Reliability of Parallel Systems under Imposed Uniform Strain

By M. Hohenbichler[1] and R. Rackwitz[1]

Abstract: An imposed strain approach is used for the reliability analysis of brittle parallel systems with arbitrary stress-strain behavior of its components. It leads to a formal description of the failure event as a parallel system. The failure probability of the strongest component overestimates the system failure probability, so that the consideration of the other components can be recommended for which first-order reliability techniques supply an efficient tool. Numerical comparisons with certain special exact results show that the correlation structure is taken fairly well into account by this first-order method. Essential improvements result from better estimates of the component reliability. This induces the definition of so called equivalent components, which might also be useful for the evaluation of more general systems. The accuracy of the method appears sufficient for many engineering applications.

Introduction

Very few analytical or numerical results are available for the reliability of redundant structural systems, although the majority of structural (sub) systems belong to this type of system. Moreover, the well-known bounds on system reliability respectively failure probability (2 or 4), i.e., the minimum of componential failure probabilities as an upper bound and their product as a lower bound, become too wide to be of any practical use. In addition, any, even very moderate, dependence between componential failures can have significant influence on the result (19). Only recently, certain numerical techniques based on so-called first-order reliability concepts have been proposed for the treatment of general systems (M. Hohenbichler, "Approximate Evaluation of the Multivariate Normal Distribution Function," (submitted for publication) and Hohenbichler, M., and Rackwitz, R., "First-Order Concepts in System Reliability," (to be published in "Structural Safety")) which are the result of a sequence of various earlier investigations (11,14,20,21). In this paper, a new method is developed and applied to an elementary structural system for which an exact result is available under certain idealizing assumptions.

In fact, apart from the well-known solutions for redundant systems whose components behave perfectly ductile and, thus, the determination of system strength results in summing up (or integrating) componential strength values with predetermined weights (which is a standard operation in probability theory with nice asymptotic properties), the only known alternative solution is for systems with perfectly brittle, equi-deformable components with an equal sharing of the applied load among unbroken components. As early as 1945 this second result was derived by Daniels (6), under the additional assumption that componential strengths are independent and identically distributed. Unfortunately, his formula is less feasible numerically for larger systems, but a better algorithm has been proposed by McCartney and Smith (16). Perhaps more importantly, Daniels found an asymptotic result which states that if $\lim_{x\to\infty} x[1 - F(x)] = 0$, in which $F(x)$ = the left continuous distribution function of elemental strength, X, then the distribution function of system strength, $R_n = R(X_1, \ldots, X_n)$, tends, with the number, n, of components approaching infinity, to the normal distribution with mean $nx_0([1 - F(x_0)]$ and variance $x_0^2 nF(x_0)[1 - F(x_0)]$, in which x_0 = the unique solution of $x[1 - F(x)] = \max$.

In the last few years, Daniels' relatively restrictive assumptions have been relaxed by various authors, all of them aiming at similar asymptotes. For example, Sen (23) and the writers (13) assumed different types of stochastic dependencies between componential strength values, while Phoenix and Taylor (18) kept the independence assumption but adopted relatively general stress-strain behavior [see also Kersken-Bradley (15)]. Harlow and Phoenix introduced local load-sharing laws (10), and Phoenix found an asymptotic distribution for the time of system failure under fatigue loading (17). Nevertheless, convergence of these asymptotic results is extremely slow. Better asymptotes have only been found for the original ideal assumptions by Daniels (5) and Barbour (1).

An essential prerequisite to a numerically feasible method for general structural system reliability will be the imposed deformation approach already used in Refs. 15 and 18 and generalized further by B. Peintinger and R. Rackwitz ("Structural System Reliability—An Imposed Deformation Approach," (to be published in "Structural Mechanics")).

In the following, the reliability problem of a "Daniels system" is formulated under slightly more general conditions. It is then solved by the aforementioned first-order reliability technique. The approximate results will be compared with exact ones. Also, a general formulation for Daniels' systems with rather arbitrary mechanical and stochastic assumptions will be derived.

The primary purpose of this paper is to demonstrate the feasibility and the accuracy of the first-order approach to parallel structural systems in connection with the concept of imposed deformations rather than to analyze in detail the special Daniels system, although some new results are presented.

