Build Your Own Reachability Analyzer with CORA

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Abstract. The COntinuous Reachability Analyzer (CORA) is a collection of MATLAB classes for the formal verification of cyber-physical systems using reachability analysis. This tutorial presents various vector and matrix set representations and operations on them, as well as reachability algorithms of various dynamic system classes. CORA is designed so that set representations can be exchanged without having to modify the code for reachability analysis. Since CORA is written in MATLAB, the installation and use is platform-independent. We demonstrate how easily CORA can be used by coding our own reachability analyzer based on CORA classes. Also, we demonstrate how to load SpaceEx files and how one can use already integrated reachability analysis approaches in CORA.

Keywords. Reachability analysis, formal verification, continuous systems, hybrid systems, CORA.

1. Introduction

The **CO**ntinuous **R**eachability **A**nalyzer (CORA)[2] is a MATLAB toolbox for prototypical design of algorithms for reachability analysis. The toolbox is designed for various kinds of systems with purely continuous dynamics (linear systems, nonlinear systems, differential-algebraic systems, parameter-varying systems, etc.) and hybrid dynamics combining the aforementioned continuous dynamics with discrete dynamics. Let us denote the continuous part of the solution of a hybrid system for a given initial discrete state by $\chi(t;x_0,u(\cdot),p)$, where $t \in \mathbb{R}$ is the time, $x_0 \in \mathbb{R}^n$ is the continuous initial state, $u(t) \in \mathbb{R}^m$ is the system input at t, $u(\cdot)$ is the input trajectory, and $p \in \mathbb{R}^p$ is a parameter vector. The continuous reachable set at time $t = r$ can be defined for a set of initial states $\mathscr{X}_0$, a set of input values $\mathscr{U}(t)$, and a set of parameter values $\mathscr{P}$, as

$$\mathscr{R}^e(r) = \left\{ \chi(r;x_0,u(\cdot),p) \in \mathbb{R}^n \middle| x_0 \in \mathscr{X}_0, \forall t : u(t) \in \mathscr{U}(t), p \in \mathscr{P} \right\}.$$

In its current version, CORA solely supports over-approximative computation of reachable sets since (a) exact reachable sets cannot be computed for most system classes [10] and (b) over-approximative computations qualify for formal verification. Thus, CORA computes over-approximations for particular points in time $\mathscr{R}(t) \supseteq \mathscr{R}^e(t)$ and for time intervals: $\mathscr{R}([t_0,t_f]) = \bigcup_{t\in[t_0,t_f]} \mathscr{R}(t)$.

CORA enables one to construct one's own reachable set computation in a relatively short amount of time. This is achieved by the following design choices:

- CORA is built for MATLAB, which is a script-based programming environment. Since the code does not have to be compiled, one can stop the program at any time and directly see the current values of variables. This makes it especially easy to understand the workings of the code and to debug new code.

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[2] `https://www6.in.tum.de/Main/SoftwareCORA`

- CORA is an object-oriented toolbox that uses modularity, operator overloading, inheritance, and information hiding. One can safely use existing classes and just adapt classes one is interested in without redesigning the whole code. Operator overloading makes it possible to write formulas that look almost identical to the ones derived in scientific papers and thus reduce programming errors. Most of the information for each class is hidden and is not relevant to users of the toolbox. Most classes use identical methods so that set representations and dynamic systems can be effortlessly replaced.
- CORA interfaces with the established toolbox MPT[3], which is also written in MATLAB. Results of CORA can be easily transferred to this toolbox and vice versa. We are currently supporting version 2 and 3 of the MPT.

Of course, it is also possible to use CORA as it is to conduct reachability analysis.

> Please be aware of the fact that outcomes of reachability analysis heavily depend on the chosen parameters for the analysis. Improper choice of parameters can result in an unacceptable over-approximation although reasonable results could be achieved by using appropriate parameters. Thus, self-tuning of the parameters for reachability analysis is investigated as part of future work.

Since this tutorial focuses on the presentation of the capabilities of CORA, no other tools for reachability analysis of continuous and hybrid systems are reviewed. A list of related tools is presented in [3–5].

2. Set Operations

Efficient set operations are crucial for efficient reachability analysis. We will start with intervals as one of the simplest set representations and finish with polynomial zonotopes, which are a non-convex set representation.

2.1. Intervals

A real-valued interval $[x] = [\underline{x},\overline{x}] = \{x \in \mathbb{R} | \underline{x} \leq x \leq \overline{x}\}$ is a connected subset of $\mathbb{R}$ and can be specified by a left bound $\underline{x} \in \mathbb{R}$ and right bound $\overline{x} \in \mathbb{R}$, where $\underline{x} \leq \overline{x}$. A detailed description of how intervals are treated in CORA can be found in [4]. This class has a lot of methods, and almost all operators and functions defined in MATLAB can be applied to intervals, such as `abs`, `cos`, `power`, etc.

```
I1 = interval([0; -1], [3; 1]); % create interval I1
I2 = interval([-1; -1.5], [1; -0.5]); % create interval I2
Z1 = zonotope([1 1 1; 1 -1 1]); % create zonotope Z1

%% basic set operations
r = rad(I1) % obtain and display radius of I1
is_intersecting = isIntersecting(I1, Z1) % Z1 intersecting I1?
I3 = I1 & I2; % computes the intersection of I1 and I2
c = mid(I3) % returns and displays the center of I3

figure; hold on
plot(I1); % plot I1
plot(I2); % plot I2
plotFilled(I3,[1 2],[.6 .6 .6],'EdgeColor','none'); % plot I3

%% mathematical functions
A_inf = [-1 -1; 0.5 -2]; % specify infimum
A_sup = [1 2; 1 -1]; % specify supremum
```

