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Time-variant reliability-oriented structural optimization and a renewal model for life-cycle costing

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Abstract

Objective functions for discounted cost optimization based on a continuous renewal model for a series of cases are presented. They include failures and subsequent renewals by crossings of loading processes or random disturbances out of safe states of structural components, failures due to aging, non-constant benefit and damage functions, finite renewal times, repeated reconstructions at renewals and inspection and repair. A method for reliability-oriented time-variant structural optimization of separable (independent) series systems using first order reliability methods in standard space is developed generalizing theories proposed earlier for component problems and time-invariant series system problems in a special one-level approach. Certain improvements by taking account of dependencies among failure modes are also given. Numerical Laplace transforms are proposed for the treatment of aging components. A newly developed gradient-based algorithm solves the optimization problem. Some algorithmic details are discussed. The approach is demonstrated at two examples.

Keywords: Structural reliability; One-level optimization; Time-variant; Outcrossing approach; Series systems

1. Introduction

The calculation of failure probabilities or reliability indices for given sets of basic variables or random processes, limit state functions and deterministic parameters is well known. It requires solution of an optimization task if modern reliability method FORM/SORM is used. The determination of a certain design parameter set, e.g. initial cost or weight of a structure, in order to maximize benefits or to make efficient use of resources is much more difficult and involves another optimization task. Reliability-oriented optimization of design parameters is more expensive than simple reliability analysis. Both tasks can, however, be combined in the inverse problem of finding optimal designs with or without reliability restrictions. In this one-level approach the first-order Kuhn–Tucker optimality conditions of the reliability problem(s) are added as constraints to the overall cost optimization problem. Techniques have been developed so far for time-invariant and time-variant component problems [18,19,21] and for time-invariant series system problems [20] based on this concept. Time-variant series systems are first dealt with in Ref. [32]. Although conceptionally similar, each of the cases requires special handling of the details.

Time-variant optimization concepts making use of a simple renewal model have been proposed as early as 1971 by Rosenblueth/Mendoza [27] with special reference to earthquake resistant design. More generality has been added by Hasofer [13] and Rosenblueth [28] and lately in Refs. [14, 25]. In this paper the classical renewal model is briefly reviewed. The tools of Laplace transforms are found to be extremely useful. An attempt is then made to further generalize the model to cover new fields of application, for example, finite renewal times, repeated reconstruction at renewal, series systems and inspection and repair. This requires new computational methods which are developed to a certain extent.

2. Objective functions

A structural or any other technical facility is optimal if the following objective is maximized:

$$Z(\mathbf{p}) = B(\mathbf{p}) - C(\mathbf{p}) - D(\mathbf{p}) \quad (1)$$

Without loss of generality it is assumed that all quantities in Eq. (1) can be measured in monetary units. $\mathbf{p}$ is the vector of all safety relevant parameters. $B(\mathbf{p})$ is the benefit derived

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from the existence of the facility, $C(\mathbf{p})$ is the cost of design and construction, usually decomposed into a cost C_0 independent of $\mathbf{p}$ and cost dependent on $\mathbf{p}$, and $D(\mathbf{p})$ is the cost in case of failure. Statistical decision theory dictates that expected values are to be taken [36]. In the following it is assumed that $B(\mathbf{p})$, $C(\mathbf{p})$ and $D(\mathbf{p})$ are differentiable in each component of $\mathbf{p}$. And it is reasonably assumed that $C(\mathbf{p})$ increases whereas $D(\mathbf{p})$ decreases in each component of $\mathbf{p}$.

The structure which eventually will fail after some time will have to be optimized at the decision point, i.e. at time $t=0$. Therefore, all cost need to be discounted. A continuous discounting function is assumed which is accurate enough for all practical purposes

$$\delta(t) = \exp[-\gamma t] \tag{2}$$

where γ is the (tax-free) interest rate. For example, if failure with consequences D_0 occurs at time t (in years) the discounted damage is $D(t) = D_0 \exp[-\gamma t]$. If a yearly discount rate γ' is defined we have $\gamma = \ln(1+\gamma')$. Also, it is assumed that the construction cost $C(\mathbf{p})$ are without cost of financing. They can, however, be included easily. In consideration of the time horizon for structural and other technical facilities of 20 to more than 100 years the interest rate used should be a long term average net of in/deflation.

In general, one has to distinguish between at least three replacement strategies; one where the facility is given up after service or failure, one where the facility is systematically replaced after failure and one where it is repaired (renewed) after some time or after inspection. Further, we distinguish between structures which fail upon completion or never and structures which fail at a random point in time much later due to service loads, extreme external disturbances or deterioration. The first option implies that loads on the structure are time-invariant. At first sight there is no particular preference for either of the replacement strategies. For infrastructure facilities the second category is a natural strategy. Structures used only once, e.g. special auxiliary construction structures, boosters for space transport vehicles or devices exploiting limited deposits, might fall into the first category. In this paper focus is on time-variant problems and systematic reconstruction. For one-mission structures the reader is referred to Refs. [14,25].

2.1. Standard case

For easy reference the standard case is rederived for systematic reconstruction. For the moment, assume reconstruction times to be negligibly short. The times between failure (renewal) events have distribution function $F(t,\mathbf{p})$ with probability density $f(t,\mathbf{p})$ and are independent. For constant benefit per time unit $b(t) = b$ and $f_n(t,\mathbf{p})$ the density of the time to the nth renewal an objective function can be derived by making use of the convolution theorem for Laplace transforms (Appendix A)

$$\begin{aligned} Z(\mathbf{p}) &= \int_0^\infty b\mathrm{e}^{-\gamma t}\mathrm{d}t - C(\mathbf{p}) - (C(\mathbf{p})+H)\sum_{n=1}^{\infty}\int_0^\infty \mathrm{e}^{-\gamma t} f_n(t,\mathbf{p})\mathrm{d}t \\ &= \int_0^\infty b\mathrm{e}^{-\gamma t}\mathrm{d}t - C(\mathbf{p}) - (C(\mathbf{p})+H)\sum_{n=1}^{\infty} f^*(\gamma,\mathbf{p})^{n-1} f^*(\gamma,\mathbf{p}) \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p})+H)\frac{f^*(\gamma,\mathbf{p})}{1-f^*(\gamma,\mathbf{p})} \\ &= \frac{b}{\gamma} - C(\mathbf{p}) - (C(\mathbf{p})+H)h^*(\gamma,\mathbf{p}) \end{aligned} \tag{3}$$

where $h^*(\gamma,\mathbf{p})$ is the Laplace transform of the renewal density (renewal intensity) $h(t,\mathbf{p}) = \sum_{n=1}^{\infty} f_n(t,\mathbf{p})$. H is the monetary loss in case of failure including direct failure cost, loss of business and, of course, the cost to reduce the risk to human life and limb. We may also include the cost of demolition in H. In principle, renewal theory also allows for the case that the time to the first renewal is different from all other. This refinement is not done here for the sake of easy notation.

Laplace transforms are analytic only for a few failure models. The one for a normal failure time distribution, i.e. $f^*(\gamma,\mathbf{p}) = \exp[(1/2)\gamma(\gamma\sigma(\mathbf{p})^2 - 2\mu(\mathbf{p}))]$, is especially important because it is also approximately the Laplace transform for an arbitrary failure time distribution with known mean μ and standard deviation σ provided that $V = \sigma/\mu$ is small and $\gamma \le 2\mu/\sigma^2$ [13]. In addition, there is an important asymptotic result for the renewal density and its Laplace transform [4, p. 55]

$$\lim_{t\to\infty} h(t,\mathbf{p}) = \lim_{\gamma\to 0} \gamma h^*(\gamma,\mathbf{p}) = \frac{1}{E[T_f(\mathbf{p})]} = \lambda(\mathbf{p}), \tag{4}$$

for $f(t) \underset{t\to\infty}{\longrightarrow} 0$

where $E[T_f(\mathbf{p})]$ is the mean time between renewals (or failures) and $\lambda(\mathbf{p})$ the occurrence rate. If the failure times have an exponential distribution one obtains

$$h^*(\gamma,\mathbf{p}) = \frac{\lambda(\mathbf{p})}{\gamma} \tag{5}$$

since $f^*(\gamma,\mathbf{p}) = \lambda(\mathbf{p})/(\gamma + \lambda(\mathbf{p}))$. This result is especially relevant because the parameter $\lambda(\mathbf{p})$ may be replaced asymptotically by the outcrossing rate $\nu^+(\mathbf{p})$ frequently used in time-variant structural reliability analysis. If $\nu^+(\mathbf{p})$ depends on an uncertain parameter vector $\mathbf{R}$ one should use $E_{\mathbf{R}}[\nu^+(\mathbf{p},\mathbf{R})]$ instead.

It is seen that continuous discounting and continuous failure models lead to relatively simple results. Completely parallel results, however, can be obtained for discrete failure models and discrete discounting [34].