Imposed Strain Approach for Brittle Material.—A "Daniels system" consists of n components of unit area in a parallel arrangement such that all components suffer the same value of "imposed strain," ϵ (Fig. 1). The applied load is denoted by L, and, without loss of generality, it is assumed constant throughout the paper. Assume further the componential stress-strain behavior is perfectly elastic-brittle (compare Fig. 2) so that the stress, S_i, in the ith component becomes

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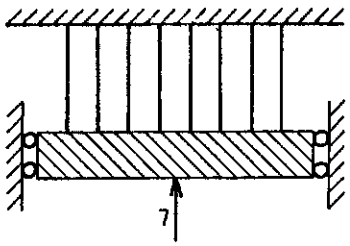

FIG. 1.—Daniels' Parallel System

$$S_i(\epsilon) = \begin{cases} \epsilon \dfrac{X_i}{Y_i} & \text{for} \quad 0 \leqq \epsilon \leqq Y_i \\ 0 & \text{elsewhere} \end{cases} \quad \ldots\ldots (1)$$

in which X_i = the random strength of the ith component; and Y_i = the corresponding random ultimate strain. This is slightly more general than Daniels' original assumption of a constant ratio, $e = X_i/Y_i$, for all components and which is recovered as a special case in Eq. 1, implying full stochastic dependence between X_i and Y_i. Obviously, the system capacity at any given deformation ϵ is the sum

$$R_n(\epsilon) = \Sigma S_i(\epsilon) \quad \ldots\ldots (2)$$

over all (unbroken) components, and system strength is the maximum of $R_n(\epsilon)$ with respect to the deformation parameter, ϵ:

$$R_n = \max_{\epsilon \geqq 0} \{\Sigma S_i(\epsilon)\} \quad \ldots\ldots (3)$$

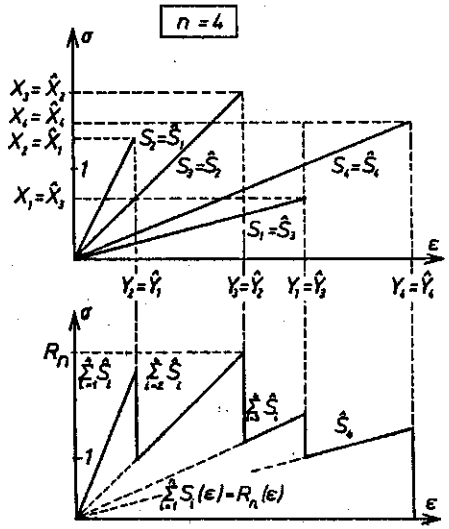

FIG. 2.—Elastic-Brittle Stress-Strain Diagrams for Components and for System

The failure probability of the system under load L simply is

$$P_f = P(R_n \leqq L) = P\{\max_{\epsilon \geqq 0} [\Sigma S_i(\epsilon)] \leqq L\}; \quad = P\left[\bigcap_{\epsilon \geqq 0} \{\Sigma S_i(\epsilon) - L \leqq 0\}\right]$$

$$\leqq \min_{\epsilon \geqq 0} \{P[\Sigma S_i(\epsilon) - L \leqq 0]\} \quad \ldots\ldots (4)$$

in which the last inequality simply derives from the well-known probability inequality for the intersection of events, i.e., $P(\cap_\epsilon A_\epsilon) \leqq \min_\epsilon \{P(A_\epsilon)\}$.

From Fig. 2 it is clear that one component attains its maximum strain at deformations at which one component attains its maximum strain. Ordering the Y_i's in ascending order yields their order statistics $\hat{Y}_1 \leqq \hat{Y}_2 \leqq \ldots \leqq \hat{Y}_n$. Therefore, Eq. 2 can be rewritten as

$$R_n(\epsilon = \hat{Y}_k) = \sum_{i=k}^{n} \hat{S}_i(\epsilon) = \sum_{i=k}^{n} \hat{Y}_k \cdot \frac{\hat{X}_i}{\hat{Y}_i} \quad \ldots\ldots (5)$$

with the $\hat{X}_i$'s and $\hat{S}_i$'s corresponding to the $\hat{Y}_i$'s. Equation 3 becomes

$$R_n = \max_{k=1}^{n} \left\{\sum_{i=k}^{n} \hat{Y}_k \frac{\hat{X}_i}{\hat{Y}_i}\right\} \quad \ldots\ldots (6)$$

and specializes under Daniels' original assumptions with a constant $e = X_i/Y_i = \hat{X}_i/\hat{Y}_i$ to

$$R_n = \max_{k=1}^{n} \left\{\sum_{i=k}^{n} \hat{Y}_k e\right\} = \max_{k=1}^{n} \left\{\sum_{i=k}^{n} \hat{X}_k\right\}; \quad = \max_{k=1}^{n} \{(n-k+1)\hat{X}_k\} \quad \ldots\ldots (7)$$

