$

where c_1 is from (60). Note that γ does not depend on ω. □

We finally need to show that the lower bound extends also to the Radon–Nikodym derivative of the limiting measures.

Corollary 5.8. *The Radon–Nikodym derivative $\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}$ is uniformly bounded from below: $\frac{d\mathbb{Q}^\rho}{d\mathbb{P}} \ge \gamma$, where γ is from* (81).

Proof. Take any $f \ge 0$ continuous and bounded. Lemma 5.7 and the fact that R_n^ω has support in $\{\tau_k\omega : k = 1, \dots, m(n)\}$ guarantee that, for all n large enough,

$$E_0^{\omega,\rho}\big[Q_n^\omega(f)\big] \ge \gamma R_n^\omega(f) \quad \text{for } \mathbb{P}\text{-a.e. } \omega.$$

Passing to the limit $n \to \infty$, and observing that, by Proposition 5.6, $E_0^{\omega,\rho}[Q_n^\omega(f)] \to \mathbb{Q}^\rho(f)$ and $R_n^\omega(f) \to \mathbb{P}(f)$ for $\mathbb{P}$-a.e. ω, we have that $\mathbb{Q}^\rho(f) \ge \gamma\mathbb{P}(f)$. The claim follows from the arbitrariness of f. □

5.3. *The weak limit of $\mathbb{Q}^\rho$ as $\rho \to \infty$*

Recall the definition of the function F given in (65) and of the constant γ given in Corollary 5.8.

Lemma 5.9. *Suppose $\mathbb{E}[e^{(1-\lambda)x_1}] < \infty$. Then the following holds*:

(i) *The family of probability measures $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$ is tight*;
(ii) *Any subsequential limit $\mathbb{Q}^\infty$ of $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$ is absolutely continuous to $\mathbb{P}$ and*

$$0 < \gamma \le \frac{d\mathbb{Q}^\infty}{d\mathbb{P}} \le F \quad \mathbb{P}\text{-a.s.}$$

Proof. For proving part (i), fix an increasing sequence of compact subsets K_n exhausting all of Ω. Thanks to Proposition 5.4 we have

$$\mathbb{Q}^\rho\big(K_n^c\big) = \mathbb{E}\bigg[\frac{d\mathbb{Q}^\rho}{d\mathbb{P}}\mathbb{1}_{K_n^c}\bigg] \le \mathbb{E}[F\mathbb{1}_{K_n^c}].$$

Setting $f_n := F\mathbb{1}_{K_n^c}$ we have that $0 \le f_n \le F$ and $f_n(\omega) \to 0$ everywhere. By the dominated convergence theorem, given $\varepsilon > 0$ we conclude that $\mathbb{Q}^\rho(K_n^c) \le \varepsilon$ uniformly in ρ eventually in n, hence the tightness.

We turn now to (ii). Let $\mathbb{Q}^\infty$ be any subsequential limit of $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$. In particular, there exists a sequence $\rho_k \to \infty$ such that $\mathbb{Q}^{\rho_k}$ converges weakly to $\mathbb{Q}^\infty$. Let $A \subset \Omega$ be measurable. Due to Corollary 5.8 and Proposition 5.4 we have $\gamma\mathbb{P}(C) \le \mathbb{Q}^{\rho_k}(C)$ and $\mathbb{Q}^{\rho_k}(G) \le \mathbb{E}[F\mathbb{1}_G]$ for any $C \subset A \subset G$ with C closed and G open. Hence, by the Portmanteau Theorem (cf. [4, Theorem 2.1]), we conclude that

$$\gamma\mathbb{P}(C) \le \limsup_{k\to\infty} \mathbb{Q}^{\rho_k}(C) \le \mathbb{Q}^\infty(C) \le \mathbb{Q}^\infty(A) \le \mathbb{Q}^\infty(G) \le \liminf_{k\to\infty} \mathbb{Q}^{\rho_k}(G) \le \mathbb{E}[F\mathbb{1}_G]. \tag{82}$$

By the the regularity of $\mathbb{P}$ (cf. [4, Theorem 1.1]), one can choose C and G so that the extreme terms in (82) are arbitrary close to $\gamma\mathbb{P}(A)$ and $\mathbb{E}[F\mathbb{1}_A]$ respectively, from which Item (ii) follows. □

6. Proof of Theorem 1(i): Transience to the right

By the discussion at the end of Section 3, it is enough to show the a.s. transience to the right of X_n^∞ and $\mathbb{X}_t^\infty$. Since the former is the jump chain associated to the latter, we only need to derive the a.s. transience to the right of X_n^∞. To this aim, it is sufficient to show that, for any $m \in \mathbb{N}$, there exists some $n(m,\omega) < \infty$ such that $X_n^\infty > m$ for all $n \ge n(m,\omega)$.

First of all notice that, by Proposition 3.11, for $\mathbb{P}$-almost every $\omega \in \Omega$ and $i \in \mathbb{Z}$ we have

$$\begin{aligned} E_i^{\omega,\infty}\big[N_\infty\big((-\infty,i]\big)\big] &\le \sum_{k=0}^{\infty} g_{\tau_i\omega}(k) \\ &= K_0\left(\sum_{k=0}^{\infty} K_0\pi^1(-k)[\tau_i\omega]\right)\cdot\left(\sum_{j=0}^{\infty} \mathrm{e}^{-2\lambda x_j(\tau_i\omega)+(1-\lambda)(x_{j+1}(\tau_i\omega)-x_j(\tau_i\omega))}\right), \end{aligned}$$

which is $\mathbb{P}$-almost surely finite (see (37) and the discussion after Proposition 3.11). Hence

$$P_i^{\omega,\infty}\big(N_\infty\big((-\infty,i]\big) < \infty\big) = 1. \tag{83}$$

Now fix $m \in \mathbb{N}$ and consider T_m, the first time the random walk is larger or equal than m. Applying the Markov property at time T_m and using (83) one gets the claim.

7. Proof of Theorem 1(ii): The ballistic regime

In this section we assume that $\mathbb{E}[\mathrm{e}^{(1-\lambda)x_1}] < +\infty$ and that $u : \mathbb{R}\times\mathbb{R} \to \mathbb{R}$ is continuous. Recall that $(\mathbb{Y})_{t\ge 0}$ and $(Y_n)_{n\ge 0}$ denote the continuous-time Mott random walk and the associated jump process, respectively. Recall also the definition of the Markov chains $(\mathbb{X}_t^\infty)_{t\ge 0}$ and $(X_n^\infty)_{n\in\mathbb{N}}$, given in Section 3 and that $\mathcal{P}_Q^\rho$ is the law of the process *environment viewed from the ρ-walker* $(\tau_{X_n^\rho}\omega)_{n\in\mathbb{N}}$ when started with some initial distribution Q.

Given $\rho \in \mathbb{N}_+ \cup \{+\infty\}$, by writing $(X_n^\rho)_{n\in\mathbb{N}}$ as a functional of $(\tau_{X_n^\rho}\omega)_{n\in\mathbb{N}}$ (which is possible for $\mathbb{P}$-a.e. ω to due Assumption (A3)) and using the ergodicity of $\mathbb{Q}^\rho$ (cf. Lemma 5.1 and Proposition 5.3) we get that the asymptotic velocity of $(X_n^\rho)_{n\ge 0}$ exists $\mathcal{P}_{\mathbb{Q}^\rho}^\rho$-a.s. and therefore $\mathcal{P}_{\mathbb{P}}^\rho$-a.s. since $\mathbb{Q}^\rho$ and $\mathbb{P}$ are mutually absolutely continuous:

$$v_{X^\rho}(\lambda) := \lim_{n\to\infty} \frac{X_n^\rho}{n} \quad \mathcal{P}_{\mathbb{Q}^\rho}^\rho\text{-a.s. and } \mathcal{P}_{\mathbb{P}}^\rho\text{-a.s.} \tag{84}$$

Moreover, $v_{X^\rho}(\lambda)$ does not depend on ω and can be characterized as

$$v_{X^\rho}(\lambda) := \mathbb{E}^\rho\big[E_0^{\omega,\rho}[X_1]\big] = \mathbb{E}^\rho\left[\sum_{m\in\mathbb{Z}} m P_0^{\omega,\rho}(X_1 = m)\right], \quad \forall\rho \in \mathbb{N}_+ \cup \{+\infty\}. \tag{85}$$

Here, $\mathbb{E}^\rho$ denotes the expectation with respect to $\mathbb{Q}^\rho$. Recall that for $\rho < \infty$ we have also an alternative representation for $v_{X^\rho}(\lambda)$ (see Proposition 4.5).

We now prove that

$$\lim_{\rho\to\infty} v_{X^\rho}(\lambda) = v_{X^\infty}(\lambda). \tag{86}$$

By the exponential decay of the jump probabilities (see (24)), for all $\delta > 0$ there exists $m_0 \in \mathbb{N}$ such that, for all ρ,

$$\sum_{|m|>m_0} |m| P_0^{\omega,\rho}(X_1 = m) < \delta \quad \mathbb{P}\text{-a.s.}$$

We now observe that, for $\rho > |m| > 0$, we have

$$\mathbb{P}_0^{\omega,\rho}(X_1 = m) = P_0^{\omega,\infty}(X_1 = m) = \frac{c_{0,m}(\omega)}{\sum_{k\in\mathbb{Z}} c_{0,k}(\omega)}, \tag{87}$$

and the r.h.s. of (87) is continuous in ω due to the continuity assumption on u and since $\|c_{0,k}(\cdot)\|_\infty \le \mathrm{e}^{-(1-\lambda)dk+\|u\|_\infty}$. Since $\mathbb{Q}^\rho \xrightarrow{w} \mathbb{Q}^\infty$, it is now simple to get (86).

Finally, we also have that $v_{X^\infty}(\lambda) \in [c_1, c_2]$ because of the limit (86) and since, by Proposition 4.5, $v_{X^\rho}(\lambda) \in (c_1, c_2)$ for suitable strictly positive constants c_1, c_2.

