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Constrained Adaptive Control of Uncertain Switched Systems

A Study with Focus on Piecewise Affine Systems

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Abstract

Numerous engineering systems are characterized by nonlinearity and switching phenomena, which pose great challenges to the analysis and control design. Considering the tremendous existing approaches and methods in the control community to analyze and control linear systems, it is natural to propose the concept to model switched systems or approximate nonlinear systems with a set of linear systems, such that the existing approaches for linear systems can also be applied to nonlinear and switched systems. Piecewise affine systems, as a class of representative switched systems, are proposed to realize this concept because of their universal approximation capability. This thesis focuses on the study of piecewise affine systems and aims to explore their adaptive control with considerations of three practically relevant constraints: limited excitations, performance constraints, and sensor constraints.

Under limited excitations, it is challenging to achieve the convergence of the estimated parameters of the uncertain piecewise affine systems to their real values in the adaptive control. To tackle this challenge, a concurrent learning-based indirect adaptive control method is developed, which exploits both the current data and the recorded history data to update the control gains and estimated parameters. Given a relatively milder excitation condition of the linear independence of the recorded data is satisfied, the control gains and the estimated parameters converge to their nominal and real values.

Performance constraints on the system output or state are imposed when the real systems have operational boundaries or safety specifications. In the presence of performance constraints, adaptive control approaches are developed for both output tracking and state tracking tasks, respectively. The use of barrier function concept endows the approaches with the feature that the tracking errors are confined within a prescribed performance bound. To enhance the robustness of the proposed approaches against disturbances, we further present projection-based modifications of the adaptation laws. Furthermore, for both output tracking and state tracking cases, direct and indirect adaptive controllers are developed, respectively. The direct adaptive control enjoys simple structure while the indirect adaptive control can achieve parameter identification in addition to the trajectory tracking task.

For switched systems with sensor constraints, namely, whose states are not available for feedback measurement, we investigate their adaptive control and adaptive observer, respectively. The common difficulty of these two tasks lies in how to cope with the transient terms caused after each switch, which depend on the unknown states at switching instants. An output feedback-based adaptive control is developed for uncertain piecewise affine systems. It treats the transient terms as disturbances and incorporates projection-based adaptation laws to guarantee the closed-loop stability. As a result, the proposed adaptive control enforces the output of the piecewise affine system to track the output of a linear reference system with the tracking error being small in the mean square sense. Besides, the convergence analysis of the estimated control parameters is also provided. Moreover, an adaptive observer is developed for uncertain switched systems. By exploiting the known information contained in the transient terms as excitation sources, the proposed adaptive observer estimates the unknown state and parameters simultaneously with asymptotic convergence of the estimation errors.

Zusammenfassung

Zahlreiche technische Systeme sind durch Nichtlinearität und Schaltphänomene gekennzeichnet, die eine große Herausforderung für die Analyse und den Entwurf der Regelungen darstellen. In Anbetracht der umfangreichen bestehenden Ansätze und Methoden zur Analyse und Regelung linearer Systeme ist es naheliegend, das Konzept vorzuschlagen, mit einer Reihe von linearen Systemen schaltende Systeme zu modellieren oder nichtlineare Systeme anzunähern, so dass die bestehenden Ansätze für lineare Systeme auch auf nichtlineare und schaltende Systeme angewendet werden können. Stückweise affine Systeme, als eine Art von repräsentativen schaltenden Systemen, werden zur Umsetzung dieses Konzepts vorgeschlagen aufgrund ihrer universellen Approximationsfähigkeit. Diese Dissertation konzentriert sich auf die Untersuchung von stückweisen affinen Systemen, mit dem Ziel, ihre adaptive Regelung unter Berücksichtigung von drei praktisch relevanten Beschränkungen zu untersuchen: begrenzte Anregungen, Performancebeschränkungen und Sensorbeschränkungen.

Bei begrenzten Anregungen ist es eine Herausforderung, die Konvergenz der geschätzten Parameter der stückweise affinen Systeme zu ihren realen Werten in der adaptiven Regelung zu erreichen. Um diese Herausforderung zu bewältigen, wird eine auf Concurrent Learning basierende indirekte adaptive Regelungsmethode entwickelt, das sowohl die aktuellen Daten als auch die aufgenommenen historischen Daten zur Aktualisierung der Regelparameter und geschätzten Systemparameter nutzt. Bei der Erfüllung einer relativ milderen Anregungsbedingung von der linearen Unabhängigkeit der aufgenommenen Daten konvergieren die Regelparameter und die geschätzten Systemparameter zu ihren nominalen beziehungsweise realen Werten.

Performancebeschränkungen für den Systemausgang oder -zustand werden auferlegt, wenn die realen Systeme Betriebsgrenzen oder Sicherheitsspezifikationen haben. Bei Vorhandensein von Performancebeschränkungen werden adaptive Steuerungsansätze sowohl für die Ausgangsverfolgung als auch für die Zustandsverfolgung entwickelt. Die Verwendung des Konzepts der Barrierefunktion verleiht den Ansätzen die Eigenschaft, dass die Verfolgungsfehler innerhalb einer vorgegebenen Performancegrenze begrenzt werden. Um die Robustheit der vorgeschlagenen Ansätze gegenüber Störungen zu verbessern, stellen wir außerdem projektionsbasierte Modifikationen der Adaptionsgesetze. Außerdem werden jeweils für die Ausgangsverfolgung und für die Zustandsverfolgung sowohl direkte als auch indirekte adaptive Regler entwickelt. Die direkte adaptive Regelung hat eine einfache Struktur, während die indirekte adaptive Regelung neben der Verfolgung einer Referenztrajektorie auch eine Parameteridentifikation durchführen kann.

Für schaltende Systeme mit Sensorbeschränkungen, d.h. deren Zustände nicht für eine Rückkopplungsmessung zur Verfügung stehen, untersuchen wir ihren adaptiven Regelung und adaptive Beobachter. Die Schwierigkeit dieser beiden Aufgaben liegt in der Behandlung der Transienten, die nach jedem Schaltvorgang auftreten und von den unbekannten Zuständen zu den Schaltzeitpunkten abhängen. Eine auf Ausgangsrückkopplung basierende adaptive Regelung für stückweise affine Systeme mit Parameterunsicherheiten wird entwickelt. Sie behandelt die Transienten als Störungen und beinhaltet projektionsbasierte Adaptionsgesetze, um die Stabilität des geschlossenen Regelkreises zu gewährleisten. Die vorgeschlagene adaptive Regelung erzwingt den Ausgang des stückweisen affinen Systems dem Ausgang eines linearen Referenzsystems zu verfolgen, wobei der Verfolgungsfehler klein im Sinne des mittleren Quadrats ist. Außerdem wird die Konvergenzanalyse der geschätzten Regelparameter durchgeführt. Darüber hinaus wird ein adaptiver Beobachter für schaltende Systeme mit Parameterunsicherheiten entwickelt. Durch Ausnutzung der bekannten Informationen enthalten in den Transienten als Anregungsquellen schätzt der vorgeschlagene adaptive Beobachter den unbekannten Zustand und die Parameter gleichzeitig mit asymptotischer Konvergenz der Schätzfehler.

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Notation

Acronyms and Abbreviations

BLF Barrier Lyapunov function

CL Concurrent learning

CLF Common Lyapunov function

CQLF Common quadratic Lyapunov function

DREM Dynamic regressor extension and mixing

LRE Linear regression equation

MIMO Multi-input multi-output

MLF Multiple Lyapunov function

MRAC Model reference adaptive control

PE Persistence of excitation

PPAC Prescribed performance adaptive control

PWA Piecewise affine (systems)

PWL Piecewise linear (systems)

SASI States at switching instants

SISO Single-input single-output

s.m.s.s Small in the mean square sense

Conventions

Numbers and Number Sets

$\mathbb{R}$	Set of real numbers
$\mathbb{R}^+$	Set of positive real numbers
$\mathbb{N}$	Set of natural numbers
$\mathbb{N}^+$	Set of positive natural numbers
ι	The imaginary unit, $\iota^2 = -1$
e	Euler's number

Matrix Operations

A^T	Transpose of a matrix A
A^{-1}	Inverse of a square matrix A
$A^\dagger$	Moore-Penrose pseudoinverse of a matrix A
$\lambda_{\max}(A)$	Maximum eigenvalue of a symmetric real matrix A
$\lambda_{\min}(A)$	Minimum eigenvalue of a symmetric real matrix A
$\mathrm{tr}(A)$	Trace of a square matrix A
$\mathrm{vec}(A)$	Vectorization of a matrix A
$\mathrm{adj}(A)$	Adjugate matrix of a square matrix A
$\det(A)$	Determinant of a square matrix A
$\mathrm{diag}\{A_1, \cdots, A_n\}$	(Block) diagonal matrix with A_i being the i-th diagonal element
$\mathrm{col}_j(A)$	j-th column of a matrix A
I_n	n-dimension identity matrix
0_n	n-dimension vector with all the components being 0

Norms and $\mathcal{L}_p$ Spaces

$\|x\|$	Euclidean norm of the vector $x \in \mathbb{R}^n$ with $\|x\| = (\sum_{i=1}^{n} \|x_i\|^2)^{\frac{1}{2}}$
$\\|x\\|_P$	Weighted Euclidean norm of the vector $x \in \mathbb{R}^n$ with $P \in \mathbb{R}^{n\times n}$ being a symmetric and positive definite matrix, defined as $\\|x\\|_P = x^T P x$
$\\|A\\|$	Induced Euclidean norm of the matrix $A \in \mathbb{R}^{m\times n}$ with $\\|A\\| = (\lambda_{\max}(A^T A))^{\frac{1}{2}}$
$\\|A\\|_F$	Frobenius norm of the matrix $A \in \mathbb{R}^{m\times n}$ with $\\|A\\|_F = (\sum_{i=1}^{m} \sum_{j=1}^{n} \|a_{ij}\|^2)^{\frac{1}{2}}$
$\\|x\\|_{2\delta}$	Exponentially weighted $\mathcal{L}_2$ norm of the function $x(t)$, defined as $\\|x\\|_{2\delta} = (\int_0^t \mathrm{e}^{-\delta(t-\tau)} x^T(\tau) x(\tau) \mathrm{d}\tau)^{\frac{1}{2}}$ where $\delta \geq 0$ is a constant
$x \in \mathcal{L}_2$	For function $x(t)$, we say that $x \in \mathcal{L}_2$ when $(\int_0^\infty \|x(\tau)\|^2 \mathrm{d}\tau)^{\frac{1}{2}}$ is finite
$x \in \mathcal{L}_{2e}$	For function $x(t)$, we say that $x \in \mathcal{L}_{2e}$ when $(\int_0^t \|x(\tau)\|^2 \mathrm{d}\tau)^{\frac{1}{2}}$ is finite for any finite t
$x \in \mathcal{L}_\infty$	For function $x(t)$, we say that $x \in \mathcal{L}_\infty$ when $\sup_{t\geq 0} \|x(t)\|$ is finite

Introduction

1

In engineering practice, many systems are characterized by a mixture of continuous dynamics and discrete modes. The switching among discrete modes governs how the continuous state of the system evolves. Systems involving these two kinds of dynamics are called *switched systems* [92]. Based on how the switching signals are generated, switched systems can be categorized into two classes: state-dependent switched systems and time-dependent switched systems. For state-dependent switched systems, the continuous state space is partitioned into several operating regions by switching surfaces. A continuous differential equation is assigned to each operating region to describe the local dynamics. Each local dynamics can be viewed as a subsystem of the switched system. Whenever the state of the system evolves across the switching surface, the switching of subsystem dynamics, or in other words, operating modes is triggered. For time-dependent switched systems, the switching signals are piecewise continuous functions of the time. In most literature, the switching signals of time-dependent switched systems are treated as exogenous signals.

A representative example of state-dependent switched systems is the class of piecewise linear (PWL) or piecewise affine (PWA) systems, which exhibit universal approximation capability to approximate nonlinear systems. A PWL or PWA system can be obtained by linearizing a nonlinear system at a set of operating points. The state space (in some cases state-input space) of a PWL/PWA system is partitioned into convex polytopes with each polytope containing one operating point. In each region, the PWL/PWA system is governed by an associated linear subsystem dynamics. The hyperplanes, which determine how the state space is partitioned into polytopes, characterize the switching mechanism of the PWL/PWA system.

Since proposed by Sontag in the pioneering work [146], PWL/PWA systems have enjoyed a wide range of engineering applications. In electronic circuits, they have been utilized to model circuits with switching components such as DC-DC converters [59, 110, 111] as well as those with nonlinear components such as tunnel diode circuits [130] and chaotic circuits [36, 182]. Another favorable application field is manufacturing engineering where many plants are mechanical systems with piecewise linear characteristics such as friction [165], backlash [171], and saturation [74]. Closely related to these systems in the manufacturing, many mechanical systems in the transportation are reported to be modelled as PWL/PWA systems like automotive vehicles with nonlinear tire friction characteristics [21, 24], aerial vehicles [30], and aircraft wing models with nonlinear aeroelasticity [176]. In robotics, PWA approximation of contact dynamics around the nominal trajectory facilitates the realization of complex dexterous manipulation [63] and planar non-prehensile manipulation tasks [66, 67]. Besides, the concept to linearize nonlinear systems at multiple operating points has been widely adopted in chemical and biological systems including stirred tank reactors [44], hormone therapy for diseases [148], and gene regulatory networks [10]. In addition to all the above-mentioned physical models, researchers have exploited PWA systems to describe social phenomena such as opinion dynamics [68, 147].

Early studies of PWA systems begin with their intrinsic properties including stability analysis [53, 72, 73], controllability and observability [14, 29, 37], well-posedness [70], non-Zenoness [28] and convergence analysis [124]. Based on the stability analysis of autonomous PWA systems, the state feedback controllers are developed for the stabilization of nonautonomous PWA systems [64, 129]. For PWA systems, whose states are not available, the state estimator [75] and observer-based control synthesis are developed to fulfill regulation tasks [130] and tracking tasks [158, 165], given that the trajectory to be tracked is also generated by a reference PWA system. In cases where the trajectory is not generated by a reference PWA system, tracking tasks are also explored with the optimization concept (minimizing the tracking errors) such as the optimal control [24, 35] and model predictive control [41]. The synthesis of the aforementioned controllers focuses on the ideal case, which requires that the system parameters and region partitions are exactly known and the systems are without disturbances. To enhance the utility of the developed methods, some practical relevant factors are considered such as actuator and sensor faults [127], noisy measurements [128], time-delay, data sampling and packet losses over communication networks [109, 132]. Moreover, in the presence of parameter uncertainties or external disturbances, robust optimal control [78] and various control methods with H_∞ performance are proposed [40, 52, 54, 189]. Nevertheless, when the uncertainties and disturbances are very large, the robust controllers may not be able to stabilize the closed-loop systems.

To counter large uncertainties and disturbances, more recent works introduce the adaptation mechanism into PWA systems. One aspect is adaptive identification. Due to the hybrid nature of PWA systems, both the switching hyperplane estimation and the subsystem parameter identification need to be explored. Given offline data of the switched systems, the simultaneous estimation of switching hyperplanes and subsystem parameters has been studied in [45, 87, 159], to name a few. A more elaborated introduction can be seen in [57] and [80, App.B]. Online estimation approaches to estimate the switching hyperplanes and the subsystem parameters are respectively proposed in [81] and [79] assuming that the counterpart is known. Recently, online identification approaches are developed in [23, 46, 113] to estimate the switching hyperplanes and subsystem parameters simultaneously.

Another aspect to cope with uncertainties lies in adaptive control. Typically, adaptive control is divided into two groups: direct adaptive control and indirect adaptive control. In direct adaptive control, the controller gains are adjusted without identifying the system parameters while the controller gains of indirect adaptive control are updated based on the parallelly conducted parameter estimation. The direct model reference adaptive control (MRAC) approaches of PWL systems for state tracking and output tracking are reported in [137] and [134], respectively. For PWA systems in control canonical form, a hybrid MRAC approach based on minimal control synthesis is proposed for the continuous-time case [16] and discrete-time case [19]. By assuming the existence of a common Lyapunov function, the stability of the controlled PWA system in control canonical form without sliding mode is guaranteed. This approach is extended in [18] such that the stability is ensured even when the closed-loop system exhibits sliding mode. Moreover, the absence of the sliding mode phenomenon is proved for the MRAC of a special class of PWA system, namely, continuous bimodal PWA systems [17]. The work in [83] generalizes the MRAC approach to multivariable PWA systems. In particular, the indirect MRAC approach, which is rarely studied for PWA systems before, is also discussed. Given a reference signal, which satisfies the persistence of excitation (PE) condition, both the tracking task and the estimation of subsystem parameters of the PWA systems can be achieved. All these referenced MRAC approaches are applied to PWL and PWA systems with known region partitions (known switching signals) and unknown subsystem parameters. This is closely related to the MRAC of switched linear systems with unknown subsystem parameters as their switching signals are given externally and therefore also known. To enhance the robustness of the adaptive switched linear systems against disturbances and time-delay, some robust MRAC approaches have been proposed. These include robust MRAC with dead-zone [161] and leakage [174], robust H_∞ MRAC [166, 169, 170] as well as control approaches with asynchronous switching between subsystems and controllers [167].

1.1 Challenges and Research Goals

Despite the above-referenced research progress, practical relevant constraints are not taken into account in the existing adaptive control of PWA systems. As a control system has three basic elements: the input, the state/output, and the feedback loop, we consider in this thesis constraints imposed on each element, namely, limited excitations at the input, performance constraints on the state/output, and sensor constraints on the feedback loop.

Limited Excitations

Most adaptive control tasks primarily aim to track a given state or output trajectory by updating the controller gains. Although asymptotic tracking can be achieved without the convergence of parameter estimation errors, studying the parameter convergence (convergence of the controller gains or unknown system parameters to the nominal values) is still a topic of major interest [150]. On the one hand, parameter convergence improves the transient behavior of the closed-loop system. It is shown in [39, 118] that large parameter estimation errors may result in bad transient behavior. On the other hand, parameter convergence facilitates the monitoring of the operation conditions of the plants. By observing the estimated parameters, one can identify hazards such as component aging and actuator failures.

Despite these advantages, achieving parameter convergence in the adaptive systems requires strong conditions. The PE assumption is a common condition for the parameter convergence. With this assumption, existing results of adaptive estimation and control of switched systems [83, 137, 173] achieve the parameter convergence. Nevertheless, the PE condition requires that the input signals should contain different frequencies. This causes oscillations and vibrations in the real engineering systems, which might be harmful to the physical plants. This poses the challenge when the system is with limited excitations and the PE condition is not satisfied. The question we would like to explore is as follows:

Question 1. Can we achieve parameter convergence for adaptive control of uncertain PWA systems without the PE condition?

Performance Constraints

Most of the existing adaptive control approaches can ensure asymptotic tracking, namely, zero steady-state tracking error, whereas the transient behavior of the closed-loop systems is not guaranteed. Large differences between the true parameters and the initial values of the estimated ones may lead to aggressive transient behavior. One possible way to improve the transient behavior is to manually tune the adaptation gains by trial and error. However, tuning adaptation gains on the real systems with uncertainties is not always feasible for two reasons. First, a lot of systems in practice have state or output constraints like physical or operational boundaries, and saturation. Taking the above-mentioned examples of switched systems for instance, an electrical machine has a maximum allowed rotation speed and its coils have a maximum allowed current to avoid mechanical injuries and overheating. A chemical reactor has a nominal volume to avoid the overflowing of the chemicals. Second, some physical systems may have to interact with the environment, which induces certain performance and safety constraints such as a collision-free specification for the trajectory of an autonomous car or a service robot. As the exact system models are not available, directly tuning adaptation gains on these systems may lead to violation of the performance constraints, which further causes accidents, personal injuries or financial loss. Recently, a lot of adaptive control methods satisfying some performance constraints on the output tracking error have been proposed for non-switched systems [12, 13, 62, 69, 153]. How to design adaptive controllers for uncertain switched systems and meanwhile satisfy the output tracking performance constraints is still challenging. In light of this fact, the following question is to be studied in this thesis:

Question 2. How to design adaptive controllers for uncertain PWA systems satisfying output tracking performance constraints?

In some circumstances, constraints may not only be imposed to the output of a system, but also the full state vector. This motivates us to further consider the following question:

Question 3. How to design adaptive controllers for uncertain PWA systems satisfying state tracking performance constraints?

Sensor Constraints

The solution of many practical problems requires precise measurements of system states. However, in engineering practice, the full state measurement requires the complex placement of sensors, which is not always feasible because of physical constraints or limited budgets. Due to such sensor constraints, we consider the situations where only the output signal is available. Most of the existing adaptive control approaches for switched systems rely on the full state feedback. In [134, 155, 156], the adaptive control methods for linear time-varying systems with parameter jumps or PWL systems are proposed based on output feedback, which guarantees small output tracking errors in the mean square sense. Nevertheless, they cannot be applied to uncertain PWA systems due to the additional affine terms of PWA systems when compared with those systems. Therefore, we would like to explore the following question:

Question 4. How to design adaptive controller for uncertain PWA systems based on output feedback?

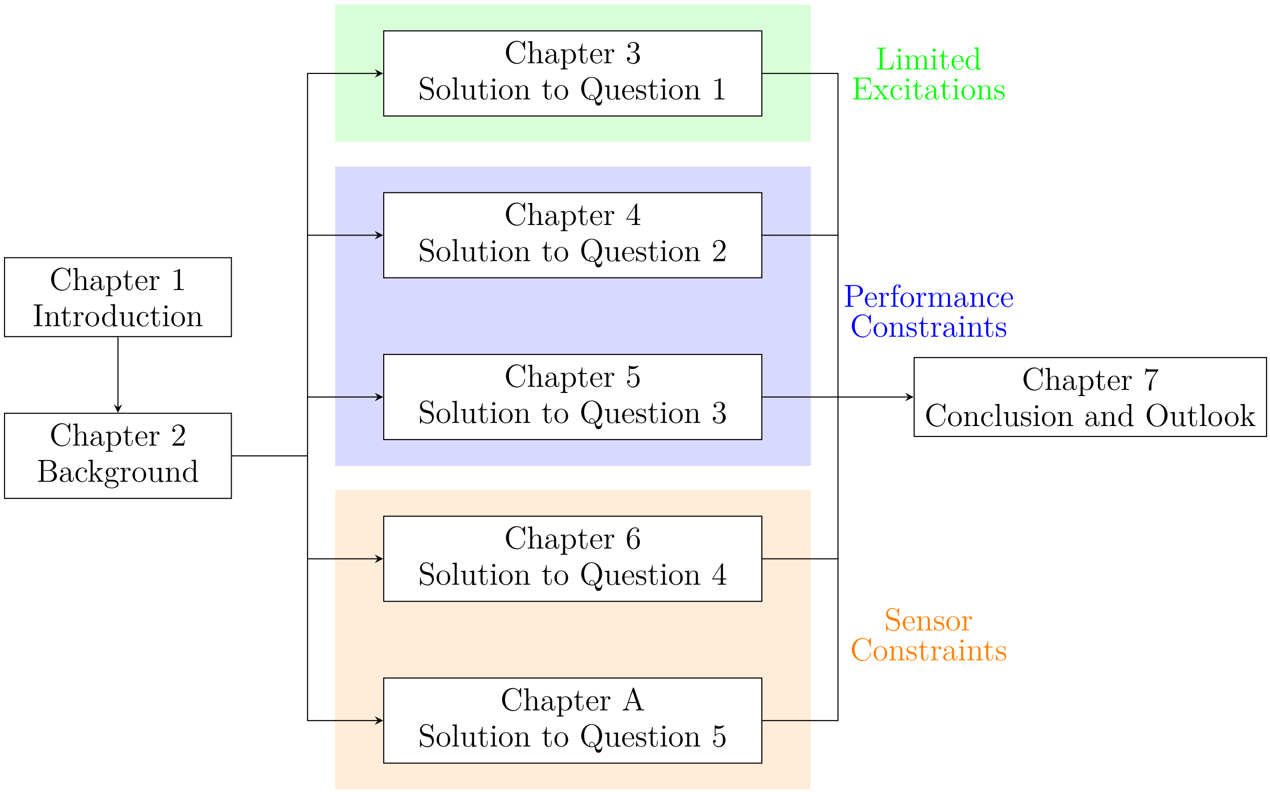


Figure 1.1: Thesis Outline

In addition to the adaptive control tasks, it is also of interest to explore the state observation tasks. To recover the state signals from the output signals, the state observers can be applied. Nevertheless, the design of state observers requires exact knowledge of system models. For systems with parameter uncertainties, it is a challenge to simultaneously estimate unknown states and system parameters. The adaptive observer is the main method to solve this problem [20, 71, 86, 108, 181]. Most existing adaptive observers focus on systems with constant unknown parameters and cannot be applied to switched systems. This gives rise to the following question:

Question 5. How to design adaptive observers for uncertain switched systems to simultaneously estimate the states and parameters?

In summary, the goal of this thesis is to study the adaptive control of uncertain PWA systems with considerations of limited excitations, performance constraints, and sensor constraints.

1.2 Contributions and Thesis Outline

As show in Figure 1.1, we begin with the background information in Chapter 2 to introduce the switched systems and the current research progress on adaptive control of these systems. In Chapter 3, the indirect adaptive control for switched systems with parameter convergence in spite of limited excitations is developed. The challenge of performance constraints is overcome in Chapter 4 for the output tracking and Chapter 5 for the state tracking. In Chapter 6 and Chapter A, the adaptive control and observation problems for switched systems subject to sensor constraints are studied, respectively. The conclusion and possible future work are given in Chapter 7. The contributions of each chapter are presented as follows.

Chapter 3: To achieve the parameter estimation under limited excitations, a concurrent learning-based indirect MRAC approach for PWA systems is proposed. The underlying concept is to exploit the current data and recorded history data concurrently to update the estimated parameters. The main advantage of this approach is that the classical PE assumption of the input signal is not required. In contrast, a relatively milder assumption of the linear independence of the recorded history data suffices for the convergence of the estimated parameters.

The content of this chapter has been published in [97].

Chapter 4: For uncertain PWA systems with output performance constraints, we develop the prescribed performance adaptive output tracking control approaches. Both direct and indirect adaptation approaches are studied. Given a desired output trajectory, both control approaches ensure the output tracking error to be confined within a performance bound, which prescribes the steady-state tracking error as well as the transient behavior such as the decaying rate and the overshoot. Based on the common Lyapunov functions, the stability of the controlled systems under arbitrary switching is established. Furthermore, the parameter convergence for both direct and indirect approaches is proved under the PE condition. The robust modifications of the adaptation laws are proposed for PWA systems with additive disturbances.

The material presented in this chapter has been published in [100].

Chapter 5: While Chapter 4 studies systems with output constraints, this chapter considers the MRAC for uncertain PWA systems with full state performance constraints. The proposed direct and indirect approaches ensure the error metric, defined as the weighted Euclidean norm of the state tracking error, to be confined within a user-defined time-varying performance bound. For the indirect approach, the parameter convergence is achieved under the PE condition. Moreover, for the uncertain PWA systems subject to unmatched disturbances, we propose the corresponding robust modifications of the adaptive controllers to ensure the robustness of the closed-loop systems.

The contributions of this chapter have been published in [98] and [101].

Chapter 6: Unlike the systems in the previous chapters, where the full state information is available, this chapter considers PWA systems with sensor constraints and only the output signal is available for the measurement. A direct MRAC of PWA systems and its parameter convergence are investigated. Under a slow switching assumption, it is shown that all the closed-loop signals are bounded and the output tracking error is small in the mean square sense. Built upon this result, the estimation error of controller parameters is proved to converge to a residual set if the input signal is sufficiently rich. Finally, the convergence of the estimated controller parameters to their nominal values can be achieved for a certain subsystem given that this subsystem is activated for infinitely long time.

The content of this chapter has been published in [99].

Chapter A: In addition to the adaptive control approach with sensor constraints, an adaptive observer is proposed for switched systems to estimate the states and parameters simultaneously. By applying the dynamic regression extension and mixing (DREM) technique, the estimation errors of system states and parameters converge to zero asymptotically. Furthermore, the robustness of the proposed adaptive observer is guaranteed in the presence of disturbances and noise.

The contribution of this chapter is scheduled to appear in [102].

Background on Switched Systems and Adaptive Control

2

In this chapter, some important background knowledge for this thesis will be revisited. We first introduce two typical switched systems: switched linear systems and piecewise affine (PWA) systems in Section 2.1. The difference between these systems and their common features will be discussed. Besides, the common tools for stability analysis of switched systems will be depicted. Then, based on Section 2.1, we revisit in Section 2.2 the existing adaptive control approaches for switched linear systems and PWA systems in three aspects: tracking performance, parameter convergence, and robustness analysis. Finally, Section 2.3 provides a summary of the existing results.

2.1 Introduction to Switched Systems

In general, switched systems can be categorized into two groups: time-dependent switched systems and state-dependent switched systems. In this section, typical examples of these two groups will be respectively introduced: switched linear systems and PWA systems.

2.1.1 Switched Linear Systems

Suppose a switched linear system has $s \in \mathbb{N}^+$ subsystems. Each subsystem can be described by the following dynamics

$$\dot{x}(t) = A_i x(t) + B_i u(t), \quad i \in \{1, 2, \cdots, s\} \tag{2.1}$$

where $x \in \mathbb{R}^n$ is the state vector, $u \in \mathbb{R}^p$ represents the input signal of the system. Let $\mathcal{I} \triangleq \{1, 2, \cdots, s\}$. The matrices $A_i \in \mathbb{R}^{n\times n}, B_i \in \mathbb{R}^{n\times p}, i \in \mathcal{I}$ represent the system parameters of subsystem i. The overall dynamics of the switched linear system can be written as

$$\dot{x}(t) = A_{\sigma(t)} x(t) + B_{\sigma(t)} u(t). \tag{2.2}$$

The switched system (2.2) has s subsystems and $A_{\sigma(t)} \in \mathbb{R}^{n\times n}, B_{\sigma(t)} \in \mathbb{R}^{n\times p}$ denote the switched system parameters with $A_{\sigma(t)} \in \{A_1, A_2, \cdots, A_s\}, B_{\sigma(t)} \in \{B_1, B_2, \cdots, B_s\}$. The switching signal $\sigma(t) : [0, \infty) \to \mathcal{I}$ is a piecewise constant function. It governs, which subsystem is activated. Namely, for $\sigma(t) = i, i \in \mathcal{I}$, we have $A_{\sigma(t)} = A_i, B_{\sigma(t)} = B_i$ and we say that i-th subsystem is activated at time t.

To characterize the switching instants, let the set of switching time instants represented by $\{t_1, t_2, \cdots, t_k, \cdots\}$ for $k \in \mathbb{N}^+$ and the initial time instant denoted by t_0.

Switched linear systems having exogenous switching signals $\sigma(t)$ are typical time-dependent switched systems. If the switching signal $\sigma(t)$ is generated following a function of the state of the switched system, then the switched system is said to be state-dependent. This brings us to the class of PWA systems.

2.1.2 Piecewise Affine Systems

The class of PWA systems belongs to state-dependent switched systems. A PWA system can be obtained by linearizing a nonlinear system at multiple operating points. Consider the nonlinear system

$$\dot{x}(t) = g(x(t), u(t)), \tag{2.3}$$

where $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state and control input of the nonlinear system, $g : \mathbb{R}^{n+p} \to \mathbb{R}^n$ denote a smooth nonlinear function. Given a set of operating points $(x_i^*, u_i^*), i \in \mathcal{I}$, the state-input space $[x^T, u^T]^T \in \mathbb{R}^{n+p}$ can be divided into s convex regions $\{\Omega_i\}_{i=1}^s$. The boundaries of the convex regions can be described by a set of hyperplanes in the state-input space, which are analytically expressed by a set of inequalities

$$\Omega_i = \left\{ \begin{bmatrix} x \\ u \end{bmatrix} \in \mathbb{R}^{n+p} \middle| H_i \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \preceq 0 \right\} \tag{2.4}$$

where a hyperplane is expressed by one row of H_i. The operator $\preceq$ represents $<$ or $\leq$ in the element-wise. Each region contains one operating point. For every time instant t, the vector $[x^T(t), u^T(t)]^T$ can only belong to one region. The regions have no overlaps, i.e., $\Omega_i \cap \Omega_j = \emptyset$ for $i \neq j$ and $i, j \in \mathcal{I}$. The linearization of the nonlinear system around the i-th operating point is given by

$$\dot{x} \approx g(x_i^*, u_i^*) + A_i(x - x_i^*) + B_i(u - u_i^*), \tag{2.5}$$

where $A_i = \frac{\partial g}{\partial x}|_{(x_i^*, u_i^*)}$ and $B_i = \frac{\partial g}{\partial u}|_{(x_i^*, u_i^*)}$. The following PWL system can be obtained by assuming zero equilibrium operating points $g(x_i^*, u_i^*) = 0$ and prior knowledge of the operating points (x_i^*, u_i^*)

$$\dot{x} \approx A_i \Delta x_i + B_i \Delta u_i \tag{2.6}$$

with $\Delta x_i = x - x_i^*$ and $\Delta u_i = u - u_i^*$ denoting the local state and input vector around the i-th operating point. The PWA model can be derived by

$$\begin{aligned} \dot{x} &\approx g(x_i^*, u_i^*) + A_i(x - x_i^*) + B_i(u - u_i^*) \\ &= A_i x + B_i u + g(x_i^*, u_i^*) - A_i x_i^* - B_i u_i^* \\ &= A_i x + B_i u + f_i \end{aligned} \tag{2.7}$$

with the affine term $f_i = g(x_i^*, u_i^*) - A_i x_i^* - B_i u_i^*$. Both PWL and PWA systems can approximate the nonlinear systems given the same operating points and partitioning of the state-input space. Comparing with PWL systems, PWA systems utilize the global state and input by introducing the affine term f_i. They allow nonzero $g(x_i^*, u_i^*)$ and do not require that the operating points are known.

Defining the switching signal $\sigma(t) : [0, \infty) \to \mathcal{I}$, whose value depends on the state and input vector

$$\sigma(t) = i, \quad \text{if } [x^T(t), u^T(t)]^T \in \Omega_i, \tag{2.8}$$

we can write the overall dynamics of the PWL system as

$$\dot{x}(t) = A_{\sigma(t)} \Delta x(t) + B_{\sigma(t)} \Delta u(t) \tag{2.9}$$

and the one of the PWA system as

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + f_{\sigma(t)}. \tag{2.10}$$

Alternatively, the indicator function can be used to describe the switching behavior, which characterizes in which region the state-input vector locates and is defined as follows

$$\chi_i(t) = \begin{cases} 1, & \text{if } [x^T(t), u^T(t)]^T \in \Omega_i \\ 0, & \text{otherwise.} \end{cases} \tag{2.11}$$

Since the regions $\{\Omega_i\}_{i=1}^s$ have no overlaps, we have $\sum_{i=1}^s \chi_i = 1$ and $\prod_{i=1}^s \chi_i = 0$. Note that the switching signal $\sigma(t)$ and the indicator function $\chi_i(t)$ have the following relationship

$$\sigma(t) = i \iff \chi_i = 1, \chi_j = 0, j \neq i \iff [x^T(t), u^T(t)]^T \in \Omega_i. \tag{2.12}$$

Thus, the PWL system can be written as

$$\dot{x}(t) = A(t)\Delta x(t) + B(t)\Delta u(t) \tag{2.13}$$

while the PWA system can be written as

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) \tag{2.14}$$

with $A(t) = \sum_{i=1}^s \chi_i(t)A_i$, $B(t) = \sum_{i=1}^s \chi_i(t)B_i$ and $f(t) = \sum_{i=1}^s \chi_i(t)f_i$.

2.1.3 Stability of Switched Systems

Consider the unforced switched system

$$\dot{x} = A_{\sigma(t)}x, \quad \sigma(t) \in \mathcal{I}, \tag{2.15}$$

with $A_i, i \in \mathcal{I}$ being Hurwitz. The following theorem of stability is given.

Theorem 2.1. *[92] "If all systems in (2.15) share a radially unbounded common Lyapunov function, then the switched system (2.15) is globally uniformly asymptotically stable."*

For the system (2.15), a widely used approach to construct a common Lyapunov function (CLF) is to find some positive definite matrices P, Q such that

$$A_i^T P + PA_i = -Q, \quad \forall i \in \mathcal{I}. \tag{2.16}$$

Then, a common quadratic Lyapunov function (CQLF) $V(x) = x^T Px$ can be constructed. In general, finding the common P matrix satisfying (2.16) is a difficult task. Necessary and sufficient conditions for the existence of the common P matrix for some special switched systems including second order switched systems [143], switched systems consisting of two stable subsystems [142, 144], and switched systems with a special parameterized family of matrices [26] are reported. In addition to these theoretical advances, some numerical methods are proposed such as algorithms based on the gradient iterations [93] and the particle swarm optimisation [49].

In case a CLF does not exist or cannot be found, one can use the multiple Lyapunov function (MLF) $V_{\sigma(t)}$, namely, one Lyapunov function for each individual subsystem, to analyze the stability of the switched systems. In general, the function $V_{\sigma(t)}$ will be discontinuous as $V_i(x(t_k)) = V_j(x(t_k)), i \neq j$ does not necessarily hold at the switching instant t_k. For i-th subsystem, $V_i, i \in \mathcal{I}$ decreases during the active phase of i-th subsystem while V_i may increase during the inactive phase. A well-known result to analyze the stability of switched systems using MLF is introduced in [92, Thm.3.1]. It shows that the switched system (2.15) is globally asymptotically stable if the values of $V_i, \forall i \in \mathcal{I}$ at each switch-in instant of subsystem i form a decreasing sequence, namely, $V_i(x(t_k)) < V_i(x(t_q))$ for $\sigma(t_k) = \sigma(t_q) = i$ and $t_k > t_q$. Nevertheless, as pointed out in [58], this result is not simple to apply due to the nonsuccessive nature of the sequence. Therefore, we recall the following stability condition.

Theorem 2.2. *[25, 172] "Let (2.15) be a finite family of globally asymptotically stable systems, and let $V_i, i \in \mathcal{I}$ be a family of corresponding radially unbounded Lyapunov functions. Suppose that for every pair of switching instants (t_{k-1}, t_k) with $\sigma(t_{k-1}) = j$ and $\sigma(t_k) = i$ for $i, j \in \mathcal{I}, i \neq j$, we have*

$$V_j(x(t_{k-1})) - V_i(x(t_k)) < 0. \tag{2.17}$$

Then, the switched system (2.15) is globally asymptotically stable."

A common choice of MLF for the autonomous switched system (2.15) is of quadratic form $V = x^T P_{\sigma(t)} x$. Under some dwell time constraints (e.g., see [65]), exponential convergence of the state can be obtained based on a combination of finite incremental jumps at switching instants ($V(t_k) \leq \mu V(t_k^-)$ for $\mu > 1$ and $V(t_k^-) \triangleq \lim_{\tau \uparrow t_k} V(\tau)$) and exponential decaying property ($\dot{V} \leq -\lambda V$ for $\lambda > 0$) in between switches.

2.2 Adaptive Control of Switched Systems

In general, adaptive control can be divided into two categories: direct adaptive control and indirect adaptive control. In direct adaptation, the controller gains are adapted directly with the error information and the estimation of system dynamics is not required. Different from the direct adaptive control, the controller gains are indirectly updated by using estimated system parameters in the indirect adaptive case.

Depending on if the system to be controlled is linear or nonlinear, various adaptive control approaches have been developed. For linear systems, model reference adaptive control (MRAC) has been extensively studied. The general idea of MRAC is to design an adaptive controller such that the closed-loop system behaves as a given reference system. For nonlinear systems in certain canonical forms, the adaptive backstepping design is a popular technique primarily for output tracking tasks. Namely, it is utilized such that the output of the controlled system tracks the desired output trajectory.

As the subsystems of switched linear systems or PWA systems are linear, most of the existing adaptive control approaches for switched linear systems and PWA systems are based on MRAC. Similar to the stability analysis of autonomous switched systems shown in Section 2.1.3, current results of adaptive control of switched linear systems and PWA systems are based on either CLFs or MLFs. Although switched linear systems and PWA systems have different switching mechanisms, it is worth pointing out that most existing adaptive control of switched linear systems and PWA systems share the same technical route. That is, the switching signals are assumed to be known and the reference switched systems are governed by the same switching signals, regardless of whether the switching is time-dependent or state-dependent. Then each subsystem is assigned with a local controller, which is updated and utilized when the corresponding subsystem is activated. Therefore, the switching among different local controllers is also determined by the same switching signals of the controlled switched systems. This section will revisit the representative results of these approaches. Furthermore, the results of the parameter convergence and robustness analysis, as two essential issues in the area of adaptive control, will also be revisited.

2.2.1 Model Reference Adaptive Control

In this section, we first revisit the reference system design and stability analysis of the reference system. Then, we review the representative results of direct and indirect MRAC of switched systems based on CQLF, respectively.

Reference Model

The goal of the MRAC is to enforce the trajectory of the controlled system to track the trajectory generated by a reference model. Consider a PWA reference system

$$\dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)r(t) + f_m(t), \tag{2.18}$$

where $x_m \in \mathbb{R}^n$ and $r \in \mathbb{R}^p$ denote the state of the reference system and the reference input. $A_m(t) = \sum_{i=1}^{s} A_{mi}\chi_i(t)$, $B_m(t) = \sum_{i=1}^{s} B_{mi}\chi_i(t)$, $f_m(t) = \sum_{i=1}^{s} f_{mi}\chi_i(t)$ are the parameters of the reference system.

For simplicity and without loss of generality, we assume that the switching of the reference system is the same as the one of the controlled PWA system, namely, it shares the common indicator function with the controlled PWA system. For time-dependent switched systems like switched linear systems, this assumption can be easily verified as both the switching signal of the reference switched linear system and the one of the controlled switched linear system are given externally. For state-dependent switched systems such as PWA systems, more explanations are needed. One may argue that if the switching of the reference PWA is governed by the switching signal of the controlled system, which further depends on the state of the controlled PWA system and cannot be determined in advance, then it is not possible for the user to design the desired behavior with the reference system. In fact, assuming the switching of the reference PWA system to be governed by the indicator function of the controlled PWA is a simplification, which can be generalized to the case, where the switching of the reference PWA system is governed by its own region partitions and is independent of the switching of the controlled PWA system. A detailed explanation will be given in Section 5.2.4.

Assume each subsystem of the reference system is stable and thus there exists a symmetric and positive definite matrix $P_i \in \mathbb{R}^{n\times n}$ for a given symmetric and positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$ such that

$$A_{mi}^T P_i + P_i A_{mi} = -Q_i, \quad \forall i \in \mathcal{I}. \tag{2.19}$$

The stability of the reference system and thus the boundedness of the reference state x_m is the prerequisite for the stability of the MRAC of the switched systems. According to [65], the stability of (2.18) can be concluded by proving the exponential stability of its homogeneous part $\dot{x}_m = \sum_{i=1}^{s} \chi_i A_{mi} x_m$. For the quadratic Lyapunov function $V_i = x_m^T P_i x_m$ for the i-th homogeneous subsystem, there exist constants $\alpha_{mi}, \lambda_{mi} > 0$ such that $\|\mathrm{e}^{A_{mi}t}\| \leq a_{mi}\mathrm{e}^{-\lambda_{mi}t}$. Therefore, we have the following lemma.

Lemma 2.1. [137] "The reference system $\dot{x} = A_m(t)x$ is exponentially stable with decay rate $\sigma \in (0, \frac{1}{2}\alpha)$ if the dwell time $T_D = \min_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\}$ satisfies

$$T_D \geq \frac{\alpha}{1-2\sigma\alpha} \ln\left(1 + \mu\Delta_{A_m}\right), \quad \mu = \frac{a_m^2}{\lambda_m \beta} \max_{i\in\mathcal{I}} \|P_i\| \tag{2.20}$$

where $\Delta_{A_m} = \max_{i,j\in\mathcal{I}} \|A_i - A_j\|$, $\alpha = \max_{i\in\mathcal{I}} \lambda_{\max}(P_i)$, $\beta = \min_{i\in\mathcal{I}} \lambda_{\min}(P_i)$, $a_m = \max_{i\in\mathcal{I}} a_{mi}$, $\lambda_m = \max_{i\in\mathcal{I}} \lambda_{mi}$."

For each subsystem, a set of controller gains is utilized. Let $K_{xi}^* \in \mathbb{R}^{p\times n}, K_{ri}^* \in \mathbb{R}^{p\times p}, K_{fi}^* \in \mathbb{R}^p, i \in \mathcal{I}$ denote the nominal controller gains for the i-th subsystem of (2.14). The controller gains and the system parameters switch synchronously. Therefore, the controller takes the form

$$u(t) = K_x^* x(t) + K_r^* r(t) + K_f^*, \tag{2.21}$$

where $K_x^*(t) = \sum_{i=1}^{s} \chi_i(t) K_{xi}^*$, $K_r^*(t) = \sum_{i=1}^{s} \chi_i(t) K_{ri}^*$, $K_f^*(t) = \sum_{i=1}^{s} \chi_i(t) K_{fi}^*$. Taking (2.21) into (2.14) yields the closed-loop system. To obtain a closed-loop system having the same behavior as the reference system, an usual assumption is that following matching equations hold:

$$A_{mi} = A_i + B_i K_{xi}^*, \quad B_{mi} = B_i K_{ri}^*, \quad f_{mi} = f_i + B_i K_{fi}^*, \quad \forall i \in \mathcal{I}. \tag{2.22}$$

As the parameters are unknown, the nominal gains are not available. The adaptive control design is based on *certainty equivalence principle* [150], namely, we use the estimated parameters in the feedback control as if they are the real system parameters in the case of uncertain or unknown system dynamics. Therefore, the adaptive controller takes the same structure as in (2.21) but with the estimated parameters

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t), \tag{2.23}$$

with

$$K_x(t) = \sum_{i=1}^{s} \chi_i K_{xi}(t), \quad K_r(t) = \sum_{i=1}^{s} \chi_i K_{ri}(t), \quad K_f(t) = \sum_{i=1}^{s} \chi_i K_{fi}(t). \tag{2.24}$$

where $K_{xi} \in \mathbb{R}^{p\times n}$, $K_{ri} \in \mathbb{R}^{p\times p}$ and $K_{fi} \in \mathbb{R}^p$ denote the estimated control gains for i-th subsystem.

Direct Adaptive Control

Inserting (2.23) into the controlled PWA system (2.14) yields the closed-loop PWA system

$$\dot{x} = \sum_{i=1}^{s} \chi_i((A_i + B_i K_{xi})x + B_i K_{ri} r + (B_i K_{fi} + f_i)). \tag{2.25}$$

Defining the state tracking error $e(t) = x(t) - x_m(t)$ and subtracting (2.18) from (2.25), we have the error equation

$$\dot{e} = A_m e + \sum_{i=1}^{s} \chi_i B_i(\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}), \tag{2.26}$$

where $\tilde{K}_{xi} = K_{xi} - K_{xi}^*, \tilde{K}_{ri} = K_{ri} - K_{ri}^*, \tilde{K}_{fi} = K_{fi} - K_{fi}^*$ represent the estimation errors of the controller gains. The adaptation laws of the estimated controller gains are given as

$$\dot{K}_{xi} = -\chi_i \Gamma_{xi} S_i^T B_{mi}^T P_i e x^T, \quad \dot{K}_{ri} = -\chi_i \Gamma_{ri} S_i^T B_{mi}^T P_i e r^T, \quad \dot{K}_{fi} = -\chi_i \Gamma_{fi} S_i^T B_{mi}^T P_i e \quad (2.27)$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling factors. $S_i \in \mathbb{R}^{p\times p}$ is a matrix such that there exists a symmetric and positive definite matrix $M_i \in \mathbb{R}^{p\times p}$ with $(K_{ri}^* S_i)^{-1} = M_i$. An usual assumption in multivariable adaptive control [150] is that S_i is known. This indicates the knowledge of control directions. The use of the indicator functions $\chi_i(t)$ in the adaptation laws (2.27) implies that the controller gains associated with a certain subsystem are updated only when this subsystem is activated. Their adaptation terminates and their values stay unchanged during the inactive phase of the corresponding subsystem. Without loss of generality, let $\Gamma_{xi} = \Gamma_{ri} = \Gamma_{fi} = 1$. By constructing the following CQLF

$$V = \frac{1}{2} e^T P e + \frac{1}{2} \sum_{i=1}^{s} (\mathrm{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \tilde{K}_{fi}^T M_i \tilde{K}_{fi}). \quad (2.28)$$

with $M_{si} = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p\times p}$, the stability and tracking performance of the closed-loop system is summarized in the following theorem.

Theorem 2.3. *[83] "Consider the reference system (2.18) for which a CQLF with $P = P_i, \forall i$ is known. Let the PWA system (2.14) with known regions Ω_i be controlled by the state feedback (2.23) with gains updated according to (2.27). Then, the state of the PWA system asymptotically tracks the state of the reference system."*

Indirect Adaptive Control

The nominal control gains can be obtained by solving the matching equations (2.22) if all the subsystem parameters A_i, B_i, f_i are known and B_i has full column rank for $i \in \mathcal{I}$

$$K_{xi}^* = B_i^\dagger (A_{mi} - A_i), \quad K_{ri}^* = B_i^\dagger B_{mi}, \quad K_{fi}^* = B_i^\dagger (f_{mi} - f_i) \quad (2.29)$$

with $(\cdot)^\dagger$ denoting the Moore-Penrose pseudoinverse.

The classical indirect adaptive control approach updates the control gains by replacing the system parameters in (2.29) with the estimated parameters. This, however, may introduce singularity by calculating $\hat{B}_i^\dagger(t)$ as $\hat{B}_i(t)$ is time-varying and may have rank deficiency at some t. To avoid this problem, the approach proposed in [83] applies the dynamic gain adjustment technique. We review this approach as follows. Define the *closed-loop estimation errors* as

$$\varepsilon_{Ai} = \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \quad \varepsilon_{Bi} = \hat{B}_i K_{ri} - B_{mi}, \quad \varepsilon_{fi} = \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}, \quad (2.30)$$

where $\hat{A}_i$, $\hat{B}_i$ and $\hat{f}_i$ denote the estimated system parameters of i-th mode. Based on the closed-loop estimation errors, the adaptation of control gains obeys

$$\dot{K}_{xi} = -S_i^T B_{mi}^T \varepsilon_{Ai}, \quad \dot{K}_{ri} = -S_i^T B_{mi}^T \varepsilon_{Bi}, \quad \dot{K}_{fi} = -S_i^T B_{mi}^T \varepsilon_{fi}. \quad (2.31)$$

Let $\hat{x}$ denote the predicted state and define its dynamics as

$$\dot{\hat{x}} = A_m \hat{x} + \sum_{i=1}^{s} ((\hat{A}_i - A_{mi}) x + \hat{B}_i u + \hat{f}_i) \chi_i. \quad (2.32)$$

This together with (2.25) leads to

$$\dot{\tilde{x}} = A_m \tilde{x} + \sum_{i=1}^{s} (\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i)\chi_i, \tag{2.33}$$

where $\tilde{x} = \hat{x} - x$ denotes the prediction error of the system state. $\tilde{A}_i = \hat{A}_i - A_i$, $\tilde{B}_i = \hat{B}_i - B_i$ and $\tilde{f}_i = \hat{f}_i - f_i$ are parameter estimation errors. The parameter update laws based on the closed-loop estimation errors and the state prediction error take the form

$$\begin{aligned} \dot{\hat{A}}_i &= -\chi_i P \tilde{x} x^T - \varepsilon_{Ai}, \\ \dot{\hat{B}}_i &= -\chi_i P \tilde{x} u^T - \varepsilon_{Ai} K_{xi}^T - \varepsilon_{Bi} K_{ri}^T - \varepsilon_{fi} K_{fi}^T, \\ \dot{\hat{f}}_i &= -\chi_i P \tilde{x} - \varepsilon_{fi}. \end{aligned} \tag{2.34}$$

Consider the following CQLF

$$\begin{aligned} V =& \frac{1}{2} \tilde{x}^T P \tilde{x} + \frac{1}{2} \sum_{i=1}^{s} (\operatorname{tr}(\tilde{A}_i^T \tilde{A}_i) + \operatorname{tr}(\tilde{B}_i^T \tilde{B}_i) + \tilde{f}_i^T \tilde{f}_i \\ &+ \operatorname{tr}(\tilde{K}_{xi}^T M_{si} \tilde{K}_{xi}) + \operatorname{tr}(\tilde{K}_{ri}^T M_{si} \tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si} \tilde{K}_{fi}). \end{aligned} \tag{2.35}$$

The stability and tracking performance of the closed-loop system is summarized in the following theorem.

Theorem 2.4. *[83] "Consider a reference system (2.18) for which a CQLF with $P_i = P, \forall i$ is known. Let the PWA system (2.14) with known regions Ω_i be controlled by the state feedback (2.23) with gains updated according to (2.31), which is based on (2.32),(2.34), and (2.30). Then, the state of the PWA system asymptotically tracks the state of the reference system."*

Theorem 2.3 and Theorem 2.4 present results based on CQLFs (CLFs in quadratic form), where the jumps of V at switching instants are avoided. Therefore, establishing $\dot{V} \leq 0$ in between switches is sufficient for the stability under arbitrary switching. For adaptive control of switched systems in the absence of CLF, the general idea is to construct MLF. Compared to CLF, the use of MLF allows more design freedom and flexibility, which leads to broader applications. In adaptive control of switched systems with MLFs, the worst-case jumps at switching instants are compensated by the exponential decrease in between switches. As V in the adaptive control contains not only the tracking error, but also the parameter estimation errors (in form of $V = e^T P_i e + \sum_i \tilde{\theta}_i^T \tilde{\theta}_i$ with $\tilde{\theta}_i$ being parameter estimation errors for i-th subsystem), establishing the exponential decrease in between switches ($\dot{V} \leq -\lambda V + d$ for $\lambda, d > 0$) requires extra conditions. In the sequel, some existing results of MLF-based direct MRAC of switched systems with PE conditions and robust modifications will be revisited in Section 2.2.2 and Section 2.2.3, respectively. For indirect MRAC of switched system, the existing result in the literature must rely on the existence of a CLF, which will also be shown in Section 2.2.2. In addition to establishing the exponential decrease of V, these conditions ensure the robustness of the closed-loop systems in the presence of disturbances and the PE condition will also lead to parameter convergence.

2.2.2 Parameter Convergence

First, we revisit some signal properties and a useful lemma, which are essential for the parameter convergence analysis in the adaptive control.

Definition 2.1 (Persistence of Excitation (PE) [71])**.** "A piecewise continuous signal vector $z : \mathbb{R}^+ \to \mathbb{R}^n$ is PE with a level of excitation α_0 if there exist constants $\alpha_1, T_0 > 0$ such that

$$\alpha_1 I \geq \frac{1}{T_0} \int_t^{t+T_0} z(\tau) z^T(\tau) \mathrm{d}\tau \geq \alpha_0 I,$$

for $\forall t \geq 0$."

The idea behind the PE property is that some internal signals should contain rich frequency components. A closely related property is sufficiently rich property [71, Def. 5.2.1], namely, a signal $u : \mathbb{R}^+ \to \mathbb{R}$ is called sufficiently rich of order $2n$, if it contains at least n distinct frequencies.

Lemma 2.2. [71] "Consider the system described by

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} A & -F^T(t) \\ P_1 F(t) P_2 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \tag{2.36}$$

where $x_1 \in \mathbb{R}^{n_1}, x_2 \in \mathbb{R}^{rn_1}$ for some integer $r, n_1 \geq 1$, A, P_1, P_2 are constant matrices and $F(t)$ is of the form

$$F(t) = \begin{bmatrix} z_1 I_{n_1} \\ z_2 I_{n_1} \\ \vdots \\ z_r I_{n_1} \end{bmatrix} \in \mathbb{R}^{rn_1 \times n_1}$$

where $z_i, i = 1, 2, \cdots, r$ are the elements of vector $z \in \mathbb{R}^r$. Suppose that z is PE and there exists a matrix $P_0 > 0$ such that

$$\dot{P}_0 + A^T P_0 + P_0 A_0 + C_0 C_0^T \leq 0 \tag{2.37}$$

where

$$A_0 = \begin{bmatrix} A & -F^T(t) \\ P_1 F(t) P_2 & 0 \end{bmatrix}, \quad C_0^T = \begin{bmatrix} I_{n_1}, 0 \end{bmatrix}.$$

Then the equilibrium $x_{1e} = 0, x_{2e} = 0$ of (2.36) is exponentially stable in the large. "

For the parameter convergence of adaptive identification and control of switched systems, it is a common approach to first study the parameter convergence of each subsystem during the active phase and then evaluate the overall convergence for the whole time interval. If the sufficiently rich input signal causes all the subsystems to be intermittently activated, then the parameter convergence of all the subsystems can be concluded for some dwell time constraints [83, 84, 137, 173].

For direct adaptive control, we rewrite the error equation (2.26) as

$$\dot{e} = A_{mi} e + \Psi_r^T \tilde{\vartheta}_i \tag{2.38}$$

with

$$\tilde{\vartheta}_i = \text{vec}(B_i[\tilde{K}_{xi} \;\; \tilde{K}_{ri} \;\; \tilde{K}_{fi}]), \qquad \Psi_r = \begin{bmatrix} x \\ r \\ 1 \end{bmatrix} \otimes I_n, \tag{2.39}$$

where vec(·) denotes the vectorization of a matrix, ⊗ represents the kronecker product. Therefore, (2.38) and the adaptation laws (2.27) can be written as the joint dynamics in a similar form as (2.36)

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\vartheta}}_i \end{bmatrix} = \begin{bmatrix} A_m & \Psi_r^T \\ -\Psi_r P_{i2} & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\vartheta}_i \end{bmatrix}, \tag{2.40}$$

where $P_{i2} = B_i M_i^{-1} B_i^T P_i$. If the signal vector $[x^T, r^T, 1]^T$ is PE, then one can invoke Lemma 2.2 to prove the exponential decrease of $z = [e^T, \tilde{\vartheta}_i^T]^T$ during each active phase of i-th subsystem. This together with some dwell time constraint would lead to the stability of the multiple Lyapunov function and the parameter convergence. A formal result is summarized in the following theorem.

Theorem 2.5. *[83] "Consider the reference system (2.18) without CQLF and let the PWA system (2.14) with known regions Ω_i be controlled by the state feedback (2.23) with gains updated according to (2.27). Let the reference signals in r be sufficiently rich of order $n+1$ with distinct frequencies. Furthermore, let the resulting switching signal be sufficiently slow with dwell time T_D and cause repeated activation of all subsystems. If the input matrices B_i have full column rank, if the system matrices A_{mi} are inertible, and if the pairs (A_{mi}, B_{mi}) are controllable, then all errors $e, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ asymptotically converge to zero for $t \to \infty$."*

For indirect adaptive control, the prediction error equation (2.33) can be written in a more compact form

$$\dot{\tilde{x}} = A_{mi}\tilde{x} + \Psi_u^T \tilde{\theta}_i \tag{2.41}$$

where

$$\tilde{\theta}_i = \text{vec}([\tilde{A}_i \;\; \tilde{B}_i \;\; \tilde{f}_i]), \qquad \Psi_u = \begin{bmatrix} x \\ u \\ 1 \end{bmatrix} \otimes I_n. \tag{2.42}$$

Based on (2.41), we can further write the joint dynamics of $\tilde{x}$ and the estimated parameters as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} A_m & \Psi_u^T \\ -\Psi_u P & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\theta}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon_i \end{bmatrix} \tag{2.43}$$

where $\varepsilon_i = -\text{vec}([\varepsilon_{Ai}, \varepsilon_{Ai} K_{xi}^T + \varepsilon_{Bi} K_{ri}^T + \varepsilon_{fi} K_{fi}^T])$. The homogeneous part of (2.43) has the same form as (2.36). Thus, invoking Lemma 2.2 leads to the convergence of $\tilde{x}$ and the parameter estimation errors.

Theorem 2.6. *[83] "Consider the reference system (2.18) for which a CQLF with $P_i = P, \forall i$ is known. Let the PWA system (2.14) with known regions Ω_i be controlled by the state feedback (2.23) with gains updated according to (2.31), which is based on (2.32),(2.34), and (2.30). Let the reference signals in r be sufficiently rich of order $n+1$ with distinct frequencies and such that all subsystems are repeatedly activated. If the input matrices B_i have full column rank, if the system matrices A_{mi} are inertible, and if the pairs (A_{mi}, B_{mi}) are controllable, then the state of the PWA system asymptotically tracks the state of the reference system and the estimated parameters $\hat{A}_i, \hat{B}_i$, and $\hat{f}_i$ as well as the estimated gains K_{xi}, K_{ri} and K_{fi} converge to their nominal values as $t \to \infty$".*

Due to the presence of ε_i in (2.43), V decreases asymptotically and the exponential decrease of V in between successive switches cannot be established. Therefore, dwell time constraints cannot be obtained when V exhibits jumps at switching instants. As a result, the reviewed indirect MRAC of PWA systems in Theorem 2.6, unlike the direct one shown in Theorem 2.5, can only be applied with the existence of CLFs. This restriction will be relaxed later in Section 5.3 of this thesis.

2.2.3 Robust Modification

In this section, we revisit two robust modifications for the direct MRAC of switched systems: projection and leakage.

