Optimum Decoding of Modified Polar Codes: From Inactivation Decoding to Tree-Search

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Polar Codes

IEEE TRANSACTIONS ON INFORMATION THEORY, VOL. 55, NO. 7, JULY 2009 3051

Channel Polarization: A Method for Constructing Capacity-Achieving Codes for Symmetric Binary-Input Memoryless Channels

Erdal Arıkan, *Senior Member, IEEE*

***Abstract*—A method is proposed, called channel polarization, to construct code sequences that achieve the symmetric capacity $I(W)$ of any given binary-input discrete memoryless channel (B-DMC) W. The symmetric capacity is the highest rate achievable subject to using the input letters of the channel with equal probability. Channel polarization refers to the fact that it is pos-**

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We write $W : \mathcal{X} \rightarrow \mathcal{Y}$ to denote a generic B-DMC with input alphabet $\mathcal{X}$, output alphabet $\mathcal{Y}$, and transition probabilities $W(y|x)$, $x \in \mathcal{X}$, $y \in \mathcal{Y}$. The input alphabet $\mathcal{X}$ will always be $\{0, 1\}$, the output alphabet and the transition probabilities may

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- But successive cancellation (SC) decoding performs poorly for small blocks.

Successive List Cancellation Decoding

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***Abstract*— We describe a successive-cancellation list decoder for polar codes, which is a generalization of the classic successive-cancellation decoder of Arıkan. In the proposed list decoder, *L* decoding paths are considered concurrently at each decoding stage, where *L* is an integer parameter. At the end of the decoding process, the most likely among the *L* paths is selected as the single codeword at the decoder output. Simulations show that the resulting performance is very close to that of maximum-likelihood decoding, even for moderate values of *L*. Alternatively, if a genie is allowed to pick the transmitted codeword from the list, the results are comparable with the performance of current state-of-the-art LDPC codes. We show that such a genie can be easily implemented using simple CRC precoding. The specific list-decoding algorithm that achieves this performance doubles the number of decoding paths for each information bit, and then uses a pruning procedure to discard all but the *L* most likely paths. However, straightforward implementation of this**

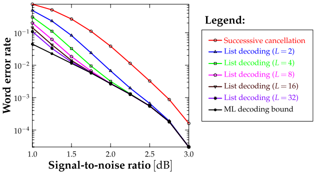

Fig. 1. List-decoding performance for a polar code of length $n = 2048$ and rate $R = 0.5$ on the BPSK-modulated Gaussian channel. The code was constructed using the methods of [15], with optimization for $E_b/N_0 = 2$ dB.

- SC list (SCL) decoding with CRC and large list-size performs very well and approaches maximum-likelihood (ML) decoding performance [TV15].

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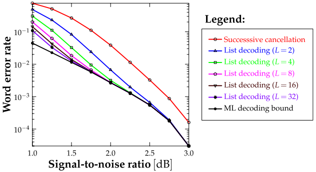

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- SC list (SCL) decoding with CRC and large list-size performs very well and approaches maximum-likelihood (ML) decoding performance [TV15].
- It can also be used to decode other codes (e.g., Reed–Muller codes).

Polar Codes with Dynamic Frozen Bits

254 IEEE JOURNAL ON SELECTED AREAS IN COMMUNICATIONS, VOL. 34, NO. 2, FEBRUARY 2016

Polar Subcodes

Peter Trifonov, *Member, IEEE*, and Vera Miloslavskaya, *Member, IEEE*

***Abstract*—An extension of polar codes is proposed, which allows some of the frozen symbols, called dynamic frozen symbols, to be data-dependent. A construction of polar codes with dynamic frozen symbols, being subcodes of extended BCH codes, is proposed. The proposed codes have higher minimum distance than classical polar codes, but still can be efficiently decoded using the successive cancellation algorithm and its extensions. The codes with Arikan, extended BCH and Reed-Solomon kernel are considered. The proposed codes are shown to outperform LDPC and turbo codes, as well as polar codes with CRC.**

RM codes, and are therefore likely to provide better finite length performance. However, there are still no efficient MAP decoding algorithms for these codes.

It was suggested in [17] to construct subcodes of RM codes, which can be efficiently decoded by a recursive list decoding algorithm. In this paper we generalize this approach, and propose a code construction "in between" polar codes and EBCH codes. The proposed codes can be efficiently decoded using the techniques developed in the area of polar coding, but provide

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- Later, polar codes were extended with the concept of dynamic frozen bits, which enabled state-of-art designs.
- It is also shown that any code can be decoded using SCL decoding, but some require very large complexity for a good performance.

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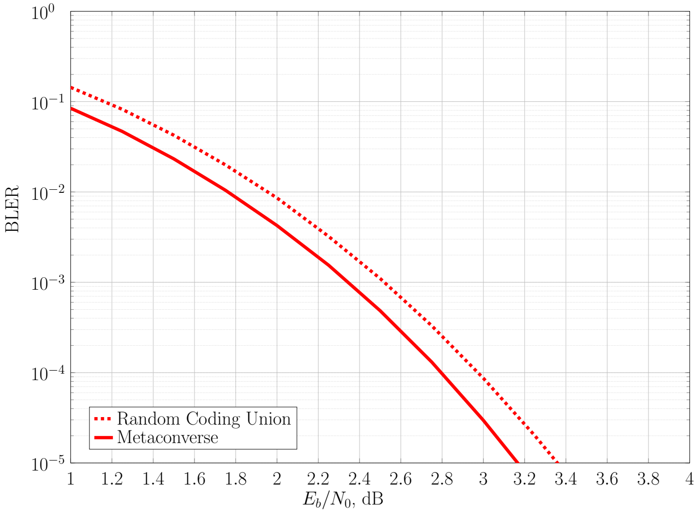
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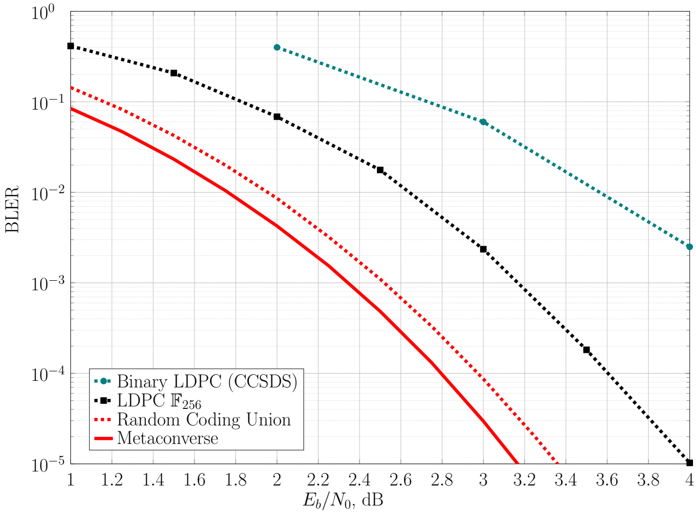
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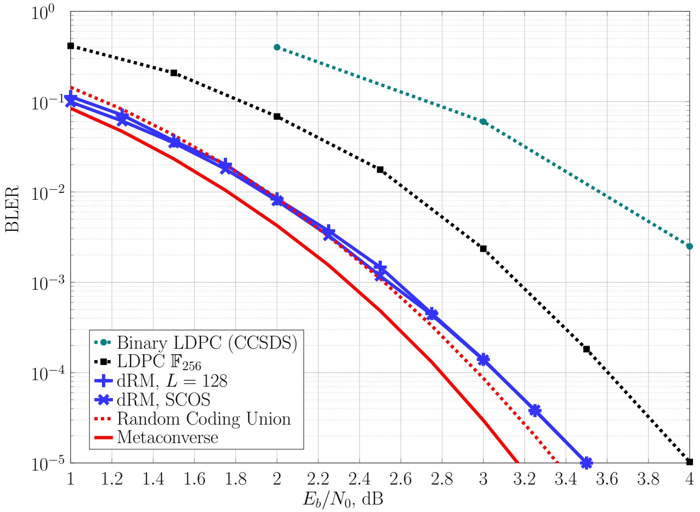
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dRM, SCOS
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Outline

1. Overview of Polar Codes
2. Successive Cancellation Inactivation Decoding
3. Successive Cancellation Ordered Search Decoding
4. Conclusions

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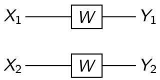
- The technique is lossless in terms of mutual information (required to achieve the capacity).
- The technique is of low complexity (there exists an encoder-decoder pair, realizing the technique with $\mathcal{O}(N \log N)$ complexity, where N is the block length).

Example: Binary Erasure Channel

Given two independent copies of a BEC(ϵ) $W : \{0,1\} \to \{0,1,?\}$, i.e.,

$$Y = \begin{cases} X & \text{w.p. } 1-\epsilon \\ ? & \text{w.p. } \epsilon \end{cases}$$



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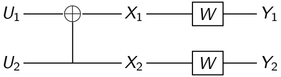

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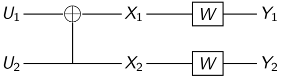

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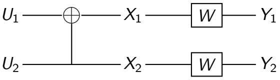

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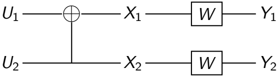

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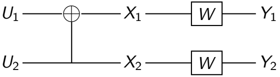

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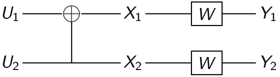

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Hence, we have

$$2\epsilon - \epsilon^2 \geq H(X_1|Y_1) = \epsilon \geq \epsilon^2 \quad \text{with equality if and only if } \epsilon \in \{0,1\}$$

Polarized Synthetic Channels: General BMSCs

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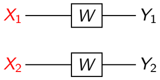

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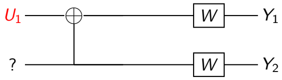

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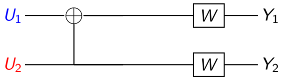

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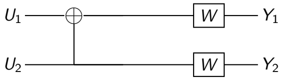

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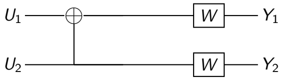

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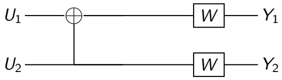

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- Then, transmit at a rate $C(W_2^{(2)})$, where the decoder uses $(Y_1^2, \hat{U}_1)$ to output $\hat{U}_2$.

Genie-Aided vs. Real Successive Decoder

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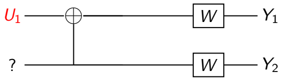

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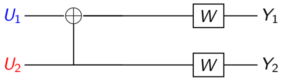

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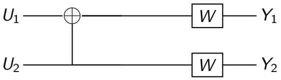

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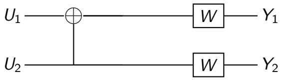

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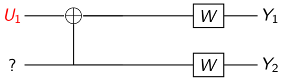

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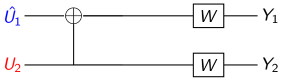

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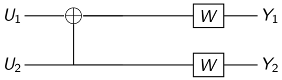

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The real decoder makes an error IF AND ONLY IF the genie-aided decoder makes an error!

Polar Transform - Recursive Application of the Basic Transform

Definition

The Kronecker product of two matrices X and Y is

$$\mathsf{X} \otimes \mathsf{Y} \triangleq \begin{bmatrix} x_{1,1}\mathsf{Y} & x_{1,2}\mathsf{Y} & \cdots \\ x_{2,1}\mathsf{Y} & x_{2,2}\mathsf{Y} & \cdots \\ \vdots & \vdots & \ddots \end{bmatrix}.$$

Then, a Kronecker power of a matrix is written as $\mathsf{X}^{\otimes n} = \mathsf{X}^{\otimes(n-1)} \otimes \mathsf{X}$, $\mathsf{X}^{\otimes 0} \triangleq 1$.