Since for any fixed k it is $R_n \geqq (n-k+1)\hat{X}_k$, we have similar to Eq. 4

$$P_f = P\left[\bigcap_k \{(n-k+1)\hat{X}_k - L \leqq 0\}\right] \leqq \min_k \{P[(n-k+1)\hat{X}_k - L \leqq 0]\} \quad (8)$$

Note that a lower bound $P(\cap_k A_k) \geqq \Pi_k P(A_k)$ can also be given which, however, is of little interest. Equation 8 allows some interesting interpretations. Obviously, the second term defines the probability of failure of a parallel system with a finite number, n, of components. By definition, a parallel system fails if all of its "components" fail, whose failure events are described by the conditions in the { }-brackets. Note here that the notion of a component is used in an abstract mathematical sense. The last term is nothing else than the failure probability of the "strongest component" and, thus, represents the well-known upper probability bound for redundant systems.

Application of First-Order Reliability Techniques.—Using Eqs. 6 yields, in analogy with Eqs. 4

$$P_f = P(R_n \leqq L) = P\left(\bigcap_k \left\{\sum_{i \geqq k} \hat{Y}_k \frac{\hat{X}_i}{\hat{Y}_i} - L \leqq 0\right\}\right) \quad \ldots\ldots (9)$$

which with obvious abbreviations can be written as

$$P_f = P\left(\bigcap_k \{g_k(\hat{\mathbf{Y}}, \hat{\mathbf{X}}) \leqq 0\}\right) \quad \ldots\ldots (10)$$

in which $\hat{\mathbf{X}} = (\hat{X}_1, \ldots, \hat{X}_n)$ and $\hat{\mathbf{Y}} = (\hat{Y}_1, \ldots, \hat{Y}_n)$ = the vectors collecting

the uncertain variables; and $g_k(\hat{y}, \hat{x}) = 0$ the failure surface for the kth component separating the failure domain $\{g_k \leqq 0\}$ from the safe domain $\{g_k > 0\}$. Let a probability distribution transformation, T, exist such that

$$(\hat{Y}, \hat{X}) = \mathbf{T}(\mathbf{U}) = [T_1(\mathbf{U}), \ldots, T_{2n}(\mathbf{U})] \quad \ldots\ldots (11)$$

in which $\mathbf{U}$ = an independent standard normal vector (12). Then, according to the first-order reliability method, it is

$$P_f = P\left[\bigcap_k \{g_k(\hat{Y}, \hat{X}) \leqq 0\}\right];$$

$$= P\left[\bigcap_k \{g_k[T_1(\mathbf{U}), \ldots, T_{2n}(\mathbf{U})] \leqq 0\}\right] = P\left[\bigcap_k \{\tilde{g}_k(\mathbf{U}) \leqq 0\}\right]$$

$$\sim P\left[\bigcap_k \{l_k(\mathbf{U}) \leqq 0\}\right] = P\left(\bigcap_k \{\alpha_k \mathbf{U} + \beta_k \leqq 0\}\right); \quad = \Phi_n(-\boldsymbol{\beta}; \mathbf{R}) \quad \ldots\ldots (12)$$

in which $l_k(\mathbf{u}) = 0$ is the linearization of $\tilde{g}_k(\mathbf{u}) = 0$ at the most likely failure point with distance (safety index), β_k, to the coordinate origin and vector of direction cosines, $\boldsymbol{\alpha}_k$; $\boldsymbol{\beta} = (\beta_1, \ldots, \beta_n)^T$ = the vector or safety indices for the components; $\mathbf{R} = \{\rho_{ij}\}$ with $\rho_{ij} = \boldsymbol{\alpha}_i^T \boldsymbol{\alpha}_j$ is the correlation matrix describing the dependence between the components; and $\Phi_n(\cdot)$ = the n-dimensional normal integral [see Hohenbichler and Rackwitz ("First-Order Concepts in System Reliability") for further details and the first-order technique to evaluate $\Phi_n(\cdot)$]. Thus, evaluation of Eq. 9 has been reduced to a standard case in first-order reliability. It will be shown later that this can be done under far more general mechanical and stochastic assumptions. While the determination of $\boldsymbol{\beta}$ and the corresponding matrix of α's is a standard procedure in first-order reliability, the transformation T in Eq. 11 requires some thoughts.