Projection

We revisit the definition of the projection operator. Let $\theta = [\theta_1, \theta_2, \cdots, \theta_n]^T \in \mathbb{R}^n$ denote the estimation of the nominal parameter $\theta^* \in \mathbb{R}^n$. Define a convex hypercube in $\mathbb{R}^n$ such that the evolution of θ should not exceed the bound of this hypercube. Specifically,

$$\Omega_\theta = \{\theta \in \mathbb{R}^n | \theta_j^{\min} \leq \theta_j \leq \theta_j^{\max}, j = 1, 2, \cdots, n\}, \tag{2.44}$$

where $(\theta_j^{\min}, \theta_j^{\max})$ represent the lower and upper bounds of the j-th component of θ. Besides, let $\Omega_v = \{\theta \in \mathbb{R}^n | \theta_j^{\min} + v \leq \theta_j \leq \theta_j^{\max} - v, j = 1, 2, \cdots, n\}$ be a second hypercube with a small constant $v \in \mathbb{R}^+$ such that $\Omega_v \subset \Omega_\theta$. For the dynamics $\dot{\theta} = \mathrm{Pr}[y]$ with $y = [y_1, y_2, \cdots, y_n] \in \mathbb{R}^n$ we define the projection operator $\mathrm{Pr}[\cdot]$ with the following element-wise operation

$$\mathrm{Pr}[y] = \begin{cases} ((\theta_j - \theta_j^{\min})/v)y_j, & \text{if } \theta_j < \theta_j^{\min} + v, \quad y_j < 0 \\ ((\theta_j^{\max} - \theta_j)/v)y_j, & \text{if } \theta_j > \theta_j^{\max} - v, \quad y_j > 0 \\ y_j, & \text{otherwise.} \end{cases} \tag{2.45}$$

Based on (2.45) we obtain the following property of the projection operator

$$(\theta - \theta^*)^T(\mathrm{Pr}[y] - y) \leq 0. \tag{2.46}$$

This inequality can be extended to the matrix case with $\Theta, Y \in \mathbb{R}^{n \times m}$. Let $\mathrm{col}_j(\cdot)$ represents the j-th column of a matrix and $\dot{\Theta} = \mathrm{Pr}[Y] = [\mathrm{Pr}[\mathrm{col}_1(Y)], \cdots, \mathrm{Pr}[\mathrm{col}_m(Y)]]$. For a given nominal parameter matrix $\Theta^* \in \mathbb{R}^{n \times m}$, we have

$$\mathrm{tr}((\Theta - \Theta^*)^T(\mathrm{Pr}[Y] - Y)) \leq 0. \tag{2.47}$$

Alternatively, we can formulate the projection operator $\mathrm{Pr}[y] = y + f, f \in \mathbb{R}^n$ with the j-th element of f, denoted by f_j, being defined as follows

$$f_j = \begin{cases} ((\theta_j - \theta_j^{\min} - v)/v)y_j, & \text{if } \theta_j < \theta_j^{\min} + v, \quad y_j < 0 \\ ((\theta_j^{\max} - \theta_j - v)/v)y_j, & \text{if } \theta_j > \theta_j^{\max} - v, \quad y_j > 0 \\ 0, & \text{otherwise.} \end{cases} \tag{2.48}$$

The projection operator presented above is widely used in robust adaptive control [6]. The concept is to construct two hypercubes. The outer one is the hard constraint for θ while the inner one serves as a soft constraint. The parameter adaptation of θ does not change within the inner hypercube whereas it terminates when reaching the outer hypercube. The area between the inner hypercube and the outer one acts as a transition area, which enables a continuous evolution of $\dot{\theta}$ to 0. Using hypercubes as bounds of the parameters corresponds to the cases, where the element-wise upper and lower bounds of the parameters are known prior. It is worth pointing out that the hypercube is not the only way to describe the bounds of parameters. A more general discussion can be seen in [89, Sec.11.4].

For PWL systems with single-input, namely, $p = 1$ and K_{ri} is scalar, Sang and Tao [137] proposed the following projection-based adaptation laws

$$\begin{aligned} \dot{K}_{xi} &= -\chi_i \text{sign}[K_{ri}^*]\Gamma_{xi} x e^T P_i B_{mi} + \chi_i f_{xi}, \\ \dot{K}_{ri} &= -\chi_i \text{sign}[K_{ri}^*]\Gamma_{ri} r e^T P_i B_{mi} + \chi_i f_{ri}, \end{aligned} \tag{2.49}$$

where f_{xi}, f_{ri} are the parameter projection laws (see (2.48)). The term K_f in (2.23) is not necessary as no affine term needs to be compensated in the PWL system. sign[$\cdot$] denotes the sign of a scalar. With $a_m, \lambda_m, \alpha, \beta, \mu, \Delta_{A_m}$ defined in Lemma 2.1, the stability result is summarized as follows.

Theorem 2.7. *[137] "Consider the closed-loop system consisting of the piecewise linear system (2.13), the reference model system (2.18) with $f_m = 0$, and the controller (2.23) with $K_f = 0$ updated by the adaptive laws (2.49). If*

$$T_D = \min_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} \geq \tau_D = \alpha(1+\kappa)\ln 1 + \mu\Delta_{A_m}, \kappa > 0, \tag{2.50}$$

then all the closed-loop signals are bounded, and the tracking error $e(t)$ is small in the sense that

$$\int_t^{t+T} e^T(\tau)e(\tau)\mathrm{d}\tau \leq \mu\Delta_{A_m} c_0 \frac{T}{T_D} + c_1, \quad t \geq t_0, \quad T > 0 \tag{2.51}$$

with $c_1 = (1+\mu\Delta_{A_m})c_0$ for some $c_0 > 0$."

The tracking performance (2.51) is called small in the mean square sense (s.m.s.s). For switched systems with disturbances, projection-based adaptation laws proposed in [166] achieve H_∞ state tracking of the closed-loop system.

Leakage

Consider the switched linear system in form of

$$\dot{x}(t) = A_{\sigma(t)}x(t) + B_{\sigma(t)}u(t) + d(t), \quad \sigma(t) \in \mathcal{I} \tag{2.52}$$

with $d(t)$ being a bounded disturbance term with the upper bound $\bar{d}$. For the reference switched linear system $\dot{x}_m(t) = A_{m\sigma(t)}x(t) + B_{m\sigma(t)}r(t)$ satisfying the following inequality

$$A_{mi}^T P_i + P_i A_{mi} + (1+\kappa_i)P_i \leq 0, \tag{2.53}$$

Yuan *et al.* proposed the following adaptive controller

$$u(t) = K_{x\sigma(t)}(t)x(t) + K_{r\sigma(t)}(t)r(t) \tag{2.54}$$

with the dwell time constraint

$$T_D > \tau_D = \frac{1}{\xi \kappa_i} \ln \mu_i, \quad \xi \in (0,1), \tag{2.55}$$

for $\mu_i = \frac{\alpha}{\lambda_{\min}(P_i)}$ and $\alpha = \max_{i\in\mathcal{I}} \lambda_{\max}(P_i)$. The adaptation laws based on leakage for the active phase of i-th subsystem are

$$\begin{aligned} \dot{K}_{xi} &= -S_i^T B_{mi}^T P_i e x^T - \delta_i M_i K_{xi}, \quad \dot{K}_{xj} = -\delta_j M_j K_{xj}, \\ \dot{K}_{ri} &= -S_i^T B_{mi}^T P_i e r^T - \delta_i M_i K_{ri}, \quad \dot{K}_{rj} = -\delta_j M_j K_{rj}. \quad j \neq i, j \in \mathcal{I}, \end{aligned} \tag{2.56}$$

where the leakage rate δ_i fulfills the constraint

$$\delta_i - \max_{i\in\mathcal{I}} \kappa_i \lambda_{\max}(M_i^{-1}) \geq 0 \tag{2.57}$$

Theorem 2.8. *[174] "With the adaptation law (2.56),(2.57) and the switching law with mode-dependent dwell time (2.55), the globally uniformly ultimately bounded stability of the unknown switched system (2.52) with controller (2.54) can be guaranteed. In addition, the tracking error e is bounded as*

$$|e(t)|^2 \leq \frac{1}{\beta^2} \max\left\{c_1, \frac{\alpha(c_2 + \alpha|\bar{d}|^2)}{(1-\xi)\max_{i\in\mathcal{I}} \kappa_i}\right\} \tag{2.58}$$

where c_1 and c_2 are two positive constants that depend on the initial estimates and on the actual values of the controller parameters."

The well-known drawback of the leakage modification is that the adaptation laws tend to drive the estimated parameters towards zero [71, Sec.8.4.1]. This problem persists in the above adaptation laws (2.56) for switched systems. Built upon these adaptation laws, the follow-up work [152] introduces novel adaptation laws, which keep the control gains in the inactive phase constant. This alleviates the problem of the estimated parameters tending to zero and improves the transient behavior of the closed-loop system.

In addition to projections and leakages, another common robust modification is the dead-zone modification [71, Sec.8.4.3]. The underlying concept of dead-zone modification is terminating parameter adaptations for small tracking errors while updating the parameters for large tracking errors. In [161], a robust adaptive controller for switched linear systems in canonical form is proposed based on the dead-zone modification, which guarantees the uniform boundedness of the tracking error and the stability of the closed-loop system.

2.3 Summary

In this chapter, the definitions of switched linear systems and PWA systems as well as their stability analysis are presented. Besides, representative adaptive control approaches for switched systems in the literature are revisited, where the switching signals are assumed to be known while the subsystem parameters are unknown. From these approaches, two typical paradigms can be summarized (see Table 2.1). The first one relies on CLF, which ensures the asymptotic stability of the closed-loop system under arbitrary switching. The

	method	extra conditions	convergence	reference
non-adaptive (autonomous)	CLF	-	$e \to 0$	[92]
	MLF + dwell time	-	$e \to 0$	[65],[186]
adaptive (direct)	CLF	-	$e \to 0$	Theorem 2.3
	MLF + dwell time	projection	s.m.s.s.	Theorem 2.7
	MLF + dwell time	PE	$e \to 0$	[137], Theorem 2.5
	MLF + dwell time	leakage	bounded e	Theorem 2.8
adaptive (indirect)	CLF	-	$\tilde{x} \to 0$	Theorem 2.4
	MLF + dwell time	according to [83] not feasible		

Table 2.1: CLF and MLF for non-adaptive and adaptive control of switched systems.

other one exploits MLF. This paradigm allows instantaneous jumps of the Lyapunov function at switching instants, which should be compensated by its exponential decrease in between sufficiently slow switches. The exponential decrease can be ensured by either introducing PE conditions or imposing robust modifications such as leakage and projection, which requires more prior information. Moreover, the robust modifications can also ensure the robustness of the closed-loop systems in the presence of disturbances. In the following chapters, these two paradigms will be adopted in the adaptive control design to solve the questions raised in Section 1.1.

3 Adaptive Control of PWA Systems with Concurrent Learning

As presented in Chapter 2, the existing results of the adaptive control of switched systems can be categorized into direct adaptive control and indirect adaptive control approaches. In direct MRAC, the controller gains are updated based on tracking errors without estimating the system parameters. Considering the case, where the identification of the uncertain system parameters is a part of the control objective, the indirect MRAC can be applied. In indirect MRAC, the control gains are updated based on the estimated system parameters. With the indirect adaptive control approach revisited in Section 2.2.1, the asymptotic convergence of the tracking error is proved by using a CLF (see Theorem 2.4). Under the PE condition, all the estimated subsystem parameters are proved to converge to the real values (see Theorem 2.6).

As pointed out in Section 1.1, the PE assumption requires the system input to contain various frequencies. A common realization of this is to exert sinusoidal waves with different frequencies on the system input. Such input signal causes oscillations and vibrations in the real engineering systems, which might be harmful to the plants. Considering the circumstance, where the PE condition cannot be satisfied and only limited excitations can be provided, how to achieve parameter convergence is a challenging task. A recently proposed approach, concurrent learning [33, 34], exploits the recorded history data of the system and replaces the restrictive PE condition with some mild assumption on the linear independence of the recorded data. This technique has been applied to the identification of PWA systems [46, 82, 84, 85] to achieve parameter convergence without PE conditions. Nevertheless, how to integrate concurrent learning into the indirect MRAC of PWA systems is still open.

This Chapter enhances the indirect MRAC approach revisited in Section 2.2.1 by integrating the concurrent learning technique to overcome the challenge of limited excitations. Without requiring the PE condition, the proposed approach guarantees the convergence of the subsystem parameters to their real values. Besides, the controller gains are converged to the nominal values. Moreover, the closed-loop system is proved to be stable when the system enters the sliding mode. Compared to the previous approach, the concurrent learning-based approach exhibits better tracking performance and guarantees parameter convergence without the PE condition.

The rest of this chapter is structured as follows. Section 3.1 defines the problem to be solved in this chapter. The concurrent learning-based indirect MRAC is displayed in Section 3.2. The stability proof and convergence analysis are provided in Section 3.3. The proposed method is validated through a numerical simulation in Section 3.4. The conclusion and discussion of this chapter are followed in Section 3.5.

3.1 Problem Formulation

In this chapter, we consider the PWA system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t), \tag{3.1}$$

with known indicator function $\chi_i(t), i \in \mathcal{I}$ and unknown parameters $A(t) = \sum_{i=1}^{s} \chi_i(t)A_i$, $B(t) = \sum_{i=1}^{s} \chi_i(t)B_i$, and $f(t) = \sum_{i=1}^{s} \chi_i(t)f_i$ for $i \in \mathcal{I}$. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state and the control input of the PWA system. The reference PWA system is described by

$$\dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)r(t) + f_m(t) \tag{3.2}$$

with $A_m = \sum_{i=1}^{s} A_{mi}\chi_i$, $B_m = \sum_{i=1}^{s} B_{mi}\chi_i$, $f_m = \sum_{i=1}^{s} f_{mi}\chi_i$ being known parameters of the reference system. Let $K_{xi}^* \in \mathbb{R}^{p\times n}, K_{ri}^* \in \mathbb{R}^{p\times p}, K_{fi}^* \in \mathbb{R}^p, i \in \mathcal{I}$ denote the nominal controller gains for the i-th subsystem of (3.1). The nominal controller gains and the system parameters switch synchronously. For the nominal controller

$$u(t) = K_x^*(t)x(t) + K_r^*(t)r(t) + K_f^*(t), \tag{3.3}$$

where $K_x^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{xi}^*$, $K_r^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{ri}^*$, $K_f^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{fi}^*$, we assume that the following matching equations hold

$$A_{mi} = A_i + B_iK_{xi}^*, \quad B_{mi} = B_iK_{ri}^*, \quad f_{mi} = f_i + B_iK_{fi}^*, \quad i \in \mathcal{I}. \tag{3.4}$$

As most existing results presented in Section 2.2, we assume here that the reference system shares the same indicator functions χ_i as the controlled PWA system (3.1). Assume each subsystem of the reference system is stable and thus there exists symmetric and positive definite matrix $P \in \mathbb{R}^{n\times n}$ for the given symmetric and positive definite matrix $Q_i \in \mathbb{R}^{n\times n}$ such that

$$A_{mi}^T P + PA_{mi} = -Q_i, \quad i \in \mathcal{I}. \tag{3.5}$$

The goal is to design the indirect adaptation laws for the adaptive controller

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t) \tag{3.6}$$

with

$$K_x = \sum_{i=1}^{s} \chi_i K_{xi}, \quad K_r = \sum_{i=1}^{s} \chi_i K_{ri}, \quad K_f = \sum_{i=1}^{s} \chi_i K_{fi}, \tag{3.7}$$

where $K_{xi} \in \mathbb{R}^{p\times n}$, $K_{ri} \in \mathbb{R}^{p\times r}$ and $K_{fi} \in \mathbb{R}^p$ denote the control gains for i-th subsystem.

The problem we would like to solve in this chapter is formulated as follows:

Problem 3.1. Given a reference system (3.2) and a PWA system (3.1) with known state space partitions Ω_i (or equivalently, known indicator functions $\chi_i(t)$) and unknown subsystem parameters A_i, B_i and f_i, design an adaptive control law $u(t)$ such that the state $x(t)$ of the PWA system (3.1) tracks the state $x_m(t)$ of the reference system (3.2) and the system parameters converge to their real values without the PE condition.

3.2 Indirect Adaptive Control Design

In this section, we incorporate the concurrent learning technique to solve Problem 3.1. Before we start, it is necessary to clarify the term “concurrent learning” as the word “learning” may create a misleading impression of a machine learning process with a nonparametric model such as Gaussian process. In fact, the problem we are studying is based on a parametric system model (PWA model), and the term “concurrent learning” means the concurrent use of current and recorded data for adaptation in the context of model-based adaptive control.

Our proposed concurrent learning-based approach combines the current data and recorded history data for the estimation of the subsystem parameters

$$\dot{\hat{A}}_i = \dot{\hat{A}}_i^C + \dot{\hat{A}}_i^R, \quad \dot{\hat{B}}_i = \dot{\hat{B}}_i^C + \dot{\hat{B}}_i^R, \quad \dot{\hat{f}}_i = \dot{\hat{f}}_i^C + \dot{\hat{f}}_i^R \tag{3.8}$$

with the superscript ‘C’ representing the parameter adaptation with the current data, while the superscript ‘R’ means the adaptation with the recorded data.

The parameter update law based on the current data takes the form

$$\begin{aligned} \dot{\hat{A}}_i^C &= -\chi_i P \tilde{x} x^T - \varepsilon_{Ai}, \\ \dot{\hat{B}}_i^C &= -\chi_i P \tilde{x} u^T - \varepsilon_{Ai} K_{xi}^T - \varepsilon_{Bi} K_{ri}^T - \varepsilon_{fi} K_{fi}^T, \\ \dot{\hat{f}}_i^C &= -\chi_i P \tilde{x} - \varepsilon_{fi}, \end{aligned} \tag{3.9}$$

where $\tilde{x} = \hat{x} - x$ denotes the prediction error of the system state and the predicted state $\hat{x}$ is generated by the following dynamics

$$\dot{\hat{x}} = A_m \hat{x} + \sum_{i=1}^{s} ((\hat{A}_i - A_{mi})x + \hat{B}_i u + \hat{f}_i)\chi_i. \tag{3.10}$$

$\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$ in (3.9) are closed-loop estimation errors with the same definition as those shown in Section 2.2.1

$$\varepsilon_{Ai} = \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \quad \varepsilon_{Bi} = \hat{B}_i K_{ri} - B_{mi}, \quad \varepsilon_{fi} = \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}. \tag{3.11}$$

Based on the closed-loop estimation errors, the control gains adaptation obeys the following adaptation laws

$$\dot{K}_{xi} = -S_i^T B_{mi}^T \varepsilon_{Ai}, \quad \dot{K}_{ri} = -S_i^T B_{mi}^T \varepsilon_{Bi}, \quad \dot{K}_{fi} = -S_i^T B_{mi}^T \varepsilon_{fi}. \tag{3.12}$$

Until now, the adaptation laws (3.9), (3.11), and (3.12) are the same as those of the classical approach revisited in Section 2.2.1. Recall that the idea of concurrent learning is to use the history data concurrently to update the estimated parameters. Suppose that $x_{i_j}, u_{i_j}, \dot{x}_{i_j}$ represent the j-th recorded state, input and derivative of state of i-th subsystem with $j \in \{1, 2, \cdots, q\}$, where $q \in \mathbb{N}^+$ denotes the number of recorded data and $q \geq n+p+1$ holds. We make the following assumption.

Assumption 3.1. The information of the state derivative $\dot{x}$ is available and precise such that the equation $\dot{x}_{i_j} = A_i x_{i_j} + B_i u_{i_j} + f_i, i \in \mathcal{I}, j \in \{1, 2, \cdots, q\}$ holds.

We define ε_{i_j} as

$$\varepsilon_{i_j}(t) = \hat{A}_i(t)x_{i_j} + \hat{B}_i(t)u_{i_j} + \hat{f}_i(t) - \dot{x}_{i_j} \tag{3.13}$$

and replace $\dot{x}_{i_j}$ based on Assumption 3.1 leading to

$$\begin{aligned}\varepsilon_{i_j}(t) &= (\hat{A}_i(t) - A_i)x_{i_j} + (\hat{B}_i(t) - B_i)u_{i_j} + (\hat{f}_i(t) - f_i) \\ &= \tilde{A}_i(t)x_{i_j} + \tilde{B}_i(t)u_{i_j} + \tilde{f}_i(t)\end{aligned} \tag{3.14}$$

with $\tilde{A}_i = \hat{A}_i - A_i, \tilde{B}_i = \hat{B}_i - B_i$ and $\tilde{f}_i = \hat{f}_i - f_i$. We propose the following update law based on the recorded data

$$\dot{\hat{A}}_i^R = -\chi_i\gamma\sum_{j=1}^{q}\varepsilon_{i_j}x_{i_j}^T, \quad \dot{\hat{B}}_i^R = -\chi_i\gamma\sum_{j=1}^{q}\varepsilon_{i_j}u_{i_j}^T, \quad \dot{\hat{f}}_i^R = -\chi_i\gamma\sum_{j=1}^{q}\varepsilon_{i_j}, \tag{3.15}$$

where $\gamma \in \mathbb{R}^+$ is a positive scaling factor.

3.3 Stability and Parameter Convergence

Compared with the previous approach proposed in [83] (and revisited in Section 2.2.1), the concurrent learning-based method of this chapter supplements the additional adaptation terms (3.15), which depend on the recorded data. Now we proceed to explore, how the modified adaptive law affects the control and parameter convergence. The state tracking and parameter identification performance are summarized in the following theorem.

Theorem 3.1. *Consider a reference system (3.2) with CQLF. The PWA system (3.1) with known region partitions and unknown subsystem parameters is controlled by (3.6) with the adaptation laws (3.8), (3.11) and (3.12) based on (3.9) and (3.15). Let the recorded data stacks $Z_i \in \mathbb{R}^{(n+p+1)\times q}$ contain $n+p+1$ linearly independent vectors $z_{i_j} = [x_{i_j}^T, u_{i_j}^T, 1]^T$. If the input matrices B_i have full column rank, the pairs (A_{mi}, B_{mi}) are controllable, then the state of the PWA system asymptotically tracks the reference state x_m. Furthermore, the estimated parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ converge to their true values and the control gains K_{xi}, K_{ri}, K_{fi} converge to the nominal gains as $t \to \infty$.*

Proof. Consider the following candidate Lyapunov function

$$\begin{aligned}V = &\frac{1}{2}\tilde{x}^T P\tilde{x} + \frac{1}{2}\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\tilde{A}_i) + \mathrm{tr}(\tilde{B}_i^T\tilde{B}_i) + \tilde{f}_i^T\tilde{f}_i \\ &+ \mathrm{tr}(\tilde{K}_{xi}^T M_{si}\tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si}\tilde{K}_{fi})\end{aligned} \tag{3.16}$$

with $M_{si} = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p \times p}$. Taking the time derivative of V yields

$$\begin{aligned}\dot{V} =& \underbrace{\frac{1}{2}(\tilde{x}^T P \dot{\tilde{x}} + \dot{\tilde{x}}^T P \tilde{x})}_{\triangleq \dot{V}_1} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T \dot{\tilde{A}}_i^C) + \mathrm{tr}(\tilde{B}_i^T \dot{\tilde{B}}_i^C) + \tilde{f}_i^T \dot{\tilde{f}}_i^C)}_{\triangleq \dot{V}_{2a}} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_{si} \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si} \dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_{si} \dot{\tilde{K}}_{fi})}_{\triangleq \dot{V}_{2b}} \\ &+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T \dot{\tilde{A}}_i^R) + \mathrm{tr}(\tilde{B}_i^T \dot{\tilde{B}}_i^R) + \tilde{f}_i^T \dot{\tilde{f}}_i^R)}_{\triangleq \dot{V}_3}.\end{aligned} \tag{3.17}$$

Inserting the parameter update law based on current data (3.9), indirect update law of control gains (3.12) and closed-loop estimation errors (3.11) into $\dot{V}_1, \dot{V}_{2a}$ and $\dot{V}_{2b}$ yields

$$\dot{V}_1 + \dot{V}_{2a} + \dot{V}_{2b} = -\tilde{x}^T(\frac{1}{2}\sum_{i=1}^{s} Q_{mi}\chi_i)\tilde{x} - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \mathrm{tr}(\varepsilon_{fi}^T \varepsilon_{fi})). \tag{3.18}$$

Detailed derivations of this step can be found in [83]. Substituting the $\dot{\tilde{A}}_i^R$, $\dot{\tilde{B}}_i^R$ and $\dot{\tilde{f}}_i^R$ in $\dot{V}_3$ with (3.15) gives

$$\dot{V}_3 = -\sum_{i=1}^{s}\chi_i\gamma(\underbrace{\mathrm{tr}(\tilde{A}_i^T \sum_{j=1}^{q}\varepsilon_{i_j} x_{i_j}^T)}_{\triangleq \dot{V}_{3ai}} + \underbrace{\mathrm{tr}(\tilde{B}_i^T \sum_{j=1}^{q}\varepsilon_{i_j} u_{i_j}^T)}_{\triangleq \dot{V}_{3bi}} + \underbrace{\mathrm{tr}(\tilde{f}_i^T \sum_{j=1}^{q}\varepsilon_{i_j})}_{\triangleq \dot{V}_{3fi}}). \tag{3.19}$$

Inserting (3.14) into $\dot{V}_{3ai}$ yields

$$\begin{aligned}\dot{V}_{3ai} &= \gamma\mathrm{tr}(\tilde{A}_i^T(\tilde{A}_i \sum_{j=1}^{q} x_{i_j} x_{i_j}^T + \tilde{B}_i \sum_{j=1}^{q} u_{i_j} x_{i_j}^T + \tilde{f}_i \sum_{j=1}^{q} x_{i_j}^T)) \\ &= \gamma\mathrm{tr}(\tilde{A}_i^T \begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix} \underbrace{\begin{bmatrix}\sum_j x_{i_j} x_{i_j}^T \\ \sum_j u_{i_j} x_{i_j}^T \\ \sum_j x_{i_j}^T\end{bmatrix}}_{\triangleq \xi_{1i}}).\end{aligned} \tag{3.20}$$

Using the property of trace $\mathrm{tr}(X^T Y) = \mathrm{vec}(X)^T \mathrm{vec}(Y)$, (3.20) can be further transformed as

$$\dot{V}_{3ai} = \gamma\mathrm{vec}(\tilde{A}_i)^T \mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix} \xi_{1i}). \tag{3.21}$$

Recalling the compatibility of vectorization with Kronecker product $\mathrm{vec}(ABC) = (C^T \otimes A)\mathrm{vec}(B)$, it follows

$$\begin{aligned}\dot{V}_{3ai} &= \gamma\mathrm{vec}(\tilde{A}_i)^T \mathrm{vec}(I_n \begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix} \xi_{1i}) \\ &= \gamma\mathrm{vec}(\tilde{A}_i)^T (\xi_{1i}^T \otimes I_n)\mathrm{vec}(\begin{bmatrix}\tilde{A}_i & \tilde{B}_i & \tilde{f}_i\end{bmatrix}).\end{aligned} \tag{3.22}$$

Similarly, we can obtain

$$\dot{V}_{3bi} = \gamma \mathrm{vec}(\tilde{B}_i)^T (\xi_{2i}^T \otimes I_n) \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix}) \tag{3.23}$$

and

$$\dot{V}_{3fi} = \gamma \mathrm{vec}(\tilde{f}_i)^T (\xi_{3i}^T \otimes I_n) \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix}) \tag{3.24}$$

with

$$\xi_{2i} = \begin{bmatrix} \sum_j x_{i_j} u_{i_j}^T \\ \sum_j u_{i_j} u_{i_j}^T \\ \sum_j u_{i_j}^T \end{bmatrix}, \quad \xi_{3i} = \begin{bmatrix} \sum_j x_{i_j} \\ \sum_j u_{i_j} \\ \sum_j 1 \end{bmatrix}. \tag{3.25}$$

Summing up $\dot{V}_{3ai}$, $\dot{V}_{3bi}$ and $\dot{V}_{3fi}$ yields

$$\begin{aligned} &\dot{V}_{3ai} + \dot{V}_{3bi} + \dot{V}_{3fi} \\ =&\gamma [\mathrm{vec}(\tilde{A}_i)^T \mathrm{vec}(\tilde{B}_i)^T \mathrm{vec}(\tilde{f}_i)^T] \begin{bmatrix} \xi_{1i}^T \otimes I_n \\ \xi_{2i}^T \otimes I_n \\ \xi_{3i}^T \otimes I_n \end{bmatrix} \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix})) \\ =&\gamma \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix})^T \begin{bmatrix} \xi_{1i}^T \otimes I_n \\ \xi_{2i}^T \otimes I_n \\ \xi_{3i}^T \otimes I_n \end{bmatrix} \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix}) \end{aligned} \tag{3.26}$$

Note that

$$\begin{aligned} \begin{bmatrix} \xi_{1i}^T \otimes I_n \\ \xi_{2i}^T \otimes I_n \\ \xi_{3i}^T \otimes I_n \end{bmatrix} &= \sum_j \begin{bmatrix} x_{i_j} x_{i_j}^T & x_{i_j} u_{i_j}^T & x_{i_j} \\ u_{i_j} x_{i_j}^T & u_{i_j} u_{i_j}^T & u_{i_j} \\ x_{i_j}^T & u_{i_j}^T & 1 \end{bmatrix} \otimes I_n \\ &= (\sum_j \begin{bmatrix} x_{i_j} \\ u_{i_j} \\ 1 \end{bmatrix} \begin{bmatrix} x_{i_j}^T & u_{i_j}^T & 1 \end{bmatrix}) \otimes I_n. \end{aligned} \tag{3.27}$$

Therefore, we obtain

$$\dot{V}_{3ai} + \dot{V}_{3bi} + \dot{V}_{3fi} = \gamma \tilde{\theta}_i^T \Xi_i \tilde{\theta}_i \tag{3.28}$$

with

$$\Xi_i = (\sum_{j=1}^{q} \begin{bmatrix} x_{i_j} \\ u_{i_j} \\ 1 \end{bmatrix} \begin{bmatrix} x_{i_j}^T & u_{i_j}^T & 1 \end{bmatrix}) \otimes I_n, \quad \tilde{\theta}_i = \mathrm{vec}(\begin{bmatrix} \tilde{A}_i & \tilde{B}_i & \tilde{f}_i \end{bmatrix}). \tag{3.29}$$

So the derivative of the candidate Lyapunov function becomes

$$\dot{V} = -\tilde{x}^T (\frac{1}{2} \sum_{i=1}^{s} Q_{mi} \chi_i) \tilde{x} - \gamma \sum_{i=1}^{s} \chi_i \tilde{\theta}_i^T \Xi_i \tilde{\theta}_i - \sum_{i=1}^{s} (\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \mathrm{tr}(\varepsilon_{fi}^T \varepsilon_{fi})). \tag{3.30}$$

The linear independence of the $n+p+1$ vectors z_{i_j} implies the full rank of the data stack Z_i, from which it follows that $Z_i Z_i^T$ is positive definite. Since the identity matrix I_n is positive definite, the Kronecker product $\Xi_i = Z_i Z_i^T \otimes I_n$ is also positive definite, which together with the positive definiteness of Q_{mi} implies the negative semidefiniteness of $\dot{V}$.

An essential issue in analyzing the stability of the adaptive control of PWA systems is that the closed-loop system may enter a sliding mode. Namely, both the vector fields of two neighbouring subsystems point towards the switching hyperplane and the trajectory of the system cannot move across the regions. According to the Filippov concept [18, 56, 83], we evaluate the convex combinations of the vector fields around the sliding surface. This can be done by substituting $\chi_i \in \{0, 1\}$ with $\bar{\chi}_i \in [0, 1]$ in the expression of $\dot{V}$. Hence, we have

$$-\tilde{x}^T(\frac{1}{2}\sum_{i=1}^{s} Q_{mi}\bar{\chi}_i)\tilde{x} \leq 0 \tag{3.31}$$

and

$$-\gamma \sum_{i=1}^{s} \bar{\chi}_i \tilde{\theta}_i^T \Xi_i \tilde{\theta}_i \leq 0, \tag{3.32}$$

which leads to the negative semidefiniteness of $\dot{V}$ even when the system enters sliding mode. This indicates the boundedness of state prediction error $\tilde{x}$, estimated subsystem parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ (and equivalently $\tilde{\theta}_i \in \mathcal{L}_\infty$) and control gains K_{xi}, K_{ri}, K_{fi}. This further implies $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty$. Moreover, from $\dot{V} \leq 0$ it follows $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_2$ and $\tilde{\theta}_i \in \mathcal{L}_2$.

From the boundedness of $\tilde{x}$ and $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$, it follows $\tilde{x}, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \to 0$ as $t \to \infty$, $x, u \in \mathcal{L}_\infty$. Therefore, $\lim_{t\to\infty}(x - x_m) = 0$. The details of this derivation step is referred to [83].

Furthermore, the recorded data $x_{i_j}, u_{i_j} \in \mathcal{L}_\infty$ due to the boundedness of x, u. This together with $\tilde{\theta}_i \in \mathcal{L}_\infty$ results in $\varepsilon_{i_j} \in \mathcal{L}_\infty$. Considering (3.15) we have $\dot{\hat{A}}_i^R, \dot{\hat{B}}_i^R, \dot{\hat{f}}_i^R \in \mathcal{L}_\infty$. From (3.9) we can obtain $\dot{\hat{A}}_i^C, \dot{\hat{B}}_i^C, \dot{\hat{f}}_i^C \in \mathcal{L}_\infty$. Therefore, let $\hat{\theta}_i = \text{vec}([\hat{A}_i \;\; \hat{B}_i \;\; \hat{f}_i])$ and it yields $\dot{\hat{\theta}}_i, \dot{\tilde{\theta}}_i \in \mathcal{L}_\infty$, which together with $\tilde{\theta}_i \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\tilde{\theta}_i \to 0$ as $t \to \infty$. Hence, $\hat{A}_i \to A_i, \hat{B}_i \to B_i$ and $\hat{f}_i \to f_i$ as $t \to \infty$.

Finally, we study the convergence of the controller gains. Considering the full column rank assumption of B_i and taking the convergence of $\hat{A}_i, \hat{B}_i, \hat{f}_i, \varepsilon_{A_i}, \varepsilon_{Bi}$ and ε_{fi} into (3.11), we can conclude that $K_{xi} \to K_{xi}^*, K_{ri} \to K_{ri}^*$ and $K_{fi} \to K_{fi}^*$ as $t \to \infty$. □

Remark 3.1. One condition to guarantee the convergence of the control and subsystem parameters is the linear independence of the sampled data vectors $\{z_{i_j}\}_{j=1}^q$. Here we use the singular value maximizing data recording algorithm proposed in [32] to maximize the singular value of the data stack Z_i and obtain rich information. By doing so the condition of linear independence can be fulfilled faster.

3.4 Numerical Validation

In this section, the proposed concurrent learning-based indirect MRAC approach is validated through a numerical example.

We take the mass-spring-damper system from [83] to validate the proposed algorithm. The system is shown in the Fig. 3.1, where $m_1 = 5\,\text{kg}, m_2 = 1\,\text{kg}$ denote the masses, $d = 1\,\text{N s/m}$ is the damping factor, p_1, p_2 represent the displacement of the two springs, F_1, F_2 are the forces operated on the masses, respectively. The left mass is connected with the static wall by a spring with static spring constant $c_0 = 1\,\text{N/m}$ whereas the two masses are connected with the right spring, which has a stiffness with a PWA characteristics

$$F_c(p_1, p_2) = \begin{cases} 10\,\text{N/m}, & \text{if } |p_2 - p_1| \leq 1\,\text{m} \\ 1\,\text{N/m}, & \text{if } p_2 - p_1 > 1\,\text{m} \\ 100\,\text{N/m}, & \text{if } p_2 - p_1 < -1\,\text{m} \end{cases} \tag{3.33}$$

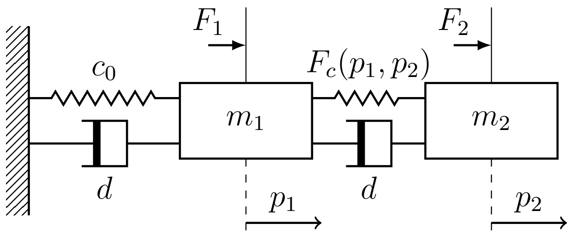


Figure 3.1: The coupled mass-spring-damper system

Defining the state vector $x = [p_1, \dot{p}_1, p_2, \dot{p}_2]^T$ and the input vector $u = [F_1, F_2]^T$, the system dynamics can be written as a PWA system in the state space form as (3.1) with

$$\dot{x} = \underbrace{\begin{bmatrix} 0 & 1 & 0 & 0 \\ -\frac{c_0+c_i}{m_1} & -\frac{2d}{m_1} & \frac{c_i}{m_1} & \frac{d}{m_1} \\ 0 & 0 & 0 & 1 \\ \frac{c_i}{m_2} & \frac{d}{m_2} & -\frac{c_i}{m_2} & -\frac{d}{m_2} \end{bmatrix}}_{A_i} x + \underbrace{\begin{bmatrix} 0 & 0 \\ \frac{1}{m_1} & 0 \\ 0 & 0 \\ 0 & \frac{1}{m_2} \end{bmatrix}}_{B_i} u + f_i \tag{3.34}$$

with the affine terms $f_i, i = \{1, 2, 3\}$ being

$$f_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ \frac{c_1-c_2}{m_1} \\ 0 \\ \frac{c_2-c_1}{m_2} \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ \frac{c_3-c_1}{m_1} \\ 0 \\ \frac{c_1-c_3}{m_2} \end{bmatrix}. \tag{3.35}$$

The partitions of the regions are given by

$$\begin{aligned} \Omega_1 &= \{x^T \in \mathbb{R}^4 | H_1^T [x, 1]^T \preceq 0\}, \\ \Omega_2 &= \{x^T \in \mathbb{R}^4 | H_2^T [x, 1]^T < 0\}, \\ \Omega_3 &= \{x^T \in \mathbb{R}^4 | H_3^T [x, 1]^T < 0\}. \end{aligned}$$

with the switching hyperplanes

$$\begin{aligned} H_1^T &= \begin{bmatrix} -1 & 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 & -1 \end{bmatrix}, \\ H_2^T &= \begin{bmatrix} 1 & 0 & -1 & 0 & 1 \end{bmatrix}, \\ H_3^T &= \begin{bmatrix} -1 & 0 & 1 & 0 & 1 \end{bmatrix}. \end{aligned}$$

The reference system is chosen as

$$\dot{x}_m = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -25 & -10 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -25 & -10 \end{bmatrix} x_m + \begin{bmatrix} 0 & 0 \\ 25 & 0 \\ 0 & 0 \\ 0 & 25 \end{bmatrix} r, \tag{3.36}$$

which exhibits a decoupling motion of the two masses. The control goal of our approach is to enforce dynamics of the controlled PWA system to track the trajectory of the reference system and identify the uncertain subsystem parameters of the controlled PWA system.

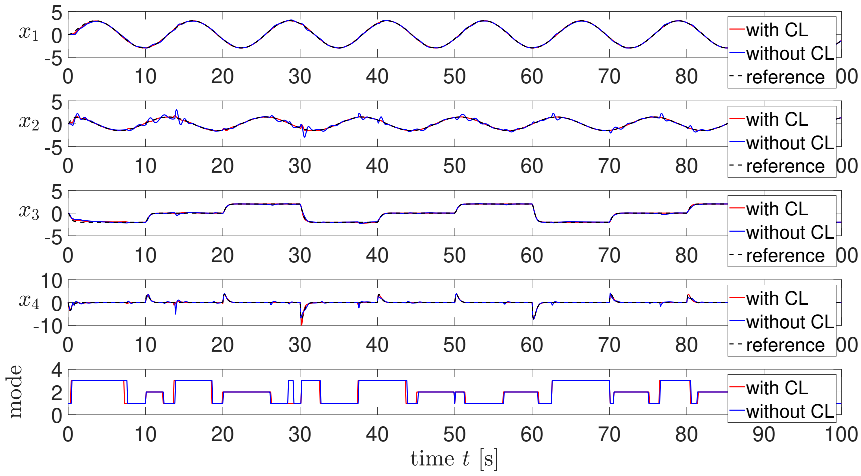


Figure 3.2: State tracking performance of indirect MRAC with and without concurrent learning

We use the reference signal $r = [r_1, r_2]^T$, where $r_1 = 3\sin(0.5t)$ and r_2 is a periodic rectangular wave switching among the values $\{-2, 0, 2\}$ with time interval $T = 30\,\mathrm{s}$. The scaling factor γ is specified to be 20. The same common P matrix is utilized as in [83]. Besides, the singular value maximizing data recording algorithm is utilized to manage the data for concurrent learning.

Fig. 3.2 shows the state tracking performance of the indirect MRAC approach with and without concurrent learning. ‘CL’ in the legends stands for ‘concurrent learning’. The black dashed lines depict the states of the reference model. The red lines and blue lines show the states of the controlled PWA system with and without concurrent learning, respectively. No significant difference between the performance of the two approaches is observed for the positions (x_1 and x_3). However, we can see that the red trajectories of the velocity components (x_2 and x_4) exhibit fewer peaks compared to the corresponding blue lines. Hence, using concurrent learning improves the state tracking performance of the controlled system.

In Figure 3.3, the norm of the parameter estimation errors $\tilde{\theta}_i$ by using algorithms with and without concurrent learning are displayed in red and blue lines, respectively. By using concurrent learning, $|\tilde{\theta}_i|$ converges to zero for $\forall i \in \{1, 2, 3\}$. Compared to the concurrent learning-based approach, the classical approach exhibits unsatisfactory convergence performance of the parameter estimation errors.

Fig. 3.4 displays the convergence of the controller gains of subsystem 2 (the controller gains for other subsystems are similar and thus not shown because of clarity) by applying the concurrent learning-based indirect MRAC approach. The dashed lines represent the nominal gains and the solid lines stand for the adaptation gains. The elements in the gain matrices are distinguished by different colors. We can see that the controller gains converge to their nominal values, which validates the conclusion of Theorem 3.1.

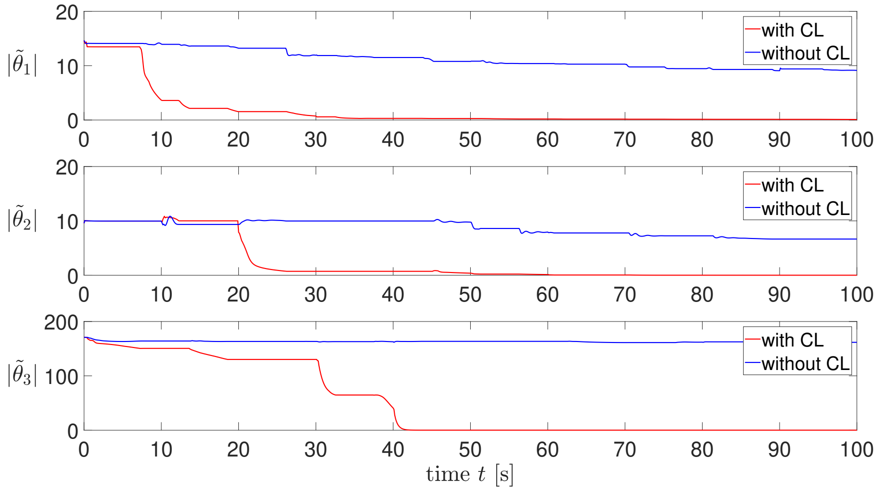


Figure 3.3: Parameter convergence of indirect MRAC with and without concurrent learning

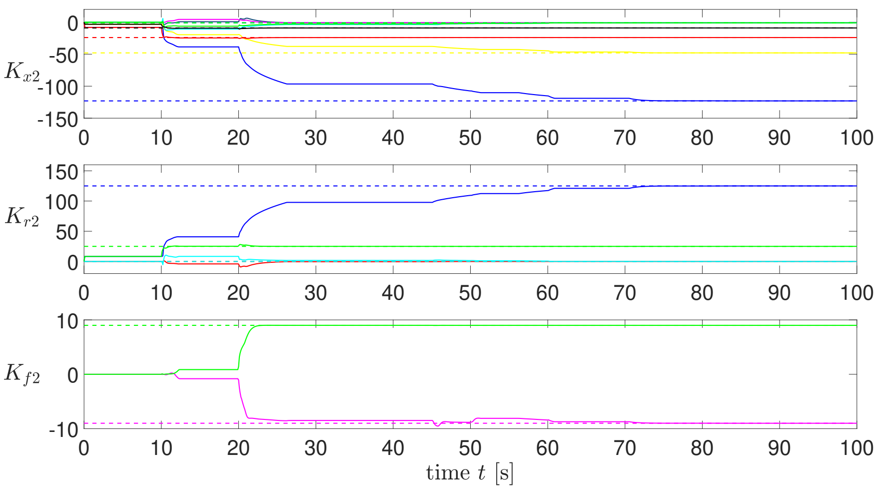


Figure 3.4: Convergence of controller gains with concurrent learning-based indirect MRAC

3.5 Summary

In this chapter, we proposed a concurrent learning-based indirect MRAC approach for uncertain PWA systems. With the proposed approach, the controlled PWA system tracks the trajectory of the reference system asymptotically. Based on the CQLF, the closed-loop system is stable under arbitrary switching and in sliding mode. Furthermore, if the recorded data of concurrent learning is linearly independent, the system parameters converge to their real values and the control gains converge to the nominal gains, which eliminates the need for the conventional PE condition.

Compared to the classical adaptive control approaches of switched linear systems and piecewise affine systems, the advantage of our approach is that it enables the identification of unknown system parameters without disturbing the primary control task due to its PE-free feature. This advantage is essential in the case where the estimation or monitoring of system parameters is required in addition to the nominal operation of a plant.

The proposed approach is validated through a numerical simulation example, a coupled mass-spring-damper system with piecewise linear spring characteristics. The simulation result validates an improvement in the state tracking and parameter estimation performance when compared with the classical indirect adaptive controller revisited in Section 2.2.1. Although the numerical example might appear simple, it models a large class of real plants in the practice such as lateral vehicle dynamics with piecewise linear tire friction characteristics [21, 24], two-inertia motor systems with backlash [171], and aircraft wing models with nonlinear aeroelasticity [176]. This indicates the applicability of the proposed method in a wide range of practical domains.

One limitation of the concurrent learning-based approach is that the state derivative is required to be known. In real applications, the state derivative information can either be obtained by placing extra sensors or be estimated by differentiating the state vector numerically at the expense of extra costs or estimation errors. Existing approaches to alleviate the estimation errors of the state derivatives include the dynamic state-derivative estimator [76], integral concurrent learning [123], and fixed point smoothing [80, Sec.5.2.1]. In future work, it is of practical interest to further investigate the assumption of known state derivatives, as recent results [119, 131] show that the need for the state derivatives can actually be removed by incorporating filtering techniques. Applying the filtering techniques to our approach to omit the assumption of known state derivatives deserves further studies.

Another limitation of the approach presented in this chapter is that it requires the existence of the CLF, which might restrict the design flexibility of the reference system. However, we will show later in Section 5.3 that such restriction can be lifted by constructing a special type of MLF, which exploits barrier functions and exhibits only non-increasing jumps at switching instants. This will inspire further research on the concurrent learning-based indirect MRAC of PWA without the existence of the CLF, with which one will have more flexibility on the design of the reference system and therefore, expand the scope of application scenarios.

Adaptive Control of PWA Systems with Output Performance Guarantees

4

Most of the MRAC approaches for PWA systems reviewed in Section 2.2 ensure asymptotic tracking, namely, zero steady-state tracking error. However, the transient behavior of the closed-loop system is not guaranteed and can only be improved by manually tuning the adaptation gains or imposing additional PE conditions. As mentioned in Section 1.1, the analysis and improvement of the transient behavior is an essential issue in adaptive control [116], because an aggressive transient response may result in saturation, oscillation, or even damage to the physical plants in real applications. In this chapter, we would like to explore the adaptive control of PWA systems with the output tracking performance constraints on the transient behavior and the steady-state tracking error.

Prescribed performance control, proposed in [12, 13], is a popular tool to guarantee the element-wise performance of adaptive systems. With this approach, the steady-state tracking error and the transient response such as the decaying rate as well as the overshoot are confined within a predefined bound. The core idea is to transform the ratio of the tracking error and a performance function into an unbounded value, the so-called transformed error. By designing the controller to ensure the boundedness of the transformed error, the original tracking error is guaranteed to stay within the performance bound. This approach has been incorporated into different areas such as multi-agent systems [38, 55, 106, 139], helicopter/satellite attitude control [187, 188], underwater vehicles [50] and robot manipulators [77]. In addition to these applications, prescribed performance control has also been introduced to the field of switched systems. For switched nonlinear systems in strict-feedback form, the method proposed in [177] combines the prescribed performance control with dynamic surface control. The tracking performance satisfies the prescribed performance with average dwell time constraints. The approaches reported in [90, 91] can also be applied to switched nonlinear systems in nonstrict-feedback form. These methods are suitable for systems with known input matrices and cannot be applied to the PWA systems with unknown subsystem parameters. Besides, the parameter convergence, which is a topic of major interest in the area of adaptive control [150], is not considered in these works.

The main contribution of this chapter lies in tackling the direct and indirect adaptive output tracking control problem of uncertain PWA systems with performance constraints. Specifically, we cast the dynamics of the transformed error metric into linear form, where the nonlinearity and switching are captured as its exogenous input. Based on that, we construct novel common Lyapunov functions, which do not rely on the solution of the conventional Lyapunov equations shown in Chapter 2, and prove the closed-loop stability under arbitrary switching. We further prove that the estimated controller and system parameters converge to their nominal values under PE conditions. Moreover, a robust modification is developed for the direct adaptation case to ensure the robustness of the closed-loop system in the presence of disturbances.

This chapter is structured as follows: in Section 4.1 the problem to be solved in this chapter is defined. Some preliminaries of the prescribed performance technique is revisited in Section 4.2. The design of nominal control is introduced in Section 4.3, which is followed by direct adaptive control in Section 4.4 and indirect adaptive control in Section 4.5. A robustness modification is presented in Section 4.6. The approaches are validated through numerical examples in Section 4.7. Finally, the discussion and conclusion of this chapter is given in Section 4.8.

4.1 Problem Formulation

In this chapter, we consider a special class of multi-input multi-output (MIMO) PWA system with *strict relative degree* $r \in \mathbb{N}^+$ described as follows

$$\begin{aligned} x_1^{(r)} =& a_{1i}^T x + b_{1i}^T u + f_{1i} \\ &\vdots \\ x_p^{(r)} =& a_{pi}^T x + b_{pi}^T u + f_{pi}, \quad i \in \mathcal{I} \\ y =& [x_1, x_2, \cdots, x_p]^T \end{aligned} \tag{4.1}$$

where

$$x_j^{(r)} = \frac{d^r x_j}{dt^r} \tag{4.2}$$

and $x = [x_1, \cdots, x_1^{(r-1)}, \cdots, x_p, \cdots, x_p^{(r-1)}]^T \in \mathbb{R}^n$ denotes the overall state vector with $n = pr$. $u, y \in \mathbb{R}^p$ represent the control input and system output, respectively. The output y and its derivatives up to order $r-1$ constitute the state vector x. They are available for the control design. $a_{ji} \in \mathbb{R}^n, b_{ji} \in \mathbb{R}^p, f_{ji} \in \mathbb{R}, j = 1, \cdots, p$ denote the system parameters of i-th subsystem. We write system (4.1) into compact form and obtain

$$\begin{aligned} \dot{x} &= A_i x + B_i u + f_i, \quad i \in \mathcal{I} \\ y &= Cx, \end{aligned} \tag{4.3}$$

where $A_i \in \mathbb{R}^{n \times n}$, $B_i \in \mathbb{R}^{n \times p}$, $C \in \mathbb{R}^{p \times n}$ and $f_i \in \mathbb{R}^n$ denote the system parameters of i-th subsystem. $a_{ji}, b_{ji}, f_{ji}, j = 1, \cdots, p, i \in \mathcal{I}$ are contained in A_i, B_i, f_i in the corresponding positions, respectively and thus, A_i, B_i, C, f_i are in control canonical form. Since a large class of physical systems can be modeled [43, 176] and transformed [15] into canonical form, the control design for PWA systems in canonical form is essential and attracts a lot of interests such as [16, 18, 19]. In this chapter, we focus on the prescribed performance adaptive control of MIMO PWA systems in control canonical form.

With the help of the indicator functions $\chi_i(t)$, we can rewrite the PWA system as

$$\begin{aligned} \dot{x} &= A(t)x + B(t)u + f(t) \\ y &= Cx, \end{aligned} \tag{4.4}$$

where $A(t) = \sum_{i=1}^{s} \chi_i A_i$, $B(t) = \sum_{i=1}^{s} \chi_i B_i$ and $f(t) = \sum_{i=1}^{s} \chi_i f_i$. Since the system has strict relative degree r, we have

$$\begin{aligned} CB_i = CA_iB_i = \cdots = CA_i^{r-2}B_i = 0, \quad CA_i^{r-1}B_i \neq 0 \\ Cf_i = CA_if_i = \cdots = CA_i^{r-2}f_i = 0, \quad CA_i^{r-1}f_i \neq 0 \end{aligned} \tag{4.5}$$

for $i \in \mathcal{I}$, which leads to

$$\begin{aligned} y &= Cx \\ \dot{y} &= CA_i x \\ &\cdots \\ y^{(r)} &= CA_i^r x + CA_i^{r-1} B_i u + CA_i^{r-1} f_i. \end{aligned} \tag{4.6}$$

In this chapter, the system input and output have the same dimension p and the system is a square system. Nevertheless, this will not necessarily restrict our approach, since square systems cover broad applications [138]. Some non-square systems can also be transformed into square systems [61, 126].

Remark 4.1. The state x in PWA system (4.1) is continuous, also on the switching hyperplanes. This leads to the continuity of $y, \dot{y}, \cdots, y^{(r-1)}$ according to the definition of y (see (4.1)). This in turn, implies $CA_i = CA_j, \cdots, CA_i^{r-1} = CA_j^{r-1}$ for $\forall i, j \in \mathcal{I}$. If the PWA system is not in control canonical form, this property does not hold and the output derivative may exhibit jump behavior on the switching hyperplanes.

Recall that we would like to study the output tracking of the PWA system (4.3). We assume that the reference signals $y_d \in \mathbb{R}^p$ and its derivatives $\dot{y}_d, \cdots, y_d^{(r)} \in \mathbb{R}^p$ are bounded and continuous. To study the output tracking with prescribed performance, we first introduce the performance function and study its properties in control systems.

Definition 4.1 (Performance function [13])**.** A smooth positive function $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ is defined as the performance function if it is decreasing and satisfies $\lim_{t\to\infty} \rho(t) = \rho_\infty > 0$.

A commonly used performance function is

$$\rho(t) = (\rho_0 - \rho_\infty)e^{-lt} + \rho_\infty \tag{4.7}$$

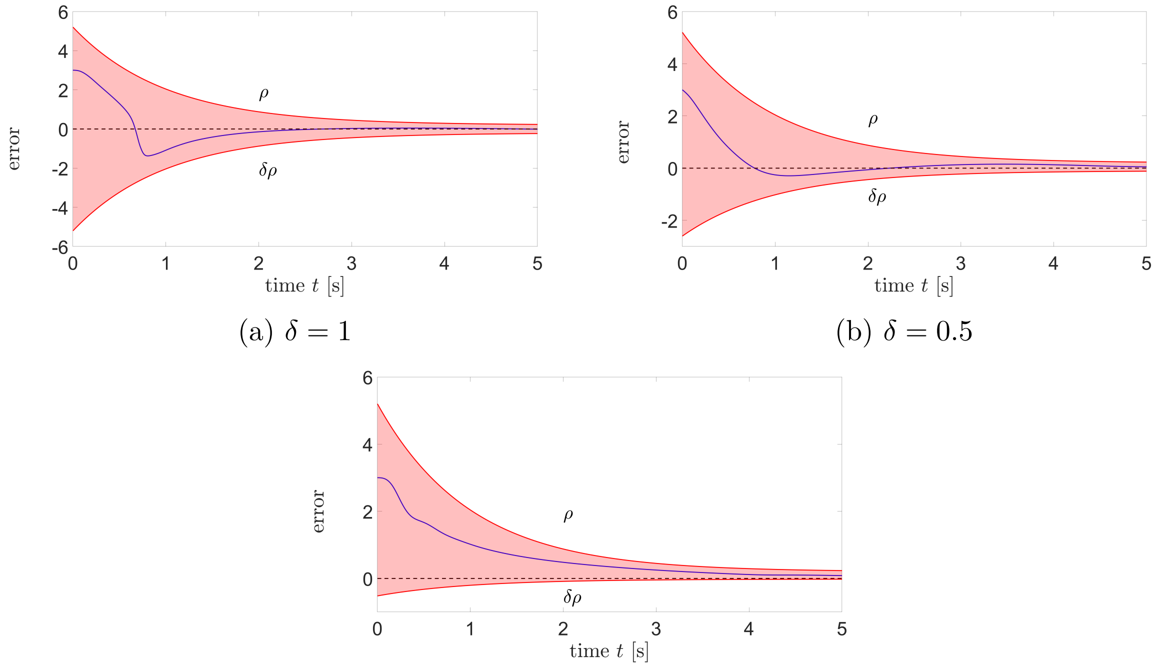
with $\rho_0, \rho_\infty, l \in \mathbb{R}^+$ and $\rho_0 > \rho_\infty$. We see that $\rho(t)$ is decreasing with $\rho(t = 0) = \rho_0$ and $\rho(t \to \infty) = \rho_\infty$.

Given the reference output y_d and a vector performance function $\rho(t) \in \mathbb{R}^p$, let $e = [e_1, e_2, \cdots, e_p]^T \in \mathbb{R}^p$ be the output tracking error $e = y - y_d$, $\rho_j(t)$ be the performance function of the j-th component of ρ. The control objective that the tracking error is confined within a prescribed performance bound can be expressed by the following inequalities

$$\begin{aligned} -\delta_j \rho_j(t) < e_j(t) < \rho_j(t), \quad \text{if } e_j(0) > 0, \\ -\rho_j(t) < e_j(t) < \delta_j \rho_j(t), \quad \text{if } e_j(0) < 0 \end{aligned} \tag{4.8}$$

for $j = 1, \cdots, p$, for $\delta_j \in [0, 1]$ and $\forall t > 0$. The variable δ_j is introduced in (4.8) such that the user could have more flexibility to design the performance bound. We take the case $e(0) > 0$ for instance. Fig. 4.1 shows the performance bounds and possible error transients with 3 different δ values (we remove the subscript j in the figure for simplicity). The performance bounds are displayed by red lines and the tracking errors are presented by blue lines. As illustrated in Fig. 4.1, the overshoot of e can be reduced by choosing smaller δ_j. If δ_j is set to be 0, then the overshoot in the error transient behavior can be completely avoided.

Problem 4.1. Given a PWA system (4.3) with known subsystem partitions Ω_i, unknown subsystem parameters A_i, B_i, f_i, design an adaptive control law $u(t)$ to enforce the output of the system $y(t)$ to track the given reference signal $y_d(t)$ with prescribed error performance (4.8). Besides, explore the conditions, under which the estimated gains or estimated parameters converge to their nominal or real values.