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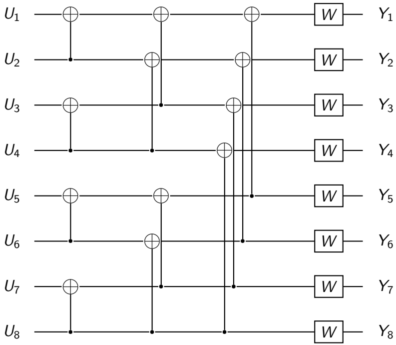
Example

Recall the matrix representing the basic transform $\mathsf{G}_2 \triangleq \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$. Then, we write

$$\mathsf{G}_2^{\otimes 2} = \mathsf{G}_2 \otimes \mathsf{G}_2 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix}.$$

Polar Transform (N=8)

$$U_1^8 \mathsf{G}_2^{\otimes \log_2 8} = X_1^8$$



Polar Transform (N=32)

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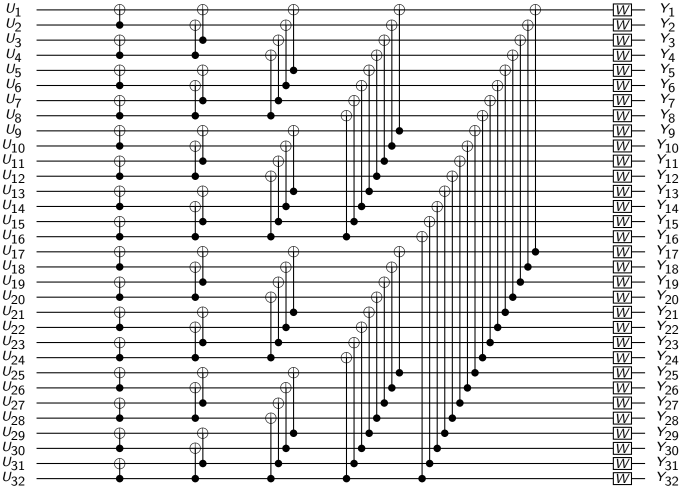

Channel Polarization

For any fixed $\delta > 0$, the fraction of the mediocre channels vanishes as $N \to \infty$, i.e., we have

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ i \in \{1, \ldots, N\} : \delta < H(W_N^{(i)}) < 1 - \delta \right\} \right| = 0.$$

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Since the transform is information-lossless, we can write

$$\lim_{N\to\infty} \frac{1}{N} \left| \left\{ i \in \{1, \ldots, N\} : H(W_N^{(i)}) \le \delta \right\} \right| = C(W)$$

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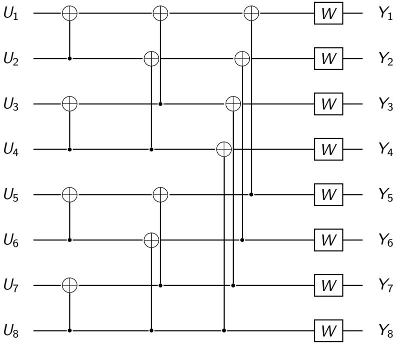
A Capacity-Achieving Scheme: Polar Codes

- Transmit uniformly distributed information bits over the good synthesized channels ($k \to N \cdot C(W)$).
- Set the inputs of the bad synthesized channels to the constant values known to the decoder.
- Decode the bits from U_1 to U_N successively.
- $P_B \leq \sum_{i \in A} \delta = N \cdot C(W) \cdot \delta \leq N \cdot C(W) \cdot 2^{-\sqrt{N}}$, resulting in $P_B \to 0$.
- Indeed, polarization holds for $\delta = \mathcal{O}(2^{-\sqrt{N}})$ [AT09] (i.e., faster than $1/N$).

Code Design

We want to design an (N, k) code, where $N = 2^n$ with $n \geq 1$.

.



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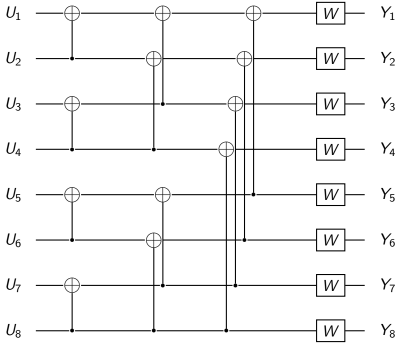

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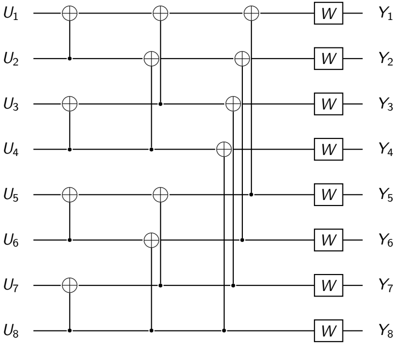

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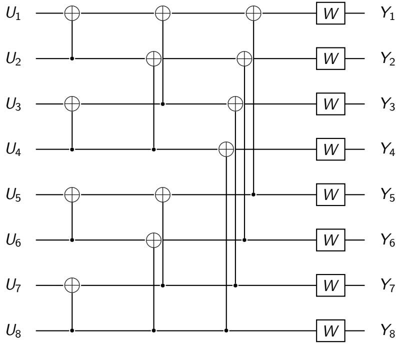

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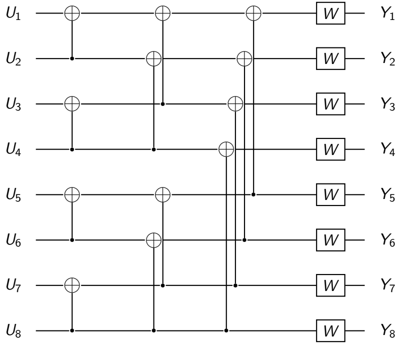
The polar rule minimizes a tight upper bound on the error probability under SC decoding while the RM rule maximizes the minimum Hamming distance.



A Historical Remark

Rekursive Codes mit der Plotkin-Konstruktion und ihre Decodierung

Vom Fachbereich
Elektrotechnik und Informationstechnik
der Technischen Universität Darmstadt
zur Erlangung des Grades
Doktor-Ingenieur genehmigte

Dissertation

von

Dipl.-Ing. Norbert Stolte
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- Observation: Reed-Muller (RM) codes perform poorly under low-complexity SC decoding.

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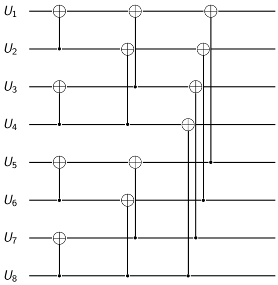
Encoding

Let V_1^k denote the random information bits to be encoded:



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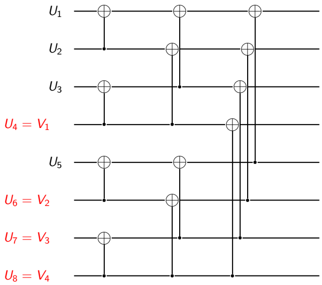

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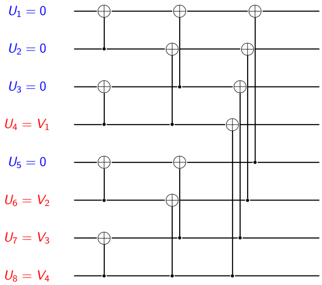

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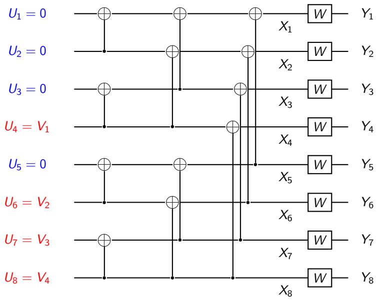

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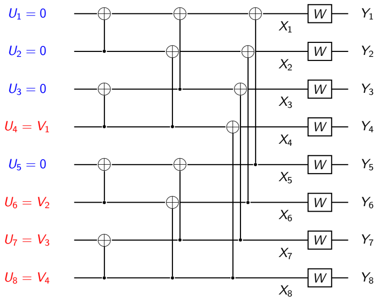
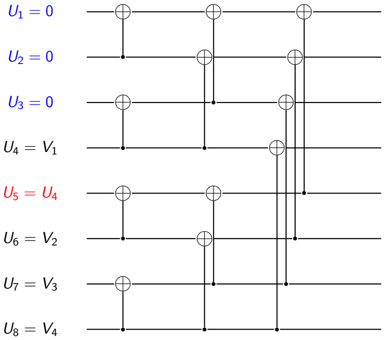
4. This can be done with a complexity of $\mathcal{O}(N \log N)$ instead of $\mathcal{O}(N^2)$.



Dynamic Frozen Bits

- The value of a frozen bit can also be set to a linear combination of previous information bits (rather than a fixed 0 or 1 value) [TM16]



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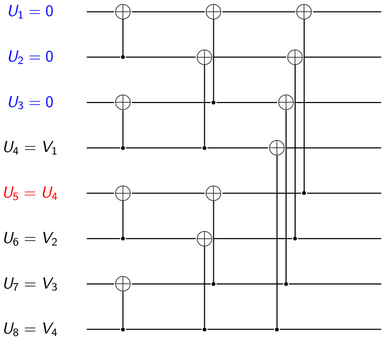

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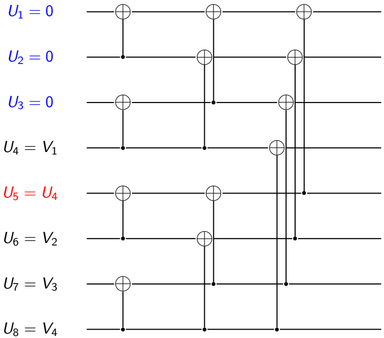

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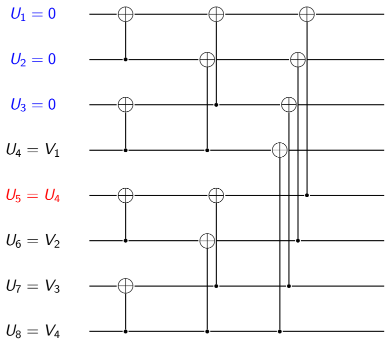
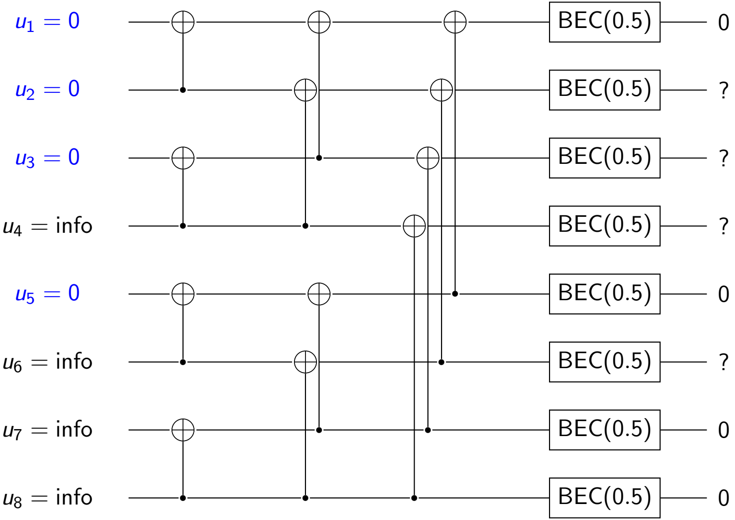
- SC/SCL decoding easily modified for polar codes with dynamic frozen bits.
- Any binary linear block code can be represented as a polar code with dynamic frozen bits!