Following the lines suggested in Ref. 12, an appropriate set of (conditional) cumulative distribution functions for the $\hat{Y}_i$'s must be determined. For the first breaking element, the distribution function of $\hat{Y}_1 = \min\{Y_1, \ldots, Y_n\}$ is

$$H_{\hat{Y},1}(\hat{y}_1) = P(\hat{Y}_1 < \hat{y}_1) = 1 - [1 - F_Y(\hat{y}_1)]^n \quad \ldots\ldots (13)$$

Here, F_Y = the distribution function of the Y_i's (prior to their ordering). Given $\hat{Y}_1 = \hat{y}_1$, all the other Y_i's are greater or equal than $\hat{y}_1$. The distribution of such a Y_i, given $\hat{Y}_1 = \hat{y}_1$, therefore, is its original distribution truncated from below at $\hat{y}_1$:

$$P(Y_i < y | Y_i \geqq \hat{y}_1) = \frac{F_Y(y) - F_Y(\hat{y}_1)}{1 - F_Y(\hat{y}_1)} \quad \ldots\ldots (14)$$

Thus, given $\hat{Y}_1 = \hat{y}_1$, $\hat{Y}_2$ = the minimum of $n - 1$ variables with distributions as defined in Eq. 14, and in analogy to Eq. 13, its conditional distribution function is (for $\hat{y}_2 \geqq \hat{y}_1$)

$$H_{\hat{Y},2}(\hat{y}_2 | \hat{y}_1) = P(\hat{Y}_2 < \hat{y}_2 | \hat{Y}_1 = \hat{y}_1); \quad = 1 - \left[1 - \frac{F_Y(\hat{y}_2) - F_Y(\hat{y}_1)}{1 - F_Y(\hat{y}_1)}\right]^{n-1} \quad \ldots (15)$$

In general, given $(\hat{Y}_1 = \hat{y}_1, \ldots, \hat{Y}_{i-1} = \hat{y}_{i-1})$, the variable $\hat{Y}_i$ is the minimum of $n - i + 1$ variables truncated at $\hat{y}_{i-1}$:

$$H_{\hat{Y},i}(\hat{y}_i | \hat{y}_1, \ldots, \hat{y}_{i-1}) = P(\hat{Y}_i < \hat{y}_i | \hat{Y}_1 = \hat{y}_1, \ldots, \hat{Y}_{i-1} = \hat{y}_{i-1});$$

$$= P(\hat{Y}_i < \hat{y}_i | \hat{Y}_{i-1} = \hat{y}_{i-1}) = 1 - \left[1 - \frac{F_Y(\hat{y}_i) - F_Y(\hat{y}_{i-1})}{1 - F_Y(\hat{y}_{i-1})}\right]^{n-i+1} \quad \ldots\ldots (16)$$

From Eqs. 13 and 16 the transformation of the $\hat{Y}_i$'s is easily derived by setting

$$H_{\hat{Y},i}(\hat{Y}_i | \hat{Y}_1, \ldots, \hat{Y}_{i-1}) = \Phi(U_i) \quad (i = 1, \ldots, n) \quad \ldots\ldots (17)$$

in which $\Phi(u)$ = the standard normal distribution function. Solving these equations recursively yields

$$\hat{Y}_i = F_Y^{-1}\left\{1 - \prod_{j=1}^{i} [\Phi(-U_j)]^{1/n-j+1}\right\} \quad \ldots\ldots (18)$$

which is the required representation of the $\hat{Y}_i$ by independent standard normal variables, U_i.

The $\hat{X}_i$'s are handled much easier. Since different components are statistically independent, the conditional distribution of $\hat{X}_i$ given $(\hat{Y}_1, \ldots, \hat{Y}_n, \hat{X}_1, \ldots, \hat{X}_{i-1})$ depends only on $\hat{Y}_i$ and not on any other element. Consequently, with $F_X(\cdot | y)$ denoting the conditional distribution function of X_i given $Y_i = y$, the required transformation of the $\hat{X}_i$'s results from solving the equations

$$F_X(\hat{X}_i | \hat{Y}_i) = \Phi(U_{n+i}) \quad \ldots\ldots (19)$$

with respect to $\hat{X}_i$:

$$\hat{X}_i = F_X^{-1}[\Phi(U_{n-i}) | \hat{Y}_i] \quad \ldots\ldots (20)$$

Further, by substituting Eq. 18 into Eq. 20, $\hat{X}_i$ is a function of $(U_1, \ldots U_i, U_{n+i})$, or, more generally, of $\mathbf{U} = (U_1, \ldots, U_{2n})$. Now, all of the uncertain variables $\hat{Y}_i$, $\hat{X}_i$ are represented as functions of the standard normal independent vector $\mathbf{U} = (U_1, \ldots, U_{2n})$:

$$\hat{Y}_i = T_i(\mathbf{U}), \quad \hat{X}_i = T_{n+i}(\mathbf{U}) \quad \ldots\ldots (21)$$

to be used in Eq. 10.