(a) $\delta = 1$

(b) $\delta = 0.5$

(c) $\delta = 0.1$

Figure 4.1: Graphical representation of performance bounds with different δ values

4.2 Prescribed Performance Technique

The concept of prescribed performance control is to transform the constrained error (4.8) into an unconstrained one, and thus the classical stability theory can be applied to design the controller for the unconstrained transformed error. Let σ_j be the transformed error and define $e_j = \rho_j(t)G_j(\sigma_j)$, where $G_j(\sigma_j)$ is a smooth and strictly increasing function of transformed error σ_j. Note that inequalities in (4.8) are equivalent to

$$\begin{aligned} -\delta_j < G_j(\sigma_j) < 1, &\quad \text{if } e_j(0) > 0, \\ -1 < G_j(\sigma_j) < \delta_j, &\quad \text{if } e_j(0) < 0, \end{aligned} \tag{4.9}$$

so the strictly increasing function $G_j(\sigma_j)$ needs to be designed such that (4.9) holds for $\sigma_j \in (-\infty, +\infty)$. We choose the following function as the most references suggested

$$\begin{aligned} G_j(\sigma_j) &= \frac{\exp(\sigma_j) - \delta_j \exp(-\sigma_j)}{\exp(\sigma_j) + \exp(-\sigma_j)}, \quad \text{if } e_j(0) > 0, \\ G_j(\sigma_j) &= \frac{\delta_j \exp(\sigma_j) - \exp(-\sigma_j)}{\exp(\sigma_j) + \exp(-\sigma_j)}, \quad \text{if } e_j(0) < 0. \end{aligned} \tag{4.10}$$

The transformed error σ_j can thus be solved by

$$\sigma_j = \begin{cases} G_j^{-1}(\frac{e_j(t)}{\rho_j(t)}) = \frac{1}{2}\ln\frac{\delta_j + G_j}{1 - G_j}, & \text{if } e_j(0) > 0, \\ G_j^{-1}(\frac{e_j(t)}{\rho_j(t)}) = \frac{1}{2}\ln\frac{1 + G_j}{\delta_j - G_j}, & \text{if } e_j(0) < 0, \end{cases} \tag{4.11}$$

from which we can see, if σ_j is bounded, then (4.9) holds, which further implies that (4.8) holds. To relate the transformed error σ_j with the tracking error e_j, we take for instance the time derivative of σ_j for $e_j(0) > 0$ and it yields

$$\dot{\sigma}_j = q^1_{0,j} e_j + q^1_{1,j} \dot{e}_j \tag{4.12}$$

with

$$\begin{aligned} q^1_{0,j} &= -\frac{\dot{\rho}_j}{2\rho_j^2} \frac{\delta_j + 1}{(1 - \frac{e_j}{\rho_j})(\delta_j + \frac{e_j}{\rho_j})} \\ q^1_{1,j} &= \frac{1}{2\rho_j} \frac{\delta_j + 1}{(1 - \frac{e_j}{\rho_j})(\delta_j + \frac{e_j}{\rho_j})}, \end{aligned}$$

and similarly, the k-th derivative of σ_j is

$$\sigma_j^{(k)} = \sum_{l=0}^{k-1} q^k_{l,j}(\rho_j, \cdots, \rho_j^{k-l}) e_i^{(l)} + \frac{\partial G_j^{-1}}{\partial(\frac{e_j}{\rho_j})} \frac{1}{\rho_j} e_j^{(k)} \tag{4.13}$$

where $q^k_{l,j}(\rho_j, \cdots, \rho_j^{k-l})$ represents a term depends on $\rho_j, \cdots, \rho_j^{(k-l)}$ for some given $k < r, k \in \mathbb{N}^+$ and $l = 1, 2, \cdots, k-1$. Define the error metric E_j

$$E_j = \sigma_j + \sum_{k=1}^{r-1} \lambda_k \sigma_j^{(k)}, \tag{4.14}$$

where $\lambda_k \in \mathbb{R}^+$ are parameters to be chosen, $\sigma_j^{(k)}$ is the k-th derivative of σ_j. E_j is utilized to describe the dynamics of the transformed error system. The derivative of E_j follows

$$\dot{E}_j = \sum_{k=0}^{r-1} \sum_{l=k-1}^{r-1} \lambda_l q^{l+1}_{k,j} e_j^{(k)} + \lambda_{r-1} q^r_{r,j} e_j^{(r)} \tag{4.15}$$

with $\lambda_{-1} = q^0_0 = 0, \lambda_0 = 1$. We can write the vector form

$$\dot{E} = \sum_{k=0}^{r-1} \sum_{l=k-1}^{r-1} \lambda_l R^{l+1}_k e^{(k)} + \lambda_{r-1} R^r_r e^{(r)} \tag{4.16}$$

with $E = [E_1, \cdots, E_p]^T \in \mathbb{R}^p$ and

$$R^l_k = \begin{bmatrix} q^l_{k,1} & & 0 \\ & \ddots & \\ 0 & & q^l_{k,p} \end{bmatrix}. \tag{4.17}$$

Since $\rho(t)$ and y_d are known, each component of their derivative up to r-th order can be calculated. The system state x is assumed to be available and thus $y, \dot{y}, \cdots, y^{(r-1)}$ are also available. Substituting $e^{(r)}$ in (4.16) with $y^{(r)} - y_d^{(r)}$ and inserting (4.6) yields

$$\begin{aligned} \dot{E} =& \underbrace{\sum_{k=0}^{r-1} \sum_{l=k-1}^{r-1} \lambda_l R^{l+1}_k e^{(k)} - \lambda_{r-1} R^r_r y_d^{(r)} + }_{\triangleq K} \lambda_{r-1} R^r_r C A^r x \\ &+ \lambda_{r-1} R^r_r C A^{r-1} B u + \lambda_{r-1} R^r_r C A^{r-1} f. \end{aligned} \tag{4.18}$$

This step associates the system input u with the error metric E. If the control input u is designed such that E is bounded, then the boundedness of $\sigma_j^{(k)}$ are ensured for $k = 1, \cdots, r-1, j = 1, \cdots, p$. This further implies the achievement of prescribed performance described by the inequalities in (4.8).

For the purpose of clarity, we replace R_r^r with R and λ_{r-1} with λ in the rest of this chapter and express $\dot{E}$ as

$$\dot{E} = K + \lambda RCA^r x + \lambda RCA^{r-1}Bu + \lambda RCA^{r-1}f. \tag{4.19}$$

4.3 Nominal Control

We start with the nominal control design, where the subsystem parameters and switching hyperplanes are known exactly.

The following control law, which is suggested by the Lyapunov stability analysis (will be shown in Theorem 4.1), is proposed

$$u = K_x^* x + K_r^* \xi + K_f^*, \tag{4.20}$$

where

$$\xi = \frac{1}{\lambda} R^{-1} E + \frac{1}{\lambda} R^{-1} K \tag{4.21}$$

and

$$\begin{aligned} K_x^* &= \sum_{i=1}^{s} \chi_i K_{xi}^* = -\sum_{i=1}^{s} \chi_i (C\Psi_i)^{-1} C\Phi_i \\ K_r^* &= \sum_{i=1}^{s} \chi_i K_{ri}^* = -\sum_{i=1}^{s} \chi_i (C\Psi_i)^{-1} \\ K_f^* &= \sum_{i=1}^{s} \chi_i K_{fi}^* = -\sum_{i=1}^{s} \chi_i (C\Psi_i)^{-1} C\Upsilon_i \end{aligned} \tag{4.22}$$

are nominal controller gains with

$$\Phi_i = A_i^r, \quad \Psi_i = A_i^{r-1} B_i, \quad \Upsilon_i = A_i^{r-1} f_i. \tag{4.23}$$

Note that $C\Psi_i$ is assumed to be invertible for $i \in \mathcal{I}$. The controller structure (4.20), the definition of ξ (4.21) as well as the nominal controllers (4.22) are determined by the Lyapunov-based stability analysis. The performance analysis of the proposed nominal control law and the closed-loop stability are summarized in the following theorem.

Theorem 4.1. *Given the reference signal y_d and predefined performance function ρ, let the PWA system (4.3) with known partition regions Ω_i and known subsystem parameters A_i, B_i, f_i be controlled by the feedback controller (4.20). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (4.8).*

Proof. Substituting u in (4.19) with (4.20) and inserting (4.22), we obtain

$$\dot{E} = K + \lambda RC\Phi x + \lambda RC\Psi u + \lambda RC\Upsilon = -E \tag{4.24}$$

where $\Phi = \sum_{i=1}^{s} \chi_i \Phi_i, \Psi = \sum_{i=1}^{s} \chi_i \Psi_i, \Upsilon = \sum_{i=1}^{s} \chi_i \Upsilon_i$. This means that the closed-loop dynamics of E can be described by the homogeneous system $\dot{E} = -E$ by applying the nominal controller (4.20). Define the following Lyapunov function

$$V = \frac{1}{2} E^T E. \tag{4.25}$$

Taking the derivative along the trajectory (4.19) yields

$$\dot{V} = -E^T E \leq 0. \tag{4.26}$$

From (4.26) it follows $E \in \mathcal{L}_\infty$ and $E \to 0$ as $t \to \infty$. This further implies the boundedness of $\sigma_j, \sigma_j^{(k)}$ with $\sigma_j, \sigma_j^{(k)} \to 0$ as $t \to \infty$, $\forall j = 1, \cdots, p$, which leads to $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \cdots, r-1$, and thus $x \in \mathcal{L}_\infty$. From the definition of K and (4.21), we also have $K \in \mathcal{L}_\infty$ and $\xi \in \mathcal{L}_\infty$. From (4.11) and the boundedness of σ_j we can conclude that the tracking error is within the performance bound, i.e., (4.8) holds. □

Remark 4.2. Asymptotic tracking can be achieved under certain conditions. From Theorem 4.1 we have $\sigma_j \to 0$. Given certain δ_j, $\lim_{t\to\infty} G_j$ can be obtained by solving $\lim_{t\to\infty} \sigma_j = 0$ according to (4.11). If $\delta_j = 1$, then we obtain $G_j \to 0$ for $t \to \infty$. Since $G_j = \frac{e_j}{\rho_j}$ and $\rho_j \neq 0$, the j-th component of the tracking error $e_j \to 0$ as $t \to \infty$.

Remark 4.3. The controller (4.20) shares the common structure as the controller of MRAC, i.e., $u = K_x^* x + K_r^* r + K_f^*$ (see (2.21)). The difference is that the reference signal r of the MRAC is replaced by ξ in this context. Unlike the reference signal r, which is given as an external signal in MRAC, ξ also contains internal signals. As shown by (4.21), ξ contains the error metric E and output tracking errors as well as their higher-order derivatives (captured by K). Therefore, its boundedness needs to be specially checked, as shown in the proof of Theorem 4.1.

Remark 4.4. According to (4.14) we have that E depends on σ_j and its derivatives, which in turn relates to the tracking error e and its derivatives. Since y_d and its derivatives as well as $y, \dot{y}, \cdots, y^{(r-1)}$ are continuous (see Remark 4.1), E is also continuous even on switching hyperplanes. Therefore, the Lyapunov function (4.25) is shared by all the subsystems and it decreases independent of which subsystem is activated. This implies that the Lyapunov function (4.25) is a CLF and the closed-loop stability can be concluded under arbitrary switching.

The nominal control design provides the basis for the further design and analysis of adaptive control. In particular, the controller structure (4.20) as well as the expressions of the nominal controller gains (4.22) are utilized in adaptive controllers, as will be shown in the next sections.

4.4 Direct Adaptive Control Design

In this section, we introduce the direct prescribed performance adaptive control for PWA systems and provide stability analysis as well as parameter convergence analysis.

4.4.1 Controller Design

The adaptive control design is based on *certainty equivalence principle* [150], namely, we use the estimated parameters in the feedback control as if they are the real system parameters in the case of uncertain or unknown system dynamics. Therefore, the controller takes the same structure as in (4.20) but with the estimated parameters

$$u = K_x x + K_r \xi + K_f \tag{4.27}$$

where

$$K_x = \sum_{i=1}^{s} \chi_i K_{xi}, \quad K_r = \sum_{i=1}^{s} \chi_i K_{ri}, \quad K_f = \sum_{i=1}^{s} \chi_i K_{fi}$$

are estimated controller gains. We propose the following adaptation law to update the estimated controller gains

$$\dot{K}_{xi} = \chi_i \Gamma_{xi} S_i^T R^T E x^T, \quad \dot{K}_{ri} = \chi_i \Gamma_{ri} S_i^T R^T E \xi^T, \quad \dot{K}_{fi} = \chi_i \Gamma_{fi} S_i^T R^T E, \tag{4.28}$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling factors. Define the estimation errors of the controller gains as

$$\tilde{K}_{xi} = K_{xi} - K_{xi}^*, \quad \tilde{K}_{ri} = K_{ri} - K_{ri}^*, \quad \tilde{K}_{fi} = K_{fi} - K_{fi}^*. \tag{4.29}$$

We insert (4.27) in (4.19) and obtain

$$\begin{aligned} \dot{E} &= K + \sum_{i=1}^{s} \chi_i (\lambda RC\Phi_i x + \lambda RC\Psi_i u + \lambda RC\Upsilon_i) \\ &= K + \sum_{i=1}^{s} \chi_i (\lambda RC\Phi_i x + \lambda RC\Psi_i K_{xi}^* x \\ &+ \lambda RC\Psi_i \tilde{K}_{xi} x + \lambda RC\Psi_i K_{ri}^* \xi + \lambda RC\Psi_i \tilde{K}_{ri} \xi \\ &+ \lambda RC\Psi_i K_{fi}^* + \lambda RC\Psi_i \tilde{K}_{fi} + \lambda RC\Upsilon_i). \end{aligned} \tag{4.30}$$

Inserting the nominal controller gains (4.22) yields

$$\dot{E} = -E + \lambda R \sum_{i=1}^{s} \chi_i C\Psi_i (\tilde{K}_{xi} x + \tilde{K}_{ri} \xi + \tilde{K}_{fi}). \tag{4.31}$$

This equation describes the dynamics of the error metric E when the adaptive controller (4.27) is utilized. The estimation errors of controller gains $\tilde{K}_x, \tilde{K}_r, \tilde{K}_f$ constitute the external inputs of the dynamics. The state transition matrix of E is $-I$ and thus is not affected by switching.

4.4.2 Stability Analysis

We study the stability and the tracking performance of the closed-loop system. The result is summarized in the following theorem.

Theorem 4.2. *Given the reference signal $y_d(t)$ and predefined performance function ρ, let the PWA system (4.3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (4.27) with the update law (4.28). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (4.8).*

Proof. We define the following Lyapunov function

$$V = \frac{E^T E}{2\lambda} + \frac{1}{2}\sum_{i=1}^{s}(\Gamma_{xi}^{-1}\text{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \Gamma_{ri}^{-1}\text{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \Gamma_{fi}^{-1}\text{tr}(\tilde{K}_{fi}^T M_i \tilde{K}_{fi})), \tag{4.32}$$

where $M_i = (K_{ri}^* S_i)^{-1} \in \mathbb{R}^{p\times p}$. Taking the time derivative of V yields

$$\dot{V} = \frac{E^T \dot{E}}{\lambda} + \sum_{i=1}^{s}(\Gamma_{xi}^{-1}\text{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \Gamma_{ri}^{-1}\text{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \Gamma_{fi}^{-1}\text{tr}(\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi})). \tag{4.33}$$

We replace $\dot{E}$ in the first part of $\dot{V}$ with (4.31) and obtain

$$\frac{E^T \dot{E}}{\lambda} = -\frac{E^T E}{\lambda} + E^T R C \Psi(\tilde{K}_x x + \tilde{K}_r \xi + \tilde{K}_f), \tag{4.34}$$

where $\tilde{K}_x = \sum_{i=1}^{s} \chi_i \tilde{K}_{xi}$, $\tilde{K}_r = \sum_{i=1}^{s} \chi_i \tilde{K}_{ri}$, $\tilde{K}_f = \sum_{i=1}^{s} \chi_i \tilde{K}_{fi}$. Now we analyze the second summand in V. Considering that

$$M_i S_i^T = M_i S_i^T(-C\Psi_i K_{ri}^*)^T = -M_i S_i^T K_{ri}^{*T}(C\Psi_i)^T = -M_i M_i^{-1}(C\Psi_i)^T = -(C\Psi_i)^T, \tag{4.35}$$

we have

$$\begin{aligned}\Gamma_{xi}^{-1}\text{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) &= \chi_i \text{tr}(\tilde{K}_{xi}^T M_i S_i^T R^T E x^T) \\ &= -\chi_i \text{tr}(\tilde{K}_{xi}^T \Psi_i^T C^T R^T E x^T) \\ &= -\chi_i \text{tr}(x E^T R C \Psi_i \tilde{K}_{xi}) \\ &= -\chi_i \text{tr}(E^T R C \Psi_i \tilde{K}_{xi} x).\end{aligned} \tag{4.36}$$

Because the term $E^T R C \Psi_i \tilde{K}_x x$ is a scalar, its trace is equal to itself. Thus,

$$\Gamma_{xi}^{-1}\text{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) = -\chi_i E^T R C \Psi_i \tilde{K}_{xi} x. \tag{4.37}$$

Similarly, we obtain

$$\begin{aligned}\Gamma_{ri}^{-1}\text{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) &= -\chi_i E^T R C \Psi_i \tilde{K}_{ri} \xi, \\ \Gamma_{fi}^{-1}\text{tr}(\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}) &= -\chi_i E^T R C \Psi_i \tilde{K}_{fi}.\end{aligned} \tag{4.38}$$

Inserting (4.34), (4.37) and (4.38) into (4.33) yields

$$\dot{V} = -\frac{1}{\lambda} E^T E \leq 0. \tag{4.39}$$

The negative semidefiniteness of $\dot{V}$ confirms the stability of the closed-loop adaptive system. More precisely, $E, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$. Considering $E \in \mathcal{L}_\infty$, (4.11) and (4.14), we have $\sigma, \sigma^{(k)}, e, e^{(k)} \in \mathcal{L}_\infty$, which further indicates $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \cdots, r-1$, and thus $x \in \mathcal{L}_\infty$.

The boundedness of $e^{(k)}$ leads to $R_k^l \in \mathcal{L}_\infty$ with $k = 0, 1, \cdots, r, l = 1, 2, \cdots, r$, from which we can obtain $K, \xi \in \mathcal{L}_\infty$ and hence, $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. The boundedness of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}, x, \xi$ gives $u \in \mathcal{L}_\infty$ and $\dot{E} \in \mathcal{L}_\infty$. (4.39) also implies that $E \in \mathcal{L}_2$, which together with $E, \dot{E} \in \mathcal{L}_\infty$ gives $\lim_{t\to\infty} E \to 0$. This together with the boundedness of σ_j

implies that the tracking error e is confined within the prescribed performance bound, i.e., (4.8) holds.

To analyse the stability in sliding mode, we follow the concept in Chapter 3 and observe the derivative of V along the sliding mode solutions, which can be achieved by replacing the indicator function $\chi_i \in \{0,1\}$ with $\bar{\chi}_i \in [0,1]$, where $\sum_{i=1}^{s} \bar{\chi}_i = 1$. Specifically, the transformed error dynamics (4.31) is convexified as

$$\dot{E} = -E + \lambda R \sum_{i=1}^{s} \bar{\chi}_i C \Psi_i (\tilde{K}_{xi} x + \tilde{K}_{ri} \xi + \tilde{K}_{fi}). \tag{4.40}$$

Equation (4.40) holds due to the synchronous switching of the plant and the controller. As a part of the closed-loop dynamics, the adaptation gains during the sliding motion are

$$\begin{aligned} \dot{K}_{xi} &= \bar{\chi}_i \Gamma_{xi} S_i^T R^T E x^T \\ \dot{K}_{ri} &= \bar{\chi}_i \Gamma_{ri} S_i^T R^T E \xi^T \\ \dot{K}_{fi} &= \bar{\chi}_i \Gamma_{fi} S_i^T R^T E. \end{aligned} \tag{4.41}$$

Inserting (4.40) and (4.41) into $\dot{V}$, we still obtain the same expression as in (4.39), which implies the stability of the controlled system also in sliding mode. □

Remark 4.5. Theorem 4.2 shows that the tracking error stays within the prescribed performance bound. Note that $E, \sigma, \sigma^{(k)} \to 0, k = 1, \cdots, r-1$ as $t \to \infty$, the time limit of the tracking error can thus be calculated by solving (4.11). Similar to Remark 4.2, for $\delta_j = 1, j \in \{1, \cdots, p\}$, we have the solution $\lim_{t\to\infty} e_j(t) = 0$.

Remark 4.6. Benefit from the property that the state transition matrix of E is independent of the switching (as shown in (4.31)), the Lyapunov function (4.32) is a CLF. It ensures the closed-loop stability under arbitrary switching. A similar concept to construct the CLF can be found in the adaptive control for switched systems in Brunovsky form [4], where an error metric is constructed based on the tracking error and its derivatives (see (11) in [4]). When comparing to the approach in [4], the distinctive feature of our approach is that the error metric E is expressed in terms of the transformed error σ_j and thus the transient behavior evolves within the prescribed performance bound if E is bounded.

Remark 4.7. The stability analysis of classical MRAC of PWA systems revisited in Section 2.2.1 also relies on the CLF. It requires the existence and the knowledge of a common Lyapunov matrix P such that the Lyapunov equation $A_{mi}^T P + P A_{mi} < 0$ holds for all the state matrices A_{mi} of the reference PWA system. Differing from this requirement, the construction of the CLF in this chapter only requires the continuity of the reference signal and its derivatives, which is less restrictive. This advantage is obtained at the expense of confining the applications to the output tracking of PWA systems in control canonical form.

Remark 4.8. Theorem 4.2 shows that the tracking error e satisfies the prescribed performance condition, i.e., (4.8) holds. If the performance function is chosen as (4.7), the tracking error e decays exponentially. In the classical direct MRAC of PWA systems, the PE condition of the reference signals must be introduced to ensure the exponential decaying of tracking errors in between switches (see Theorem 2 in [137] and Theorem 2 in [83]). Besides, the decaying rate depends on the excitation level of the reference signals. Expressing it explicitly is not straightforward (see (26), (27) in [83]). In contrast, the exponential decaying of the tracking error in our approach does not require PE conditions and the decaying rate can be specified directly in the performance function (4.7) by choosing the value of l.

4.4.3 Parameter Convergence

Theorem 4.2 shows the boundedness of the controller gains K_{xi}, K_{ri}, K_{fi}. In this section, we discuss if the adaptive controller gains converge to the nominal gains under the classical PE conditions. First of all, we explore if the signal vector $z = [x^T, \xi^T, 1]^T$ is PE given a sufficiently rich reference signal y_d. This is summarized in the following lemma.

Lemma 4.1. Let the system (4.3) be controlled by the controller (4.27). If the closed-loop system has $E \in \mathcal{L}_\infty$, $E \to 0$ for $t \to \infty$, if the reference signal y_d is sufficiently rich of order $r+1$, and if $\delta_j = 1, j = 1, \cdots, p$, then the vector $z = [x^T, \xi^T, 1]^T$ is PE.

Proof. The signal vector z can be written as

$$z = \begin{bmatrix} x \\ \xi \\ 1 \end{bmatrix} = \begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(r-1)} \\ \xi \\ 1 \end{bmatrix} + \begin{bmatrix} e \\ \dot{e} \\ \vdots \\ e^{(r-1)} \\ 0 \\ 0 \end{bmatrix} = \underbrace{\begin{bmatrix} y_d \\ \dot{y}_d \\ \vdots \\ y_d^{(r-1)} \\ -y_d^{(r)} \\ 1 \end{bmatrix}}_{\triangleq \zeta} + \underbrace{\begin{bmatrix} e \\ \dot{e} \\ \vdots \\ e^{(r-1)} \\ \nu \\ 0 \end{bmatrix}}_{\triangleq \epsilon'}$$

with

$$\nu = \frac{1}{\lambda} R^{-1} E + \frac{1}{\lambda} R^{-1} (\sum_{k=0}^{r-1} \sum_{l=k-1}^{r-1} \lambda_l R_k^{l+1} e^{(k)}). \tag{4.42}$$

We have $E \in \mathcal{L}_\infty$ and $E \to 0$ for $t \to \infty$. This results in $e, e^{(k)} \in \mathcal{L}_\infty$. From $\delta_j = 1, j = 1, \cdots, p$, it follows $e, e^{(k)} \to 0, k = 1, \cdots, r-1$ as $t \to \infty$, which further leads to $\nu \to 0$ as $t \to \infty$. Therefore, $\epsilon' \in \mathcal{L}_\infty$ and $\epsilon' \to 0$ as $t \to \infty$. In order to obtain the PE property of z, it suffices to show the PE property of ζ [71, Lemma 4.8.3].

Writing ζ in the frequency domain related to y_d yields

$$\zeta(s) = \sum_{j=1}^{p} \underbrace{\begin{bmatrix} l_j \\ s l_j \\ \vdots \\ -s^r l_j \\ 0 \end{bmatrix}}_{\triangleq H_j} y_{dj}(s) + \underbrace{\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}}_{\triangleq H_f} 1(s). \tag{4.43}$$

where $y_{dj}(s)$ is the j-th element of the reference signal y_d. $l_j = \mathrm{col}_j(I_p) \in \mathbb{R}^p$ is the j-th column of the identity matrix $I_p \in \mathbb{R}^{p \times p}$. The auto-covariance of ζ is given by

$$\begin{aligned} R_\zeta(0) = &\frac{1}{2\pi} \int_{-\infty}^{\infty} H_f(-\iota\omega) S_1(\omega) H_f^T(\iota\omega) \mathrm{d}\omega \\ &+ \frac{1}{2\pi} \sum_{j=1}^{p} \int_{-\infty}^{\infty} H_j(-\iota\omega) S_{y_j}(\omega) H_j^T(\iota\omega) \mathrm{d}\omega, \end{aligned} \tag{4.44}$$

where ι is the imaginary unit ($\iota^2 = -1$), $S_{yj}(\omega)$ and $S_1(\omega)$ are the spectral distribution of the j-th component of the desired output y_d and the constant input 1, respectively. Given that y_d is sufficiently rich of order $r+1$, each of its element has $r+1$ distinct peaks at $\omega_{lj}, l = 1, \cdots, r+1$. Therefore, we have

$$S_{y_j}(\omega) = \sum_{l=1}^{r+1} f_{y_j}(\omega_{lj})\delta(\omega - \omega_{lj}), \tag{4.45}$$

where $f_{y_j}(\omega_{lj})$ is the l-th peak of y_{dj} in the frequency domain and the constant 1 leads to a single unit delta function at zero

$$S_1(\omega) = \delta(\omega). \tag{4.46}$$

Inserting (4.45) and (4.46) into (4.44) yields

$$\begin{aligned} R_\zeta(0) = &\frac{1}{2\pi} H_f(-\iota 0) H_f^T(\iota 0) \\ &+ \frac{1}{2\pi} \sum_{j=1}^{p} \sum_{l=1}^{r+1} f_{y_j}(\omega_{lj}) H_j(-\iota\omega_{lj}) H_j(\iota\omega_{lj}). \end{aligned} \tag{4.47}$$

To prove the PE property of ζ, it suffices to prove the positive definiteness of $R_\zeta(0)$, namely, the only solution of the equation

$$\mathcal{X}^T R_\zeta(0)\mathcal{X} = 0 \tag{4.48}$$

is $\mathcal{X} = 0_{n+p+1}$ for $\mathcal{X} \in \mathbb{R}^{n+p+1}$. Considering that each summand in (4.47) is positive semidefinite, (4.48) can hold if and only if

$$H_f^T(0)\mathcal{X} = 0, \quad H_j^T(\iota\omega_{lj})\mathcal{X} = 0, \tag{4.49}$$

for $l = 1, 2, \cdots, r+1$ and $j = 1, \cdots, p$.

Suppose $\mathcal{X} = [\mathcal{Y}, \mathcal{Z}]^T$ with $\mathcal{Y} = [\mathcal{Y}_1, \cdots, \mathcal{Y}_{n+p}]^T \in \mathbb{R}^{n+p}$ and $\mathcal{Z} \in \mathbb{R}$. $H_f^T(0)\mathcal{X} = 0$ indicates $\mathcal{Z} = 0$.

For a certain $j \in \{1, \cdots, p\}$, $H_j^T(\iota\omega_{lj})\mathcal{X} = 0$ with $r+1$ distinct frequencies ω_{lj} results in

$$\underbrace{\begin{bmatrix} 1 & \iota\omega_{1j} & \cdots & (\iota\omega_{1j})^{r-1} & -(\iota\omega_{1j})^r \\ 1 & \iota\omega_{2j} & \cdots & (\iota\omega_{2j})^{r-1} & -(\iota\omega_{2j})^r \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 1 & \iota\omega_{r+1j} & \cdots & (\iota\omega_{r+1j})^{r-1} & -(\iota\omega_{r+1j})^r \end{bmatrix}}_{\triangleq \mathcal{H}_j} \begin{bmatrix} \mathcal{Y}_j \\ \mathcal{Y}_{p+j} \\ \vdots \\ \mathcal{Y}_{rp+j} \end{bmatrix} = 0.$$

$\mathcal{H}_j$ has full rank for $\omega_{hj} \neq \omega_{lj}$ with $h, l \in \{1, 2, \cdots, r+1\}$, which leads to

$$[\mathcal{Y}_j, \mathcal{Y}_{p+j}, \cdots, \mathcal{Y}_{rp+j}]^T = 0_{r+1}, \quad \forall j \in \{1, \cdots, p\}. \tag{4.50}$$

Note that $rp = n$, we have $\mathcal{Y} = 0_{n+p}$. Therefore, $\mathcal{X}^T R_\zeta(0)\mathcal{X} = 0$ holds if and only if $\mathcal{X} = 0$, so $R_\zeta(0)$ is positive definite, from which it follows ζ is PE and hence, z is also PE.

□

For the case, where the adaptive systems have to fulfill the desired tracking task y_d, which does not contain a sufficient amount of frequencies, the sufficiently rich condition can be fulfilled by superposing some periodic signals with the required amount of frequencies and small enough amplitudes upon the desired trajectory. By doing so, the sufficiently rich condition can be fulfilled without significantly disturbing the primary tracking task.

Since a PWA system has multiple subsystems, the controller gains of all the subsystems need to be estimated. To this end, we require that the reference signal y_d to be sufficiently rich and repeatedly activate all the subsystems as also suggested in other works of adaptive switched systems [83, 84, 173]. The conclusion is depicted by the following theorem.

Theorem 4.3. *Let the PWA system (4.3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (4.27) with the update law (4.28). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. Let the reference signals y_d be sufficiently rich of order $r + 1$ and cause repeated activation of all subsystems. If the matrices $C\Psi_i$ are invertible, and $\delta_j = 1$ for $j = 1, \cdots, p$, then $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ for $t \to \infty$.*

Proof. According to Theorem 4.2, the closed-loop system is stable under arbitrary switching. For clarity we first study a single subsystem and suppose the i-th subsystem to be activated during some time interval, i.e., $\chi_i(t) = 1$. We rewrite $\dot{E}$ as

$$\dot{E} = -E + \lambda RC\Psi_i(\tilde{K}_{xi}x + \tilde{K}_{ri}\xi + \tilde{K}_{fi}), \tag{4.51}$$

which can be further simplified by using Kronecker product

$$\dot{E} = -E + \lambda R\Xi^T\tilde{\theta}_i \tag{4.52}$$

with

$$\Xi = \begin{bmatrix} x \\ \xi \\ 1 \end{bmatrix} \otimes I_p, \quad \tilde{\theta}_i = \mathrm{vec}(C\Psi_i[\tilde{K}_{xi} \quad \tilde{K}_{ri} \quad \tilde{K}_{fi}]), \tag{4.53}$$

where $\otimes$ denotes the Kronecker product, $I_p \in \mathbb{R}^{p\times p}$ is an identity matrix, the operator $\mathrm{vec}(\cdot)$ represents the vectorization of a matrix.

Note that

$$\begin{aligned} \dot{\tilde{\theta}}_i &= \mathrm{vec}(C\Psi_i[\dot{\tilde{K}}_{xi} \quad \dot{\tilde{K}}_{ri} \quad \dot{\tilde{K}}_{fi}]) \\ &= \mathrm{vec}(C\Psi_i S^T R^T E[x^T \quad \xi^T \quad 1]) \\ &= \Xi \cdot \mathrm{vec}(C\Psi_i S^T R^T E) \\ &= -\Xi \cdot W_i R^T E, \end{aligned} \tag{4.54}$$

where $W_i = C\Psi_i M_i^{-1}(C\Psi_i)^T$. We write E and $\tilde{\theta}_i$ in a form of a new dynamical system and obtain

$$\begin{bmatrix} \dot{E} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} -I_p & \lambda R\Xi^T \\ -\Xi W_i R^T & 0 \end{bmatrix} \begin{bmatrix} E \\ \tilde{\theta}_i \end{bmatrix}. \tag{4.55}$$

From Theorem 4.2 and $\delta_j = 1, j = 1, \cdots, p$ we have $e_j(t) \to 0$. This leads to $R \to R^*$ as $t \to \infty$, where $R^* \in \mathbb{R}^{p\times p}$ is some constant diagonal matrix. R^* can be calculated by going through the derivation shown in Section 4.2. Let r_j^* denote j-th diagonal element of R^* and we have

$$r_j^* = \frac{1}{2\rho_j(t)} \frac{\delta_j + 1}{(1 - \frac{e_j(t)}{\rho_j(t)})(\delta_j + \frac{e_j(t)}{\rho_j(t)})}\Bigg|_{t\to\infty} = \frac{1}{\rho_{\infty j}} \tag{4.56}$$

with $\rho_{\infty j} = \rho_j(t \to \infty)$ being the predefined static bound of j-th error component. For $R = R^*$ we have the dynamical system

$$\begin{bmatrix} \dot{E} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} -I_p & \lambda R^* \Xi^T \\ -\Xi W_i R^{*T} & 0 \end{bmatrix} \begin{bmatrix} E \\ \tilde{\theta}_i \end{bmatrix}, \tag{4.57}$$

which has the same structure as the one of Lemma 2.2. Applying this lemma with the PE property of z (obtained by invoking Lemma 4.1) we have that $E \to 0$ and $\tilde{\theta}_i \to 0$ exponentially for system (4.57), which together with $R \to R^*$ implies that $E \to 0$ and $\tilde{\theta}_i \to 0$ as $t \to \infty$ for (4.55). Note that the exponential convergence property of $[E, \tilde{\theta}_i]$ in (4.57) is not retained in (4.55) due to the time varying R. So $[E, \tilde{\theta}_i]$ converges towards zero asymptotically during the inverval, when i-th subsystem is activated. Since all the subsystems are activated repeatedly, we have $\tilde{\theta}_i \to 0, \forall i \in \mathcal{I}$ as $t \to \infty$.

The convergence of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ cannot be directly concluded from the convergence of $\tilde{\theta}_i$. Further steps of analysis are needed. Note that

$$\begin{aligned} \tilde{\theta}_i &= \mathrm{vec}(C\Psi_i[K_{xi} - K_{xi}^* \quad K_{ri} - K_{ri}^* \quad K_{fi} - K_{fi}^*]) \\ &= \mathrm{vec}([C\Psi_i K_{xi} - C\Phi_i \quad C\Psi_i K_{ri} - I_p \quad C\Psi_i K_{fi} - C\Upsilon_i]), \end{aligned}$$

Since $C\Psi_i$ are invertible, $\tilde{\theta}_i \to 0$ implies $K_{xi} \to (C\Psi_i)^{-1} C\Phi_i = K_{xi}^*$, $K_{ri} \to (C\Psi_i)^{-1} = K_{ri}^*$, and $K_{fi} \to (C\Psi_i)^{-1} C\Upsilon_i = K_{fi}^*$. Hence, $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ as $t \to \infty$. □

4.5 Indirect Adaptive Control Design

If the estimation of the system parameters is also a part of the control objective, the indirect adaptation can be applied.

4.5.1 Controller Design

The indirect adaptive control use the same control structure as (4.27). The concept of indirect adaptation suggests the following update law

$$K_{xi} = -(C\hat{\Psi}_i)^{-1} C\hat{\Phi}_i, \quad K_{ri} = -(C\hat{\Psi}_i)^{-1}, \quad K_{fi} = -(C\hat{\Psi}_i)^{-1} C\hat{\Upsilon}_i \tag{4.58}$$

where $\hat{\Phi}_i, \hat{\Psi}_i, \hat{\Upsilon}_i$ denote the estimated i-th subsystem parameters. The main difficulty by using this method is the singularity of $(C\hat{\Psi}_i)^{-1}$, which is also known as loss of controllability issue. Since $\hat{\Psi}_i$ is updated by some adaptation law, it cannot be ruled out that the smallest singular value of $C\hat{\Psi}_i$ may go across zero or become some small value around zero, which leads to unbounded controller gains.

To solve this singularity problem, we use the dynamic gain adjustment technique revisited in Section 2.2.1. This concept is originally introduced by [48] and extended to MRAC of PWA systems in [83]. We extend this method to the context of adaptive control of PWA systems with prescribed performance. Specifically, the dynamic gain adjustment in MRAC starts with defining the closed-loop estimation errors, which capture the matching errors between the reference system and the controlled closed-loop system with estimated parameters. Unlike the MRAC, there exists no reference system in our context and thus we propose the following novel closed-loop estimation errors

$$\varepsilon_{\Phi i} = C\hat{\Phi}_i + C\hat{\Psi}_i K_{xi}, \quad \varepsilon_{\Psi i} = C\hat{\Psi}_i K_{ri} + I_p, \quad \varepsilon_{fi} = C\hat{\Upsilon}_i + C\hat{\Psi}_i K_{fi}. \tag{4.59}$$

These closed-loop estimation errors are obtained by multiplying both sides of (4.58) with $C\hat{\Psi}_i$ and taking the difference between the left and right-hand sides. The controller gains are updated by using the closed-loop estimation errors

$$\begin{aligned} \dot{K}_{xi} &= \chi_i \Gamma_{xi} S_i^T R^T E x^T + \Gamma_{xi} S_i^T \varepsilon_{\Phi i} \\ \dot{K}_{ri} &= \chi_i \Gamma_{ri} S_i^T R^T E \xi^T + \Gamma_{ri} S_i^T \varepsilon_{\Psi i} \\ \dot{K}_{fi} &= \chi_i \Gamma_{fi} S_i^T R^T E + \Gamma_{fi} S_i^T \varepsilon_{fi} \end{aligned} \tag{4.60}$$

and the estimated system parameters are updated by

$$\begin{aligned} \dot{\Phi}_i &= -\Gamma_{\Phi i} C^T \varepsilon_{\Phi i} \\ \dot{\Psi}_i &= -\Gamma_{\Psi i} (C^T \varepsilon_{\Phi i} K_{xi}^T + C^T \varepsilon_{\Psi i} K_{ri}^T + C^T \varepsilon_{fi} K_{fi}^T) \\ \dot{\Upsilon}_i &= -\Gamma_{\Upsilon i} C^T \varepsilon_{fi} \end{aligned} \tag{4.61}$$

with $\Gamma_{\Phi i}, \Gamma_{\Psi i}, \Gamma_{\Upsilon i} \in \mathbb{R}^+$ being positive scaling factors. The update laws (4.60) and (4.61) are derived based on the stability analysis. We can see from (4.60) and (4.61) that the inverse calculation shown in (4.58) is avoided through the utilization of closed-loop estimation errors.

4.5.2 Stability Analysis

The stability of the closed-loop system by using the indirect adaptive laws is summarized in the following theorem.

Theorem 4.4. *Given the reference signal y_d and predefined performance function ρ, let the PWA system (4.3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (4.27) with the update laws (4.59), (4.60) and (4.61). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (4.8).*

Proof. For clarity and without loss of generality, we let the scaling factors in (4.59) and (4.60) be 1 and propose the Lyapunov function

$$\begin{aligned} V =& \frac{E^T E}{2\lambda} + \frac{1}{2}\sum_{i=1}^{s} (\mathrm{tr}(\tilde{\Phi}_i^T \tilde{\Phi}_i) + \mathrm{tr}(\tilde{\Psi}_i^T \tilde{\Psi}_i) + \mathrm{tr}(\tilde{\Upsilon}_i^T \tilde{\Upsilon}_i)) \\ & + \mathrm{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \mathrm{tr}(\tilde{K}_{fi}^T M_i \tilde{K}_{fi})). \end{aligned} \tag{4.62}$$

where

$$\tilde{\Phi}_i = \hat{\Phi}_i - \Phi_i, \quad \tilde{\Psi}_i = \hat{\Psi}_i - \Psi_i, \quad \tilde{\Upsilon}_i = \hat{\Upsilon}_i - \Upsilon. \tag{4.63}$$

Taking the derivative yields

$$\begin{aligned} \dot{V} = & \underbrace{\frac{E^T \dot{E}}{\lambda}}_{\triangleq \dot{V}_1} + \sum_{i=1}^{s} (\underbrace{\mathrm{tr}(\tilde{\Phi}_i^T \dot{\tilde{\Phi}}_i) + \mathrm{tr}(\tilde{\Psi}_i^T \dot{\tilde{\Psi}}_i) + \mathrm{tr}(\tilde{\Upsilon}_i^T \dot{\tilde{\Upsilon}}_i)}_{\triangleq \dot{V}_{2i}} \\ & + \underbrace{\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \mathrm{tr}(\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi})}_{\triangleq \dot{V}_{3i}}). \end{aligned} \tag{4.64}$$

Replacing $\dot{E}$ with (4.31) we have

$$\dot{V}_1 = \frac{E^T\dot{E}}{\lambda} = -\frac{E^TE}{\lambda} + E^TRC\Psi(\tilde{K}_x x + \tilde{K}_r\xi + \tilde{K}_f). \tag{4.65}$$

After inserting (4.60) into $\dot{V}_{3i}$, we observe the first summand in $\dot{V}_{3i}$

$$\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) = \mathrm{tr}(\tilde{K}_{xi}^T M_i S_i^T \varepsilon_{\Phi i}) + \chi_i \mathrm{tr}(\tilde{K}_{xi}^T M_i S_i^T R^T E x^T). \tag{4.66}$$

We insert (4.35), transform the second summand of (4.66) according to (4.36) and obtain

$$\begin{aligned}\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) &= \mathrm{tr}(\tilde{K}_{xi}^T M_i S_i^T \varepsilon_{\Phi i}) + \chi_i \mathrm{tr}(\tilde{K}_{xi}^T M_i S_i^T R^T E x^T)\\ &= -\mathrm{tr}(\tilde{K}_{xi}^T \Psi_i^T C^T \varepsilon_{\Phi i}) - \chi_i E^T RC\Psi_i \tilde{K}_{xi} x.\end{aligned} \tag{4.67}$$

Similarly, the second and third summand in $\dot{V}_{3i}$ can be transformed as

$$\begin{aligned}\mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) &= -\mathrm{tr}(\tilde{K}_{ri}^T \Psi_i^T C^T \varepsilon_{\Psi i}) - \chi_i E^T RC\Psi_i \tilde{K}_{ri}\xi\\ \mathrm{tr}(\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}) &= -\mathrm{tr}(\tilde{K}_{fi}^T \Psi_i^T C^T \varepsilon_{fi}) - \chi_i E^T RC\Psi_i \tilde{K}_{fi},\end{aligned} \tag{4.68}$$

which gives

$$\begin{aligned}\sum_{i=1}^{s}\dot{V}_{3i} = &-\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T \Psi_i^T C^T \varepsilon_{\Phi i}) + \mathrm{tr}(\tilde{K}_{ri}^T \Psi_i^T C^T \varepsilon_{\Psi i}) + \mathrm{tr}(\tilde{K}_{fi}^T \Psi_i^T C^T \varepsilon_{fi}))\\ &-\sum_{i=1}^{s}(\chi_i E^T RC\Psi_i \tilde{K}_{xi} x + \chi_i E^T RC\Psi_i \tilde{K}_{ri}\xi + \chi_i E^T RC\Psi_i \tilde{K}_{fi}).\end{aligned} \tag{4.69}$$

Noticing that the second summand in (4.69) can be canceled out with the second term of $\dot{V}_1$ in (4.65), we have

$$\begin{aligned}\dot{V}_1 + \sum_{i=1}^{s}\dot{V}_{3i} = &-\frac{E^TE}{\lambda} - \sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T \Psi_i^T C^T \varepsilon_{\Phi i})\\ &+ \mathrm{tr}(\tilde{K}_{ri}^T \Psi_i^T C^T \varepsilon_{\Psi i}) + \mathrm{tr}(\tilde{K}_{fi}^T \Psi_i^T C^T \varepsilon_{fi})),\end{aligned} \tag{4.70}$$

which further leads to

$$\begin{aligned}\dot{V} = -\frac{E^TE}{\lambda} &+ \sum_{i=1}^{s}\mathrm{tr}(\tilde{\Phi}_i^T \dot{\tilde{\Phi}}_i - \tilde{K}_{xi}^T \Psi_i^T C^T \varepsilon_{\Phi i})\\ &+ \sum_{i=1}^{s}\mathrm{tr}(\tilde{\Psi}_i^T \dot{\tilde{\Psi}}_i - \tilde{K}_{ri}^T \Psi_i^T C^T \varepsilon_{\Psi i})\\ &+ \sum_{i=1}^{s}\mathrm{tr}(\tilde{\Upsilon}_i^T \dot{\tilde{\Upsilon}}_i - \tilde{K}_{fi}^T \Psi_i^T C^T \varepsilon_{fi}).\end{aligned} \tag{4.71}$$

Now we insert (4.61) into the trace operators and simplify the expressions in the trace operators. We have

$$\begin{aligned}&\tilde{\Phi}_i^T \dot{\tilde{\Phi}}_i - \tilde{K}_{xi}^T \Psi_i^T C^T \varepsilon_{\Phi i}\\ =&-(\tilde{\Phi}_i^T C^T + \tilde{K}_{xi}^T \Psi_i^T C^T)\varepsilon_{\Phi i}\\ =&-(\hat{\Phi}_i^T C^T - \Phi_i^T C^T + K_{xi}^T \Psi_i^T C^T - K_{xi}^{*T} \Psi_i^T C^T)\varepsilon_{\Phi i}\\ =&-(\hat{\Phi}_i^T C^T + K_{xi}^T \hat{\Psi}_i^T C^T - K_{xi}^T \tilde{\Psi}_i^T C^T)\varepsilon_{\Phi i}\\ =&-(\varepsilon_{\Phi i}^T - K_{xi}^T \tilde{\Psi}_i^T C^T)\varepsilon_{\Phi i},\end{aligned} \tag{4.72}$$

and

$$\tilde{\Psi}_i^T\dot{\tilde{\Psi}}_i - \tilde{K}_{ri}^T\Psi_i^T C^T\varepsilon_{\Psi i} = -(\tilde{\Psi}_i^T C^T(\varepsilon_{\Phi i}K_{xi}^T + \varepsilon_{fi}K_{fi}^T) + \tilde{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T + \tilde{K}_{ri}^T\Psi_i^T C^T\varepsilon_{\Psi i}). \quad (4.73)$$

Note that the second and third terms in (4.73) can be further simplified as

$$\begin{aligned}
&\tilde{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T + \tilde{K}_{ri}^T\Psi_i^T C^T\varepsilon_{\Psi i}\\
=&\hat{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T - \Psi_i^T C^T\varepsilon_{\Psi i}K_{ri}^T + (K_{ri}^T - K_{ri}^{*T})\Psi_i^T C^T\varepsilon_{\Psi i}\\
=&\hat{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T - K_{ri}^{*T}\Psi_i^T C^T\varepsilon_{\Psi i}\\
=&\hat{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T + \varepsilon_{\Psi i},
\end{aligned} \quad (4.74)$$

which in turn leads to

$$\begin{aligned}
&\operatorname{tr}(\tilde{\Psi}_i^T\dot{\tilde{\Psi}}_i - \tilde{K}_{ri}^T\Psi_i^T C^T\varepsilon_{\Psi i})\\
=&-\operatorname{tr}(\tilde{\Psi}_i^T C^T(\varepsilon_{\Phi i}K_{xi}^T + \varepsilon_{fi}K_{fi}^T) + \hat{\Psi}_i^T C^T\varepsilon_{\Psi i}K_{ri}^T + \varepsilon_{\Psi i})\\
=&-\operatorname{tr}(\tilde{\Psi}_i^T C^T(\varepsilon_{\Phi i}K_{xi}^T + \varepsilon_{fi}K_{fi}^T) + (K_{ri}^T\hat{\Psi}_i^T C^T + I_p)\varepsilon_{\Psi i})\\
=&-\operatorname{tr}(\tilde{\Psi}_i^T C^T(\varepsilon_{\Phi i}K_{xi}^T + \varepsilon_{fi}K_{fi}^T) + \varepsilon_{\Psi i}^T\varepsilon_{\Psi i}).
\end{aligned} \quad (4.75)$$

Similarly, we have

$$\begin{aligned}
&\tilde{\Upsilon}_i^T\dot{\tilde{\Upsilon}}_i - \tilde{K}_{fi}^T\Psi_i^T C^T\varepsilon_{fi}\\
=&-(\tilde{\Upsilon}_i^T C^T + \tilde{K}_{fi}^T\Psi_i^T C^T)\varepsilon_{fi}\\
=&-(\hat{\Upsilon}_i^T C^T + \Upsilon_i^T C^T + K_{fi}^T\Psi_i^T C^T - K_{fi}^{*T}\Psi_i^T C^T)\varepsilon_{fi}\\
=&-(\hat{\Upsilon}_i^T C^T + K_{fi}^T\hat{\Psi}_i^T C^T - K_{fi}^T\tilde{\Psi}_i^T C^T)\varepsilon_{fi}\\
=&-(\varepsilon_{fi}^T - K_{fi}^T\tilde{\Psi}_i^T C^T)\varepsilon_{fi}.
\end{aligned} \quad (4.76)$$

Therefore, we obtain

$$\begin{aligned}
\dot{V} = -\frac{E^TE}{\lambda} &- \sum_{i=1}^{s}\operatorname{tr}(\varepsilon_{\Phi i}^T - K_{xi}^T\tilde{\Psi}_i^T C^T)\varepsilon_{\Phi i})\\
&- \sum_{i=1}^{s}\operatorname{tr}(\tilde{\Psi}_i^T C^T(\varepsilon_{\Phi i}K_{xi}^T + \varepsilon_{fi}K_{fi}^T) + \varepsilon_{\Psi i}^T\varepsilon_{\Psi i})\\
&- \sum_{i=1}^{s}\operatorname{tr}((\varepsilon_{fi}^T - K_{fi}^T\tilde{\Psi}_i^T C^T)\varepsilon_{fi}),
\end{aligned} \quad (4.77)$$

which after further simplification leads to

$$\dot{V} = -\frac{E^TE}{\lambda} - \sum_{i=1}^{s}\operatorname{tr}(\varepsilon_{\Phi i}^T\varepsilon_{\Phi i} + \varepsilon_{\Psi i}^T\varepsilon_{\Psi i} + \varepsilon_{fi}^T\varepsilon_{fi}) \le 0. \quad (4.78)$$

From the negative semidefiniteness of $\dot{V}$ it follows that $E, \hat{\Phi}_i, \hat{\Psi}_i, \hat{\Upsilon}_i, K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$, which together with (4.59) implies $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_\infty$. Thus, we have $\dot{\hat{\Phi}}_i, \dot{\hat{\Psi}}_i, \dot{\hat{\Upsilon}}_i \in \mathcal{L}_\infty$. Moreover, (4.78) also indicates $E, \varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_2$. Following the same analysis as in the direct adaptation case, one can conclude that $\sigma, \sigma^{(k)}, e, e^{(k)} \in \mathcal{L}_\infty$, which further results in $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \cdots, r-1$, and hence, $x, \xi, K \in \mathcal{L}_\infty$. This in turn, implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in$

$\mathcal{L}_\infty$. The boundedness of $u, \dot{E}$ can be concluded from the boundedness of $K_{xi}, K_{ri}, K_{fi}, x, \xi$. Furthermore, $\dot{E} \in \mathcal{L}_\infty$ as well as $E \in \mathcal{L}_\infty \cap \mathcal{L}_2$ results in $\lim_{t\to\infty} E \to 0$ and thus $\sigma_j, \sigma_j^{(k)} \to 0$ as $t \to \infty$, $\forall j = 1, \cdots, p, k = 1, \cdots, r-1$. Therefore, we conclude that the tracking error e stays within the performance bound, i.e., inequalities in (4.8) hold.

Observe that the same expression as (4.78) can be obtained by replacing χ_i with $\bar{\chi}_i$ in the transformed error dynamics (4.31) and in adaptation laws (4.60), we thus can conclude the closed-loop stability when the closed-loop system enters sliding mode. □

Remark 4.9. Two other methods used to avoid singularity (or loss of controllability) problem can be found in [13] and [12], respectively. While calculating the inverse of a matrix F_G using the formula $F_G^{-1} = \text{adj}(F_G)/\det(F_G)$, the method in [13] adds a positive desgin number $\delta_D \in \mathbb{R}^+$ to the denominator to prevent the division by zero (see (12) in [13]). The method in [12] replaces the denominator with a positive constant if its norm is smaller than a threshold (see (12) in [12]). With these two methods, the transformed tracking error and the parameter estimation error converge only to a bounded set. Differing from these results, one key feature of our approach is that the convergence of the tracking error $e_j \to 0$ is achieved by specifying $\delta_j = 1$. Furthermore, the parameter estimation errors, as will be shown later, also converge to 0 under PE conditions. Nevertheless, more prior knowledge (S_i matrix and the the structure of canonical form) is required compared to [12].

Remark 4.10. In the classical indirect MRAC of PWA systems revisited in Section 2.2.1, the utilization of the dynamic gain adjustment technique has the disadvantage that the convergence of the tracking error is not exponential. This still persists when the PE condition of the reference signals is introduced. This issue, however, could be bypassed in our approach by choosing an exponentially decreasing performance function (such as the performance function (4.7)).

4.5.3 Parameter Convergence

Theorem 4.4 shows the boundedness of the parameter estimation error $\tilde{\Phi}_i, \tilde{\Psi}_i, \tilde{\Upsilon}_i$. If one of the control objectives is the estimation of the real system parameters, the PE property of the reference signal y_d should be added to ensure the convergence of the estimated parameters to their real values. This is summarized as follows.

Theorem 4.5. *Let the PWA system (4.3) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (4.27) with the update laws (4.59), (4.60) and (4.61). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. Let the reference signals in y_d be sufficiently rich of order $r+1$ and cause repeated activation of all subsystems. If the matrices $C\Psi_i$ are invertible, and $\delta_j = 1$ for $j = 1, \cdots, p$, then $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \to 0$ and $\tilde{A}_i, \tilde{B}_i, \tilde{f}_i \to 0$ as $t \to \infty$.*

Proof. Let $\tilde{\theta}_i = \text{vec}(C\Psi_i[\tilde{K}_{xi} \;\; \tilde{K}_{ri} \;\; \tilde{K}_{fi}])$. From (4.60) we have (for unit scaling factors)

$$\begin{aligned}
\dot{\tilde{\theta}}_i &= \text{vec}(C\Psi_i[\dot{\tilde{K}}_{xi} \quad \dot{\tilde{K}}_{ri} \quad \dot{\tilde{K}}_{fi}]) \\
&= \text{vec}(C\Psi_i S^T(R^T E[x^T \quad \xi^T \quad 1] + [\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}])) \\
&= \Xi \cdot \text{vec}(C\Psi_i S^T R^T E) + \text{vec}(C\Psi_i S^T [\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}]) \\
&= -\Xi W_i R^T E + \text{vec}(C\Psi_i S^T[\varepsilon_{\Phi i} \quad \varepsilon_{\Psi i} \quad \varepsilon_{fi}]).
\end{aligned} \tag{4.79}$$

Combining it with (4.52), we have the dynamical system with the state $[E, \tilde{\theta}_i]^T$

$$\begin{bmatrix} \dot{E} \\ \dot{\tilde{\theta}}_i \end{bmatrix} = \begin{bmatrix} -I_p & \lambda R \Xi^T \\ -\Xi W_i R^T & 0 \end{bmatrix} \begin{bmatrix} E \\ \tilde{\theta}_i \end{bmatrix} + \begin{bmatrix} 0 \\ \epsilon_i \end{bmatrix}. \tag{4.80}$$

with $\epsilon_i = \mathrm{vec}(C\Psi_i S^T[\varepsilon_{\Phi i}\ \ \varepsilon_{\Psi i}\ \ \varepsilon_{fi}])$. Considering (4.59) and the property $\dot{\Phi}_i, \dot{\Psi}_i, \dot{\Upsilon}_i \in \mathcal{L}_\infty$ and $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$, we have $\dot{\varepsilon}_{\Phi i}, \dot{\varepsilon}_{\Psi i}, \dot{\varepsilon}_{fi} \in \mathcal{L}_\infty$, which together with $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\varepsilon_{\Phi i}, \varepsilon_{\Psi i}, \varepsilon_{fi} \to 0$ as $t \to \infty$. Therefore, the convergence property can be shown through the homogeneous part of (4.80).

It has already been shown in Theorem 4.3 that $E, \tilde{\theta}_i \to 0$ asymptotically if $\epsilon_i = 0$ and all subsystems are activated repeatedly, from which one can conclude that $\tilde{K}_{xi} \to 0, \tilde{K}_{ri} \to 0, \tilde{K}_{fi} \to 0$ as $t \to \infty$, namely, the adaptive controller gains converge to the nominal gains $K_{xi} \to K^*_{xi}, K_{ri} \to K^*_{ri}, K_{fi} \to K^*_{fi}, \forall i \in \mathcal{I}$ as $t \to \infty$. Considering $\varepsilon_{\Psi i} \to 0$ and the expression of $\varepsilon_{\Psi i}$ in (4.59), it follows $C\hat{\Psi}_i \to -(K^*_{ri})^{-1} = C\Psi_i$. Taking this into the expression of $\varepsilon_{\Phi i}$ and ε_{fi} in (4.59), we have $C\hat{\Phi}_i \to C\Phi_i$ and $C\hat{\Upsilon}_i \to C\Upsilon_i$ as $t \to \infty$.

Note that

$$\begin{aligned} C\hat{\Phi}_i &= [\hat{a}_{1i}, \hat{a}_{2i}, \cdots, \hat{a}_{pi}]^T \\ C\hat{\Psi}_i &= [\hat{b}_{1i}, \hat{b}_{2i}, \cdots, \hat{b}_{pi}]^T \\ C\hat{\Upsilon}_i &= [\hat{f}_{1i}, \hat{f}_{2i}, \cdots, \hat{f}_{pi}]^T \end{aligned} \tag{4.81}$$

where $\hat{a}_{ji}, \hat{b}_{ji}, \hat{f}_{ji}$ represent the estimated values of a_{ji}, b_{ji}, f_{ji} in (4.1) for $j = 1, \cdots, p, i \in \mathcal{I}$. The convergence of $C\hat{\Phi}_i, C\hat{\Psi}_i, C\hat{\Upsilon}_i$ implies $\hat{a}_{ji} \to a_{ji}$, $\hat{b}_{ji} \to b_{ji}$ and $\hat{f}_{ji} \to f_{ji}$. Considering that the system is in control canonical form, it follows from the convergence of $\hat{a}_{ji}, \hat{b}_{ji}$ and $\hat{f}_{ji}$ that $\hat{A}_i \to A_i$, $\hat{B}_i \to B_i$ and $\hat{f}_i \to f_i$ as $t \to \infty$. □

The advantage of our indirect adaptive controller over the direct adaptive controller is that the indirect adaptive controller exhibits the capability to identify the subsystem parameters. This is, however, achieved at the expense of imposing more complexity into the closed-loop system. Specifically, the update law of the controller gains of the indirect adaptive controller (4.60) is obtained by fusing the closed-loop estimation errors to the update law of the direct adaptive controller (4.28). Meanwhile, the subsystem parameters are updated through the information of the closed-loop estimation errors and the estimated controller gains. Therefore, more computational costs must be tolerable when applying the indirect adaptive controller.

Remark 4.11. (Parameter tuning guidelines) Larger adaptation gains $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ and $\Gamma_{\Phi i}$, $\Gamma_{\Psi i}$, $\Gamma_{\Upsilon i}$ speed up the parameter adaptation while too large adaptation gains may lead to numerical instability and high control effort. λ is the coefficient of $\sigma_j^{(r-1)}$ and serves as the input gain of the dynamics of E (see (4.31)). A larger λ amplifies the sensibility introduced by the higher order derivative and results in aggressive response of E, whereas a too small λ leads to "stiff" descent of the Lyapunov function (see (4.39) and (4.78)), which is numerically difficult to solve.

The concept to convert a constrained error into an unconstrained one to satisfy a prescribed performance requirement has been studied for hybrid systems and switching systems in [91, 95, 107, 177]. These approaches are based on backstepping design and require either input gains to be completely known [91, 177] or the control direction as well as lower bounds of input gains to be known [95, 107]. Compared to these approaches, only the control direction is assumed to be known in this chapter. Another feature that differentiates our approaches from these approaches is that the convergence of gain and parameter estimation errors is achieved under PE conditions.

In prescribed performance control, there also exist approximation-free control methods [11, 163, 188], where no adaptation mechanism is introduced. Such approaches have low controller complexity and computational costs. Compared to these approximation-free methods, our approaches are based on adaptations and can achieve unknown parameter estimation in addition to the tracking task. This is especially useful for monitoring systems with parameter drifts and component aging as well as for joint control and identification tasks.

4.6 Robust Modification

Since the PWA system is utilized to approximate a nonlinear system, there may exist approximation errors and external disturbances. To achieve the provable robust stability of the proposed methods for PWA systems with disturbances and approximation errors, the adaptation laws need to be modified. Here we demonstrate a projection-based robust modification for the direct adaptive control. So the original PWA system (4.1) becomes now

$$\begin{aligned} x_1^{(r)} =& a_{1i}^T x + b_{1i}^T u + f_{1i} + d_{1i}(x,t) \\ &\vdots \\ x_p^{(r)} =& a_{pi}^T x + b_{pi}^T u + f_{pi} + d_{pi}(x,t), \quad i \in \mathcal{I} \\ y =& [x_1, x_2, \cdots, x_p]^T \end{aligned} \tag{4.82}$$

where $d_{1i}, \cdots, d_{pi}$ represent bounded continuous error terms (approximation errors or disturbances). The compact form (4.3) becomes now

$$\begin{aligned} \dot{x} &= A_i x + B_i u + f_i + d_i(x,t), \quad i \in \mathcal{I} \\ y &= Cx, \end{aligned} \tag{4.83}$$

with $d_i(x,t) \in \mathbb{R}^n$ being the vector of the error terms. It contains p nonzero elements. Due to the structure of control canonical form, there exists $w_i(x,t) \in \mathbb{R}^p$ such that $d_i(x,t) = B_i w_i(x,t)$. We assume that $w_i, i \in \mathcal{I}$ are bounded and satisfy $|w_i| \leq \bar{w}, i \in \mathcal{I}$. Therefore, the transformed error (4.19) becomes

$$\begin{aligned} \dot{E} &= K + \lambda RCA^r x + \lambda RCA^{r-1}Bu + \lambda RCA^{r-1}f + \lambda RCA^{r-1}d. \\ &= K + \lambda RCA^r x + \lambda RCA^{r-1}Bu + \lambda RCA^{r-1}f + \lambda RCA^{r-1}Bw \\ &= K + \lambda RC\Phi x + \lambda RC\Psi u + \lambda RC\Upsilon + \lambda RC\Psi w \end{aligned} \tag{4.84}$$

for $d = \sum_{i=1}^s \chi_i d_i, w = \sum_{i=1}^s \chi_i w_i$. We add an additional term $v = \sum_{i=1}^s \chi_i v_i = \sum_{i=1}^s \chi_i S_i^T R^T E$ to the controller (4.27) and obtain the updated controller

$$u = K_x x + K_r \xi + K_f + v. \tag{4.85}$$

The following robust adaptation laws are proposed

$$\begin{aligned} \dot{K}_{xi} &= \chi_i \mathrm{Pr}[\Gamma_{xi} S_i^T R^T E x^T] \\ \dot{K}_{ri} &= \chi_i \mathrm{Pr}[\Gamma_{ri} S_i^T R^T E \xi^T] \\ \dot{K}_{fi} &= \chi_i \mathrm{Pr}[\Gamma_{fi} S_i^T R^T E], \end{aligned} \tag{4.86}$$

where Pr[·] denotes the projection operator depicted in Section 2.2.3, it terminates the adaptation when a predefined bound of the corresponding estimated gain is reached.