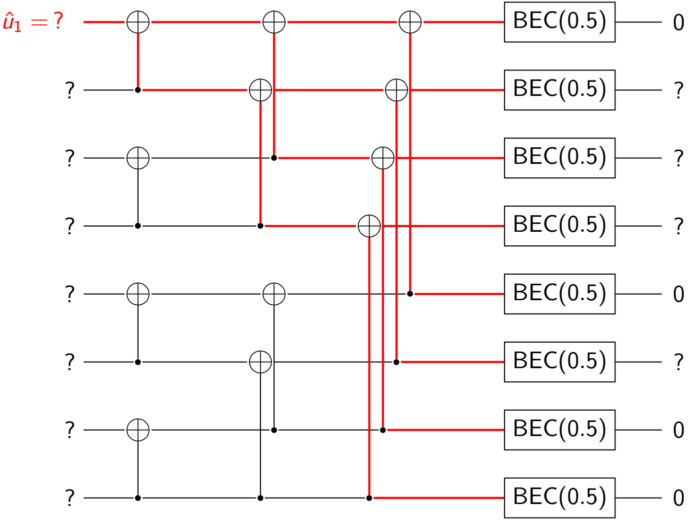


Successive Cancellation Decoding: BEC Example



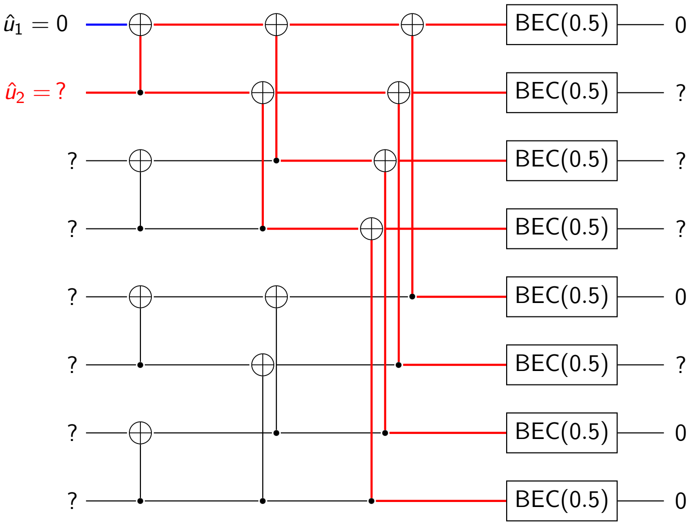
Successive Cancellation Decoding: BEC Example

$\hat{u}_1 = 0$ (frozen)



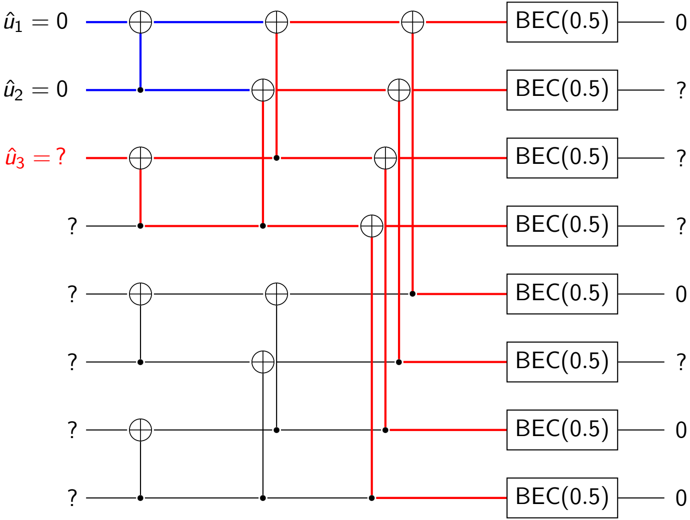
Successive Cancellation Decoding: BEC Example

$\hat{u}_2 = 0$ (frozen)



Successive Cancellation Decoding: BEC Example

$\hat{u}_3 = 0$ (frozen)



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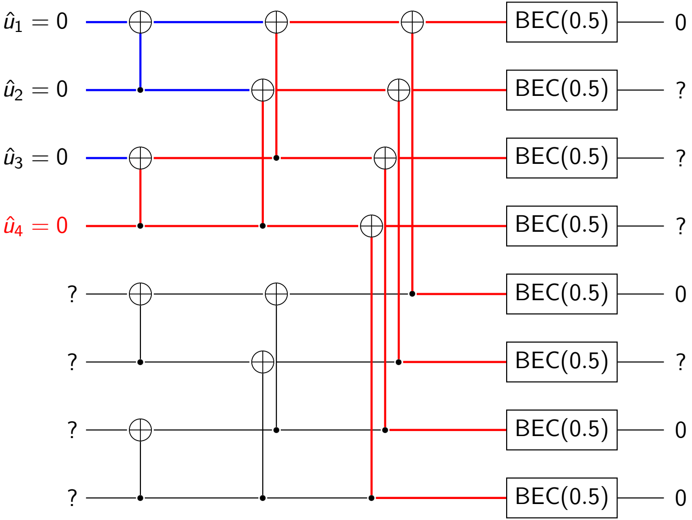
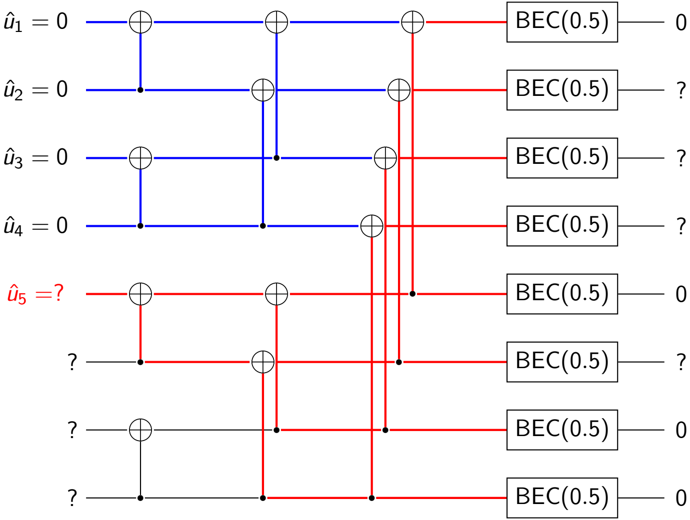


Successive Cancellation Decoding: BEC Example

$\hat{u}_5 = 0$ (frozen)



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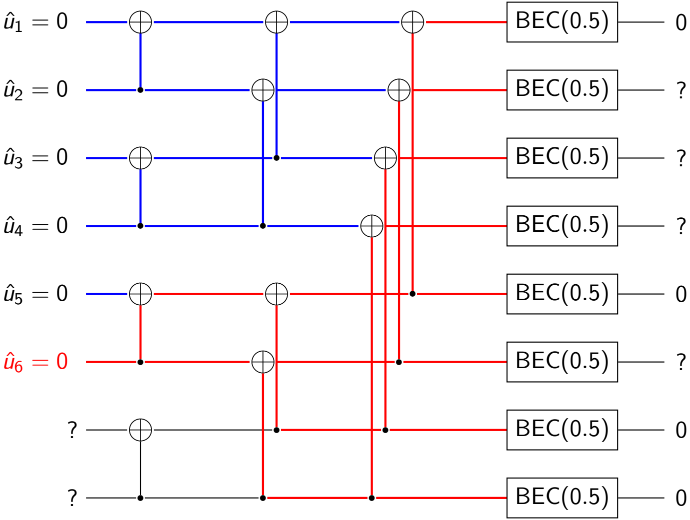

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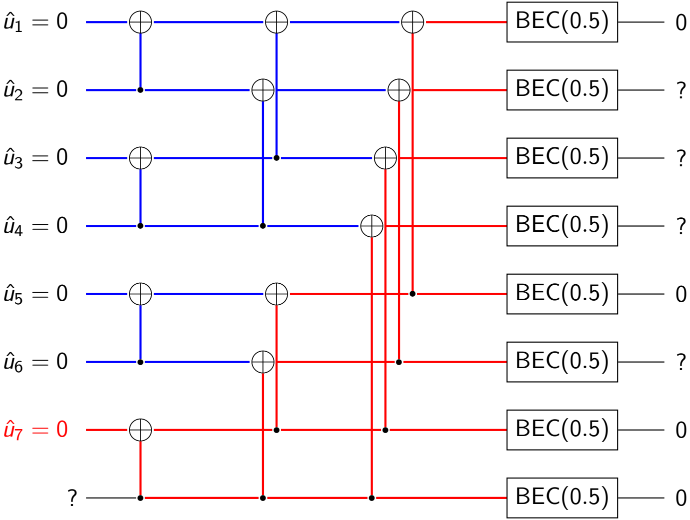

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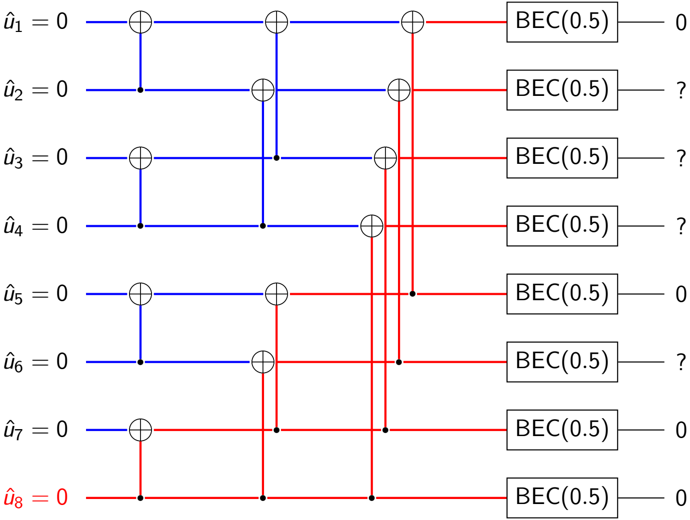

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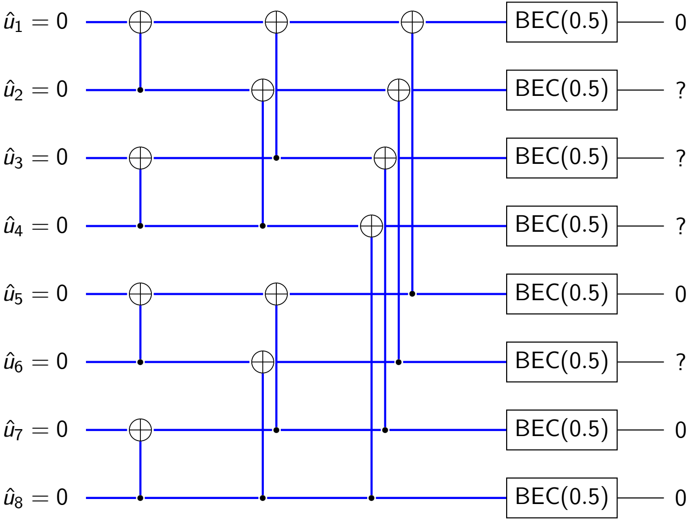

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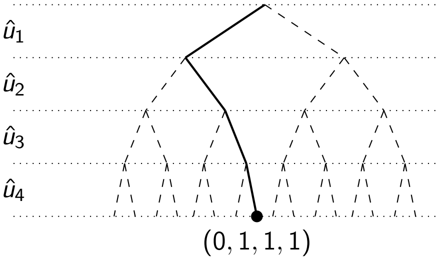

$$\hat{u}_i = \begin{cases} u_i & \text{if } i \in \mathcal{F} \\ f_i\left(y_1^N, \hat{u}_1^{i-1}\right) & \text{if } i \in \mathcal{A}. \end{cases} \quad (1)$$

$$f_i\left(y_1^N, \hat{u}_1^{i-1}\right) \triangleq \begin{cases} 0 & \text{if } P_{U_i|Y^N,U^{i-1}}(0|y_1^N, \hat{u}_1^{i-1}) = 1 \\ ? & \text{if } P_{U_i|Y^N,U^{i-1}}(0|y_1^N, \hat{u}_1^{i-1}) = \frac{1}{2} \\ 1 & \text{otherwise} \end{cases} \quad (2)$$

where a frame error occurs if $\hat{u}_i = ?$ for any $i \in \mathcal{A}$!