For example, let $(X, Y)^T$ be jointly normally distributed:

$$F_Y(y) = P(Y_i \leqq y) = \Phi\left(\frac{y - \mu_Y}{\sigma_Y}\right) \quad \ldots\ldots (22a)$$

$$F_X(x|y) = \Phi\left[\frac{\frac{x - \mu_X}{\sigma_X} - \rho\frac{y - \mu_Y}{\sigma_Y}}{\sqrt{1 - \rho^2}}\right] \quad \ldots\ldots (22b)$$

Equations 18 and 20 read in this case

$$\hat{Y}_i = \mu_Y - \sigma_Y \Phi^{-1}\left\{\prod_{j=1}^{i} [\Phi(-U_j)]^{1/n-j+1}\right\} \quad \ldots\ldots (23a)$$

$$\hat{X}_i = \mu_X + \rho\sigma_X \frac{\hat{Y}_i - \mu_Y}{\sigma_Y + \sigma_X} + \sigma_X\sqrt{1 - \rho^2}\, U_{n+i} \quad \ldots\ldots (23b)$$

Note further that the upper bound corresponding to the last part of Eq. 8 is

$$P_f \leqq \min_k \{\Phi(-\beta_k)\} \quad \ldots\ldots (24)$$

If $\rho = \rho(X_i, Y_i] = 1$ and $\mu_X = \mu_Y$, $\sigma_X = \sigma_Y$, Daniels' original assumptions are regained. Therefore, in Fig. 3 we shall first compare the first-order solution as given earlier with the exact solution according to Daniels. ($X_i = Y_i$ are modelled as normal variables with $\rho = 1$, $\mu = 1$, and $\sigma = 0.2$). In addition, the improved asymptotes of Daniels and Barbour are drawn in.

It is recognized that the improvement achieved by considering more than only the strongest component is substantial, but the first-order solution for the strongest component is conservatively in error by a non-negligible amount and, presumably, the same argument applies for all other components. One might conclude that the first-order method is already inadequate for the componential probabilities and, of course, even more so for parallel systems since errors in the componential probabilities somehow multiply and, therefore, system probability estimates are in error by an amount which grows rapidly with system size. A more detailed analysis (not presented herein) shows that the failure surfaces, $\{\tilde{g}_k(\mathbf{u}) = 0)\}$, obtained after transforming the original variables into standard normal variables, deviate significantly, in fact, from plane surfaces required for the first-order method to be exact.

However, various techniques exist to improve componential failure probabilities (3, 8, and 9). Such improved estimates can be used with advantage in the framework of the concept of equivalent components, apparently put forward first in Ref. 22, and which is related to the "generalized" safety index as defined by Ditlevsen (7). In particular, an equivalent component in the context of first-order reliability is defined

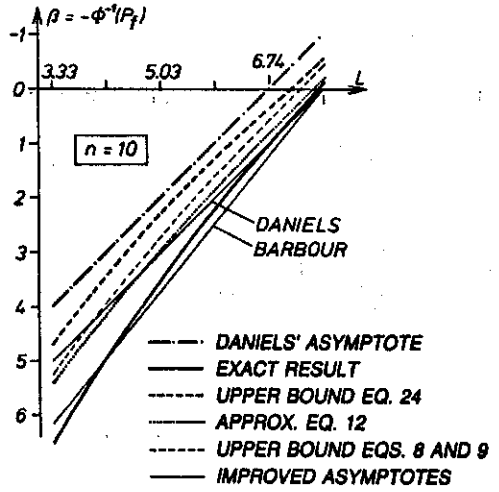

FIG. 3.—Comparison of Various Approximations and Asymptotes

as a component having a linear failure surface which cuts off the same failure probability as the original, generally non-linear failure surface and carries over as many stochastic characteristics of the original problem as possible. Beyond the same failure probability, the equivalent plane should at least have the same sensitivity against changes in one of the basic uncertainty variables, i.e., should have the same vector of direction cosines. In other words, the equivalent hyperplane is defined by the "equivalent" safety index