By the previous observations and by the second identity in (16), we also obtain that the limit

$$v_Y(\lambda) := \lim_{n\to\infty} \frac{Y_n}{n} \tag{88}$$

exists $\mathcal{P}_{\mathbb{P}}^\infty$-a.s. and equals $\mathbb{E}[Z_0] v_{X^\infty}(\lambda)$. As a consequence, $v_Y(\lambda)$ is deterministic, finite and strictly positive.

By a suitable time change we can recover the LLN for $(\mathbb{X}_t^\infty)_{t\ge 0}$ from the LLN for $(X_n^\infty)_{n\ge 0}$ as follows. By enlarging the probability space $(\Omega^{\mathbb{N}}, \mathcal{P}_{\mathbb{Q}^\infty}^\infty)$ with a product space, we introduce a sequence of i.i.d. exponential random variables $(\beta_n)_{n\ge 0}$ of mean one, all independent from the process *environment viewed from the* ∞*-walker* $(\tau_{X_n^\infty}\omega)_{n\in\mathbb{N}}$. We call $(\Omega^{\mathbb{N}} \otimes \mathbb{R}_+^{\mathbb{N}}, \bar{\mathcal{P}}_{\mathbb{Q}^\infty}^\infty)$ the resulting probability space. Note that $\bar{\mathcal{P}}_{\mathbb{Q}^\infty}^\infty$ is stationary and ergodic with respect to shifts. On $(\Omega^{\mathbb{N}} \otimes \mathbb{R}_+^{\mathbb{N}}, \bar{\mathcal{P}}_{\mathbb{Q}^\infty}^\infty)$ we define the random variable

$$S_n := \sum_{k=0}^{n-1} \frac{\beta_k}{r(\tau_{X_k^\infty}\omega)}, \qquad r(\omega) := \pi^\infty(0)[\omega] = \sum_{k\in\mathbb{Z}} c_{0,k}(\omega).$$

We note that $r(\omega)$ coincides with $r_0^\lambda(\omega)$ of Section 2. By the ergodicity of $\bar{\mathcal{P}}_{\mathbb{Q}^\infty}^\infty$ we have

$$\lim_{n\to\infty} \frac{S_n}{n} = \mathbb{E}^\infty[1/r] \quad \bar{\mathcal{P}}_{\mathbb{Q}^\infty}^\infty\text{-a.s.} \tag{89}$$

Since, by Proposition 5.3, $\mathbb{Q}^\infty \ll \mathbb{P}$ and $\frac{d\mathbb{Q}^\infty}{d\mathbb{P}} \le F$ with F defined in (65), using Lemma 3.6, Assumption (A4) and the hypothesis $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}] < +\infty$ we get

$$\begin{aligned} 0 < \mathbb{E}^\infty[1/r] &\le K\mathbb{E}\left[\frac{\pi^1(0)}{r} \sum_{j=0}^\infty (j+2)^2 \mathrm{e}^{-2\lambda x_j + (1-\lambda)(x_{j+1}-x_j)}\right] \\ &\le K' \sum_{j=0}^\infty (j+2)^2 \mathrm{e}^{-2\lambda d} \mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0}\big] < +\infty. \end{aligned} \tag{90}$$

For any $t \geq 0$ we define $n(t)$ on $(\Omega^{\mathbb{N}} \otimes \mathbb{R}_+^{\mathbb{N}}, \bar{\mathcal{P}}^{\infty}_{\mathbb{Q}^\infty})$ as the only integer n such that $S_n \leq t < S_{n+1}$. By (89) and (90) we get that $n(t) \to \infty$ as $t \to \infty$, $\bar{\mathcal{P}}^{\infty}_{\mathbb{Q}^\infty}$-a.s. As a byproduct of the above limit, of (89) and the bound

$$\frac{S_{n(t)}}{n(t)} \leq \frac{t}{n(t)} < \frac{S_{n(t)+1}}{n(t)}, \tag{91}$$

we conclude that

$$\lim_{n\to\infty} \frac{n(t)}{t} = \frac{1}{\mathbb{E}^\infty[1/r]} \quad \bar{\mathcal{P}}^{\infty}_{\mathbb{Q}^\infty}\text{-a.s.} \tag{92}$$

By writing $\frac{X^\infty_{n(t)}}{t} = \frac{X^\infty_{n(t)}}{n(t)} \frac{n(t)}{t}$, from (84) and (92) we get that

$$\lim_{t\to\infty} \frac{X^\infty_{n(t)}}{t} = \frac{v_{X^\infty}(\lambda)}{\mathbb{E}^\infty[1/r]}, \quad \bar{\mathcal{P}}^{\infty}_{\mathbb{Q}^\infty}\text{-a.s.} \tag{93}$$

At this point it is enough to observe that the process $(X^\infty_{n(t)})_{t\geq 0}$ defined on the probability space $(\Omega^{\mathbb{N}} \otimes \mathbb{R}_+^{\mathbb{N}}, \bar{\mathcal{P}}^{\infty}_{\mathbb{Q}^\infty})$ has the same law as the process $(\mathbb{X}^\infty_t)_{t\geq 0}$. Using also (90) and the fact that $\mathcal{P}^\infty_{\mathbb{P}} \ll \mathcal{P}^\infty_{\mathbb{Q}^\infty}$, we conclude that

$$v_{\mathbb{X}^\infty}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{X}^\infty_t}{t} = \frac{v_{X^\infty}(\lambda)}{\mathbb{E}^\infty[1/r]} \in (0, +\infty) \tag{94}$$

holds $P_0^{\omega,\infty}$-a.s., for $\mathbb{P}$-a.e. ω. Finally, using (16), we conclude that

$$v_{\mathbb{Y}}(\lambda) := \lim_{t\to\infty} \frac{\mathbb{Y}_t}{t} = \frac{\mathbb{E}[Z_0]}{\mathbb{E}^\infty[1/r]} v_{X^\infty}(\lambda) \in (0, +\infty) \tag{95}$$

holds for almost all trajectories of the Mott random walk, for $\mathbb{P}$-a.e. ω. As already observed, the r.h.s. of (95) is deterministic and this concludes the proof of Theorem 1(ii) and its counterpart for the jump process $(Y_n)_{n\geq 0}$ (cf. (88)).

8. Proof of theorem 1(iii): The sub-ballistic regime

First we point out that it will be sufficient to prove that $v_{X^\infty}(\lambda) = 0$ a.s., for $\mathbb{P}$-a.e. realization of the environment ω: Recall the identities (15) and (16) of Section 3. By Assumptions (A1) and (A2), $\lim_{i\to\infty} \psi(i)/i = \mathbb{E}[Z_0] < \infty$, $\mathbb{P}$-a.s. On the other hand, as proved in Section 6, the random walks X^∞_n and $\mathbb{X}^\infty_t$ are a.s. transient to the right. As a byproduct, due to (15) and (16), we have $v_Y(\lambda) = 0$, $v_{\mathbb{Y}}(\lambda) = 0$ whenever $v_{X^\infty}(\lambda) = 0$, $v_{\mathbb{X}^\infty}(\lambda) = 0$, respectively. But we also have that $v_{X^\infty}(\lambda) = 0$ implies $v_{\mathbb{X}^\infty}(\lambda) = 0$. Indeed, the continuous-time random walk $(\mathbb{X}^\infty_t)_{t\geq 0}$ is obtained from the discrete-time random walk $(X^\infty_n)_{n\geq 0}$ by the rule that, when site k is reached, $\mathbb{X}^\infty$ remains at k for an exponential time with parameter $r_k^\lambda(\omega)$. Since $\sup_{k\in\mathbb{Z},\omega\in\Omega} r_k^\lambda(\omega) =: C < \infty$ (cf. Section 2), we can speed up $\mathbb{X}^\infty$ by replacing all parameters $r_k^\lambda(\omega)$ by C. The resulting random walk can be realized as $t \mapsto X^\infty_{n(t)}$ where $(n(t))_{t\geq 0}$ is a Poisson process with intensity C. Hence, its velocity is zero whenever $v_{X^\infty}(\lambda) = 0$.

We first show in Proposition 8.1 a sufficient condition for $v_{X^\infty}(\lambda) = 0$. In Lemma 8.2 we prove that this condition is equivalent to the hypothesis (5) of Theorem 1(iii) and in Corollary 8.3 we discuss some stronger conditions corresponding to the last statement in Theorem 1(iii).

Proposition 8.1. *Suppose that*

$$\mathbb{E}\Big[\Big(\sup_{z\leq 0} P_z^{\omega,\infty}(X_1 \geq 1)\Big)^{-1}\Big] = \infty. \tag{96}$$

Then $v_{X^\infty}(\lambda) = 0$.

We postpone the proof of the above proposition to Section 8.1.

Lemma 8.2. *Condition* (96) *is equivalent to*

$$\mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0-(1+\lambda)Z_{-1}}\big]=\infty. \tag{97}$$

Proof. First of all, we claim that for all $\omega\in\Omega$ and $z\le 0$ we have

$$P_0^\omega(X_1\ge 1)\ge \mathrm{e}^{2(u_{\min}-u_{\max})}P_z^\omega(X_1\ge 1). \tag{98}$$

In fact,

$$P_0^\omega(X_1\ge 1)\ge \mathrm{e}^{(u_{\min}-u_{\max})}\frac{\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}}{\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}+\sum_{j\le -1}\mathrm{e}^{(1+\lambda)x_j}}$$

and

$$P_z^\omega(X_1\ge 1)\le \mathrm{e}^{(u_{\max}-u_{\min})}\frac{\mathrm{e}^{(1-\lambda)x_z}\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}}{\sum_{j\ge z+1}\mathrm{e}^{-(1-\lambda)(x_j-x_z)}+\sum_{j\le z-1}\mathrm{e}^{(1+\lambda)(x_j-x_z)}}.$$

Hence, (98) is satisfied if

$$\frac{\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}}{\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}+\sum_{j\le -1}\mathrm{e}^{(1+\lambda)x_j}}\ge \frac{\mathrm{e}^{(1-\lambda)x_z}\sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}}{\sum_{j\ge z+1}\mathrm{e}^{-(1-\lambda)(x_j-x_z)}+\sum_{j\le z-1}\mathrm{e}^{(1+\lambda)(x_j-x_z)}},$$

which is true if and only if

$$\mathrm{e}^{-(1-\lambda)x_z}\bigg(\sum_{j\ge z+1}\mathrm{e}^{-(1-\lambda)(x_j-x_z)}+\sum_{j\le z-1}\mathrm{e}^{(1+\lambda)(x_j-x_z)}\bigg)\ge \sum_{j\ge 1}\mathrm{e}^{-(1-\lambda)x_j}+\sum_{j\le -1}\mathrm{e}^{(1+\lambda)x_j}.$$

Simplifying the expression (the terms with $j\ge 1$ cancel out), the last display is equivalent to

$$\sum_{z+1\le j\le -1}\mathrm{e}^{-(1-\lambda)x_j}+1+\mathrm{e}^{-2x_z}\sum_{j\le z-1}\mathrm{e}^{(1+\lambda)x_j}\ge \sum_{z+1\le j\le -1}\mathrm{e}^{(1+\lambda)x_j}+\mathrm{e}^{(1+\lambda)x_z}+\sum_{j\le z-1}\mathrm{e}^{(1+\lambda)x_j}$$

and the last inequality clearly holds since the l.h.s. terms dominate one by one the r.h.s. ones.