2.2. Random disturbances

If, at extreme loading events (e.g. flood, wind storm, earthquake, explosion) having a density $f(t)$ of interarrival

times, (independent) failure occurs with probability $P_f(\mathbf{p})$, the density of times between failures is

$$g(t,\mathbf{p}) = \sum_{k=1}^{n} f_k(t) P_f(\mathbf{p}) R_f(\mathbf{p})^{k-1} \tag{6}$$

and after taking Laplace transforms [13,28]:

$$g^*(\gamma,\mathbf{p}) = \sum_{n=1}^{\infty} f^*(\gamma) P_f(\mathbf{p}) [f^*(\gamma) R_f(\mathbf{p})]^{n-1} = \frac{P_f(\mathbf{p}) f^*(\gamma)}{1 - R_f(\mathbf{p}) f^*(\gamma)} \tag{7}$$

with $R_f(\mathbf{p}) = 1 - P_f(\mathbf{p})$. The damage term becomes:

$$D(p) = (C(\mathbf{p}) + H)\frac{g^*(\gamma,\mathbf{p})}{1 - g^*(\gamma,\mathbf{p})} = (C(\mathbf{p}) + H) h^*(\gamma,\mathbf{p})$$

If, in particular, the loading events follow a stationary Poisson process with intensity λ we have

$$h^*(\gamma,\mathbf{p}) = \frac{\lambda P_f(\mathbf{p})}{\gamma} \tag{8}$$

2.3. Non-constant benefit function

Assume that the benefit rate is not constant but an arbitrary function of time. At each failure (and renewal) it starts at $b(0)$. Following Ref. [14], let U_n be the time between the $(n-1)$th and the nth arrival and let

$$T_n = \sum_{r=1}^{n} U_r \tag{9}$$

be the time to the nth arrival. We recollect that U_n is independent of T_{n-1} for $n = 2, 3, \ldots$. Given the U_n, the total discounted benefit B_T is given by

$$\begin{aligned} B_T &= \int_0^{U_1} \mathrm{e}^{-\gamma t} b(t)\mathrm{d}t + \sum_{n=2}^{\infty} \int_0^{U_n} b(t) \mathrm{e}^{-\gamma(T_{n-1}+t)}\mathrm{d}t \\ &= \int_0^{U_1} \mathrm{e}^{-\gamma t} b(t)\mathrm{d}t + \sum_{n=2}^{\infty} \mathrm{e}^{-\gamma T_{n-1}} \int_0^{U_n} b(t)\mathrm{e}^{-\gamma t}\mathrm{d}t \end{aligned} \tag{10}$$

and with

$$B_D(t) = \int_0^t \mathrm{e}^{-\gamma u} b(u)\mathrm{d}u \tag{11}$$

$$B_T = B_D(U_1) + \sum_{n=2}^{\infty} \mathrm{e}^{-\gamma T_{n-1}} B_D(U_n) \tag{12}$$

Taking expectations it follows that

$$\begin{aligned} B = E(B_T) &= E[B_D(U_1)] + \sum_{n=2}^{\infty} E(\mathrm{e}^{-\gamma T_{n-1}}) E[B_D(U_n)] \\ &= \int_0^\infty B_D(t) f(t)\mathrm{d}t + \left(\sum_{n=2}^{\infty}\int_0^\infty \mathrm{e}^{-\gamma t} f_{n-1}(t)\mathrm{d}t\right)\int_0^\infty B_D(t) f(t)\mathrm{d}t \end{aligned}$$

Using the results on Laplace transforms, we obtain

$$\begin{aligned} B &= \int_0^\infty B_D(t) f(t)\mathrm{d}t + \left(\sum_{n=2}^{\infty} f^*(\gamma)[f^*(\gamma)]^{n-2}\right)\int_0^\infty B_D(t) f(t)\mathrm{d}t \\ &= \int_0^\infty B_D(t) f(t)\mathrm{d}t + \left[\frac{f^*(\gamma)}{1-f^*(\gamma)}\right]\int_0^\infty B_D(t) f(t)\mathrm{d}t \\ &= \frac{1}{1-f^*(\gamma)}\int_0^\infty B_D(t) f(t)\mathrm{d}t \end{aligned} \tag{13}$$

and for a homogeneous Poissonian failure process with rate $\lambda(\mathbf{p})$:

$$B = \left(1 + \frac{\lambda(\mathbf{p})}{\gamma}\right)\int_0^\infty B_D(t)\lambda(\mathbf{p})\exp[-\lambda(\mathbf{p})t]\mathrm{d}t \tag{14}$$

2.4. Non-constant damage

Also, the damage term may depend on t. For example, the damage cost $H(t)$ can accumulate over time due to gradual storage of valuable goods or the reconstruction cost net of inflation can increase over time. Van Noortwijk [34] generalized a result in Ref. [9, p. 329 et sqq.], to consider time-dependent damage cost $K(t,\mathbf{p}) = C(t,\mathbf{p}) + H(t)$.

$$D(\mathbf{p}) = \frac{\int_0^\infty \exp[-\gamma t] K(t,\mathbf{p}) f(t,\mathbf{p})\mathrm{d}t}{1 - f^*(\gamma,\mathbf{p})} \tag{15}$$

Clearly, the numerator is no more the classical Laplace transform of a failure density.

2.5. Finite renewal times

Next we consider finite renewal times, i.e. finite reconstruction times, ignoring the rare case of failure under an external extreme loading event. During these times the facility cannot be used and it cannot fail. Let T_W be the (random) renewal times and T_N be the (random) times of use. Therefore, $T = T_W + T_N$ is the time between failures (or renewals). An exact consideration is complicated. However, renewal theory shows that the availability of a system asymptotically equals:

$$A(\infty) = \frac{E[T_N]}{E[T_W] + E[T_N]} \tag{16}$$

It follows that the benefit and the damage term have to be multiplied by $A(\infty)$ so that:

$$Z(\mathbf{p}) \approx \frac{b}{\gamma}A(\infty) - C(\mathbf{p}) - (C(\mathbf{p}) + H) h_A^*(\gamma,\mathbf{p}) A(\infty) \tag{17}$$

The renewal intensity $h_A^*(\gamma,\mathbf{p})$ is most easily be determined from the density of times between renewals $f_A(t)$ as a convolution of $f_W(t)$ and $f_N(t)$ whose Laplace transform simply is $f_A^*(\gamma) = f_W^*(t) f_N^*(t)$. During a finite renewal the structure is supposed not to fail. Therefore, in first approximation the damage term is also multiplied by $A_W(\infty)$.

2.6. Possibly repeated reconstruction at renewal

One also can consider possibly repeated failure at construction (or reconstruction), i.e. the facility is reconstructed after failure in the construction phase according to the same rules until it can be put into service. It is sufficient to consider the additional cost of reconstruction at a time, which are

$$D_W(\mathbf{p}) = (C_W(\mathbf{p}) + H_W)\sum_{i=1}^{\infty} iP_{f,W}(\mathbf{p})^i R_{f,W}(\mathbf{p})$$

$$= (C_W(\mathbf{p}) + H_W)\frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})}$$

with $R_{f,W}(\mathbf{p}) = 1 - P_{f,W}(\mathbf{p})$. The result is:

$$Z(\mathbf{p}) = \frac{b}{\gamma} - \left(C(\mathbf{p}) + \frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})}(C_W(\mathbf{p}) + H_W)\right) - (C(\mathbf{p}) + H)\left(1 + \frac{P_{f,W}(\mathbf{p})}{R_{f,W}(\mathbf{p})}\frac{(C_W(\mathbf{p}) + H_W)}{(C(\mathbf{p}) + H)}\right)h^*(\gamma, \mathbf{p}) \quad (18)$$

The additional factor reflects the fact that the reconstruction and damage cost $(C_W(\mathbf{p}) + H_W)$ can happen multiply with probability $P_{f,W}(\mathbf{p})$. It is useful to distinguish between reconstruction cost $C_W(\mathbf{p})$ and damage cost H_W in the erection phase and the reconstruction and damage cost $C(\mathbf{p}) + H$ during use of the facility. Multiple reconstruction can, of course, also happen during first construction.

This model also enables to estimate the length of the finite renewal time. If $E[T_{W,1}]$ is the expectation of independent, identically distributed reconstruction times, then, since $E[Y] = E[\sum_{i=1}^{N} X_i] = E[X]E[N]$ (N random and geometrically distributed according to $p(n) = P_{f,W}(\mathbf{p})^{n-1}R_{f,W}(\mathbf{p})$) we have $E[T_W] = E[T_{W,1}]/(1 - P_{f,W}(\mathbf{p}))$. In general, this time is only insignificantly larger than $E[T_{W,1}]$ for $(1 - P_{f,W}(\mathbf{p})) \lesssim 1$.