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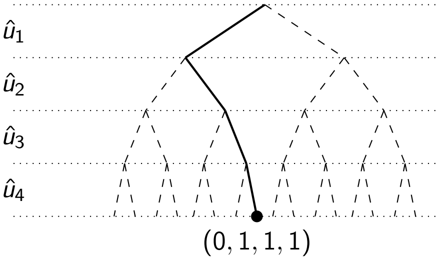

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where a frame error occurs if $\hat{u}_i = ?$ for any $i \in \mathcal{A} \rightarrow$ successive cancellation list (SCL) decoding!

Successive Cancellation List Decoding: BEC Example

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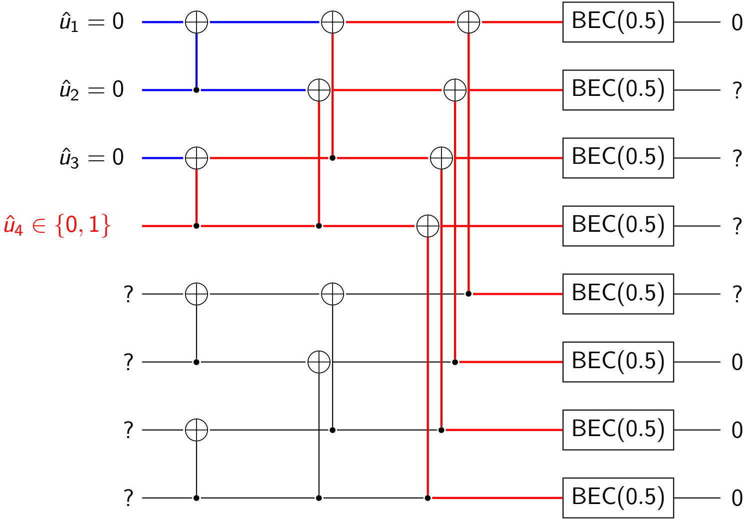
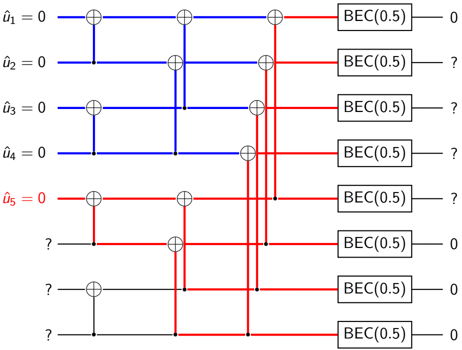
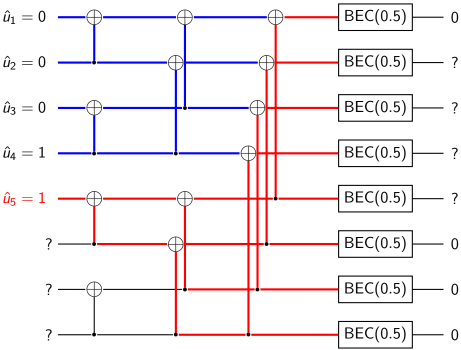


Successive Cancellation List Decoding: BEC Example

$$\mathcal{L}_4 = \{0000, 0001\}$$





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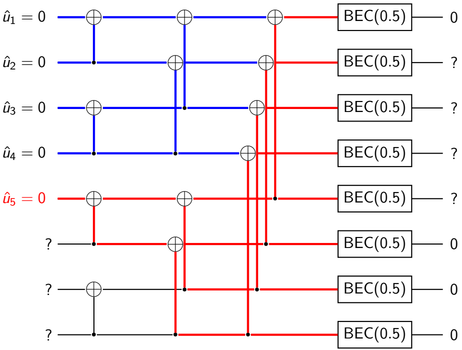

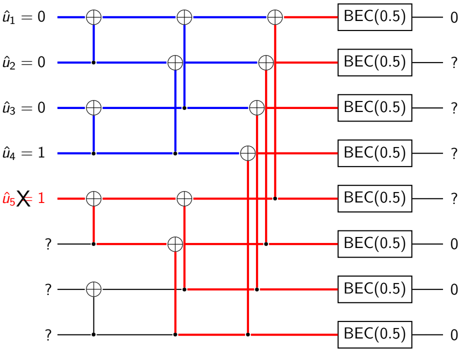

$\hat{u}_5 = 0$ (frozen) $\rightarrow P_{U_5|Y_1^8,U_1^4}\left(0|y_1^8,(0,0,0,1)\right) = 0$ tree pruning

Successive Cancellation List Decoding: BEC Example

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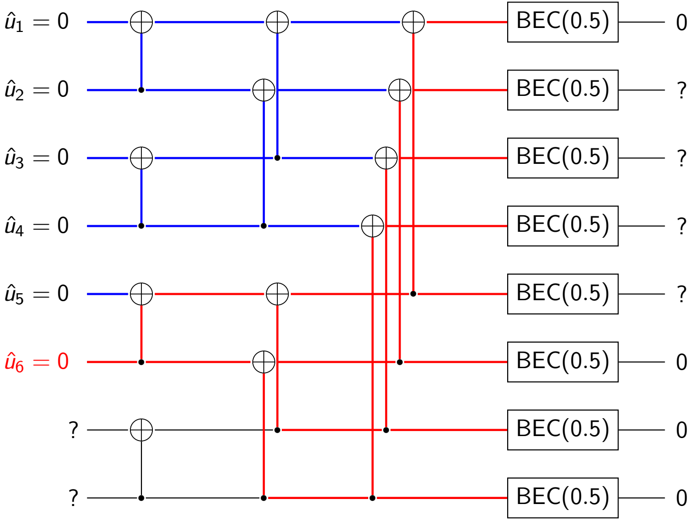
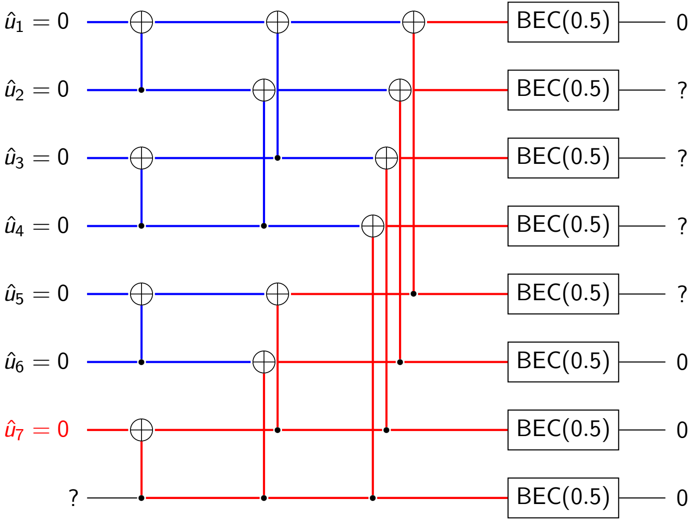


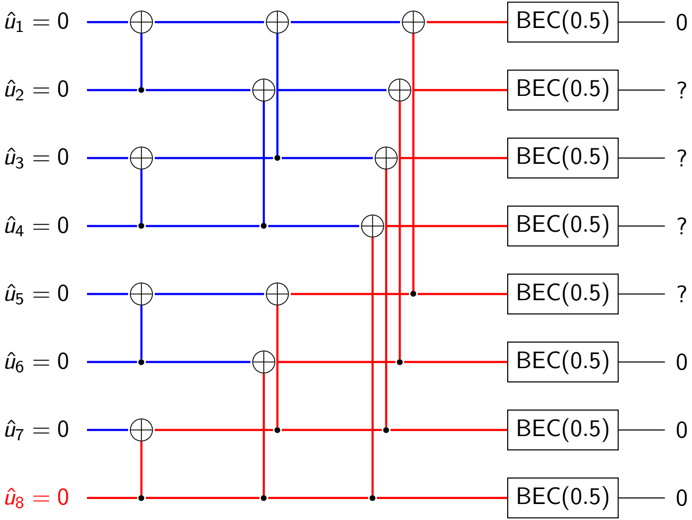
Successive Cancellation List Decoding: BEC Example

$|\mathcal{L}_6| = 1$

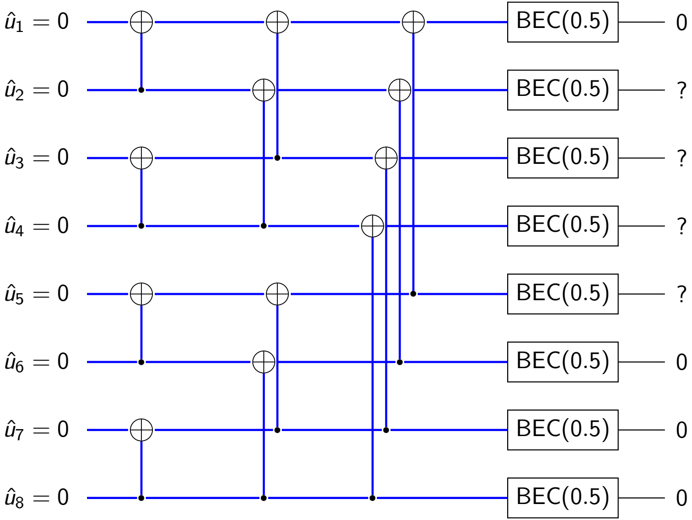


Successive Cancellation List Decoding: BEC Example

$|\mathcal{L}_7| = 1$



Successive Cancellation List Decoding: BEC Example



$|\mathcal{L}_8| = 1$

Successive Cancellation List Decoding: General BMSCs

Key idea: Each time a decision is needed on $\hat{u}_i$, both options, i.e., $\hat{u}_i = 0$ and $\hat{u}_i = 1$, are stored. This doubles the number of partial input sequences (paths) at each decoding stage.

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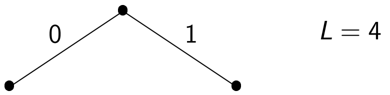

- When the number of paths exceeds a predefined list size L, discard the least likely paths.