Equation (98) shows that $P_0^\omega(X_1\ge 1)\le \sup_{z\le 0}P_z^\omega(X_1\ge 1)\le C\cdot P_0^\omega(X_1\ge 1)$ for a constant C which does not depend on ω. On the other hand, using estimates (22) and (23),

$$P_0^\omega(X_1\ge 1)=\frac{\sum_{j>1}c_{0,j}}{\sum_{j\ne 0}c_{0,j}}\le K_1\cdot\frac{c_{0,1}}{c_{0,-1}}=K_1'\cdot \mathrm{e}^{-(1-\lambda)Z_0+(1+\lambda)Z_{-1}},$$

$$P_0^\omega(X_1\ge 1)\ge K_2\cdot\frac{c_{0,1}}{c_{0,-1}+c_{0,1}}=K_2'\cdot\frac{\mathrm{e}^{-(1-\lambda)Z_0}}{\mathrm{e}^{-(1+\lambda)Z_{-1}}+\mathrm{e}^{-(1-\lambda)Z_0}}$$

for constants K_1, K_1', K_2, K_2' which do not depend on ω.

Hence, we have (96) $\Longleftrightarrow \mathbb{E}[\frac{1}{P_0^\omega(X_1^\lambda\ge 1)}]=\infty \Longleftrightarrow \mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0-(1+\lambda)Z_{-1}}]=\infty$. □

Corollary 8.3. *Suppose that* $\mathbb{E}[Z_{-1}|Z_0]\le C$ *for some constant which does not depend on* ω *(e.g. if the* $(Z_i)_{i\in\mathbb{Z}}$ *are i.i.d.) and that* $\mathbb{E}[\mathrm{e}^{(1-\lambda)Z_0}]=\infty$. *Then condition* (97) *is satisfied and in particular* $v_{X^\infty}(\lambda)=0$.

Proof. Conditioning on Z_0 and using Jensen's inequality, we get

$$\begin{aligned}\mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0-(1+\lambda)Z_{-1}}\big]&=\mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0}\mathbb{E}\big[\mathrm{e}^{-(1+\lambda)Z_{-1}}|Z_0\big]\big]\ge \mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0}\mathrm{e}^{-(1+\lambda)\mathbb{E}[Z_{-1}|Z_0]}\big]\\&\ge \mathrm{e}^{-(1+\lambda)C}\mathbb{E}\big[\mathrm{e}^{(1-\lambda)Z_0}\big]=\infty.\end{aligned}$$

□

8.1. *Proof of Proposition* 8.1

The proof goes through the construction of several couplings, including the so called quantile coupling that we recall in Lemma 8.4 below.

Before entering into the technical details we give a sketch of the proof and its main ideas. We fix an environment $\omega \in \Omega$. In *Step 1* below we define T_1 as the first time the random walk $(X_n^{\omega,\infty})_{n\in\mathbb{N}}$ goes to the right of the origin and couple $W_1 := X_{T_1}^{\omega,\infty}$ with a suitable finite-mean random variable ξ_1 such that ξ_1 is independent of ω and $\xi_1 \geq W_1$ almost surely (see Claim 8.6). We also couple (see the paragraph containing (112)) T_1 with a geometric random variable S_0 of parameter $s_0 = \sup_{z\leq 0} P_z^{\omega,\infty}(X_1 \geq 1)$ in a way that guarantees $S_0 \leq T_1$ almost surely.

At this point, we can inductively define the random variables T_{k+1}, W_{k+1}, ξ_{k+1} (see *Step k+1*) and S_k (see the paragraph containing (112)) for $k \geq 1$ in the following way:

- T_{k+1} is the first time $(X_n^{\omega,\infty})_{n\in\mathbb{N}}$ goes to the right of the point $\xi_1 + \cdots + \xi_k$ (see Fig. 1);
- W_{k+1} is the overshoot $X_{T_{k+1}}^{\omega,\infty} - (\xi_1 + \cdots + \xi_k)$;
- ξ_{k+1} is a random variable with the same law of ξ_1 that is independent of all the previous $\xi_\cdot$'s and of ω. We can couple ξ_{k+1} and W_{k+1} so that $\xi_{k+1} \geq W_{k+1}$ almost surely (see Claim 8.8). As a consequence, $X_{T_{k+1}}^{\infty} \leq \xi_1 + \cdots + \xi_{k+1}$ almost surely;
- S_k is a geometric random variable (with parameter given in (112)) coupled with the difference $T_{k+1} - T_k$ in a way that guarantees $S_k \leq T_{k+1} - T_k$ almost surely.

We point out that, to have a unique probability space where all the above infinite random objects are defined, we will use the Ionescu–Tulcea Extension Theorem as discussed in *Step* $+\infty$. In particular, to be precise, in the kth step we will work on a k-fold product probability space, and the random walk built on such a space, and behaving as $(X_n^{\omega,\infty})_{n\in\mathbb{N}}$ when ω is fixed, will be denoted by $(X_n^{(k)})_{n\in\mathbb{N}}$.

Notice now that, by construction, for $T_k \leq n < T_{k+1}$ we have

$$\frac{X_n^{\omega,\infty}}{n} \leq \frac{X_{T_{k+1}}^{\omega,\infty}}{T_k} \leq \frac{\xi_1 + \cdots + \xi_{k+1}}{S_1 + \cdots + S_k} \tag{99}$$

and we would like to conclude by applying the LLN to the $\xi_\cdot$'s and $S_\cdot$'s sequences. While $(\xi_k)_{k\in\mathbb{N}_+}$ is an i.i.d. sequence, the $S_\cdot$'s are unfortunately not independent since their parameters depend on the $\xi_\cdot$'s. Nevertheless, it can be proven that they still constitute a stationary ergodic sequence (see Lemma B.3 in Appendix B). Hence, we are allowed to apply the LLN and derive the sub-ballisticity of $(X_n^{\infty})_{n\in\mathbb{N}}$ by noticing that ξ_1 has finite mean, but $\mathbb{E}[S_1] = +\infty$ because of our assumption (96).

Finally, we point out that the above strategy can be implemented in other contexts to derive sub-ballisticity of random walks in random environment, possibly with long jumps. On the other hand, the construction of the above mentioned couplings require some care also from a notational viewpoint. In order not to introduce further notation we have restricted the exposition to our random walk $(X_n^{\omega,\infty})_{n\in\mathbb{N}}$.

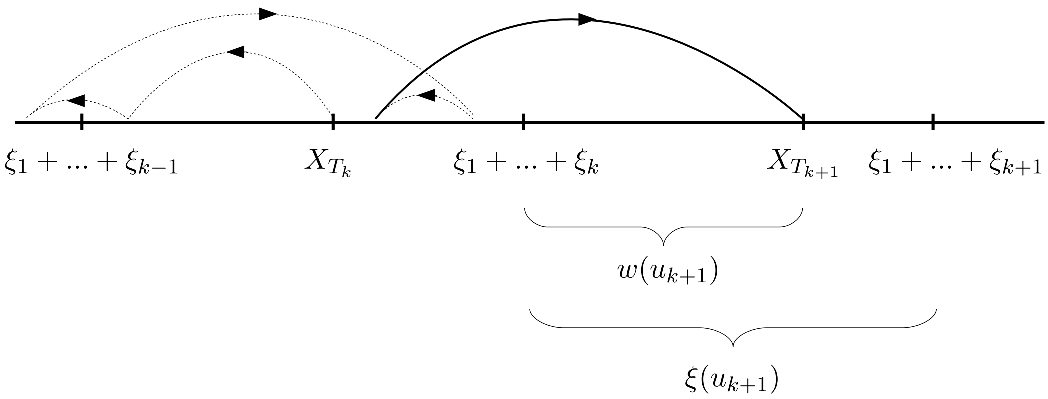

Fig. 1. T_{k+1} is the first time the random walk overjumps the point $\xi_1 + \cdots + \xi_k$. The overshoot $w(u_{k+1})$ is dominated by $\xi(u_{k+1})$ by construction.

Before giving the proof of Proposition 8.1 we describe in the next lemma a basic tool to build couplings:

Lemma 8.4 (Quantile coupling). *For a distribution function G and a value $u \in [0,1]$, define the function*

$$\phi(G,u) := \inf\{x \in \mathbb{R} : G(x) > u\}.$$

Let F and F' be two distribution functions such that $F(x) \le F'(x)$ for all $x \in \mathbb{R}$. Take U to be a uniform random variable on $[0,1]$ and let $Y := \phi(F,U)$ and $Y' := \phi(F',U)$. Then Y is distributed according to F, Y' is distributed according to F' and $Y \ge Y'$ almost surely.

The proof of the above lemma can be found in [22]. Usually, as in [22], the quantile coupling is defined with $\phi_q(G,u)$ instead of $\phi(G,u)$, where $\phi_q(G,u)$ is the quantile function $\phi_q(G,u) := \inf\{x \in \mathbb{R} : G(x) \ge u\}$. One can easily prove that $\phi(G,U) = \phi_q(G,U)$ a.s.