2.7. Multiple failure modes and different failure causes

Assume for the moment two independent failure modes with possibly different reconstruction and damage cost, denoted by $V1$ and $V2$, respectively, each requiring renewal after failure. The times between renewals then are distributed as $F(t) = 1 - (1 - F_{V_1}(t))(1 - F_{V_2}(t)) = 1 - \bar{F}_{V_1}(t)\bar{F}_{V_2}(t)$. The corresponding density is $f(t) = f_{V_1}(t)\bar{F}_{V_2}(t) + f_{V_2}(t)\bar{F}_{V_1}(t)$ and its Laplace transform is $f^{**}(\gamma, \mathbf{p}) = f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma)$. It follows that

$$D(\mathbf{p}) = \frac{(C_1(\mathbf{p}) + H_1)f^{**}_{V_1|\bar{V}_2}(\gamma) + (C_2(\mathbf{p}) + H_2)f^{**}_{V_2|\bar{V}_1}(\gamma)}{1 - (f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma))} \quad (19)$$

This equation is derived as follows: Let $\theta_i = t_i - t_{i-1}$ be the times between renewals with density $f(t)$ and, for example, C_{V_1} and C_{V_2} the cost associated with the two types of renewals. Then, the expected cost is

$$D = E\left[\sum_{n=1}^{\infty}(C_{V_1} + C_{V_2})\exp\left[-\gamma\sum_{k=1}^{n}\theta_k\right]\right]$$

$$= \sum_{n=1}^{\infty} E[\exp(-\gamma\theta)]^{n-1}E[(C_{V_1} + C_{V_2})\exp[-\gamma\theta]]$$

$$= \frac{E[(C_{V_1} + C_{V_2})\exp[-\gamma\theta]]}{1 - E[\exp(-\gamma\theta)]} = \frac{C_{V_1}f^{**}_{V_1|\bar{V}_2} + C_{V_2}f^{**}_{V_2|\bar{V}_1}(\gamma)}{1 - (f^{**}_{V_1|\bar{V}_2}(\gamma) + f^{**}_{V_2|\bar{V}_1}(\gamma))}$$

Here, we distinguish between ordinary Laplace transforms $f^*(\gamma)$ for densities and modified Laplace transforms $f^{**}(\gamma)$ for which $f^{**}(\gamma) \le f^*(\gamma)$. One can generalize to more (independently) caused renewals:

$$D(\mathbf{p}) = \frac{\sum_{i=1}^{s} C_{V_i}f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma)}{1 - \sum_{i=1}^{s} f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma)} \le \frac{\sum_{i=1}^{s} C_{V_i}f^{*}_{V_i}(\gamma)}{1 - \sum_{i=1}^{s} f^{*}_{V_i}(\gamma)} \quad (20)$$

with $C_{V_i} = (C_i(\mathbf{p}) + H_i)$, $f(t) = \sum_{i=1}^{s} f_i(t)\prod_{j\neq i}\bar{F}_j(t)$ and, therefore, $f^{**}_{V_i|\cap_{j\neq i}\bar{V}_j}(\gamma) = \int_0^\infty \exp[-\gamma t]f_i(t)\prod_{j\neq i}\bar{F}_j(t)\mathrm{d}t$. Eq. (20) is an upper bound with $f^*_{V_i}(\gamma) = \int_0^\infty \exp[-\gamma t]f_i(t)\,\mathrm{d}t$. Equality in Eq. (20) implies independent modes with exponentially distributed failure times.

Generalizing Eq. (20) to arbitrary failure models gives

$$D = \sum_{k=1}^{s}(C_k(\mathbf{p}) + H_k)\frac{f_k^{**}(\gamma)}{1 - \sum_{k=1}^{s} f_k^{**}(\gamma)}$$

$$\le \sum_{k=1}^{s}(C_k(\mathbf{p}) + H_k)\frac{f_k^{*}(\gamma)}{1 - \sum_{k=1}^{s} f_k^{*}(\gamma)} \quad (21)$$

Eq. (21) holds allowing for mixtures of different types of failure including extreme value failure and failure by aging. $f_k^{**}(\gamma)$ is the corresponding Laplace transform for the dependent arbitrary failure models. $f_k^*(\gamma)$ is the approximation for independent failure models neglecting that failure occurred already before in another failure mode. A derivation for $f_k^{**}(\gamma)$ and the upper bound $f_k^*(\gamma)$ is given in Appendix B.

2.8. Inspection and repair of aging components

In the literature maintenance cost frequently have been assumed to increase continuously with time. More realistic in the structures area is the case where maintenance cost are the sum of inspection and possible repair cost. Assume inspections at regular intervals $a, 2a, 3a, \ldots$ Inspections and repairs occur only if renewals have not occurred before due to obsolescence or failure. Assume further that repairs, if undertaken, restore the properties of a component to its

original (stochastic) state, i.e. repairs are equivalent to renewals. Inspection and repair times are assumed negligibly short. Of course, it makes only sense to consider aging components with increasing risk function $r(t)$. Denote the failure model for the aging component by V whereas A stands for any other (independent) failure mode (or obsolescence as another cause for renewal).

Consider first the case with only one failure mode. A renewal occurs either after failure or at times $a, 2a, 3a, \ldots$ and renewal (repair) times are negligibly short. In Ref. [1] this is denoted by age replacement. Then, we obviously have [11]:

$$Z(\mathbf{p}, a) = B - C(\mathbf{p}) - \frac{(C(\mathbf{p}) + H)(f_V^{***}(\gamma, \mathbf{p}, a) + I_1(\mathbf{p})\exp[-\gamma a]\bar{F}_V(\mathbf{p}, a))}{1 - (f_V^{***}(\gamma, \mathbf{p}, a) + \exp[-\gamma a]\bar{F}_V(\mathbf{p}, a))} \quad (22)$$

with $I_1(\mathbf{p})$ the cost of repair and $I_1(\mathbf{p}) < (C(\mathbf{p}) + H)$.

If there is an inspection there is not necessarily a repair because inspections are uncertain. Then, inspection and repair cost must be included in the damage term:

$$Z(\mathbf{p}, a) = B - C(\mathbf{p}) - D(\mathbf{p}, a) \quad (23)$$

with

$$D(\mathbf{p}, a) = \frac{N}{D} \quad (24)$$

$$N = (C(\mathbf{p}) + A)(f^{***}_{A|\bar{V}}(\gamma, a) + A11) + (C(\mathbf{p}) + H)(f^{***}_{V|\bar{A}}(\gamma, \mathbf{p}, a) + A12) + I_0((1 - P_R(a))\exp[-\gamma a]\bar{F}_A(a)\bar{F}_V(\mathbf{p}, a) + A21) + (I_0 + I_1(\mathbf{p}))(P_R(a)\exp[-\gamma a]\bar{F}_A(a)\bar{F}_V(\mathbf{p}, a) + A22)$$

$$D = 1 - (f^{***}_{A|\bar{V}}(\gamma, a) + f^{***}_{V|\bar{A}}(\gamma, \mathbf{p}, a) + A11 + A12 + P_R(a) \times \exp[-\gamma(a)]\bar{F}_A(a)\bar{F}_V(\mathbf{p}, a) + A22)$$

$$A11 = \sum_{n=2}^{\infty}\prod_{j=1}^{n-1}(1 - P_R(ja))f^{***}_{A|\bar{V}}(\gamma, \mathbf{p}, (n-1)a \le t \le na)$$

$$A12 = \sum_{n=2}^{\infty}\prod_{j=1}^{n-1}(1 - P_R(ja))f^{***}_{V|\bar{A}}(\gamma, \mathbf{p}, (n-1)a \le t \le na)$$

$$A21 = \sum_{n=2}^{\infty}(1 - P_R(na))\prod_{j=1}^{n-1}(1 - P_R(ja)) \times \exp[-\gamma na]\bar{F}_A(na)\bar{F}_V(\mathbf{p}, na)$$

$$A22 = \sum_{n=2}^{\infty}P_R(na)\prod_{j=1}^{n-1}(1 - P_R(ja))\exp[-\gamma na]\bar{F}_A(na)\bar{F}_V(\mathbf{p}, na)$$

and where

$P_R(a)$: probability of repair after inspection
$\bar{P}_R(a) = 1 - P_R(a)$: probability of no repair after inspection
a: deterministic inspection interval
I_0: cost per inspection
$I_1(\mathbf{p})$: repair cost
$f^{***}_{X|\bar{Y}}(\gamma, a, \mathbf{p}) = \int_0^a \exp[-\gamma t]f_X(t)\bar{F}_Y(t)\mathrm{d}t \le \int_0^a \exp[-\gamma t]f_X(t)\,\mathrm{d}t$: incomplete, modified Laplace transform of $f_X(t)$.

The repair probability depends on the magnitude of a suitable damage indicator. For cumulative damage phenomena $P_R(a, \mathbf{p})$ increases with a. For example, $P_R(a, \mathbf{p}) = P(S(a, X, \mathbf{p}) > s_c)$ with $S(a, X, \mathbf{p})$ a monotonically increasing damage indicator and X a random variable taking into account of all uncertainties during inspection. Frequently, the length of inspection intervals is taken as an optimization parameter. The case without inspection and $P_R(a, \mathbf{p}) = 1$ is already dealt with in the literature [11, 34]. Repair after inspection is interpreted as preventive renewal (replacement of an aging component after a finite time of use a). Renewal after failure is called corrective renewal. It must be mentioned that optimal inspection/repair intervals do not always exist. Preventive renewals must, in fact, be substantially cheaper than corrective renewals. Also, the repair probability must be sufficiently high at a.

A similar derivation holds for a non-constant benefit function $b(t)$ [33]. If it is assumed, that the benefit B is constant with $b(t) = b$, it follows that $B = \lim_{t\to\infty}\int_0^t b \exp(-\gamma t) = b/\gamma$. Additionally, at the expense of additional computations a delay Δ of repairs can be assumed, so that they occur at times $a + \Delta$, $2a + \Delta$, $3a + \Delta$.