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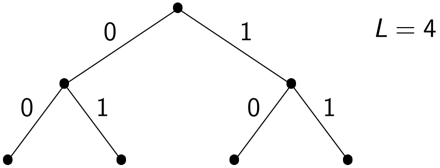

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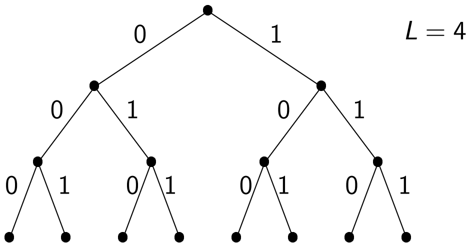

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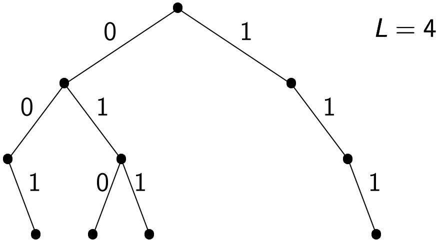

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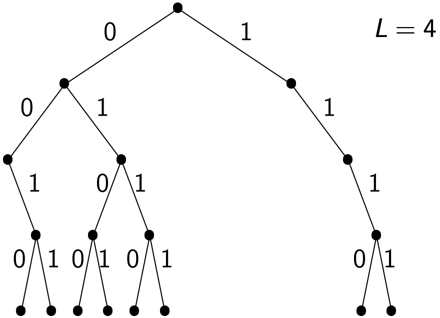

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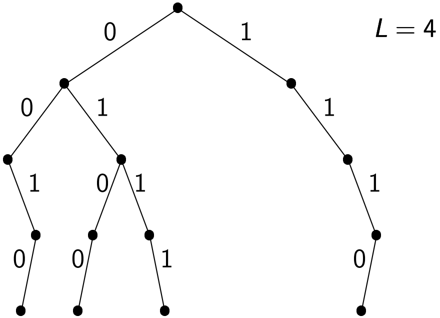

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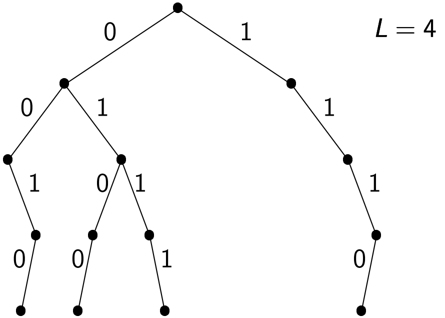
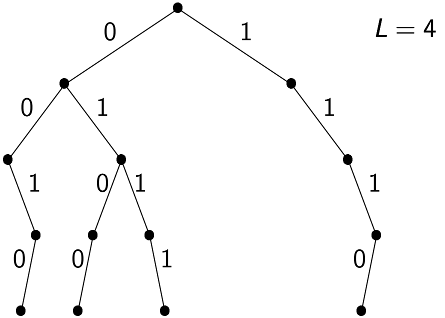


- When the number of paths exceeds a predefined list size L, discard the least likely paths.
- After N-th stage,
  $\hat{u}_1^N = \arg\max_{u_1^N \in \mathcal{L}_N} P_{U_N|Y_1^N, U_1^{N-1}}(u_N | y_1^N, u_1^{N-1}) = \arg\max_{u_1^N \in \mathcal{L}_N} \Pr(u_1^N | y_1^N)$.

Successive Cancellation List Decoding: General BMSCs

Key idea: Each time a decision is needed on $\hat{u}_i$, both options, i.e., $\hat{u}_i = 0$ and $\hat{u}_i = 1$, are stored. This doubles the number of partial input sequences (paths) at each decoding stage.



- When the number of paths exceeds a predefined list size L, discard the least likely paths.
- After N-th stage,
  $\hat{u}_1^N = \arg\max_{u_1^N \in \mathcal{L}_N} P_{U_N|Y_1^N, U_1^{N-1}}(u_N|y_1^N, u_1^{N-1}) = \arg\max_{u_1^N \in \mathcal{L}_N} \Pr(u_1^N|y_1^N)$.
- Very similar ideas were applied to RM codes (see, e.g., [Sto02, DS06]).

Outline

1 Overview of Polar Codes

2 Successive Cancellation Inactivation Decoding

3 Successive Cancellation Ordered Search Decoding

4 Conclusions

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- For linear codes on the BEC, any uncertainty in the information bits always takes the form of an affine subspace.

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Based on joint a work with Joachim Neu (Stanford) and Henry D. Pfister (Duke) [CNP20]

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- Similar methods have been proposed for low-density parity-check (LDPC) [RU01, PF04, MMU08, PLMC12] and raptor [Sho06, LLB17] codes.
- For polar codes, a BP decoder with inactivations was proposed [EP10], but it does not use SC decoding schedule. More activity this year [UB21, UMB21].

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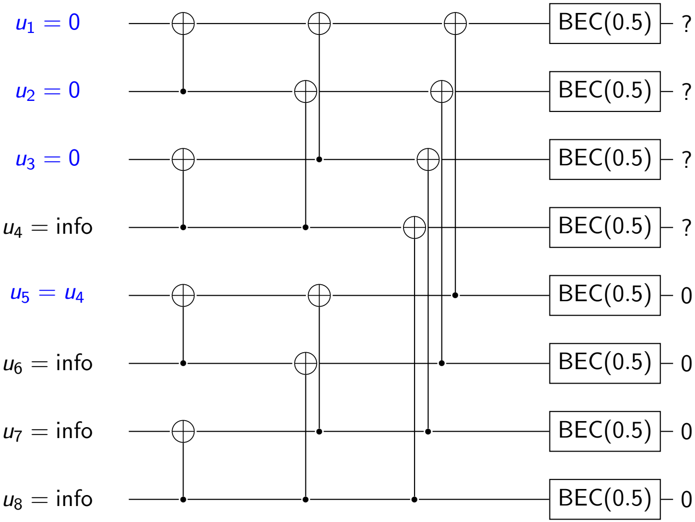
- It continues decoding using SC decoding for the BEC, where the message values are allowed to be functions of all inactivated variables.
- The inactivated bits are resolved, later, using linear equations derived from decoding frozen bits.

Successive Cancellation Inactivation Decoding

Example: $u_1 = u_2 = u_3 = 0$, $u_5 = u_4$ (frozen bits)

$g \triangleq$ total number of inactivations during a decoding attempt



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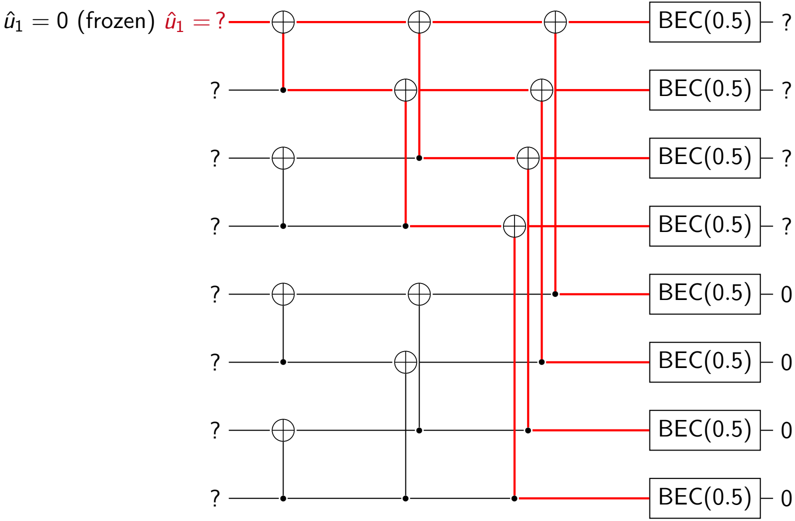

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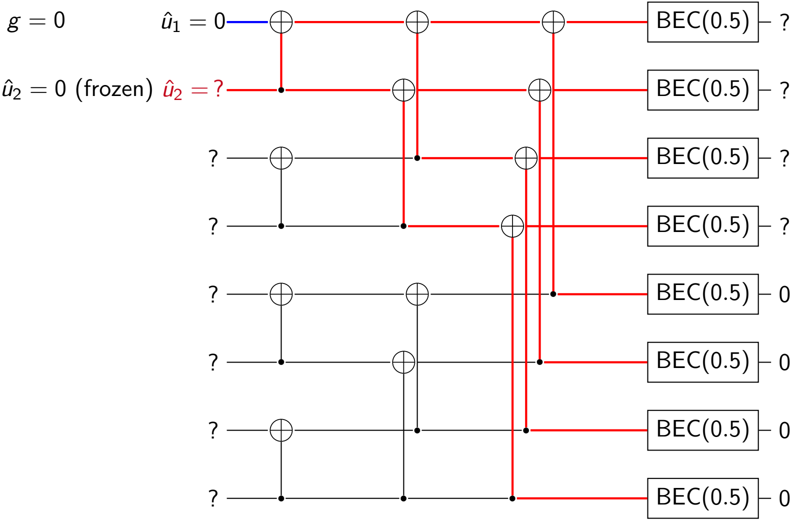

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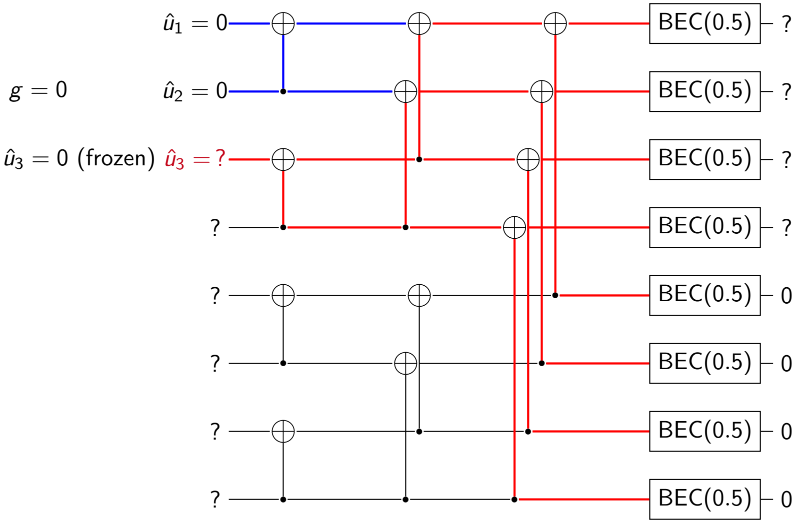

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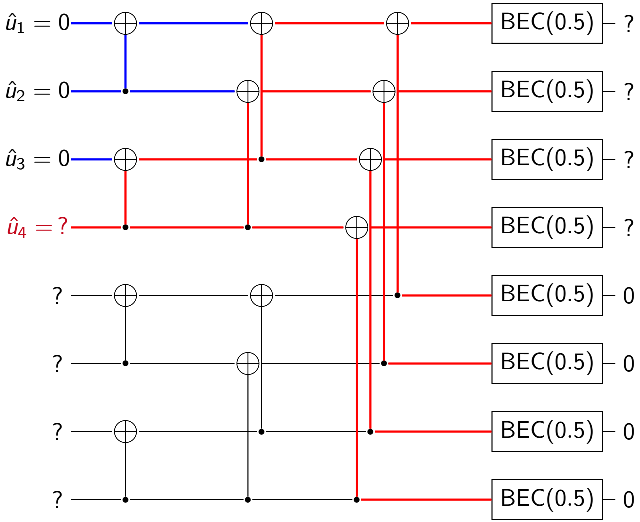

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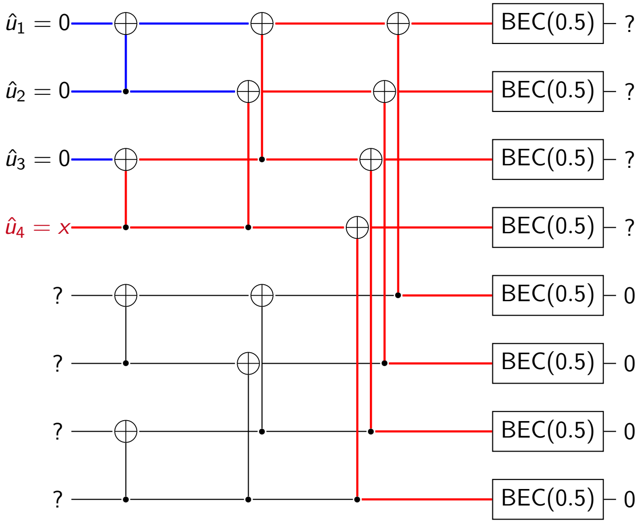