Proof of Proposition 8.1. Call F_ξ the distribution function of the random variable $\xi := L + G$, where $L \in \mathbb{N}$ is some constant such that

$$\frac{\mathrm{e}^{u_{\max}-u_{\min}}\mathrm{e}^{-(1-\lambda)dL}}{1-\mathrm{e}^{-(1-\lambda)d}} < 1, \tag{100}$$

and G is a geometric random variable with parameter $\gamma = 1-\mathrm{e}^{-(1-\lambda)d}$. Note that given an integer a it holds

$$1 - F_\xi(a) = \begin{cases} 1 & \text{if } a - L \le 0, \\ (1-\gamma)^{a-L} = \mathrm{e}^{-(1-\lambda)d(a-L)} & \text{if } a - L \ge 1. \end{cases} \tag{101}$$

In particular, given an integer $M \ge L+2$, due to (100) we have

$$\frac{\mathrm{e}^{u_{\max}-u_{\min}}\mathrm{e}^{-(1-\lambda)d(M-1)}}{1-\mathrm{e}^{-(1-\lambda)d}} < \mathrm{e}^{-(1-\lambda)d(M-1-L)} = 1 - F_\xi(M-1). \tag{102}$$

We will now inductively construct a sequence of probability spaces $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^n, P^{(n)})$, on which we will define some random variables.

Step 1. We first consider the space $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]$, the generic element of which is denoted by $(\omega, \bar{x}, u_1)$.

We introduce a probability $P^{(1)}$ on $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]$ by the following rules. The marginal of $P^{(1)}$ on Ω is $\mathbb{P}$, its marginal on $[0,1]$ is the uniform distribution and, under $P^{(1)}$, the coordinate functions $(\omega,\bar{x},u_1) \mapsto \omega$ and $(\omega,\bar{x},u_1) \mapsto u_1$ are independent random variables. Finally, we require that

$$P^{(1)}\big(X^{(1)}_\cdot \in A \mid \omega, u_1\big) = P_0^{\omega,\infty}\big(X_\cdot \in A \mid X_{T_1} = \phi\big(F^{(1)}_\omega, u_1\big)\big) \tag{103}$$

for any measurable set $A \subseteq \mathbb{Z}^{\mathbb{N}}$, where $(X^{(1)}_n)_{n\in\mathbb{N}}$ is the second-coordinate function $(\omega,\bar{x},u_1) \mapsto \bar{x}$ and

$$F^{(1)}_\omega(y) = P_0^{\omega,\infty}(X_{T_1} \le y), \qquad T_1 = \inf\{n \in \mathbb{N} : X^\infty_n > 0\}.$$

From now on we consider the space $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]$ endowed with the probability $P^{(1)}$.

It is convenient to introduce the random variables U_1, ξ_1, W_1 defined as follows:[3]

$$U_1(\omega,\bar{x},u_1) := u_1, \qquad \xi_1(\omega,\bar{x},u_1) := \phi(F_\xi, u_1), \qquad W_1(\omega,\bar{x},u_1) := \phi\big(F^{(1)}_\omega, u_1\big).$$

Note that, by the quantile coupling (cf. Lemma 8.4), ξ_1 is distributed as ξ and W_1 under $P^{(1)}(\cdot|\omega)$ is distributed as $X^\infty_{T_1}$ under $P_0^{\omega,\infty}$.

The interpretation to keep in mind is the following: $(X^{(1)}_n)_{n\in\mathbb{N}}$ plays the role of our initial random walk in environment ω; W_1 is the overshoot at time T_1, i.e. how far from 0 the random walk will land the first time it jumps beyond the point 0; ξ_1 is a positive random variable that dominates W_1 (see Claim 8.6) and that is distributed like ξ.

[3] We will denote the first-coordinate function again by ω, without introducing a new symbol.

Claim 8.5. *For any integer $M \geq 1$ it holds*

$$P_0^{\omega,\infty}(X_{T_1} \geq M) \leq \sup_{z \leq 0} P_z^{\omega,\infty}(X_1 \geq M | X_1 \geq 1). \tag{104}$$

Proof of Claim 8.5. Given $j \geq 1$ and integers $z_1, z_2, \dots, z_{j-1} \leq 0$ we denote by $E(z_1, z_2, \dots, z_{j-1})$ the event $\{X_1^\infty = z_1, \dots, X_{j-1}^\infty = z_{j-1}\}$. Note that, by the Markov property,

$$\frac{P_0^{\omega,\infty}(X_j \geq M, E(z_1, \dots, z_{j-1}))}{P_0^{\omega,\infty}(X_j \geq 1, E(z_1, \dots, z_{j-1}))} = \frac{P_{z_{j-1}}^{\omega,\infty}(X_1 \geq M)}{P_{z_{j-1}}^{\omega,\infty}(X_1 \geq 1)} = P_{z_{j-1}}^{\omega,\infty}(X_1 \geq M | X_1 \geq 1).$$

By the above identity we can write

$$\begin{aligned}
&P_0^{\omega,\infty}(X_{T_1} \geq M) \\
&\quad = \sum_{j=1}^{\infty} \sum_{z_1, \dots, z_{j-1} \leq 0} P_0^{\omega,\infty}\big(X_j \geq M | X_j \geq 1, E(z_1, \dots, z_{j-1})\big) P_0^{\omega,\infty}\big(X_j \geq 1, E(z_1, \dots, z_{j-1})\big) \\
&\quad \leq \sup_{z \leq 0} P_z^{\omega,\infty}(X_1 \geq M | X_1 \geq 1) \sum_{j=1}^{\infty} \sum_{z_1, \dots, z_{j-1} \leq 0} P_0^{\omega,\infty}\big(X_j \geq 1, E(z_1, \dots, z_{j-1})\big) \\
&\quad \leq \sup_{z \leq 0} P_z^{\omega,\infty}(X_1 \geq M | X_1 \geq 1).
\end{aligned}$$

□

Claim 8.6. *The following holds*:

(i) $P^{(1)}(\xi_1 \geq W_1) = 1$;
(ii) ξ_1 *is independent of* ω *under* $P^{(1)}$;
(iii) $P^{(1)}(X_\cdot^{(1)} \in B | \omega) = P_0^{\omega,\infty}(X_\cdot \in B)$ *for each measurable set* $B \subset \mathbb{Z}^{\mathbb{N}}$.

Proof of Claim 8.6. In order to show (i), we just have to prove that $F_\omega^{(1)}(x) \leq F_\xi(x)$ for all $\omega \in \Omega$ and $x \in \mathbb{R}$ (in fact, it is enough to prove it for all $x \in \mathbb{N}$) thanks to Lemma 8.4. To this aim, recall the definition of L (see (100)) and notice that for all $\omega \in \Omega$ and all integers $M \geq L + 2$, one has

$$\begin{aligned}
1 - F_\omega^{(1)}(M-1) = P_0^{\omega,\infty}(X_{T_1} \geq M) &\leq \sup_{z \leq 0} P_z^{\omega,\infty}(X_1 \geq M | X_1 \geq 1) \\
&= \sup_{z \leq 0} \frac{\sum_{j \geq M} \mathrm{e}^{-(1-\lambda)(x_j - x_z) + u(E_z, E_j)}}{\sum_{j \geq 1} \mathrm{e}^{-(1-\lambda)(x_j - x_z) + u(E_z, E_j)}} \\
&\leq \mathrm{e}^{u_{\max} - u_{\min}} \sup_{z \leq 0} \frac{\sum_{j \geq M} \mathrm{e}^{-(1-\lambda)(x_j - x_z)}}{\mathrm{e}^{-(1-\lambda)(x_1 - x_z)}} \\
&= \mathrm{e}^{u_{\max} - u_{\min}} \sum_{j \geq M} \mathrm{e}^{-(1-\lambda)(x_j - x_1)} \leq \mathrm{e}^{u_{\max} - u_{\min}} \sum_{j \geq M} \mathrm{e}^{-(1-\lambda)d(j-1)} \\
&= \mathrm{e}^{u_{\max} - u_{\min}} \frac{\mathrm{e}^{-(1-\lambda)d(M-1)}}{1 - \mathrm{e}^{-(1-\lambda)d}} \leq 1 - F_\xi(M-1),
\end{aligned} \tag{105}$$

where in the first line we have used Claim 8.5 and in the last bound we have used (102) and the fact that $M \geq L + 2$. This proves that $F_\omega^{(1)}(a) \geq F_\xi(a)$ for all $a \in \mathbb{N}$ with $a \geq L + 1$. The same inequality trivially holds also for $a \leq L$ since in this case $F_\xi(a) = 0$ (because $\xi > L$).

Part (ii) is clear since ξ_1 is determined only by U_1, while U_1 and ω are independent by construction.

For part (iii) take some measurable set $B \subset \mathbb{Z}^{\mathbb{N}}$ and notice that (recalling (103) and the independence of ω and $U-1$)

$$
\begin{aligned}
P^{(1)}\big(X_\cdot^{(1)} \in B|\omega\big) &= \int_{[0,1]} P^{(1)}\big(X_\cdot^{(1)} \in B|\omega, U_1 = u_1\big) P^{(1)}(U_1 \in du_1) \\
&= \int_{[0,1]} P_0^{\omega,\infty}\big(X_\cdot \in B | X_{T_1} = \phi\big(F_\omega^{(1)}, u_1\big)\big)\, du_1 \\
&= \sum_{j=1}^{\infty} P_0^{\omega,\infty}(X_\cdot \in B|X_{T_1} = j) P_0^{\omega,\infty}(X_{T_1} = j) = P_0^{\omega,\infty}(X_\cdot \in B).
\end{aligned}
$$

□

Step $k+1$. Suppose now we have achieved our construction up to step k. In particular, we have built the probability $P^{(k)}$ on the space $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^k$ and several random variables on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^k, P^{(k)})$ that we list:

- $U_1, \dots, U_k$ are independent and uniformly distributed random variables such that $(U_1, \dots, U_k)$ is the projection function on $[0,1]^k$;
- $\xi_1, \dots, \xi_k$ is defined as $\xi_j = \phi(F_\xi, U_j)$, $j = 1, \dots, k$;
- $(X_n^{(k)})_{n \ge 0}$, defined as the projection function on $\mathbb{Z}^{\mathbb{N}}$, whose law under $P^{(k)}(\cdot|\omega)$ is $P_0^{\omega,\infty}$;
- $W_1, W_2, \dots, W_k$ such that $P^{(k)}(\xi_i \ge W_i \text{ for all } i = 1, \dots, k) = 1$.