Theorem 4.6. *Given the reference signal y_d and predefined performance function ρ, let the PWA system (4.83) with known partition regions Ω_i and unknown subsystem parameters be controlled by the feedback controller (4.85) with the update law (4.86). Let ρ be designed such that the inequality (4.8) holds at initial time instant $t = 0$. The closed-loop system is stable and the output tracking error satisfies the prescribed performance (4.8).*

Proof. Based on the controller (4.85) and following the same derivations as (4.30), the error dynamics of the transformed error E becomes

$$\dot{E} = -E + \lambda R \sum_{i=1}^{s} \chi_i (C\Psi_i(\tilde{K}_{xi}x + \tilde{K}_{ri}\xi + \tilde{K}_{fi}) + C\Psi_i w_i + C\Psi_i v_i). \tag{4.87}$$

For stability analysis, we use the same Lyapunov function as (4.32). Taking its derivative and inserting (4.87) and (4.86) yields

$$\dot{V} \leq -\frac{1}{\lambda} E^T E + E^T R \sum_{i=1}^{s} \chi_i C\Psi_i w_i + E^T R \sum_{i=1}^{s} \chi_i C\Psi_i v_i. \tag{4.88}$$

Since $K^*_{ri} = -(C\Psi_i)^{-1}$ and $M_i = (K^*_{ri}S_i)^{-1}$, we have

$$C\Psi_i = -S_i M_i. \tag{4.89}$$

Inserting (4.89) and $v_i = S_i^T R^T E$ into (4.88) yields

$$\dot{V} \leq -\frac{1}{\lambda} E^T E - \sum_{i=1}^{s} \chi_i E^T R S_i M_i w_i - \sum_{i=1}^{s} \chi_i E^T R S_i M_i S_i^T R^T E \tag{4.90}$$

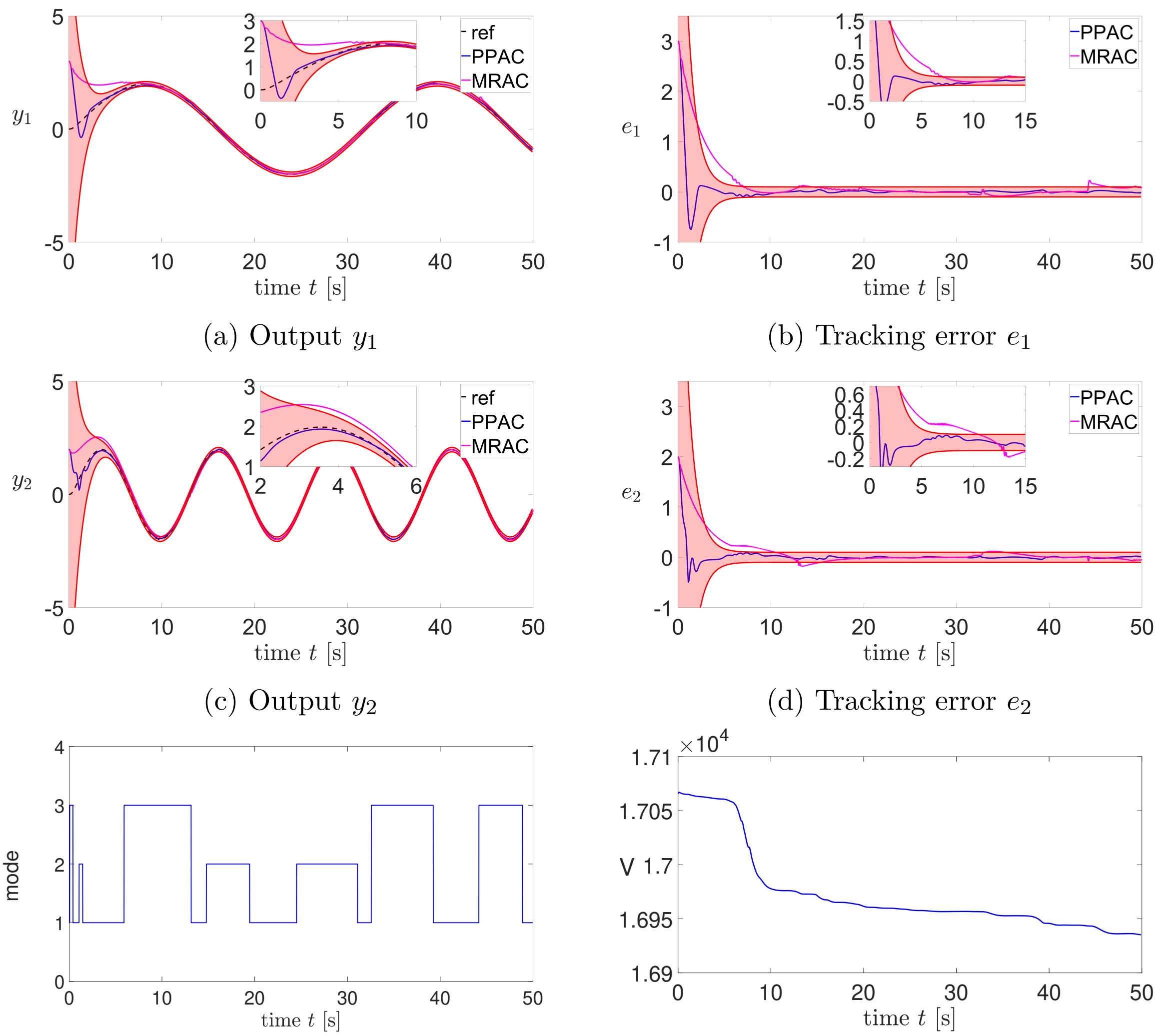
Because of the inequality $-X^T MX - X^T MY \leq \frac{1}{4} Y^T MY$ for positive definite M, we have

$$\begin{aligned} \dot{V} &\leq -\frac{1}{\lambda} E^T E + \frac{1}{4} \sum_{i=1}^{s} \chi_i w_i^T M_i w_i \\ &\leq -\frac{1}{\lambda} E^T E + \frac{1}{4} \mathrm{max}_i \lambda_{\max}(M_i) \bar{w}^2, \end{aligned} \tag{4.91}$$

where $\lambda_{\max}(M_i)$ denotes the maximum eigenvalue of M_i. Let $V_K = \sum_{i=1}^{s}(\Gamma_{xi}^{-1}\mathrm{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \Gamma_{ri}^{-1}\mathrm{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \Gamma_{fi}^{-1}\mathrm{tr}(\tilde{K}_{fi}^T M_i \tilde{K}_{fi}))$, it follows from (4.32) that $2V = \frac{1}{\lambda} E^T E + V_K$. Thus, (4.91) can be further transformed as

$$\begin{aligned} \dot{V} &\leq -\frac{1}{\lambda} E^T E - V_K + V_K + \frac{1}{4} \mathrm{max}_i \lambda_{\max}(M_i) \bar{w}^2 \\ &= -2V + V_K + \frac{1}{4} \mathrm{max}_i \lambda_{\max}(M_i) \bar{w}^2 \\ &= -2V + \mathcal{B} \end{aligned} \tag{4.92}$$

with $\mathcal{B} = V_K + \frac{1}{4}\mathrm{max}_i \lambda_{\max}(M_i)\bar{w}^2$. Due to the use of projection in (4.86), V_K is bounded and $\mathcal{B}$ also bounded. Therefore, we can conclude the boundedness of V. Based on the similar reasoning as in Theorem 4.2, we have $E, R \in \mathcal{L}_\infty$, $\sigma, \sigma^{(k)}, e, e^{(k)} \in \mathcal{L}_\infty$, $y, y^{(k)} \in \mathcal{L}_\infty, k = 1, \cdots, r-1$, $x \in \mathcal{L}_\infty$ and the prescribed performance constraint (4.8) holds. □



(a) Output y_1 (b) Tracking error e_1

(c) Output y_2 (d) Tracking error e_2

(e) Switching signal (f) Lyapunov function V

Figure 4.2: Output tracking performance of direct adaptation case.

4.7 Numerical Validation

In this section, the proposed adaptive approaches of PWA systems with prescribed performance are validated through two numerical examples. The mass-spring-damper system studied in Chapter 3 and the aeroelastic model of aircraft wings.

4.7.1 Mass-Spring-Damper System

In the following simulation, we validate the proposed methods with the mass-spring-damper system shown in Section 3.4. The region partitions are assumed to be known and the subsystem parameters are unknown. Both direct and indirect adaptation cases depicted in Section 4.4 and Section 4.5 are analyzed as follows.

Direct Adaptation

We start by testing the tracking performance of the direct prescribed performance adaptive control approach, abbreviated as PPAC. To compare this performance with the one of MRAC [83], we let the desired trajectory y_d be the output of the reference system $y_d = W_m(s)r$, where $W_m(s) = \text{diag}\{\frac{1}{(0.2s+1)^2}, \frac{1}{(0.2s+1)^2}\}$ denotes the transfer matrix of the reference system (see Sec.V in [83]), the input signal r is chosen as $r = [2\sin(0.2t), 2\sin(0.5t)]^T$. We define the performance bounds by specifying $\rho_0 = [10, 10]^T$ and $\rho_\infty = [0.1, 0.1]^T$ with the decaying rates $l = [l_1, l_2]^T = [1, 1]^T$. The error bounds in (4.8) are chosen to be symmetric by letting $\delta_1 = \delta_2 = 1$. λ is selected to be 0.04. Besides, we use unit scaling factors for controller gains adaptation, $\Gamma_{xi} = \Gamma_{ri} = \Gamma_{fi} = 1, \forall i = 1, 2, 3$ and we specify $S_i = -I_2, \forall i = 1, 2, 3$.

The output tracking performance of PPAC and MRAC are shown in Figure 4.2. In Figure 4.2a and Figure 4.2c, the red regions represent the prescribed performance bounds of the output. Blue solid lines indicate the real system output of PPAC and the black dashed lines depict the desired output. In Figure 4.2b and Figure 4.2d, the tracking errors as well as the performance bound of errors are displayed in blue lines and red regions, respectively. Besides, the mode information is given in Figure 4.2e and the CLF in Figure 4.2f. The Lyapunov function is continuous at each switching instant and strictly decreasing. It can be seen from the figures that both components of the output tracking error of the controlled system stay within the prescribed performance bounds. For comparison purpose, the tracking performance of the MRAC approach is displayed with magenta lines. We observe that the transients of MRAC converge slower than the one of PPAC and violate the prescribed performance constraints.

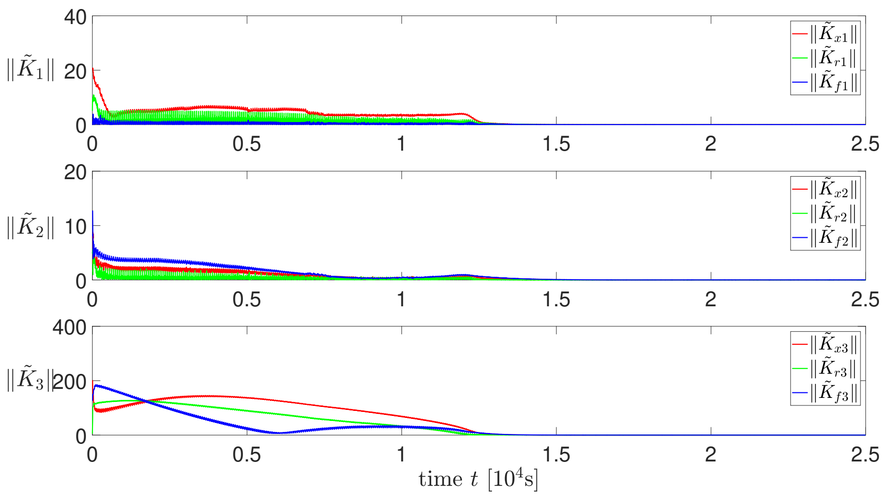
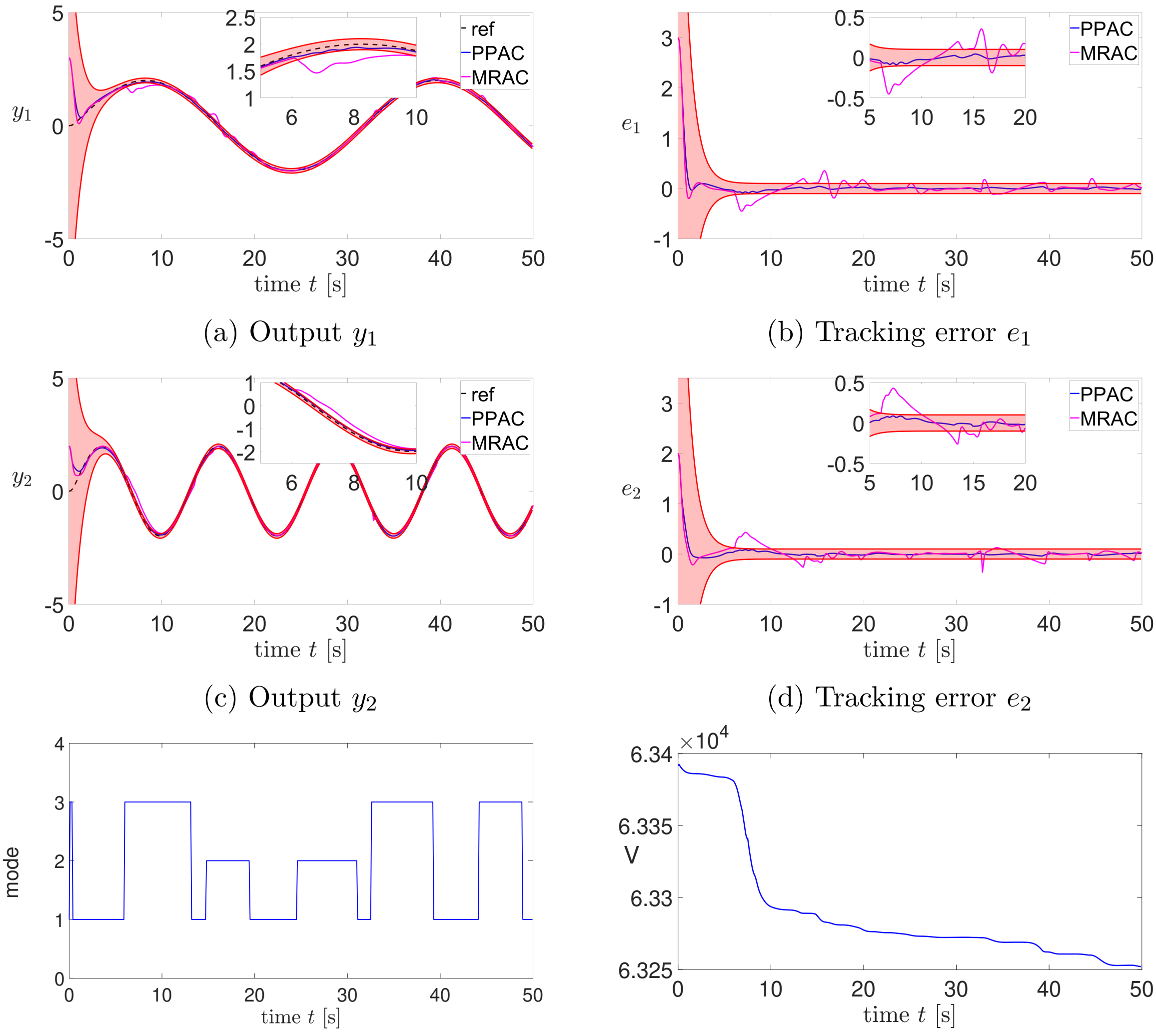


Figure 4.3: Convergence of estimation errors of controller gains of direct adaptation case.

To validate the convergence of the controller gains under PE conditions, the desired output signal is chosen as $y_d = [2\sin(0.2t) - 0.2\sin(3t), 2\sin(0.5t) - 0.2\sin(7t)]^T$. The relative degree of the system is $r = 2$. According to Theorem 4.2, y_d should be sufficiently rich of order 3 to guarantee the convergence of the controller gains to their nominal values. Since each component of y_d contains 2 distinct frequencies, the sufficiently rich condition is satisfied. Besides, the chosen desired output signal ensures that all the subsystems are activated repeatedly. The scaling factors are chosen as $\Gamma_{xi} = \Gamma_{ri} = \Gamma_{fi} = 5, \forall i = 1, 2, 3$ and λ

is specified as 0.01. The performance bounds are specified by $\rho_0 = [10, 10]^T$ and $\rho_\infty = [0.15, 0.15]^T$ with the decaying rates $l = [l_1, l_2]^T = [0.5, 0.5]^T$.



(a) Output y_1 (b) Tracking error e_1

(c) Output y_2 (d) Tracking error e_2

(e) Switching signal (f) Lyapunov function V

Figure 4.4: Output tracking performance of indirect adaptation case.

Figure 4.3 shows the convergence of the errors between estimated controller gains and nominal controller gains (the norm operators $\|\cdot\|$ in the figures of this chapter represent the Frobenius norms $\|\cdot\|_F$ for clarity purpose). We use $\tilde{K}_i$ on the vertical axis to represent the set of estimation errors of the controller gains for i-th subsystem, i.e., $\tilde{K}_i = \{\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}\}$. As we can conclude from the figure, the estimated controller gains of all the subsystems converge to their nominal values. This validates the theoretical results of Theorem 4.3.

Indirect Adaptation

The tracking performance of the indirect adaptation case is tested with the same parameters as in the direct adaptation case. Figure 4.4a and Figure 4.4c display the desired output in black dashed lines, the real output of PPAC in blue solid lines as well as the performance bound of output in red lines. The tracking errors, as well as the performance bound of the

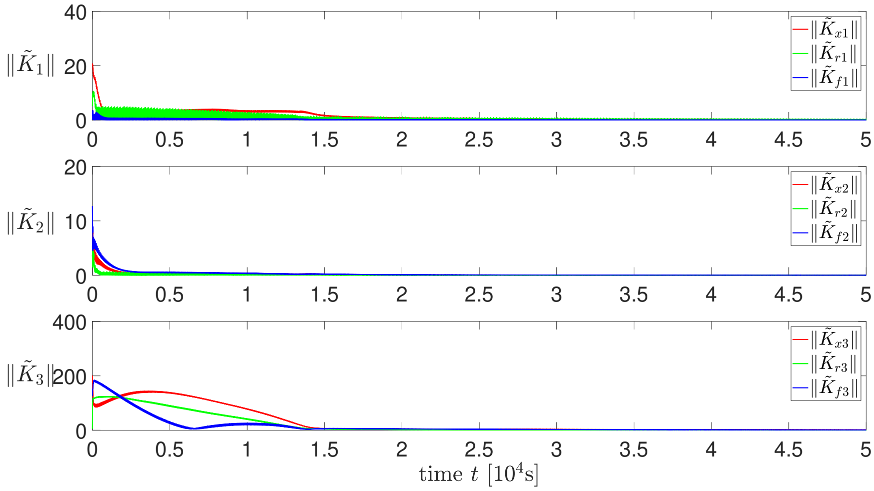


Figure 4.5: Convergence of estimation errors of controller gains of indirect adaptation case.

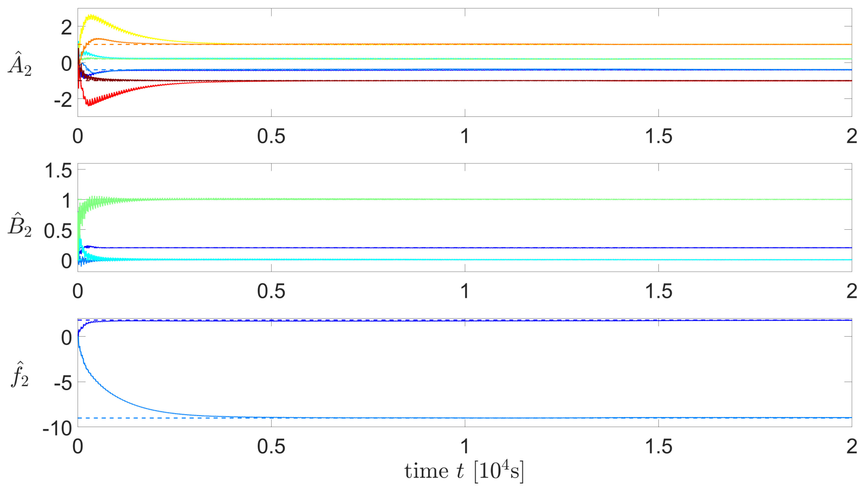


Figure 4.6: Convergence of estimated parameters of indirect adaptation case.

errors, are presented in Fig. 4.4b and Fig. 4.4d with blue and red colors, respectively. The switches are displayed in Fig. 4.4e and the CLF in Fig. 4.4f, which is continuous at each switching instant and strictly decreasing. As we can see, the output of the controlled system is enclosed by the performance bound and the prescribed transient performance is satisfied. In comparison to this, the tracking performance of the MRAC approach, displayed with magenta lines, violates the prescribed performance constraints.

The convergence of the controller gains and the estimated parameters is tested by applying the same PE input signal with the same setting of parameters as in the direct case. In addition, $\Gamma_{\Phi i}, \Gamma_{\Psi i}, \Gamma_{\Upsilon i} = 1, \forall i = 1, 2, 3$. As Figure 4.5 shows, the estimation error of the controller gains $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ converge to zero. The parameter estimation of subsystem 2 is displayed in Figure 4.6. Note that only the to be estimated components rather than all the components in the parameter matrices are displayed, see (4.81). The dashed lines represent the real values and the solid lines depict the estimated values. As can be seen from the figure, the estimated system parameters converge to the real values.

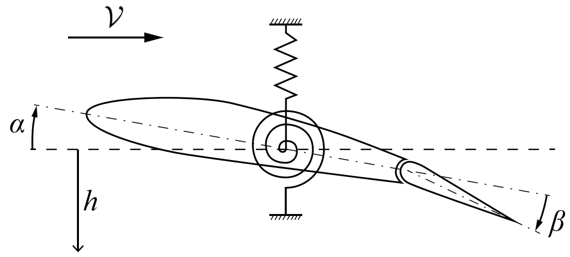


Figure 4.7: The aeroelastic model of aircraft wings [176].

4.7.2 Aeroelastic Model

In this section, the proposed approaches of Section 4.4 and Section 4.5 are tested with an engineering application example, the aeroelastic model of aircraft wings [1, 176]. The wing fluctuation is simplified as the dynamics of an airfoil with linear and torsional spring, which is illustrated in Fig. 4.7. The airfoil has two degrees of freedom, plunging and pitching. h denotes the plunging deflection and α represents the pitch angle about the elastic axis. $\beta = [\beta_1, \beta_2]^T$ serves as the input signal and denotes the left and right flap deflection angles, which are not distinguished from each other in Fig. 4.7 due to the side view. $\mathcal{V}$ denotes the constant airspeed. Let $y = [h, \alpha]^T$ be the system output. The motion of the aeroelastic model can be described by the equation

$$\mathcal{M}\ddot{y} + \mathcal{C}\dot{y} + \mathcal{K}y + \mathcal{W}_q = \mathcal{B}_\mu \beta, \tag{4.93}$$

where $\mathcal{M}$ denotes the mass and inertia matrix, $\mathcal{B}_\mu$ represents the control gain. The structural damping effect, stiffness, aerodynamic lift and moment effect are included in matrices $\mathcal{C}$ and $\mathcal{K}$. Their values are known and detailed derivations can be seen in [1]. $\mathcal{W}_q = [0, \bar{K}\alpha]^T$ constitutes the source of uncertainties with $\bar{K}$ being the nonlinear torsional stiffness

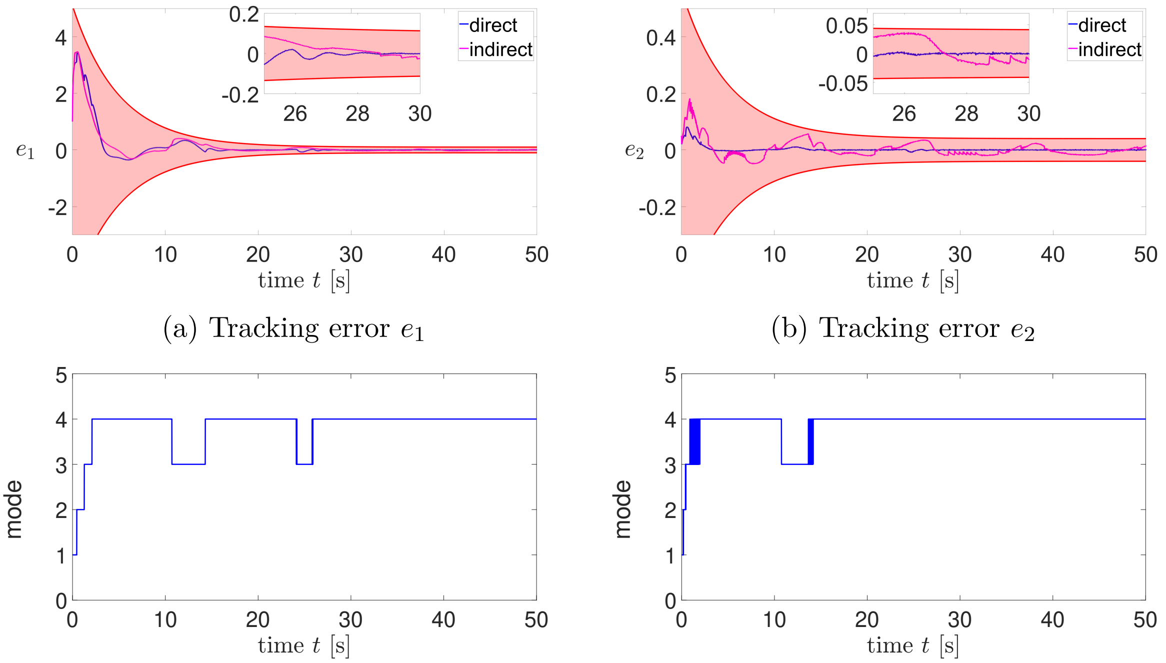
$$\bar{K} = 2.82 - 62.322\alpha + 3709.71\alpha^2 - 24195.6\alpha^3 + 48756.954\alpha^4.$$

The characteristics of the nonlinear term $\bar{K}\alpha$ in the interval $\alpha \in [-0.38, 0.38]$ can be divided into 4 regions and its piecewise linear approximation in form of $\bar{a}_i\alpha + \bar{b}_i, i = 1, \cdots, 4$ is given in Tab. 4.1. Let the state be $x = [h, \alpha, \dot{h}, \dot{\alpha}]^T$. The dynamics (4.93) can be approximated by the PWA system in form of (4.3) with

$$\begin{aligned} A_i &= \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -293.27 & -100.59 + 0.66\bar{a}_i & -5.9027 & -0.40542 \\ 1885.9 & 743.79 - 19.65\bar{a}_i & 34.728 & 2.4687 \end{bmatrix}, \\ B_i &= \begin{bmatrix} 0 & 0 \\ 0 & 0 \\ -7606.8 & -7642.6 \\ 14250 & 9021.9 \end{bmatrix}, f_i = \begin{bmatrix} 0 \\ 0 \\ 0.66\bar{b}_i \\ -19.65\bar{b}_i \end{bmatrix}, i = 1, \cdots, 4 \end{aligned} \tag{4.94}$$

mode	1	2	3	4
$\bar{a}_i$	10044	5992	2482.1	19.141
$\bar{b}_i$	2732.9	1377.8	422.78	2.8463
Region	[-0.38,-0.33]	[-0.33,-0.27]	[-0.27,-0.17]	[-0.17, 0.38]

Table 4.1: Piecewise linear approximation of $\bar{K}\alpha$



(a) Tracking error e_1 (b) Tracking error e_2

(c) Switching signal (direct adaptation) (d) Switching signal (indirect adaptation)

Figure 4.8: Output tracking performance of direct and indirect adaptation cases.

Now we test the tracking performance of both direct and indirect PPAC approaches on the nonlinear system (4.93), which is equivalent to the PWA system (4.94) with approximation errors as external disturbances. Gaussian noise with zero mean and 0.001 variance is added to the state measurements. We define the performance bounds by specifying $\rho_0 = [5, \pi/6]^T$ and $\rho_\infty = [0.1, 0.04]^T$ with the decaying rates $l = [0.2, 0.2]^T$. The error bounds are symmetric with $\delta_1 = \delta_2 = 1$. λ is selected to be 0.01. The adaptation gains are $\Gamma_{xi} = \Gamma_{fi} = 1, \Gamma_{ri} = 0.001, \forall i = 1, \cdots, 4$ and we specify the reference signal as $y_d = [0, -0.4\mathrm{e}^{-0.03t}\sin(0.5t + \frac{\pi}{2})]^T$. The initial state of the system reads $x(0) = [1, -0.35, 0, 0]^T$. The initial guess of the parameters for each subsystem is specified by letting $\bar{a}_i = \bar{b}_i = 0$ in (4.94). The following S_i matrices are applied

$$S_i = \begin{bmatrix} 0.7607 & 0.7643 \\ -1.4250 & -0.9022 \end{bmatrix}, \forall i = 1, \cdots, 4. \tag{4.95}$$

The output tracking performance of direct and indirect PPAC are shown in Fig. 4.8. In Fig. 4.8a and Fig. 4.8b, the blue lines and magenta lines depict the output tracking errors of direct and indirect approaches. The mode switches by using direct and indirect PPAC are shown in Fig. 4.8c and Fig. 4.8d, respectively. It can be seen from the figures that the output tracking errors of both direct and indirect approaches stay within the prescribed performance bounds. This also suggests some degree of robustness of our approaches against noise and disturbances.

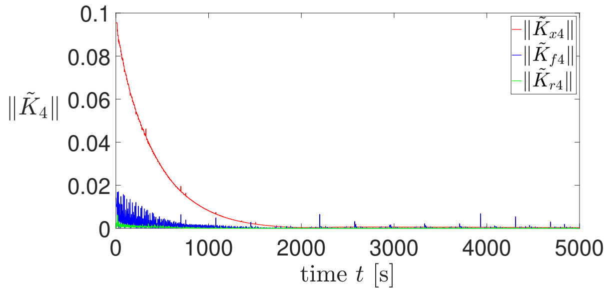
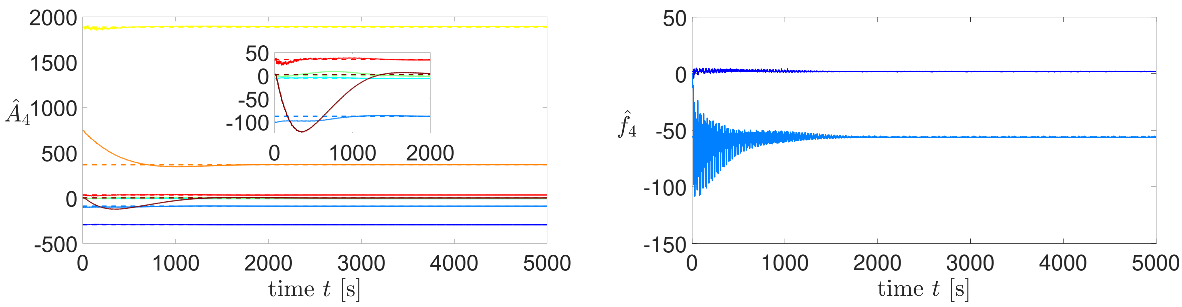


Figure 4.9: Convergence of estimated controller gains of indirect adaptation case.

The parameter convergence property is tested on the PWA system (4.94) with indirect adaptation approach. The reference signal is $y_d = [0.5\sin(0.2t) + 0.05\sin(0.9t), 0.2\sin(0.5t) + 0.05\sin(1.2t)]^T$ without Gaussian noise. The adaptation gains, the performance bound, and the initial guess of parameters are chosen the same as those of the tracking case. Besides, we specify $\lambda = 0.04$.



(a) Parameter convergence $\hat{A}_4$

(b) Parameter convergence $\hat{f}_4$

Figure 4.10: Convergence of estimated parameters of indirect adaptation case.

Figure 4.9 shows the convergence of the estimation errors of the controller gains of subsystem 4. The red line, the green line and the blue line represent the (Frobenius) norms of the estimation errors $\tilde{K}_{x4}, \tilde{K}_{r4}, \tilde{K}_{f4}$, respectively. The figure validates the convergence of the estimated controller gains to the nominal ones.

Similarly, the component-wise convergence of the estimated parameters of subsystem 4 by using the indirect PPAC approach is shown in Fig. 4.10. As can be seen from the figure, the estimated system parameters, displayed by solid lines, converge to the real values (dashed lines).

4.8 Summary

In this chapter, we have investigated the adaptive control approaches for PWA systems satisfying prescribed output tracking performance constraints in terms of both direct and indirect adaptations, respectively. For both control approaches, we have shown that the output tracking errors stay within the prescribed performance bounds. Based on the novel CLFs, closed-loop stability is achieved under arbitrary switching. The controller gains and estimated subsystem parameters are proved to converge to their nominal and real values if the desired trajectory is PE. Moreover, a robust modification is developed to enhance the robustness of the closed-loop system against disturbances.

Compared to the classical MRAC of PWA systems reviewed in Chapter 2, our methods ensure that the transient behavior of the closed-loop systems fulfills prescribed output tracking performance constraints and guarantees safety during adaptations. Besides, the existing prescribed performance control methods proposed for switched systems [91, 177] are based on completely known input coefficients/matrices while our approaches are eligible for uncertain PWA systems, whose input matrices are unknown. Furthermore, parameter convergence is achieved in our approaches, which fills the gap that the existing prescribed performance control approaches either do not consider parameter convergence [91, 177] or only partially achieve parameter convergence in the sense that the parameter estimation error converges to a bounded set[12, 13, 55].

One limitation of the approaches presented in this chapter is that the use of the prescribed performance technique may result in a large control input. In future work, how to improve the current approaches for systems having saturation constraints on the input signal is of practical interest. In our current setting, we treat the entries a_{ji}, b_{ji}, f_{ji} in matrices A_i, B_i, f_i as unknown for the purpose of generality except that some directional information of the input matrices is known. In practice, the number of unknown parameters can be reduced by performing offline measurements of some physical quantities. Furthermore, it is mostly also possible to determine the upper and lower bounds of the unknown parameters. For instance, in the PWA model of lateral vehicle dynamics [24], which can be well represented by the numerical example in Sec. 3.4, the vehicle mass and yaw inertia can be determined while the tire friction coefficients are unknown as they depend on the road conditions. Nevertheless, reasonable upper and lower bounds can be well determined by testing the interaction of the tire and different types of roads. With a reduced number of unknown parameters with certain upper and lower bounds, it would be possible to investigate how the input saturation constraints are related to the bounds of the uncertain parameters, which is of great significance in the area of prescribed performance for not only PWA systems but also a more general class of systems.

Another limitation is that the current approaches are only eligible for PWA systems in canonical form. How to extend these approaches to generalized PWA systems (PWA systems without structural restrictions) remains to be explored in future work. Methods proposed in [91, 177] can be applied to switched systems in strict feedback form and non-strict feedback form. However, as mentioned above, the input coefficients/matrices are assumed to be completely known and parameter convergence is not achieved. In the direct next step, it is interesting to study, how to extend our approaches to PWA systems in strict and non-strict feedback form while retaining our key features that the input matrices do not have to be completely known and the parameter convergence is guaranteed under the PE condition.

Adaptive Control of PWA Systems with State Performance Guarantees 5

In the previous chapter, we explored the adaptive control of PWA systems satisfying output tracking performance constraints. Considering the cases where not only the output but also the whole state vector need to satisfy some performance constraints, we would like, in this chapter, to explore the adaptive control for PWA systems with state tracking performance guarantees.

As also reviewed in Chapter 4, notable progress has been made in the field of adaptive control with performance guarantees. These include funnel control [62, 69], barrier Lyapunov function-based approach [153], and prescribed performance control [12, 13]. All of these methods are proposed to confine the output tracking error within the predefined constraints. Although some recent barrier Lyapunov function-based controllers achieve the full state constraints [103, 104, 117, 184], they are built upon the backstepping concept, which requires the controlled system to be in strict feedback form or pure feedback form. Thus, they cannot be applied to generalized PWA systems. Recently, a set-theoretic MRAC for linear systems is developed [8]. It uses the barrier Lyapunov function concept to confine the weighted Euclidean norm of the state tracking error within a predefined bound. The controller does not rely on the backstepping-type analysis and therefore does not impose restrictions on the system structure. This method is extended to the cases with time-varying performance bounds [6], systems with actuator faults [168], and systems with unstructured uncertainties [7]. However, extending this method from linear systems to switched systems is nontrivial and challenging. Specifically, if the barrier Lyapunov function is constructed with the user-defined performance bound being the barrier, as it is done in the linear system case, then the discontinuity of the weighted Euclidean norm of the tracking error at switching instants may cause transgression of the barrier, which makes the barrier Lyapunov function invalid. Besides, only matched uncertainties (uncertainties, which can be compensated with an additional input term) are addressed in the work of set-theoretic MRAC approaches. Since the PWA systems are mostly approximations of nonlinear systems, their approximation errors are not necessarily matched, let alone other kinds of external disturbances. How to enhance the robustness against unmatched uncertainties/disturbances when applying the set-theoretic MRAC to PWA systems is still open.

The main contribution of this chapter is threefold. First, direct and indirect set-theoretic MRAC approaches for uncertain PWA systems with state tracking performance guarantees are developed. Second, parameter convergence is achieved for the indirect adaptation case. Finally, a robust modification of the proposed method is developed for PWA systems subject to unmatched disturbances. In addition to these achievements, another highlight of the approaches proposed in this chapter is that the multiple Lyapunov functions are non-increasing even at switching instants. Therefore, closed-loop stability and asymptotic state tracking in the disturbance-free case is achieved for the MLF setting without introducing extra conditions, while the classical methods (see Table 2.1) require robust modifications or PE conditions. This enables the application of the proposed methods with fewer prior knowledge requirements and less excitation. Furthermore, the non-increasing property of the MLF is especially essential for the indirect adaptive control of PWA systems as it overcomes the limitation of the classical indirect adaptive control of PWA systems, which requires the existence of the CLF and cannot be applied to the MLF setting.

This chapter is structured as follows. The problem to be solved in this chapter is formulated in Section 5.1. The direct adaptive control is presented in Section 5.2, in which the stability analysis is also provided. In Section 5.3 and Section 5.4, two variants of indirect adaptive control are introduced with the analysis of the stability as well as the parameter convergence. The robust modification is shown in Section 5.5. Numerical examples are illustrated in Section 5.6, which is followed by the summary in Section 5.7.

5.1 Problem Formulation

In this chapter, we consider the PWA system

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) \tag{5.1}$$

with known indicator functions $\chi_i(t), i \in \mathcal{I}$, unknown parameters $A(t) = \sum_{i=1}^{s} \chi_i(t)A_i$, $B(t) = \sum_{i=1}^{s} \chi_i(t)B_i$, and $f(t) = \sum_{i=1}^{s} \chi_i(t)f_i$ for $i \in \mathcal{I}$. $x \in \mathbb{R}^n$ and $u \in \mathbb{R}^p$ represent the state and the control input of the PWA system. The reference PWA system

$$\dot{x}_m(t) = A_m(t)x_m(t) + B_m(t)r(t) + f_m(t), \tag{5.2}$$

with $A_m(t) = \sum_{i=1}^{s} \chi_i(t)A_{mi}$, $B_m(t) = \sum_{i=1}^{s} \chi_i(t)B_{mi}$, $f_m(t) = \sum_{i=1}^{s} \chi_i(t)f_{mi}$ shares the same indicator functions with the controlled PWA system (5.1). There exist some positive definite matrices P_i and $Q_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}$ such that

$$A_{mi}^T P_i + P_i A_{mi} = -Q_i, \quad i \in \mathcal{I}. \tag{5.3}$$

Let the nominal controller be

$$u(t) = K_x^* x(t) + K_r^* r(t) + K_f^*, \tag{5.4}$$

where $K_x^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{xi}^*$, $K_r^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{ri}^*$, $K_f^*(t) = \sum_{i=1}^{s} \chi_i(t)K_{fi}^*$ with $K_{xi}^* \in \mathbb{R}^{p\times n}, K_{ri}^* \in \mathbb{R}^{p\times p}, K_{fi}^* \in \mathbb{R}^p, i \in \mathcal{I}$ denoting the nominal controller gains for the i-th subsystem of (5.1). We make the usual assumption as in Chapter 2 and Chapter 3 that the following matching equations hold:

$$A_{mi} = A_i + B_i K_{xi}^*, \quad B_{mi} = B_i K_{ri}^*, \quad f_{mi} = f_i + B_i K_{fi}^*, \quad i \in \mathcal{I}. \tag{5.5}$$

The adaptive controller takes the form

$$u(t) = K_x(t)x(t) + K_r(t)r(t) + K_f(t) \tag{5.6}$$

with $K_x(t) = \sum_{i=1}^{s} \chi_i(t) K_{xi}(t)$, $K_r(t) = \sum_{i=1}^{s} \chi_i(t) K_{ri}(t)$ and $K_f(t) = \sum_{i=1}^{s} \chi_i(t) K_{fi}(t)$. Inserting (5.6) into the controlled PWA system (5.1) and defining the state tracking error $e(t) = x(t) - x_m(t)$, we have

$$\dot{e} = A_m e + \sum_{i=1}^{s} \chi_i B_i (\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}), \tag{5.7}$$

where $\tilde{K}_{xi} = K_{xi} - K_{xi}^*$, $\tilde{K}_{ri} = K_{ri} - K_{ri}^*$, $\tilde{K}_{fi} = K_{fi} - K_{fi}^*$ represent the estimation errors of the control gains.

In this chapter, we would like to design an adaptive controller for PWA systems such that the norm of the state tracking error e is enforced within a predefined performance bound and hence the closed-loop system satisfies performance constraints. The performance bound can be chosen the same as the one in Chapter 4 (see (4.7))

$$\rho(t) = (\rho_0 - \rho_\infty) \mathrm{e}^{-l(t-t_0)} + \rho_\infty, \tag{5.8}$$

where $\rho_0, \rho_\infty, l \in \mathbb{R}^+$ and $\rho_0 > \rho_\infty$. The performance constraint to be satisfied can be formulated as

$$\|e(t)\|_P < \rho(t), \tag{5.9}$$

where $\|e(t)\|_P$ is the weighted Euclidean norm of $e(t)$ with the weighting matrix P, i.e., $\|e(t)\|_P = (e^T(t) P e(t))^{\frac{1}{2}}$. The error metric $\|e(t)\|_P$ serves as a performance measure reflecting the difference between the state of the controlled system and the reference system. P is equal to P_i if subsystem i is activated, i.e., $P = \sum_{i=1}^{s} \chi_i(t) P_i$, where the weighting matrices P_i satisfy (5.3). So the error metric $\|e(t)\|_P$ and the system parameters switch synchronously.

Remark 5.1. Some questions may arise regarding (5.9): is it feasible to specify a global weighting matrix for the error metric instead of the switching one? What if the user would like to define a performance constraint with an arbitrary weighting matrix, which does not necessarily satisfy the Lyapunov equation (5.3)? In fact, these requirements can be transformed into the formulation (5.9). We explain this point in the following.
Suppose that a global performance measure, which should hold for every subsystem, is desired by the user, i.e., $\|e(t)\|_S < \rho^*(t)$, where $S \in \mathbb{R}^{n \times n}$ is an arbitrary user-defined positive definite matrix and $\rho^*(t)$ represents a user-defined performance function in form of (5.8). Then, we can choose $P_i, i \in \mathcal{I}$ matrices based on (5.3). We know $\|e\|_S \leq \frac{1}{\gamma} \|e\|_P$ with $\gamma = \min_{i \in \mathcal{I}} \sqrt{\frac{\lambda_{\min}(P_i)}{\lambda_{\max}(S)}}$. To satisfy $\|e(t)\|_S < \rho^*(t)$, it suffices to let $\|e\|_P < \gamma \rho^*(t)$ hold, which is equivalent to (5.9) by letting $\rho(t) = \gamma \rho^*(t)$. A graphical illustration of this explanation can be seen in Fig. 5.1.

The problem to be studied in this chapter is formulated as follows:

Problem 5.1. Given a performance function (5.8), a reference model (5.2) and a PWA system (5.1) with unknown subsystem parameters A_i, B_i, f_i and known regions Ω_i (or equivalently, known indicator functions $\chi_i(t)$), design the adaptive controller $u(t)$ such that the state $x(t)$ of (5.1) tracks the state $x_m(t)$ of (5.2) with the tracking error $e(t)$ satisfying the performance constraint (5.9). Besides, ensure that the estimated gains or estimated parameters converge to their nominal or real values.

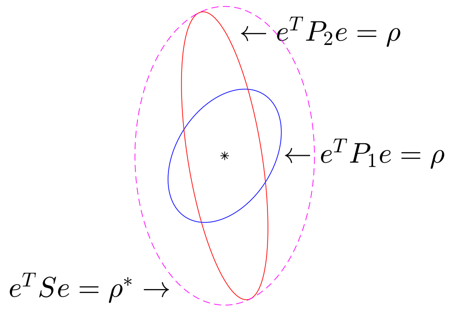


Figure 5.1: Graphical illustration of Remark 5.1

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5.2 Direct Adaptive Control Design

In this section, we propose the direct adaptive controller to solve the given problem in the disturbance-free case. First, we introduce the auxiliary performance bound and explain the solution concept. Then the proposed adaptation laws are presented, which are followed by the stability analysis of the closed-loop system.

5.2.1 Auxiliary Performance Bound

We define a generalized restricted potential function (barrier Lyapunov function) [6] $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ on the set $\mathcal{D}_\theta \triangleq \{e \mid \|e\|_P \in [0, \theta)\}$

$$\phi(\|e\|_P) = \frac{\|e\|_P^2}{\theta^2(t) - \|e\|_P^2}, \quad \|e\|_P < \theta(t). \tag{5.10}$$

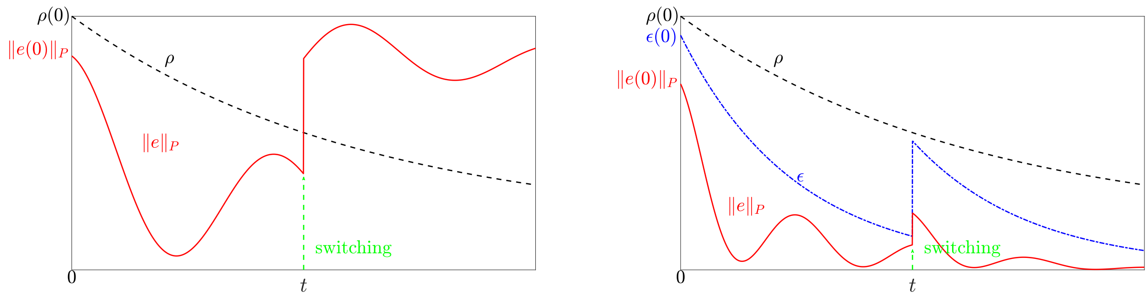
By properly initializing the reference system or designing the performance function, we can let $\|e(t_0)\|_P < \rho(t_0)$. The set-theoretic MRAC approach for linear systems [6] suggests specifying the barrier θ to be $\rho(t)$ and designing the adaptation laws such that $\phi(\|e\|_P)$ is bounded $\forall t \in [t_0, \infty)$, then it would be obtained that $\|e(t)\|_P < \rho(t), \forall t \in [t_0, \infty)$.

The difficulty in switched systems is that $P = \sum_{i=1}^{s} \chi_i(t) P_i$ leads to the jumps of $\|e(t)\|_P$ at switching instants. Suppose $\chi_i(t) = 1$ for $t \in [t_{k-1}, t_k)$ and $\chi_j(t) = 1$ for $t \in [t_k, t_{k+1})$ for $i \neq j, i, j \in \mathcal{I}$ and recall that $e(t_k^-) \triangleq \lim_{\tau \uparrow t_k} e(\tau)$, we have

$$\|e(t_k)\|_P^2 = e^T(t_k) P_j e(t_k) \leq \lambda_{\max}(P_j) |e(t_k)|^2 \leq \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)} \|e(t_k^-)\|_P^2, \tag{5.11}$$

which may result in $\|e(t_k)\|_P > \rho(t_k)$ for $\frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)} > 1$ and $\|e(t_k^-)\|_P < \rho(t_k^-)$, as shown in Figure 5.2a. This further makes the barrier function $\phi(\|e\|_P)$ invalid. We call this *barrier transgression* problem.

To overcome this problem, our idea is to introduce an auxiliary performance bound, denoted by $\epsilon(t)$, which decays faster than the user-defined performance bound $\rho(t)$. $\epsilon(t)$ is reset at each switching instant such that $\|e(t_k)\|_P < \epsilon(t_k)$ for $k \in \mathbb{N}^+$, see Fig. 5.2b. If the adaptive controller ensures $\|e\|_P < \epsilon(t)$ and if $\epsilon(t)$ is designed such that $\epsilon(t) < \rho(t)$ for $t \in [t_0, \infty)$,



(a) The barrier transgression problem (b) Introducing auxiliary performance bound

Figure 5.2: Graphical illustration of the barrier transgression problem and the concept to introduce auxiliary performance bound

then the control objective (5.9) is achieved. We propose the auxiliary performance bound $\epsilon(t)$ with the following dynamics

$$\dot{\epsilon}(t) = -h\epsilon(t) + g, \quad \epsilon(t_0) \in (\frac{g}{h}, \rho_0), \quad \epsilon(t_k) = G(\epsilon(t_k^-)), \quad k \in \mathbb{N}^+ \tag{5.12}$$

where $h, g \in \mathbb{R}^+$. $G : \mathbb{R}^+ \to \mathbb{R}^+$ is a state reset map. It resets the value of ϵ at each switching instant. Note that ϵ shares the same switching instants with the controlled PWA system $t_k, k \in \mathbb{N}^+$, i.e., when the switch of the controlled PWA system occurs, ϵ is reset by the state reset map simultaneously. We specify the state reset map G to be

$$G(\epsilon(t_k^-)) = \sqrt{\mu}\epsilon(t_k^-), \quad \mu \triangleq \max_{i,j\in\mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}. \tag{5.13}$$

with $\mu > 1$. The parameters h, g, μ control the evolution of the auxiliary performance bound $\epsilon(t)$. Specifically, h represents the decreasing rate of $\epsilon(t)$. g serves as a input for the dynamics of ϵ and affects its minimum value. $\sqrt{\mu}$ reflects the increment of $\epsilon(t)$ at each switching instant. As stated before, $\epsilon(t)$ should be smaller than $\rho(t), \forall t \in [t_0, \infty)$. To achieve this, the state reset of $\epsilon(t)$ needs to satisfy some dwell time constraints, i.e., $T_D = \min_{k\in\mathbb{N}^+}\{t_k - t_{k-1}\} > \tau_D$ for some $\tau_D \in \mathbb{R}^+$. We have the following lemma:

Lemma 5.1. Given the performance function (5.8) and the auxiliary performance bound (5.12) with the reset map (5.13), if $h > l$, $\rho_\infty > \sqrt{\mu}\frac{g}{h}$ and if the dwell time of $\epsilon(t)$ satisfies

$$T_D > \tau_D = \frac{1}{h-l} \ln \frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}} \tag{5.14}$$

for $\mu > 1$, then the following inequality holds

$$\frac{g}{h} \le \epsilon(t) < \rho(t), \quad \forall t \in [t_0, \infty). \tag{5.15}$$

Proof. The initial value of ϵ satisfies $\epsilon(t_0) > \frac{g}{h}$, meaning that ϵ decreases exponentially towards $\frac{g}{h}$ if no switch occurs. Since $\sqrt{\mu} > 1$, ϵ increases at each switching time instant and $\epsilon(t_k) > \frac{g}{h}$ for $\forall k \in \mathbb{N}^+$. If the switch terminates from some time on, then $\epsilon \to \frac{g}{h}$ for $t \to \infty$, otherwise, $\epsilon > \frac{g}{h}$ for $t \in [t_0, \infty)$. Therefore, we have $\epsilon(t) \ge \frac{g}{h}, \forall t \in [t_0, \infty)$.

Now, we explore the relationship between $\epsilon(t)$ and $\rho(t)$. We have for the time interval $[t_0, t_1)$

$$\epsilon(t) = \epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + g\int_{t_0}^{t} \mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau = (\epsilon(t_0) - \frac{g}{h})\mathrm{e}^{-h(t-t_0)} + \frac{g}{h}. \tag{5.16}$$

Since $\epsilon(t_0) \in (\frac{g}{h}, \rho_0)$, $h > l$ and $\rho_\infty > \sqrt{\mu}\frac{g}{h}$, we have $\epsilon(t) < \rho(t)$ for $t \in [t_0, t_1)$. For $t = t_1$ it gives

$$\epsilon(t_1) = \sqrt{\mu}\epsilon(t_1^-) = \sqrt{\mu}(\epsilon(t_0) - \frac{g}{h})\mathrm{e}^{-h(t_1-t_0)} + \sqrt{\mu}\frac{g}{h}. \tag{5.17}$$

Let $\Delta t_1 \triangleq t_1 - t_0$, we have

$$\begin{aligned}
\rho(t_1) - \epsilon(t_1) =&(\rho_0 - \rho_\infty)\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}(\epsilon(t_0) - \frac{g}{h})\mathrm{e}^{-h\Delta t_1} + (\rho_\infty - \sqrt{\mu}\frac{g}{h}) \\
\geq&(\rho_0 - \rho_\infty)\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}(\epsilon(t_0) - \frac{g}{h})\mathrm{e}^{-h\Delta t_1} + (\rho_\infty - \sqrt{\mu}\frac{g}{h})\mathrm{e}^{-l\Delta t_1} \\
=&(\rho_0 - \sqrt{\mu}\frac{g}{h})\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}(\epsilon(t_0) - \frac{g}{h})\mathrm{e}^{-h\Delta t_1} \\
\geq&(\rho_0 - \sqrt{\mu}\frac{g}{h})\mathrm{e}^{-l\Delta t_1} - \sqrt{\mu}(\rho_0 - \frac{g}{h})\mathrm{e}^{-h\Delta t_1}.
\end{aligned} \tag{5.18}$$

If the inequality

$$(\rho_0 - \sqrt{\mu}\frac{g}{h})\mathrm{e}^{-l\Delta t_1} > \sqrt{\mu}(\rho_0 - \frac{g}{h})\mathrm{e}^{-h\Delta t_1} \tag{5.19}$$

holds, we will immediately have $\rho(t_1) > \epsilon(t_1)$. Since $\rho_0 > \rho_\infty > \sqrt{\mu}\frac{g}{h} > \frac{g}{h}$, we have $\rho_0 - \sqrt{\mu}\frac{g}{h} > 0$ and $\sqrt{\mu}(\rho_0 - \frac{g}{h}) > 0$. Therefore, (5.19) is equivalent to

$$\frac{\rho_0 - \sqrt{\mu}\frac{g}{h}}{\sqrt{\mu}(\rho_0 - \frac{g}{h})} > \mathrm{e}^{-(h-l)\Delta t_1} \tag{5.20}$$

Taking the logarithm of both sides we obtain

$$\Delta t_1 > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho_0 - \frac{g}{h}\sqrt{\mu}}{\rho_0 - \frac{g}{h}\sqrt{\mu}}. \tag{5.21}$$

Following the above analysis we can obtain $\epsilon(t) < \rho(t)$ for $t \in [t_{k-1}, t_k)$ and $\epsilon(t_k) < \rho(t_k)$ for $k \in \mathbb{N}^+$ if

$$\Delta t_k > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}}{\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}} = \frac{1}{h-l}\ln(\sqrt{\mu} + \frac{(\mu - \sqrt{\mu})\frac{g}{h}}{\rho(t_{k-1}) - \frac{g}{h}\sqrt{\mu}}). \tag{5.22}$$

If the dwell time T_D is no smaller than the maximal required interval length $\max\{\Delta t_k\}$, then $\epsilon(t) < \rho(t)$ holds for $\cup[t_{k-1}, t_k), k \in \mathbb{N}^+$. Because $\rho(t_{k-1}) \geq \rho_\infty$ for $k \in \mathbb{N}^+$, we have

$$T_D \geq \max\{\Delta t_k\} > \frac{1}{h-l}\ln\frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}} \tag{5.23}$$

So we can conclude that if (5.14) holds, then $\epsilon(t) < \rho(t)$ for $t \in [t_0, \infty)$. □

Lemma 5.1 tells the dwell time constraint to be fulfilled. We will further discuss how this dwell time constraint can be satisfied later in Section 5.2.4. Since ϵ, the reference system (5.2) and the closed-loop system share the same switching signal, the first question to ask is, if the reference system is stable with the dwell time constraint (5.14)? This is answered by the following lemma.

Lemma 5.2. The reference system (5.2) satisfying (5.3) is stable with the dwell time constraint (5.14) and h satisfying $h < \frac{1}{2}\min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}$.

Proof. Consider the Lyapunov function $V_m = x_m^T(\sum_{i=1}^s \chi_i P_i)x_m$ for the homogeneous part of (5.2). The increment of V_m at each switching instant satisfies $V_m(t_k) \le \mu V_m(t_k^-)$. In the interval $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$, we have $\dot{V}_m \le -\alpha_m V_m$ with

$$\alpha_m = \min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}. \tag{5.24}$$

If the switching satisfies $t_k - t_{k-1} > \frac{\ln\mu}{\alpha_m}, \forall k \in \mathbb{N}^+$, the homogeneous system $\dot{x}_m = A_m x_m$ is exponentially stable and the stability of the reference system (5.2) can be concluded for bounded input r [65, 112]. Since $h < \frac{1}{2}\min_{i\in\mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P_i)}$, we have $h - l < h < \frac{1}{2}\alpha_m$. This together with $\mu > 1$ leads to

$$T_D > \tau_D > \frac{2}{\alpha_m}\ln\frac{\sqrt{\mu}\rho_\infty - \frac{g}{h}\sqrt{\mu}}{\rho_\infty - \frac{g}{h}\sqrt{\mu}} > \frac{2}{\alpha_m}\ln\frac{\sqrt{\mu}(\rho_\infty - \frac{g}{h})}{\rho_\infty - \frac{g}{h}} = \frac{\ln\mu}{\alpha_m}. \tag{5.25}$$

So this tells that the reference system is stable and $x_m \in \mathcal{L}_\infty$ if the dwell time constraint (5.14) is satisfied. □

5.2.2 Adaptation Laws

Based on the auxiliary performance bound proposed in Section 5.2.1, we define the following generalized restricted potential function (barrier Lyapunov function) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$

$$\phi(\|e\|_P) = \frac{\|e\|_P^2}{\epsilon^2(t) - \|e\|_P^2}, \quad \|e\|_P < \epsilon(t) \tag{5.26}$$

with $P = \sum_{i=1}^s \chi_i(t)P_i$. Since $\|e\|_P^2$ and $\epsilon^2(t)$ are piecewise continuous and piecewise differentiable, the partial derivative of ϕ with respect to $\|e\|_P^2$ over the time interval $[t_k, t_{k+1}), k \in \mathbb{N}$ takes the form

$$\phi_d(\|e\|_P) \triangleq \frac{\partial\phi}{\partial\|e\|_P^2} = \frac{\epsilon^2(t)}{(\epsilon^2(t) - \|e\|_P^2)^2} > 0. \tag{5.27}$$

ϕ and ϕ_d have the property that $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$.

The direct adaptation laws of the estimated controller gains are given as

$$\begin{aligned} \dot{K}_{xi} &= -\chi_i\Gamma_{xi}\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e x^T, \\ \dot{K}_{ri} &= -\chi_i\Gamma_{ri}\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e r^T, \\ \dot{K}_{fi} &= -\chi_i\Gamma_{fi}\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e \end{aligned} \tag{5.28}$$

where $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} \in \mathbb{R}^+$ are positive scaling factors. $S_i \in \mathbb{R}^{p\times p}$ is a matrix such that there exists a symmetric and positive definite matrix $M_i \in \mathbb{R}^{p\times p}$ with $(K_{ri}^* S_i)^{-1} = M_i$. Here we make the usual assumption that S_i is known. The use of the indicator functions $\chi_i(t)$ in the adaptation laws (5.28) implies that the controller gains associated with a certain subsystem are updated only when this subsystem is activated. Their adaptation terminates and their values stay unchanged during the inactive phase of the corresponding subsystem. Note that ϕ_d in (5.28) can also be viewed as an error-dependent gain, whose effect can be weakened or amplified by tuning the constant gains $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$. They are chosen by trial and error in the simulation. If $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ are too small, the effect of ϕ_d on the adaptation speeds $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi}$ is weakened. Consequently, ϕ and ϕ_d may have every small denominators and become ill-conditioned. If $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi}$ are too large, the differential equations may become "stiff" and difficult to solve numerically.