3. Reliability models

It is assumed that classical FORM/SORM is used. Safe and failure domains are separated by differentiable limit state surfaces $h(\mathbf{x}, \mathbf{p}, t) = 0$ where $\mathbf{x}$ is a n-dimensional vector of uncertain (process) variables with continuous distribution function $F_X(\mathbf{x}, \mathbf{p}, t)$, $\mathbf{p}$ is a parameter vector and t is the time. Also, it is assumed that a unique probability distribution transformation $\mathbf{x} = \mathbf{T}(\mathbf{u})$ exists where $\mathbf{u}$ is an independent standard normal vector so that $g(\mathbf{u}, \mathbf{p}, t) = 0$ (see Ref. [15] or approximate transformations in Refs. [6, 37]). Finally, it is assumed that a unique β-point for each failure mode exists, i.e. for which $\beta_k = \|\mathbf{u}^*\| = \max\{\|\mathbf{u}\|$ for $\{\mathbf{u} : g_k(\mathbf{u}, \mathbf{p}, t) \le 0\}$. Note that $\beta_k > 0$ for $g_k(\mathbf{0}, \mathbf{p}, t) > 0$ and $\beta_k \le 0$ for $g_k(\mathbf{0}, \mathbf{p}, t) \le 0$.

As mentioned, failure (or renewal) rates can be replaced by outcrossing rates under asymptotic conditions [5]. In the stationary case, arbitrary limit state function and loading by a combination of a vectorial rectangular wave renewal Gaussian process with jump rates λ_j and a vectorial differentiable Gaussian process with covariance function matrix $\mathbf{R}(\tau)$ the outcrossing rate is [2,3, 23–25,35]:

$$\nu^+(\mathbf{p}) = \left(\sum_{i=1}^{n_J}\lambda_i\Phi_2(\beta(\mathbf{p}), -\beta(\mathbf{p}); \rho_i(\mathbf{p})) + \omega_0\frac{\varphi(\beta(\mathbf{p}))}{\sqrt{2\pi}}\right) \quad (25)$$

$\Phi_2(\cdot,\cdot;\cdot)$ is the bivariate standard normal integral with correlation coefficient $\rho_i = 1 - \alpha_i^2$, $\beta(\mathbf{p}) = \|\mathbf{u}^*\|$ and ω_0 is the central frequency with which the process outcrosses the limit state function (i.e. $\omega_0^2 \approx \mathbf{n}(\mathbf{u}^*,\mathbf{p})^{\mathrm{T}}\ddot{\mathbf{R}}(0)\mathbf{n}(\mathbf{u}^*,\mathbf{p}), \mathbf{n}(\mathbf{u}^*,\mathbf{p}) = -\alpha^{\mathrm{T}}(\mathbf{u}^*,\mathbf{p}) = -(\mathbf{u}^*/\beta(\mathbf{p}))$, $\ddot{\mathbf{R}}(0) = \mathbf{E}[\dot{\mathbf{U}}\dot{\mathbf{U}}^{\mathrm{T}}]$). Eq. (25) may be multiplied by a SORM-correction factor $C_{\mathrm{SORM}}(\mathbf{p})$ involving curvature information of $g(\mathbf{u},\mathbf{p},t) = 0$ in $\mathbf{u}^*$. For brevity of notation this is not done herein. Certain non-normal processes can also be handled after a suitable probability distribution transformation.

In many cases the distribution of times between failure due to aging cannot be given directly but has to be computed numerically. For strictly monotonic cumulative damage phenomena $F_{\mathrm{T}}(t)$ can be computed from

$$F_{\mathrm{T}}(t) = P_{\mathrm{f}}(t) = P(g(\mathbf{X},t) \le 0)) \approx \Phi(-\beta(t))C_{\mathrm{SORM}}(t) \tag{26}$$

where $\beta(t)$ is the usual geometric reliability index and $C_{\mathrm{SORM}}(t)$ is a second-order correction factor. Quite generally, one ignores the second-order correction. The failure density is $f(t) = -\varphi(\beta(t))(\mathrm{d}\beta(t)/\mathrm{d}t)$. The Laplace transform must be computed numerically (see below). Mean and variance of the failure times can be computed from

$$E[T^k] = \int_0^\infty kt^{k-1}(1 - F_{\mathrm{T}}(t))\mathrm{d}t \tag{27}$$

for $T \ge 0$. For $k = 1$ we obtain the mean $E[T]$ and for $k = 2$ the second moment $E[T^2]$, respectively, and therefore $\mathrm{var}[T] = E[T^2] - E[T]^2$.

If in some application one is forced to use a time-variant reliability method for non-stationary problems, particularly the outcrossing method, the asymptotic life time distribution is

$$F(t) = 1 - \exp\left[-\int_0^t \nu^+(\tau)\mathrm{d}\tau\right] \tag{28}$$

with density

$$f(t) = \nu^+(t)\exp\left[-\int_0^t \nu^+(\tau)\mathrm{d}\tau\right] \tag{29}$$

Here again, the Laplace transform must be computed numerically. Instead of Eq. (28) it is sometimes better to use the upper bound

$$F_{\mathrm{T}}(t) = P_{\mathrm{f}}(t) \le P_{\mathrm{f}}(0) + \int_0^t \nu^+(\tau)\mathrm{d}\tau \le 1 \tag{30}$$

where $P_{\mathrm{f}}(0) = 0$ in many cases. The corresponding density is $f_{\mathrm{T}}(\tau) = P_{\mathrm{f}}(0)\delta(0) + \nu^+(\tau)$. This density should be close to the exact one for aging problems but is less suitable for the stationary case. The advantage of these formulations is that well-known FORM/SORM-methodology is applicable, at least for more complicated problems and $\nu^+(\tau)$ depending on a random vector $\mathbf{R}$ [29].

4. Numerical techniques

Let $\mathbf{p}$ be a parameter vector which enters in both the cost function and the limit state function $g(\mathbf{u},\mathbf{p}) = 0$. Benefit, construction and damage function as well as the limit state function(s) are differentiable in $\mathbf{p}$ and $\mathbf{u}$. The conditions for the application of FORM/SORM hold. In the so-called β-point $\mathbf{u}^*$ the optimality conditions (Kuhn–Tucker conditions) are [17]:

$$g(\mathbf{u},\mathbf{p}) = 0$$

$$\frac{\mathbf{u}}{\|\mathbf{u}\|} = -\frac{\nabla_{\mathbf{u}}g(\mathbf{u},\mathbf{p})}{\|\nabla_{\mathbf{u}}g(\mathbf{u},\mathbf{p})\|} \tag{31}$$

The geometrical meaning of Eq. (31) is that the gradient of $g(\mathbf{u},\mathbf{p}) = 0$ is perpendicular to the vector of direction cosines of $\mathbf{u}^*$. The basic idea mentioned first in Ref. [10] and elaborated in Ref. [17] now is to use these conditions as constraints in the cost optimization problem thus avoiding a bi-level optimization. It will turn out that this concept is crucial for further numerical analysis.

This leads to:

$$Z(\mathbf{p}) = B - C(\mathbf{p}) - (C(\mathbf{p}) + H)\,\frac{\nu^+(\mathbf{p})}{\gamma} \tag{32}$$

subject to:

$$g(\mathbf{u},\mathbf{p}) = 0$$

$$u_i\|\nabla_{\mathbf{u}}g(\mathbf{u},\mathbf{p})\| + \nabla_{\mathbf{u}}g(\mathbf{u},\mathbf{p})_i\|\mathbf{u}\| = 0, \qquad i = 1,\dots,n-1$$

$$h_k(\mathbf{p}) \le 0, \qquad k = 1,\dots,q$$

$$\nu^+(\mathbf{p}) \le \nu^+_{\mathrm{admissible}}$$

It is important to reduce the set of the gradient conditions in the Kuhn–Tucker conditions by one. Otherwise, the system of Kuhn–Tucker conditions is overdetermined. It is also important that the remaining Kuhn–Tucker conditions are retained under all circumstances, for example, if one or more gradient Kuhn–Tucker conditions become colinear with one or more of the other constraints. Otherwise, the so-called β-point conditions are not fulfilled.

If there are multiple failure modes $\nu^+(\mathbf{p})/\gamma$ must simply be replaced by $\sum_{i=1}^{s}\nu^+(\mathbf{p})/\gamma$ (Eq. (20)). In this case generalizing ideas in Ref. [19]

$$Z(\mathbf{p}) \le B - C(\mathbf{p}) - (C(\mathbf{p}) + H)\,\frac{\sum_{k=1}^{s}\nu_k^+(\mathbf{p})}{\gamma} \tag{33}$$

subject to

$$g_k(\mathbf{u}_k,\mathbf{p}) = 0, \qquad k = 1,\dots,s$$

$$u_{i,k}\|\nabla_{\mathbf{u}}g_k(\mathbf{u}_k,\mathbf{p})\| + \nabla_{\mathbf{u}}g_k(\mathbf{u}_k,\mathbf{p})_i\|\mathbf{u}_k\| = 0,$$

$$i = 1,\dots,n_k - 1; \quad k = 1,\dots,s$$

$$h_\ell(\mathbf{p}) \le 0, \qquad \ell = 1,\dots,q$$

$$\sum_{k=1}^{s}\nu_k^+(\mathbf{p}) \le \nu^+_{\mathrm{admissible}}; \qquad k = 1,\dots,s$$

where the Kuhn–Tucker conditions have to be fulfilled separately for each failure mode. Note that there are s distinct independent vectors $\mathbf{u}_k$.