We introduce a probability $P^{(k+1)}$ on $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^{k+1}$ by the following rules. The marginal of $P^{(k+1)}$ on Ω is $\mathbb{P}$, its marginal on $[0,1]^{k+1}$ is the uniform distribution and, under $P^{(k+1)}$, the projection functions $(\omega, \bar{x}, u_1, \dots, u_{k+1}) \mapsto \omega$ and $(\omega, \bar{x}, u_1, \dots, u_{k+1}) \mapsto (u_1, \dots, u_{k+1})$ are independent random variables. Finally, we require that

$$
\begin{aligned}
&P^{(k+1)}\big(X_\cdot^{(k+1)} \in A|\omega, u_1, \dots, u_k, u_{k+1}\big) \\
&\quad = P^{(k)}\big(X_\cdot^{(k)} \in A|\omega, u_1, \dots, u_k, X_{T_{k+1}}^{(k)} = \xi_1 + \dots + \xi_k + \phi\big(F_{\omega,u_1,\dots,u_k}^{(k+1)}, u_{k+1}\big)\big)
\end{aligned}
\tag{106}
$$

for any measurable set $A \subseteq \mathbb{Z}^{\mathbb{N}}$, where

$$
\begin{aligned}
&F_{\omega,u_1,\dots,u_k}^{(k+1)}(y) := P^{(k)}\big(X_{T_{k+1}}^{(k)} \le \xi_1 + \dots + \xi_k + y|\omega, u_1, \dots, u_k\big), \\
&T_{k+1} := \inf\big\{n \in \mathbb{N} : X_n^{(k)} > \xi_1 + \dots + \xi_k\big\}.
\end{aligned}
$$

Note that T_{k+1} is a random variable on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^k, P^{(k)})$. We stress that the conditional probability in the r.h.s. of (8.1) has to be thought of as the regular conditional probability $P^{(k)}(\cdot|\omega, u_1, \dots, u_k)$ further conditioned on the event $\{X_{T_{k+1}}^{(k)} = \xi_1 + \dots + \xi_k + \phi(F_{\omega,u_1,\dots,u_k}^{(k+1)}, u_{k+1})\}$.

Claim 8.7. *The marginal of* $P^{(k+1)}$ *on* $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^k$ *is exactly* $P^{(k)}$.

Proof of Claim 8.7. Since the marginal of $P^{(k+1)}$ along the coordinate u_{k+1} is the uniform distribution, by integrating (8.1) over u_{k+1}, we get

$$
\begin{aligned}
&P^{(k+1)}\big(X_\cdot^{(k+1)} \in A|\omega, u_1, \dots, u_k\big) \\
&\quad = \sum_{j=1}^{\infty} P^{(k)}\big(X_\cdot^{(k)} \in A|\omega, u_1, \dots, u_k, X_{T_{k+1}}^{(k)} = \xi_1 + \dots + \xi_k + j\big) \int_0^1 \mathbb{1}\big(\phi\big(F_{\omega,u_1,\dots,u_k}^{(k+1)}, u\big) = j\big)\, du.
\end{aligned}
\tag{107}
$$

Above we have used Lemma 8.4 to deduce that $\phi(F_{\omega,u_1,\dots,u_k}^{(k+1)}, u)$ has integer values. Applying again Lemma 8.4 and the definition of $F_{\omega,u_1,\dots,u_k}^{(k+1)}$ we have

$$
\int_0^1 \mathbb{1}\big(\phi\big(F_{\omega,u_1,\dots,u_k}^{(k+1)}, u\big) = j\big)\, du = P^{(k)}\big(X_{T_{k+1}}^{(k)} = \xi_1 + \dots + \xi_k + j|\omega, u_1, \dots, u_k\big).
\tag{108}
$$

Plugging (108) into (8.1), we get

$$P^{(k+1)}\big(X_\cdot^{(k+1)} \in A|\omega, u_1, \dots, u_k\big) = P^{(k)}\big(X_\cdot^{(k)} \in A|\omega, u_1, \dots, u_k\big). \tag{109}$$

On the other hand, the projections of $P^{(k+1)}$ and $P^{(k)}$ on $\Omega \times [0,1]^k$, i.e. along the coordinates $\omega, u_1, \dots, u_k$, are equal by construction, thus concluding the proof of our claim. □

Due to the above claim, any random variable Y defined on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^k, P^{(k)})$ can be thought of as a random variable on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^{k+1}, P^{(k+1)})$, by considering the map $(\omega, \bar{x}, u_1, \dots, u_k, u_{k+1}) \mapsto Y(\omega, \bar{x}, u_1, \dots, u_k)$. With some abuse of notation, we denote by Y also the last random variable.

As a consequence, $U_1, \dots, U_k, \xi_1, \dots, \xi_k, W_1, \dots, W_k$ can be thought as random variables on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^{k+1}, P^{(k+1)})$. Finally, we introduce the new random variables $U_{k+1}, \xi_{k+1}, W_{k+1}$ on $(\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^{k+1}, P^{(k+1)})$ defined as

$$\begin{aligned}
&U_{k+1}(\omega, \bar{x}, u_1, \dots, u_{k+1}) := u_{k+1}, \\
&\xi_{k+1}(\omega, \bar{x}, u_1, \dots, u_{k+1}) := \phi(F_\xi, u_{k+1}), \\
&W_{k+1}(\omega, \bar{x}, u_1, \dots, u_{k+1}) := \phi\big(F^{(k+1)}_{\omega, u_1, \dots, u_k}, u_{k+1}\big).
\end{aligned}$$

The interpretation is similar as in *Step* 1: W_{k+1} is the overshoot at time T_{k+1}, i.e. how far from $\xi_1 + \dots + \xi_k$ the random walk will land the first time it jumps beyond that point; ξ_{k+1} is a positive random variable that dominates W_{k+1} (see Claim 8.8) and that is distributed as ξ.

Claim 8.8. *The following three facts hold true*:

(i) $P^{(k+1)}(\xi_{k+1} \geq W_{k+1}) = 1$;
(ii) ξ_{k+1} *is independent of* $\omega, U_1, \dots, U_k$ *under* $P^{(k+1)}$;
(iii) *For each measurable set* $B \subset \mathbb{Z}^{\mathbb{N}}$,

$$P^{(k+1)}\big(X_\cdot^{(k+1)} \in B|\omega\big) = P_0^{\omega,\infty}(X_\cdot \in B).$$

Proof of Claim 8.8. The three facts can be proved in a similar way as Claim 8.6. We give the proof for completeness.

For Part (i) we want to show that $F^{(k+1)}_{\omega, u_1, \dots, u_k}(M-1) \geq F_\xi(M-1)$ for all $M \geq L+2$, with $M \in \mathbb{N}$. In fact, as for Claim 8.6, this inequality can easily be extended to all $M \in \mathbb{N}$ and the conclusion follows.

First of all we notice that, by iteratively applying (8.1) and using Claim 8.6(iii), we have

$$\begin{aligned}
1 - F^{(k+1)}_{\omega, u_1, \dots, u_k}(M-1) &= P^{(k)}\big(X^{(k)}_{T_{k+1}} \geq \xi_1 + \dots + \xi_k + M|\omega, u_1, \dots, u_k\big) \\
&= P_0^{\omega,\infty}\big(X_{\inf\{n: X_n > \xi(u_1) + \dots + \xi(u_k)\}} \geq \xi(u_1) + \dots + \xi(u_k) + M|D_k\big),
\end{aligned} \tag{110}$$

where we have used the shortened notation $\xi(u) := \phi(F_\xi, u)$ and D_k is the event

$$\begin{aligned}
D_k : = \big\{&X_{T_1} = \phi\big(F^{(1)}_\omega, u_1\big), X_{\inf\{n: X_n > \xi(u_1)\}} = \xi(u_1) + \phi\big(F^{(2)}_{\omega, u_1}, u_2\big), \dots, \\
&X_{\inf\{n: X_n > \xi(u_1) + \dots + \xi(u_{k-1})\}} = \xi(u_1) + \dots + \xi(u_{k-1}) + \phi\big(F^{(k)}_{\omega, u_1, \dots, u_{k-1}}, u_k\big)\big\}.
\end{aligned}$$

For convenience we call

$$\begin{aligned}
&D'_k := \{X_{\inf\{n: X_n > \xi(u_1) + \dots + \xi(u_{k-1})\}} = y_k\}, \\
&y_k := y_k(u_1, \dots, u_k) := \xi(u_1) + \dots + \xi(u_{k-1}) + \phi\big(F^{(k)}_{\omega, u_1, \dots, u_{k-1}}, u_k\big), \\
&w_k = w_k(u_1, \dots, u_k) := \phi\big(F^{(k)}_{\omega, u_1, \dots, u_{k-1}}, u_k\big).
\end{aligned}$$

We also note that $\xi(u_k) \geq w_k$ $P^{(k)}$-a.s. (see the list of properties at the beginning of *Step* $k+1$). Coming back to (110), by using the strong Markov Property, we obtain (see also the proof of Claim 8.5)

$$\begin{aligned}
&P^{(k)}\big(X^{(k)}_{T_{k+1}} \geq \xi_1 + \cdots + \xi_k + M|\omega, u_1, \dots, u_k\big)\\
&\quad = P_0^{\omega,\infty}\big(X_{\inf\{n: X_n > \xi(u_1)+\cdots+\xi(u_k)\}} \geq \xi(u_1) + \cdots + \xi(u_k) + M|D'_k\big)\\
&\quad = P_0^{\tau_{y_k}\omega,\infty}\big(X_{\inf\{n: X_n > \xi(u_k)-w_k\}} \geq \xi(u_k) - w_k + M\big)\\
&\quad = \sum_{i\in\mathbb{N}_+} P_0^{\tau_{y_k}\omega,\infty}\big(X_i \geq \xi(u_k) - w_k + M|\inf\{n : X_n > \xi(u_k) - w_k\} = i\big)\\
&\qquad \times P_0^{\tau_{y_k}\omega,\infty}\big(\inf\{n : X_n > \xi(u_k)\} = i\big)\\
&\quad \leq \sup_{z\leq\xi(u_k)-w_k} P_z^{\tau_{y_k}\omega,\infty}(X_1 \geq M|X_1 \geq 1).
\end{aligned} \tag{111}$$

The last inequality follows by conditioning to the position of the random walk at time $i-1$. Knowing this, we can proceed as in (105) getting that the last term in (8.1) is bounded from above by $1 - F_\xi(M-1)$. This concludes the proof of Part (i).