5.2.3 Stability Analysis

The tracking performance and the stability of the closed-loop system are summarized in the following theorem.

Theorem 5.1. *Given the reference PWA system (5.2) satisfying (5.3) and the predefined performance function (5.8), let the PWA system (5.1) with known regions $\Omega_i, i \in \mathcal{I}$ and unknown subsystem parameters $A_i, B_i, f_i, i \in \mathcal{I}$ be controlled by the feedback controller (5.6) with the adaptation laws (5.28). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. The closed-loop system is stable and the state tracking error $e(t)$ fulfills the prescribed performance constraint (5.9) if the time constant h in (5.12) satisfies*

$$h < \frac{1}{2} \min_{i \in \mathcal{I}} \frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)} \tag{5.29}$$

and if the switching signal of the controlled PWA system obeys the dwell time constraint in (5.14).

Proof. Without loss of generality, we let the scaling factors in (5.28) be 1. Consider the following Lyapunov function

$$V = \phi(\|e\|_P) + \underbrace{\sum_{i=1}^{s} (\mathrm{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \tilde{K}_{fi}^T M_i \tilde{K}_{fi})}_{\triangleq V_K}. \tag{5.30}$$

V is piecewise continuous and piecewise differentiable. In particular, V is continuous and differentiable in between any two consecutive switching instants $[t_{k-1}, t_k), k \in \mathbb{N}^+$ (including t_0 for the initial instant), while it is non-differentiable and (possibly) discontinuous at each switching instant $t_k, k \in \mathbb{N}^+$. The mixture of the continuous evolution and the discontinuous jumps of V constitutes the main challenge of the stability analysis of switched systems. The overall idea is to prove $\dot{V} \leq 0$ in between switches and evaluate the incremental or decremental jumps at each switching instant. First of all, we would like to study the evolution of V in the continuous phase (named *phase 1*), namely, in between two consecutive switches:

phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

V is continuous in the intervals between two successive switches. Without loss of generality, we suppose that the i-th subsystem is activated for $t \in [t_{k-1}, t_k)$ and $e(t_{k-1})$ satisfies

$\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$. The time-derivative of V in $[t_{k-1}, t_k)$ is given by

$$\dot{V} = \dot{\phi}(\|e\|_{P_i}) + 2\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}). \tag{5.31}$$

First, we simplify the second term of $\dot{V}$. Taking the adaptation laws (5.28) into the first summand of the second term of $\dot{V}$ gives

$$\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) = -\chi_i \phi_d \mathrm{tr}(\tilde{K}_{xi}^T M_i S_i^T B_{mi}^T P_i e x^T). \tag{5.32}$$

Since $(K_{ri}^* S_i)^{-1} = M_i$ and $B_i K_{ri}^* = B_{mi}$, we have $M_i S_i^T B_{mi}^T = M_i S_i^T (B_i K_{ri}^*)^T = M_i M_i^{-1} B_i^T = B_i^T$, which further gives

$$\begin{aligned}\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) &= -\chi_i \phi_d \mathrm{tr}(\tilde{K}_{xi}^T B_i^T P_i e x^T) = -\chi_i \phi_d \mathrm{tr}(x e^T P_i B_i \tilde{K}_{xi}) \\ &= -\chi_i \phi_d \mathrm{tr}(e^T P_i B_i \tilde{K}_{xi} x) = -\chi_i \phi_d e^T P_i B_i \tilde{K}_{xi} x.\end{aligned} \tag{5.33}$$

Doing the same simplification for $\mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri})$ and $\tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}$ we have

$$2\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}) = -2\sum_{i=1}^{s} \chi_i \phi_d e^T P_i B_i (\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}).$$

$\dot{\phi}$ can be further simplified as

$$\dot{\phi} = \frac{\partial \phi}{\partial \|e\|_{P_i}^2} \frac{\mathrm{d}\|e\|_{P_i}^2}{\mathrm{d}t} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = 2\phi_d(\|e\|_{P_i}) e^T P_i \dot{e} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}. \tag{5.34}$$

Substituting $\dot{e}$ with (5.7) yields

$$\begin{aligned}\dot{\phi} &= \phi_d(e^T(A_m^T P_i + P_i A_m)e + 2e^T P_i \sum_{i=1}^{s} \chi_i B_i (\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi})) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \\ &= -\phi_d e^T Q_i e + 2\sum_{i=1}^{s} \chi_i \phi_d e^T P_i B_i (\tilde{K}_{xi} x + \tilde{K}_{ri} r + \tilde{K}_{fi}) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}.\end{aligned} \tag{5.35}$$

Therefore, $\dot{V}$ can be simplified as

$$\dot{V} = -\phi_d e^T Q_i e + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \tag{5.36}$$

with

$$\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = \frac{-2\epsilon\|e\|_{P_i}^2}{(\epsilon^2 - \|e\|_{P_i}^2)^2}\dot{\epsilon} = -2\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2 \frac{\dot{\epsilon}}{\epsilon} \le 2\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2 \frac{|\dot{\epsilon}|}{\epsilon}. \tag{5.37}$$

Invoking Lemma 5.1, we have $\epsilon(t) \ge \frac{g}{h}, \forall t \in [t_0, \infty)$. Therefore,

$$\frac{|\dot{\epsilon}|}{\epsilon} = \frac{h\epsilon - g}{\epsilon} = h - \frac{g}{\epsilon} \le h, \tag{5.38}$$

which leads to

$$\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \le 2h\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2. \tag{5.39}$$

Taking this into (5.36) yields

$$\dot{V} \leq -\phi_d |e|^2 \lambda_{\min}(Q_i) + 2h\phi_d |e|^2 \lambda_{\max}(P_i) = -\phi_d |e|^2 (\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)). \tag{5.40}$$

From the condition (5.29) it follows $\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i) > 0$, which together with the property $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$ gives

$$\dot{V} \leq -\frac{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)}{2\lambda_{\max}(P_i)} \phi \leq 0. \tag{5.41}$$

The fact $\dot{V} \leq 0$ in intervals $[t_{k-1}, t_k), k \in \mathbb{N}^+$ implies that the Lyapunov function decreases between two consecutive switches. ϕ and ϕ_d are bounded in $[t_{k-1}, t_k)$. Since $\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$, we have $\|e(t)\|_{P_i} < \epsilon(t)$ for $\forall t \in [t_{k-1}, t_k)$.

The property $\dot{V} \leq 0$ for each $[t_{k-1}, t_k)$ does not imply the global stability of the closed-loop system over the whole $t \in [t_0, \infty)$. It is necessary to evaluate the discontinuity of V at each switching instant (*phase 2*):

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$

Now we analyse the behavior of the Lyapunov function at the switching time instants. Suppose that i-th subsystem is activated in $[t_{k-1}, t_k)$ and j-th subsystem is activated in $[t_k, t_{k+1})$, where $i, j \in \mathcal{I}, i \neq j$. From the adaptation laws of the estimated controller gains (5.28), we see that the estimated controller gains are continuous, i.e., $\tilde{K}_{xi}(t_k) = \tilde{K}_{xi}(t_k^-)$, $\tilde{K}_{ri}(t_k) = \tilde{K}_{ri}(t_k^-)$ and $\tilde{K}_{fi}(t_k) = \tilde{K}_{fi}(t_k^-)$ for $\forall i \in \mathcal{I}$, from which it follows $V_K(t_k^-) = V_K(t_k)$. To study the relationship between $V(t_k)$ and $V(t_k^-)$, it remains to analyse $\phi(\|e(t_k)\|_P)$ and $\phi(\|e(t_k^-)\|_P)$. Since $e(t)$ is also continuous, $e(t_k) = e(t_k^-)$. This results in

$$\begin{aligned} \|e(t_k)\|_P^2 = e^T(t_k)P_j e(t_k) \leq \lambda_{\max}(P_j)|e(t_k)|^2 \leq \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)} e^T(t_k)P_i e(t_k) \\ = \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)} \|e(t_k^-)\|_P^2 \leq \mu \|e(t_k^-)\|_P^2. \end{aligned} \tag{5.42}$$

From the analysis of *phase 1*, we already know that $\|e(t_k^-)\|_P < \epsilon(t_k^-)$. ϵ is reset at t_k and we have

$$\|e(t_k)\|_P \leq \sqrt{\mu}\|e(t_k^-)\|_P < \sqrt{\mu}\epsilon(t_k^-) = \epsilon(t_k), \tag{5.43}$$

which makes the potential function $\phi(\|e(t_k)\|_P)$ also valid at t_k. Recalling the dynamics of ϵ (5.12) and the above inequalities (5.42), we have

$$\begin{aligned} \phi(\|e(t_k)\|_P) = \frac{\|e(t_k)\|_P^2}{\epsilon^2(t_k) - \|e(t_k)\|_P^2} \leq \frac{\mu\|e(t_k^-)\|_P^2}{\epsilon^2(t_k) - \mu\|e(t_k^-)\|_P^2} \\ = \frac{\mu\|e(t_k^-)\|_P^2}{\mu\epsilon^2(t_k^-) - \mu\|e(t_k^-)\|_P^2} = \phi(\|e(t_k^-)\|_P). \end{aligned} \tag{5.44}$$

Combining the facts $\phi(\|e(t_k)\|_P) \leq \phi(\|e(t_k^-)\|_P)$ and $V_K(t_k^-) = V_K(t_k)$, we have

$$V(t_k) = \phi(\|e(t_k)\|_P) + V_K(t_k) \leq \phi(\|e(t_k^-)\|_P) + V_K(t_k^-) = V(t_k^-). \tag{5.45}$$

Therefore, the Lyapunov function is non-increasing at every switching time instant. This together with the fact $\dot{V} \leq 0$ in $[t_{k-1}, t_k)$ for $\forall k \in \mathbb{N}^+$ implies that $V(t)$ is non-increasing for $\forall t \in [t_0, \infty)$. The discontinuity of the Lyapunov function does not introduce extra dwell time constraints.

Combining the analysis of *phase 1* and *phase 2*, we have $\phi, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$, which further leads to $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Besides, $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds for $\forall t \in [t_0, \infty)$ and $\phi_d \in \mathcal{L}_\infty$.

Invoking Lemma 5.2 we have $x_m \in \mathcal{L}_\infty$. This property and $\|e(t)\|_P < \epsilon(t) < \rho(t)$ lead to $x \in \mathcal{L}_\infty$, which together with $r, \phi_d \in \mathcal{L}_\infty$ implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. □

Theorem 5.1 shows the tracking performance and the stability of the closed-loop system under the dwell time constraint (5.14). Now we study the case with arbitrary switching. For the PWA reference systems with common Lyapunov matrix P, i.e., if positive definite matrices P and $Q_i, i \in \mathcal{I}$ exist such that

$$A_{mi}^T P + P A_{mi} = -Q_i, \quad i \in \mathcal{I}, \tag{5.46}$$

the error metric $\|e(t)\|_P$ exhibits no jumps at the switching instants. We can construct the potential function with the user-defined performance function directly

$$\phi_0(\|e\|_P) = \frac{\|e\|_P^2}{\rho^2(t) - \|e\|_P^2}, \quad \|e\|_P < \rho(t). \tag{5.47}$$

Corollary 5.1. For the reference PWA system (5.2) with a common Lyapunov matrix P, if the adaptation laws

$$\dot{K}_{xi} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e x^T, \quad \dot{K}_{ri} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e r^T, \quad \dot{K}_{fi} = -\chi_i \phi_{d0} S_i^T B_{mi}^T P e \tag{5.48}$$

are used with $\phi_{d0} \triangleq \frac{\partial \phi_0}{\partial \|e\|_P^2}$, and if the decaying rate of ρ satisfies

$$l < \frac{1}{2} \min_{i \in \mathcal{I}} \frac{\lambda_{\min}(Q_i)}{\lambda_{\max}(P)}, \tag{5.49}$$

the closed-loop system is stable under arbitrary switching and the state tracking error $e(t)$ satisfies the prescribed performance guarantees (5.9).

Proof. We propose the following CLF

$$V = \phi_0(\|e\|_P) + \sum_{i=1}^{s} (\mathrm{tr}(\tilde{K}_{xi}^T M_i \tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \tilde{K}_{ri}) + \tilde{K}_{fi}^T M_i \tilde{K}_{fi}). \tag{5.50}$$

V is continuous not only within each interval $[t_k, t_{k+1}), k \in \mathbb{N}$ but also at switch instants $t_k, k \in \mathbb{N}^+$. Taking its time derivative and inserting (5.48) and (5.7), we obtain

$$\dot{V} = -\phi_{d0} e^T (\sum_{i=1}^{s} \chi_i Q_i) e + \frac{\partial \phi_0}{\partial \rho} \dot{\rho}. \tag{5.51}$$

Since $\frac{\partial \phi_0}{\partial \rho} \dot{\rho} \leq 2\phi_{d0}(\|e\|_P) \|e\|_P^2 \frac{|\dot{\rho}|}{\rho}$ and $\frac{|\dot{\rho}|}{\rho} \leq l$, we have

$$\dot{V} \leq -\phi_{d0} |e|^2 \min_{i \in \mathcal{I}} \lambda_{\min}(Q_i) + 2l\phi_{d0} |e|^2 \lambda_{\max}(P) \leq -\frac{\min_{i \in \mathcal{I}} \lambda_{\min}(Q_i) - 2l\lambda_{\max}(P)}{2\lambda_{\max}(P)} \phi_0 \leq 0$$

given that (5.49) holds. $\dot{V} \leq 0$ is negative semidefinite. Therefore, we have $\phi_0, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$ for arbitrary switching. The boundedness of $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ implies $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Furthermore, $\|e(t)\|_P < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. This leads to $x \in \mathcal{L}_\infty$ and $\phi_{d0} \in \mathcal{L}_\infty$, which together with $r \in \mathcal{L}_\infty$ implies that $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. □

It is worth comparing the proposed method with other control approaches for switched systems with performance guarantees. The bang-bang funnel controller [94] enforces the output tracking error of systems, which can be transformed into Byrnes-Isidori normal form, within a predefined funnel. The backstepping-based approaches can achieve output tracking with performance guarantees for systems with special structures (strict-feedback form [22, 177] and non-strict-feedback form [91]). In contrast, our approach achieves performance-guaranteed full state tracking without special structural requirements provided that the matching conditions (5.5) hold. Nevertheless, extra efforts are needed in our case for the design of auxiliary performance bound to bypass the barrier transgression problem. The fault-tolerant approach [179] solves the barrier transgression problem by modifying the performance function when actuator failure occurs. Compared to this concept, our method imposes the auxiliary performance bound with certain dwell time constraints such that the modification of the original performance function $\rho(t)$ is not necessary.

Remark 5.2. The classical MRAC approaches for switched systems [83, 137, 166] (see also the review in Section 2.2) suggest using $e^T(\sum_{i=1}^{s} \chi_i P_i)e$ as the error-related term (the first summand) of the Lyapunov function V. This leads to potential increases of V at switching instants. The dwell time constraints are then derived by formulating an inequality in form of $\dot{V} < -\alpha V + \beta$ for some constant $\alpha, \beta > 0$ to keep V exponentially decreasing in between the switches. To achieve this, the projection operator needs to be introduced (see Theorem 2.7 as well as work by Wu and Zhao [166]) or the input signal must be PE (see Theorem 2.5) in the disturbance-free case. One key feature of our approach is that the Lyapunov function V is non-increasing even at the switching instants and does not impose extra dwell time constraints. This omits the need for introducing the projection or the PE condition in the disturbance-free case.

Remark 5.3. The non-increasing property at switching instants of Lyapunov functions is also achieved in the recently proposed adaptive control approaches for switched systems [173, 175], which employ time-varying gains for adaptation laws. These time-varying gains are either obtained by interpolating a set of pre-calculated $P_{i,k}$ matrices satisfying certain linear matrix inequalities [173] or generated by an auxiliary piecewise continuous dynamical system [175]. Compared to these approaches, our method can be viewed as an error-dependent dynamic gain approach (see ϕ_d in adaptation laws (5.28)) and endows the closed-loop system with a user-defined performance guarantee.

Remark 5.4. Introducing the auxiliary performance bound ϵ has the advantage that the barrier transgression problem can be avoided. Nevertheless, this imposes one technical challenge: how its parameters are related to the dwell time constraint and the system stability. We resolve this challenge by deriving a novel dwell time constraint in terms of the parameters of ϵ in Lemma 5.1, which differs from the existing dwell time constraints [105, 114] and proving that the resulted Lyapunov function does not impose extra dwell time constraints.

5.2.4 Independent Switching

So far, the theoretical results are obtained with the assumption that the reference PWA system (5.2) and the controlled PWA system (5.1) switch synchronously, where the switches depend on the state of the controlled PWA system. To show how the dwell time constraint (5.14) can be satisfied, we consider a more general case, where the reference PWA system switches based on its own state space partitions $x_m \in \{\Omega_i^*\}_{i=1}^{s^*}$. For $x \in \Omega_i$ and $x_m \in$

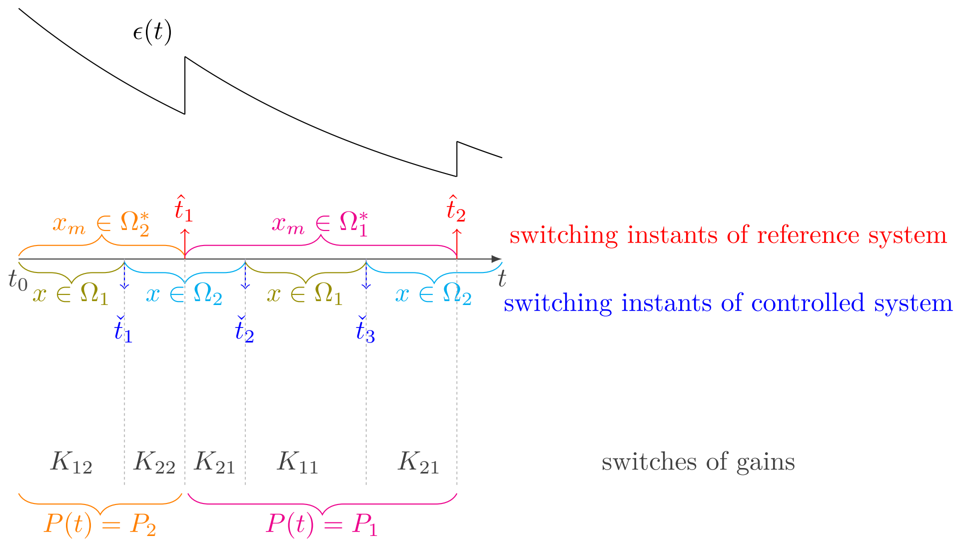


Figure 5.3: Reset of $\epsilon(t)$ for independent switching of the reference PWA system and the controlled PWA system. Switching instants of the reference PWA system and the controlled PWA system are displayed in red and blue, respectively. The value of $P(t)$ and the reset of $\epsilon(t)$ depend on the switches of the reference PWA system $\hat{t}_1, \hat{t}_2$, which can be pre-determined and checked when designing the reference system and the reference input. The switching of the controlled PWA system (see e.g. $\check{t}_2, \check{t}_3$), which cannot be determined in advance, does not affect the value of $P(t)$. So even when the controlled PWA system switches within the interval $[\hat{t}_1, \hat{t}_2)$, the analysis of the closed-loop stability in this interval can be conducted with a common Lyapunov function framework based on Corollary 5.1 with ϵ being the jump-free barrier.

Ω_j^*, a set of controllers $K_{xij}, K_{rij}, K_{fij}$ is activated for adaptations, whose nominal values $K_{xij}^*, K_{rij}^*, K_{fij}^*$ satisfy the matching equations for $\{A_i, B_i, f_i\}$ and $\{A_{mj}, B_{mj}, f_{mj}\}$. At the switching instants $\{\hat{t}_k\}_{k\in\mathbb{N}^+}$ of the reference PWA system, i.e., $x(\hat{t}_k^-)$ and $x(\hat{t}_k) \in \Omega_i, x_m(\hat{t}_k^-) \in \Omega_j, x_m(\hat{t}_k) \in \Omega_l, j \neq l$, we have $P(\hat{t}_k^-) = P_j, P(\hat{t}_k) = P_l$. The reset of ϵ is triggered; At the switching instants $\{\check{t}_k\}_{k\in\mathbb{N}^+}$ of the controlled PWA system, i.e., $x(\check{t}_k^-) \in \Omega_i, x(\check{t}_k) \in \Omega_l, i \neq l, x_m(\check{t}_k^-)$ and $x_m(\check{t}_k) \in \Omega_j$, we have a common $P(\check{t}_k^-) = P(\check{t}_k) = P_j$. ϵ is not reset at $\check{t}_k$. So within each interval $[\hat{t}_{k-1}, \hat{t}_k)$, the analysis follows a common Lyapunov setting shown in Corollary 5.1; Over the whole time interval $\cup_k[\hat{t}_{k-1}, \hat{t}_k)$, the stability argumentation follows Theorem 5.1. The above analysis shows that only $\{\hat{t}_k\}_{k\in\mathbb{N}^+}$ of the reference system have to satisfy the dwell time constraint. Since the reference PWA system is designed by the user, the dwell time constraint can be fulfilled by properly designing the reference input and the reference PWA system offline and can be checked in advance. Fig. 5.3 also provides a graphical illustration of the above explanations.

5.3 Indirect Adaptive Control Design: Variant 1

The core idea of the direct MRAC for PWA systems with state tracking performance guarantees proposed in the previous section lies in the use of the barrier function (5.26) to replace the conventional quadratic error term $e^T(\sum_{i=1}^{s} \chi_i P_i)e$ (see e.g. the conventional Lyapunov function (18) in [137]), which may exhibit incremental jump behavior at switching instants. In this section, we explore the indirect counterpart of this method. A natural idea is to inherit the barrier function concept to replace the quadratic prediction error term of the

Lyapunov function of the indirect adaptation case (see (2.35)). As it will show later in this section, this idea would not guarantee the constraint on the state tracking performance to be satisfied. Nevertheless, compared with other indirect MRAC approaches ([83, Thm. 3] and the approach shown in Chapter 3), where the existence of a CLF is required, the approach presented in this section fills the theoretical gap of the indirect adaptive control of PWA systems in the MLF setting with provable stability guarantee. Based on the result presented in this section, we will further present an alternative to fulfill the state tracking performance constraint in Section 5.4.

5.3.1 Adaptation Laws

Assume that for the reference system (5.2), there exist positive definite matrices $P_i, Q_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}$ such that

$$A_{mi}^T P_i + P_i A_{mi} + 2hP_i = -Q_i, \quad i \in \mathcal{I} \tag{5.52}$$

where the positive constant $h \in \mathbb{R}^+$ is defined in (5.12). In indirect adaptive control, the unknown system parameters need to be identified while tackling the tracking task. So recall that $\hat{A}_i, \hat{B}_i, \hat{f}_i$ denote the estimated values of A_i, B_i and f_i. The estimated parameters are updated based on the state information x and the predicted state, denoted by $\hat{x} \in \mathbb{R}^n$, whose dynamics can be written as

$$\dot{\hat{x}} = A_m\hat{x} + \sum_{i=1}^{s}((\hat{A}_i - A_{mi})x + \hat{B}_i u + \hat{f}_i)\chi_i. \tag{5.53}$$

Define $\tilde{A}_i = \hat{A}_i - A_i$, $\tilde{B}_i = \hat{B}_i - B_i$, and $\tilde{f}_i = \hat{f}_i - f_i$ to be the parameter estimation errors. By (5.1) and (5.53) we obtain

$$\dot{\tilde{x}} = A_m\tilde{x} + \sum_{i=1}^{s}(\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i)\chi_i \tag{5.54}$$

with $\tilde{x} = \hat{x} - x$ representing the state prediction error. Equation (5.54) relates $\tilde{x}$ with the parameter estimation errors $\tilde{A}_i, \tilde{B}_i, \tilde{f}_i$.

Before we derive the adaptation laws for parameters and control gains, we first define the prediction error metric

$$\|\tilde{x}\|_P = \tilde{x}^T(\sum_{i=1}^{s}\chi_i P_i)\tilde{x}. \tag{5.55}$$

$\|\tilde{x}\|_P$ is piecewise continuous and piecewise differentiable. Moreover, the following property holds at each switching instant t_k

$$\|\tilde{x}(t_k)\|_P^2 \le \mu\|\tilde{x}(t_k^-)\|_P^2, \quad \text{with } \mu = \max_{i,j\in\mathcal{I}} \frac{\lambda_{\max}(P_j)}{\lambda_{\min}(P_i)}. \tag{5.56}$$

$\lambda_{\max}(\cdot)$ and $\lambda_{\min}(\cdot)$ denotes the maximum and minimum eigenvalue of a symmetric matrix, respectively. Next, we define the following generalized restricted potential function (barrier Lyapunov function) $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ on the domain $\mathcal{D}_\epsilon(t) = \{\tilde{x} \mid \|\tilde{x}\|_P \in [0, \epsilon(t))\}$

$$\phi(\|\tilde{x}\|_P) = \frac{\|\tilde{x}\|_P^2}{\epsilon^2(t) - \|\tilde{x}\|_P^2}, \tag{5.57}$$

where $\epsilon(t)$, as the "barrier" of ϕ, is the time-varying piecewise continuous signal we used for the direct adaptation generated by the dynamics (5.12) with h satisfying (5.52). At each switching instant t_k, ϵ is reset by the reset map $\epsilon(t_k) = \sqrt{\mu}\epsilon(t_k^-)$. Considering the jump of $\|\tilde{x}\|_P$ at t_k shown in (5.56), the reset map of ϵ guarantees $\|\tilde{x}(t_k)\|_P < \epsilon(t_k)$ if $\|\tilde{x}(t_k^-)\|_P < \epsilon(t_k^-)$, i.e., the increment of $\|\tilde{x}\|_P$ at switching instants will not lead to invalidity of $\phi(\|\tilde{x}\|_P)$.

In every interval between two successive switching instants $[t_k, t_{k+1}), k \in \mathbb{N}^+$, ϕ is differentiable and its partial derivative with respect to $\|\tilde{x}\|_P^2$ is

$$\phi_d(\|\tilde{x}\|_P) \triangleq \frac{\partial \phi}{\partial \|\tilde{x}\|_P^2} = \frac{\epsilon^2(t)}{(\epsilon^2(t) - \|\tilde{x}\|_P^2)^2} > 0 \tag{5.58}$$

Furthermore, ϕ and ϕ_d have the relation that $2\phi_d(\|\tilde{x}\|_P)\|\tilde{x}\|_P^2 - \phi > 0$.

The indirect adaptation laws for estimated system parameters and control gains are again based on dynamic gain adjustment technique, which has been exploited in Chapter 3 and Chapter 4. Define the closed-loop estimation errors

$$\varepsilon_{Ai} = \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \quad \varepsilon_{Bi} = \hat{B}_i K_{ri} - B_{mi}, \quad \varepsilon_{fi} = \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}. \tag{5.59}$$

The adaptation of control gains is driven by these closed-loop estimation errors

$$\dot{K}_{xi} = -S_i^T B_{mi}^T \varepsilon_{Ai}, \quad \dot{K}_{ri} = -S_i^T B_{mi}^T \varepsilon_{Bi}, \quad \dot{K}_{fi} = -S_i^T B_{mi}^T \varepsilon_{fi}, \tag{5.60}$$

where $S_i, i \in \mathcal{I}$ are known matrices such that $K_{ri}^* S_i$ are symmetric and positive definite.

On the other hand, the estimation of system parameters is adjusted based on closed-loop estimation errors, the state prediction error $\tilde{x}$, the state x, and input u. Specifically,

$$\begin{aligned} \dot{\hat{A}}_i &= -\chi_i \phi_d(\|\tilde{x}\|_P) P_i \tilde{x} x^T - \varepsilon_{Ai}, \\ \dot{\hat{B}}_i &= -\chi_i \phi_d(\|\tilde{x}\|_P) P_i \tilde{x} u^T - \varepsilon_{Ai} K_{xi}^T - \varepsilon_{Bi} K_{ri}^T - \varepsilon_{fi} K_{fi}^T, \\ \dot{\hat{f}}_i &= -\chi_i \phi_d(\|\tilde{x}\|_P) P_i \tilde{x} - \varepsilon_{fi}. \end{aligned} \tag{5.61}$$

In (5.61), the partial derivative $\phi_d(\|\tilde{x}\|_P)$ is used as a time-varying gain of the first summand of each adaptation law, which differs from the classical adaptation laws (see (2.34)). Such arrangement as well as the adaptation of control gains (5.60) are suggested by Lyapunov-based stability analysis, which is shown in the next section.

5.3.2 Stability Analysis

We start by exploring under which condition the auxiliary "barrier" signal $\epsilon(t)$ is bounded. Unlike the fixed dwell time constraint introduced in Lemma 5.1, we explore here an average dwell time constraint, which is summarized in the following lemma.

Lemma 5.3. Given the piecewise continuous signal $\epsilon(t)$ defined in (5.12), if the number of switches within an interval (τ, t), denoted by $N(t, \tau)$, satisfies $N(t, \tau) \le N_0 + \frac{t-\tau}{\tau_D'}$ with $N_0 = \frac{2\alpha}{\ln\mu}$ and $\tau_D' = \frac{\ln\mu}{2(h-l)}$ for any positive constants $\alpha \in \mathbb{R}^+, l \in (0, h)$, then $\epsilon(t)$ is bounded.

Proof. For an arbitrary time interval $[t_0, t)$ containing the number of switches $N(t, t_0) = k$, namely, $t_0 < t_1 < \cdots < t_k < t, k \in \mathbb{N}^+$, we have

$$\begin{aligned}\epsilon(t) &= \epsilon(t_k)\mathrm{e}^{-h(t-t_k)} + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ &= \sqrt{\mu}\epsilon(t_k^-)\mathrm{e}^{-h(t-t_k)} + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau.\end{aligned} \tag{5.62}$$

Replacing $\epsilon(t_k^-)$ in (5.62) with

$$\epsilon(t_k^-) = \epsilon(t_{k-1})\mathrm{e}^{-h(t_k-t_{k-1})} + \int_{t_{k-1}}^{t_k^-} g\mathrm{e}^{-h(t_k-\tau)}\mathrm{d}\tau \tag{5.63}$$

leads to

$$\epsilon(t) = \mu^{\frac{1}{2}}\epsilon(t_{k-1})\mathrm{e}^{-h(t-t_{k-1})} + \mu^{\frac{1}{2}}\int_{t_{k-1}}^{t_k^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau.$$

Recursively doing the same derivation shown above yields

$$\begin{aligned}\epsilon(t) = {} & \mu^{\frac{k}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \mu^{\frac{k}{2}}\int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ & + \mu^{\frac{k-1}{2}}\int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau + \cdots + \mu^{\frac{0}{2}}\int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau.\end{aligned}$$

Note that $k = N(t, t_0)$, $k - 1 = N(t, t_1)$, etc. Therefore,

$$\begin{aligned}\epsilon(t) = {} & \mu^{\frac{N(t,t_0)}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \mu^{\frac{N(t,t_0)}{2}}\int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau \\ & + \mu^{\frac{N(t,t_1)}{2}}\int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau + \cdots + \mu^{\frac{N(t,t_k)}{2}}\int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mathrm{d}\tau.\end{aligned}$$

Since $N(t, \tau) = N(t, t_j)$ for $\tau \in [t_j, t_{j+1})$, the $\mu^{\frac{N(t,t_j)}{2}}, j \in \{0, 1, \cdots, k\}$ terms can be put into the integral operators, which yields

$$\begin{aligned}\epsilon(t) = {} & \mu^{\frac{N(t,t_0)}{2}}\epsilon(t_0)\mathrm{e}^{-h(t-t_0)} + \int_{t_0}^{t_1^-} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau \\ & + \int_{t_1}^{t_2^-} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau + \cdots + \int_{t_k}^{t} g\mathrm{e}^{-h(t-\tau)}\mu^{\frac{N(t,\tau)}{2}}\mathrm{d}\tau.\end{aligned}$$

Merging all the integral terms yields

$$\epsilon(t) = \epsilon(t_0)\mathrm{e}^{-h(t-t_0)+\frac{N(t,t_0)}{2}\ln\mu} + g\int_{t_0}^{t} \mathrm{e}^{-h(t-\tau)+\frac{N(t,\tau)}{2}\ln\mu}\mathrm{d}\tau.$$

To ensure the boundedness of $\epsilon(t)$, it suffices to let $-h(t-\tau) + \frac{N(t,\tau)}{2}\ln\mu \le \alpha - l(t-\tau)$ for some positive constants $\alpha \in \mathbb{R}^+$ and $l \in (0, h)$. This further leads to

$$N(t, \tau) \le N_0 + \frac{t-\tau}{\tau_D'} \tag{5.64}$$

with $N_0 \triangleq \frac{2\alpha}{\ln\mu}$ and $\tau_D' \triangleq \frac{\ln\mu}{2(h-l)}$. □

The average dwell time means that the switches do not necessarily fulfill a fixed dwell time constraint but are constrained in an average sense. Despite having the same form as the one in [65], (5.64) is derived based on the nonautonomous switched system (5.12), where g acts as a constant input. For nonautonomous switched systems with $\mathcal{L}_2$ input, this derivation can also be adjusted to the H_∞ performance analysis [178, 185]. Furthermore, applying Theorem 2 of [65], the reference model (5.2) is stable and x_m is bounded under the constraint (5.64).

Theorem 5.2. *Consider the reference system (5.2) satisfying (5.52) and the PWA system (5.1) with known regions Ω_i and unknown subsystem parameters A_i, B_i, f_i. Let the PWA system (5.1) be controlled by the adaptive controller (5.6) with adaptation laws (5.59), (5.60), and (5.61). If $\|\tilde{x}(t_0)\|_P < \epsilon(t_0)$ and the switch of the controlled PWA system satisfies the dwell time constraint (5.64), then the state tracking error $e = x - x_m \to 0$ as $t \to \infty$.*

Proof. Consider the following candidate Lyapunov function

$$V = \frac{1}{2}\phi(\|\tilde{x}\|_P) + \frac{1}{2}V_K + \frac{1}{2}V_\theta$$

where

$$\begin{aligned}
V_K &\triangleq \sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_{si}\tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si}\tilde{K}_{fi}) \\
V_\theta &\triangleq \sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\tilde{A}_i) + \mathrm{tr}(\tilde{B}_i^T\tilde{B}_i) + \tilde{f}_i^T\tilde{f}_i).
\end{aligned}$$

Similar as the proof of Theorem 5.1, we analyse the closed-loop stability with the following two phases:

phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

Suppose i-th subsystem is activated in the interval $[t_{k-1}, t_k)$ and the time-derivative of V in this interval is

$$\begin{aligned}
\dot{V} =& \frac{1}{2}\dot{\phi}(\|\tilde{x}\|_{P_i}) + \sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i) + \tilde{f}_i^T\dot{\tilde{f}}_i) \\
&+ \underbrace{\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i\dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i\dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i\dot{\tilde{K}}_{fi})}_{\triangleq v_k}
\end{aligned}$$

Considering (5.61), we expand the second summand of $\dot{V}$ as

$$\begin{aligned}
&\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T\dot{\tilde{B}}_i) + \tilde{f}_i^T\dot{\tilde{f}}_i) \\
=& -\phi_d(\mathrm{tr}(\tilde{A}_i^T P_i\tilde{x}x^T) + \mathrm{tr}(\tilde{B}_i^T P_i\tilde{x}u^T) + \tilde{f}_i^T P_i\tilde{x}) \\
&\underbrace{-\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\varepsilon_{A_i} + \tilde{B}_i^T(\varepsilon_{Ai}K_{xi}^T + \varepsilon_{Bi}K_{ri}^T + \varepsilon_{fi}K_{fi}^T)) + \tilde{f}_i^T\varepsilon_{fi})}_{\triangleq v_\varepsilon}
\end{aligned}$$

Inserting (5.60) into v_k we obtain

$$v_k - v_\varepsilon = -\sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T\varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T\varepsilon_{Bi}) + \varepsilon_{fi}^T\varepsilon_{fi}).$$

Detailed derivations of this step can be found in [83, Sec. IV]. Thus, we have

$$\begin{aligned}\dot{V} =&\frac{1}{2}\dot{\phi}(\|\tilde{x}\|_{P_i}) - \phi_d(\mathrm{tr}(\tilde{A}_i^T P_i \tilde{x} x^T) + \mathrm{tr}(\tilde{B}_i^T P_i \tilde{x} u^T) + \tilde{f}_i^T P_i \tilde{x}) \\ &- \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}).\end{aligned} \tag{5.65}$$

The time-derivative of ϕ can be further simplified as

$$\dot{\phi} = \frac{\partial \phi}{\partial \|\tilde{x}\|_{P_i}^2} \frac{\mathrm{d}\|\tilde{x}\|_{P_i}^2}{\mathrm{d}t} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = 2\phi_d \tilde{x}^T P_i \dot{\tilde{x}} + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \tag{5.66}$$

Substituting $\dot{\tilde{x}}$ in (5.66) with (5.54) yields

$$\dot{\phi} = \phi_d \tilde{x}^T (A_{mi}^T P_i + P_i A_{mi})\tilde{x} + 2\phi_d \tilde{x}^T P_i (\tilde{A}_i x + \tilde{B}_i u + \tilde{f}_i) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}. \tag{5.67}$$

Inserting (5.67) into (5.65) we obtain after some cancellations

$$\dot{V} = \frac{1}{2}\phi_d \tilde{x}^T (A_{mi}^T P_i + P_i A_{mi})\tilde{x} + \frac{1}{2}\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}).$$

We know that

$$\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} = \frac{-2\epsilon \|\tilde{x}\|_{P_i}^2}{(\epsilon^2 - \|\tilde{x}\|_{P_i}^2)^2}\dot{\epsilon} = -2\phi_d \|\tilde{x}\|_{P_i}^2 \frac{\dot{\epsilon}}{\epsilon} \le 2\phi_d \|\tilde{x}\|_{P_i}^2 \frac{|\dot{\epsilon}|}{\epsilon}.$$

From the dynamics of ϵ (5.12) we have $|\dot{\epsilon}|/\epsilon \le h$, which gives $\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \le 2h\phi_d \|\tilde{x}\|_{P_i}^2 = \phi_d \tilde{x}^T (2hP_i)\tilde{x}$. This together with (5.52) leads to

$$\dot{V} \le -\frac{1}{2}\phi_d \tilde{x}^T Q_i \tilde{x} - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}).$$

Therefore, V is non-increasing in between two successive switching instants $[t_{k-1}, t_k), k \in \mathbb{N}^+$.

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$

Now we analyse the behavior of V at switching instant t_k. Without loss of generality, we suppose that i-th subsystem is activated in $[t_{k-1}, t_k)$ and j-th subsystem is activated in $[t_k, t_{k+1})$, where $i, j \in \mathcal{I}, i \neq j$. From (5.56) we have

$$\begin{aligned}\phi(\|\tilde{x}(t_k)\|_P) &= \frac{\|\tilde{x}(t_k)\|_{P_j}^2}{\epsilon^2(t_k) - \|\tilde{x}(t_k)\|_{P_j}^2} \le \frac{\mu\|\tilde{x}(t_k^-)\|_{P_i}^2}{\epsilon^2(t_k) - \mu\|\tilde{x}(t_k^-)\|_{P_i}^2} \\ &= \frac{\mu\|\tilde{x}(t_k^-)\|_{P_i}^2}{\mu\epsilon^2(t_k^-) - \mu\|\tilde{x}(t_k^-)\|_{P_i}^2} = \phi(\|\tilde{x}(t_k^-)\|_P).\end{aligned}$$

Since V_K and V_θ remain unchanged at t_k, we have $V(t_k) \le V(t_k^-), k \in \mathbb{N}^+$.

According to the above analysis, $\dot{V}$ is negative semidefinite for $t \in [t_0, \infty)$. Thus, we have $\phi \in \mathcal{L}_\infty$, $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$, and $\hat{A}_i, \hat{B}_i, \hat{f}_i \in \mathcal{L}_\infty$, which, according to the definition (5.59), leads to $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty$. The boundedness of ϕ implies that $\|\tilde{x}(t)\|_P < \epsilon(t)$ for $\forall t \in [t_0, \infty)$. Since ϵ is bounded, we have $\tilde{x} \in \mathcal{L}_\infty$ and $\phi_d \in \mathcal{L}_\infty$. Integrating $\dot{V}$ over $[t_0, \infty)$, we obtain

$\int_{t_0}^{\infty} \dot{V} \mathrm{d}t = V(\infty) - V(t_0) \le -\int_{t_0}^{\infty} (\frac{1}{2}\phi_d \tilde{x}^T Q_i \tilde{x} + \sum_{i=1}^{s} (\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}))\mathrm{d}t$. Because ϕ_d, $V(\infty)$, and $V(t_0)$ are bounded, we conclude $\tilde{x}, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_2$. Letting $\varepsilon_A = \sum_{i=1}^{s} \chi_i \varepsilon_{Ai}$, $\varepsilon_B = \sum_{i=1}^{s} \chi_i \varepsilon_{Bi}$, $\varepsilon_f = \sum_{i=1}^{s} \chi_i \varepsilon_{fi}$ and inserting (5.6) and (5.59) into (5.53) yields

$$\dot{\hat{x}} = (A_m + \varepsilon_A)\hat{x} - \varepsilon_A \tilde{x} + (B_m - \varepsilon_B)r + f_m - \varepsilon_f.$$

This equation together with stable A_m, $r \in \mathcal{L}_\infty$, $\tilde{x} \in \mathcal{L}_\infty$, $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $\hat{x}, x \in \mathcal{L}_\infty$. According to (5.6) we have $u \in \mathcal{L}_\infty$. Bounded x, u imply bounded $\dot{\hat{A}}_i, \dot{\hat{B}}_i, \dot{\hat{f}}_i, \dot{\tilde{x}}$ and further $\dot{\varepsilon}_{Ai}, \dot{\varepsilon}_{Bi}, \dot{\varepsilon}_{fi} \in \mathcal{L}_\infty$.

Therefore, $\tilde{x}, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \to 0$ as $t \to \infty$. This together with $\hat{x} \to x_m$ leads to $e \to 0$ as $t \to \infty$. □

Remark 5.5. Similar to the discussion in Remark 5.2, the indirect adaptive control approach of this section achieves asymptotic tracking for switched systems without introducing extra conditions when compared to the classical MLF-based direct adaptive control approaches for switched systems (see Theorem 2.5 and Theorem 2.7).

Remark 5.6. The conventional MLF concept introduced in [65, 186] does not work for indirect adaptive control of PWA systems without the CLF. Due to the presence of $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$ in $\dot{V}$, the exponential decaying property of V in between successive switches cannot be established. No dwell time constraint can be found to compensate the increment of V at switches and the closed-loop stability cannot be obtained (see [83, Thm. 4]). Therefore, the indirect adaptive control reviewed in Section 2.2.1 and the approach developed in Chapter 3 require the existence of CLF ($P_i = P_j, \forall i \neq j$) to avoid jumps of V at each t_k. In contrast, the proposed Lyapunov function in this section is non-increasing at each switch instant and provable closed-loop stability can be established without the CLF.

5.3.3 Parameter Convergence

We explore here the parameter convergence property when applying the proposed approach.

Theorem 5.3. *Consider the reference system (5.2) satisfying (5.52) and the PWA system (5.1) with known regions Ω_i and unknown subsystem parameters A_i, B_i, f_i. Let the PWA system (5.1) be controlled by the adaptive controller (5.6) with adaptation laws (5.59), (5.60), and (5.61). Let $\|\tilde{x}(t_0)\|_P < \epsilon(t_0)$ and the switch of the controlled PWA system satisfies the dwell time constraint (5.64). If the input matrices B_i have full column rank, the pairs (A_{mi}, B_{mi}) are controllable, the system matrices A_{mi} are invertible, if the reference input r is sufficiently rich of order $n+1$ such that all subsystems are repeatedly activated, then the state tracking error $e \to 0$ and the estimated parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ as well as the estimated gains K_{xi}, K_{ri}, K_{fi} converge to their real or nominal values as $t \to \infty$.*

Proof. The stability and the asymptotic convergence of e has been proved in Theorem 5.2. In this proof, we remove the subscript i and let the following steps refer to the activated subsystem. Since all subsystems are activated intermittently, the convergence of estimated parameters of all the subsystems can be concluded.

Define $\tilde{\theta} = \mathrm{vec}([\tilde{A}\ \ \tilde{B}\ \ \tilde{f}])$ and $\Psi_u = [x^T, u^T, 1]^T \otimes I_n$ with $I_n \in \mathbb{R}^{n \times n}$ being the identity matrix. From (5.54) and (5.61), we can write the dynamics of the prediction error $\tilde{x}$ and parameter estimation error $\tilde{\theta}$ in compact form as

$$\begin{bmatrix} \dot{\tilde{x}} \\ \dot{\tilde{\theta}} \end{bmatrix} = \begin{bmatrix} A_m & \Psi_u^T \\ -\phi_d \Psi_u P & 0 \end{bmatrix} \begin{bmatrix} \tilde{x} \\ \tilde{\theta} \end{bmatrix} + \begin{bmatrix} 0 \\ \varepsilon \end{bmatrix} \tag{5.68}$$

where $\varepsilon = -\text{vec}([\varepsilon_A, \varepsilon_A K_x^T + \varepsilon_B K_r^T + \varepsilon_f K_f^T, \varepsilon_f])$. Define $X \triangleq [\tilde{x}^T, \tilde{\theta}^T]^T$, we can rewrite (5.68) as

$$\dot{X} = \bar{A}X + L\tilde{x} + d, \quad \tilde{x} = CX \tag{5.69}$$

where

$$\bar{A} = \begin{bmatrix} A_m & \Psi_u^T \\ -\Psi_u P & 0 \end{bmatrix}, \quad L = \begin{bmatrix} 0 \\ (1-\phi_d)\Psi_u P \end{bmatrix} \quad C^T = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

and $d = [0, \varepsilon^T]^T$. (5.69) reveals that the dynamics of X can be decomposed into a homogeneous part $\bar{A}X$, an output injection part $L\tilde{x}$ and a disturbance term d. It is proved in Theorem 5.2 that $\varepsilon \to 0$ and $\tilde{x} \to 0$ as $t \to \infty$, so $L\tilde{x} \to 0, d \to 0$. We can focus on proving the convergence property of the homogeneous part of (5.69): $\dot{X} = \bar{A}X$. Let $\bar{P} = \text{diag}\{P, I_{n(n+p+1)}\}$. We construct the Lyapunov function $V = X^T \bar{P} X$, whose derivative along the solution $\dot{X} = \bar{A}X$ is given by

$$\dot{V} = X^T(\bar{A}^T\bar{P} + \bar{P}\bar{A})X = -\tilde{x}(Q + 2hP)\tilde{x} \le -\nu \tilde{x}^T\tilde{x}$$

with $\nu = \lambda_{\min}(Q + 2hP) \in \mathbb{R}^+$. This further leads to $\dot{V} \le -\nu X^T C^T C X$. Invoking Lemma 2 of [83] and considering the conclusion of Theorem 5.2 that $e \to 0$, we have the signal vector $z = [x^T, u^T, 1]^T$ is PE if the reference signal r is sufficiently rich. Applying Lemma 2.2 we obtain $\tilde{\theta} \to 0$ and therefore, $\hat{A} \to A, \hat{B} \to B, \hat{f} \to f$ as $t \to \infty$. This together with $\varepsilon_A, \varepsilon_B, \varepsilon_f \to 0$ and B_i having full column rank gives $\tilde{K}_x, \tilde{K}_r, \tilde{K}_f \to 0$ as $t \to \infty$. □

Remark 5.7. The challenge of parameter convergence analysis of the barrier function-based approach proposed in this section lies in the presence of the time-varying gain ϕ_d in the joint dynamics of $[\tilde{x}, \tilde{\theta}]$ in (5.68), where the conventional analysis (see Theorem 2.6) cannot be applied. To bypass this issue, we transform the joint dynamics of $[\tilde{x}, \tilde{\theta}]$ into an output injection form (5.69), where ϕ_d is shifted into the injected output $\tilde{x}$. Since $\tilde{x} \to 0$ is proved in Theorem 5.2, the effect of ϕ_d on the parameter convergence decays to zero.

5.4 Indirect Adaptive Control Design: Variant 2

Variant 1 shown in the previous section enables the indirect MRAC for PWA systems in the MLF setting. However, it can only ensure the performance of the prediction error $\|\tilde{x}\|_P < \epsilon$ while it can make no statement about the relationship between the error metric of the tracking error $\|e\|_P$ and the auxiliary performance bound ϵ. To endow the indirect MRAC of PWA systems with state tracking performance guarantees, we explore an alternative approach in this section to satisfy the state tracking performance constraint.

5.4.1 Adaptation Laws

The same as Variant 1, we assume that for the reference system (5.2), there exist a set of positive definite matrices $P_i, Q_i \in \mathbb{R}^{n\times n}, i \in \mathcal{I}$ such that

$$A_{mi}^T P_i + P_i A_{mi} + 2hP_i = -Q_i, \quad i \in \mathcal{I} \tag{5.70}$$

where the positive constant $h \in \mathbb{R}^+$ is defined in (5.12). We propose the following adaptation laws for the controller gains

$$\begin{aligned}\dot{K}_{xi} &= -\chi_i\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e x^T - S_i^T B_{mi}^T \varepsilon_{Ai},\\ \dot{K}_{ri} &= -\chi_i\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e r^T - S_i^T B_{mi}^T \varepsilon_{Bi},\\ \dot{K}_{fi} &= -\chi_i\phi_d(\|e\|_P)S_i^T B_{mi}^T P_i e - S_i^T B_{mi}^T \varepsilon_{fi}.\end{aligned} \tag{5.71}$$

Note that the time-varying gain ϕ_d in (5.71) refers to $\phi_d(\|e\|_P)$ and should be distinguished from $\phi_d(\|\tilde{x}\|_P)$ in (5.61). Recall that $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$ are closed-loop estimation errors

$$\varepsilon_{Ai} = \hat{A}_i + \hat{B}_i K_{xi} - A_{mi}, \quad \varepsilon_{Bi} = \hat{B}_i K_{ri} - B_{mi}, \quad \varepsilon_{fi} = \hat{f}_i + \hat{B}_i K_{fi} - f_{mi}. \tag{5.72}$$

The adaptation laws for the parameter estimation are

$$\begin{aligned}\dot{\hat{A}}_i &= -\varepsilon_{Ai},\\ \dot{\hat{B}}_i &= -\varepsilon_{Ai}K_{xi}^T - \varepsilon_{Bi}K_{ri}^T - \varepsilon_{fi}K_{fi}^T,\\ \dot{\hat{f}}_i &= -\varepsilon_{fi}.\end{aligned} \tag{5.73}$$

5.4.2 Stability Analysis

The tracking performance and the stability of the closed-loop system using the indirect adaptive control are summarized in the following theorem.

Theorem 5.4. *Given the reference PWA system (5.2) satisfying (5.70) and the predefined performance function (5.8), let the PWA system (5.1) with known regions $\Omega_i, i \in \mathcal{I}$ and unknown subsystem parameters $A_i, B_i, f_i, i \in \mathcal{I}$ be controlled by the feedback controller (5.6) with the adaptation laws (5.71), (5.72), and (5.73). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. The closed-loop system is stable and the state tracking error $e(t)$ fulfills the prescribed performance guarantees (5.9) if the time constant h in (5.12) satisfies (5.70) and if the switching signal of the controlled PWA system obeys the dwell time constraint in (5.14).*

Proof. Consider the following Lyapunov function

$$V = \frac{1}{2}\phi(\|e\|_P) + \frac{1}{2}V_K + \frac{1}{2}V_\theta. \tag{5.74}$$

with

$$\begin{aligned}V_K &\triangleq \sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_{si}\tilde{K}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_{si}\tilde{K}_{ri}) + \tilde{K}_{fi}^T M_{si}\tilde{K}_{fi})\\ V_\theta &\triangleq \sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T\tilde{A}_i) + \mathrm{tr}(\tilde{B}_i^T\tilde{B}_i) + \tilde{f}_i^T\tilde{f}_i).\end{aligned}$$

Similar to the proof of Theorem 5.1, the stability analysis can be divided into two phases:
phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

Suppose that the i-th subsystem is activated for $t \in [t_{k-1}, t_k)$ and $e(t_{k-1})$ satisfies the inequality $\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$. The time-derivative of V in $[t_{k-1}, t_k)$ is given by

$$\begin{aligned}\dot{V} = \frac{1}{2}\dot{\phi}(\|e\|_{P_i}) + \sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}) \\ + \sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T \dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T \dot{\tilde{B}}_i) + \tilde{f}_i^T \dot{\tilde{f}}_i).\end{aligned} \tag{5.75}$$

Taking the adaptation laws (5.71) into the second summand of (5.75) gives

$$\begin{aligned}&\sum_{i=1}^{s}(\mathrm{tr}(\tilde{K}_{xi}^T M_i \dot{\tilde{K}}_{xi}) + \mathrm{tr}(\tilde{K}_{ri}^T M_i \dot{\tilde{K}}_{ri}) + \tilde{K}_{fi}^T M_i \dot{\tilde{K}}_{fi}) \\ &= -\sum_{i=1}^{s}\chi_i \phi_d e^T P_i B_i(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{K}_{fi}) + \underbrace{\sum_{i=1}^{s}(-\mathrm{tr}(\tilde{K}_{xi}^T B_i^T \varepsilon_{Ai}) - \mathrm{tr}(\tilde{K}_{ri}^T B_i^T \varepsilon_{Bi}) - \tilde{K}_{fi}^T B_i^T \varepsilon_{fi})}_{\triangleq v_k}.\end{aligned} \tag{5.76}$$

Inserting the adaptation laws (5.73) into the third summand of (5.75) yields

$$\begin{aligned}&\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T \dot{\tilde{A}}_i) + \mathrm{tr}(\tilde{B}_i^T \dot{\tilde{B}}_i) + \tilde{f}_i^T \dot{\tilde{f}}_i) \\ &= -\sum_{i=1}^{s}(\mathrm{tr}(\tilde{A}_i^T \varepsilon_{A_i} + \tilde{B}_i^T(\varepsilon_{Ai}K_{xi}^T + \varepsilon_{Bi}K_{ri}^T + \varepsilon_{fi}K_{fi}^T)) + \tilde{f}_i^T \varepsilon_{fi}) \triangleq -v_\varepsilon.\end{aligned} \tag{5.77}$$

Note that

$$v_k - v_\varepsilon = -\sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}). \tag{5.78}$$

Detailed derivations of this step can again be found in [83, Sec. IV]. Therefore, (5.75) can be rewritten as

$$\begin{aligned}\dot{V} &= \frac{1}{2}\dot{\phi}(\|e\|_{P_i}) - \sum_{i=1}^{s}\chi_i \phi_d e^T P_i B_i(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{K}_{fi}) + v_k - v_\varepsilon \\ &= \frac{1}{2}\dot{\phi}(\|e\|_{P_i}) - \sum_{i=1}^{s}\chi_i \phi_d e^T P_i B_i(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{K}_{fi}) - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}).\end{aligned}$$

Following the same derivation as (5.34) and (5.35), $\dot{\phi}$ can be further simplified as

$$\dot{\phi} = -\phi_d e^T(A_{mi}^T P_i + P_i A_{mi})e + 2\sum_{i=1}^{s}\chi_i \phi_d e^T P_i B_i(\tilde{K}_{xi}x + \tilde{K}_{ri}r + \tilde{K}_{fi}) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon}. \tag{5.79}$$

Therefore, $\dot{V}$ can be further simplified as

$$\dot{V} = -\frac{1}{2}\phi_d e^T(A_{mi}^T P_i + P_i A_{mi})e - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}) + \frac{1}{2}\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \tag{5.80}$$

Because $\frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \leq 2h\phi_d(\|e\|_{P_i})\|e\|_{P_i}^2 = \phi_d(\|e\|_{P_i})e^T(2hP_i)e$ (see (5.39)), inserting (5.70) into (5.80) we have

$$\dot{V} \leq -\frac{1}{2}\phi_d e^T Q_i e - \sum_{i=1}^{s}(\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}). \tag{5.81}$$

Therefore, V decreases between two consecutive switches. ϕ and ϕ_d are bounded in $[t_{k-1}, t_k)$. Since $\|e(t_{k-1})\|_{P_i} < \epsilon(t_{k-1})$, we have $\|e(t)\|_{P_i} < \epsilon(t)$ for $\forall t \in [t_{k-1}, t_k)$.

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$

As the estimated controller gains and estimated parameters are continuous, we have $V_K(t_k^-) = V_K(t_k)$ and $V_\theta(t_k^-) = V_\theta(t_k)$. Similar as the analysis of *phase 2* in Theorem 5.1, we have $V(t_k) \leq V(t_k^-)$.

Therefore, the Lyapunov function is non-increasing at every switching time instant. This together with the fact $\dot{V} \leq 0$ in $[t_{k-1}, t_k)$ for $\forall k \in \mathbb{N}^+$ implies that $V(t)$ is non-increasing for $\forall t \in [t_0, \infty)$. This leads to $\phi, \tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$, and $\tilde{A}_i, \tilde{B}_i, \tilde{f}_i \in \mathcal{L}_\infty$, which further leads to $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$, $\hat{A}_i, \hat{B}_i, \hat{f}_i \in \mathcal{L}_\infty$ and $\dot{e} \in \mathcal{L}_\infty$. Besides, $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. Since ϵ is bounded, $e \in \mathcal{L}_\infty$, $\phi_d \in \mathcal{L}_\infty$. Integrating $\dot{V}$ over $[t_0, \infty)$, we obtain $\int_{t_0}^{\infty} \dot{V} \mathrm{d}t = V(\infty) - V(t_0) \leq -\int_{t_0}^{\infty} (\frac{1}{2}\phi_d e^T Q_i e + \sum_{i=1}^{s} (\mathrm{tr}(\varepsilon_{Ai}^T \varepsilon_{Ai}) + \mathrm{tr}(\varepsilon_{Bi}^T \varepsilon_{Bi}) + \varepsilon_{fi}^T \varepsilon_{fi}))\mathrm{d}t$. Because ϕ_d, $V(\infty)$, and $V(t_0)$ are bounded, we conclude $e, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_2$. Recall that $x_m \in \mathcal{L}_\infty$. This property and $\|e(t)\|_P < \epsilon(t) < \rho(t)$ lead to $x \in \mathcal{L}_\infty$, which together with $r, \phi_d \in \mathcal{L}_\infty$ implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$, $\dot{\hat{A}}_i, \dot{\hat{B}}_i, \dot{\hat{f}}_i \in \mathcal{L}_\infty$ and $u \in \mathcal{L}_\infty$. This further leads to $\dot{\varepsilon}_{Ai}, \dot{\varepsilon}_{Bi}, \dot{\varepsilon}_{fi} \in \mathcal{L}_\infty$, which together with $e, \varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \in \mathcal{L}_\infty \cap \mathcal{L}_2$ leads to $e \to 0$, $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi} \to 0$ as $t \to \infty$. □

5.4.3 Parameter Convergence

The parameter convergence property of Variant 2 is summarized in the following theorem.

Theorem 5.5. *Consider the reference system (5.2) satisfying (5.70) and the PWA system (5.1) with known regions Ω_i and unknown subsystem parameters A_i, B_i, f_i. Let the PWA system (5.1) be controlled by the adaptive controller (5.6) with adaptation laws (5.71), (5.72), and (5.73). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. If the input matrices B_i have full column rank, the pairs (A_{mi}, B_{mi}) are controllable, the system matrices A_{mi} are invertible, if the switch of the controlled PWA system satisfies the dwell time constraint (5.14) and if the reference input r is sufficiently rich of order $n+1$ such that all subsystems are repeatedly activated, then the state tracking error $e \to 0$ and the estimated parameters $\hat{A}_i, \hat{B}_i, \hat{f}_i$ as well as the estimated gains K_{xi}, K_{ri}, K_{fi} converge to their real or nominal values as $t \to \infty$.*

Proof. The stability and the asymptotic convergence of e has been proved in Theorem 5.4. Similar to the proof of Variant 1, we remove the subscript i for clarity purpose and let the following steps refer to the activated subsystem.