[3] http://control.ee.ethz.ch/~mpt/2/

```
A = interval(A_inf, A_sup); % generate matrix
B_inf = [0 -1; 0 1]; % specify infimum
B_sup = [1 3; 0.5 1.2]; % specify supremum
B = interval(B_inf, B_sup); % generate matrix

A+B % addition
A*B % multiplication
A.*B % pointwise multiplication
A/interval(1,2) % division
A./B % pointwise division
A^3 % power function
sin(A) % sine function
sin(A(1,1)) + A(1,1)^2 - A(1,1)*B(1,1) % scalar combination of functions
sin(A) + A^2 - A*B % matrix combination of functions
```

2.2. Zonotopes

A zonotope is a point-symmetric set parameterized by a center $c \in \mathbb{R}^n$ and generators $g^{(i)} \in \mathbb{R}^n$:

$$\mathscr{Z} = \left\{ c + \sum_{i=1}^{p} \beta_i g^{(i)} \middle| \beta_i \in [-1,1] \right\}. \tag{1}$$

We write in short $\mathscr{Z} = (c, g^{(1)}, \ldots, g^{(p)})$. A zonotope can be interpreted as the Minkowski addition of line segments $l^{(i)} = [-1,1]g^{(i)}$ and is visualized step-by-step in a two-dimensional vector space in Fig. 1. Zonotopes are a compact way of representing sets in high dimensions. More importantly, operations required for reachability analysis, such as linear maps $M \otimes \mathscr{Z} := \{Mz | z \in \mathscr{Z}\}$ ($M \in \mathbb{R}^{q \times n}$) and Minkowski addition $\mathscr{Z}_1 \oplus \mathscr{Z}_2 := \{z_1 + z_2 | z_1 \in \mathscr{Z}_1, z_2 \in \mathscr{Z}_2\}$ can be computed efficiently and exactly, and others such as convex hull computation can be tightly over-approximated [8].

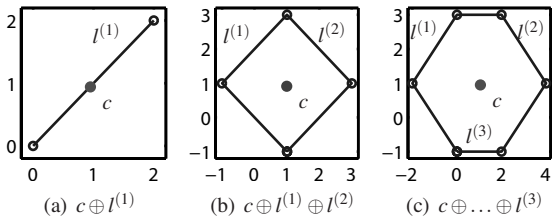


Figure 1. Step-by-step construction of a zonotope.

```
Z1 = zonotope([1 1 1; 1 -1 1]); % create zonotope Z1
Z2 = zonotope([-1 1 0; 1 0 1]); % create zonotope Z2
A = [0.5 1; 1 0.5]; % numerical matrix A

%% set operations
Z3 = Z1 + Z2; % Minkowski addition
Z4 = A*Z3; % linear map

figure; hold on
plot(Z1,[1 2],'b'); % plot Z1 in blue
plot(Z2,[1 2],'g'); % plot Z2 in green
plot(Z3,[1 2],'r'); % plot Z3 in red
plot(Z4,[1 2],'k'); % plot Z4 in black

%% conversion to other set representations
P = polytope(Z4) % convert to and display halfspace representation
I = interval(Z4) % convert to and display interval

figure; hold on
```

```
plot(Z4); % plot Z4
plot(I,[1 2],'g'); % plot intervalhull in green
```

2.3. Zonotope Bundles

A disadvantage of zonotopes is that they are not closed under intersection, i.e., the intersection of two zonotopes does not return a zonotope in general. In order to overcome this disadvantage, zonotope bundles are introduced in [6]. Given a finite set of zonotopes $\mathscr{Z}_i$, a zonotope bundle is $\mathscr{Z}^{\cap} = \bigcap_{i=1}^{s} \mathscr{Z}_i$, i.e., the intersection of zonotopes $\mathscr{Z}_i$. Note that the intersection is not computed, but the zonotopes $\mathscr{Z}_i$ are stored in a list, which we write as $\mathscr{Z}^{\cap} = \{\mathscr{Z}_1, \ldots, \mathscr{Z}_s\}^{\cap}$.

```
Z{1} = zonotope([1 1 1; 1 -1 1]); % create zonotope Z1;
Z{2} = zonotope([-1 1 0; 1 0 1]); % create zonotope Z2;
Zb = zonotopeBundle(Z); % instantiate zonotope bundle from Z1, Z2
vol = volume(Zb) % compute and display volume of zonotope bundle

figure; hold on
plot(Z{1}); % plot Z1
plot(Z{2}); % plot Z2
plotFilled(Zb,[1 2],[.675 .675 .675],'EdgeColor','none'); % plot Zb in gray

% zonotope bundles are closed under intersections, but Minkowski addition
% is no longer exact

%% Example 1
% convert to polytopes
Pb = polytope(Zb);
P1 = polytope(Z{1});

% compute exact Minkowski addition
Pres = Pb + P1;

% compute over-approximate addition
Zres = Zb + Z{1};

% compare results
figure; hold on
plot(Pres); % plot Pres
plotFilled(Zres,[1 2],[.675 .675 .675],'EdgeColor','none'); % plot Zres

%% Example 2
% generate new zonotopes
Z{1} = zonotope([0 0.5 1 2; 0 0.5 2 1]);
Z{2} = zonotope([0 3 -0.7; 0 3 0.7]);

Zadd = zonotope([0 2 0; 0 0 1]); % create zonotope to be added;
Padd = polytope(Zadd); % convert zonotope to polytope

Zbundle = zonotopeBundle(Z); % create zonotope bundle
Pmpt = polytope(Zbundle); % convert zonotope to polytope

P_final = Pmpt + Padd; % compute exact Minkowski addition for polytopes
Zbundle_final = Zbundle + Zadd; % compute overapprox. Minkowski addition

%plot results
figure; hold on
plot(Zbundle_final); % plot Zbundle_final
plotFilled(P_final,[1 2],[.675 .675 .675],'EdgeColor','none');% plot P_final
```