Part (ii) is clear by the construction of ξ_{k+1}. Finally, we prove Part (iii). Since the projections of $P^{(k+1)}$ and of $P^{(k)}$ on $[0,1]^k$, i.e. along the coordinates $u_1, \dots, u_k$, are both the uniform distribution on $[0,1]^k$, integrating (109) over $u_1, \dots, u_k$ we get $P^{(k+1)}(X^{(k+1)}_\cdot \in A|\omega) = P^{(k)}(X^{(k)}_\cdot \in A|\omega)$. The claim then follows by the induction hypothesis (see the discussion at the beginning of *Step* $k+1$). □

Due to the results discussed above, the list of properties at the beginning of *Step* $k+1$ is valid also for $P^{(k+1)}$.

Step $+\infty$: By the Ionescu–Tulcea Extension Theorem, there exists a measure $P^{(\infty)}$ on the space $\Omega \times \mathbb{Z}^{\mathbb{N}} \times [0,1]^{\mathbb{N}}$, random variables $\xi_1, \xi_2, \dots, W_1, W_2, \dots, T_1, T_2, \dots$ and a random walk $(X_n^{(\infty)})_{n\in\mathbb{N}}$, such that: For all measurable $A \subset \Omega$, $P^{(\infty)}(\omega \in A) = \mathbb{P}(\omega \in A)$; the ξ_k's are i.i.d., distributed like ξ and independent of ω; $P^{(\infty)}(X^{(\infty)}_{T_k} = \xi_1 + \cdots + \xi_{k-1} + W_k) = 1$; $P^{(\infty)}(\xi_k \geq W_k) = 1$; for all measurable $B \subset \mathbb{Z}^{\mathbb{N}}$, $P^{(\infty)}((X_n^{(\infty)})_{n\in\mathbb{N}} \in B|\omega) = P_0^{\omega,\infty}((X_n)_{n\in\mathbb{N}} \in B)$.

We are now ready to finish the proof. Notice that, under $P^{(\infty)}(\cdot|\omega)$, the differences $(T_{k+1} - T_k)_{k=0,1,\dots}$ have a rather complicated structure, but they stochastically dominate a sequence of pretty simple objects, call them $(S_k)_{k=0,1,\dots}$. Each S_k is a geometric random variable of parameter

$$s_k = \sup_{z\leq 0} P_z^{\tau_{\xi_1+\cdots+\xi_k}\omega,\infty}(X_1 \geq 1). \tag{112}$$

In fact, due to Lemma 3.16, we can imagine that for each $n \geq T_k$ the random walk "attempts" to overjump $\xi_1 + \cdots + \xi_k$ and manages to do so with a probability that is clearly smaller than s_k. By Strassen's Theorem, on an enlarged probability space with new probability $\tilde{P}^{(\infty)}$, we can couple each S_k with $T_{k+1} - T_k$ so that $S_k \leq T_{k+1} - T_k$ almost surely. Moreover, due to the strong Markov property of the random walk, all the S_k's can be taken independent once we have fixed the parameters s_k's. Now note the key fact that, since the $\xi_\cdot$'s are independent of the environment and that the GCD of the values attained with positive probability by the $\xi_\cdot$'s is 1, the shifts $(\tau_{\xi_1+\cdots+\xi_k}\omega)_{k\in\mathbb{N}}$ form a stationary ergodic sequence under $P^{(\infty)}$. We refer to Appendix B for a proof of this fact (see Lemma B.1). This observation allows to prove that $(S_j)_{j\in\mathbb{N}}$ is a stationary ergodic sequence with respect to shifts under $\tilde{P}^{(\infty)}$ (see Lemma B.3 in Appendix B).

We now take $\omega \in \Omega$ such that $\lim_{n\to\infty} X_n = +\infty$ P_0^ω-a.s. (which holds for $\mathbb{P}$-a.a. ω by Theorem 1(i)). This implies that $\liminf_{n\to\infty} \frac{X_n}{n} \geq 0$, P_0^ω-a.s.

We can bound (see (1))

$$\begin{aligned}
P_0^\omega\left(\limsup_{n\to\infty} \frac{X_n}{n} > 0\right) &= P^{(\infty)}\left(\limsup_{n\to\infty} \frac{X_n}{n} > 0\middle|\omega\right)\\
&\leq P^{(\infty)}\left(\limsup_{k\to\infty} \frac{X_{T_{k+1}}}{T_k} > 0\middle|\omega\right)
\end{aligned}$$

$$\leq P^{(\infty)}\Big(\limsup_{k\to\infty}\frac{\xi_1+\cdots+\xi_{k+1}}{\sum_{j=0}^{k-1}(T_{j+1}-T_j)}>0\Big|\omega\Big)$$

$$\leq \tilde P^{(\infty)}\Big(\limsup_{k\to\infty}\Big(\frac{\sum_{i=1}^{k+1}\xi_i}{k}\Big)\Big(\frac{\sum_{j=0}^{k-1}S_j}{k}\Big)^{-1}>0\Big|\omega\Big).$$

Let us concentrate on the last line. The arithmetic mean of $\xi_1,\dots,\xi_{k+1}$ converges almost surely to $L+1/\gamma$, the mean of ξ, by the law of large numbers. The arithmetic mean of $S_0,\dots,S_{k-1}$ converges instead to $\mathbb{E}[S_0]$ because of the ergodic theorem (for simplicity, we write simply $\mathbb{E}$ for the expectation with respect to $\tilde P^{(\infty)}$). Since $\mathbb{E}[S_0]=\mathbb{E}[\mathbb{E}[S_0|s_0]]=\mathbb{E}[\frac{1}{s_0}]=\infty$ by assumption (96) and by (112), we obtain that $P_0^{\omega,\infty}(\limsup_{n\to\infty}\frac{X_n}{n}>0)=0$ for almost all $\omega\in\Omega$. Taking into account that $\liminf_{n\to\infty}\frac{X_n}{n}\geq 0$, $P_0^{\omega,\infty}$-a.s., we get that $\lim_{n\to\infty}\frac{X_n}{n}=0$, $P_0^{\omega,\infty}$-a.s. □

Appendix A: Proof of Proposition 5.3

By the tightness stated in Lemma 5.9 and by Prohorov's theorem, $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$ admits some limit point and any limit point $\mathbb{Q}^\infty$ is absolutely continuous to $\mathbb{P}$, with Radon–Nikodym derivative $\frac{d\mathbb{Q}^\infty}{d\mathbb{P}}$ bounded by F from above and by γ from below.

We now show that any limit point is an invariant distribution of the process given by the environment viewed from the walker without truncation $(\tau_{X_n^\infty}\omega)_{n\in\mathbb{N}}$. To this end, let $(\mathbb{Q}^{\rho_k})_{k\geq1}$ be a subsequence weakly converging to some probability $\mathbb{Q}^\infty$ on Ω. We take a bounded continuous function f on Ω (without loss of generality we assume $\|f\|_\infty\leq 1$) and we write

$$\begin{aligned}\big|\mathbb{E}^\infty\big[f(\omega)\big]-\mathbb{E}^\infty E_0^{\omega,\infty}\big[f(\tau_{X_1}\omega)\big]\big| &\leq \big|\mathbb{E}^\infty\big[f(\omega)\big]-\mathbb{E}^{\rho_k}\big[f(\omega)\big]\big|\\ &\quad+\big|\mathbb{E}^{\rho_k}E_0^{\omega,\rho_k}\big[f(\tau_{X_1}\omega)\big]-\mathbb{E}^\infty E_0^{\omega,\rho_k}\big[f(\tau_{X_1}\omega)\big]\big|\\ &\quad+\big|\mathbb{E}^\infty E_0^{\omega,\rho_k}\big[f(\tau_{X_1}\omega)\big]-\mathbb{E}^\infty E_0^{\omega,\infty}\big[f(\tau_{X_1}\omega)\big]\big|\\ &=: B_1+B_2+B_3.\end{aligned}\tag{113}$$

Above, $\mathbb{E}^\infty$ is the expectation with respect to the measure $\mathbb{Q}^\infty$ and in the second line we have used the fact that $\mathbb{E}^{\rho_k}$, the expectation with respect to the measure $\mathbb{Q}^{\rho_k}$, is invariant for the process $(\tau_{X_n^{\rho_k}}\omega)_{n\in\mathbb{N}}$. The term B_1 goes to zero as $k\to\infty$ since $\mathbb{Q}^{\rho_k}\to\mathbb{Q}^\infty$. To deal with term B_2 we observe that, by Lemma 3.9, for any $\delta>0$ there exists h_0 such that, for any $\rho\in\mathbb{N}_+\cup\{\infty\}$,

$$P_0^{\omega,\rho}\big(|X_1|>h_0\big)<\delta,\quad \mathbb{P}\text{-a.s.}\tag{114}$$