If the problem is not stationary it is sufficient to determine the asymptotic renewal intensity or the mean value of time between renewals in many cases (Eq. (4)). Several but not always successful methods have been studied in Ref. [26]. However, for (locally) non-stationary problems, especially aging problems and for problems with non-Poissonian failures, it is possible to propose a numerical solution. More precisely, the Laplace transform is taken numerically and each value of the failure density is computed by FORM/SORM. A first model makes use of the asymptotic result in Eq. (4), i.e. requires the computation of the mean failure time. A better failure model certainly is a model where mean and standard deviation of the failure times are determined. As mentioned this is also an asymptotic approximation for arbitrary failure models being identical to the Gaussian model. Both models may be used as approximations. The first two moments of an arbitrary failure model then need to be computed from Eq. (27). The integrals are represented as sums of equi-distant values of the integrand

$$I_k(\mathbf{p}) = \Delta\sum_{j=0}^{m} w_j i_k(t_j,\mathbf{p}) \tag{34}$$

where w_j are the weights (for example, according to Simpson or Newton) and $i(t_j)$ are the values of the integrands, that is $\Phi(\beta(t_j,\mathbf{p}))$ and $t_j\Phi(\beta(t_j,\mathbf{p}))$ according to Eq. (27), respectively, for $k = 1$ and 2 (SORM-factor neglected). Since the integrand function $\exp[-\gamma t]f_{\mathrm{T}}(t,\mathbf{p})$ is bell-shaped other integration schemes, for example, Gaussian quadrature schemes, are less suitable. Then, with $f^*(\gamma,\mathbf{p}) = \exp[(1/2)\gamma(\gamma\sigma(\mathbf{p})^2 - 2\mu(\mathbf{p}))]$ and $\mu(\mathbf{p}) = E[T(\mathbf{p})] = \Delta\sum_{j=0}^{n} w_j i_1(t_j,\mathbf{p})$ as well as $\sigma(\mathbf{p})^2 = \Delta\sum_{j=0}^{n} w_j i_2(t_j,\mathbf{p}) - \mu(\mathbf{p})^2$ the Kuhn–Tucker-conditions must be fulfilled at each t_j and one can write similar to the procedure for series systems

$$Z(\mathbf{p}) \approx B - C(\mathbf{p}) - (C(\mathbf{p}) + H)\frac{\exp[\frac{1}{2}\gamma(\gamma\sigma(\mathbf{p})^2 - 2\mu(\mathbf{p}))]}{1 - \exp[\frac{1}{2}\gamma(\gamma\sigma(\mathbf{p})^2 - 2\mu(\mathbf{p}))]} \tag{35}$$

subject to

$$g(\mathbf{u}_j,\mathbf{p},t_j) = 0, \quad \text{for } j = 0,1,\dots,m$$

$$u_{i,j}\|\nabla_{\mathbf{u}}g(\mathbf{u}_j,\mathbf{p},t_j)\| + \nabla_{\mathbf{u}}g(\mathbf{u}_j,\mathbf{p},t_j)_i\|\mathbf{u}_j\| = 0,$$

$$i = 1,\dots,n-1; \; j = 0,\dots,m$$

$$h_k(\mathbf{p}) \le 0, \quad k = 1,\dots,q$$

$$\frac{1}{E_{\mathrm{T}}[T(\mathbf{p})]} \le \nu^+_{\mathrm{admissible}}$$

where $\beta(t_j,\mathbf{p}) = \|\mathbf{u}_j^*\|$. The vectors $\mathbf{u}_j$, $j = 0,1,\dots,m$ are mutually independent. Therefore, the size of the optimization problem grows as $n \times m$.

The same scheme, however, applies to the full Laplace transform of non-stationary problems as well

$$Z(\mathbf{p}) \approx B - C(\mathbf{p}) - (C(\mathbf{p}) + H)\,\frac{f^*(\gamma,\mathbf{p})}{1 - f^*(\gamma,\mathbf{p})} \tag{36}$$

$$g(\mathbf{u}_j,\mathbf{p},t_j) = 0, \qquad \text{for } j = 0,1,\dots,m$$

$$u_{i,j}\|\nabla_{\mathbf{u}}g(\mathbf{u}_j,\mathbf{p},t_j)\| + \nabla_{\mathbf{u}}g(\mathbf{u}_j,\mathbf{p},t_j)_i\|\mathbf{u}_j\| = 0,$$

$$i = 1,\dots,n-1; \; j = 0,\dots,m$$

$$h_\ell(\mathbf{p}) \le 0, \qquad \ell = 1,\dots,q$$

$$\frac{1}{E_{\mathrm{T}}[T(\mathbf{p})]} \le \nu^+_{\mathrm{admissible}}$$

where

$$f^*(\gamma,\mathbf{p}) \approx \Delta\sum_{j=0}^{m} w_j \exp[-\gamma t_j]f_{\mathrm{T}}(t_j,\mathbf{p}) \tag{37}$$

For the case in Eq. (26) it is

$$f^*(\gamma,\mathbf{p}) \approx \Delta\sum_{j=0}^{m} w_j \exp[-\gamma t_j](-\varphi(\beta(t_j,\mathbf{p})))\frac{\mathrm{d}\beta(t_j,\mathbf{p})}{\mathrm{d}t} \tag{38}$$

with $\mathrm{d}\beta(t_j,\mathbf{p})/\mathrm{d}t = ((\partial/\partial t)g(\mathbf{u}_j,t_j,\mathbf{p}))/\|\nabla_{\mathbf{u}}g(\mathbf{u}_j,t_j,\mathbf{p})\|$ [16] and in the case Eq. (30):

$$f^*(\gamma,\mathbf{p}) \approx \Delta\sum_{j=0}^{m} w_j \exp[-\gamma t_j]\Bigg(P_{\mathrm{f}}(0)\delta(0) + \Bigg(\sum_{i=1}^{n_j}\lambda_i\Phi_2(\beta(t_j,\mathbf{p}),-\beta(t_j,\mathbf{p});\rho_i(t_j,\mathbf{p})) + \omega_0\frac{\varphi(\beta(t_j,\mathbf{p}))}{\sqrt{2\pi}}\Bigg)\Bigg) \tag{39}$$

If $f_{\mathrm{T}}(t,\mathbf{p})$ depends on a random parameter $\mathbf{R}$ one has to use the approximation

$$E_{\mathbf{R}}\left[\frac{f^*(\gamma,\mathbf{p},\mathbf{R})}{1 - f^*(\gamma,\mathbf{p},\mathbf{R})}\right] \approx \frac{f^*(\gamma,\mathbf{p},E[\mathbf{R}])}{1 - f^*(\gamma,\mathbf{p},E[\mathbf{R}])}$$

It can be shown that usually the approximation

$$E_{\mathbf{R}}\left[\frac{f^*(\gamma,\mathbf{p},\mathbf{R})}{1 - f^*(\gamma,\mathbf{p},\mathbf{R})}\right] \approx \frac{E_{\mathbf{R}}[f^*(\gamma,\mathbf{p},\mathbf{R})]}{1 - f^*(\gamma,\mathbf{p},E[\mathbf{R}])}$$

is closest if

$$f^*(\gamma,\mathbf{p},\mathbf{R}) \ll 1$$

A similar computation scheme can, of course, be used if obsolescence and/or inspections and repairs are included.

Finally, the case of multiple failure mode system is given for arbitrary failure models:

$$Z(\mathbf{p}) \le B - C(\mathbf{p}) - \sum_{k=1}^{s}(C_k(\mathbf{p}) + H_k)\,\frac{f_k^*(\gamma,\mathbf{p})}{1 - f_k^*(\gamma,\mathbf{p})} \tag{40}$$

subject to:

$$g_k(\mathbf{u}_{j,k},\mathbf{p},t_j) = 0, \qquad \text{for } k = 1,\dots,s; \; j = 0,1,\dots,m$$

$$u_{i,j,k}\|\nabla_{\mathbf{u}} g_k(\mathbf{u}_{j,k},\mathbf{p},t_j)\| + \nabla_{\mathbf{u}} g_k(\mathbf{u}_{j,k},\mathbf{p},t_j)_i \|\mathbf{u}_{j,k}\| = 0$$

$$i = 1,\dots,n_k - 1;\; j = 0,\dots,m;\; k = 1,\dots,s$$

$$h_\ell(\mathbf{p}) \le 0, \qquad \ell = 1,\dots,q$$

$$\sum_{k=1}^{s} \frac{1}{E_T[T_k(\mathbf{p})]} \le \nu^+_{\text{admissible}}$$

We note that the problem now can be rather large, i.e. there are $s \times m$ independent random vectors of length n_k. Clearly, the most difficult part in such calculations is the assessment of t_m and m. However, the exponent term in Eq. (37) usually lets the integrand decay sufficiently fast. If reliability restrictions are imposed it is necessary in all practical cases to use Eq. (4) because the inversion of the Laplace transform of the renewal density is numerically extremely difficult. It is further noted that the scheme proposed above can also be used when the benefit is non-constant as in Eq. (13) or the damage term is non-constant as in Eq. (15).