2.4. Constrained Zonotopes

An extension of zonotopes described in Sec. 2.2 are constrained zonotopes, which are introduced in [11]. A constrained zonotope is defined as a zonotope with additional equality constraints on the factors β_i:

$$\mathcal{Z}_c = \left\{ c + G\beta \,\middle|\, \|\beta\|_\infty \leq 1, A\beta = b \right\}, \tag{2}$$

where $c \in \mathbb{R}^n$ is the zonotope center, $G \in \mathbb{R}^{n \times p}$ is the zonotope generator matrix and $\beta \in \mathbb{R}^p$ is the vector of zonotope factors. The equality constraints are parametrized by the matrix $A \in \mathbb{R}^{q \times p}$ and the vector $b \in \mathbb{R}^q$. Constrained zonotopes are able to describe arbitrary polytopes, and are therefore a more general set representation than zonotopes. The main advantage compared to a polytope representation using inequality constraints is that constrained zonotopes inherit the excellent scaling properties of zonotopes for increasing state space dimensions, since constrained zonotopes are also based on a generator representation for sets.

```
Z{1} = zonotope([1 1 1; 1 -1 1]); % create zonotope Z1;
Z{2} = zonotope([-1 1 0; 1 0 1]); % create zonotope Z2;
Zb = zonotopeBundle(Z); % instantiate zonotope bundle from Z1, Z2
Zc = conZonotope(Zb); % convert to constrained zonotope

figure; hold on
plot(Z{1}); % plot Z1
plot(Z{2}); % plot Z2
plotFilled(Zc,[1 2],[.675 .675 .675],'EdgeColor','none'); % plot gray Zb

% constr. zonotopes are closed under intersection and Minkowski addition

%% Example 1
% convert to polytopes
Pc = mptPolytope(Zc);
P1 = polytope(Z{1});

% compute Minkowski addition for polytopes
Pres = Pc + P1;

% compute Minkowski addition for constrained zonotopes
Zres = Zc + Z{1};

% compare results
figure; hold on
plot(Pres); % plot P
plotFilled(Zres,[1 2],[.675 .675 .675],'EdgeColor','none'); % plot Zres

%% Example 2
% generate new zonotopes
Z{1} = zonotope([0 0.5 1 2; 0 0.5 2 1]);
Z{2} = zonotope([0 3 -0.7; 0 3 0.7]);

Zadd = zonotope([0 2 0; 0 0 1]); % create zonotope to be added;
Padd = polytope(Zadd); % convert zonotope to polytope

Zb = zonotopeBundle(Z); % create zonotope bundle
Zc = conZonotope(Zb); % convert to constrained zonotope
P = mptPolytope(Zc); % convert zonotope to polytope

P_final = P + Padd; % compute Minkowski addition for polytopes
Zc_final = Zc + Zadd; % compute Minkowski addition for zonotopes

%plot results
figure; hold on
```

```
46 plot(Zc_final); % plot Zbundle_final
47 plotFilled(P_final,[1 2],[.675 .675 .675],'EdgeColor','none');%plot P_final
```

2.5. Polynomial Zonotopes

Zonotopes are a very efficient representation for reachability analysis of linear systems [8] and of nonlinear systems that can be well abstracted by linear differential inclusions [1]. However, more advanced techniques, such as in [2], abstract more accurately to nonlinear difference inclusions. As a consequence, linear maps of reachable sets are replaced by nonlinear maps. Zonotopes are not closed under nonlinear maps and are not particularly good at over-approximating them. For this reason, polynomial zonotopes are introduced in [2]. Polynomial zonotopes are a new non-convex set representation and can be efficiently stored and manipulated. The new representation shares many similarities with Taylor models [9] and is a generalization of zonotopes. Please note that a zonotope cannot be represented by a Taylor model.