Then, for $\rho_k\geq h_0$, we write

$$\begin{aligned}B_2&\leq\Big|\mathbb{E}^{\rho_k}\Big[\sum_{|j|\leq h_0}P_0^{\omega,\rho_k}(X_1=j)f(\tau_j\omega)\Big]-\mathbb{E}^\infty\Big[\sum_{|j|\leq h_0}P_0^{\omega,\rho_k}(X_1=j)f(\tau_j\omega)\Big]\Big|+2\delta\\ &\leq\Big|\mathbb{E}^{\rho_k}\Big[\sum_{|j|\leq h_0}P_0^{\omega,\infty}(X_1=j)f(\tau_j\omega)\Big]-\mathbb{E}^\infty\Big[\sum_{|j|\leq h_0}P_0^{\omega,\infty}(X_1=j)f(\tau_j\omega)\Big]\Big|\\ &\quad+\mathbb{E}^{\rho_k}\big[P_0^{\omega,\infty}\big(|X_1|>h_0\big)\big]+\mathbb{E}^\infty\big[P_0^{\omega,\infty}\big(|X_1|>h_0\big)\big]+2\delta\\ &\leq\Big|\mathbb{E}^{\rho_k}\Big[\sum_{|j|\leq h_0}P_0^{\omega,\infty}(X_1=j)f(\tau_j\omega)\Big]-\mathbb{E}^\infty\Big[\sum_{|j|\leq h_0}P_0^{\omega,\infty}(X_1=j)f(\tau_j\omega)\Big]\Big|+4\delta.\end{aligned}$$

Note that we have used (114) in the first and third estimates. For the second bound we have used that $h_0\leq\rho_k$, $P_0^{\omega,\rho_k}(X_1=j)=P_0^{\omega,\infty}(X_1=j)$ for $0<|j|\leq\rho_k$, while $P_0^{\omega,\rho_k}(X_1=0)=1-\sum_{j:0<|j-x|\leq\rho_k}P_0^{\omega,\infty}(X_1=j)$ and $P_0^{\omega,\infty}(X_1=0)=0$ (cf. (14)).

By the continuity assumption on u and since $\|c_{0,k}(\cdot)\|_\infty \le e^{-(1-\lambda)dk+u_{\max}}$, the map $\Omega \ni \omega \mapsto P_0^{\omega,\infty}(X_1 = j) = \frac{c_{0,j}(\omega)}{\sum_{i\in\mathbb{Z}} c_{0,i}(\omega)} \in \mathbb{R}_+$ is continuous. Hence, using that $\mathbb{Q}^{\rho_k}$ converges to $\mathbb{Q}^\infty$ as $k \to \infty$, we can choose k large enough so that $B_2 \le 5\delta$. B_3 is also smaller than δ for k big enough, again by (114). Altogether, letting $\rho \to \infty$, (113) implies that $\mathbb{Q}^\infty$ is invariant for $(\tau_{X_n^\infty}\omega)_{n\in\mathbb{N}}$ with transition mechanism induced by $P_0^{\omega,\infty}$.

Having that $\mathbb{Q}^\infty \ll \mathbb{P}$, the ergodicity of $\mathbb{Q}^\infty$ can be proved in the same way as Lemma 5.1.

It remains to prove uniqueness of the limit point. To this aim, take two limit points $\mathbb{Q}^\infty$ and $\mathbb{Q}'^\infty$ of $(\mathbb{Q}^\rho)_{\rho\in\mathbb{N}_+}$. Recall that we write $\mathcal{P}^\infty_{\mathbb{Q}^\infty}$ and $\mathcal{P}^\infty_{\mathbb{Q}'^\infty}$ for the law on the path space $\Omega^{\mathbb{Z}}$ of the Markov chains $(\tau_{X_n^\infty}\omega)_{n\in\mathbb{N}}$, induced by $P_0^{\omega,\infty}$, with initial distributions $\mathbb{Q}^\infty$ and $\mathbb{Q}'^\infty$, respectively. As proved above, $\mathcal{P}^\infty_{\mathbb{Q}^\infty}$ and $\mathcal{P}^\infty_{\mathbb{Q}'^\infty}$ are stationary and ergodic with respect to shifts. In particular, they must be either singular or the same. They cannot be singular, since $\mathbb{Q}^\infty$ and $\mathbb{Q}'^\infty$ are both mutually absolutely continuous with respect to $\mathbb{P}$ by Lemma 5.9 and therefore absolutely continuous with respect to each other. Hence, $\mathcal{P}^\infty_{\mathbb{Q}^\infty}$ and $\mathcal{P}^\infty_{\mathbb{Q}'^\infty}$ are equal, and therefore $\mathbb{Q}^\infty = \mathbb{Q}'^\infty$.

Appendix B: Ergodic issues

In Lemmas B.1 and B.3 we prove the results we used in the proof of Proposition 8.1, see the discussion after equation (112). In Lemma B.4 we prove an assertion on assumption (A1) made in Section 2.1.

For the first technical result, we slightly change the notation to make it lighter: Take $\Omega := \mathbb{R}^{\mathbb{Z}}$, the space of two-sided sequences with real values, and let μ be a stationary measure on Ω, ergodic with respect to the usual shift τ_1 for sequences. We indicate by ω an element in Ω. Let $\Xi := \mathbb{N}^{\mathbb{N}}$ and P be a probability measure on it. $\eta = (\eta_i)_{i\in\mathbb{N}} \in \Xi$ is an i.i.d. sequence of natural numbers under the measure P. We assume that the η_i's are independent of the ω's.

On the space $\Omega \times \Xi$ endowed with the product measure $\mathbb{L} = \mu \otimes P$, we define the transformation $T : \Omega \times \Xi \to \Omega \times \Xi$, with $T(\omega, \eta) = (\tau_{\eta_1}\omega, \tau_1\eta)$.

Lemma B.1. *Assume that the greatest common divisor of* $\{k : P(\eta_1 = k) > 0\}$ *equals* 1. *Assume also* (*just for simplicity*) *that the* η_i*'s have finite expectation. Then, the transformation* T *is ergodic.*

Remark B.2. The statement is not true in general without the GCD condition. Indeed, take the very simple space with only two elements, $\omega_1 = (\dots, 0, 1, 0, 1, 0, 1, \dots)$ and $\omega_2 = \tau_1\omega_1$, and take μ putting $1/2$ probability to each of the two elements. Then μ is ergodic with respect to τ_1. But, if we take η_i's that can attain only even values, then the sequence $(\tau_{\eta_1+\dots+\eta_j}\omega)_{j\in\mathbb{N}}$ is not ergodic under $\mathbb{L} = \mu \times P$.

Proof. Take a function $f = f(\omega, \eta)$ which is invariant under T and bounded. We are going to show that f is constant, $\mathbb{L}$-almost surely, hence proving the claim.

Assume we have, for two sequences $\eta^{(1)}, \eta^{(2)}$,

$$\sum_{k=1}^n \eta_k^{(1)} = \sum_{k=1}^n \eta_k^{(2)} \tag{115}$$

for some n and $\eta_k^{(1)} = \eta_k^{(2)}$ for $k \ge n$. Then $T^n(\omega, \eta^{(1)}) = T^n(\omega, \eta^{(2)})$ and hence $f(\omega, \eta^{(1)}) = f(\omega, \eta^{(2)})$.

We define $\mathcal{F}_n$ as the σ-algebra generated by $\omega, \eta_1, \dots, \eta_n$. By the above observation we get

$$\mathbb{E}_{\mathbb{L}}[f|\mathcal{F}_n]\big(\omega, \eta^{(1)}\big) = \mathbb{E}_{\mathbb{L}}[f|\mathcal{F}_n]\big(\omega, \eta^{(2)}\big) \tag{116}$$

if (115) holds true for some n (where $\mathbb{E}_{\mathbb{L}}$ denotes the expectation with respect to the measure $\mathbb{L}$). On the other hand, $f = \lim_{n\to\infty} \mathbb{E}_{\mathbb{L}}[f|\mathcal{F}_n]$ $\mathbb{L}$-a.s. As a byproduct, we get that $f(\omega, \eta^{(1)}) = f(\omega, \eta^{(2)})$ for $\mu \otimes P \otimes P$ a.e. $(\omega, \eta^{(1)}, \eta^{(2)})$ such that (115) happens for infinitely many n (note that this event has probability one due to the Chung–Fuchs Theorem [10] applied to the random walk $Z_n := \sum_{j=1}^n (\eta_j^{(1)} - \eta_j^{(2)})$). Hence,

$$f\big(\omega, \eta^{(1)}\big) = f\big(\omega, \eta^{(2)}\big), \quad \mu \otimes P \otimes P\text{-a.s.} \tag{117}$$

We now claim that for μ-a.e. ω the function $f(\omega,\cdot)$ is constant P-a.s. To this aim, it is enough to show that for μ-a.e. ω the P-variance of $f(\omega,\cdot)$ is zero, and this follows from (117) and the identity

$$\mathrm{Var}_P\big(f(\omega,\cdot)\big)=\frac{1}{2}\int dP\big(\eta^{(1)}\big)\int dP\big(\eta^{(2)}\big)\big[f\big(\omega,\eta^{(1)}\big)-f\big(\omega,\eta^{(2)}\big)\big]^2.$$

Now let $A_{\ell,m}:=\{\eta:\sum_{i=1}^m \eta_i=\ell\}$. Since f is invariant under T, $f(\omega,\eta)=f(\tau_\ell\omega,\tau_m\eta)$ for $\eta\in A_{\ell,m}$. If $P(A_{\ell,m})>0$, we conclude that $f(\omega,\cdot)=f(\tau_\ell\omega,\cdot)$ P-almost surely, for μ-a.e. ω. Since the greatest common divisor of $\{k:P(\eta_1=k)>0\}$ equals 1, we conclude that there is some finite L such that $f(\omega,\cdot)=f(\tau_\ell\omega,\cdot)$ for all $\ell\geq L$, for μ-a.e. ω. Since the law of ω is ergodic with respect to τ_1, this implies easily that $f(\cdot,\cdot)$ is constant $\mathbb{L}$-almost surely. □

Now recall the definition of the random sequence $(S_k)_{k\geq 0}$ introduced at the end of the proof of Proposition 8.1, and the notation therein.