To obtain the joint dynamics of the tracking error e and parameter estimation errors of controller gains in form of (2.40), we first rewrite the error dynamics (5.7) as

$$\dot{e} = A_m e + \Psi_r^T \tilde{\vartheta} \tag{5.82}$$

with

$$\tilde{\vartheta} = \mathrm{vec}(B[\tilde{K}_x \;\; \tilde{K}_r \;\; \tilde{K}_f]), \quad \Psi_r = \begin{bmatrix} x \\ r \\ 1 \end{bmatrix} \otimes I_n \tag{5.83}$$

with $I_n \in \mathbb{R}^{n\times n}$ being the identity matrix. Besides, the adaptation law (5.71) leads to

$$\begin{aligned}\dot{\tilde{\vartheta}} &= \mathrm{vec}(B[\dot{\tilde{K}}_x \;\; \dot{\tilde{K}}_r \;\; \dot{\tilde{K}}_f]) \\ &= -\phi_d \Psi_r B M^{-1} B^T P e - \mathrm{vec}(B S^T B_m^T [\varepsilon_A \;\; \varepsilon_B \;\; \varepsilon_f] \\ &= -\phi_d \Psi_r \underbrace{B M^{-1} B^T P}_{\triangleq P_b} e - \underbrace{\mathrm{vec}(B M^{-1} B^T [\varepsilon_A \;\; \varepsilon_B \;\; \varepsilon_f])}_{\triangleq \varepsilon_e}\end{aligned} \tag{5.84}$$

From (5.82) and (5.84), we obtain the following joint dynamics

$$\begin{bmatrix} \dot{e} \\ \dot{\tilde{\vartheta}} \end{bmatrix} = \begin{bmatrix} A_m & \Psi_r^T \\ -\phi_d \Psi_r P_b & 0 \end{bmatrix} \begin{bmatrix} e \\ \tilde{\vartheta} \end{bmatrix} + \begin{bmatrix} 0 \\ -\varepsilon_e \end{bmatrix}. \tag{5.85}$$

Define $z_e \triangleq [e^T, \tilde{\vartheta}^T]^T$, we can rewrite (5.85) as

$$\dot{z}_e = \bar{A}_r z_e + L_r e + d_r, \quad e = C z_e \tag{5.86}$$

where

$$\bar{A}_r = \begin{bmatrix} A_m & \Psi_r^T \\ -\Psi_r P_b & 0 \end{bmatrix}, \quad L_r = \begin{bmatrix} 0 \\ (1-\phi_d)\Psi_r P_b \end{bmatrix}, \quad C^T = \begin{bmatrix} I_n \\ 0 \end{bmatrix},$$

and $d_r = [0, \varepsilon_e^T]^T$. Similar to the one shown in Variant 1, the joint dynamics of z_e (5.86) is decomposed into a homogeneous part $\bar{A}_r z_e$, an output injection part $L_r z_e$ and a disturbance term d_r. It is proved in Theorem 5.4 that $\varepsilon_e \to 0$ and $e \to 0$ as $t \to \infty$, so $L_r z_e \to 0, d_r \to 0$. We can focus on proving the convergence property of the homogeneous part of (5.86): $\dot{z}_e = \bar{A}_r z_e$. In fact, $\dot{z}_e = \bar{A}_r z_e$ is exactly the same as the equation shown in (2.40). According to Theorem 2.5 we have $\tilde{\vartheta} \to 0$. As B_i have full column rank, $\tilde{K}_x, \tilde{K}_r, \tilde{K}_f \to 0$ as $t \to \infty$. This together with $\varepsilon_A, \varepsilon_B, \varepsilon_f \to 0$ gives $\hat{A} \to A, \hat{B} \to B, \hat{f} \to f$ as $t \to \infty$. □

Two variants of the indirect MRAC for PWA systems are developed in this chapter, both of them use the intermediate variables, the closed-loop system errors $\varepsilon_{Ai}, \varepsilon_{Bi}, \varepsilon_{fi}$, to dynamically update the subsystem parameters and controller gains. This concept originates from the so-called *combined direct and indirect adaptive control* proposed in [47], where both the prediction error $\tilde{x}$ and the tracking error e are included in the Lyapunov function. One follow-up version developed in [48] renders the combined approach into the indirect approach by considering only the prediction error $\tilde{x}$ in the Lyapunov function. This approach constitutes the basis for the indirect MRAC for PWA systems (see the approach reviewed in Section 2.2.1, [83], our approach proposed in Chapter 3, and Variant 1 of the indirect approach in Section 5.3). Unlike Variant 1 presented in Section 5.3, Variant 2 is derived based on the error equation of the tracking error e instead of the prediction error $\tilde{x}$. On the one hand, Variant 2 preserves the advantages of Variant 1 that the MLF setting is allowed and the parameter convergence can be achieved. On the other hand, Variant 2 ensures the closed-loop system to satisfy the performance constraint of the state tracking, which is not guaranteed in Variant 1. Nevertheless, as the auxiliary performance bound ϵ needs to locate within the performance function ρ, a fixed dwell time constraint (5.14) is required to be satisfied in Variant 2 whereas a more flexible dwell time constraint, the average dwell time (5.64) is allowed in Variant 1.

5.5 Robust Modification

In the previous sections, the adaptive control approaches and the stability of the closed-loop systems are studied in the disturbance-free case. Since the PWA systems are commonly used as approximations of nonlinear systems, approximation errors exist. Besides, unmodeled dynamics and external disturbances cannot be neglected in real applications. In this section, we focus on the robust adaptive control design for PWA systems with approximation errors, unmodeled dynamics, and external disturbances, i.e., we consider

$$\dot{x}(t) = A(t)x(t) + B(t)u(t) + f(t) + d(x,u,t), \tag{5.87}$$

where $d(x,u,t) \in \mathbb{R}^n$ can denote the approximation error of the linearization, unmodeled dynamics or external disturbances. d is continuous and its norm is upper bounded, i.e., $|d| \leq \bar{d}$, where $\bar{d}$ is known. We propose the following robust adaptation laws

$$\begin{aligned} \dot{K}_{xi} &= -\chi_i \phi_d(\|e\|_P) S_i^T B_{mi}^T P_i e x^T + \chi_i F_{xi}, \\ \dot{K}_{ri} &= -\chi_i \phi_d(\|e\|_P) S_i^T B_{mi}^T P_i e r^T + \chi_i F_{ri}, \\ \dot{K}_{fi} &= -\chi_i \phi_d(\|e\|_P) S_i^T B_{mi}^T P_i e + \chi_i F_{0i} \end{aligned} \tag{5.88}$$

where $F_{xi} \in \mathbb{R}^{p\times n}, F_{ri} \in \mathbb{R}^{p\times p}, F_{0i} \in \mathbb{R}^p$ represent the projection terms to confine the estimated controller gains K_{xi}, K_{ri}, K_{fi} within some given bounds. The projection terms have no effect on the adaptation if K_{xi}, K_{ri}, K_{fi} are within their bounds, otherwise, the adaptation terminates (see Section 2.2.3 for detailed explanations). Here we make the assumption that a known matrix $S_i \in \mathbb{R}^{p\times p}$ as well as an unknown *diagonal* and positive definite matrix $M_i \in \mathbb{R}^{p\times p}$ exist such that $(K_{ri}^* S_i)^{-1} = M_i$.

Remark 5.8. For the robust adaptive control design, more prior information is required compared with the disturbance-free case. For our projection-based approach, M_i must be diagonal and the element-wise bounds of K_{xi}, K_{ri}, K_{fi} need to be known (see also work by Sang and Tao [133]). The leakage-based approach proposed by Yuan (see Theorem 2.8) requires M_i to be completely known because they are used in the leakage terms. The follow-up work [152] requires $\lambda_{max}(M_i^{-1})$ to satisfy some constraints associated with the leakage rates.

Remark 5.9. Regarding the input matrix, there is another popular formulation $\dot{x} = A_p x + B_p \Lambda u$ for linear systems appearing in many works inspired by aerospace applications [6, 7, 88], where B_p is known and Λ is an unknown diagonal matrix with strictly positive diagonal elements. Such arrangement of the input matrix is equivalent to our formulation. Specifically, we have $B = B_m S M$ (we remove the subscript i) in our notations. The unknown diagonal matrix Λ with strictly positive diagonal elements corresponds to the diagonal and positive definite matrix M in our case, while the known control direction B_p corresponds to the multiplication $B_m S$.

Besides, instead of the Lyapunov equation (5.3), we assume for the reference system (5.2) that positive definite matrices $P_i, Q_i, i \in \mathcal{I}$ exist such that

$$A_{mi}^T P_i + P_i A_{mi} + P_i = -Q_i, \quad i \in \mathcal{I}. \tag{5.89}$$

Before we proceed with the robustness analysis, another property of the potential function (5.26), which is useful for the analysis in this chapter, is given in the following lemma.

Lemma 5.4. For a positive constant $c \in \mathbb{R}^+$ and $c < \min_t \epsilon^2(t)$, the function $\phi(\|e\|_P)$ defined in (5.26) and its partial derivative ϕ_d with respect to $\|e\|_P^2$ satisfy

(1) $2\phi_d \cdot (\|e\|_P^2 - c) - \phi > 0$ for $\zeta < \|e\|_P^2 < \epsilon^2$

(2) $2\phi_d \cdot (\|e\|_P^2 - c) - \phi \leq 0$ for $\|e\|_P^2 \leq \zeta$

with $\zeta \triangleq \frac{-\epsilon^2+\sqrt{\epsilon^4+8\epsilon^2 c}}{2}$.

Proof. From the definition of ϕ given in (5.26) we have

$$2\phi_d \cdot (\|e\|_P^2 - c) - \phi = \frac{\|e\|_P^4 + \epsilon^2\|e\|_P^2 - 2c\epsilon^2}{(\epsilon^2 - \|e\|_P^2)^2}. \tag{5.90}$$

The denominator of (5.90) is positive and the sign of $2\phi_d \cdot (\|e\|_P^2 - c) - \phi$ is determined by the numerator, which can be viewed as a quadratic function $f(z) = z^2 + \epsilon^2 z - 2c\epsilon^2$ with $z = \|e\|_P^2$. We have $f(z) \leq 0$ for $z \in [\frac{-\epsilon^2-\sqrt{\epsilon^4+8\epsilon^2 c}}{2}, \frac{-\epsilon^2+\sqrt{\epsilon^4+8\epsilon^2 c}}{2}]$ and $f(z) > 0$ otherwise. Since ϕ, ϕ_d are defined over $\|e\|_P^2 \in [0, \epsilon^2)$ and $\frac{-\epsilon^2-\sqrt{\epsilon^4+8\epsilon^2 c}}{2} < 0$, it can be obtained that $2\phi_d \cdot (\|e\|_P^2 - c) - \phi > 0$ for $\zeta < \|e\|_P^2 < \epsilon^2$ and $2\phi_d \cdot (\|e\|_P^2 - c) - \phi \leq 0$ for $\|e\|_P^2 \leq \zeta$ with $\zeta = \frac{-\epsilon^2+\sqrt{\epsilon^4+8\epsilon^2 c}}{2}$. □

The control performance and the closed-loop stability by using the robust adaptive controller are summarized in the following theorem.

Theorem 5.6. *Given the reference PWA system (5.2) satisfying (5.89) and the predefined performance function (5.8), let the PWA system (5.1) with known regions $\Omega_i, i \in \mathcal{I}$ and unknown subsystem parameters $A_i, B_i, f_i, i \in \mathcal{I}$ be controlled by the feedback controller (5.6) with the adaptation laws (5.88). Let the initial state of ϵ satisfy $\|e(t_0)\|_P < \epsilon(t_0)$. The closed-loop system is stable and the state tracking error $e(t)$ satisfies the prescribed performance guarantees (5.9) if the time constant h in (5.12) satisfies*

$$h < \frac{1}{2}\min_{i\in\mathcal{I}} \frac{\lambda_{min}(Q_i)}{\lambda_{max}(P_i)}, \qquad \max_{i\in\mathcal{I}} \frac{\lambda_{max}(P_i)\bar{d}}{\sqrt{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)}} < \frac{g}{h}, \tag{5.91}$$

and if the switching signal of the controlled PWA system obeys the dwell time constraint in (5.14).

Proof. We propose the same Lyapunov function as (5.30). The stability analysis can also be divided into two phases as the one in Theorem 5.1.

phase 1: $t \in [t_{k-1}, t_k), k \in \mathbb{N}^+$

Following the same steps from (5.31) to (5.35) as in Theorem 5.1, we have

$$\begin{aligned}\dot{V} = -\phi_d e^T(A_{mi}^T P_i + P_i A_{mi})e + \phi_d(e^T P_i d + d^T P_i e) + \frac{\partial \phi}{\partial \epsilon}\dot{\epsilon} \\ + 2\phi_d(\text{tr}(\tilde{K}_{xi}^T M_i F_{xi}) + \text{tr}(\tilde{K}_{ri}^T M_i F_{ri}) + \tilde{K}_{fi}^T M_i F_{0i})\end{aligned} \tag{5.92}$$

Since M_i is diagonal, we have

$$\begin{aligned}&\phi_d(\text{tr}(\tilde{K}_{xi}^T M_i F_{xi}) + \text{tr}(\tilde{K}_{ri}^T M_i F_{ri}) + \tilde{K}_{fi}^T M_i F_{0i}) \\ =&\phi_d(\sum_{j=1}^{p}\sum_{l=1}^{n} m_i^{(j)}\tilde{k}_{xi}^{(jl)} f_{xi}^{(jl)} + \sum_{j=1}^{p}\sum_{l=1}^{p} m_i^{(j)}\tilde{k}_{ri}^{(jl)} f_{ri}^{(jl)} + \sum_{j=1}^{p} m_i^{(j)}\tilde{k}_{fi}^{(j)} f_{0i}^{(j)})\end{aligned} \tag{5.93}$$

with $\tilde{K}_{xi} = [\tilde{k}_{xi}^{(jl)}]$, $\tilde{K}_{ri} = [\tilde{k}_{ri}^{(jl)}]$, $\tilde{K}_{fi} = [\tilde{k}_{fi}^{(j)}]$, $F_{xi} = [f_{xi}^{(jl)}]$, $F_{ri} = [f_{ri}^{(jl)}]$, and $F_{0i} = [f_{0i}^{(j)}]$. $M_i = \text{diag}\{m_i^{(1)}, \cdots, m_i^{(p)}\}$. It can be verified that $\tilde{k}_{xi}^{(jl)} f_{xi}^{(jl)} \leq 0$, $\tilde{k}_{ri}^{(jl)} f_{ri}^{(jl)} \leq 0$ and $\tilde{k}_{fi}^{(jl)} f_{0i}^{(jl)} \leq 0$, which together with the fact that $m_i^{(j)} > 0, i \in \mathcal{I}, j = 1, \cdots, p$ leads to

$$\dot{V} \leq -\phi_d e^T (A_{mi}^T P_i + P_i A_{mi}) e + \frac{\partial \phi}{\partial \epsilon} \dot{\epsilon} + \phi_d (e^T P_i d + d^T P_i e). \tag{5.94}$$

Since P_i is positive definite, it can be written as $P_i = H_i H_i^T$ with H_i being a nonsingular matrix. The inequality (5.94) can be further transformed as

$$\begin{aligned} \dot{V} &\leq -\phi_d e^T (A_{mi}^T P_i + P_i A_{mi}) e + \frac{\partial \phi}{\partial \epsilon} \dot{\epsilon} + 2\phi_d e^T H_i H_i^T d \\ &\leq -\phi_d e^T (Q_i + P_i) e + \frac{\partial \phi}{\partial \epsilon} \dot{\epsilon} + \phi_d (e^T H_i H_i^T e + d^T H_i H_i^T d) \\ &= -\phi_d e^T Q_i e + \frac{\partial \phi}{\partial \epsilon} \dot{\epsilon} + \phi_d d^T H_i H_i^T d \\ &\leq -\phi_d |e|^2 (\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)) + \phi_d d^T P_i d \\ &\leq -\phi_d |e|^2 \kappa_i + \phi_d \lambda_{\max}(P_i) \bar{d}^2 \end{aligned} \tag{5.95}$$

with $\kappa_i \triangleq \lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)$. For P_i, Q_i and h satisfying the condition (5.91), we have $\kappa_i > 0$. Further analysis can be divided into two cases: $\|e\|_P^2 > \zeta_i$ and $\|e\|_P^2 \leq \zeta_i$, where

$$\zeta_i = \frac{-\epsilon^2 + \sqrt{\epsilon^4 + 8\epsilon^2 c_i}}{2}, \quad i \in \mathcal{I} \tag{5.96}$$

with $c_i \triangleq \frac{\lambda_{\max}^2(P_i)}{\kappa_i} \bar{d}^2$. From (5.91) we obtain

$$\epsilon(t)^2 \geq \frac{g^2}{h^2} > \max_{i \in \mathcal{I}} \frac{\lambda_{\max}^2(P_i) \bar{d}^2}{\lambda_{\min}(Q_i) - 2h\lambda_{\max}(P_i)} = \max_{i \in \mathcal{I}} \{ \frac{\lambda_{\max}^2(P_i)}{\kappa_i} \bar{d}^2 \} \geq c_i, \tag{5.97}$$

which further leads to

$$\zeta_i < \frac{-\epsilon^2 + \sqrt{\epsilon^4 + 8\epsilon^2 \cdot \epsilon^2}}{2} = \epsilon^2. \tag{5.98}$$

Case 1 $\|e\|_P^2 > \zeta_i$: invoking Lemma 5.4, inequality (5.95) can be further derived as

$$\dot{V} \leq -\frac{\kappa_i \phi_d}{\lambda_{\max}(P_i)} (\|e\|_P^2 - \frac{\lambda_{\max}^2(P_i)}{\kappa_i} \bar{d}^2) < -\frac{\kappa_i}{2\lambda_{\max}(P_i)} \phi < 0 \tag{5.99}$$

Case 2 $\|e\|_P^2 \leq \zeta_i$: defining $\kappa \triangleq \min_{i \in \mathcal{I}} \{\kappa_i\}$, $\alpha = \max_{i \in \mathcal{I}} \lambda_{\max}(P_i)$ and considering the property that $2\phi_d(\|e\|_P)\|e\|_P^2 - \phi > 0$, we have

$$\dot{V} \leq -\frac{\kappa}{2\alpha} \phi + \phi_d \alpha \bar{d}^2 = -\frac{\kappa}{2\alpha} (\phi + V_K) + \frac{\kappa}{2\alpha} V_K + \phi_d \alpha \bar{d}^2 \leq -\frac{\kappa}{2\alpha} V + \frac{\kappa}{2\alpha} V_K + \phi_{d_{\max}} \alpha \bar{d}^2 \tag{5.100}$$

with $\phi_{d_{\max}} = \max_{\|e\|_P^2 \leq \zeta} \phi_d(\|e\|_P^2) = \phi_d(\max_t \zeta) \in \mathcal{L}_\infty$ for $\zeta = \sum_{i=1}^s \chi_i \zeta_i$. V_K is defined in (5.30). $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi}$ are bounded due to the utilization of the projection, which leads to

$V_K \in \mathcal{L}_\infty$. Suppose $\bar{V}_K$ to be the maximum of V_K and let the positive number $\mathcal{B} \in \mathbb{R}^+$ be defined as

$$\mathcal{B} \triangleq \bar{V}_K + \frac{2\phi_{d_{\max}}\alpha^2\bar{d}^2}{\kappa}. \tag{5.101}$$

For $V \le \mathcal{B}$, V may increase. For $V > \mathcal{B}$, we have $\dot{V} < 0$ and therefore, V is decreasing. Combing *Case 1* and *Case 2*, we know that V is bounded for the interval $[t_{k-1}, t_k)$.

phase 2: jump at switch instant $t_k, k \in \mathbb{N}^+$

Following the same steps as shown in Theorem 5.1 and we have $V(t_k) \le V(t_k^-)$.

Based on the analysis of phase 1 and phase 2, we can conclude that

$$V(t) \le \max\{V(t_0), \mathcal{B}\}, \forall t \in [t_0, \infty), \tag{5.102}$$

from which we obtain $\phi, \phi_d \in \mathcal{L}_\infty$. The projection leads to $\tilde{K}_{xi}, \tilde{K}_{ri}, \tilde{K}_{fi} \in \mathcal{L}_\infty$, which further leads to $K_{xi}, K_{ri}, K_{fi} \in \mathcal{L}_\infty$. Besides, $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds for $\forall t \in [t_0, \infty)$. The prescribed performance guarantee (5.9) is satisfied.

With similar steps in the proof of Lemma 5.2, one can prove the stability of the reference system with (5.89), so we have $x_m \in \mathcal{L}_\infty$. This leads to $x \in \mathcal{L}_\infty$, which together with $r, \phi_d \in \mathcal{L}_\infty$ implies $\dot{K}_{xi}, \dot{K}_{ri}, \dot{K}_{fi} \in \mathcal{L}_\infty$. □

Remark 5.10. The leakage-based robust MRAC approach for switched linear systems reviewed in Section 2.2.3 obtains the boundedness of the Lyapunov function V by formulating the inequality $\dot{V} \le -\alpha V + \beta$, where $\alpha > 0$ and β is a disturbance-related term. This, however, does not apply to our approach, because the disturbance-related term in our case has a time-varying coefficient ϕ_d (see the term $\phi_d\lambda_{\max}(P_i)\bar{d}^2$ in (5.95)). The boundedness of ϕ_d cannot be concluded without proving the boundedness of V, while the boundedness of V requires $\phi_d\lambda_{\max}(P_i)\bar{d}^2$ to be bounded. This potential circular reasoning constitutes one of the main technical challenges of the robust modification. Our solution concept is employing the property of ϕ shown in Lemma 5.4 to discuss the stability in two separate cases. When $\|e(t)\|_P \le \zeta$, V may increase with ϕ_d and V upper bounded. V is strictly decreasing if $\|e(t)\|_P > \zeta$ for $\zeta = \sum_{i=1}^{s} \chi_i\zeta_i$.

Remark 5.11. In work about set-theoretic MRAC by Arabi and Yucelen [6, 7, 8], disturbances flow into the system through the same input matrix as the control signal. The fault-tolerant set-theoretic MRAC approach proposed by Xiao and Dong [168] also assumes the actuator fault and external disturbances to be matched, i.e., they can be compensated by designing additive terms in the control signal. Compared with these works, a distinctive feature of our approach is that the disturbance term d is also allowed to be unmatched.

Remark 5.12. According to (5.14), the length of the dwell time is governed by $\sqrt{\mu}$, the reset map of the auxiliary performance signal $\epsilon(t)$ (see (5.13)). By reducing $\sqrt{\mu}$, a less conservative dwell time constraint can be obtained. In the adaptive controllers introduced in this chapter, the reset map is defined with $\mu = \max_{i,j\in\mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_j)}$, which indicates the maximal possible jump of $\|e(t)\|_P^2$ at each switching instant. Since the current activated subsystem is known (supposed to be p), the maximal jump of $\|e(t)\|_P^2$ at next switching instant is $\mu_p = \max_{i\in\mathcal{I}} \frac{\lambda_{\max}(P_i)}{\lambda_{\min}(P_p)} \le \mu$. For the case where both current subsystem (supposed to be p) and the next subsystem to be switched on (supposed to be q) are known in advance, the maximal jump of $\|e(t)\|_P^2$ at this switching instant is $\mu_{pq} = \frac{\lambda_{\max}(P_q)}{\lambda_{\min}(P_p)} \le \mu$. Adopting $\sqrt{\mu_p}$ or $\sqrt{\mu_{pq}}$ instead of $\sqrt{\mu}$ as the reset map of ϵ yields a less conservative dwell time constraint. The

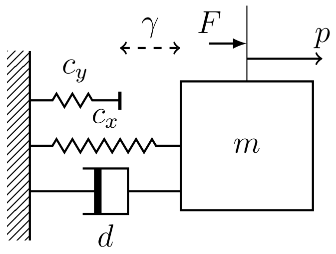


Figure 5.4: The mass-spring-damper system

corresponding stability properties of the reference system (5.2) and the closed-loop system are still retained. Such dwell time constraints are known as *mode-dependent dwell time* [42] (when μ_p is adopted) and *mode-mode-dependent dwell time* [174] (when μ_{pq} is utilized).

5.6 Numerical Validation

In this section, the proposed direct and indirect adaptive controllers will be validated through numerical examples.

5.6.1 Direct Adaptive Control

In this section, the proposed direct MRAC approach is validated through a numerical example modified based on the example of Section 3.4, a mass-spring-damper system, which is shown in Fig. 5.4. The displacement of the mass is denoted by p and the force operated on the mass is F, respectively. The mass is $m = 1\,\text{kg}$ and the damping factor is $d = 1\,\text{N s/m}$. The mass is connected to the static wall with the spring c_x and the damper d. For $|p| \leq \gamma = 1\,\text{m}$, the spring factor $c_x = 10\,\text{N/m}$. If it is extended beyond γ, i.e., $p > 1\,\text{m}$, the spring factor c_x is reduced to $c_x = 1\,\text{N/m}$. The spring $c_y = 90\,\text{N/m}$ is a floating spring with one end connected to the wall. The distance between the mass and the tip of the spring c_y is γ when c_x is in its resting position. The system is equivalent to a classical mass-spring-damper system with the spring exhibiting a PWA stiffness characteristics

$$F_c(p) = \begin{cases} c_1 = 10\,\text{N/m}, & \text{if } |p| \leq 1\,\text{m} \\ c_2 = 1\,\text{N/m}, & \text{if } p > 1\,\text{m} \\ c_3 = 100\,\text{N/m}, & \text{if } p < -1\,\text{m} \end{cases} \tag{5.103}$$

Let the state $x = [x_1, x_2]^T = [p, \dot{p}]^T$ and the input $u = F$. The system dynamics can be described by a PWA system in form of

$$\dot{x} = \begin{bmatrix} 0 & 1 \\ -\frac{c_i}{m} & -\frac{d}{m} \end{bmatrix} x + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} u + \begin{bmatrix} 0 \\ \bar{f}_i \end{bmatrix}, \quad i \in \{1, 2, 3\} \tag{5.104}$$

with $\bar{f}_1 = 0, \bar{f}_2 = (c_2 - c_1)/m, \bar{f}_3 = (c_1 - c_3)/m$. The region partitions are given as

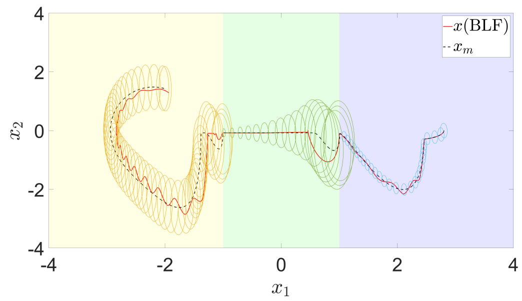
$$\Omega_1 = \{x \in \mathbb{R}^2 || x_1| \leq 1\}, \quad \Omega_2 = \{x \in \mathbb{R}^2 | x_1 > 1\}, \quad \Omega_3 = \{x \in \mathbb{R}^2 | x_1 < -1\}.$$

The reference system is a PWA system with the following subsystem matrices

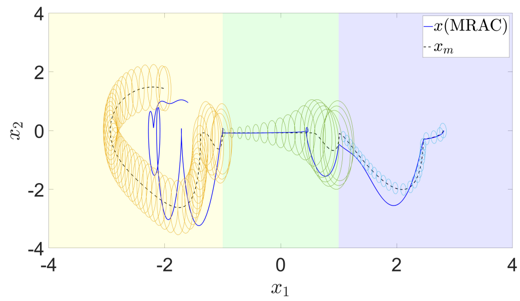
$$A_{m1} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix}, \quad B_{m1} = \begin{bmatrix} 0 \\ 25 \end{bmatrix}, \quad f_{m1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.105}$$

$$A_{m2} = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix}, \quad B_{m2} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}, \quad f_{m2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}, \tag{5.106}$$

$$A_{m3} = \begin{bmatrix} 0 & 1 \\ -49 & -14 \end{bmatrix}, \quad B_{m3} = \begin{bmatrix} 0 \\ 49 \end{bmatrix}, \quad f_{m3} = \begin{bmatrix} 0 \\ -10 \end{bmatrix}. \tag{5.107}$$



(a) Proposed approach



(b) Classical MRAC approach

Figure 5.5: Closed-loop system's trajectory by applying proposed direct adaptation method and the classical direct MRAC approach.

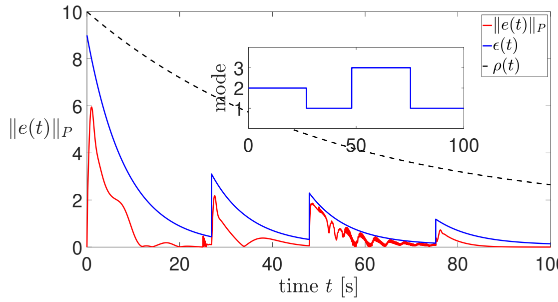
Ideal Case:

The adaptive controller in the ideal case with the adaptation laws (5.28) is tested. The P_i and Q_i matrices satisfying (5.3) are chosen as

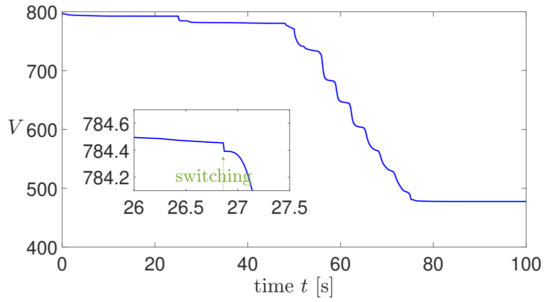
$$P_1 = \begin{bmatrix} 140 & 2 \\ 2 & 5.2 \end{bmatrix}, P_2 = \begin{bmatrix} 121.25 & 3.125 \\ 3.125 & 6.64 \end{bmatrix}, P_3 = \begin{bmatrix} 182.857 & 1.02 \\ 1.02 & 3.644 \end{bmatrix},$$
$$Q_1 = Q_2 = Q_3 = \begin{bmatrix} 100 & 10 \\ 10 & 100 \end{bmatrix},$$

which gives $\sqrt{\mu} = 7.1$. The scaling factors are $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} = 0.1$. The performance function is designed with $\rho_0 = 10, \rho_\infty = 1.5, l = 0.02$. We choose $\epsilon(t_0) = 9, h = 0.12$ and $g = 0.01$ such that the condition (5.29) and further conditions stated in Lemma 5.1 hold. Let the initial values of the reference system and the controlled PWA system be $[2, 0]^T$. The initial values of the estimated controller gains are specified as $K_{xi}(t_0) = 0.5K^*_{xi}, K_{ri}(t_0) = 0.5K^*_{ri}, K_{fi}(t_0) = 0.5K^*_{fi}, i \in \{1, 2, 3\}$. We use the following input signal r

$$r(t) = \begin{cases} 2 + 0.5\sin(0.2\pi t), & \text{for } 0\,\text{s} \le t < 25\,\text{s} \\ -0.08t + 2.8, & \text{for } 25\,\text{s} \le t < 50\,\text{s} \\ -2 + 0.8\sin(2t - 100 - \pi)), & \text{for } 50\,\text{s} \le t < 75\,\text{s} \\ 0, & \text{for } t \ge 75\,\text{s} \end{cases} \tag{5.108}$$



(a) State tracking error and performance bound



(b) Lyapunov function V

Figure 5.6: Tracking performance of the direct adaptive controller in ideal case.

The state-space trajectories of the reference system and the closed-loop system in the time interval [23 s, 52 s] are displayed in Fig. 5.5a with black dashed and red solid lines, respectively. The light blue, light green, and light yellow regions refer to Ω_2, Ω_1, and Ω_3. The ellipses centered at the state trajectory of the reference system represent $\|e(t)\|_P = \epsilon(t)$ and indicate the bounds of the state of the closed-loop PWA system. The colors of the ellipses distinguish $\|e(t)\|_{P_1}, \|e(t)\|_{P_2}$, and $\|e(t)\|_{P_3}$. We can observe that the state of the closed-loop system always stays within the auxiliary performance bound. For comparison, the state trajectory of the closed-loop system by using the direct MRAC approach [83] is displayed with blue solid lines in Fig. 5.5b, from which the violation of the performance bound can be observed.

According to Lemma 5.1, the dwell time of the closed-loop system should satisfy $T_D > 24$ s. The small window of Figure 5.6a shows the mode information of the closed-loop system. We can observe that the dwell time constraint is satisfied. In Figure 5.6a, the prescribed performance bound $\rho(t)$, the auxiliary performance bound $\epsilon(t)$ and the weighted norm of the state tracking error $\|e(t)\|_P$ are displayed with the black dashed line, the blue solid line and the red solid line, respectively. We can see that $\|e(t)\|_P < \epsilon(t) < \rho(t)$. The weighted norm of the state tracking error $\|e(t)\|_P$ and the auxiliary performance bound $\epsilon(t)$ jump at the switching instants, where the relation $\|e(t)\|_P < \epsilon(t)$ is still satisfied. This guarantees the potential function $\phi(t)$ to be valid and the control objective (5.9) to be fulfilled.

The Lyapunov function V is displayed in Figure 5.6b. We observe that the Lyapunov function V is non-increasing, also at the switching instants. This validates the theoretical statement given in Theorem 5.1.

Robust Case:

Now we test the performance of the robust adaptive controller with the adaptation laws (5.88). The PWA system is subject to an unmatched disturbance term $d = [0.036\cos(0.7t) + 0.072\sin(0.2t) + 0.018\sin(t), 0]^T$. The P_i, Q_i matrices satisfying (5.89) are chosen as

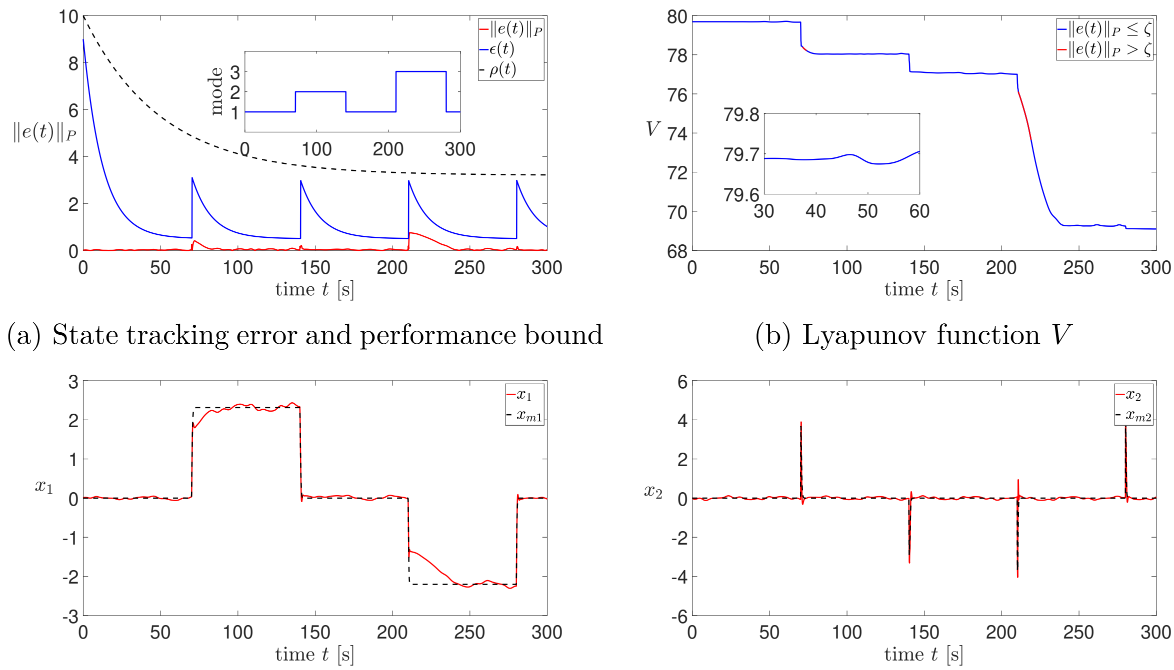
$$\begin{aligned} P_1 &= \begin{bmatrix} 0.7627 & 0.0353 \\ 0.0353 & 0.0458 \end{bmatrix}, P_2 = \begin{bmatrix} 0.6140 & 0.0504 \\ 0.0504 & 0.0601 \end{bmatrix}, P_3 = \begin{bmatrix} 0.7932 & 0.0183 \\ 0.0183 & 0.0236 \end{bmatrix}, \\ Q_1 &= Q_2 = \begin{bmatrix} 1 & 0.7 \\ 0.7 & 0.8 \end{bmatrix}, Q_3 = \begin{bmatrix} 1 & 0.6 \\ 0.6 & 0.6 \end{bmatrix}, \end{aligned} \tag{5.109}$$

which gives $\sqrt{\mu} = 5.86$. The scaling factors are $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} = 1$. The performance function is designed with $\rho_0 = 10, \rho_\infty = 3.2, l = 0.02$. The auxiliary performance signal is designed with $\epsilon(t_0) = 9, h = 0.08$ and $g = 0.04$ to fulfill the conditions in Lemma 5.1 and Theorem 5.6. The dwell time of the closed-loop system must satisfy $T_D > 67.7\,\mathrm{s}$. Let the initial values of the reference system and the controlled PWA system be $[0, 0]^T$. The initial values of the estimated controller gains are specified as $K_{xi}(t_0) = 0.5K^*_{xi}, K_{ri}(t_0) = 0.5K^*_{ri}, K_{fi}(t_0) = 0.5K^*_{fi}, i \in \{1, 2, 3\}$. The input signal r is

$$r(t) = \begin{cases} 0, & \text{for } 0\,\mathrm{s} + \frac{KT}{2} \leq t < 70\,\mathrm{s} + \frac{KT}{2} \\ 2, & \text{for } 70\,\mathrm{s} + KT \leq t < 140\,\mathrm{s} + KT \\ -2, & \text{for } 210\,\mathrm{s} + KT \leq t < 280\,\mathrm{s} + KT \end{cases} \tag{5.110}$$

with $K \in \mathbb{N}$ and $T = 280\,\mathrm{s}$.

The small window of Fig. 5.7a shows the switching information of the closed-loop system. It can be observed that the dwell time constraint $T_D > 67.7\,\mathrm{s}$ is satisfied. In Figure 5.7a, the black dashed line, the blue solid line, and the red solid line represent the prescribed performance bound $\rho(t)$, the auxiliary performance bound $\epsilon(t)$ and the weighted norm of the state tracking error $\|e(t)\|_P$, respectively. It can be seen that $\|e(t)\|_P < \epsilon(t) < \rho(t)$ holds. The element-wise tracking performance of the closed-loop system is displayed in Fig. 5.7c and Fig. 5.7d, where the black dashed lines represent the reference signals and the red solid lines represent the state signals. Despite the existence of the disturbance, the closed-loop state tracks the one of the reference system with the prescribed performance.



(a) State tracking error and performance bound

(b) Lyapunov function V

(c) state x_1

(d) state x_2

Figure 5.7: Tracking performance of the robust direct adaptive controller with disturbances.

The Lyapunov function V is shown in Fig. 5.7b. According to the proof of Theorem 5.6, V may increase when $\|e\|_P \leq \zeta$. In Fig. 5.7b, V is shown in red for $\|e\|_P > \zeta$ and in blue for

$\|e\|_P \leq \zeta$. We observe that V is decreasing for $\|e\|_P > \zeta$ whereas it may increase (as shown in the small window) but remain bounded for $\|e\|_P \leq \zeta$. This validates the theoretical result given in Theorem 5.6.

Independent Switching Case:

Now we validate the direct adaptive controller when the controlled PWA system and the reference PWA system switch independently, depicted in Section 5.2.4. The controlled PWA system has the same parameters and the same partitions ($\Omega_1 = \{x^T \in \mathbb{R}^2 | |x_1| \leq 1\}, \Omega_2 = \{x^T \in \mathbb{R}^2 | x_1 > 1\}, \Omega_3 = \{x^T \in \mathbb{R}^2 | x_1 < -1\}$) as those shown before. Now the reference PWA system has the subsystem matrices

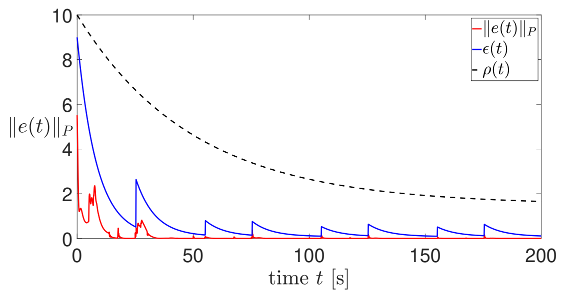
$$A_{m1} = \begin{bmatrix} 0 & 1 \\ -25 & -10 \end{bmatrix}, \quad B_{m1} = \begin{bmatrix} 0 \\ 25 \end{bmatrix}, \quad f_{m1} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \tag{5.111}$$

$$A_{m2} = \begin{bmatrix} 0 & 1 \\ -16 & -8 \end{bmatrix}, \quad B_{m2} = \begin{bmatrix} 0 \\ 16 \end{bmatrix}, \quad f_{m2} = \begin{bmatrix} 0 \\ 5 \end{bmatrix}. \tag{5.112}$$

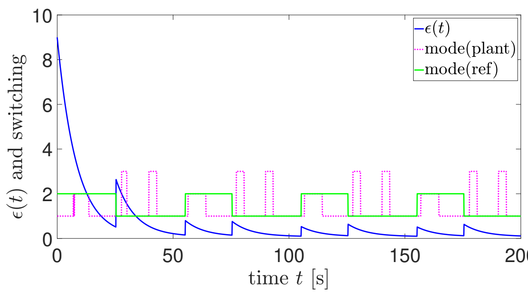
and its own state-space partitions

$$\Omega_1^* = \{x_m^T \in \mathbb{R}^2 | x_{m1} < 0\}, \quad \Omega_2^* = \{x_m^T \in \mathbb{R}^2 | x_{m1} \geq 0\}.$$

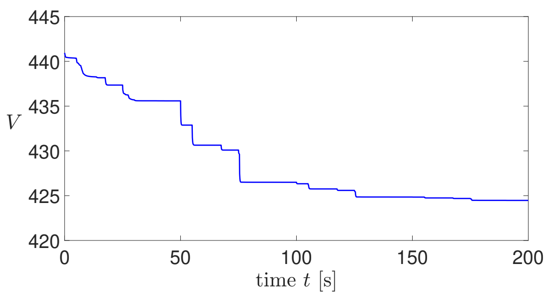
It switches independent of the controlled PWA system. The P_i and Q_i matrices satisfying



(a) State tracking error and performance bound



(b) $\epsilon(t)$ and switching information



(c) Lyapunov function V

Figure 5.8: Tracking performance of the independent switching case.

(5.5) are chosen as

$$P_1 = \begin{bmatrix} 140 & 2 \\ 2 & 5.2 \end{bmatrix}, P_2 = \begin{bmatrix} 121.25 & 3.125 \\ 3.125 & 6.64 \end{bmatrix}, Q_i = \begin{bmatrix} 100 & 10 \\ 10 & 100 \end{bmatrix} \quad \text{for } i \in \{1, 2\}, \tag{5.113}$$

which leads to $\sqrt{\mu} = 5.2$. The scaling factors are $\Gamma_{xi}, \Gamma_{ri}, \Gamma_{fi} = 1$. The performance parameters ρ_0, ρ_∞, l and $\epsilon(t_0), h, g$ are the same as those specified before. The barrier function is $\phi(\|e\|_P) = \frac{\|e\|_P^2}{\epsilon^2(t) - \|e\|_P^2}$, where $\epsilon(t)$, $P(t)$ and the reference PWA system switch synchronously. The resulted dwell time $T_D > 19.3\,\mathrm{s}$. Let the initial values of the controlled PWA system be $[-0.5, 0.5]^T$. The initial values of the estimated controller gains are specified as $K_{xij}(t_0) = 0.5K^*_{xij}, K_{rij}(t_0) = 0.5K^*_{rij}, K_{fij}(t_0) = 0.5K^*_{fij}, i \in \{1,2,3\}, j \in \{1,2\}$. The reference signal is given as $r = \bar{r} + 0.3\sin(0.5t + \pi)$ with

$$\bar{r}(t) = \begin{cases} 2, & \text{for } 5\,\mathrm{s} + KT \le t < 17.5\,\mathrm{s} + KT \\ -2, & \text{for } 25\,\mathrm{s} + KT \le t < 50\,\mathrm{s} + KT \\ 0, & \text{otherwise} \end{cases} \tag{5.114}$$

for $K \in \mathbb{N}$ and $T = 50\,\mathrm{s}$. This reference signal is designed such that it will result in a switching sequence of the reference PWA system satisfying the dwell time constraint.

From Fig. 5.8a we observe that the tracking performance is as expected that the error metric $\|e\|_P$ is confined within the performance bound (marked by black dashed line). Fig. 5.8b presents the switching information and how ϵ depends on it. Specifically, ϵ is shown in blue solid line. The mode information of the reference PWA system is displayed in green solid line and the mode information of the controlled PWA system is given in magenta dashed line. The switches of the reference system (the green solid line) can be tested in advance to determine that the dwell time constraint is satisfied before applying the controller to the controlled PWA system. The reset (jumps) of $\epsilon(t)$ is triggered only when the reference PWA system changes its mode and is not affected by the switching of the controlled PWA system.

Fig. 5.8c shows that the Lyapunov function V is non-increasing during the whole simulated time interval. These three figures validate the statements in Section 5.2.4.

5.6.2 Indirect Adaptive Control

Now we validate the two variants of the indirect adaptive control. Both of them will be applied to the PWA model of the pitch control of a helicopter system [132]. This PWA model can be written in form of (5.1) with the state $x = [x_1, x_2]^T \in \mathbb{R}^2$ denoting the vector of pitch angle and pitch rate. The state space for $x_1 \in [-\frac{3\pi}{5}, \frac{3\pi}{5}]$ is divided into 3 regions,

$$\Omega_1 = \{x| -\frac{\pi}{5} \le x_1 \le \frac{\pi}{5}\}, \quad \Omega_2 = \{x|\frac{\pi}{5} < x_1 < \frac{3\pi}{5}\}, \quad \Omega_3 = \{x| -\frac{3\pi}{5} < x_1 < -\frac{\pi}{5}\}. \tag{5.115}$$

The associated system parameters are from [132] and presented as follows.

$$A_1 = \begin{bmatrix} 0 & 1 \\ -10.5751 & -0.1447 \end{bmatrix}, \quad f_1 = \begin{bmatrix} 0 \\ -4.6265 \end{bmatrix},$$
$$A_2 = \begin{bmatrix} 0 & 1 \\ 1.9210 & -0.1447 \end{bmatrix}, \quad f_2 = \begin{bmatrix} 0 \\ 12.4780 \end{bmatrix},$$
$$A_3 = \begin{bmatrix} 0 & 1 \\ -8.1786 & -0.1447 \end{bmatrix}, \quad f_3 = \begin{bmatrix} 0 \\ -3.1208 \end{bmatrix},$$

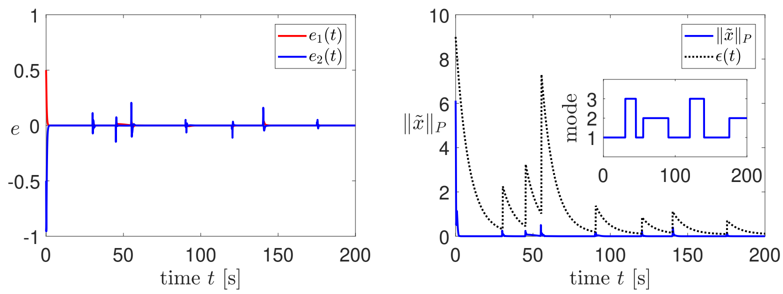
and $B_1 = B_2 = B_3 = [0, 35.3012]^T$. The reference PWA model has the same subsystem parameters as the one shown in (5.105) while its switching follows the same switching signal as the controlled PWA determined by the region partitions (5.115).

Variant 1

The auxiliary signal $\epsilon(t)$ is generated with $h = 0.12$, $g = 0.01$, and $\epsilon(0) = 9$. Numerical experiments show that larger h and smaller g may improve the tracking and parameter convergence rate but lead to smaller denominator of ϕ_d, which may cause ill-conditioned problems when solving it numerically. P_i and Q_i matrices satisfying (5.52) are chosen as

$$P_1 = \begin{bmatrix} 149.9448 & 2.7197 \\ 2.7197 & 5.3360 \end{bmatrix}, \quad P_2 = \begin{bmatrix} 131.7956 & 4.1135 \\ 4.1135 & 6.8672 \end{bmatrix},$$
$$P_3 = \begin{bmatrix} 192.2987 & 1.4913 \\ 1.4913 & 3.7098 \end{bmatrix}, \quad Q_1 = Q_2 = Q_3 = \begin{bmatrix} 100 & 10 \\ 10 & 100 \end{bmatrix}.$$

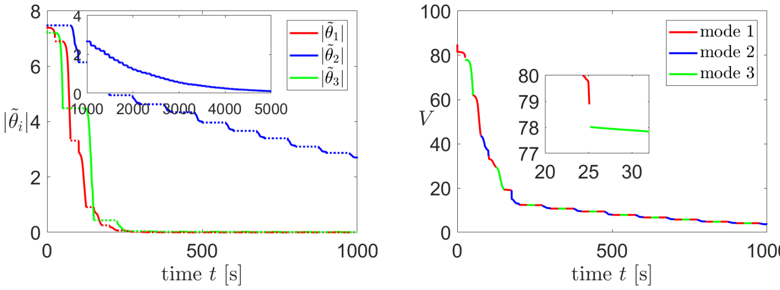
Therefore, by (5.56), $\mu = 52.0045$ and $\tau_D' > 16.5s$ for $l \in (0, h)$ (see (5.64)). Let $x(0) = [0.5, -0.5]^T$ and $x_m(0) = \hat{x}(0) = [0, 0]^T$. The initial values of the estimated system parameters are specified as 80% of their real values with zero initial gains. Given the reference input rectangular signal switching among $\{-1, 0, 1\}$, the tracking performance is shown in Fig. 5.9. One can observe that the tracking error $e \to 0$. Besides, each jump of $\epsilon(t)$ (marked by the black dashed line) indicates a switch instant. The overall switch (7 switches within 200s) is slower than τ_D' and the decrease of $\epsilon(t)$ together with the decaying $\|\tilde{x}\|_P$ validates Theorem 5.2.



(a) State tracking error e (b) Error metric $\|\tilde{x}\|_P$

Figure 5.9: Tracking performance of the indirect adaptive controller (Variant 1).

To test the parameter convergence property, let the input signal be $r = \bar{r} + 0.4\sin(1.2t) + 0.2\sin(11t)$ for $\bar{r}$ switching among $\{-1, 0, 1\}$ with a fixed interval 25s. It contains 2 distinct frequencies and is sufficiently rich of order $4 > n + 1 = 3$. $\bar{r}(t)$ is exerted such that the state of the closed-loop system is driven through all the partitioned regions. The estimation errors of system parameters are shown in Fig. 5.10a, where the parameters of different subsystems are distinguished with difference colors. The dashed sections represent the phase where the corresponding subsystem is inactive and the solid ones displaying the active phase. As we can see, $|\tilde{\theta}_1|$ and $|\tilde{\theta}_3|$ converge quite close to 0 within 300 s, while $|\tilde{\theta}_2|$ decreases relatively slower. The Lyapunov function V is displayed in Fig. 5.10b. Different colors indicate which mode is active. Although no common Lyapunov matrix is applied (because $P_i \neq P_j, i \neq j$), V is non-increasing at each switching instant and decreasing in between every two consecutive switches. This indicates the stability of the closed-loop system with the proposed method and validates the theoretical results.

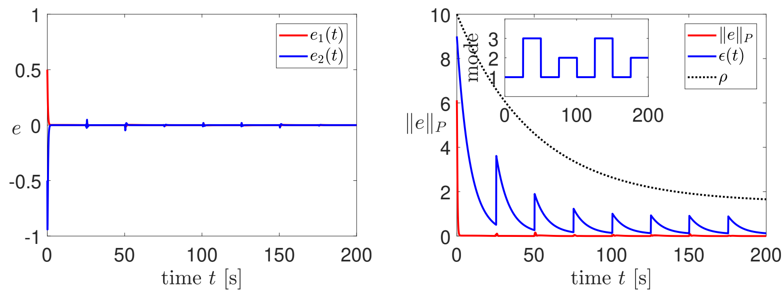


(a) Parameter convergence (b) Lyapunov function V

Figure 5.10: Parameter convergence and the Lyapunov function of the indirect adaptive controller (Variant 1).

Variant 2

Now we validate Variant 2 of the indirect MRAC of PWA systems using the same helicopter system. The initial states of the controlled PWA system and the reference system are $x(0) = [0.5, -0.5]^T$ and $x_m(0) = [0, 0]^T$. The initial values of the estimated subsystem parameters and the controller gains are the same as those in Variant 1. Besides, the configuration of ϵ as well as the selection of P_i, Q_i matrices are identical to those in Variant 1. As we aim to validate if Variant 2 is able to satisfy the performance constraint, we let the performance function designed with $\rho_0 = 10, \rho_\infty = 1.5, l = 0.02$, which leads to $T_D > 24\,\mathrm{s}$.



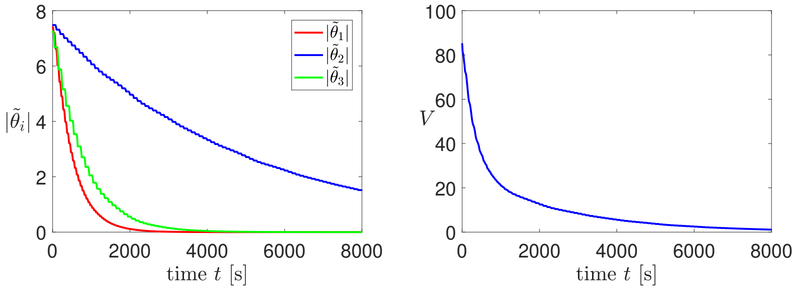
(a) State tracking error e (b) Error metric $\|e\|_P$

Figure 5.11: Tracking performance of the indirect adaptive controller (Variant 2).

First, we validate the tracking performance of Variant 2 using a reference input rectangular signal switching among $\{-1, 0, 1\}$ with a fixed interval $25s$. In Fig. 5.11a, each element of e is displayed. One can observe that $e \to 0$. Besides, as shown in Fig. 5.11b, the error metric $\|e\|_P$ satisfies $\|e\|_P < \epsilon < \rho$ and therefore, the performance constraint (5.9) is fulfilled.

Then, we test the parameter convergence of Variant 2 using the same input signal as the one in Variant 1: $r = \bar{r} + 0.4\sin(1.2t) + 0.2\sin(11t)$ for $\bar{r}$ switching among $\{-1, 0, 1\}$ with a fixed interval $25s$. From Fig. 5.12a, we can observe the convergence of $|\tilde{\theta}_i|, i \in \{1, 2, 3\}$ while $|\tilde{\theta}_2|$, similar as it is in Variant 1, decreases relatively slower than $|\tilde{\theta}_1|$ and $|\tilde{\theta}_3|$.

The Lyapunov function V displayed in Fig. 5.12b also validates the theoretical statement that V is non-increasing at each switching instant and decreasing in between every two consecutive switches and the closed-loop system is stable.



(a) Parameter convergence (b) Lyapunov function V

Figure 5.12: Parameter convergence and the Lyapunov function of the indirect adaptive controller (Variant 2).

5.7 Summary

In this chapter, we explored the direct and indirect MRAC approach for PWA systems with time-varying performance guarantees on the state tracking error. The proposed methods are based on barrier Lyapunov functions. In the direct MRAC design, to solve the barrier transgression problem caused by the discontinuity of the weighted Euclidean norm of the state tracking error, we introduce an auxiliary performance bound with a state reset map at switching instants to construct the barrier Lyapunov function. This auxiliary performance bound resides within the user-defined performance bound if some dwell time constraints are satisfied. The Lyapunov function is non-increasing at and in between the switching instants, which ensures the weighted Euclidean norm of the state tracking error to fulfill the performance guarantee. To enhance the robustness of the closed-loop system against unmatched disturbances, the projection-based robust modification of the proposed method is presented.

The non-increasing property of the Lyapunov function also enables a MLF setting for the indirect MRAC design for PWA systems. This overcomes the limitation of the classical indirect MRAC approaches which require the presence of the CLFs. Two variants of indirect MRAC are developed. Variant 1 allows an average dwell time constraint, which, compared to the fixed dwell time constraint of the direct MRAC, provides more design freedom. The drawback of this variant is that no statement can be made for the fulfillment of the performance constraints of the state tracking. To endow the indirect MRAC with the performance guarantees, Variant 2 is proposed, which combines the advantage of performance guarantees of the direct adaptation approach and the advantage of the parameter estimation of Variant 1 of the indirect adaptation approach.

Compared to the classical MRAC of PWA systems revisited in Chapter 2, one key feature of the proposed approaches is that the state tracking error satisfies a user-defined performance constraint in form of a time-varying performance bound, which prescribes both the transient and steady-state behaviors. This is essential for the safe operation of systems under uncertainties. Another key feature is that the Lyapunov functions exhibit non-increasing jumps at switching instants, which enjoys the advantage that, in the disturbance-free cases, no extra conditions need to be introduced to ensure closed-loop stability in the MLF setting. In contrast, the classical approaches require e.g., projections (see Theorem 2.7) or PE conditions (see Theorem 2.5) to establish the exponential decrease of the Lyapunov function in between switches such that possible increasing jumps at switching instants can be compensated given that the switches are slow enough.

The proposed approaches are validated through two-dimensional numerical examples for the purpose of better visualization of the safety constraints on the phase plane. Applications to PWA systems with higher dimensions are straightforward and thus, as pointed out in Chapter 3, the numerical examples reveal the applicability to a large class of real plants.

Similar to the output tracking case shown in Chapter 4, one drawback of the approaches presented in this chapter is that the saturation of the control input signal is not considered. Extending the current methods to the systems with input saturation would be an interesting topic for future work. Besides, it is also worth due to practical interest investigating the reduction of the jumps of the input signal of the PWA systems (or equivalently the output of the adaptive controllers u) at switching instants and taking into account the actuator dynamics [5] in future work.

Adaptive Control of PWA Systems with Output Feedback

6

In the previous chapters, the adaptive control problems for uncertain PWA systems with limited excitations and performance constraints are explored. Now we consider the challenge of sensor constraints, namely, only the output signal is available for the feedback. This chapter deals with sensor constraints by designing adaptive controller based on output feedback such that the output of the controlled system tracks the trajectory generated by a reference system.

Adaptive control methods based on output feedback is well-established for linear systems [71, 115]. Built upon this result, few works [134, 155, 156] develop the adaptive control methods for linear time-varying systems with parameter jumps or PWL systems based on output feedback, which guarantees small output tracking errors in the mean square sense. Nevertheless, these approaches cannot be applied to PWA systems due to the existence of the unknown affine terms. Moreover, the convergence of the controller parameters remains unexplored. To fill this gap, we investigate the output feedback MRAC for PWA counterparts with special focuses on the analysis of controller parameter convergence. It is a challenge to analyze the effect of the special controller structure for PWA systems on the excitation of the estimated parameters. Besides, the influence of the tracking error, as well as the switching behavior on the parameter convergence needs to be evaluated.

In this chapter, we first extend the controller proposed in [134] to the context of PWA systems and prove that the tracking error is small in the mean square sense under slow switching. Based on this result, we prove that the controller parameter estimation error converges to a bounded set given a PE reference signal. We establish the relationship between the size of the set and the switching frequency. Finally, we show that the convergence of the controller parameters to the nominal values can be achieved in a special case where the trajectory is kept staying in one subsystem for infinitely long.

The rest of the chapter is structured as follows. The problem formulation is presented in section 6.1. In section 6.2, our proposed control law is depicted. The tracking error as well as parameter convergence are investigated. The approach is validated by a numerical example presented in section 6.3, which is followed by the conclusion in section 6.4.

6.1 Problem Formulation

In this chapter, we focus on the SISO PWA system with the i-th subsystem described by

$$\begin{aligned} \dot{x} &= A_i x + B_i u + f_i, \quad i \in \mathcal{I} \\ y &= Cx, \end{aligned} \tag{6.1}$$

where $x \in \mathbb{R}^n$ represents the state. $y \in \mathbb{R}$ and $u \in \mathbb{R}$ represent the output and the control input. $A_i \in \mathbb{R}^{n\times n}$, $B_i, C_i^T \in \mathbb{R}^{n\times 1}$ and $f_i \in \mathbb{R}^n$ denote the unknown system parameters of i-th subsystem.

The input-output mapping of the PWA system when system i is activated is given by

$$y(t) = G_{pi}(s)[u](t) + G_{fi}(s)[1](t), \tag{6.2}$$

where

$$\begin{aligned} G_{pi}(s) &= k_{pi}\frac{Z_{pi}(s)}{R_{pi}(s)} = C(sI - A_i)^{-1}B_i, \\ G_{fi}(s) &= k_{fi}\frac{Z_{fi}(s)}{R_{pi}(s)} = C(sI - A_i)^{-1}f_i. \end{aligned} \tag{6.3}$$

The notation $y(t) = G(s)[u](t)$ represents the output in time-domain at time t of a system, which is characterized by transfer function $G(s)$ and input $u(t)$ [149]. Given a reference system

$$y_m(t) = W_m(s)[r](t), \quad W_m(s) = k_m\frac{Z_m(s)}{R_m(s)}, \tag{6.4}$$

where y_m is the reference output trajectory, $W_m(s)$ denotes the transfer function of the reference system and $r(t)$ represents the reference input signal.

The problem we would like to solve is formulated as follows:

Problem 6.1. Given a PWA system (6.1) with known subsystem partition Ω_i and unknown subsystem parameters, design a feedback control law based on output feedback, such that the plant output $y(t)$ tracks the reference trajectory $y_m(t)$.

Assumption 6.1. The assumptions are summarized as follows, which apply to the entire chapter.

- $Z_{pi}(s)$ is a monic strict Hurwitz polynomial of degree m
- The sign of $k_{pi}, i \in \mathcal{I}$ is assumed to be known
- $Z_m(s)$, $R_m(s)$ are monic strict Hurwitz polynomials
- $Z_{pi}(s)$ and $R_{pi}(s), i \in \mathcal{I}$ are coprime
- $Z_{fi}(s)/Z_{pi}(s), i \in \mathcal{I}$ is proper
- The relative degree of the plant is equal to that of the reference model
- The number of switches within time interval $[t, t+T)$, which is denoted by $N(t+T, t)$, satisfies $N(t+T, t) \le \bar{c} + \mu T, \forall t, T \ge 0$ for some positive constants $\bar{c}, \mu$
- Each polyhedral region Ω_i only depends on y and u and is assumed to be known

Assuming $Z_{pi}(s)$ and $Z_m(s)$ to be strict Hurwitz requires the reference model and each subsystem of the PWA to be minimum phase. The reference model is also stable since $R_m(s)$ is assumed to be strict Hurwitz. The term strict Hurwitz polynomial implies that the real parts of the roots are strictly negative.