Given a *starting point* $c \in \mathbb{R}^n$, multi-indexed *generators* $f^{([i],j,k,\ldots,m)} \in \mathbb{R}^n$, and single-indexed *generators* $g^{(i)} \in \mathbb{R}^n$, a polynomial zonotope is defined as

$$\mathscr{PZ} = \Big\{ c + \sum_{j=1}^{p} \beta_j f^{([1],j)} + \sum_{j=1}^{p} \sum_{k=j}^{p} \beta_j \beta_k f^{([2],j,k)} + \ldots + \tag{3}$$

$$\sum_{j=1}^{p} \sum_{k=j}^{p} \ldots \sum_{m=l}^{p} \underbrace{\beta_j \beta_k \ldots \beta_m}_{\eta \text{ factors}} f^{([\eta],j,k,\ldots,m)} + \sum_{i=1}^{q} \gamma_i g^{(i)} \Big| \beta_i, \gamma_i \in [-1,1] \Big\}.$$

The scalars β_i are called *dependent factors*, since changing their values does not only affect the multiplication with one generator, but with other generators too. On the other hand, the scalars γ_i only affect the multiplication with one generator, so they are called *independent factors*. The number of dependent factors is p, the number of independent factors is q, and the polynomial order η is the maximum power of the scalar factors β_i. The order of a polynomial zonotope is defined as the number of generators ξ divided by the dimension, which is $\rho = \frac{\xi}{n}$. For a concise notation and later derivations, we introduce the matrices

$$E^{[i]} = [\underbrace{f^{([i],1,1,\ldots,1)}}_{=:e^{([i],1)}} \ldots \underbrace{f^{([i],p,p,\ldots,p)}}_{=:e^{([i],p)}}] \text{ (all indices are the same value)},$$

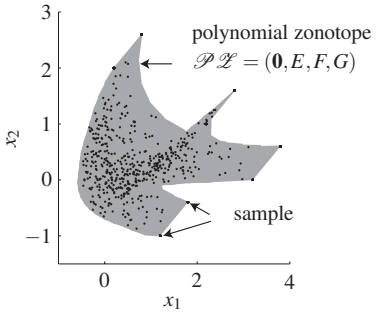
$$F^{[i]} = [f^{([i],1,1,\ldots,1,2)}\ f^{([i],1,1,\ldots,1,3)} \ldots f^{([i],1,1,\ldots,1,p)}$$
$$f^{([i],1,1,\ldots,2,2)}\ f^{([i],1,1,\ldots,2,3)} \ldots f^{([i],1,1,\ldots,2,p)}$$
$$f^{([i],1,1,\ldots,3,3)} \ldots] \text{ (not all indices are the same value)},$$

$$G = [g^{(1)} \ldots g^{(q)}],$$

and $E = \left[E^{[1]} \ldots E^{[\eta]}\right]$, $F = \left[F^{[2]} \ldots F^{[\eta]}\right]$ ($F^{[i]}$ is only defined for $i \geq 2$). Note that the indices in $F^{[i]}$ are ascending due to the nested summations in (2.5). In short form, a polynomial zonotope is written as $\mathscr{PZ} = (c, E, F, G)$.

For a given polynomial order i, the total number of generators in $E^{[i]}$ and $F^{[i]}$ is derived using the number $\binom{p+i-1}{i}$ of combinations of the scalar factors β with replacement (i.e., the same factor can be used again). Adding the numbers for all polynomial orders and adding the number of independent generators q, results in $\xi = \sum_{i=1}^{\eta} \binom{p+i-1}{i} + q$ generators, which is in $\mathscr{O}(p^\eta)$ with respect to p. The non-convex shape of a polynomial zonotope with polynomial order 2 is shown in Fig. 2.

```
1 c = [0;0]; % starting point
2 E1 = diag([-1,0.5]); % generators of factors with identical indices
```



$$E^{[1]} = \begin{bmatrix} -1 & 0 \\ 0 & 0.5 \end{bmatrix}$$
$$E^{[2]} = \begin{bmatrix} 1 & 1 \\ 0.5 & 0.3 \end{bmatrix}$$
$$F^{[2]} = \begin{bmatrix} -0.5 \\ 1 \end{bmatrix}$$
$$G = \begin{bmatrix} 0.3 \\ 0.3 \end{bmatrix}$$

Figure 2. Over-approximative plot of a polynomial zonotope as specified in the figure. Random samples of possible values demonstrate the accuracy of the over-approximative plot.

```
E2 = [1 1; 0.5 0.3]; % generators of factors with identical indices
F = [-0.5; 1]; % generators of factors with different indices
G = [0.3; 0.3]; % independent generators

qZ = quadZonotope(c,E1,E2,F,G); % instantiate quadratic zonotope
Z = zonotope(qZ) % over-approximate by a zonotope

figure; hold on
plot(Z); % plot Z
plotFilled(qZ,[1 2],7,[],[.6 .6 .6],'EdgeColor','none'); % plot qZ
```

2.6. Further Set Representations

These set representations are also implemented in CORA, but not covered in this tutorial:

- affine representations
- capsules
- ellipsoids
- halfspaces
- polytopes
- probabilistic zonotopes
- Taylor models
- zoo (mixture of presentations)