Lemma B.3. *The random sequence $(S_k)_{k\in\mathbb{N}}$ is stationary and ergodic with respect to shifts.*

Proof. We first show that the sequence $(s_k)_{k\geq 0}$ (see (112)) is stationary and ergodic with respect to shifts, under $P^{(\infty)}$. Indeed, writing (112) in a compact form as $(s_k)_{k\geq 0}=G(\omega,(\xi_k)_{k\geq 1})$, it holds $(s_k)_{k\geq 1}=G(\tau_{\xi_1}\omega,(\xi_k)_{k\geq 2})$. Then stationarity and ergodicity of $(s_k)_{k\geq 0}$ under $P^{(\infty)}$ follow from the stationarity and ergodicity of $(\omega,(\xi_k)_{k\geq 1})$ under $P^{(\infty)}$ as in Lemma B.1.

We move to $(S_k)_{k\geq 0}$. Since $(s_k)_{k\geq 0}$, under $P^{(\infty)}$, is stationary, one gets easily the stationarity of $(S_k)_{k\geq 0}$ under $\tilde{P}^{(\infty)}$. Take now a shift invariant Borel set $A\subset\mathbb{N}^{\mathbb{N}_0}$ (i.e. $A=\{(x_0,x_1,\dots)\in\mathbb{N}^{\mathbb{N}_0}:(x_1,x_2,\dots)\in A\}$). We claim that

$$\tilde{P}^{(\infty)}\big((S_0,S_1,\dots)\in A\big)\in\{0,1\}. \tag{118}$$

We define $f:\mathbb{N}^{\mathbb{N}_0}\to\mathbb{R}$ as the Borel function such that

$$f(s_0,s_1,s_2,\dots)=\tilde{P}^{(\infty)}\big((S_0,S_1,\dots)\in A|s_0,s_1,\dots\big).$$

Since A is shift invariant, A belongs to the tail σ-algebra of $\mathbb{N}^{\mathbb{N}_0}$. By Kolmogorov's 0–1 law and due to the independence of $S_0,S_1,\dots$ under $\tilde{P}^{(\infty)}(\cdot|s_0,s_1,\dots)$, we get that f has values in $\{0,1\}$.

Below, for the sake of intuition we condition to events of zero probability although all can be formalized by means of regular conditional probabilities. Using that $\{(S_0,S_1,\dots)\in A\}=\{(S_1,S_2,\dots)\in A\}$ due to the shift invariance of A and using the definition of $(S_k)_{k\geq 0}$, we get

$$\begin{aligned}f(a_0,a_1,\dots)&=\tilde{P}^{(\infty)}\big((S_0,S_1,\dots)\in A|s_0=a_0,s_1=a_1,\dots\big)\\&=\tilde{P}^{(\infty)}\big((S_1,S_2,\dots)\in A|s_0=a_0,s_1=a_1,s_2=a_2,\dots\big)\\&=\tilde{P}^{(\infty)}\big((S_0,S_1,\dots)\in A|s_0=a_1,s_1=a_2,\dots\big)=f(a_1,a_2,\dots).\end{aligned}$$

Hence f is shift invariant. By the ergodicity of $(s_k)_{k\geq 0}$, we conclude that the $0/1$-function $f(s_0,s_1,\dots)$ is constant $P^{(\infty)}$-a.s. An integration over $(s_0,s_1,\dots)$ allows to get (118). □

Lemma B.4. *Consider two independent random sequences $(Z_k)_{k\in\mathbb{Z}}$ and $(E_k)_{k\in\mathbb{Z}}$, the former stationary and ergodic with respect to shifts, the latter given by i.i.d. random variables. Then the random sequence $(Z_k,E_k)_{k\in\mathbb{Z}}$ is stationary and ergodic with respect to shifts.*

The above statement can be derived also from more general results on ergodic theory for dynamical systems, see [13]. We give an independent proof for completeness.

Proof. Call P the law of $((Z_k)_{k\in\mathbb{Z}},(E_k)_{k\in\mathbb{Z}})$, which is a probability measure on the space $\mathbb{R}^{\mathbb{Z}}\times\mathbb{R}^{\mathbb{Z}}$, whose generic element will be denoted by $(\underline{z},\underline{e})$. We write T for the shift $[T(\underline{z},\underline{e})]_k=(z_{k+1},e_{k+1})$. Let A be a shift-invariant Borel subset of $\mathbb{R}^{\mathbb{Z}}\times\mathbb{R}^{\mathbb{Z}}$. We want to show that $P(A)\in\{0,1\}$.

We first claim that, given $r \geq 1$, A is independent of any set B in the σ-algebra generated by e_i with $|i| \leq r$. To this aim, given $\varepsilon > 0$, we fix a Borel set $A_n \subset \mathbb{R}^{\mathbb{Z}} \times \mathbb{R}^{\mathbb{Z}}$ belonging to the σ-algebra generated by e_i, z_i with $|i| \leq n$, and such that $P(A \Delta A_n) \leq \varepsilon$. We take m large enough so that $[-r, r] \cap [-n+m, n+m] = \varnothing$. We observe that

$$P(A \cap B) = P(A_n \cap B) + O(\varepsilon), \tag{119}$$

$$P(A \cap B) = P\big(T^m A \cap B\big) = P\big(T^m A_n \cap B\big) + O(\varepsilon) = P\big(T^m A_n\big) P(B) + O(\varepsilon). \tag{120}$$

Indeed, the first identity in (120) follows from the shift invariance of A, while the second identity follows from the shift stationarity of P implying that $P(T^m A_n \Delta T^m A) \leq \varepsilon$. To get the third identity in (120) we observe that $T^m A_n$ belongs to the σ-algebra generated by e_i, z_i with $i \in [-n+m, n+m]$. By our choice of m and due to the properties of P, we get that $T^m A_n$ and B are independent, thus implying the third identity.

As a byproduct of (119) and (120) and the fact that $P(T^m A_n) = P(A) + O(\varepsilon)$, we get that $P(A \cap B) = P(A)P(B) + O(\varepsilon)$. By the arbitrariness of ε we conclude the proof of our claim.

Due to our claim, $\mathbb{1}_A = P(A|\mathcal{F})$, $\mathcal{F}$ being the σ-algebra generated by z_i, $i \in \mathbb{Z}$. We can think of $P(A|\mathcal{F})$ as function of $\underline{z} \in \mathbb{R}^{\mathbb{Z}}$. Due to the shift invariance of A, $P(A|\mathcal{F})$ is shift invariant in $\mathbb{R}^{\mathbb{Z}}$ except on an event of probability zero. Due to the ergodicity of the marginal of P along $\underline{z}$, we conclude that $P(A|\mathcal{F})$ is constant a.s. Since $\mathbb{1}_A = P(A|\mathcal{F})$, $\mathbb{1}_A$ is constant a.s., hence $P(A) \in \{0, 1\}$. □

Appendix C: The nearest neighbor random walk $(X_n^\rho)_{n\geq 1}$, $\rho = 1$

The biased Mott random walk $(\mathbb{Y}_t)_{t\geq 0}$ can be compared to the nearest neighbor random walk obtained by considering only nearest neighbor jumps on $\{x_j\}_{j\in\mathbb{Z}}$ with probability rate for a jump from x to y given by (3) when x, y are nearest neighbors. By the same arguments as in Section 7, it is simple to show that this random walk is ballistic/subballistic if and only if the same holds for $(X_n^\rho)_{n\in\mathbb{N}}$, $\rho = 1$. The latter can be easily analyzed and the following holds:

Proposition C.1. *The limit $v_{X^1}(\lambda) := \lim_{n\to\infty} \frac{X_n^1}{n}$ exists $\mathbb{P}_0^{\omega,1}$-a.s. for $\mathbb{P}$-a.a. ω, and it does not dependent on ω. Moreover, the velocity $v_{X^1}(\lambda)$ is positive if and only if condition (7) is fulfilled, otherwise it is zero.*

Proof. We apply Theorem 2.1.9 in [24] using the notations therein. Since $\rho_i = c_{i,i-1}/c_{i,i+1}$ we get that $\bar{S} = \frac{1}{c_{0,1}} \sum_{i=0}^{\infty} (c_{-i,-i-1} + c_{-i,-i+1})$. Therefore, $\mathbb{E}(\bar{S}) < \infty$ if and only if $\sum_{i=0}^{\infty} \mathbb{E}(c_{-i,-i-1}/c_{0,1}) < \infty$. The last condition is equivalent to (7) since the energy marks are bounded. On the other hand $\bar{F} = \frac{1}{c_{-1,0}} \sum_{i=1}^{\infty} (c_{i,i-1} + c_{i,i+1})$. Hence, $\mathbb{E}(\bar{F}) = \infty$ if and only if $\sum_{i=0}^{\infty} \mathbb{E}(c_{i,i+1}/c_{-1,0}) = \infty$. Since, when $u \equiv 0$, $c_{i,i+1}/c_{-1,0} = \exp\{(1+\lambda)Z_{-1} + 2\lambda(Z_0 + \cdots + Z_{i-1}) - (1-\lambda)Z_i\}$, by Assumption (A4) it follows that $\mathbb{E}(\bar{F}) = +\infty$ always. The claim then follows since, by Theorem 2.1.9 in [24], $v_{X^1}(\lambda) > 0$ if $\mathbb{E}(\bar{S}) < \infty$, while $v_{X^1}(\lambda) = 0$ if $\mathbb{E}(\bar{S}) = \infty$ and $\mathbb{E}(\bar{F}) = \infty$. □

Acknowledgements

The authors are very much indebted to Noam Berger and thank him for many useful and inspiring discussions. A.F. kindly acknowledges the Short Visit Grant of the European Science Foundation (ESF) in the framework of the ESF program "Random Geometry of Large Interacting Systems and Statistical Physics", and the Department of Mathematics of the Technical University Munich for the kind hospitality. M.S. thanks the Department of Mathematics of University La Sapienza in Rome for the kind hospitality, the Department of Mathematics of the Technical University Munich where he has started to work on this project as a Post Doc, and the funding received from the European Union's Horizon 2020 research and innovation programme under the Marie Sklodowska-Curie Grant agreement No. 656047.

We would like to thank an anonymous referee for his/her extremely careful reading and his/her very helpful suggestions and corrections.

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