The assumption on the number of the switches $N(t+T,t)$ over a time interval with the length of T constraints the average frequency of the switches among subsystems, which is characterized by μ. A small μ reveals slow switching. Limiting the frequency of switches is essential to ensure closed-loop stability and is, as also shown in the previous chapters, widely adopted in the area of adaptive control for switched systems [134, 136, 137].

Definition 6.1 (small in the mean square sense [71])**.** Let $x : [0,\infty) \mapsto \mathbb{R}^n$ with $x \in \mathcal{L}_{2e}$, and consider the set

$$\mathcal{S}(\mu) = \Big\{x| \int_t^{t+T} x^T(\tau)x(\tau)\mathrm{d}\tau \leq c_0\mu T + c_1, \forall t, T \geq 0\Big\}$$

for a given positive constant μ, where $c_0, c_1 \geq 0$ are some finite constants, and c_0 is independent of μ. x is said to be μ-small in the mean square sense, if $x \in \mathcal{S}(\mu)$.

6.2 Controller Design

In this section, the plant parameters are firstly assumed to be known in order to derive the nominal controller. The nominal controller parameters are determined by solving algebraic matching equations. Then the adaptive controller is discussed in the case where the plant parameters are unknown.

6.2.1 Nominal Control Design

Consider the feedback control law for i-th subsystem

$$u(t) = {\theta_{1i}^*}^T \frac{\alpha(s)}{\Lambda(s)}[u](t) + {\theta_{2i}^*}^T \frac{\alpha(s)}{\Lambda(s)}[y](t) + \theta_{3i}^* y(t) + c_{0i}^* r(t) + d_{0i}^*, \tag{6.5}$$

where $c_{0i}^*,\ d_{0i}^*,\ \theta_{3i}^* \in \mathbb{R}$, $\theta_{1i}^*,\ \theta_{2i}^* \in \mathbb{R}^{n-1}$ represent the nominal controller parameters to be designed, $\alpha(s) = [s^{n-2}, s^{n-3}, ..., s, 1]^T$, $\Lambda(s)$ is an arbitrary monic Hurwitz polynomial of degree $n-1$, which can be designed by the user, e.g., $\Lambda(s) = s^{n-1} + \lambda_{n-2}s^{n-2} + \cdots + \lambda_1 s + \lambda_0$. Inserting the control law into (6.2) yields the closed-loop behavior of i-th subsystem

$$y(t) = G_{ci}(s)[r](t) + F_{ci}(s)[1](t) \tag{6.6}$$

with

$$G_{ci}(s) = \frac{k_{pi} Z_{pi} c_{0i}^* \Lambda}{R_{pi}(\Lambda - {\theta_{1i}^*}^T \alpha) - k_{pi} Z_{pi}({\theta_{2i}^*}^T \alpha + \theta_{3i}^* \Lambda)} \tag{6.7}$$

representing the transfer function which relates the input and output signals and

$$F_{ci}(s) = \frac{k_{pi} Z_{pi} \Lambda d_{0i}^* + k_{fi} Z_{fi}(\Lambda - {\theta_{1i}^*}^T \alpha)}{R_{pi}(\Lambda - {\theta_{1i}^*}^T \alpha) - k_{pi} Z_{pi}({\theta_{2i}^*}^T \alpha + \theta_{3i}^* \Lambda)} \tag{6.8}$$

denoting the behavior caused by the affine term. The control goal is that the output of the closed-loop system tracks the output of the reference model. So we let the transfer function of the closed-loop system equal to the one of the reference model and use the final value theorem to enforce the affine term to decay to zero

$$\begin{aligned} G_{ci}(s) &= k_m \frac{Z_m(s)}{R_m(s)} \\ \lim_{s\to 0} sF_{ci}(s)\frac{1}{s} &= \lim_{s\to 0} F_{ci}(s) = 0, \end{aligned} \tag{6.9}$$

which leads to the matching equations

$$R_{pi}(\Lambda - {\theta_{1i}^*}^T\alpha) - k_{pi}Z_{pi}({\theta_{2i}^*}^T\alpha + \theta_{3i}^*\Lambda) = Z_{pi}\Lambda_0 R_m, \tag{6.10a}$$

$$k_{pi}Z_{pi}\Lambda d_{0i}^* + k_{fi}Z_{fi}(\Lambda - {\theta_{1i}^*}^T\alpha)|_{s\to 0} = 0. \tag{6.10b}$$

Here c_{0i} is chosen as $c_{0i} = \frac{k_m}{k_{pi}}$ and $\Lambda(s) = \Lambda_0(s)Z_m(s)$. The nominal control parameters are obtained by solving the algebraic matching equations.

Remark 6.1. Since the relative degree of the plant is equal to that of the reference model and $Z_{pi}(s)$ and $R_{pi}(s)$ are coprime, the left and right sides of (6.10a) have the same degree $2n-1$ without cancellation. This ensures the uniqueness of its solution.

Remark 6.2. Note that the final value theorem is utilized to eliminate the biasing effect of the affine term. The conditions of final value theorem are that the non-zero roots of the denominator of F_{ci} must have negative real parts and it must not have more than one zero-pole, which requires that Z_{pi}, Λ_0 and R_m are strict Hurwitz polynomials.

Remark 6.3. The second equation of (6.9) can be expanded as

$$\lim_{s\to 0} \frac{k_{pi}Z_m}{R_m}(d_{0i}^* + \frac{k_{fi}Z_{fi}(\Lambda - {\theta_{1i}^*}^T\alpha)}{k_{pi}Z_{pi}\Lambda_0 Z_m}) = 0. \tag{6.11}$$

Since Z_{fi}/Z_{pi} is proper, the second summand in the brackets is also proper. To ensure the existence of the solution of d_{0i}^*, it requires that $Z_m|_{s\to 0} \neq 0, R_m|_{s\to 0} \neq 0$ and $k_{pi}Z_{pi}\Lambda_0 Z_m|_{s\to 0} \neq 0$. These are achieved by applying the assumptions Z_{pi}, R_m, Z_m being strict Hurwitz polynomials and designing Λ_0 to be strict Hurwitz. Simplifying (6.11) further gives the equation (6.10b).

6.2.2 Error Model

We rewrite the nominal control law for the PWA system as

$$u(t) = \sum_{i=1}^{s} \chi_i {\theta_i^*}^T \omega(t), \tag{6.12}$$

where $\theta_i^* = [{\theta_{1i}^*}^T, {\theta_{2i}^*}^T, \theta_{3i}^*, c_{0i}^*, d_{0i}^*]^T$ is the control parameter vector and $\omega = [{\omega_1}^T, {\omega_2}^T, y, r, 1]^T$ with

$$\omega_1 = \frac{\alpha(s)}{\Lambda(s)}[u](t), \quad \omega_2 = \frac{\alpha(s)}{\Lambda(s)}[y](t). \tag{6.13}$$

Applying the nominal controller, the closed-loop system can be written in state-space form

$$\begin{aligned} \dot{x}_c &= A_{ci}x_c + B_{ci}r + f_{ci} \\ y &= C_c x_c, \end{aligned} \tag{6.14}$$

where $x_c = [x^T, \omega_1^T, \omega_2^T]^T$, $C_c = [C, 0_{2n-2}^T]$.

Lemma 6.1. For the closed-loop system (6.14), the equation $C_c A_{ci}^{-1} f_{ci} = 0$ holds for $\forall i \in \mathcal{I}$.

Proof. The effect of the i-th closed-loop affine term can be expressed by $F_{ci}(s)\frac{1}{s} = C_c(sI - A_{ci})^{-1} f_{ci}\frac{1}{s}$. Recalling (6.9) yields $\lim_{s\to 0} C_c(sI - A_{ci})^{-1} f_{ci} = 0$ and it follows $C_c A_{ci}^{-1} f_{ci} = 0$. □

We study the tracking error behavior when the nominal controller is applied. The following theorem states the smallness property of the tracking error.

Theorem 6.1. *Let the PWA system (6.1) with known subsystem partitioning Ω_i and known subsystem parameters be controlled by output feedback nominal controller (6.5). There exists $\mu_0 \in \mathbb{R}^+$ such that $\forall \mu \in (0, \mu_0)$, the output tracking error $e = y - y_m \in \mathcal{S}(\mu)$. Furthermore, $\lim_{t\to\infty} \sup_{\tau > t} |e(\tau)| \leq c\bar{r} + d$ for $|r(t)| \leq \bar{r}$ and some constants $c, d \in \mathbb{R}^+$.*

Proof. Let $(C_c, A_{ci_k}, B_{ci_k}, f_{ci_k})$ denote the active system over time interval $[t_k, t_{k+1})$, $k \in \mathbb{Z}^+$, $i_k \in \mathcal{I}$. The trajectory of y over time interval $[t_k, t_{k+1})$ is given by

$$\begin{aligned} y(t) = &\int_{t_k}^{t} C_c \Phi_c(t,\tau) B_{ci_k} r(\tau) \mathrm{d}\tau + C_c \Phi_c(t, t_k) x_c(t_k) \\ &+ \int_{t_k}^{t} C_c \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau, \end{aligned} \tag{6.15}$$

where $\Phi_c(t,\tau)$ denotes the associated closed-loop state transition matrix. The matching equation (6.9) ensures

$$\int_{t_k}^{t} C_c \Phi_c(t,\tau) B_{ci_k} r(\tau) \mathrm{d}\tau = \int_{t_k}^{t} C_m \Phi_m(t,\tau) B_m r(\tau) \mathrm{d}\tau, \tag{6.16}$$

which yields the tracking error at time $t \in [t_k, t_{k+1})$

$$\begin{aligned} e(t) =& y(t) - y_m(t) \\ =& C_c \Phi_c(t, t_k) x_c(t_k) - C_m \Phi_m(t, t_k) x_m(t_k) + \int_{t_k}^{t} C_c \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau. \end{aligned} \tag{6.17}$$

The eigenvalues of A_{ci} depend on Λ, Z_{pi}, R_m, so A_{ci} is stable and

$$\eta_{i_k}(t) \triangleq C_c^T \Phi_c(t, t_k) x_c(t_k) - C_m^T \Phi_m(t, t_k) x_m(t_k) \tag{6.18}$$

is exponentially decaying. Furthermore, according to the matching equation (6.10b), we have that the term

$$\varrho_{i_k}(t) \triangleq \int_{t_k}^{t} C_c^T \Phi_c(t,\tau) f_{ci_k} \mathrm{d}\tau, \tag{6.19}$$

which is the deviation caused by the affine term, decays to zero exponentially. The general expression of the tracking error e over an arbitrary time interval $[t, t+T)$ is

$$e(t) = \eta(t) + \varrho(t) \tag{6.20}$$

with $\eta(t) = \eta_{i_k}(t)$ and $\varrho(t) = \varrho_{i_k}(t)$ when $t \in [t_k, t_{k+1})$. It is proved in [134] that there exists $\mu_1 > 0$ such that $\forall \mu \in [0, \mu_1)$, $\eta \in \mathcal{S}(\mu)$. This indicates that if the switching is sufficiently slow, the error term η is small in the mean square sense. Following the same concept, there exists $\mu_2 > 0$, such that $\forall \mu \in [0, \mu_2)$, $\varrho \in \mathcal{S}(\mu)$, which together with (6.20) leads to $e \in \mathcal{S}(\mu)$ for $\forall \mu \in [0, \mu_0)$ with $\mu_0 = \min\{\mu_1, \mu_2\}$.

From (6.18)-(6.20) we have

$$|e| \leq |\eta| + |\varrho| \text{ with } |\eta| \leq |C_c| \|\Phi_c\| \max_k |x_c(t_k)| + |C_m| \|\Phi_m\| \max_k |x_m(t_k)|$$

and $|\varrho| \leq \max_k \int_{t_k}^{t_{k+1}} |C_c| \|\Phi_c\| |f_{ci}| \mathrm{d}\tau$. Based on the slow switching assumption and Theorem 2 in [65], we have $\|\Phi_c(t)\| \leq \lambda_c \mathrm{e}^{-\alpha_c t}$ for some $\lambda_c, \alpha_c > 0$. For the reference system we have $\|\Phi_m(t)\| \leq \lambda_m \mathrm{e}^{-\alpha_m t}$ for some $\lambda_m, \alpha_m > 0$. These lead to $|x_m(t)| \leq c_m \bar{r} + \epsilon_m$ and $|x_c(t)| \leq c_c \bar{r} + d_c + \epsilon_c$ for some $c_m, c_c, d_c > 0$ and exponentially decaying terms ϵ_m, ϵ_c, which in turn gives $\lim_{t \to \infty} \sup_{\tau > t} |e(\tau)| \leq c\bar{r} + d$ for some $c, d > 0$. □

Theorem 6.1 reveals that the tracking error exists even if the nominal control parameters are utilized and the matching equations for every subsystem hold. Once the system switches, the output deviates from the reference one. The deviation decays to zero provided that the trajectory stays in the subsystem for sufficiently long time (characterized by μ) until the next switch occurs.

6.2.3 Adaptive Control Design

Now consider the case where the plant parameters are unknown. In this case, the nominal control parameters cannot be determined by solving matching equations. The estimation of the controller parameters is utilized to implement the adaptive controller

$$u(t) = \sum_{i=1}^{l} \chi_i {\theta_i}^T \omega(t), \tag{6.21}$$

where $\theta_i = [{\theta_{1i}}^T, {\theta_{2i}}^T, \theta_{3i}, c_{0i}, d_{0i}]^T$ denotes the estimated parameter vector for i-th subsystem. The output of the system can then be expressed by the output of the reference system perturbed by the error of control parameters $\tilde{\theta}_i = \theta_i - \theta_i^*$ and the transient terms η, ϱ caused by switching

$$y(t) = W_m[r + \sum_{i=1}^{s} \chi_i \rho_i^* \tilde{\theta}_i^T \omega](t) + \eta(t) + \varrho(t) \tag{6.22a}$$

$$= W_m[r](t) + \sum_{i=1}^{s} \chi_i \rho_i^* W_m[\sum_{i=1}^{s} \chi_i \tilde{\theta}_i^T \omega](t) + \eta(t) + \varrho(t) \tag{6.22b}$$

with $\rho_i^* = 1/c_{0i}^*$. From (6.22a) to (6.22b), a swapping error term $W_m[\sum_{i=1}^{s} \chi_i \rho_i^* \tilde{\theta}_i^T \omega](t) - \sum_{i=1}^{s} \chi_i \rho_i^* W_m[\sum_{i=1}^{s} \chi_i \tilde{\theta}_i^T \omega](t)$ is neglected without loss of generality as the normalized version of this error term has the same property as η and ϱ, namely, it is also small in the mean square sense [134]. Define the estimation error for i-th subsystem as

$$\epsilon_i(t) = e(t) + \rho_i(t) \xi_i(t) \tag{6.23}$$

where

$$\xi_i(t) = \theta_i^T(t) \zeta(t) - W_m(s)[\theta_i^T \omega](t), \tag{6.24a}$$

$$\zeta(t) = W_m(s)[\omega](t) \tag{6.24b}$$

with $e(t) = y - y_m$ denoting the tracking error. The following update law is proposed

$$\dot{\theta}_i(t) = -\chi_i \text{Pr}[\frac{\text{sign}[k_{pi}]\Gamma_i \epsilon_i(t)\zeta(t)}{m^2(t)}],$$
$$\dot{\rho}_i(t) = -\chi_i \text{Pr}[\frac{\gamma_i \epsilon_i(t)\xi_i(t)}{m^2(t)}], \tag{6.25}$$

where $\text{Pr}[\cdot]$ is, as revisited in Section 2.2.3, the projection operator to confine the parameters within a bounded convex set, which is known as prior information. $\Gamma_i > 0$ and $\gamma_i > 0$ are adaptation gains, $m(t)$ is a dynamic normalizing signal defined by $m^2 = 1 + m_s$ with

$$\dot{m}_s(t) = -\delta_0 m_s + u^2 + y^2, \quad m_s(0) = 0, \tag{6.26}$$

where δ_0 is a positive constant. The following theorem describes the property of the tracking error in the adaptive case.

Theorem 6.2. *Let the PWA system (6.1) with known subsystem partitioning Ω_i and unknown subsystem parameters be controlled by output feedback controller (6.21) with the adaptation law (6.25). There exists $\mu_0 \in \mathbb{R}^+$ such that $\forall \mu \in (0, \mu_0)$, the output tracking error $e \in \mathcal{S}(\mu)$.*

Proof. From (6.23) and (6.24), it can be derived that

$$\epsilon_i = \rho_i^* \tilde{\theta}_i^T \zeta + \tilde{\rho}_i \xi_i + \eta_i + \varrho_i \tag{6.27}$$

with $\eta_i = \chi_i \eta$, $\varrho_i = \chi_i \varrho$. Consider the Lyapunov function

$$V(\tilde{\theta}, \tilde{\rho}) = \sum_{i=1}^{s} \chi_i (|\rho_i^*| \tilde{\theta}_i^T \Gamma_i^{-1} \tilde{\theta}_i + \gamma_i^{-1} \tilde{\rho}_i^2). \tag{6.28}$$

Taking the piecewise derivative of V along the trajectories of (6.25) yields

$$\dot{V} = -\sum_{i=1}^{s} \chi_i \frac{2\epsilon_i}{m^2} (\rho_i^* \tilde{\theta}_i^T \zeta + \tilde{\rho}_i \xi_i). \tag{6.29}$$

Inserting (6.27) into (6.29) yields

$$\dot{V} = -2\sum_{i=1}^{s} \frac{\chi_i \epsilon_i^2}{m^2} + 2\sum_{i=1}^{s} \frac{\chi_i \epsilon_i}{m^2} (\eta_i + \varrho_i). \tag{6.30}$$

Integrating over an arbitrary interval $[t, t+T)$, in which multiple switching may occur, yields

$$\int_t^{t+T} (\frac{\epsilon}{m})^2 \text{d}t \leq (V(t) - V(t+T)) + \frac{1}{m^2} \int_t^{t+T} (\eta + \varrho)^2, \tag{6.31}$$

with $\epsilon = \sum_{i=1}^{s} \chi_i \epsilon_i$. Because $\frac{\eta}{m} + \frac{\varrho}{m} \in \mathcal{S}(\mu)$, it follows $\frac{\epsilon}{m} \in \mathcal{S}(\mu)$. The rest of the proof can be divided into several steps as follows:

step 1: Express the input, output signals in terms of $\tilde{\theta}_{i_k}^T \omega$. Based on (6.22), y over the time interval $[t_k, t_{k+1})$ is expressed as

$$y(t) = W_m[r + \rho_{i_k}^* \tilde{\theta}_{i_k}^T \omega](t) + \eta_{i_k}(t) + \varrho_{i_k}(t). \tag{6.32}$$

Ignoring the effect of the exponentially decaying terms $\eta_{i_k}, \varrho_{i_k}$, the control signal u can be expressed by

$$u(t) = G_{pi_k}^{-1} W_m [r + \rho_{i_k}^* \tilde{\theta}_{i_k}^T \omega](t) - G_{pi_k}^{-1} [G_{fi_k}[1]](t) \tag{6.33}$$

$G_{pi_k}^{-1} G_{fi_k}$ and $G_{pi_k}^{-1} W_m$ are stable and proper. Define a fictitious normalizing signal $m_f^2 = 1 + \mathrm{e}^{-\delta(t-t_k)} m^2(t_k) + \|u\|_{2\delta}^2 + \|y\|_{2\delta}^2$, it follows from (6.32), (6.33) and Lemma 3.3.2 in [71] that

$$m_f^2 \le c + c\mathrm{e}^{-\delta(t-t_k)} m^2(t_k) + c\|\tilde{\theta}_{i_k}^T \omega\|_{2\delta}^2, \tag{6.34}$$

where $\|\cdot\|_{2\delta}$ denotes the $\mathcal{L}_{2\delta}$-norm, $\delta \in (0, \delta_0]$, $c \ge 0$ denotes any finite constant.

step 2: Use the Swapping Lemma to establish the boundedness of $\|\tilde{\theta}_{i_k}^T \omega\|_{2\delta}$. The following inequality is obtained by applying Swapping Lemma [71]

$$\begin{aligned}\|\tilde{\theta}_{i_k}^T \omega\|_{2\delta} \le{}& c\mathrm{e}^{-\delta(t-t_k)} m^2(t_k) + \frac{c}{\alpha_0}(m_f + \|\dot{\theta}_{i_k} m_f\|_{2\delta}) \\ &+ c\alpha_0^{n^*}(\|\epsilon_{i_k}\|_{2\delta} + \|\dot{\theta}_{i_k} m_f\|_{2\delta} + \|\eta_{i_k}\|_{2\delta} + \|\varrho_{i_k}\|_{2\delta}),\end{aligned} \tag{6.35}$$

for some $\alpha_0 > 0$. Since $\delta \in (0, \delta_0]$, we have $m \le m_f$, thus

$$\begin{aligned}\|\tilde{\theta}_{i_k}^T \omega\|_{2\delta} \le{}& c\mathrm{e}^{-\delta(t-t_k)} m^2(t_k) + \frac{c}{\alpha_0}(m_f + \|\dot{\theta}_{i_k} m_f\|_{2\delta}) \\ &+ c\alpha_0^{n^*}(\|\frac{\epsilon_{i_k}}{m} m_f\|_{2\delta} + \|\dot{\theta}_{i_k} m_f\|_{2\delta} + \|\frac{\eta_{i_k}}{m} m_f\|_{2\delta} + \|\frac{\varrho_{i_k}}{m} m_f\|_{2\delta}).\end{aligned} \tag{6.36}$$

step 3: Prove the boundedness of closed-loop signals. From (6.34) and (6.36) it follows

$$m_f^2 \le c + c\mathrm{e}^{-\delta(t-t_k)} m^2(t_k) + c\alpha_0^{2n^*} \|\tilde{g}_{i_k} m_f\|_{2\delta}^2 \tag{6.37}$$

for large α_0 with $\tilde{g}_{i_k}^2 = (\frac{\epsilon_{i_k}}{m})^2 + \dot{\theta}_{i_k}^2 + (\frac{\eta_{i_k}}{m})^2 + (\frac{\varrho_{i_k}}{m})^2$. Now consider an arbitrary time interval $[t, t+T)$, within which switches occur at time instants $t \le t_{k_1}, t_{k_2}, \cdots, t_{k_N} \le t+T$. The normalizing signal m_f over this interval is then expressed by

$$m_f^2 \le c + c\mathrm{e}^{-\delta T} m_q^2 + c\int_t^{t+T} \mathrm{e}^{-\delta(t-\tau)} \tilde{g}^2(\tau) m_f^2(\tau) \mathrm{d}\tau, \tag{6.38}$$

where $m_q = \max\{m(t_{k_1}), \cdots, m(t_{k_N})\}$ and $\tilde{g}^2 = (\frac{\epsilon}{m})^2 + \sum_{i=1}^s \chi_i \dot{\theta}_i^2 + (\frac{\eta}{m})^2 + (\frac{\varrho}{m})^2, \tilde{g} \in \mathcal{S}(\mu)$.

Applying Bellman-Gronwall Lemma [71] yields

$$m_f^2 \le c\mathrm{e}^{-\delta T}(1 + m_q^2)\mathrm{e}^{c\int_t^{t+T} \tilde{g}^2(\tau)\mathrm{d}\tau} + c\delta \int_t^{t+T} \mathrm{e}^{-\delta(t-s)} e^{c\int_s^t \tilde{g}^2(\tau)\mathrm{d}\tau} \mathrm{d}s. \tag{6.39}$$

To obtain the boundness of m_f, $c\mu < \delta$ should be hold for some positive constant c. This condition can be achieved by letting μ sufficiently small, which implies slow switching. Since $m_f \in \mathcal{L}_\infty$, following from Lemma 6.8.1 in [71], it can be concluded that $u, y, \omega, m \in \mathcal{L}_\infty$.

step 4: Study the property of the tracking error. It follows from (6.24) and the boundedness of ω that $\xi_i, \zeta \in \mathcal{L}_\infty$. $\frac{\epsilon}{m} \in \mathcal{S}(\mu)$ together with $m \in \mathcal{L}_\infty$ yields $\epsilon \in \mathcal{S}(\mu)$. From (6.23) we write the general expression for e

$$e = \sum_{i=1}^s \chi_i(\epsilon_i - \rho_i \xi_i) = \epsilon - \sum_{i=1}^s \chi_i \rho_i \xi_i. \tag{6.40}$$

With the boundedness of ρ_i, ξ_i and $\epsilon \in \mathcal{S}(\mu)$ we can conclude that $e \in \mathcal{S}(\mu)$. □

Remark 6.4. Compared with the counterpart for PWL systems [134], the controller (6.21) introduces a constant term in ω to cancel out the biasing effect caused by the affine term. Since the affine term can also be viewed as input disturbance [164], non-equilibrium offset [141], actuator failure [151], and system damage [61], the controller for each subsystem has the common structure as the output feedback-based controllers proposed in [164, Sec. 4] and [151, Ch. 4]. Note that these two cases exhibit either no switching or switching only once and thus the disturbance or the actuator failure compensation error decays to zero as $t \to \infty$. This further gives asymptotic output tracking. Different from this result, the tracking error e in the PWA context is small in the mean square sense due to the switch-dependent property of η as well as ϱ, as discussed in Theorem 6.1.

Remark 6.5. Note that the operator $W_m[\cdot](t)$ in (6.24) reveals the input-output relationship without specifying the initial conditions (at initial instant t_0 and switching instants t_k for $k \in \mathbb{N}^+$). Depending on how this operator is implemented for the signals ζ and ξ_i in practice, the expression of ϵ_i in (6.27) may vary. Specifically, if ζ and ξ_i are implemented with a reset at each switching instant t_k, then equation (6.27) remains unchanged. If ζ and ξ_i are implemented without the reset at each switching instant t_k, then for some interval $t \in [t_k, t_{k+1})$, in which i-th subsystem is activated, (6.24) would become

$$\xi_i(t) = \theta_i^T(t)\zeta(t) - (W_m(s)[\theta_i^T\omega](t) + \varphi_{\xi i}(t)), \tag{6.41a}$$

$$\zeta(t) = W_m(s)[\omega](t) + \varphi_\zeta(t) \tag{6.41b}$$

where $\varphi_\zeta(t) \in \mathbb{R}^{2n+1}$ and $\varphi_{\xi i}(t) \in \mathbb{R}$ for $t \in [t_k, t_{k+1})$ represent some transient zero-input response terms triggered at t_k. Therefore, based on (6.41), the error equation (6.27) would become $\epsilon_i = \rho_i^*\tilde{\theta}_i^T\zeta + \tilde{\rho}_i\xi_i + \eta_i + \varrho_i + \rho_i^*{\theta_i^*}^T\varphi_\zeta - \rho_i^*\varphi_{\xi i}$ after some algebraic manipulation. Since the normalized version of φ_ζ and $\varphi_{\xi i}$ have the same property as η_i, ϱ_i (small in the mean square sense) under slow switching, the above result (Theorem 6.2) and the following analysis based on the error equation (6.27) can also be applied to the implementation without resets.

6.2.4 Control Parameter Convergence

Now we study the convergence of the control parameters. We extend the analysis method for linear systems in [71, p. 757] to the PWA systems. In particular, the proposed controller (6.21) contains a constant term, which is reflected in ω or equivalently ζ. The effect of this controller structure on the PE property of ω, ζ needs to be specifically analyzed. Furthermore, the tracking error e is small in the mean square sense, whose influence on the parameter convergence needs to be discussed. In addition, how the switching frequency affects the parameter convergence remains to be explored. The following theorem shows our result.

Theorem 6.3. *Let the PWA system (6.1) with known subsystem partitioning Ω_i and unknown subsystem parameters be controlled by output feedback controller (6.21) with the adaptation law (6.25). If the reference signal r is sufficiently rich of order $2n$ with distinct frequencies and activates all the subsystems repeatedly, i.e., $\forall i \in \mathcal{I}$ and $\forall t_s \in \mathbb{R}^+$, there exists $t_d > t_s$ and $\delta t \in \mathbb{R}^+$ such that $\chi_i(t) = 1$ for $t \in [t_d, t_d + \delta t)$ and if the projection in (6.25) is not activated, then $|e|$ and $|\tilde{\theta}_i|$ converge to a residual set*

$$\mathcal{S}_{\theta_i} = \left\{ e \in \mathbb{R}, \tilde{\theta}_i \in \mathbb{R}^{2n+1} \middle| |e| + |\tilde{\theta}_i| \le c_0(\nu_0 + \sqrt{\mu}) \right\}$$

for some positive constants $c_0, \nu_0 \in \mathbb{R}^+$ and $\mu \in (0, \mu_0)$.

Proof. First, as each subsystem is activated intermittently and $\hat{\theta}_i$ is frozen when the i-th subsystem is not activated, we remove the subscript i for simplicity and we show that ζ is PE.

$$\zeta(t) = W_m(s)\begin{bmatrix}[\omega_1](t)\\ [\omega_2](t)\\ [y](t)\\ [r](t)\\ [1](t)\end{bmatrix} = W_m(s)\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)}[u](t)\\ \frac{\alpha(s)}{\Lambda(s)}[y](t)\\ [y](t)\\ [r](t)\\ [1](t)\end{bmatrix}. \tag{6.42}$$

Inserting (6.2) into (6.42) and substituting u yields

$$\zeta(t) = \zeta_m + \zeta_e, \tag{6.43}$$

where

$$\begin{aligned}\zeta_m =& \underbrace{W_m(s)(\underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)}G_p^{-1}W_m(s)\\ \frac{\alpha(s)}{\Lambda(s)}W_m(s)\\ W_m(s)\\ 1\\ 0\end{bmatrix}}_{\triangleq H(s)}[r](t) + \underbrace{\begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}}_{\triangleq H_f(s)}[1](t))}_{\triangleq z},\\ \zeta_e =& W_m(s)(\underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)}G_p^{-1}\\ \frac{\alpha(s)}{\Lambda(s)}\\ 1\\ 0\\ 0\end{bmatrix}}_{\triangleq H_e(s)}[e](t) + \underbrace{\begin{bmatrix}\frac{\alpha(s)}{\Lambda(s)}G_p^{-1}G_f\\ 0\\ 0\\ 0\\ 0\end{bmatrix}}_{\triangleq H_{fe}(s)}[1](t)).\end{aligned} \tag{6.44}$$

To prove the PE property of ζ, we start by showing that z is PE. The auto-covariance of z is given by

$$R_z(0) = \underbrace{\frac{1}{2\pi}H_f(0)H_f(0)^T}_{\triangleq R_{z1}(0)} + \underbrace{\frac{1}{2\pi}\sum_{l=1}^{2n}F_r(\bar{\Omega}_l)H(-\iota\bar{\Omega}_l)H(\iota\bar{\Omega}_l)^T}_{\triangleq R_{z2}(0)}, \tag{6.45}$$

where ι is the imaginary unit ($\iota^2 = -1$), $F_r(\bar{\Omega}_l)$ denotes the spectral peak associated with frequency $\bar{\Omega}_l, l \in \{1, \cdots, 2n\}$. Note that the constant input 1 in ω leads to an unit spectral peak at zero frequency (see $R_{z1}(0)$) while the frequencies contained in r build $2n$ distinct peaks $F_r(\bar{\Omega}_l)$ (see $R_{z2}(0)$). We rewrite $H(s)$ as

$$H(s) = \begin{bmatrix}H_-(s)\\ 0\end{bmatrix}, \tag{6.46}$$

with

$$H_{-}(s) = \begin{bmatrix} \frac{\alpha(s)}{\Lambda(s)} G_p^{-1} W_m(s) \\ \frac{\alpha(s)}{\Lambda(s)} W_m(s) \\ W_m(s) \\ 1 \end{bmatrix}. \tag{6.47}$$

It is proved in [71] that $H_{-}(\iota\bar{\Omega}_1), H_{-}(\iota\bar{\Omega}_2), \ldots H_{-}(\iota\bar{\Omega}_{2n})$ are linearly independent.

If $R_z(0)$ is positive definite, then equation

$$\mathcal{X}^T R_z(0) \mathcal{X} = 0 \tag{6.48}$$

only has solution $\mathcal{X} = 0_{2n+1}, \mathcal{X} \in \mathbb{R}^{2n+1}$. Because $R_{z1}(0)$ and $R_{z2}(0)$ are positive semidefinite, we have

$$\mathcal{X}^T R_{z1}(0) \mathcal{X} \geq 0, \quad \mathcal{X}^T R_{z2}(0) \mathcal{X} \geq 0, \tag{6.49}$$

which together with (6.48) implies that

$$\mathcal{X}^T R_{z1}(0) \mathcal{X} = 0, \quad \mathcal{X}^T R_{z2}(0) \mathcal{X} = 0 \tag{6.50}$$

only if $\mathcal{X} = 0_{2n+1}$. Suppose $\mathcal{X} = [\mathcal{X}_C^T, \mathcal{X}_d]^T$ with $\mathcal{X}_C \in \mathbb{R}^{2n}, \mathcal{X}_d \in \mathbb{R}$. From $\mathcal{X}^T R_{z2}(0) \mathcal{X} = 0$ follows

$$L^T \mathcal{X}_C = 0 \tag{6.51}$$

with $L = [H_{-}(\iota\bar{\Omega}_1), H_{-}(\iota\bar{\Omega}_2), \ldots, H_{-}(\iota\bar{\Omega}_{2n})]$. Because L has full rank, $\mathcal{X}_C$ must be 0_{2n}. From $\mathcal{X}^T R_{z1}(0) \mathcal{X} = 0$ it follows that $\mathcal{X}_d$ must be 0, which implies the positive definiteness of $R_z(0)$, thus z is PE, which together with $\zeta_m = W_m[z](t)$ yields ζ_m being PE. Hence, there exist $T_0 > 0, \alpha_0 > 0$ such that

$$\frac{1}{T_0} \int_t^{t+T_0} \zeta_m(\tau) \zeta_m^T(\tau) \mathrm{d}\tau \geq \alpha_0 I, \forall t \geq 0. \tag{6.52}$$

Next, we would like to prove that ζ is also PE. Note that

$$\begin{aligned} &\frac{1}{\bar{n}T_0} \int_t^{t+\bar{n}T_0} \zeta(\tau)\zeta^T(\tau) \mathrm{d}\tau \\ &\geq \frac{1}{2\bar{n}T_0} \int_t^{t+\bar{n}T_0} \zeta_m(\tau)\zeta_m^T(\tau) \mathrm{d}\tau - \frac{1}{\bar{n}T_0} \int_t^{t+\bar{n}T_0} \zeta_e(\tau)\zeta_e^T(\tau) \mathrm{d}\tau, \end{aligned} \tag{6.53}$$

where $\bar{n}$ is an arbitrary positive integer. Because $G_p(s)$ and $W_m(s)$ have the same relative degree, $W_m(s)H_e(s)$ is strictly proper. $G_f(s)$ is also proper, which implies that $W_m(s)H_{fe}(s)$ is strictly proper. Because it is established that $e \in \mathcal{S}(\mu)$, we have $W_m(s)H_e(s)[e](t) \in \mathcal{S}(\mu)$. Considering $W_m(s)H_{fe}(s)[1](t) \in \mathcal{L}_\infty$, we have $\zeta_e \in \mathcal{S}(\mu)$, which together with the PE property of ζ_m yields

$$\frac{1}{\bar{n}T_0} \int_t^{t+\bar{n}T_0} \zeta(\tau)\zeta^T(\tau) \mathrm{d}\tau \geq \frac{\alpha_0}{2} I - (K_0\mu + \frac{C_0}{\bar{n}T_0}) I \tag{6.54}$$

for some $C_0, K_0 \geq 0$. If $\bar{n}$ is chosen such that $C_0 < \frac{\alpha_0}{8}\bar{n}T_0$, then for $K_0\mu < \frac{\alpha_0}{8}$, we have

$$\frac{1}{\bar{n}T_0} \int_t^{t+\bar{n}T_0} \zeta(\tau)\zeta^T(\tau) \mathrm{d}\tau \geq \frac{\alpha_0}{4} I. \tag{6.55}$$

So ζ is PE. Based on the obtained PE property of ζ, we continue to explore the convergence property of $\tilde{\theta}_i$. Insert (6.27) into (6.25) yields

$$\dot{\tilde{\theta}}_i(t) = -\chi_i \text{sign}[k_{pi}]\Gamma_i(\frac{\rho_i^* \zeta\zeta^T}{m^2}\tilde{\theta}_i + \frac{\tilde{\rho}_i \xi_i \zeta}{m^2} + \frac{(\eta_i + \varrho_i)\zeta}{m^2}). \tag{6.56}$$

Considering that the homogeneous part of (6.56) is exponentially stable due to the PE property of ζ and $\frac{\zeta}{m} \in \mathcal{L}_\infty$, we have

$$\begin{aligned} |\tilde{\theta}_i| &\le \beta_0 \mathrm{e}^{-\beta_2(t-t_k)} + \beta_1 \int_{t_k}^{t} \mathrm{e}^{-\beta_2(t-\tau)}(\frac{|\tilde{\rho}_i \xi_i|}{m} + \frac{|\eta_i| + |\varrho_i|}{m})\mathrm{d}\tau \\ &\le \beta_0 \mathrm{e}^{-\beta_2(t-t_k)} + \bar{\beta} + \beta_1 \int_{t_k}^{t} \mathrm{e}^{-\beta_2(t-\tau)}(\frac{|\eta_i| + |\varrho_i|}{m})\mathrm{d}\tau, \end{aligned} \tag{6.57}$$

where $\beta_0, \beta_1, \beta_2 \in \mathbb{R}^+$ are some positive constants, $\bar{\beta} = \frac{\beta_1}{\beta_2}\sup_t \frac{|\tilde{\rho}_i \xi_i|}{m}$. Because $\eta_i, \varrho_i \in \mathcal{S}(\mu)$, we apply [71, Corollary 3.3.3] and have

$$\beta_1 \int_{t_k}^{t} \mathrm{e}^{-\beta_2(t-\tau)}(\frac{|\eta_i| + |\varrho_i|}{m})\mathrm{d}\tau \le \beta^{'}(\sqrt{C'} + \sqrt{K'\mu}) \tag{6.58}$$

for some constants $C', K' \in \mathbb{R}^+$ with $\beta^{'} = 2\sqrt{\frac{\beta_1^2}{\beta_2}\frac{e^{\beta_2}}{1-e^{-\beta_2}}}$. This implies that $\tilde{\theta}_i$ converges to a residual set

$$|\tilde{\theta}_i| \le c'(\nu + \sqrt{\mu}) + \epsilon_t, \tag{6.59}$$

where $\mu \in (0, \mu_0)$, $\nu = \frac{\bar{\beta}}{\beta'\sqrt{K'}} + \sqrt{\frac{C'}{K'}}$, $c' = \beta'\sqrt{K'}$ and ϵ_t is an exponentially decaying term. Invoking Lemma 3.3.2 of [71] we have from $e = y - W_m[r]$ and (6.22) that

$$|e| \le \max_i |\rho_i^*| \|W_m(s)\|_{2\delta} \|\tilde{\theta}_i^T \omega\|_{2\delta} + \bar{d} \tag{6.60}$$

with $\bar{d} = \sup_t(|\eta| + |\varrho|)$ and $\|W_m(s)\|_{2\delta}$ denoting the δ-shifted H_2 norm of $W_m(s)$ for some $\delta > 0$. Inserting (6.59) into the $\mathcal{L}_{2\delta}$-norm $\|\tilde{\theta}_i^T\omega\|_{2\delta}$ in (6.60) leads to

$$|e| \le \bar{\omega}(c'(\nu + \sqrt{\mu})) + \bar{d} + \epsilon' \tag{6.61}$$

for $\bar{\omega} = \max_i |\rho_i^*| \|W_m(s)\|_{2\delta} \frac{\sup_t |\omega|}{\sqrt{\delta}}$ and ϵ' being a decaying to zero term. Combining (6.59) and (6.61) we have that $|e|$ and $|\tilde{\theta}_i|$ converge to the residual set

$$\mathcal{S}_{\theta_i} = \left\{ e \in \mathbb{R},\, \tilde{\theta}_i \in \mathbb{R}^{2n+1} \middle| |e| + |\tilde{\theta}_i| \le c_0(\nu_0 + \sqrt{\mu}) \right\}$$

for $c_0 = c'(1 + \bar{\omega}), \nu_0 = \nu + \frac{\bar{d}}{c_0}$.

□

In our proof, we first decompose ζ into ζ_m and ζ_e. ζ_m can be further decomposed into one component depending on input frequencies and one constant term representing zero frequency. These constitute the excitation source. ζ_e contains all the error terms and is proved to be $\mathcal{S}(\mu)$. We show that its effect on the excitation can be eliminated by carefully balancing the switching frequency μ and excitation level α_0 of ζ_m. Finally, we establish the relationship between the switching frequency μ and the size of the bounded set $\mathcal{S}_{\theta_i}$ by expressing $|\theta_i|$ in terms of an inequality of μ.

Theorem 6.3 indicates that the bound of the residual set relates to the switching frequency. Fast switching results in a large residual set. The convergence to the nominal value is, however, possible and discussed as follows:

Corollary 6.1. Let the PWA system (6.1) with known subsystem partitioning Ω_i and unknown subsystem parameters be controlled by output feedback controller (6.21) with the adaptation law (6.25) without the projection. The reference signal r is sufficiently rich of order $2n$. If for a certain $i \in \mathcal{I}$ and a certain time instant $t_\mu \geq 0$, we have $\chi_i(t) = 1$ for $\forall t \in [t_\mu, \infty)$, then $e(t) \to 0, \tilde{\theta}_i(t) \to 0$ as $t \to \infty$.

Proof. Since the system output remains in i-th subsystem, we focus on i-th subsystem and remove the subscript i for simplicity.

Let $\eta = \eta_c + \eta_m$ with $\eta_c = C_c \Phi_c(t, t_\mu) x_c(t_\mu)$ and $\eta_m = -C_m \Phi_m(t, t_\mu) x_m(t_\mu)$. η_c and η_m satisfy

$$\begin{aligned} \dot{\omega}_c &= A_c \omega_c, \quad \omega_c(t_\mu) = x_c(t_\mu) \\ \eta_c &= C_c \omega_c \end{aligned} \tag{6.62}$$

and

$$\begin{aligned} \dot{\omega}_m &= A_m \omega_m, \quad \omega_m(t_\mu) = x_m(t_\mu) \\ \eta_m &= -C_m \omega_m, \end{aligned} \tag{6.63}$$

respectively. Besides, ϱ satisfies the equation

$$\begin{aligned} \dot{\omega}_\delta &= A_c \omega_\delta + F_c, \quad \omega_\delta(t_\mu) = 0 \\ \varrho &= C_c \omega_\delta. \end{aligned} \tag{6.64}$$

Define the Lyapunov-like function

$$\begin{aligned} V =& |\rho^*| \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \gamma^{-1} \tilde{\rho}^2 + \omega_c^T P_c \omega_c + \omega_m^T P_m \omega_m \\ &+ (\omega_\delta - A_c^{-1} F_c)^T P_c (\omega_\delta - A_c^{-1} F_c). \end{aligned} \tag{6.65}$$

Since A_m and A_c are stable, there exist positive definite matrices P_c and P_m such that

$$A_c^T P_c + P_c A_c = -\gamma_c I, \quad A_m^T P_m + P_m A_m = -\gamma_m I \tag{6.66}$$

for some constants $\gamma_c, \gamma_m > 0$ to be chosen. Take the derivative of V and insert (6.27), (6.62), (6.63), (6.64) and (6.66), we have

$$\dot{V} = -\frac{2\epsilon^2}{m^2} + \frac{2\epsilon\eta_c}{m^2} - \gamma_c |\omega_c|^2 + \frac{2\epsilon\eta_m}{m^2} - \gamma_m |\omega_m|^2 + \frac{2\epsilon\varrho}{m^2} - \gamma_c |\varpi_\delta|^2, \tag{6.67}$$

where $\varpi_\delta = \omega_\delta - A_c^{-1} F_c$. Substituting η_c, η_m, ϱ with (6.62), (6.63), (6.64) and invoking Lemma 6.1, it follows

$$\begin{aligned} \dot{V} \leq& -\frac{2\epsilon^2}{m^2} + \frac{2}{m^2} |\epsilon| |C_c| |\omega_c| - \gamma_c |\omega_c|^2 + \frac{2}{m^2} |\epsilon| |C_m| |\omega_m| \\ &- \gamma_m |\omega_m|^2 + \frac{2}{m^2} |\epsilon| |C_c| |\varpi_\delta| - \gamma_c |\varpi_\delta|^2 \\ =& -\frac{\epsilon^2}{2m^2} + \phi_1 + \phi_2 + \phi_3 \end{aligned} \tag{6.68}$$

where

$$\begin{aligned}\phi_1 =&\frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_c||\omega_c| - \gamma_c|\omega_c|^2 \\ =& -\frac{\epsilon^2 + (\epsilon - 4|C_c||\omega_c|)^2}{4m^2} - |\omega_c|^2(\gamma_c - \frac{4|C_c|^2}{m^2}),\end{aligned} \tag{6.69}$$

$$\begin{aligned}\phi_2 =& -\frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_m||\omega_m| - \gamma_m|\omega_m|^2 \\ =& -\frac{\epsilon^2 + (\epsilon - 4|C_m||\omega_m|)^2}{4m^2} - |\omega_m|^2(\gamma_m - \frac{4|C_m|^2}{m^2}),\end{aligned} \tag{6.70}$$

and

$$\begin{aligned}\phi_3 =& -\frac{\epsilon^2}{2m^2} + \frac{2}{m^2}|\epsilon||C_c||\overline{\omega}_\delta| - \gamma_c|\overline{\omega}_\delta|^2 \\ =& -\frac{\epsilon^2 + (\epsilon - 4|C_c||\overline{\omega}_\delta|)^2}{4m^2} - |\overline{\omega}_\delta|^2(\gamma_c - \frac{4|C_c|^2}{m^2}).\end{aligned} \tag{6.71}$$

We obtain $\phi_1, \phi_2, \phi_3 \leq 0$ by choosing $\gamma_c \geq 4|C_c|^2$ and $\gamma_m \geq 4|C_m|^2$, which indicates $\dot{V} \leq 0$.

It follows that $\tilde{\theta}, \tilde{\rho} \in \mathcal{L}_\infty$ and $\frac{\epsilon}{m} \in \mathcal{L}_2$. Following the derivation of Theorem 6.2 yields $\tilde{g} \in \mathcal{L}_2$ and $\omega, m, \xi, \zeta \in \mathcal{L}_\infty$, which together with (6.27) and $\eta, \varrho \in \mathcal{L}_2 \cap \mathcal{L}_\infty$ gives $\epsilon \in \mathcal{L}_2 \cap \mathcal{L}_\infty$. From (6.25) we have $\dot{\theta} \in \mathcal{L}_2$ and thus $\xi \in \mathcal{L}_2$. It follows from (6.20) and $\epsilon, \xi \in \mathcal{L}_2, \rho \in \mathcal{L}_\infty$ that $e \in \mathcal{L}_2$, which combined with $\dot{e} \in \mathcal{L}_\infty$ reveals $e(t) \to 0$ as $t \to \infty$.

Since ζ is PE, the homogeneous part of (6.56) is exponentially stable, which together with $\xi, \eta, \varrho \in \mathcal{L}_2$ implies $\tilde{\theta}(t) \to 0$ as $t \to \infty$. □

As Corollary 6.1 shows, if the output trajectory is kept staying in a certain subsystem, the periodic deviations caused by switching are avoided and this further results in both the convergence of the tracking error and control parameter estimation error.

6.3 Numerical Validation

A numerical example taken from [80] is utilized to validate the proposed control algorithm. The plant parameters of the PWA system are given by

$$\begin{aligned} A_1 &= \begin{bmatrix} 0 & 1 \\ -2 & -1 \end{bmatrix}, \quad A_2 = \begin{bmatrix} 0 & 1 \\ -2.5 & -1 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 1 \\ -1.5 & -1 \end{bmatrix}, \\ f_1 &= \begin{bmatrix} 0 \\ 0.4 \end{bmatrix}, \qquad\quad f_2 = \begin{bmatrix} 0 \\ 0.2 \end{bmatrix}, \qquad\qquad f_3 = \begin{bmatrix} 0 \\ -0.3 \end{bmatrix}, \end{aligned}$$

with the common input matrix $B = [0, 1.5]^T$ and the output matrix $C = [1, 0]$. The sign of each subsystem is 1 and known as prior. The switching hyperplanes depend on the system output and are given by

$$\begin{aligned} \Omega_1 &= \{y \in \mathbb{R} | -2 \leq y \leq 2\}, \\ \Omega_2 &= \{y \in \mathbb{R} | y > 2\}, \\ \Omega_3 &= \{y \in \mathbb{R} | y < -2\}. \end{aligned}$$

The reference model is chosen as

$$W_m = \frac{1}{(s+1)^2} \tag{6.72}$$

The relative degree of the reference system is 2, which is equal to the one of all the subsystems of the PWA system. Selecting $\Lambda(s) = \frac{1}{s+1}$ and $\alpha = 1$, the nominal control parameters are obtained by matching equations (6.9)

$$\begin{aligned}
\theta_1^* &= [-1, 1.33, -0.67, 0.67, -0.53]^T, \\
\theta_2^* &= [-1, 1.67, 1, 0.67, -0.27]^T, \\
\theta_3^* &= [-1, 1, 0.33, 0.67, 0.4]^T,
\end{aligned}$$

Given an input signal $r = 4\sin(0.05t)$, the output tracking performance of the closed-loop system by applying the nominal controller is displayed in Fig. 6.1. It shows that the

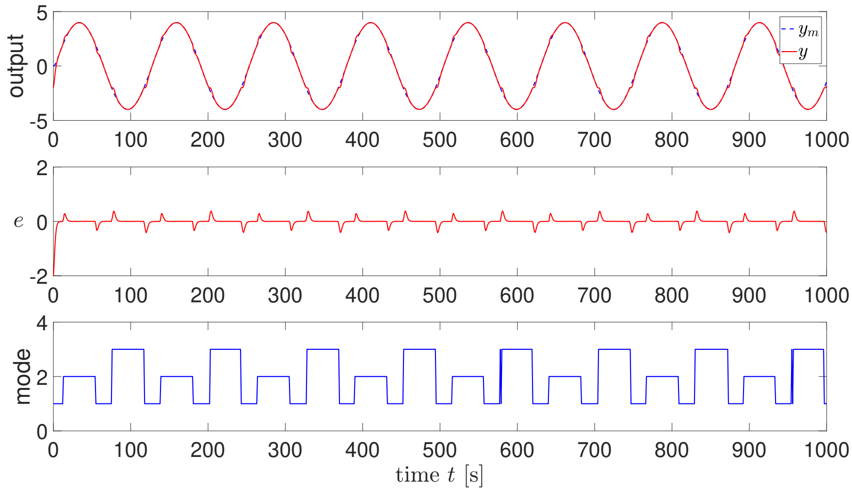


Figure 6.1: Output tracking performance with input signal $r = 4\sin(0.05t)$, the nominal controller is applied

output tracking error exists even when the nominal control parameters are employed. When the system switches, the output of the closed-loop system deviates from the output of the reference system, as depicted in equation (6.20). This deviation vanishes given a sufficiently slow switching. The overall output tracking error over the whole time interval is thus small in the mean square sense.

Given the adaptation gains $\Gamma_i = \gamma_i = 10$, the output tracking performance of the adaptive system is displayed in Fig. 6.2. It can be seen that the desired performance is achieved by applying adaptive controller. The deviation from the reference output occurs due to the switches among subsystems. The smallness of the tracking error in the mean square sense validates the theory derivation. Compared with adaptive controller, the nominal controller exhibits better transient performance. This motivates us to study the convergence property of the controller parameters. To validate the control parameter convergence, the input signal is required to be sufficiently rich of order 4. Define the input signal $r = \sin(0.9t) + \sin(0.1t) + \bar{r}$,

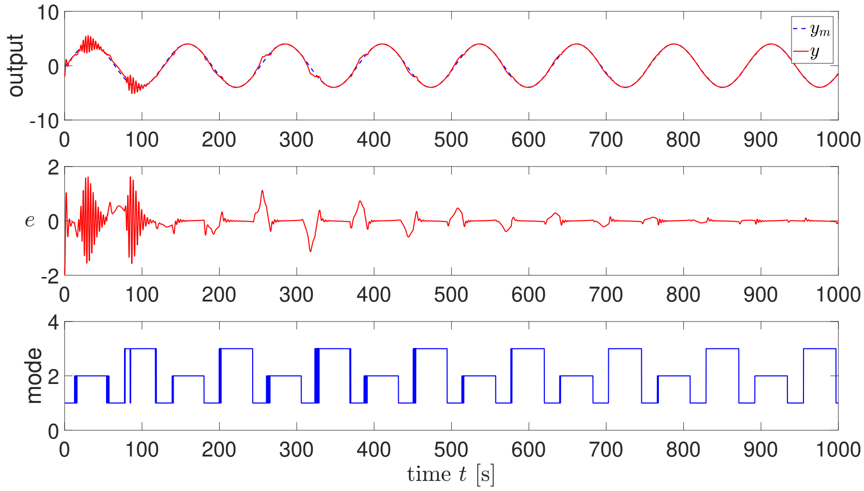


Figure 6.2: Output tracking performance with input signal $r = 4\sin(0.05t)$, the proposed adaptive controller is applied

with a periodic offset signal

$$\bar{r}(t) = \begin{cases} 4, & 1000 + kT \leq t < 3000 + kT\text{s} \\ -4, & 4000 + kT \leq t < 6000 + kT\text{s} \\ 0, & \text{otherwise} \end{cases} \tag{6.73}$$

where $k \in \mathbb{N}, T = 6000s$. $\bar{r}$ drives the trajectory into all subsystems periodically. In Fig. 6.3,

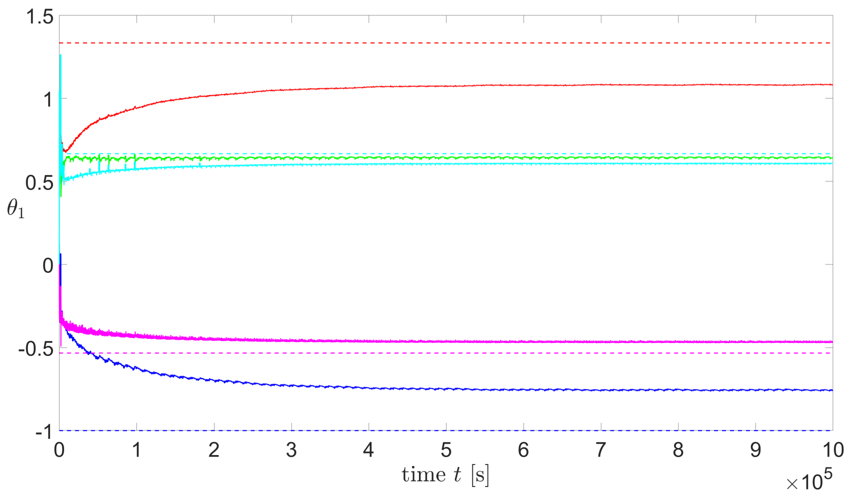


Figure 6.3: Control parameters converge to a residual set around the nominal values with slow switching

the dashed lines represent the nominal control parameters and the solid lines the adaptive control parameters. It reveals that the adaptive control parameters converge to a set around the nominal values under slow switching. Since the convergence of $\tilde{\theta}_i$ is similar, so only $\tilde{\theta}_1$ is displayed for clarity.

To show the parameter convergence stated in Corollary 6.1, the trajectory of the closed-loop system must be kept within a certain subsystem of the PWA system from a certain time instant t_μ on. Here we remove $\bar{r}$ from r at $t_\mu = 40000s$, which leads to $\chi_1 = 1, \chi_2 = \chi_3 = 0, \forall t \in [t_\mu, \infty)$, the adaptive control parameters in θ_1 converge to the nominal control values in θ_1^*, as shown in Fig. 6.4. The simulation validates the theory derivation.

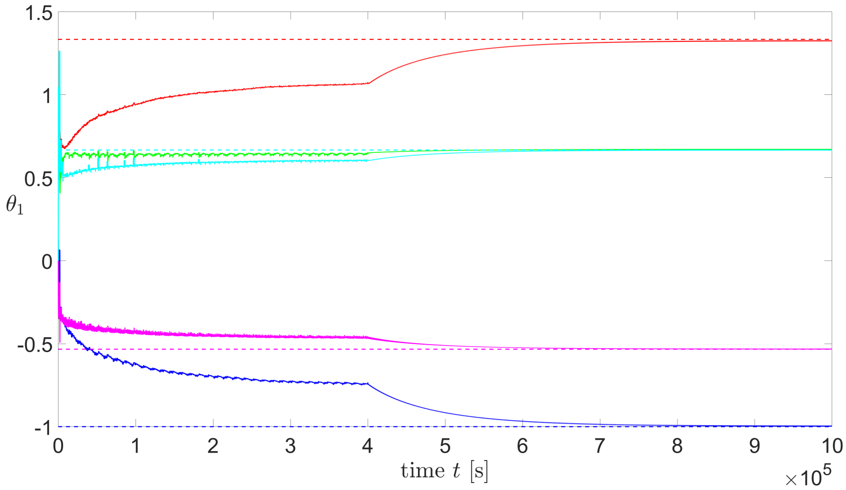


Figure 6.4: Control parameters converge to the nominal values for $\chi_1 = 1$ after $40000s$

6.4 Summary

In this chapter, we have developed the output feedback-based direct MRAC for PWA systems for output tracking and explored the controller parameter convergence. With the proposed approach, all the signals in the closed-loop are bounded and the output tracking error is small in the mean square sense with sufficiently slow switching. If the input signal is sufficiently rich, the control parameters converge to a residual set around the nominal values for slow switches.

When comparing with the output feedback-based direct MRAC for PWL counterparts, one highlight of our approach is that a novel adaptation law is proposed, which introduces a constant term and compensates the biasing effect of the affine term. Furthermore, we fill the gap that the controller parameter convergence is not explored for output feedback-based MRAC of switched systems by establishing the relationship between the switching frequency and the size of the residual set, to which the controller parameters converge, provided that the input signal is sufficiently rich.

The effectiveness of the proposed approach is validated through a two-dimensional numerical example with relative degree 2. As the theoretical analysis shows, the proposed approach can also be applied to PWA systems with higher dimensions and higher relative degree of each subsystem, so long as the Assumption 6.1 on the system classes holds.

The proposed approach is limited to be applied to SISO PWA systems with measurable input and output signals, whose region partitions only depend on the known input and output signals, in contrast to the more common cases where region partitions depend on the state vector. Considering this limitation, the following directions for future work are suggested.

Regarding the extension of the feedback mechanism, it would be interesting to study a more general case where the partial state feedback [145] instead of the output feedback is available for the adaptive controller design. Regarding the extension of the switching mechanism, it would be worth investigating the output feedback-based direct MRAC for PWA systems, whose region partitions depend on the system state and thus are unknown. Furthermore, the current approach is based on a slow-switching assumption. It is also an interesting topic to study how to avoid frequent switching and sliding mode.

7 Conclusion and Outlook

The motivation of this thesis is to explore the adaptive control for uncertain switched systems with a particular focus on three aspects of constraints: limited excitations, performance constraints, and sensor constraints.

7.1 Conclusion

The background knowledge presented in Chapter 2 reveals two common concepts in the literature to solve the adaptive control problem of switched systems with known switching signals: CLF-based approaches and MLF-based approaches. This thesis adopts these two concepts to solve the constrained adaptive control problem of switched systems. As most of the chapters focus on uncertain PWA systems, whose region partitions are assumed to be known, the proposed methods are also eligible for time-dependent switched linear systems.

7.1.1 Solution to Limited Excitations

In Chapter 3, an indirect MRAC approach based on concurrent learning is proposed for uncertain PWA systems. The concept of this approach is that it exploits not only the current data, but also the recorded history data for the adaptation of the parameter estimation. Specifically, each subsystem is assigned with a stack for the storage of history data. The recorded data of a stack is updated and utilized for the parameter adaptation when the corresponding subsystem is activated. The key advantage of the proposed method is that the parameter convergence does not require the excitation to fulfill the conventional PE condition. Instead, a relatively mild excitation condition of the linear independence of the recorded data suffices to ensure the convergence of the controller gains and system parameter estimation errors. This provides a solution to achieve parameter convergence in the indirect MRAC of PWA systems under limited excitations. In light of the limitations of the known state derivative assumption and the existence of the CLF, we suggest incorporating filtering techniques to eliminate the assumption of known state derivatives and leveraging the non-increasing property of barrier function-based MLF introduced in Chapter 5 to relax the assumption of the existing CLF as future work.

7.1.2 Solution to Performance Constraints

In Chapter 4 and Chapter 5, adaptive control approaches of uncertain PWA systems satisfying performance constraints are developed, where Chapter 4 deals with constraints on the output tracking and Chapter 5 deals with constraints on the state tracking. In addition to the performance constraints, both chapters achieve parameter convergence and present modified adaptation laws to guarantee the robustness of the closed-loop systems in the presence of disturbances.