3. Linear Discrete-Time Systems

One of the simplest problems in reachability analysis is to compute the reachable set of a linear discrete time system

$$x(t_{k+1}) = Ax(t_k) + u(t_k),$$

where $x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n$, $u(t_k) \in \mathcal{U} \subset \mathbb{R}^m$, and $A \in \mathbb{R}^{n \times n}$. The reachable set $\mathcal{R}(t_k)$ can be trivially computed as

$$\mathcal{R}(t_{k+1}) = A\mathcal{R}(t_k) + u(t_k),$$

where $\mathcal{R}(0) = \mathcal{X}_0$. Since the representation size of the reachable set grows with each iteration, we additionally have to reduce the size of the reachable set in each iteration.

```
function Rcont = tutorial_reachLinDiscrete(A,B,options)

% Input to be added
Uadd = B*options.U;

% Initialize reachable set
Rnext = options.R0;

%time period
tVec = options.tStart:options.timeStep:options.tFinal;

% init Rcont
Rcont = cell(length(tVec)-1,1);

% loop over all reachability steps
for i = 2:length(tVec)

    % order reduction
    Rcont{i-1} = reduce(Rnext,options.reductionTechnique,...
    options.zonotopeOrder);
    %compute next reachable set
    Rnext = A*Rnext + Uadd;
end
```

It remains to specify the system and plot the results

```
% system matrix
A = [0.95 -0.15 0 0 0;...
    0.15 0.95 0 0 0; ...
    0 0 0.9 0.05 0; ...
    0 0 -0.05 0.9 0; ...
    0 0 0 0 0.92];

% input matrix
B = 1;

%% set reachability options
options.tStart=0; %start time
options.tFinal=5; %final time
options.timeStep=0.04; %time step size for reachable set computation
options.R0=zonotope([ones(length(A),1),0.1*eye(length(A))]); %initial set
options.reductionTechnique='girard'; %technique for order reduction
options.zonotopeOrder=200; %zonotope order
options.U=zonotope([zeros(5,1),0.02*diag([0.1, 0.3, 0.1, 0.3, 0.3])]);

% compute reachable set
Rcont = tutorial_reachLinDiscrete(A,B,options);

%% plot results
for plotRun=1:2
    % plot different projections
    if plotRun==1
        projDimensions=[1 2];
    elseif plotRun==2
        projDimensions=[3 4];
    end

    figure;
    hold on

    %plot reachable sets
    for i=1:length(Rcont)
        plotFilled(Rcont{i},projDimensions,[.8 .8 .8],'EdgeColor','none');
    end
```

```
39
40      %plot initial set
41      plot(options.R0,projDimensions,'b-','lineWidth',2);
42
43      %label plot
44      xlabel(['x_{',num2str(projDimensions(1)),'}']);
45      ylabel(['x_{',num2str(projDimensions(2)),'}']);
46  end
47
```

4. Linear Continuous-Time Systems

While reachable sets can be easily computed for linear discrete time systems, one requires more advanced algorithms for linear continuous time systems

$$\dot{x}(t) = Ax(t) + u(t), \tag{4}$$

where $x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n$, $u(t) \in \mathcal{U} \subset \mathbb{R}^m$, and $A \in \mathbb{R}^{n \times n}$. Since one can apply the superposition principle to linear systems, we first compute the homogeneous solution ($u(t) = 0$) followed by the particular solution ($x(0) = 0$).

4.1. Homogeneous Solution for Points in Time

It is well known that the homogeneous solution of (4) is

$$x(t) = e^{At}x(0),$$

where

$$\begin{aligned} e^{At} &= \sum_{i=0}^{\infty} \frac{(At)^i}{i!} \\ &= I + \frac{At}{1!} + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \dots \\ \frac{\mathrm{d}}{\mathrm{d}t}e^{At} &= \mathbf{0} + A + 2\frac{At}{2!}A + 3\frac{(At)^2}{3!}A + \dots \\ &= A(I + \frac{At}{1!} + \frac{(At)^2}{2!} + \dots) = Ae^{At} \end{aligned}$$

Test of solution:

$$\begin{aligned} x(t) &= e^{At}x(0) \\ \frac{\mathrm{d}}{\mathrm{d}t}x(t) &= \frac{\mathrm{d}}{\mathrm{d}t}e^{At}x(0) = A\underbrace{e^{At}x(0)}_{x(t)} \end{aligned} \qquad \checkmark$$

How to compute the infinite series?
Answer: make it finite

$$e^{At} = \sum_{i=0}^{l} \frac{(At)^i}{i!} + \underbrace{\sum_{i=l+1}^{\infty} \frac{(At)^i}{i!}}_{=:E}$$

$$\|E\| \leq \sum_{i=l+1}^{\infty} \frac{(\|A\|t)^i}{i!}$$

Ratio of terms:

$$\frac{\text{“2nd term”}}{\text{“1st term”}} = \frac{(\|A\|t)^{l+2}}{(l+2)!}\frac{(l+1)!}{(\|A\|t)^{l+1}} = \frac{\|A\|t}{l+2} =: \omega$$
$$\frac{\text{“3rd term”}}{\text{“2nd term”}} = \frac{\|A\|t}{l+3} < \omega$$

Assume ω as worst-case ratio for all terms:

$$\|E\| \leq \sum_{i=l+1}^{\infty} \frac{(\|A\|t)^i}{i!} \leq \frac{(\|A\|t)^{l+1}}{(l+1)!} \underbrace{(1+\omega+\omega^2+\omega^3+\ldots)}_{=\frac{1}{1-\omega} \text{ for } \omega<1}$$

$$\implies e^{At} \in \sum_{i=0}^{l} \frac{(At)^i}{i!} + \underbrace{\begin{bmatrix}[-1,1] & \cdots \\ \vdots & [-1,1]\end{bmatrix}}_{[-\mathbf{1},\mathbf{1}]} \frac{(\|A\|t)^{l+1}}{(l+1)!}\frac{1}{1-\omega}$$