In order to solve the optimization problem a suitable optimization algorithm is required. Based on sequential quadratic programming methods a new optimization algorithm JOINT 5 has been developed from an earlier algorithm proposed by Enevoldsen/Sorensen [8]. This turned out necessary because the tasks in Eqs. (32), (33), (35), (36) and (40) require special precautions which are not necessarily available in most of the off-shelf algorithms. For example, the algorithm includes a reliable and robust slow down strategy to improve stability of the algorithm instead of an exact (or approximate) line search which too often is the reason for non-convergence [22]. A special 'extended' equation system is solved in case of failure in the quadratic subalgorithm, e.g. due to linear dependence of the linearized constraints. In addition, the algorithm contains a careful active set strategy (for further details see Ref. [31]).

Gradient-based methods need first derivatives of the objective and all active constraints. In case of cost optimization under reliability constraints first order Kuhn–Tucker optimality conditions for a design point are restrictions to the optimization problem. These equations are given in terms of the first derivatives of the limit state function. The gradients of these conditions involve second derivatives. Thus, the solution of the quadratic subproblem needs second derivatives, i.e. the complete Hessian of $g(\mathbf{u}, \mathbf{p})$. The determination of the Hessian in each iteration step is laborious and can be numerically inexact. In order to avoid this, an approximation by iteration is proposed. The Hessian is first preset with zeros. Note that linear limit state functions always have a zero Hessian matrix. This implies loss of efficiency, but the overall numerical effort needs not to rise, because calculation of the Hessian is no more necessary. In order to improve the results in case of non-linear limit state functions, it is possible to evaluate the Hessian after the first optimization run and restart the algorithm. The solution is the new starting point and the Hessian matrix is fixed for the whole run. This iterative improvement with subsequent restarts continues until the results differ only with respect to a given precision, which is usually after very few steps. The results can be simultaneously improved by including second-order corrections during reiteration [21]. Any other more exact improvement can be taken into account in a similar manner.

All in all, the techniques proposed enable the solution of quite general problems. They are still based on a one-level optimization but rather strong requirements on differentiability of the objective, limit state functions and other restrictions must be made. Also, a possibly substantial increase of the problem dimension must be expected in extreme case and, hence, much computing time will be necessary.

5. Illustrating examples

5.1. Example 1: rigid plastic two-bay frame

In this simple example a double-bay frame as shown in Fig. 1 using rigid-plastic theory with random horizontal and vertical loading and random plastic moments at nodes 1–10 will be optimized under reliability constraints.

The structure can fail in eight different failure modes as shown in Fig. 2. The first three failure events are elementary mechanisms, the others combined mechanisms. A limit state function for each failure mode is available using the energy theorem:

$$G_1(x,p) = X_2 + 2X_3 + X_4 - X_{12}\cdot\frac{h}{2}$$

$$G_2(x,p) = X_6 + 2X_7 + X_8 - X_{13}\cdot\frac{h}{2}$$

$$G_3(x,p) = X_1 + X_2 + X_5 + X_8 + X_9 + X_{10} - X_{11}\cdot h$$

$$G_4(x,p) = X_2 + 2X_3 + X_4 + X_6 + 2X_7 + X_8 - X_{12}\cdot\frac{h}{2} - X_{13}\cdot\frac{h}{2}$$

$$G_5(x,p) = X_1 + X_2 + X_5 + X_6 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{13}\cdot\frac{h}{2}$$

$$G_6(x,p) = X_1 + X_2 + X_4 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{13}\cdot\frac{h}{2}$$

$$G_7(x,p) = X_1 + 2X_3 + X_4 + X_5 + X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{12}\cdot\frac{h}{2}$$

$$G_8(x,p) = X_1 + 2X_3 + 2X_4 + 2X_7 + 2X_8 + X_9 + X_{10} - X_{11}\cdot h - X_{12}\cdot\frac{h}{2} - X_{13}\cdot\frac{h}{2}$$

where X_i, $i = 1,\dots,10$ are the plastic moments of the frame at node i. X_{11}, X_{12} and X_{13} are stochastic loads at node 2, 3 and 7. The stochastic properties of the random variables X_i are given in the following table. The random plastic moments at node X_i, $i = 1,\dots,10$ are assumed to be lognormally distributed, the stochastic loads X_{11}, X_{12} and X_{13} are normally distributed. All variables are assumed to be independent

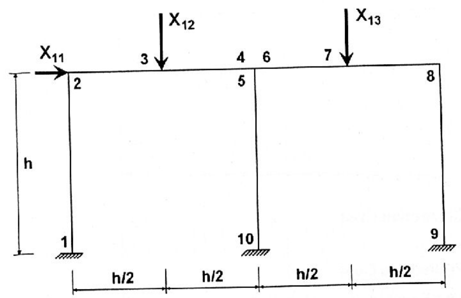

Fig. 1. Loads and statically system of the frame.

Stochastic variable	Unit	Mean/SD
Plastic moment at nodes 1, 2, 5 and 8–10	X_i (kN m)	$p_1/0.1p_1$
Plastic moment at nodes 3, 4, 6 and 7	X_i (kN m)	$p_2/0.1p_2$
Load at node 2	X_{11} (kN)	2/0.6
Load at node 3	X_{12} (kN)	4/1.2
Load at node 7	X_{13} (kN)	6/1.8

The loads at nodes 3 and 7 are modeled as stationary rectangular wave renewal processes with jump rates $\lambda_{12} = \lambda_{13} = 0.5$ [1/year]. The load at node 2 is modeled as stationary differentiable Gaussian process with autocorrelation function $\rho_{ij}(\tau) = \exp(-\tau^2)$. The design parameters p_1 and p_2 are the mean values of the appropriate stochastic variables. The bounds for p_1 and p_2 are as follows: $p_1 \in [5.0; 80.0]$ kN m, $p_2 \in [5.0; 80.0]$ kN m. The objective function, which will be minimized in the optimization program, is defined as construction cost depending on the mean values of the plastic moments at nodes 1,…, 10 as $C(\mathbf{p}) = p_1 + 2.0p_2$. The failure cost are $H = 1000$ and the interest rate is $\gamma = 0.02$. The optimization problem contains of 106 optimization variables. The optimal cost parameter for time- variant cost optimization under reliability constraints of this series system with $h = 20$ m and a time interval of one year are

$$p_1^* = 16.95, \quad p_2^* = 38.45$$

and the optimal cost are $C_{\text{tot}}(\mathbf{p}) = 93.85$ [CU]. Note that due to the common resistance variables the crossings are dependent. The time-variant upper bound failure probability

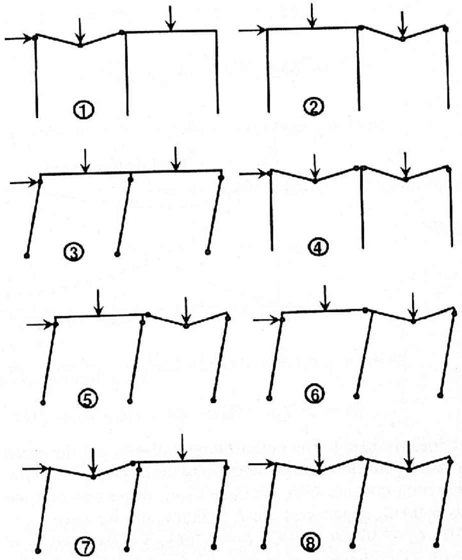

Fig. 2. Failure modes of the frame.

in each mode is computed as: $(P_{f,1}(\mathbf{p}^*), P_{f,2}(\mathbf{p}^*), P_{f,3}(\mathbf{p}^*), P_{f,4}(\mathbf{p}^*), P_{f,5}(\mathbf{p}^*), P_{f,6}(\mathbf{p}^*), P_{f,7}(\mathbf{p}^*), P_{f,8}(\mathbf{p}^*)) = (9.76\times10^{-11}, 2.21\times10^{-4}, 2.53\times10^{-6}, 2.37\times10^{-11}, 4.13\times10^{-8}, 1.82\times10^{-6}, 9.66\times10^{-10}, 1.38\times10^{-9})$. The system failure probability is 2.25×10^{-4} with corresponding equivalent reliability index 3.51.