$$\beta_{E,k} = -\Phi^{-1}[P(a_k(\mathbf{U}) \leqq 0)] \quad \ldots\ldots (25)$$

in which $a_k(\mathbf{u}) = 0$ is the original failure surface, $\tilde{g}_k(\mathbf{u}) = 0$, or an approximation to it; and $P(\cdot)$ = the corresponding "failure probability" and, by the vector of direction cosines, $\boldsymbol{\alpha}_k$, of the original failure point. The system failure probability now becomes in accordance with Eq. 12

$$P_f \approx \Phi_n(-\boldsymbol{\beta}_E; \mathbf{R}) \leqq \min_k \Phi(-\beta_{E,k}) \quad \ldots\ldots (26)$$

In Daniels' special case ($X/Y = e$ = constant), $P(a_k(\mathbf{U}) \leqq 0)$ can be given exactly by applying Eq. 9. If this concept is used together with the foregoing results, excellent agreement is reached with the exact results as indicated in Fig. 4. For system sizes $n = 20$ and $n = 50$, the exact system failure probability, P_f [respectively its safety index $\beta = -\Phi^{-1}(P_f)$] under load L is compared with the just proposed approximation, Eq. 26, with Daniels' asymptote and the upper bound Eq. 26 (which equals here the bound Eqs. 8 plus 9).

Daniels' System with Arbitrary Brittle Material.—While the exact solution and the improved asymptote are restricted to Daniels' original assumptions, the same first-order concept also applies, if the parallel elements are no longer perfectly elastic brittle. Let, in general, the stress-strain relationship, $S_i(\epsilon) = S(\epsilon, \boldsymbol{\Theta}_i)$, of the ith component depend on an uncertain parameter vector, $\boldsymbol{\Theta}_i = (Y_i, \ldots)$, containing the ultimate element strain, Y_i, and other parameters like yield stress and arbitrary shape parameters. Brittleness implies

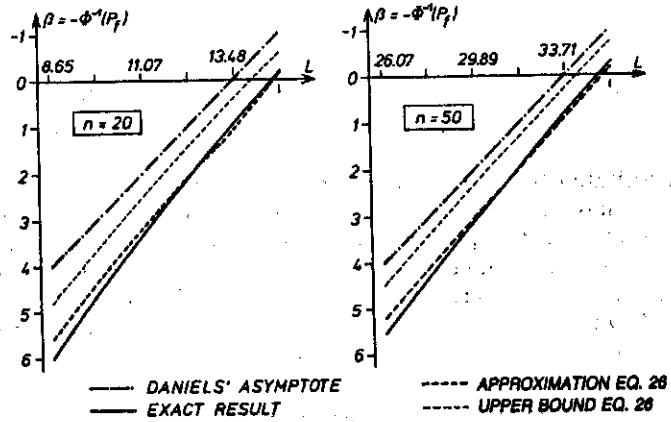

FIG. 4.—Example of Eq. 27 for Perfectly Elastic-Brittle Material

$$S(\epsilon, \Theta_i) = 0 \quad \text{for} \quad \epsilon > Y_i \quad \ldots\ldots (27)$$

Furthermore, let $S(\epsilon, \Theta_i)$ be nondecreasing for $0 \leqq \epsilon \leqq Y_i$. Then, in analogy with Eqs. 3 and 7, system strength is

$$R_n = \max_{\epsilon \geq 0} \sum_{i=1}^{n} S(\epsilon, \Theta_i) = \max_{1 \leq k \leq n} \sum_{i=1}^{n} S(Y_k, \Theta_i) \quad \ldots\ldots (28)$$

and after reordering the elements with respect to their ultimate strains, $\hat{Y}_1 \leqq \ldots \leqq \hat{Y}_n$, one obtains (compare Eq. 6)

$$R_n = \max_{1 \leq k \leq n} \left\{ \sum_{i=1}^{n} S(\hat{Y}_k, \hat{\Theta}_i) \right\} = \max_{1 \leq k \leq n} \left\{ \sum_{i=k}^{n} S(\hat{Y}_k, \hat{\Theta}_i) \right\} \quad \ldots\ldots (29)$$

$$\text{and} \quad P_f = P(R_n < L) = P\left[\bigcap_{k=1}^{n} \left\{ \sum_{i=k}^{n} S(\hat{Y}_k, \hat{\Theta}_i) - L \geqq 0 \right\} \right]$$

$$\leqq \min_{1 \leq k \leq n} \left\{ P\left[\sum_{i=k}^{n} S(\hat{Y}_k, \hat{\Theta}_i) - L \leqq 0 \right] \right\} \quad \ldots\ldots (30)$$

Some details are shown in the next example.