4.2. Homogeneous Solution for Time Intervals

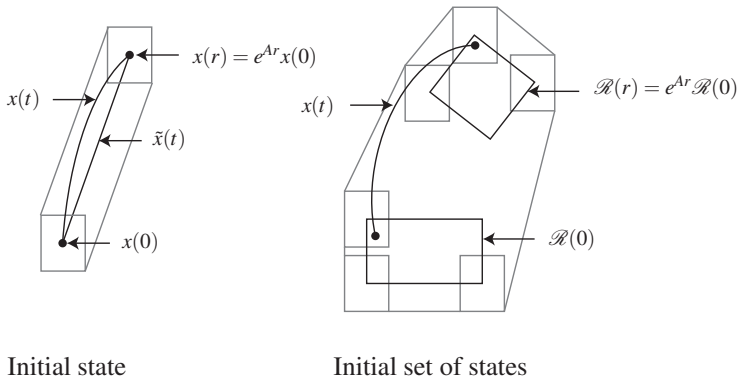


Figure 3. Enclosure of a trajectory for an initial state and a set of initial states.

$$\tilde{x}(t) = x(0) + \frac{t}{r}(x(r) - x(0)) \text{ for } t \in [0, r]$$

$$\begin{aligned} x(t) - \tilde{x}(t) =& x(t) - x(0) - \frac{t}{r}(x(r) - x(0)) \\ =& e^{At}x(0) - x(0) - \frac{t}{r}(e^{Ar}x(0) - x(0)) \\ =& \underbrace{\left[e^{At} - I - \frac{t}{r}(e^{Ar} - I)\right]}_{\subseteq F} x(0) \end{aligned}$$

$$\implies x(t) \in \tilde{x}(t) + Fx(0)$$

Insert Taylor series:

$$\begin{aligned}
F =& I + At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots + E(t) - I - \frac{t}{r}\left(I + Ar + \frac{(Ar)^2}{2!} + \ldots + E(r) - I\right) \\
=& At + \frac{(At)^2}{2!} + \frac{(At)^3}{3!} + \ldots + E(t) - At - \frac{A^2rt}{2!} + \frac{A^3r^2t}{3!} - \ldots - \frac{t}{r}E(r) \\
=& \frac{A^2(t^2 - tr)}{2!} + \frac{A^3(t^3 - tr^2)}{3!} + \ldots + E(t) - \frac{t}{r}E(r) \\
=& \sum_{i=2}^{l} \frac{A^i(t^i - tr^{i-1})}{i!} + E(t) - \frac{t}{r}E(r)
\end{aligned}$$

What is the bound for $\gamma(t) := t^i - tr^{i-1}$ for $t \in [0, r]$?

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}(t^i - tr^{i-1}) = it^{i-1} - r^{i-1} &= 0 \\
t^{i-1} &= \frac{1}{i} r^{i-1} \\
t &= i^{\frac{-1}{i-1}} r \\
\frac{\mathrm{d}^2}{\mathrm{d}t^2}\gamma(t) = \frac{\mathrm{d}}{\mathrm{d}t} it^{i-1} - r^{i-1} &= i(i-1)t^{i-2} > 0
\end{aligned}$$

Thus, $\gamma(t)$ is a convex function and thus we obtain the minimum from above as $t_{min} = i^{\frac{-1}{i-1}} r$. Inserting t_{min} results in

$$\gamma(t_{min}) = i^{-\frac{i}{i-1}} r^i - i^{-\frac{1}{i-1}} r r^{i-1} = \underbrace{\underbrace{\left(i^{-\frac{i}{i-1}} - i^{-\frac{1}{i-1}}\right)}_{\frac{1}{i^{\frac{i}{i-1}}} - \frac{1}{i^{\frac{1}{i-1}}}}}_{<0} \underbrace{r^i}_{>0} < 0$$

Since the function is concave, the maximum is to be found at the boundaries ($t = 0$, $t = r$):

$$\begin{aligned}
&\gamma(0) = 0, \quad \gamma(r) = r^i - rr^{i-1} = 0 \\
\Longrightarrow\ &\forall t \in [0, r] : \gamma(t) \in [i^{-\frac{i}{i-1}} - i^{-\frac{1}{i-1}}, 0] r^i
\end{aligned}$$

It remains to bound $E(t) - \frac{t}{r}E(r)$ for $t \in [0, r]$, where $E(t) \in [-\mathbf{1}, \mathbf{1}]^{n \times n} \phi(t)$.