5.2. Example 2: reinforced concrete structure subject to chloride corrosion in warm sea water

Following Ref. [26] a simplified failure criterion for chloride corrosion in the splash zone in warm sea water is:

$$C_{\text{cr}} - C_s\left(1 - \text{erf}\left(\frac{c}{2\sqrt{Dt}}\right)\right) \le 0$$

where C_{cr} is the critical chloride content; C_s, the surface chloride content; c, the concrete cover; D is the diffusion parameter. The stochastic model is

Variable	Distrib. function	Parameters
C_{cr}	Uniform	0.125, 0175
C_s	Uniform	0.2, 0.4
c	Log–normal	m_c, 1
D	Uniform	0.1, 0.315

The uniform distributions reflect the large uncertainty in the variables. The units are chosen such that t is in years. Inspection are performed at regular intervals a. They are followed by renewals (repairs) with probability $P_R(a) =$

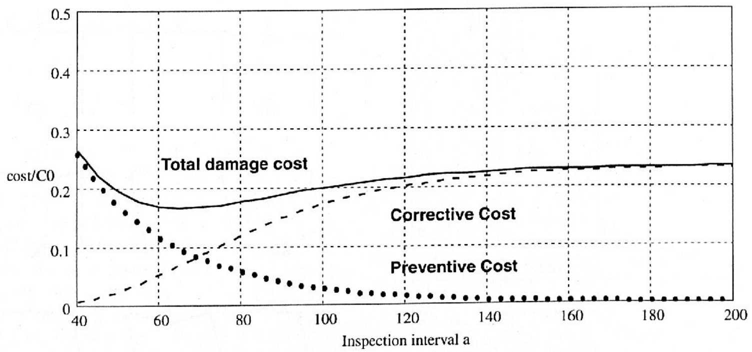

Fig. 3. Total cost for regular inspections and renewals.

$1 - \exp[-a_R a^2]$. The optimization variables are the mean concrete cover m_c and the length a of the inspection interval. Erection cost are $C(m_c) = C_0 + C_1 m_c^2$, inspection cost are $I_0 = 0.1C_0$, repair cost are $I_1 = 0.5C_0$ and we have $C_0 = 10^6$, $C_1 = 10^4$, $H = 10C_0$, $b = 0.15C_0$, $\gamma = 0.03$ and $a_R = 0.01$. The solution is $a^* = 66$ and $m_c^* = 6.5$. It turns out that preventive repairs should be performed every 66 years which saves up to 30% of the cost. These results agree well with the practical experience with such structures. The contributions to the total damage cost are shown in Fig. 3. Relatively small variations in the repair model or in the cost factors will, however, result in cases where it is better not to inspect and repair but just wait for failure. It is noted that for the given failure model no mean time to failure exists.

6. Conclusions

Objective functions based on a continuous renewal model for a series of cases frequently met in practice are formulated. In particular, they include failures by outcrossings of loading processes and by random disturbances, non-constant benefit and damage functions, finite renewal times, repeated reconstructions and inspection and repair. A method for reliability-oriented time-variant structural optimization of series systems using FORM in standard space is developed generalizing theories proposed earlier for component problems and time-invariant series system problems in a special one-level approach. Certain improvements by taking account of dependencies among failure modes are also proposed. Approximations for time-variant failure probabilities are computed via the outcrossing method for locally stationary rectangular wave renewal and differentiable Gaussian processes. Numerical Laplace transforms are proposed for the treatment of aging components.

The optimization problem is solved by the newly developed gradient-based algorithm JOINT 5. It requires second derivatives of the limit state functions. This can be avoided by iteration. In the first iteration the Hessian is approximated by a zero matrix corresponding to linear limit state functions. In the second iteration the Hessian is determined once and kept fixed. The results can, thus, be improved by reiteration of the complete optimization task. The same reiteration loop can be used to update the results by SORM or any other suitable method.

Appendix A. Laplace transforms

Laplace transforms are defined by $f^*(\gamma) = \int_0^\infty e^{-\gamma t} f(t)\,dt$. If $f(t)$, $t \ge 0$, is a probability density, it is $f^*(0) = 1$, $f^*(\infty) = 0$ and $0 < f^*(\gamma) \le 1$ for all $\gamma \ge 0$. In the transformed space one can easily show that there is $h^*(\gamma) = f(\gamma)^* g^*(\gamma)$ for $h(t) = \int_0^\infty f(t-\tau)g(\tau)\,d\tau$. For independent, identically distributed interarrival times of failures the density of the time to the n-th event $f_n(t)$ then is $f_n(t) = \int_0^\infty f_{n-1}(t-\tau)f(\tau)\,d\tau$ and, therefore, $f_n^*(\gamma) = f_{n-1}^*(\gamma)f^*(\gamma) = f^*(\gamma)^{n-1}f^*(\gamma)$. The simplicity of this operation is the main reason why a continuous discounting function is used.

Appendix B. Improvement for dependent failure modes

An improvement for dependent failure modes is easiest for the case in Eq. (8). Here, one replaces $\lambda\sum_{k=1}^{s} P_{f,k}(\mathbf{p})$ by the discount factor involving either one minus the probability of survival in all modes or the upper and/or lower bound for a union of failure events [7], i.e. by

$$P\left(\bigcup_{k=1}^{s} V_k\right) \approx \lambda(1 - \Phi_s(\boldsymbol{\beta}; \mathbf{R})) \le \lambda\sum_{k=1}^{s}\left(P_{f,k} - \max_{j<k}\{P_{f,k\cap j}\}_{k>1}\right) \ge \lambda\sum_{k=1}^{s}\max\left(0, P_{f,k} - \sum_{j=1}^{k}\{P_{f,k\cap j}\}\right) \tag{B1}$$

with $P_{f,k} = P(\{\mathbf{U} \in V_k\} = P(g_k(\mathbf{U}) \le 0) \approx P(\boldsymbol{\alpha}_k^T\mathbf{U} + \beta_k \le 0)$ and $P_{f,k\cap j} \approx P(\{\boldsymbol{\alpha}_k^T\mathbf{U} + \beta_k \le 0\} \cap \{\boldsymbol{\alpha}_j^T\mathbf{U} + \beta_j \le 0\})$ requiring either the computation of the s-dimensional multinormal integral or the computation of $(s-1)^2/2$ bivariate normal integrals.

Consider next a non-redundant series system under stationary renewal rectangular wave loading, i.e. a system where before the jump the process must be in the safe domain of all components and in the failure domain of at least one of the components after the jump and $V_S = \{\bigcup_{k=1}^{s} g_k(\mathbf{u}) \le 0\}$. The outcrossing rate is:

$$\begin{aligned}\nu_J^+(V_S) &= \sum_{i=1}^{n}\lambda_i\left[P\left(\bigcup_{k=1}^{s}\left\{F_{ik}^+ \cap \bigcap_{j=1}^{s}\bar{F}_{ij}\right\}\right)\right] \\ &= \sum_{i=1}^{n}\lambda_i\left[P\left(\bigcup_{k=1}^{s}F_{ik}^+\right) - P\left(\bigcup_{k=1}^{s}F_{ik}^+ \cap \bigcap_{j=1}^{s}F_{ij}\right)\right] \\ &\le \sum_{i=1}^{n}\lambda_i\left[\sum_{k=1}^{s}(P(F_{ik}^+) - \max_{j<k}\{P(\{F_{ik}^+\}\cap\{F_{ij}^+\})\}_{k>1}) - A\right] \\ &\le \sum_{i=1}^{n}\lambda_i\sum_{k=1}^{s}P(F_{ik}^+)\end{aligned} \tag{B2}$$

with

$$A = \sum_{k=1}^{s}\max\left\{P\left(F_{ik}^+ \cap \bigcap_{j=1}^{s}F_{ij}\right) - \sum_{\ell<k}P\left(F_{i\ell}^+ \cap F_{ik}^+ \cap \bigcap_{j=1}^{s}F_{ij}\right)_{k>1}, 0\right\} \tag{B3}$$

and where $F_{ik}^+ = \{\mathbf{U}_i^+ \in V_k\}$, $F_{ik} = \{\mathbf{U}_i \in V_k\}$, $\bar{F}_{ij} = \{\mathbf{U}_i \in \bar{V}_j\}$ and where the subscript '$k>1$' indicates that this term equals zero for $k=1$. The computation of the correction term involves $s+1$- and $s+2$-dimensional normal integrals, respectively, if $V_k = \{g_k(\mathbf{u}) \approx \boldsymbol{\alpha}_k^T\mathbf{u} + \beta_k \le 0\}$ [12].