Example: "Imperfect Loading."—In addition to the assumptions underlying Eq. 1, let the elements exhibit uncertain slack caused by imperfect anchorage (threads in a bundle) or by unavoidable tolerances (screwed joints). Denoting $\Theta_i^* = (Y_i^*, X_i^*)$ as the intrinsic material parameters, and $S^*(\epsilon, \Theta^*)$ the "material" stress-strain law, the effective stress-strain law, including slack Γ_i, becomes

$$S(\epsilon, \Theta_i) = \begin{cases} 0 & \text{for} \quad 0 \leqq \epsilon < \Gamma_i \\ S^*(\epsilon - \Gamma_i, \Theta_i^*) & \text{for} \quad \epsilon \geqq \Gamma_i \end{cases} \quad \ldots\ldots (31)$$

In particular, the effective ultimate element strain is $Y_i = Y_i^* + \Gamma_i$, and with $\Theta_i = (Y_i, X_i^*, \Gamma_i)$ and the material law, Eq. 1, one obtains

$$S(\epsilon, \Theta_i) = \begin{cases} 0 & \text{for} \quad 0 \leqq \epsilon < \Gamma_i \\ (\epsilon - \Gamma_i) \dfrac{X_i^*}{Y_i - \Gamma_i} & \text{for} \quad \Gamma_i < \epsilon < Y_i \\ 0 & \text{for} \quad \epsilon > Y_i \end{cases} \quad \ldots\ldots (32)$$

which is an extremely nonlinear, brittle stress-strain law. Assume further that the material parameters (Y_i^*, X_i^*) are binormally distributed with mean values, μ_Y^*, μ_X^*, standard deviations, σ_Y^*, σ_X^*, and correlation coefficient, ρ^*. The slack variable, Γ_i, is independent of (Y_i^*, X_i^*) and normally distributed $(\mu_\Gamma, \sigma_\Gamma)$. Consequently, $\Theta_i = (Y_i, X_i^*, \Gamma_i)$ follows a trinormal distribution with $\mu_Y = \mu_Y^* + \mu_\Gamma$, $\sigma_Y^2 = (\sigma_Y^*)^2 + \sigma_\Gamma^2$, and correlations $\rho_{12} = \rho[Y_i, X_i^*] = \rho^* \sigma_Y^* / \sigma_Y$, $\rho_{13} = \rho[Y_i, \Gamma_i] = \sigma_\Gamma / \sigma_Y$, $\rho_{23} = \rho[X_i^*, \Gamma_i] = 0$. The distribution function of Y and the conditional distribution function of X^* given Y, therefore equal Eqs. 22 (with $\rho = \rho_{12}$ and $\mu_X = \mu_X^* \sigma_X = \sigma_X^*$, but μ_Y, σ_Y as defined earlier), while the conditional distribution function of Γ given $Y = y$ and $X^* = x$ is

$$F_\Gamma(\gamma | y, x) = \Phi\left[\frac{\dfrac{\gamma - \mu_\Gamma}{\sigma_\Gamma} - a \dfrac{y - \mu_Y}{\sigma_Y} - b \dfrac{x - \mu_X^*}{\sigma_X^*}}{c} \right] \quad \ldots\ldots (33)$$

$$\text{with} \quad a = \rho_{13} + \frac{\rho_{12}^2 \rho_{13}}{1 - \rho_{12}^2} \quad \ldots\ldots (34a)$$

$$b = -\frac{\rho_{12} \rho_{13}}{1 - \rho_{12}^2} \quad \ldots\ldots (34b)$$

$$c = \sqrt{1 - \frac{\rho_{13}^2}{1 - \rho_{12}^2}} \quad \ldots\ldots (34c)$$

Consequently, Eqs. 23 describe the transformation of $\hat{Y}_i$ and $\hat{X}_i^*$. The transformation of $\hat{\Gamma}_i$ derives from

$$F_\Gamma(\hat{\Gamma}_i | \hat{Y}_i, \hat{X}_i^*) = \Phi(U_{2n+i}) \quad \ldots\ldots (35)$$

and results in

$$\hat{\Gamma}_i = \mu_\Gamma + \sigma_\Gamma \left[c U_{2n+i} + a \frac{\hat{Y}_i - \mu_Y}{\sigma_Y} + b \frac{\hat{X}_i^* - \mu_X^*}{\sigma_X^*} \right] \quad \ldots\ldots (36)$$