$\phi(t)$ is monotonically increasing

$$\begin{aligned}
&\Longrightarrow \forall t \in [0, r] : \phi(t) \in [0, 1]\phi(r) \\
&\Longrightarrow \forall t \in [0, r] : \phi(t) - \frac{t}{r}\phi(r) \in [0, 1]\phi(r) - [0, 1]\phi(r) = [-1, 1]\phi(r) \\
&\Longrightarrow \forall t \in [0, r] : E(t) - \frac{t}{r}E(r) \in [-\mathbf{1}, \mathbf{1}]\phi(r)
\end{aligned}$$

Final result:

$$F = \sum_{i=2}^{l} [i^{-\frac{i}{i-1}} - i^{-\frac{1}{i-1}}, 0] r^i \frac{A^i}{i!} + [-\mathbf{1}, \mathbf{1}] \frac{(\|A\| r)^{l+1}}{(l+1)!} \frac{1}{1 - \omega}$$

4.3. Input Solution

The derivation of the input solution is more complicated so that we just provide the result:

$$\mathscr{R}^p(r) = \sum_{i=0}^{l} \left(\frac{A^i r^{i+1}}{(i+1)!} \mathscr{U} \right) + E(r) r \mathscr{U}$$

4.4. Realization in CORA

We specify the system similarly to the one for discrete time:

```
%% set reachability options
options.tStart=0; %start time
options.tFinal=5; %final time
options.R0=zonotope([ones(5,1),0.1*eye(5)]); %initial set

options.timeStep=0.04; %time step size for reachable set computation
options.taylorTerms=4; %number of taylor terms for reachable sets
options.zonotopeOrder=200; %zonotope order
options.originContained=0; %specify whether input set contains origin
options.reductionTechnique='girard'; %specify order reduction method

uTrans=[1; 0; 0; 0.5; -0.5];
options.uTrans=uTrans; %center of input set
options.U=0.5*zonotope([zeros(5,1),diag([0.2, 0.5, 0.2, 0.5, 0.5])]);

%% set simulation options
simOptions.tStart=0; %start time
simOptions.tFinal=5; %final time
simOptions.R0=zonotope([ones(5,1),0.1*eye(5)]);

simOptions.uTrans=[1; 0; 0; 0.5; -0.5]; %center of input set
simOptions.U=0.5*zonotope([zeros(5,1),diag([0.2, 0.5, 0.2, 0.5, 0.5])]);

%% specify continuous dynamics
A=[-1 -4 0 0 0; 4 -1 0 0 0; 0 0 -3 1 0; 0 0 -1 -3 0; 0 0 0 0 -2];
B=1;
fiveDimSys=linearSys('fiveDimSys',A,B); %initialize system


%% compute reachable set using zonotopes
tic
Rcont = reach(fiveDimSys, options);
tComp = toc;
disp(['computation time of reachable set: ',num2str(tComp)]);

%% create random simulations
runs = 60;
fractionVertices = 0.5;
fractionInputVertices = 0.5;
inputChanges = 6;
simRes = simulate_random(fiveDimSys, simOptions, runs,  ...
    fractionVertices, fractionInputVertices, inputChanges);

%% plot results
for plotRun=1:2
    % plot different projections
    if plotRun==1
        projDim=[1 2];
    elseif plotRun==2
        projDim=[3 4];
    end

    figure;
    hold on
```

```
55
56      %plot reachable sets
57      for i=1:length(Rcont)
58          plotFilled(Rcont{i},projDim,[.8 .8 .8],'EdgeColor','none');
59      end
60
61      %plot initial set
62      plot(options.R0,projDim,'w-','lineWidth',2);
63
64      %plot simulation results
65      for i=1:length(simRes.t)
66          plot(simRes.x{i}(:,projDim(1)),simRes.x{i}(:,projDim(2)),'k');
67      end
68
69      %label plot
70      xlabel(['x_{',num2str(projDim(1)),'}']);
71      ylabel(['x_{',num2str(projDim(2)),'}']);
72  end
```

5. Loading Models from SpaceEx

To load SpaceEx models (stored as XML files) into CORA, one only has to execute a simple command:

```
spaceex2cora('model.xml');
```

This command creates a CORA model in `/models/SpaceExConverted` under a folder with the identical name as the SpaceEx model. If the SpaceEx model contains nonlinear differential equations, additional dynamics files are stored in the same folder. Below, we present as an example the converted model of the bouncing ball model from SpaceEx:

```
1  function HA = bball(~)
2
3
4  %% Generated on 07-Aug-2018
5
6  %---------------Automaton created from Component 'system'------------------
7
8  %% Interface Specification:
9  %   This section clarifies the meaning of state & input dimensions
10 %   by showing their mapping to SpaceEx variable names.
11
12 % Component 1 (system.ball):
13 %  state x := [x; v]
14 %  input u := [uDummy]
15
16 %-------------------------Component system.ball----------------------------
17
18 %-----------------------------State always---------------------------------
19
20 %% equation:
21 %   x' == v & v' == -g
22 dynA = ...
23 [0,1;0,0];
24 dynB = ...
25 [0;0];
26 dync = ...
27 [0;-1];
28 dynamics = linearSys('linearSys', dynA, dynB, dync);
29
```

```
%% equation:
%   x >= 0
invA = ...
[-1,0];
invb = ...
[-0];
invOpt = struct('A', invA, 'b', invb);
inv = mptPolytope(invOpt);

trans = {};
%% equation:
%   v' := -c*v
resetA = ...
[1,0;0,-0.75];
resetb = ...
[0;0];
reset = struct('A', resetA, 'b', resetb);

%% equation:
%   x <= eps & v < 0
guardA = ...
[1,0;0,1];
guardb = ...
[-0;-0];
guardOpt = struct('A', guardA, 'b', guardb);
guard = mptPolytope(guardOpt);

trans{1} = transition(guard, reset, 1, 'dummy', 'names');

loc{1} = location('S1',1, inv, trans, dynamics);



HA = hybridAutomaton(loc);


end
```

At the beginning of each automatically created model, we list the state and inputs so that the created models can be interpreted more easily using the variable names from the SpaceEx model. These variable names are later replaced by the state vector x and the input vector u to make use of matrix multiplications in MATLAB for improved efficiency. Next, the dynamic equations, guard sets, invariants, transitions, and locations are created.