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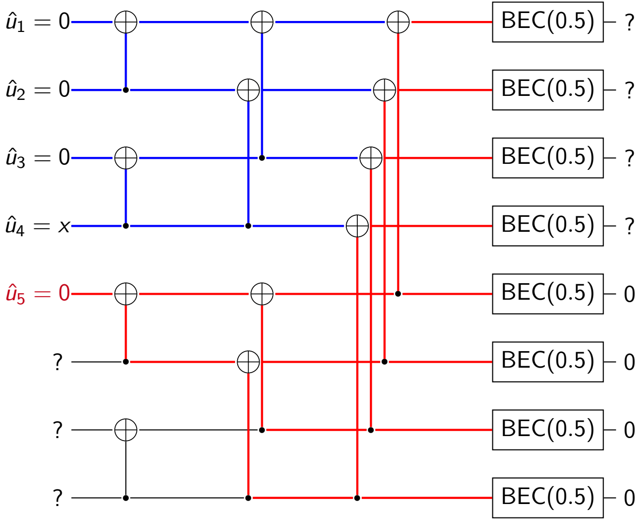

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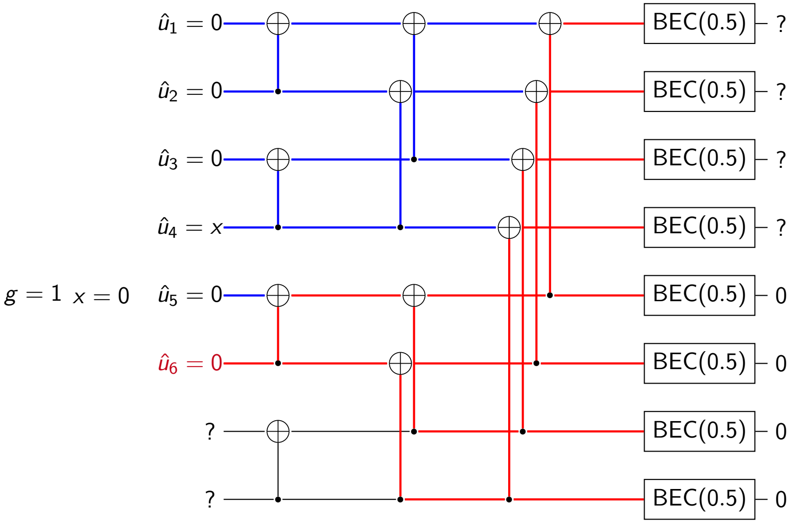

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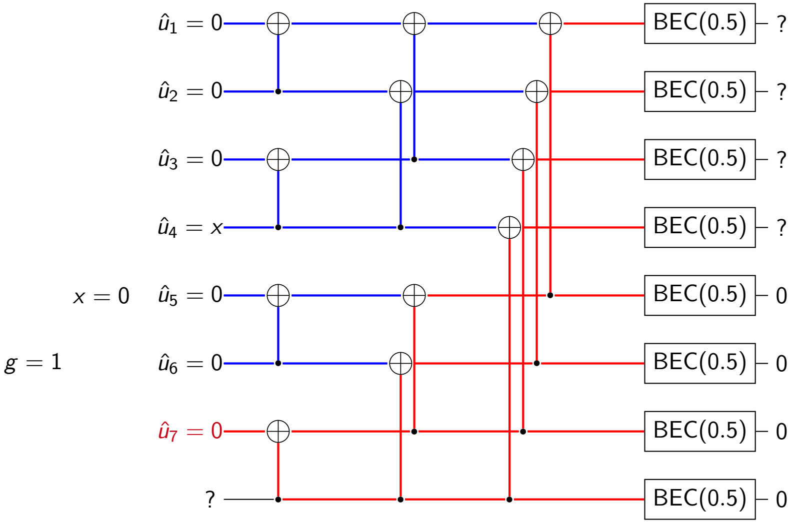

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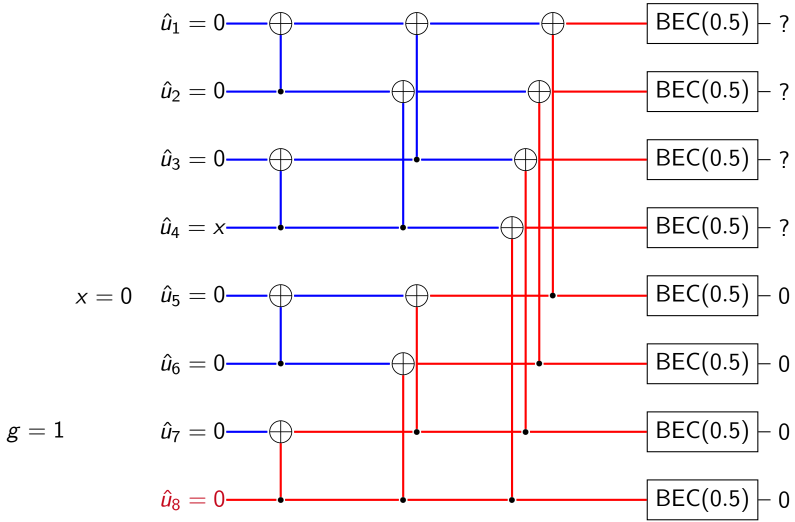

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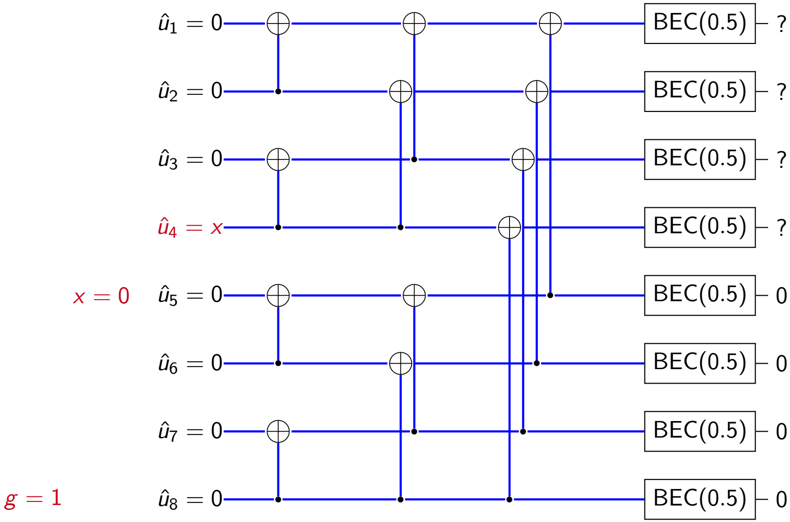

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- The SCI decoder with consolidations mimics the path pruning stage of SCL decoding.

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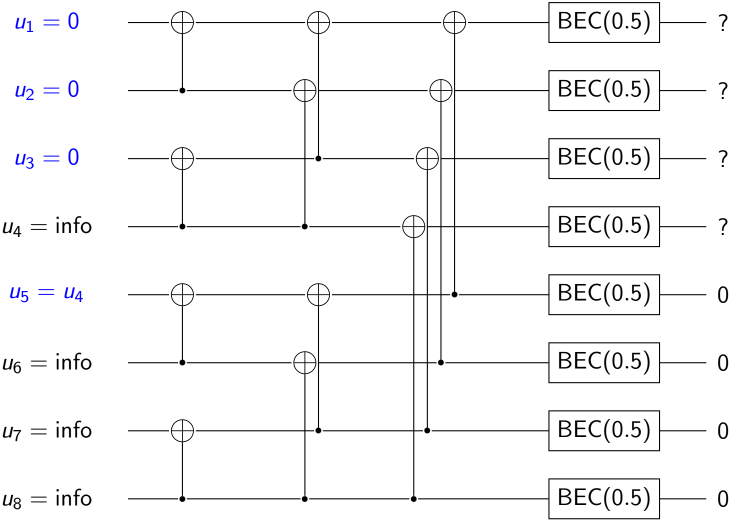
Example (Cont'd): $u_1 = u_2 = u_3 = 0$, $u_5 = u_4$ (frozen bits)

$d_i \triangleq$ number of unresolved inactivations (subspace dimension) at i-th decoding stage



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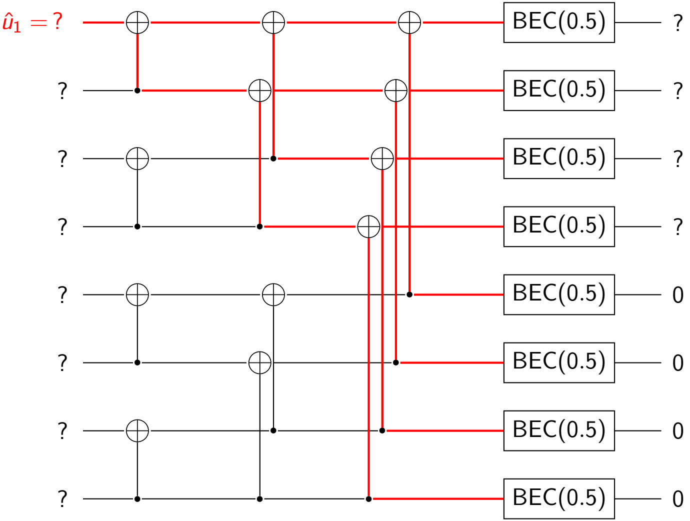

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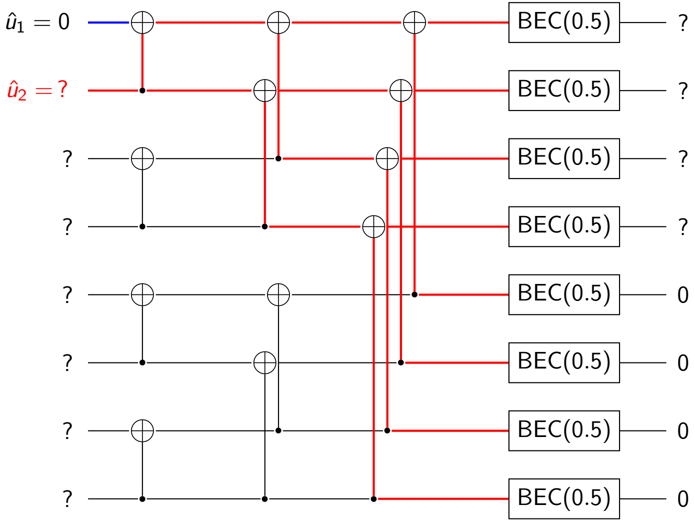

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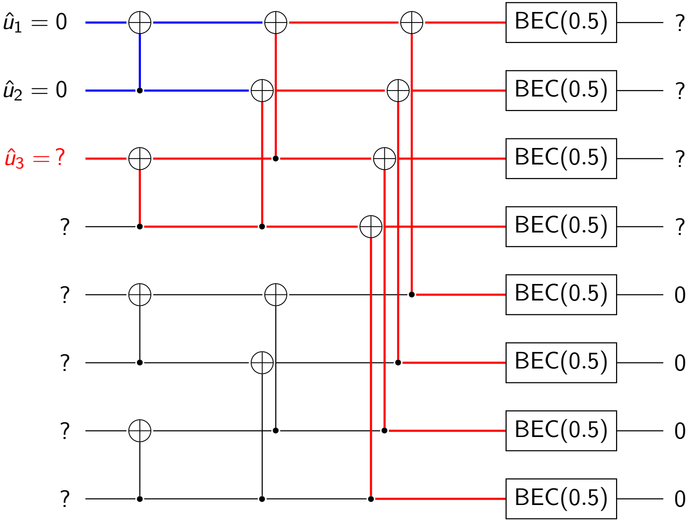