In Chapter 4, the adaptive control approaches for uncertain PWA systems in control canonical form with both direct and indirect adaptation options are developed. The key idea is to introduce an error transform, which transforms the constrained output tracking error into an unconstrained one and then design the adaptive controller to ensure the boundedness of the unconstrained transformed error. Given a desired trajectory, both control approaches ensure the output tracking error to be confined within a performance bound, which prescribes the steady-state tracking error as well as the transient behavior such as decaying rate and overshoot. Based on CLFs, we prove the stability of the closed-loop system under arbitrary switching. Besides, the parameter convergence for both direct and indirect approaches is proved under the PE condition. Furthermore, a projection-based robust modification of the direct adaptation laws is provided, which ensures the closed-loop stability as well as the fulfillment of the performance constraint when the PWA system is subject to matched disturbances.

While Chapter 4 deals with performance constraints on the output tracking, Chapter 5 focuses on MRAC for uncertain PWA systems with performance constraints on the state tracking, which is formulated as an inequality in terms of an error metric, defined as the weighted Euclidean norm of the state tracking error. The core concept is to construct a barrier function to confine the error metric to stay within a certain “barrier”. In the absence of a CLF, this error metric exhibits jumps at the switching instants and may lead to the barrier transgression problem. To solve this problem, we introduce an auxiliary performance bound with a state reset map at switching instants to construct the “barrier” of the barrier function. On the one hand, this auxiliary performance bound resides within the user-defined performance bound if some dwell time constraints are satisfied. On the other hand, by resetting the auxiliary performance bound at each switching instant, the error metric with jumps is guaranteed to be confined within the auxiliary performance bound at switching instants. We construct MLFs for both direct and indirect adaptation cases, which enjoy the non-increasing property at as well as in between the switching instants such that the error metric fulfills the performance constraint. The parameter convergence is proved for the indirect adaptation case. Moreover, a projection-based robust modification of the proposed method is developed to enhance the robustness of the closed-loop system against unmatched disturbances.

In addition to the capability of the performance guarantee, another advantage of the approaches developed in Chapter 5 is that the constructed MLFs are non-increasing at switching instants such that extra conditions like projections or PE are not needed to guarantee the stability in the ideal case. Based on this property, two variants of indirect MRAC approaches for uncertain PWA systems are developed, which relax the assumption of the existence of a CLF, which is known as one key limitation of the previous indirect MRAC approaches for PWA systems.

To better understand the connections and the difference between Chapter 4 and Chapter 5, it is worth discussing the following points. First, both of them follow a barrier function concept, namely, mapping a constrained error term to an unconstrained term. The performance constraints can be fulfilled by ensuring the boundedness of the unconstrained term. However, the value of the the unconstrained term may become very large if the error approaches the boundary of the constraint. This would lead to large control inputs, which is the common limitation of the approaches in both chapters. Second, approaches of Chapter 4 belong to adaptive control with state feedback for output tracking and those of Chapter 5 belong to adaptive control with state feedback for state tracking. The former can only be applied to uncertain PWA systems in control canonical form, while the latter ones do not have structural requirements on the PWA systems. The canonical form in Chapter 4 leads to the existence of CLFs. Nevertheless, the existence of CLFs in Chapter 5 depends on the choice of the reference models. Finally, the robust modification in Chapter 4 is limited to cope with matched disturbances whereas the one in Chapter 5 can also handle unmatched disturbances.

Considering the common limitation of large control inputs, extending the current approaches to systems having input saturation constraints will be part of future work. Besides, it is of practical interest to explore the relationship between the input saturation constraints and the bounds of the uncertain parameters.

7.1.3 Solution to Sensor Constraints

Chapter 6 and Chapter A cope with the uncertain switched systems with sensor constraints, namely, whose states are not available. To be specific, Chapter 6 is devoted to the adaptive control of such systems based on the output feedback while Chapter A focuses on the adaptive observer (simultaneous estimation of unknown states and parameters). The convergence of the estimation errors of the control gains is also explored in Chapter 6.

In Chapter 6, the output feedback-based direct MRAC control design of PWA systems as well as its parameter convergence analysis are investigated. The transients after each switch, including the mismatch of zero-input responses and the compensation error of the affine term, are treated as disturbances. Under the slow switching assumption, it is shown that all the closed-loop signals are bounded and the output tracking error is small in the mean square sense. Built upon this result, the estimation error of the controller parameters is proved to converge to a residual set if the input signal is sufficiently rich. The relationship between the size of this residual set and the switching frequency is established. Moreover, the convergence of the estimated controller parameters to their nominal values can be achieved for a certain subsystem given that this subsystem is activated for infinitely long time. The main limitation is that the proposed approach is only eligible for SISO PWA systems, whose region partitions only depend on the known input and output signals.

Chapter A presents an adaptive observer for a class of nonlinear systems with switched unknown parameters, which covers some special classes of PWA systems. The key difficulty lies in how to deal with the disturbance effect of the intermittently appeared zero-input responses caused at each switching instant. These responses depend on the unknown SASI and constitute an additive disturbance to the parameter estimation, which obstructs the parameter convergence to zero. The solution concept is to treat the zero-input responses as excitations instead of disturbances. This is realized by first augmenting the system parameter with the SASI and then developing an estimator for the augmented parameter using the DREM technique. Thanks to its property of element-wise parameter adaptation, the system parameter estimation is decoupled from the SASI. As a result, the estimation errors of system states and parameters converge to zero asymptotically. Furthermore, the robustness of the proposed adaptive observer is guaranteed in the presence of disturbances and noise under a PE condition. The application of our approach is confined to switched systems whose uncertainties are linearly parameterized. The extension to nonlinearly parameterized switched systems is suggested as part of future work.

Although the zero-input responses caused at each switching instant constitute the common difficulty for the adaptive observer design in Chapter A and the adaptive control design in Chapter 6, the concepts to treat these responses are different. Specifically, in Chapter A, the known information contained in the zero-input responses are fully exploited to update the estimated parameters and therefore, these responses can be viewed as excitation sources. In contrast, the zero-input responses (together with other transients caused by switching) are treated as disturbance terms in Chapter 6. Such conceptual difference further leads to the following difference of results: if every subsystem is intermittently activated, then the estimation errors of the state and the parameters of the adaptive observer in Chapter A converge to zero asymptotically, while the tracking error and parameter estimation errors of the adaptive controller in Chapter 6 are only bounded. Nevertheless, if the switch terminates, then the zero-input response converge to zero. Thereby, the parameter estimation errors of the adaptive observer in Chapter A remain to be bounded whereas those of the adaptive control in Chapter 6 converge to zero.

In summary, we have developed adaptive control approaches for uncertain PWA systems with the focus on dealing with constraints from three perspectives. Specifically, from the aspect of the input of the control system, the problem of limited excitations at the input is solved by achieving the convergence of the estimated system parameters in the indirect MRAC of PWA systems without requiring the PE condition. From the aspect of the state/output of the control system, direct and indirect adaptive control approaches are developed for PWA systems to satisfy the output performance constraints and full state performance constraints, respectively. From the aspect of the feedback loop, a direct MRAC approach is developed for PWA systems, which only relies on the output feedback. Besides, a novel adaptive observer is proposed for switched systems to estimate the unknown parameters and states simultaneously.

7.2 Outlook

The adaptive control methods for PWA systems developed in this thesis can be categorized as model-based control, which enjoys the advantages of provable stability, robustness and parameter convergence. Meanwhile, the rapid growth in the computing power of the modern processors in recent years has initiated much effort on developing model-free learning methods based on big data to solve planning, control and decision making problems of agents with increasing complexity in highly dynamic and uncertain environments. Despite their ability to solve control problems with large complexity, the stability and robustness are mostly not formally ensured. Considering the complementary features of the model-based and model-free control approaches, it is an exciting topic to investigate how to combine the research achievements presented in this thesis with the learning methods to solve the current technological challenges. In light of this, we discuss the following potential research directions for future work.

Potential to fill the "sim-to-real" gap: the "sim-to-real" gap stems from the area of reinforcement learning (RL). Many existing RL methods train the control policies in the simulation environment. Compared to the training on the real agents, training in the simulation enjoys the advantages such as reduced time and financial costs as well as less danger. Nevertheless, control policies obtained from simulation-based training may fail when directly applied to real agents due to uncertainties in the real world. This reveals a "sim-to-real" gap. MRAC has the potential to fill such gap by training the reference input for a reference model with RL in the simulation while enforcing the uncertain real plant to track the state of the reference model [60]. As PWA systems have the universal approximation capability to model highly nonlinear and hybrid systems, an interesting direction would be training the reference input for the reference PWA models with RL for complex agents in the simulation and then applying the MRAC developed in this thesis to the real agents such that the behavior of the PWA model in the simulation can be transferred to the real system with identified parameters and safety guarantees. Since the reference PWA model and the real PWA system are connected by the matching conditions, the core of the successful transfer lies in the question, how to design or find the reference PWA models to ensure the fulfillment of matching conditions for complex and large-scale PWA systems?

Avoidance of the sliding mode phenomenon: the PWA systems may enter sliding mode on the switching hyperplanes, which causes the chattering and may be harmful for the actuators. It is of practical interest to explore, how to avoid the sliding mode phenomenon in the adaptive control of uncertain PWA systems. Current research results of PWA systems to rule out the sliding mode are restricted to simple cases such as bimodal autonomous PWA systems [140, 154], continuous autonomous PWA systems [27], where the system parameters and switching hyperplanes are completely known and there is no adaptation mechanism. For adaptive control of PWA systems developed in this thesis, the stability can be guaranteed with CLFs (Chapter 3, Chapter 4) if the closed-loop system enters sliding mode. However, how to enhance these adaptive controllers for PWA systems without the occurrence of sliding mode is still open.

Adaptive Observer for Switched Systems A

In Chapter 6, the MRAC of PWA systems is explored with sensor constraints. This chapter continues to cope with sensor constraints and studies the simultaneous state and parameter estimation of uncertain switched systems.

Over the last decades, a lot of efforts have been devoted to the simultaneous state and parameter estimation of dynamical systems with adaptive observers. Early results of adaptive observer design focus on systems, which can be transformed into canonical form [20, 71, 86, 108]. In [181], a new adaptive observer is proposed for a class of time-varying systems, which does not require the system to be transformed into canonical form. Extensions of this adaptive observer to nonlinearly parameterized systems [51], systems nonlinear in the parameters [157], and stochastic systems [180] are reported. These approaches also enjoy a wide range of applications, e.g., biodiesel fueled engines [183], lithium-ion batteries [162], and antilock braking systems [2]. The fundamental idea of most referenced adaptive observers is to construct a linear regression equation (LRE) by utilizing filtering operations on known signals depending on the inputs and outputs. The LRE enables the application of various adaptive parameter estimation approaches to estimate the unknown parameters. Then the state observation is conducted with the estimated parameters. It is worth pointing out that the LRE captures the forced response of the filtered system whereas the zero-input response stemming from the filtering operation is treated as a disturbance and mostly disregarded, as it is exponentially decaying and does not destroy the convergence of the estimation errors.

Despite the above-mentioned advances, the existing adaptive observers focus on systems with constant unknown parameters and cannot be applied to systems with switched unknown parameters. In practice, the operation conditions of most plants, as mentioned in Chapter 1, may change, which cannot be modeled using constant parameters. This motivates us to explore adaptive observers for systems with switched unknown parameters. The main challenge in this regard lies in the disturbance effect of zero-input responses caused by the switching. These responses are products of unknown states at switching instants (SASI) and the known transient terms. They act as a non-vanishing unknown additive disturbance term to the LRE and prevent the parameter estimation error from converging to zero. This problem cannot be solved by using conventional adaptive observers. However, the recently proposed adaptive observers using dynamic regressor extension and mixing (DREM) [120, 121, 125] provide new inspiration to cope with the zero-input responses. These approaches treat the initial state as an unknown parameter and transform the state observation problem into the parameter estimation problem of the initial state. The key feature of these DREM-based approaches is the ability to ensure the *element-wise* parameter adaptation, namely, the adaptation of each element of the estimated parameter is decoupled from each other. Although methods in [120, 121, 125] are not eligible for switched systems, we are inspired by their element-wise adaptation property and develop an adaptive observer for systems with switched unknown parameters by using DREM, which overcomes the challenge of the treatment of zero-input responses.

The main contribution of this chapter is that we develop an adaptive observer for a class of systems with switched unknown parameters. We achieve asymptotic convergence of state and parameter estimation errors despite the presence of the non-vanishing zero-input responses. Furthermore, the robustness of the proposed adaptive observer is analyzed for the case with noise and disturbances. We emphasize the novelty of the technical route through which we cope with the zero-input responses. Specifically, distinct from the most adaptive observers for systems with constant unknown parameters [2, 20, 51, 86, 108, 157, 162, 180, 181, 183], where the zero-input response is viewed as a disturbance and disregarded, we exploit the known information of the zero-input responses to construct the regressor and augment the system parameters with SASI (the unknown part of zero-input responses). To decouple the system parameter estimation and the evolution of SASI, we propose a DREM-based parameter estimator for the augmented parameter. Thanks to its element-wise adaptation property, the asymptotic convergence of state and parameter estimation errors is achieved.

The rest of this chapter is structured as follows. The problem formulation is given in Sec. A.1. The proposed adaptive observer is depicted in Sec. A.2 with the robustness analysis shown in Sec. A.3. The numerical validation is shown in Sec. A.4. In Sec. A.5, the conclusion is given and future work is discussed.

A.1 Problem Formulation

In this chapter, we consider the following uncertain nonlinear single-input single-output (SISO) switched system

$$\dot{x} = Ax + Bu + \Psi(y, u)\theta^*_{\sigma(t)}, \tag{A.1a}$$

$$y = Cx, \tag{A.1b}$$

where $x \in \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}$ denotes the output signal, and $u \in \mathbb{R}$ represents the input signal of the system. $A \in \mathbb{R}^{n\times n}, B \in \mathbb{R}^{n\times 1}, C \in \mathbb{R}^{1\times n}$ are known constant matrices. The nonlinearity $\Psi(y, u) \in \mathbb{R}^{n\times m}$ is a known time-varying matrix depending on the output y and the input u. The switched system (A.1) has $s \in \mathbb{N}^+$ subsystems and $\theta^*_{\sigma(t)} \in \mathbb{R}^m$ denotes the switched unknown parameter vector with $\theta^*_{\sigma(t)} \in \{\theta^*_1, \theta^*_2, \cdots, \theta^*_s\}$. The switching signal $\sigma(t)$ is known.

The system structure (A.1) stems from the well-known system form with constant unknown parameters studied in [2, 180, 181] with the difference that system (A.1) depicts systems with switched unknown parameters. As depicted in Chapter 1, extending estimation and control methods from linear systems to switched systems is of practical interest as switched systems can model many engineering applications operating in multiple modes such as mechanical systems with friction [165] and backlash [171].

Although system (A.1) appears to be different from the PWA systems studied in the previous chapters, it covers some special class of PWA systems. For instance, if a PWA system is utilized to approximate a Lur'e system (an interconnection between a linear system and a memoryless nonlinearity) [160], then it can be written in form of (A.1). As we will show in Section A.4, the Chua's circuit, a typical PWA system, can be expressed with the system (A.1).

The problem to be solved in this chapter is formulated as follows:

Problem A.1. Given a switched system (A.1) with the known switching signal $\sigma(t)$ and the unknown subsystem parameters $\theta_i^*, i \in \mathcal{I}$, design an adaptive observer based on the input u and the output y to simultaneously estimate the system state x and the subsystem parameters $\theta_i^*, i \in \mathcal{I}$ with asymptotic convergence of the estimation errors.

For the observer design in this chapter, we make the following assumptions:

Assumption A.1. The state $x(t)$, the output $y(t)$, the input $u(t)$ and the parameters θ_i^* are bounded. i.e., $x(t) \in X, y(t) \in Y, u(t) \in U, \forall t \geq 0$ and $\theta_i^* \in \Theta_i, i \in \mathcal{I}$ with $X \in \mathbb{R}^n, Y \in \mathbb{R}, U \in \mathbb{R}, \Theta_i \in \mathbb{R}^m$ being compact sets.

Assumption A.1 is a common assumption in adaptive observer design problem [51, 96].

A.2 Adaptive Observer Design

In this section, we introduce the proposed adaptive observer to solve the above-mentioned problem. We first derive the LRE of (A.1) and redefine the role of the transient zero-input responses caused by switching by augmenting the unknown parameters with the SASI. Then, we develop a DREM-based parameter estimator to decouple the parameter estimation from the SASI. Based on the estimated parameter, we design the state observer and conduct the robustness analysis.

A.2.1 Derivation of LRE

Let us start by transforming the system (A.1) into a LRE. The goal of this step is to establish an algebraic relation between the unknown parameters and the known signals. We rewrite (A.1a) as

$$\dot{x} = (A - KC)x + Bu + Ky + \Psi(y, u)\theta^*_{\sigma(t)} \tag{A.2}$$

with $K \in \mathbb{R}^{n \times 1}$ being an output feedback gain such that $(A - KC)$ is Hurwitz. The time response of $x(t)$ for the interval $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, in which $\theta^*_{\sigma(t)}$ remains constant, can be written as

$$\begin{aligned} x(t) = \Phi(t, t_k)x(t_k) + &\int_{t_k}^{t} \Phi(t, \tau)(Bu(\tau) + Ky(\tau))\mathrm{d}\tau \\ &+ \int_{t_k}^{t} \Phi(t, \tau)\Psi\theta^*_{\sigma(t)}\mathrm{d}\tau, \end{aligned} \tag{A.3}$$

where $\Phi(t, \tau)$ is the state transition matrix associated with $(A - KC)$. From (A.3) we can see that three components constitute the solution of x for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$: the zero-input response associated with the SASI $x(t_k)$, the forced response driven by the known signal $Bu + Ky$ and the forced response driven by the switched uncertain part $\Psi\theta^*_\sigma$. These two forced responses can also be described by two auxiliary signals x_u and x_θ generated by the following dynamics

$$\dot{x}_u = (A - KC)x_u + Bu + Ky, \quad x_u(t_k) = 0, \tag{A.4a}$$

$$\dot{x}_\theta = (A - KC)x_\theta + \Psi\theta^*_{\sigma(t)}, \qquad x_\theta(t_k) = 0, \quad k \in \mathbb{N} \tag{A.4b}$$

Therefore, for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$, equation (A.3) becomes

$$x = \Phi(t, t_k)x(t_k) + x_u + x_\theta. \tag{A.5}$$

Furthermore, let the signal matrix $\Upsilon(t) \in \mathbb{R}^{n\times m}$ be generated by the following dynamics

$$\dot{\Upsilon} = (A - KC)\Upsilon + \Psi, \quad \Upsilon(t_k) = 0, \;\; k \in \mathbb{N}, \tag{A.6}$$

from which one obtains $\Upsilon(t) = \int_{t_k}^{t} \Phi(t, \tau)\Psi \mathrm{d}\tau, t \in [t_k, t_{k+1})$. This together with the solution of x_θ (the third term in (A.3)) leads to

$$x_\theta = \Upsilon\theta^*_{\sigma(t)}. \tag{A.7}$$

The signals x_u, x_θ, Υ can be respectively viewed as filtered signals of $Bu + Ky, \Psi\theta^*_{\sigma(t)}, \Psi$ with the filter parameter $(A - KC)$. x_θ is unknown and to be estimated. x_u, Υ are known and will be used later for the adaptive observer design.

Remark A.1. In the adaptive observer design for non-switching systems [181], it suffices to specify zero initial states of the auxiliary filtered signal x_u, Υ at t_0. In contrast, these signals in our context are reset to zero at each time instant t_k ($k = 0$ for the initial instant and $k \in \mathbb{N}^+$ for switching instants, see (A.4), (A.6)). Such reset is essential for a clear decomposition of x into the zero-input response (see $\Phi(t, t_k)x(t_k)$ in (A.5)) and forced responses (see x_u, x_θ in (A.5)) for every continuous interval $[t_k, t_{k+1}), k \in \mathbb{N}$.

Recall that the goal of this section is to establish an algebraic relation between the unknown parameters and known signals by transforming the system (A.1) into a LRE. To achieve this, we take (A.7) into (A.5), move the known signal x_u to the left side of (A.5), and multiply both sides with C, which yields for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ the following LRE

$$z = C\Upsilon\theta^*_{\sigma(t)} + C\Phi(t, t_k)x(t_k). \tag{A.8}$$

with $z = y - Cx_u$.

Remark A.2. For systems without switching, the LRE would become $z = C\Upsilon\theta^* + C\Phi(t, t_0)x(t_0)$, $t \in [t_0, \infty)$ with θ^* being the constant unknown parameter. In most of the approaches of the current line of research [2, 20, 51, 86, 108, 157, 162, 180, 181, 183], only $z = C\Upsilon\theta^*$ is considered and the zero-input response $C\Phi(t, t_0)x(t_0)$ is disregard due to its exponentially decaying property. The work [9] provides rigorous asymptotic convergence analysis when $C\Phi(t, t_0)x(t_0)$ is not neglected. As oppose to the decaying property in these papers, the zero-input responses $C\Phi(t, t_k)x(t_k)$ in our case build a non-vanishing disturbance signal under intermittent switching, as each switch triggers a zero-input response depending on the SASI $x(t_k), k \in \mathbb{N}^+$. Consequently, the effect of these transients on the parameter estimation cannot be neglected.

The disturbance effect of $C\Phi(t, t_k)x(t_k), k \in \mathbb{N}$ is the main obstacle to obtain the asymptotic convergence of the estimation errors. Observe that $C\Phi(t, t_k)x(t_k)$ consists of the known part $C\Phi(t, t_k)$ and the unknown part $x(t_k)$. Our solution concept is to treat the SASI $x(t_k)$ as a part of the unknown parameters and view $C\Phi(t, t_k)$ as a part of the regressor such that we can make full use of this known signal for the parameter estimation. Namely, we rewrite (A.8) as

$$z = \nu^T\bar{\theta}^*(t) \tag{A.9}$$

with the augmented parameter and regressor

$$\begin{aligned}\nu^T &= [C\Upsilon, C\Phi(t,t_k)] \in \mathbb{R}^{1\times(m+n)},\\ \bar{\theta}^*(t) &= \begin{bmatrix}\theta^*_{\sigma(t)}\\ x^*(t)\end{bmatrix} \in \mathbb{R}^{(m+n)},\end{aligned} \tag{A.10}$$

where $x^*(t) \in \mathcal{X} \triangleq \{x(t_1), x(t_2), \cdots x(t_k), \cdots\}$ and $x^*(t) = x(t_k)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$. Viewing over the whole time interval $t \in [t_0, \infty)$, $x^*(t)$ is a piecewise constant vector.

The augmented parameter vector $\bar{\theta}^*(t)$ in (A.10) is a switched parameter, which remains constant within each interval $[t_k, t_{k+1})$ for $k \in \mathbb{N}$ and switches at each t_k. It consists of two parts: the parameters to be estimated θ^*_i, $i \in \mathcal{I}$ and the state at each switching instant $x^*(t) \in \mathcal{X}$. The cardinality of $\mathcal{I}$ is s while the cardinality of $\mathcal{X}$ is unknown. Due to the mismatch of the cardinalities, it is necessary to develop a parameter estimator, which enables separable adaptations of $\theta^*_{\sigma(t)}$ and x^*. To realize this, we propose a DREM-based parameter estimator as it can achieve element-wise adaptation of the parameter.

A.2.2 DREM-based Parameter Estimator

The LRE (A.9) is derived by incorporating the filterd signals x_u, Υ with the filter parameter $(A-KC)$. Based on this derivation, the first step of DREM is to create $m+n$ LRE by using a set of filters with distinct filter parameters $(A-K_jC), j \in \{1, 2, \cdots, m+n\}$ instead of using a single filter with $(A-KC)$. K_j are designed such that $(A-K_jC)$ are Hurwitz. So we repeat the derivation from (A.2) to (A.10) and replace K with $K_j, j \in \{1, 2, \cdots, m+n\}$. This leads to

$$z_j = \nu_j^T \bar{\theta}^*(t) \tag{A.11}$$

with

$$\begin{aligned}z_j &= y - Cx_{uj},\\ \nu_j^T &= [C\Upsilon_j, C\Phi_j(t,t_k)],\end{aligned} \tag{A.12}$$

where

$$\begin{aligned}\dot{x}_{uj} &= (A-K_jC)x_{uj} + Bu + K_jy, & x_{uj}(t_k) &= 0,\\ \dot{\Upsilon}_j &= (A-K_jC)\Upsilon_j + \Psi, & \Upsilon_j(t_k) &= 0, k \in \mathbb{N}\end{aligned}$$

and $\Phi_j(t,\tau)$ denotes the state transition matrix associated with $(A-K_jC)$. We rewrite the $m+n$ LRE in matrix form and obtain

$$Z_f = \mathcal{N}^T \bar{\theta}^*(t) \tag{A.13}$$

with

$$Z_f = \begin{bmatrix} z_1\\ \vdots\\ z_{m+n}\end{bmatrix} \in \mathbb{R}^{m+n}, \quad \mathcal{N}^T = \begin{bmatrix}\nu_1^T\\ \vdots\\ \nu_{m+n}^T\end{bmatrix} \in \mathbb{R}^{(m+n)\times(m+n)}. \tag{A.14}$$

The second step of DREM suggests multiplying the extended regression equation (A.13) with the adjoint of the extended regressor matrix $\mathcal{N}^T$, denoted by $\mathrm{adj}(\mathcal{N}^T)$. This leads to

$$\mathrm{adj}(\mathcal{N}^T)Z_f = \mathrm{adj}(\mathcal{N}^T)\mathcal{N}^T\bar{\theta}^*(t) = \det(\mathcal{N})\bar{\theta}^*(t) \tag{A.15}$$

with $\det(\cdot)$ denoting the determinant of a matrix. Let $\Delta = \det(\mathcal{N}) \in \mathbb{R}$ and $\bar{\mathcal{Z}} = \mathrm{adj}(\mathcal{N}^T)Z_f \in \mathbb{R}^{m+n}$. From (A.15) we obtain

$$\bar{\mathcal{Z}} = \Delta\bar{\theta}^*(t), \tag{A.16}$$

As Δ is a scalar, (A.16) leads to $m+n$ separate scalar regression equations

$$\bar{\mathcal{Z}} = \begin{bmatrix} \bar{\mathcal{Z}}_1 \\ \vdots \\ \bar{\mathcal{Z}}_m \\ \bar{\mathcal{Z}}_{m+1} \\ \vdots \\ \bar{\mathcal{Z}}_{m+n} \end{bmatrix} = \Delta \begin{bmatrix} \theta^*_{1\sigma(t)} \\ \vdots \\ \theta^*_{m\sigma(t)} \\ x^*_1(t) \\ \vdots \\ x^*_n(t) \end{bmatrix} = \Delta\bar{\theta}^*(t), \tag{A.17}$$

where $\bar{\mathcal{Z}}_j$ is the j-th element of the vector $\bar{\mathcal{Z}}$, θ^*_{ji} represents the j-th element of θ^*_i, and x^*_j denotes the j-th element of x^*.

Recall that the indicator function is an alternative way to describe the switching

$$\chi_i(t) = \begin{cases} 1, & \text{if } \sigma(t) = i, \\ 0, & \text{otherwise.} \end{cases} \tag{A.18}$$

Let $\hat{\theta}_{ji} \in \mathbb{R}$ be the estimated value of θ^*_{ji} and let $\hat{\theta}_i = [\hat{\theta}_{1i}, \cdots, \hat{\theta}_{mi}]^T$. The adaptation of the estimated parameter follows the adaptation law

$$\dot{\hat{\theta}}_{ji} = \gamma_i\chi_i\Delta(\bar{\mathcal{Z}}_j - \Delta\hat{\theta}_{ji}) \tag{A.19}$$

for $i \in \mathcal{I}, j \in \{1, \cdots, m\}$. $\gamma_i \in \mathbb{R}^+$ is a positive scaling factor. This adaptation law gives the parameter error equation

$$\dot{\tilde{\theta}}_{ji} = -\gamma_i\chi_i\Delta^2\tilde{\theta}_{ji} \tag{A.20}$$

for $\tilde{\theta}_{ji}$ being the j-th element of $\tilde{\theta}_i$ with $\tilde{\theta}_i = \hat{\theta}_i - \theta^*_i$. The indicator function χ_i in the adaptation law (A.19) indicates that the value of $\hat{\theta}_i$ remains constant when subsystem i is inactive and $\hat{\theta}_i$ is adapted during the active phase of subsystem i.

As underscored in the introduction of this chapter, the conceptual highlight of this chapter is to convert the role of the zero-input responses from disturbances to excitations. To better understand this concept, we observe from (A.12) and (A.14) that the known part of the zero-input response $C\Phi_j(t, t_k)$ constitutes a part of the regressor matrix $\mathcal{N}$, whose determinant Δ further drives the adaptation of the parameter estimation errors (see (A.20)). Furthermore, the element-wise adaptation property of DREM ensures that the evolution of the unknown part of the zero-input response $x(t_k)$ does not affect the adaptation of the estimated system parameters $\hat{\theta}_i$.

Remark A.3. Simulation results in [180, 181] show that adaptive observers proposed for non-switched systems have the tolerance for rare switches of the parameters at the expense of transient parameter estimation errors after each switch. Due to these transient parameter estimation errors, provable asymptotic convergence of parameter and state estimation errors cannot be established. Moreover, performance degradation may occur when the time between two successive switches of parameters is not long enough to let the transients converge. One feature that distinguishes our method from these methods is that each subsystem has its own estimated parameter $\hat{\theta}_i, i \in \mathcal{I}$. The estimated parameter $\hat{\theta}_i$ is only adapted when i-th subsystem is activated. Otherwise, $\hat{\theta}_i$ is frozen and is retained as the initial value for the next active period for i-th subsystem. Therefore, asymptotic convergence of parameter estimation errors can be achieved without suffering from transient errors after each switch.

A.2.3 Adaptive State Observer

After obtaining the parameter adaptation law (A.19) for the estimated parameters $\hat{\theta}_i$, the adaptive state observer to estimate the state x is given based on $\hat{\theta}_i$

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + \Psi(y,u)\hat{\theta}_{\sigma(t)} + K(y-\hat{y}), \\ \hat{y} &= C\hat{x} \end{aligned} \tag{A.21}$$

where $\hat{x}, \hat{y}$ denote the estimated state and output, respectively. $K \in \mathbb{R}^{n\times 1}$ is to be chosen such that $(A - KC)$ is Hurwitz.

Assumption A.2. The scalar signal $\chi_i\Delta$ satisfies $\chi_i\Delta \notin \mathcal{L}_2, \forall i \in \mathcal{I}$.

The performance of the proposed adaptive observer (A.19), (A.21) can be summarized below.

Theorem A.1. *Consider the switched system (A.1) with unknown parameters $\theta_i^*, i \in \mathcal{I}$ and the adaptive observer (A.21) with the adaptation law (A.19). If Assumption A.1 and Assumption A.2 hold, then we have the parameter estimation error $\tilde{\theta}_i(t) \to 0, \forall i \in \mathcal{I}$ as $t \to \infty$ and the state estimation error $\tilde{x}(t) = \hat{x}(t) - x(t) \to 0$ as $t \to \infty$.*

Proof. From (A.20), we have

$$\tilde{\theta}_{ji}(t) = \mathrm{e}^{-\gamma_i \int_{t_0}^{t} \chi_i(s)\Delta^2(s)\mathrm{d}s}\tilde{\theta}_{ji}(t_0). \tag{A.22}$$

Since $\chi_i\Delta \notin \mathcal{L}_2, i \in \mathcal{I}$, we have $\tilde{\theta}_{ji} \to 0$ and therefore, $\tilde{\theta}_i \to 0$ as $t \to \infty$. From (A.1) and (A.21) we obtain for $\tilde{x} = \hat{x} - x$

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + \Psi\tilde{\theta}_{\sigma(t)}. \tag{A.23}$$

Since $\tilde{\theta}_i \to 0, \forall i \in \mathcal{I}$ for $t \to \infty$ and $(A - KC)$ is Hurwitz, it leads to $\tilde{x} \to 0$ as $t \to \infty$. □

Remark A.4. Regarding the zero-input response, the underlying concept of the adaptive observers for non-switched systems (e.g., see [20, 51, 181]) or adaptive control for switched systems (see [135] and Chapter 6) is treating the zero-input response as a disturbance, regardless of whether neglecting it or including it in the stability analysis [9, 135]. Distinct from this concept, we provide a new perspective that the zero-input responses can be utilized as excitations to promote the parameter estimation such that asymptotic convergence of state and parameter estimation errors can be achieved.

Remark A.5. The DREM-based adaptive observers in [120, 125] augment the system parameters with the initial state. Then the state estimation is established based on the identified initial state through an open-loop integration [120] or a non-fragile algebraic equation [125]. As oppose to these approaches, the purpose of augmenting the system parameters with SASI in this note is to exploit the known information of the zero-input responses in the regressor whereas the estimation of SASI is not of interest and disregarded.

Remark A.6. The convergence analysis of the parameter estimation in the previous chapters requires that each subsystem is activated intermittently [84, 99, 173]. In this chapter, this condition is implicitly included in the condition $\chi_i \Delta \notin \mathcal{L}_2, \forall i \in \mathcal{I}$ (Assumption A.2). Specifically, if there exists subsystem $l \in \mathcal{I}$ that is not intermittently activated, then there exists $\check{t} \geq 0$ such that $\sigma(t) \neq l$ and $\chi_l(t) = 0, \forall t \in [\check{t}, \infty)$. This would lead to $\chi_l \Delta \in \mathcal{L}_2$, which contradicts with the condition $\chi_i \Delta \notin \mathcal{L}_2, \forall i \in \mathcal{I}$. Therefore, Assumption A.2 implies that every subsystem is activated intermittently.

To frame the current result in the state of the art, it is worth distinguishing our method from *switched adaptive observers* in the literature. Switched adaptive observers [2, 3] are proposed to deal with non-uniformly observable systems (systems in form of (A.1) but with a time-varying $A(t)$ matrix such that no constant K matrix can be found to let $(A(t)-KC)$ be Hurwitz). A switching output feedback gain K_σ is designed such that $(A(t) - K_\sigma C)$ is always Hurwitz. The reset adaptive observer [122] utilizes a reset integral term in the state observer aiming to improve the transient behavior of the adaptive observer. All the above-mentioned works [2, 3, 122] deal with systems with constant unknwon parameters. The adaptive observers therein cannot be applied to systems with switched unknown parameters. A switched adaptive observer for switched nonlinear systems is proposed in [96]. The adaptation law therein still follows the concept of [181], namely, using single parameter adaptation to estimate parameters with jumps. Therefore, it suffers from the same problem mentioned in Remark A.3.

One potential drawback of using the DREM-based parameter estimator is the computational complexity as $n+m$ LREs are involved. In view of this, one question which may arise is why not use gradient-based adaptation [181] or least square-based adaptation [51], where only one LRE (e.g., (A.8)) is needed with relatively lower computational costs? We explain this point as follows.

The reason for using DREM lies in its ability to establish element-wise relationships such that the adaptation of the parameter $\hat{\theta}_{ji}$ is decoupled from the jumps of x^*. In contrast, neither the gradient method nor the least square-based adaptation can achieve such decoupling. We take the gradient method as an example. Specifically, the gradient method would suggest for the estimated augmented parameter vector (denoted by $\hat{\bar{\theta}}_i$ for i-th subsystem) in (A.9) the following adaptation law

$$\dot{\hat{\bar{\theta}}}_i = \gamma_i \chi_i \nu (z - \nu^T \hat{\bar{\theta}}_i) = -\gamma_i \chi_i \nu \nu^T \tilde{\bar{\theta}}_i. \tag{A.24}$$

for $\tilde{\bar{\theta}}_i = \hat{\bar{\theta}}_i - \bar{\theta}^*$. Together with the excitation condition that $\chi_i \nu$ is PE, this would lead to an exponential decrease of the norm $|\tilde{\bar{\theta}}_i(t)|$ in the time interval

$$|\tilde{\bar{\theta}}_i(t)| \leq \kappa_i' \mathrm{e}^{-\gamma_i \chi_i \kappa_i (t - t_{i_k})} |\tilde{\bar{\theta}}_i(t_{i_k})|, \quad t \in [t_{i_k}, t_{i_k+1})$$

$\kappa, \kappa' \in \mathbb{R}^+$ are some positive constants, $[t_{i_k}, t_{i_k+1}), k \in \mathbb{N}^+$ denote time intervals in which i-th subsystem is active. Note that we cannot expect the state at the switch-out instant of i-th subsystem $x(t_{i_k+1})$ to be equal to the state at the next switch-in instant $x(t_{i_{k+1}})$. Namely, $x^*(t_{i_k+1}) = x^*(t_{i_{k+1}})$ does not necessarily hold. This implies a potential instantaneous jump of $\bar{\theta}^*$ at each t_{i_k}. As $\hat{\bar{\theta}}_i$ evolves continuously and $\tilde{\bar{\theta}}_i = \hat{\bar{\theta}}_i - \bar{\theta}^*$, the jump of $\bar{\theta}^*$ leads to the jump of $\tilde{\bar{\theta}}_i$ at each t_{i_k}. These jumps together with the exponential decrease of $|\tilde{\bar{\theta}}_i|$ in each

$[t_{i_k}, t_{i_k+1}), k \in \mathbb{N}^+$ does not lead to the convergence of $|\tilde{\tilde{\theta}}_i|$ to 0. In this regard, the jumps of x^* is coupled with the parameter vector $\tilde{\theta}_i$ through the norm operator and obstructs the convergence of $\tilde{\theta}_i$ to 0.

A.3 Robustness Analysis

In this section, we study the robustness of the adaptive observer when applying it to systems with disturbances and noise. Consider the system

$$\dot{x} = Ax + Bu + \Psi(y,u)\theta^*_{\sigma(t)} + \omega, \tag{A.25a}$$
$$y = Cx, \tag{A.25b}$$
$$\bar{y} = y + v, \tag{A.25c}$$

where $\omega \in \mathbb{R}^n$ represents the state disturbance. $\bar{y}$ denotes the measured output with $v \in \mathbb{R}$ being the measurement noise. ω and v are bounded, i.e., $|\omega| \le \omega_0, |v| \le v_0$ for some constants $\omega_0, v_0 \in \mathbb{R}^+$.

With regards to the nonlinear function $\Psi(y,u)$ in (A.25), we make the additional assumption as follows:

Assumption A.3. The function $\Psi(y,u)$ in (A.25) is Lipschitz with respect to y. That is, there exists positive constant $L_\Psi \in \mathbb{R}^+$ such that $\forall u \in U$ and $y, \bar{y} \in Y$ we have $\|\Psi(y,u) - \Psi(\bar{y},u)\| \le L_\Psi|y-\bar{y}| = L_\Psi|v| \le L_\Psi v_0$.

To study how ω and v affect the stability, we rederive the error equation (A.20). We start by rewriting (A.25a) as

$$\dot{x} = (A - K_jC)x + Bu + K_jy + \Psi(y,u)\theta^*_{\sigma(t)} + \omega \tag{A.26}$$

for some $K_j \in \mathbb{R}^{n\times 1}, j \in \{1, 2, \cdots, m+n\}$ such that $(A - K_jC)$ is Hurwitz. The time response of $x(t)$ for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$ can be written as

$$\begin{aligned} x(t) =& \Phi_j(t,t_k)x(t_k) + \int_{t_k}^t \Phi_j(t,\tau)(Bu + K_jy)\mathrm{d}\tau \\ &+ \int_{t_k}^t \Phi_j(t,\tau)\Psi(y,u)\theta^*_\sigma \mathrm{d}\tau + \int_{t_k}^t \Phi_j(t,\tau)\omega \mathrm{d}\tau. \end{aligned} \tag{A.27}$$

We use the measured output $\bar{y}$ to generate the filtered signals

$$\begin{aligned} \dot{x}_{uj} &= (A - K_jC)x_{uj} + Bu + K_j\bar{y}, \quad x_{uj}(t_k) = 0, \\ \dot{\Upsilon}_j &= (A - K_jC)\Upsilon_j + \Psi(\bar{y},u), \qquad \Upsilon_j(t_k) = 0, k \in \mathbb{N} \end{aligned}$$

Let $z_j = \bar{y} - Cx_{uj}$, which gives for $t \in [t_k, t_k+1), k \in \mathbb{N}$

$$z_j = Cx + v - C(\int_{t_k}^t \Phi_j(t,\tau)(Bu + K_jy + K_jv)\mathrm{d}\tau). \tag{A.28}$$

Substituting x with (A.27) yields

$$z_j = C\Phi_j(t,t_k)x(t_k) + C\Upsilon_j\theta^*_{\sigma(t)} + d_j \tag{A.29}$$

with the disturbance-related term d_j being expressed by

$$\begin{aligned} d_j =& C\int_{t_k}^{t} \Phi_j(t,\tau)(\Psi(y,u)-\Psi(\bar{y},u))\theta^*_{\sigma(t)}\mathrm{d}\tau \\ &+ C\int_{t_k}^{t} \Phi_j(t,\tau)\omega\mathrm{d}\tau + v - C\int_{t_k}^{t} \Phi_j(t,\tau)K_j v\mathrm{d}\tau. \end{aligned} \tag{A.30}$$

With the regressor $\nu_j^T = [C\Upsilon_j, C\Phi_j(t,t_k)]$, equation (A.29) can be written in the linear regression form

$$z_j = \nu_j^T\bar{\theta}^*(t) + d_j. \tag{A.31}$$

Stacking $m+n$ equations yields

$$Z_f = \mathcal{N}^T\bar{\theta}^*(t) + d \tag{A.32}$$

with $d = [d_1, d_2, \cdots, d_{m+n}]^T$. Multiplying both sides with $\mathrm{adj}(\mathcal{N}^T)$ leads to

$$\bar{\mathcal{Z}} = \Delta\bar{\theta}^*(t) + \bar{D} \tag{A.33}$$

with $\bar{D} = \mathrm{adj}(\mathcal{N}^T)d$. We apply the same parameter adaptation law as in the ideal case (A.19) and obtain the element-wise parameter error equation

$$\dot{\tilde{\theta}}_{ji} = -\gamma_i\chi_i\Delta^2\tilde{\theta}_{ji} + \gamma_i\chi_i\Delta\bar{D}_j, \tag{A.34}$$

where $i \in \mathcal{I}, j \in \{1,2,\cdots,m\}$, $\bar{D}_j$ is the j-th element of $\bar{D}$. Let D be a vector of the first m elements of $\bar{D}$, i.e., $D = [\bar{D}_1, \bar{D}_2, \cdots, \bar{D}_m]^T$. We can write (A.34) into the vector form

$$\dot{\tilde{\theta}}_i = -\gamma_i\chi_i\Delta^2\tilde{\theta}_i + \gamma_i\chi_i\Delta D. \tag{A.35}$$

The adaptive observer is constructed based on the measured output $\bar{y}$ and (A.21) now becomes

$$\begin{aligned} \dot{\hat{x}} &= A\hat{x} + Bu + \Psi(\bar{y},u)\hat{\theta}_{\sigma(t)} + K(\bar{y}-\hat{y}), \\ \hat{y} &= C\hat{x} \end{aligned} \tag{A.36}$$

Assumption A.4. The scalar signal $\chi_i\Delta$ is PE, $\forall i \in \mathcal{I}$.

The following theorem summaries the robustness of the proposed adaptive observer in the presence of disturbances and noise.

Theorem A.2. *Consider the switched system (A.25) with unknown parameters $\theta_i^*, i \in \mathcal{I}$ and the adaptive observer (A.36) with the adaptation law (A.19). If Assumption A.1, Assumption A.3, and Assumption A.4 hold, then the parameter estimation error $\tilde{\theta}_i$ and the state estimation error $\tilde{x}$ converge to the residual set*

$$\mathcal{R}_e = \{\tilde{x} \in \mathbb{R}^n, \tilde{\theta}_i \in \mathbb{R}^m \Big| |\tilde{x}| + |\tilde{\theta}_i| \leq \mu\sup_t|\Delta D| + c\} \tag{A.37}$$

for some positive constants $\mu, c \in \mathbb{R}^+$.

Proof. We start the proof by showing that $d_j, j \in \{1, 2, \cdots, n+m\}$ in (A.30) is bounded for $t \in [t_0, \infty)$. As $\Phi_j(t,\tau)$ is the state-transition matrix of the Hurwitz matrix $(A - K_jC)$, there exist constants $\beta_j, \beta_j' \in \mathbb{R}^+$ such that $\|\Phi_j(t,\tau)\| \leq \beta_j' \mathrm{e}^{-\beta_j(t-\tau)}$. Therefore, for $t \in [t_k, t_{k+1})$, we have from (A.30) the following inequality

$$\begin{aligned}|d_j| \leq \beta_j'|C| \int_{t_k}^{t} \mathrm{e}^{-\beta_j(t-\tau)} \|\Psi(y,u) - \Psi(\bar{y},u)\| |\theta^*_{\sigma(t)}| \mathrm{d}\tau \\ + \beta_j'|C| \int_{t_k}^{t} \mathrm{e}^{-\beta_j(t-\tau)} (|\omega| + |K_j||v|) \mathrm{d}\tau + |v|.\end{aligned}$$

Let $L_\theta = \max_i |\theta_i^*|$ and $\Delta t_k = t_{k+1} - t_k, k \in \mathbb{N}$. Due to Assumption A.3 we obtain for $t \in [t_k, t_{k+1}), k \in \mathbb{N}$

$$\begin{aligned}|d_j| &\leq \beta_j'|C| \int_{t_k}^{t} \mathrm{e}^{-\beta_j(t-\tau)} (L_\Psi L_\theta v_0 + \omega_0 + |K_j|v_0) \mathrm{d}\tau + v_0 \\ &\leq \frac{\beta_j'}{\beta_j}|C|(L_\Psi L_\theta v_0 + \omega_0 + |K_j|v_0)(1 - \mathrm{e}^{-\beta_j \Delta t_k}) + v_0 \\ &< \frac{\beta_j'}{\beta_j}|C|(L_\Psi L_\theta v_0 + \omega_0 + |K_j|v_0) + v_0,\end{aligned}$$

which together with $\mathrm{adj}(\mathcal{N}^T) \in \mathcal{L}_\infty$ leads to $D, \bar{D} \in \mathcal{L}_\infty$.

From (A.35), we have

$$\tilde{\theta}_i(t) = \mathrm{e}^{-\gamma_i \int_{t_0}^{t} \chi_i \Delta^2(s) \mathrm{d}s} \tilde{\theta}_i(t_0) + \gamma_i \int_{t_0}^{t} \mathrm{e}^{-\gamma_i \int_{\tau}^{t} \chi_i \Delta^2(s) \mathrm{d}s} \chi_i \Delta D \mathrm{d}\tau.$$

Since $\chi_i \Delta$ is PE, there exist constants $\alpha_i', \alpha_i \in \mathbb{R}^+$ such that

$$|\tilde{\theta}_i(t)| \leq \alpha_i' \mathrm{e}^{-\gamma_i \alpha_i (t-t_0)} |\tilde{\theta}_i(t_0)| + \alpha_i' \gamma_i \int_{t_0}^{t} \mathrm{e}^{-\gamma_i \alpha_i (t-t_0)} |\chi_i \Delta D| \mathrm{d}\tau, \tag{A.38}$$

which further leads to

$$\lim_{t\to\infty} \sup_{\tau \geq t} |\tilde{\theta}_i(\tau)| \leq \frac{\alpha_i'}{\alpha_i} \sup_t |\chi_i(t)\Delta(t)D(t)|. \tag{A.39}$$

From (A.25) and (A.36) we obtain

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + (\Psi(\bar{y},u)\hat{\theta}_{\sigma(t)} - \Psi(y,u)\theta^*_{\sigma(t)}) - w + Kv,$$

which can be further rearranged as

$$\dot{\tilde{x}} = (A - KC)\tilde{x} + (\Psi(\bar{y},u) - \Psi(y,u))\hat{\theta}_{\sigma(t)} + \Psi(y,u)\tilde{\theta}_{\sigma(t)} - w + Kv.$$

Recalling that $(A - KC)$ is Hurwitz, there exist constants $\beta, \beta' \in \mathbb{R}^+$ such that

$$\begin{aligned}|\tilde{x}(t)| \leq &\beta' \mathrm{e}^{-\beta(t-t_0)} |\tilde{x}(t_0)| + \beta' \int_{t_0}^{t} \mathrm{e}^{-\beta(t-\tau)} (L_\Psi v_0 |\hat{\theta}_{\sigma(t)}| \\ &+ \sup_t \|\Psi(y,u)\| |\tilde{\theta}_{\sigma(t)}| + w_0 + |K|v_0) \mathrm{d}\tau.\end{aligned}$$

Since $|\hat{\theta}_{\sigma(t)}| \leq |\tilde{\theta}_{\sigma(t)}| + |\theta^*_{\sigma(t)}| \leq |\tilde{\theta}_{\sigma(t)}| + L_\theta$, we obtain

$$\begin{aligned}|\tilde{x}(t)| \leq & \beta' \mathrm{e}^{-\beta(t-t_0)}|\tilde{x}(t_0)| + \beta' \int_{t_0}^{t} \mathrm{e}^{-\beta(t-\tau)}(L_\Psi v_0 |\tilde{\theta}_{\sigma(t)}| \\ & + L_\Psi L_\theta v_0 + \sup_t \|\Psi(y,u)\| |\tilde{\theta}_{\sigma(t)}| + w_0 + |K| v_0)\mathrm{d}\tau.\end{aligned}$$

This further yields

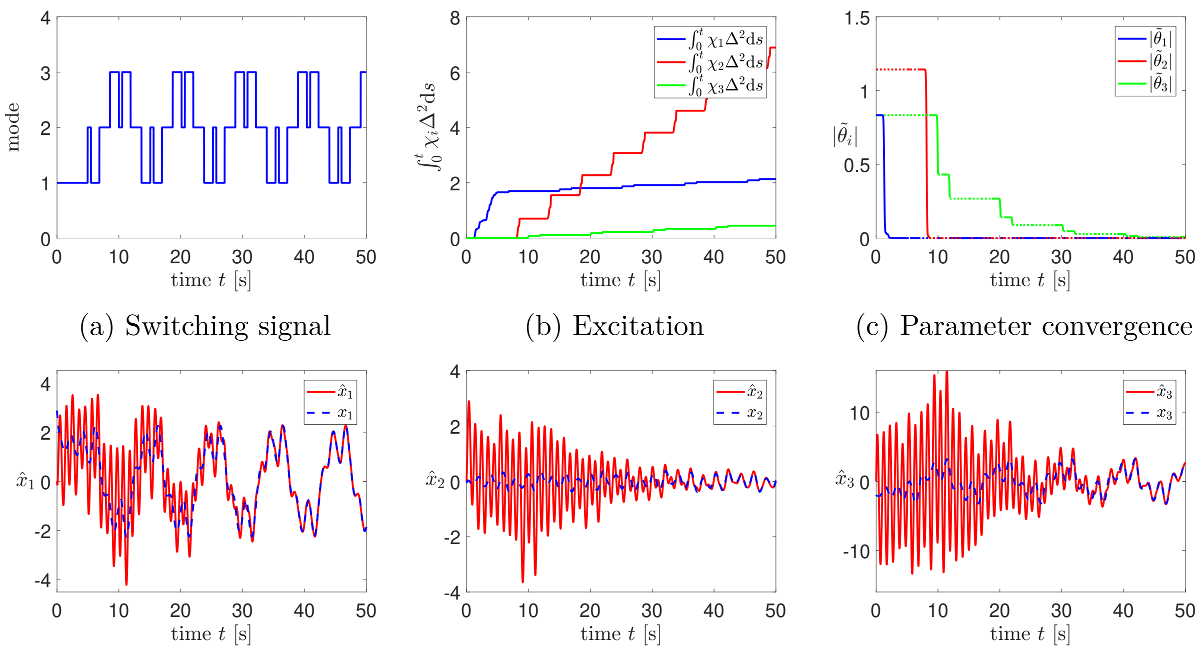
$$\lim_{t\to\infty} \sup_{\tau \geq t} |\tilde{x}(\tau)| \leq \frac{\beta'}{\beta}((L_\Psi v_0 + \sup_t \|\Psi(y,u)\|)|\tilde{\theta}_{\sigma(t)}| + L_\Psi L_\theta v_0 + w_0 + |K| v_0),$$

which together with (A.39) gives (A.37) with

$$\begin{aligned}\mu &= \max_i \frac{\alpha_i'}{\alpha_i}(1 + \frac{\beta'}{\beta}(L_\Psi v_0 + \sup_t \|\Psi(y,u)\|)), \\ c &= \frac{\beta'}{\beta}(L_\Psi L_\theta v_0 + w_0 + |K| v_0).\end{aligned} \tag{A.40}$$

This completes the proof. □

Remark A.7. Compared to the disturbance-free case, Theorem A.2 requires a stronger excitation condition that $\chi_i \Delta$ is PE, which is instrumental to ensure the boundedness of $\tilde{x}$ and $\tilde{\theta}_i$. In case the PE condition cannot be satisfied in some circumstances, robust modifications such as projections and leakages revisited in Chapter 2 can be applied to the adaptation law (A.19). The modified adaptation law together with the boundedness of D in (A.35) would lead to the boundedness of $\tilde{x}$ and $\tilde{\theta}_i$ (see [71, Ch. 9.2]).



(a) Switching signal (b) Excitation (c) Parameter convergence

(d) State estimation $\hat{x}_1$ (e) State estimation $\hat{x}_2$ (f) State estimation $\hat{x}_3$

Figure A.1: Parameter and state estimation using the proposed adaptive observer.

A.4 Numerical Validation

In this section, the proposed adaptive observer is validated through a numerical example of the chaotic oscillator, the Chua's circuit, adjusted from the literature [31, 182]. Its system equation is given by

$$\begin{cases} \dot{x}_1 = p_0(-x_1 + x_2 - g(x_1)) \\ \dot{x}_2 = x_1 - x_2 + x_3 \\ \dot{x}_3 = -q_0 x_2 - r_0 x_3 \end{cases} \tag{A.41}$$

where $p_0 = 10, q_0 = 16, r_0 = 0.0385$ are known parameters. Let $x = [x_1, x_2, x_3]^T$ be the state vector. The system output $y = x_1$ is measurable and x_2, x_3 are to be estimated. The function $g(x_1)$ is a piecewise linear function

$$g(x_1) = \begin{cases} -0.7143x_1 - 0.4286, & \text{for } x_1 \geq 1 \\ -1.1429x_1, & \text{for } |x_1| < 1 \\ -0.7143x_1 + 0.4286, & \text{for } x_1 \leq -1 \end{cases} \tag{A.42}$$

Therefore, the system (A.41) can be written in form of (A.1) with

$$A = \begin{bmatrix} -p_0 & p_0 & 0 \\ 1 & -1 & 1 \\ 0 & -q_0 & -r_0 \end{bmatrix}, \quad \Psi = -p_0 \begin{bmatrix} y & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}, \tag{A.43}$$

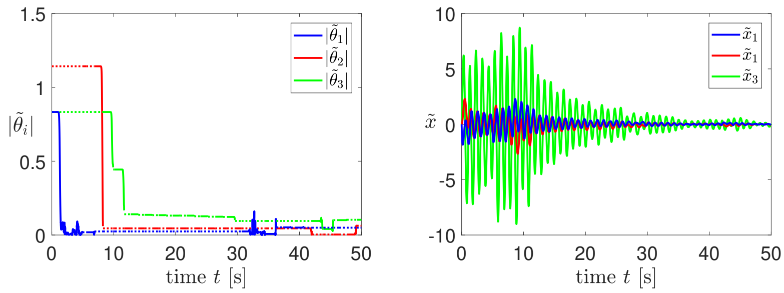
$B = [0, 0, 0]^T$ and $C = [1, 0, 0]$. The initial state of the system is $x(0) = [2.88, -0.066, -2.12]^T$. The nominal parameters to be estimated are

$$\begin{aligned} \theta_1^* &= [-0.7143, -0.4286]^T, \\ \theta_2^* &= [-1.1429, 0]^T, \\ \theta_3^* &= [-0.7143, 0.4286]^T. \end{aligned} \tag{A.44}$$

The switching signal $\sigma(t) = 1$ for $x_1(t) \in \Omega_1 = \{x_1 | \, x_1 \geq 1\}$, $\sigma(t) = 2$ for $x_1(t) \in \Omega_2 = \{x_1 | \, |x_1| < 1\}$, and $\sigma(t) = 3$ for $x_1(t) \in \Omega_3 = \{x_1 | \, x_1 \leq -1\}$.

Now we evaluate the estimation performance of our proposed adaptive observer in the ideal case. The filter parameters $\{K_j\}_{j=1}^5$ are chosen as $K_1 = [0, -1, -15]^T$, $K_2 = [-2, 2.5, 20]^T$, $K_3 = [-2, 0.1, 1]^T$, $K_4 = [-0.4, -0.4, -8]^T$, $K_5 = [-8, 6.5, 18]^T$ and K in the state observer (A.21) is $K = [-2, 2.5, 20]^T$. The initial value of the observer $\hat{x}(0) = [0, 0, 0]^T$. We specify the scaling factors $\gamma_i = 10, i = \{1, 2, 3\}$. The switching signal is show in Fig. A.1a. We can observe the intermittent switching, namely, every mode is repeatedly activated. Fig. A.1b shows the evolution of integrals $\int_0^t \chi_i(s)\Delta^2(s)\mathrm{d}s, i \in \mathcal{I}$, from which one can conclude $\chi_i \Delta \notin \mathcal{L}_2$. The norm of the parameter estimation error for each subsystem $|\tilde{\theta}_i|$ is shown in Fig. A.1c, where dashed sections represent the inactive phase and solid sections represent the active phase. As it can be seen from Fig. A.1c, the value of $|\tilde{\theta}_i|$ stays unchanged during the inactive phase whereas it, thanks to the use of DREM, decreases monotonically during the active phase. Furthermore, The estimated parameters of all subsystems converge to 0. The element-wise state estimation is shown in Fig. A.1d, Fig. A.1e, and Fig. A.1f, respectively. The red solid lines display estimated states and the blue dashed lines represent real states. One can observe that the state estimation errors also converge to 0, this together with the parameter convergence validates the theoretical results of Theorem A.1 that the proposed method is able to eliminate the disturbance effect of the zero-input responses and achieves asymptotic convergence of state and parameter estimation errors.

Finally, we show the robustness of the proposed adaptive observer in the presence of disturbances and noise. The filter parameters $\{K_j\}_{j=1}^5$, the scaling factors γ_i, as well as the initial value of the observer $\hat{x}$ are specified to be the same as those in the ideal case. The disturbance term in (A.25a) is $\omega = [0.05\sin 7t, 0.005\sin 5t, 0.1\sin 13t]$. v in (A.25c) is generated as random numbers with $|v| \le v_0 = 0.1$. In the simulation, the true switching signal $\sigma(t)$ of the plant is used for the switching of the parameter estimator and the state observer.



(a) Parameter estimation (b) State estimation

Figure A.2: Parameter and state estimation using the proposed adaptive observer in the presence of disturbances and noise.

The parameter estimation errors $|\tilde{\theta}_i|$ and the state estimation error $\tilde{x}$ are shown in Fig. A.2a and Fig. A.2b, respectively. Despite of the presence of disturbances and noise, both the parameter estimation errors and the state estimation error converge to bounded sets, which implies the robustness of the proposed adaptive observer.

A.5 Summary

In this chapter, we studied the simultaneous state and parameter estimation of uncertain switched systems with sensor constraints. Instrumental for this task is the derivation of a new LRE, which takes the intermittently appeared zero-input responses into account. We underscore the novelty that we convert the known information of zero-input responses into a part of an augmented LRE and propose a DREM-based parameter estimator to decouple the parameter adaptation from the SASI of these responses. With the proposed adaptive observer, we managed to eliminate the disturbance effect of the zero-input responses and achieve asymptotic convergence of state and parameter estimation errors. Moreover, we have proved the robustness of the proposed method by showing that the state and parameter estimation errors converge to a bounded set in the presence of disturbances and noise.

The key advantage of using DREM in our adaptive observer is that it enables element-wise adaptation and thus decouples the adaptation of the system parameters from the jumps of the SASI. In contrast, classical adaptive observers can only achieve the decrease of the *norm* of the parameter estimation error vector. As a result, the jumps of the SASI would obstruct the convergence of the system parameters via the norm operation. In addition, a byproduct advantage of using DREM in our approach is that it does not require the classical persistence of excitation condition to achieve the parameter convergence in the disturbance-free case.

The theoretical results are validated through a numerical example of Chua's circuit, a typical application example in form of A.1 with constant matrices A, B. The proposed method can be generalized to systems with switched system matrices $A_{\sigma(t)}, B_{\sigma(t)}$ (instead of the constant ones A, B in (A.1)) by following the same procedure but using switched filter parameters and a switched observer gain (see [102, Remark 6]).

One limitation of the proposed method is that the switching signal is required to be known. For PWA systems, whose region partitions depend on the input and output signals but the output measurement is corrupted with noises, the switching signal will be imprecise because it is obtained by evaluating in which region the measured output locates. How the imprecise switching signal affects the convergence property and the robustness of the state and parameter estimation remains to be explored in future work.

Moreover, the proposed adaptive observer is eligible for SISO linearly parameterized switched systems. To enlarge the application domain of system classes, future work can also focus on the extension of the proposed adaptive observer to MIMO and nonlinearly parameterized switched systems.

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