Substituting Eqs. 23, 36, and 32 into Eq. 30, yields (with obvious abbreviations) the formulation

$$P_f = P\left(\bigcap_{k=1}^{n} \{ \tilde{g}_k(\mathbf{U}) - L \leqq 0 \} \right) \leqq \min_{1 \leq k \leq n} \{ P(\tilde{g}_k(\mathbf{U}) - L \leqq 0) \} \quad \ldots\ldots (37)$$

which is evaluated with the proposed technique.

A numerical example has been performed for system sizes $n = 20$, 50, and for $\mu_Y^* = \mu_X^* = 1$, $\mu_\Gamma = 0.5$, $\rho = \rho[X_i^*, Y_i^*] = 0.5$, and $\sigma_Y^* = \sigma_X^* = \sigma_\Gamma = 0.2$. Firstly, the "equivalent" safety indices, $\beta_{E,k}$, were evaluated with first-order methods, but a special transformation technique, which avoids first-order errors due to Eq. 18, makes explicit use of Eq. 9. However, these transformations differ for different k's and do not allow for the determination of correlation-coefficients between modes. Details are not considered here. The mode correlations were determined as proposed earlier. Figure 5 compares, for various loads, L, the results for the safety

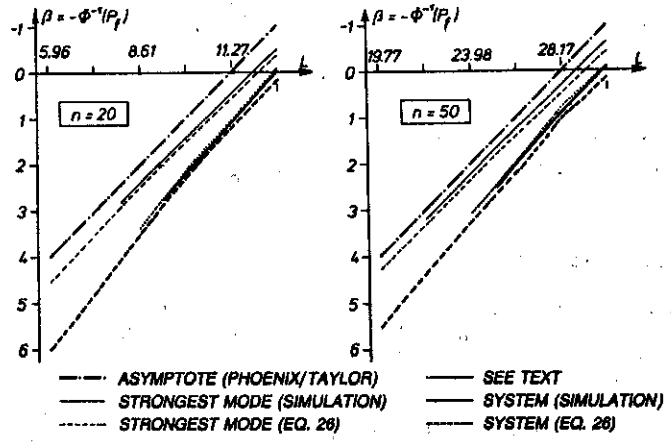

FIG. 5.—Example of Eq. 27 for Material with Uncertain Slack

index $\beta = -\Phi^{-1}(P_f)$, of the strongest mode ($\beta_{E,k}$; second part of Eq. 37) and of the system (first part of Eq. 37 respectively, Eq. 26) with the results of a simulation (about 100,000/150,000 runs) and the asymptote of Phoenix and Taylor (18). It is observed that the approximation, Eq. 26, is more in error than in Fig. 4. For the most part, this error is due to the still existing first-order error in the determination of the equivalent modal safety indices, $\beta_{E,k}$. If in the approximation, Eq. 26, β_E is determined via the same simulation, the results are considerably improved (dotted line), indicating that dependencies between the modes are properly taken into account.

Conclusions

The writers dare to draw some relatively general conclusions on the basis of the results just obtained. They are believed to apply also to less idealized structural systems:

1. The failure probability of the "strongest" component in a redundant system usually overestimates drastically the system failure probability. Similarly, lower bounds are of little use for structural systems.
2. Therefore, the consideration of the contribution of the other components is mandatory. Here, an efficient tool is available in the form of the first-order system reliability approach as expressed by Eq. 12.
3. An exact calculation of componential failure probabilities is essential. First-order estimates may be insufficient. Higher-order estimates are required.
4. Then, the use of the equivalent component concept may yield fairly accurate probability estimates. The numerical comparisons imply that the correlation structure of the components is taken into account fairly well by those equivalent components.
5. The numerical accuracy of formulation Eq. 12, in conjunction with the equivalent component technique, appears sufficient for many engineering applications.
6. The concept of imposed deformations is particularly fruitful, as proven by the examples, but may have much wider applicability.
7. Applications of the foregoing approach which is limited to methodological questions for certain practical problems, e.g., the strength of a bundle of threads (parallel wire cables), connections with shrews, nut or dowels, the stability of embankment slopes with "almost brittle" shear strength of the soil, supercritical buckling problems, etc., are straightforward.

Appendix.—References

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