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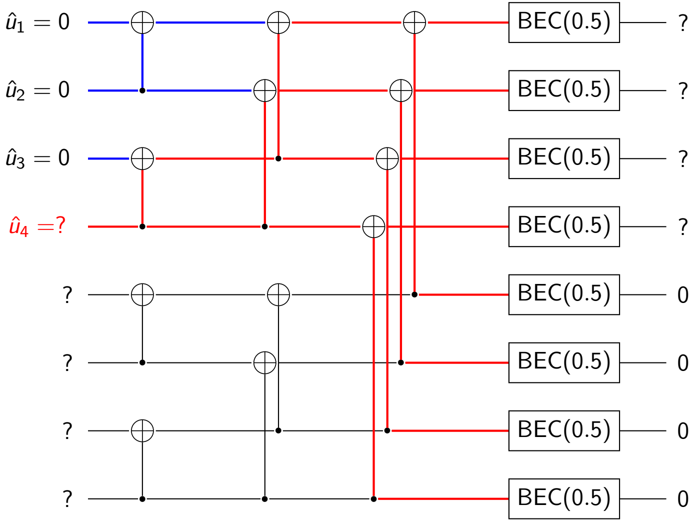

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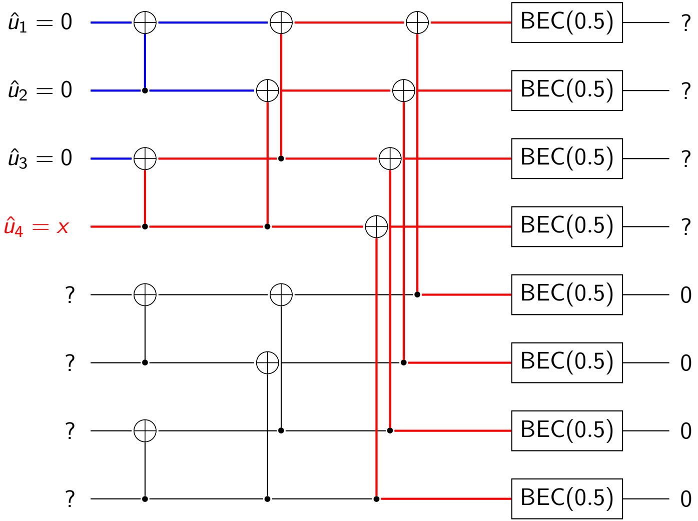

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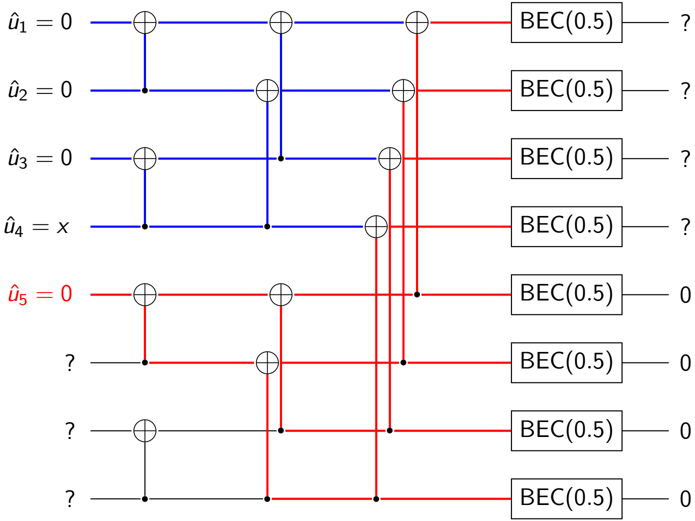

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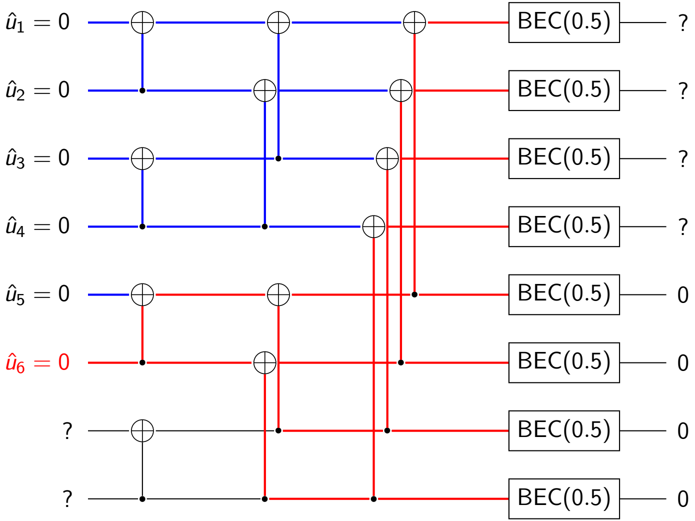

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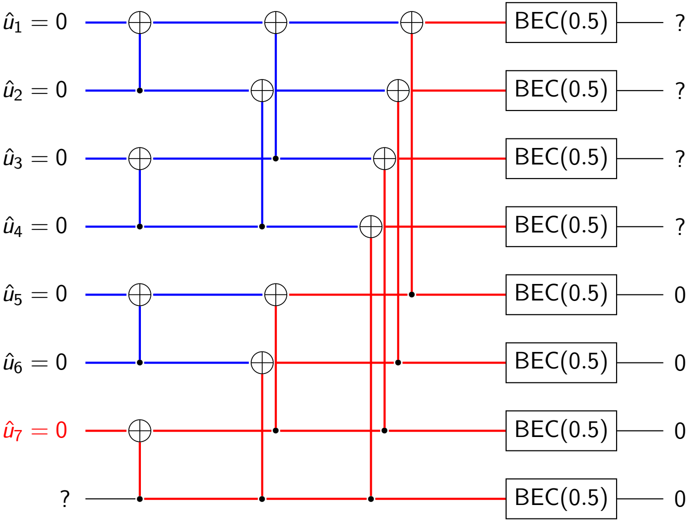

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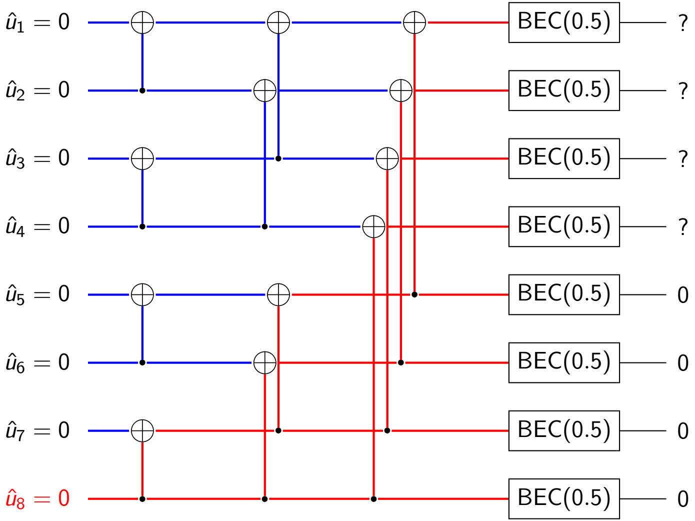

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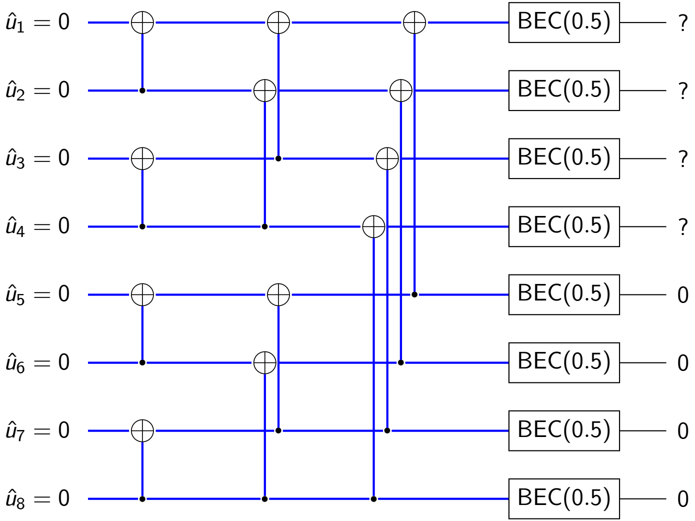


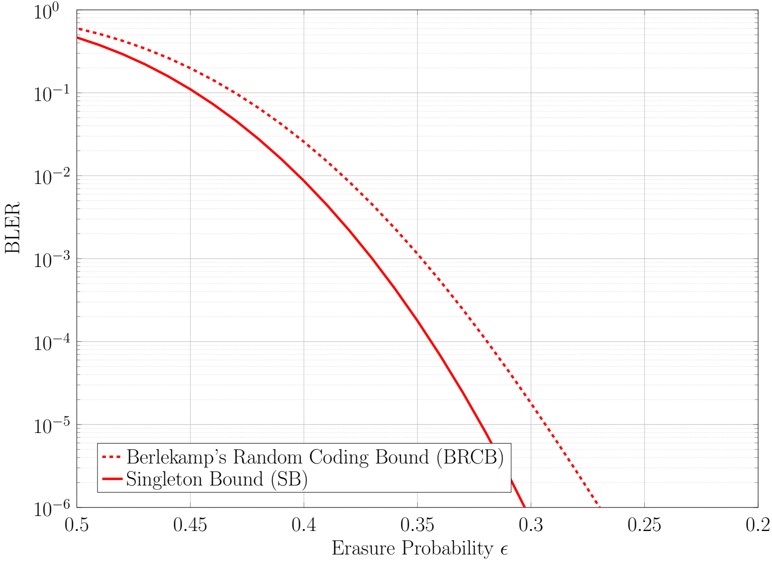
$d_8 = 0$

Analyzed Codes

We first consider the following codes with parameters $(N = 128, k = 64)$:

1. A polar code designed for $\epsilon = 0.4$
2. The RM code
3. The eBCH code
4. A uniform random linear code: $\mathcal{A} = \{1, 2, \dots, 64\}$, where each frozen bit is a uniform random linear combination of bits u_i, $i \in \mathcal{A}$