Crossings by Gaussian vector processes into componential failure domains can also be considered, at least approximately. For linearized limit state surfaces, i.e. for $V_S = \{\bigcup_{k=1}^{s} g_k(\mathbf{u}) \le 0\}$ with $\partial V_k = \{g_k(\mathbf{u}) \approx \boldsymbol{\alpha}_k^T\mathbf{u} + \beta_k = 0\}$, $g_k(\mathbf{0}) > 0$ for all $k = 1, \ldots, s$ one obtains after some computation [30]

$$\begin{aligned}\nu^+(V_S) &= \sum_{k=1}^{s}\int_{\partial V_k} E[(-\boldsymbol{\alpha}_k^T\dot{\mathbf{U}} - \dot{a}_k)^+ | \mathbf{U} = \mathbf{u}] \\ \varphi_n(\mathbf{u})ds(\mathbf{u}) &= \sum_{k=1}^{s}\int_{\partial V_k}\Psi(\dot{a}_k, m_k(\mathbf{u}), \sigma_k)\varphi_n(\mathbf{u})ds(\mathbf{u}) \\ &\approx \sum_{k=1}^{s}\Psi(\dot{a}_k, m_k(\mathbf{u}_k^*), \sigma_k)\varphi(\beta_k)[1 - \Phi_{s-1}(\mathbf{b}_k; \mathbf{B}_k)]\end{aligned} \tag{B4}$$

with

$$\mathbf{b}_k = \{\beta_r - \beta_k\boldsymbol{\alpha}_r^T\boldsymbol{\alpha}_k; 1 \le r \le s; r \ne k\}$$

$$\mathbf{B}_k = \{\boldsymbol{\alpha}_r^T\boldsymbol{\alpha}_t - (\boldsymbol{\alpha}_r^T\boldsymbol{\alpha}_k)(\boldsymbol{\alpha}_t^T\boldsymbol{\alpha}_k); 1 \le r, t \le s; r, t \ne k\}$$

$$\begin{aligned}\Psi(\dot{a}_k, m_k(\mathbf{u}^*), \sigma_k) &= E[(-\boldsymbol{\alpha}_k^T\dot{\mathbf{U}} - \dot{a}_k)^+ | \mathbf{U} = \mathbf{u}] \\ &= \sigma_k\varphi\left(\frac{\dot{a}_k - m_k(\mathbf{u}^*)}{\sigma_k}\right) - (\dot{a}_k - m_k(\mathbf{u}^*))\Phi\left(\frac{\dot{a}_k - m_k(\mathbf{u}^*)}{\sigma_k}\right)\end{aligned}$$

where $(y)^+ = \max\{0, y\}$, $\Phi_0(\cdot;\cdot) = 0$, $\beta_r = \boldsymbol{\alpha}_r^T\mathbf{u}_r^*$, $\beta_k = \boldsymbol{\alpha}_k^T\mathbf{u}_k^*$, $\sigma_k^2 = \boldsymbol{\alpha}_k^T(\ddot{\mathbf{R}} - \dot{\mathbf{R}}\dot{\mathbf{R}}^T)\boldsymbol{\alpha}_k$, $m_k(\mathbf{u}_k^*) = -\boldsymbol{\alpha}_k^T\dot{\mathbf{R}}\mathbf{u}_k^*$, $\dot{a}_k = \dot{\beta}_k - \sum_{j=1}^{n}\dot{\alpha}_{kj}$ and $2 \le s \le n$ as well as $\mathbf{u}_k^*$ the s different β-points. This equation is exact for $m_k(\mathbf{u}_k^*) = 0$. An upper bound is obtained if $m_k(\mathbf{u}_k^*)$ is made largest under the condition $g_k(\mathbf{u}) = 0 \cap \bigcap_{j\ne k} g_j(\mathbf{u}) > 0$. For $\dot{a}_k = 0$, i.e. the stationary case $\Psi(\dot{a}_k, m_k(\mathbf{u}^*), \sigma_k) = \sigma_k/\sqrt{2\pi}$.

For a combination of jump and differentiable processes we finally have due to the regularity of the crossing process:

$$\nu^+(V_S) \le \sum_{k=1}^{m}\left\{\sum_{i=1}^{n_j}\lambda_i[(P(F_{ik}^+) - \max_{j<k}\{P(F_{ik}^+ \cap F_{ij}^+)\}_{k>1}) - A] + \Psi(\dot{a}_k, m_k(\mathbf{u}_k^*), \sigma_k)\varphi(\beta_k)[1 - \Phi_{s-1}(\mathbf{b}_k; \mathbf{B}_k)]\right\} \tag{B5}$$

It is noted that the result is an upper bound for both differentiable processes and for rectangular wave renewal processes (to first order). One can then use Eq. (36) with (37).

The case of monotonically decreasing state functions can be solved as follows. Assume that there are s time-dependent failure modes and whose state functions are given by $g_k(\mathbf{u}, t) \approx \boldsymbol{\alpha}_k^T(t)\mathbf{u} + \beta_k(t)$ so that $V_k(t) = P(T_k \le t) = P(g_k(\mathbf{U}, t) \le 0) = P(Z_k \le -\beta_k(t))$. The failure probability at time t_j then is $F(t) = P(\bigcup_{k=1}^{s}\{Z_k \le -\beta_k(t)\}) = 1 - P(\bigcap_{k=1}^{s}\{Z_k \le \beta_k(t)\}) \approx 1 - \Phi_s(\boldsymbol{\beta}(t); \mathbf{R})$ where $\boldsymbol{\beta}(t) = \{\boldsymbol{\alpha}_k^T\mathbf{u}_k^*(t); k = 1, 2, \ldots, s\}$, $\|\boldsymbol{\alpha}_k\| = 1$, $k = 1, 2, \ldots, s$; $\mathbf{u}_k^*(t) = \min\{\|\mathbf{u}\|\}$ for $\{\mathbf{u} : g_k(\mathbf{u}, t) \le 0\}$ and $\mathbf{R} = E[\mathbf{Z}\mathbf{Z}^T] = \{\rho_{ij}\} = \{\boldsymbol{\alpha}_i^T\boldsymbol{\alpha}_j; i, j = 1, 2, \ldots, s\}$. In good approximation it is assumed that the matrix of correlation coefficients $\mathbf{R}$ varies little with time so that $(\partial/\partial t)\boldsymbol{\alpha}_k(t) \approx 0$ and, hence, $(\partial/\partial t)\rho_{ij}(t) \approx 0$ and there is $g_k(\mathbf{0}, t) > 0$ for all k. The general case of $(\partial/\partial t)\rho_{ij}(t) \ne 0$ is given in Ref. [33]. The failure density is

$$
\begin{aligned}
f_s(t) &= \frac{\mathrm{d}}{\mathrm{d}t}(1 - \Phi_s(\boldsymbol{\beta}(t);\mathbf{R})) \\
&= -\sum_{k=1}^{s} \frac{\partial}{\partial \beta_k(t)} \Phi_s(\boldsymbol{\beta}(t);\mathbf{R}) \frac{\partial \beta_k(t)}{\partial t} \\
&= -\sum_{k=1}^{s} \frac{\partial}{\partial \beta_k(t)} \int_{-\infty}^{\beta_k(t)} \Phi_{s-1}(\boldsymbol{\beta}(t);\mathbf{R}|Z_k = \beta_k(t))
\end{aligned}
$$

$$
\begin{aligned}
\varphi_1(z_k)\mathrm{d}z_k \frac{\partial \beta_k(t)}{\partial t} &= -\sum_{k=1}^{s} \varphi_1(\beta_k(t)) \Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k) \frac{\partial \beta_k(t)}{\partial t} \\
&= \sum_{k=1}^{s} \varphi_1(\beta_k(t)) \Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k) \left(\frac{-\frac{\partial}{\partial t} g_k(\mathbf{u}^* t)}{\|\nabla_{\mathbf{u}} g_k(\mathbf{u}^*,t)\|} \right) \\
&\le \sum_{k=1}^{s} \varphi_1(\beta_k(t)) \left(\frac{-\frac{\partial}{\partial t} g_k(\mathbf{u}^* t)}{\|\nabla_{\mathbf{u}} g_k(\mathbf{u}^*,t)\|} \right) \qquad \text{(B6)}
\end{aligned}
$$

with $\hat{\mathbf{c}}_k = \boldsymbol{\beta}^k(t) - \beta_k(t)\boldsymbol{\rho}_k^k$; and $\hat{\mathbf{R}}_k = \mathbf{R}^k - \boldsymbol{\rho}_k^k(\boldsymbol{\rho}_k^k)^{\mathrm{T}}$ where $\boldsymbol{\rho}_k$ is the kth column vector of $\mathbf{R}$ and the superscript means that the kth row and column, respectively, are deleted from the original vector and matrix, respectively. This result is obtained from regression analysis. Note that $\hat{\mathbf{R}}_k$ needs to be renormalized and therefore also $\hat{\mathbf{c}}_k$. More details can be found in Ref. [33]. Here, $s - 1$-dimensional normal integrals have to be evaluated for each t. Due to the substantial numerical effort when computing multi-normal probabilities this scheme can only be applied to smaller systems. Dropping the terms $\Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k)$, i.e. the survival probabilities in the other failure modes, corresponds to the upper bound solution in

$$
D = \sum_{k=1}^{s} (C_k(\mathbf{p}) + H_k) \frac{f_k^{**}(\gamma)}{1 - \sum_{k=1}^{s} f_k^{**}(\gamma)} \le \sum_{k=1}^{s} (C_k(\mathbf{p}) + H_k) \frac{f_k^{*}(\gamma)}{1 - \sum_{k=1}^{s} f_k^{*}(\gamma)} \qquad \text{(B7)}
$$

where

$$
f_k^{**}(\gamma) = \int_0^{\infty} \exp[-\gamma t] \varphi_1(\beta_k(t)) \Phi_{s-1}(\hat{\mathbf{c}}_k;\hat{\mathbf{R}}_k) \left(\frac{-\frac{\partial}{\partial t} g_k(\mathbf{u}^* t)}{\|\nabla_{\mathbf{u}} g_k(\mathbf{u}^*,t)\|} \right) \mathrm{d}t
$$

$$
\le f_k^{*}(\gamma) = \int_0^{\infty} \exp[-\gamma t] \varphi_1(\beta_k(t)) \mathrm{d}t
$$

The trivial upper bound may be useful but its application is limited to smaller systems because Laplace transforms of densities must remain smaller than unity.

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