$(128, 64)$ Codes - MAP Performance



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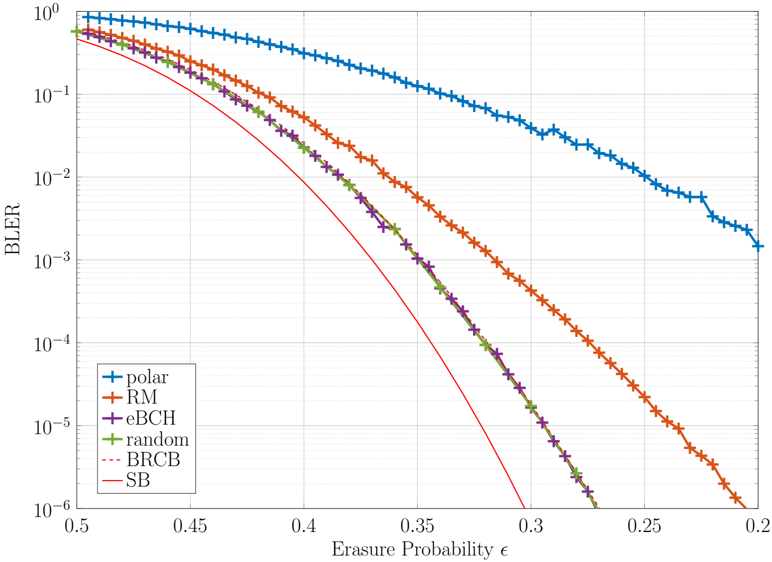
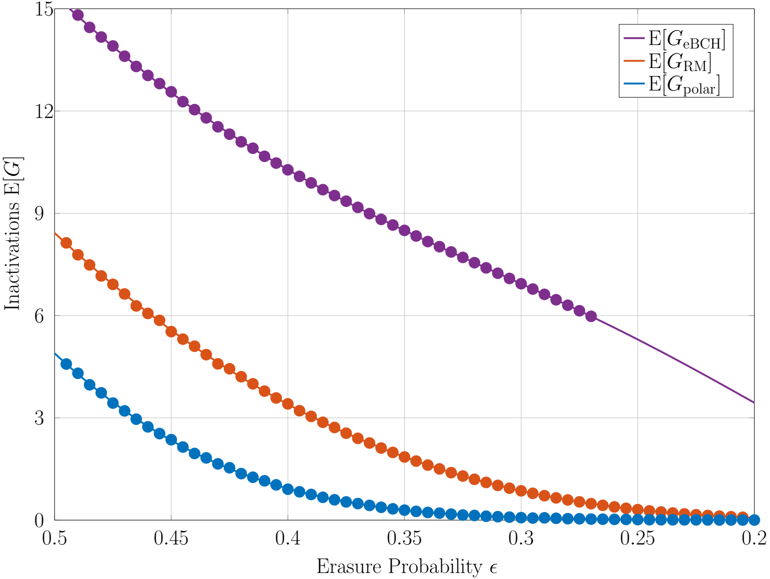
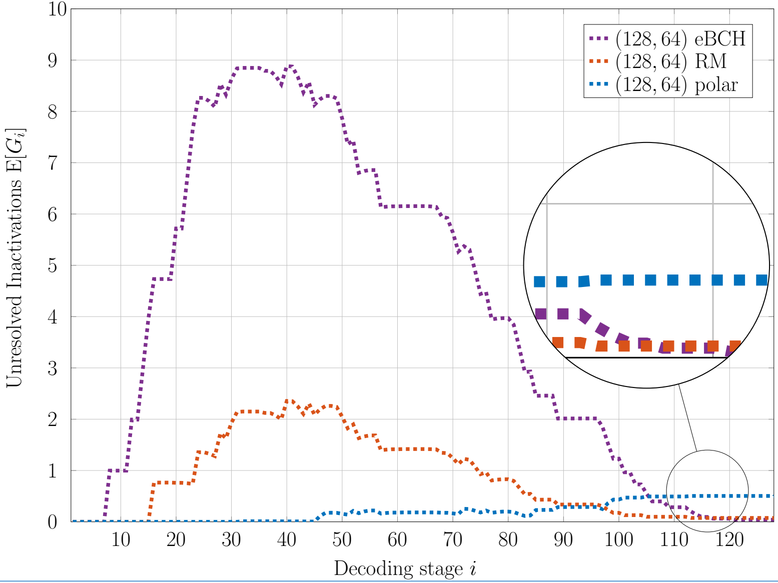


$(128, 64)$ Codes - Expected Number of Inactivations $\mathrm{E}[G]$



(128, 64) Codes - Expected Subspace Dimension $\mathrm{E}[D_i]$ for $\epsilon = 0.4$



Dynamic Reed-Muller Code Ensemble

Since the $(N = 128, k = 64)$ RM code provides a good complexity vs. performance trade-off, we propose a modification to it:

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  - PAC codes are defined by an information index set $\mathcal{A}$ and an upper-triangular Toeplitz matrix T
  - PAC codes are polar codes with dynamic frozen bits, where information index set is $\mathcal{A}$ (Arıkan chooses $\mathcal{A}$ of the RM code) and dynamic frozen bits are specified by T [RBV20, YFV20]

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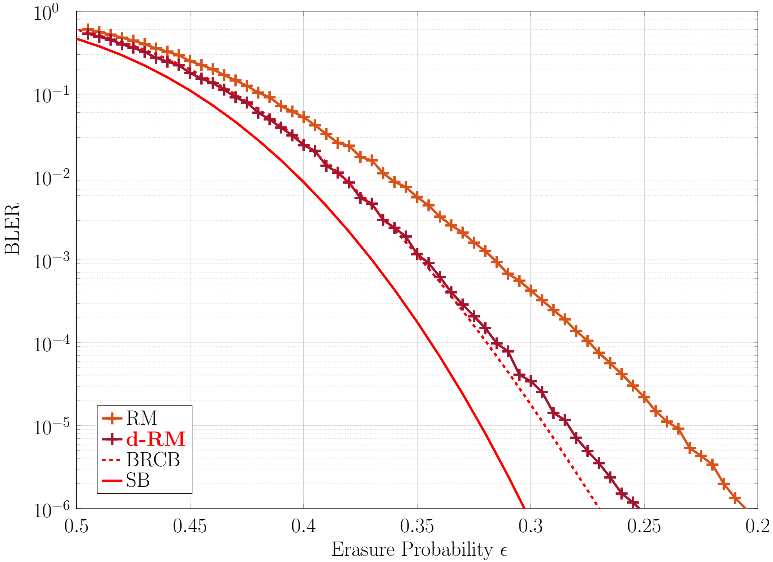
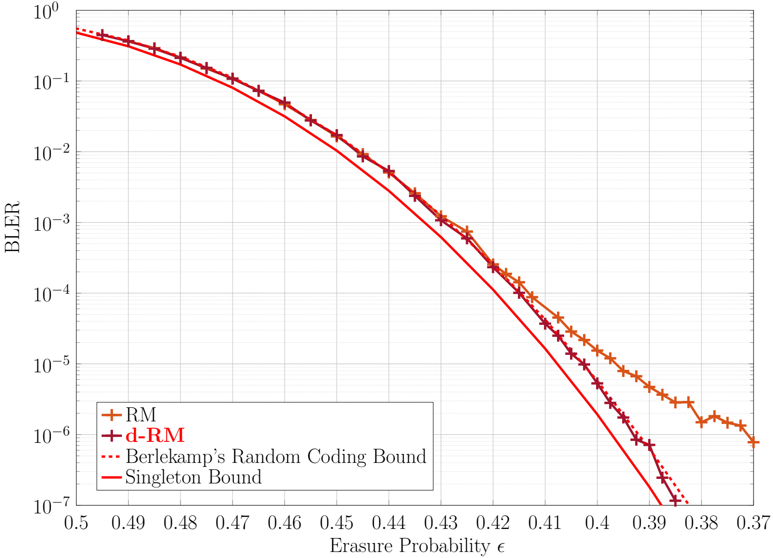


$(512, 256)$ Codes - MAP Performance



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Based on a joint work with Peihong Yuan (TUM) [YC21]

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Hence, successive cancellation ordered search decoding

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- The log-probability of a path $\tilde{u}_1^i$ via SC decoding [TV15]

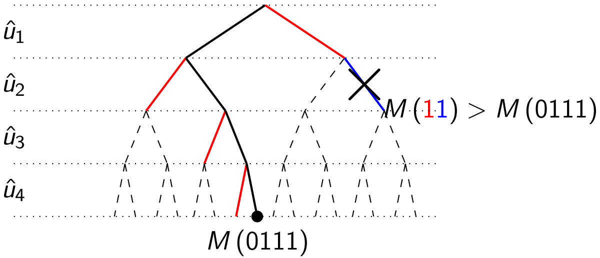
$$M_i\left(\tilde{u}_1^i\right) \triangleq -\log P_{U_1^i|Y_1^N}\left(\tilde{u}_1^i|y_1^N\right). \tag{3}$$

- A score function for a path $\tilde{u}_1^i$ [JH19]

$$S_i\left(\tilde{u}_1^i\right) \triangleq M_i\left(\tilde{u}_1^i\right) + \sum_{j=1}^{i} \log\left(1-p_j\right) \tag{4}$$

  where p_j is the probability of the event that the first bit error occurred for u_j in SC decoding and $S\left(\tilde{u}^0\right) \triangleq 0$.

Ordered Search Decoding: Example



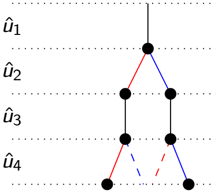
- Search priority: $S_i\left(\tilde{u}_1^i\right)$
- Decision & Pruning: $M_i\left(\tilde{u}_1^i\right)$

Ordered Search Decoding

1. SC path: 0111, $M(0111)$
   $M_1(1), M_2(00), M_3(010), M_4(0110)$
   $S(1), S(00), S(010), S(0110)$
2. Find the sub-path with lowest S.
3. Return to the Lowest Common Ancestor and re-start SC decoding.
4. $M\,(11) > M\,(0111)$
5. Repeat until it is impossible to find a more reliable path.

Decoding Complexity

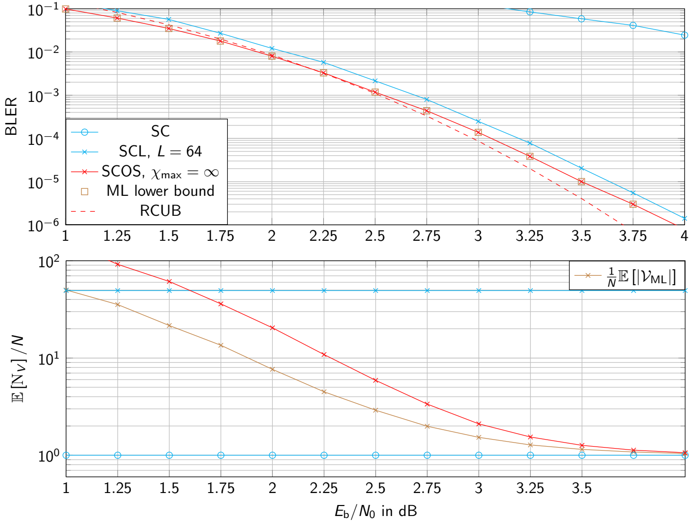
- Space complexity: $\mathcal{O}\left(N \log N\right)$
- N_V: Number of node-visits in the SC-decoding tree for SCOS decoding



$\mathrm{N}_V^{\mathrm{SCL}}\left(2, \{2,4\}\right) = 7$

- $\mathcal{V}_{\mathrm{ML}} \triangleq \bigcup_{i=1}^{N} \left\{u^i \in \{0,1\}^i : M\left(u^i\right) \leq M\left(\hat{\boldsymbol{u}}_{\mathrm{ML}}\right)\right\}$
- $\mathrm{N}_V \geq \left|\mathcal{V}_{\mathrm{ML}}\right|$ due to the redundant visits.

(128, 64) PAC code



Outline

1 Overview of Polar Codes

2 Successive Cancellation Inactivation Decoding

3 Successive Cancellation Ordered Search Decoding

4 Conclusions

Conclusions and Outlook

- Complexity-adaptive block-wise optimum decoding for BMSCs, where the complexity approaches very closely to that of SC decoding for wide range of codes (polar codes, short RM codes, their modifications, etc.) as the channel quality gets better.
- For much more, see [CNP20, CP21, YC21].
- A promising direction is to extend SCI decoding for codes over q-ary erasure channels.
- For SCOS decoding, optimizing the search schedule for the RM and/or PAC codes of larger blocklengths is promising. It has already been explored in [HDG+21] for RM codes of various lengths and rates.

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