How to obtain SpaceEx models?

- Use the SpaceEx model editor: spaceex.imag.fr/download-6
- Convert Simulink models to SpaceEx: github.com/nikos-kekatos/SL2SX

6. Nonlinear Systems

So far, reachable sets of linear continuous systems have been presented. Although a fairly large group of dynamic systems can be described by linear continuous systems, the extension to nonlinear continuous systems is an important step for the analysis of more complex systems. The analysis of nonlinear systems is much more complicated since many valuable properties are no longer valid. One of them is the superposition principle, which allows the homogeneous and the inhomogeneous solution to be obtained separately. Another is that reachable sets of linear systems can be computed by a linear map. This makes it possible to exploit that geometric representations such as ellipsoids, zonotopes, and polytopes are closed under linear transformations, i.e., they are

again mapped to ellipsoids, zonotopes and polytopes, respectively. In CORA, reachability analysis of nonlinear systems is based on abstraction. We present abstraction by linear systems as presented in [1, Section 3.4] and by polynomial systems as presented in [2]. Since the abstraction causes additional errors, the abstraction errors are determined in an over-approximative way and added as an additional uncertain input so that an over-approximative computation is ensured.

6.1. Main Principle

General nonlinear continuous systems with uncertain parameters and Lipschitz continuity are considered. In analogy to the previous linear systems, the initial state $x(0)$ can take values from a set $\mathcal{X}_0 \subset \mathbb{R}^n$ and the input u takes values from a set $\mathcal{U} \subset \mathbb{R}^m$. The evolution of the state x is defined by the following differential equation:

$$\dot{x}(t) = f(x(t),u(t)), \qquad x(0) \in \mathcal{X}_0 \subset \mathbb{R}^n, \quad u(t) \in \mathcal{U} \subset \mathbb{R}^m,$$

where $u(t)$ and $f(x(t),u(t))$ are assumed to be globally Lipschitz continuous so that the Taylor expansion for the state and the input can always be computed, a condition required for the abstraction.

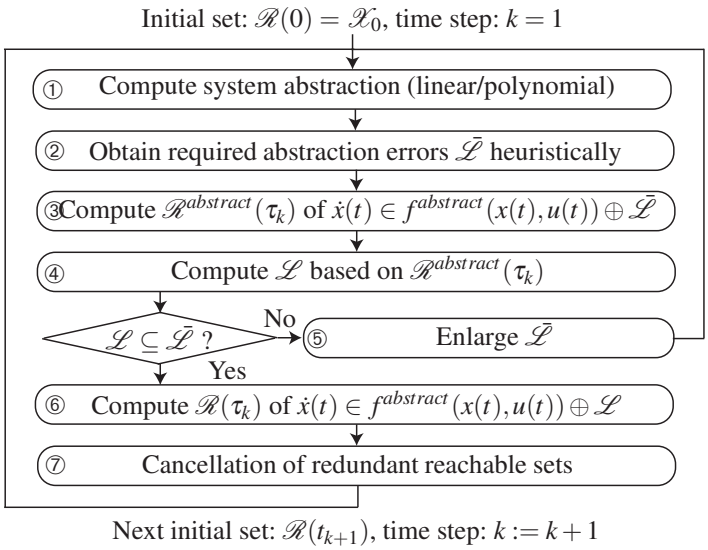


Figure 4. Computation of reachable sets – overview.

A brief visualization of the overall concept for computing the reachable set is shown in Fig. 4. As in the previous approaches, the reachable set is computed iteratively for time intervals $t \in \tau_k = [k\,r,(k+1)r]$ where $k \in \mathbb{N}^+$. The procedure for computing the reachable sets of the consecutive time intervals is as follows:

① The nonlinear system $\dot{x}(t) = f(x(t),u(t))$ is either abstracted to a linear system, or after introducing $z = [x^T, u^T]^T$, to a polynomial system of the form

$$\begin{aligned}\dot{x}_i = f^{abstract}(x,u) = w_i + \frac{1}{1!}\sum_{j=1}^{o} C_{ij} z_j(t) + \frac{1}{2!}\sum_{j=1}^{o}\sum_{k=1}^{o} D_{ijk} z_j(t) z_k(t) \\ + \frac{1}{3!}\sum_{j=1}^{o}\sum_{k=1}^{o}\sum_{l=1}^{o} E_{ijkl} z_j(t) z_k(t) z_l(t) + \ldots\end{aligned} \tag{5}$$

The set of abstraction errors $\mathcal{L}$ ensures that $f(x,u) \in f^{abstract}(x,u) \oplus \mathcal{L}$, which allows the reachable set to be computed in an over-approximative way.

② Next, the set of required abstraction errors $\bar{\mathcal{L}}$ is obtained heuristically.

③ The reachable set $\mathscr{R}^{abstract}(\tau_k)$ of $\dot{x}(t) \in f^{abstract}(x(t),u(t)) \oplus \bar{\mathscr{L}}$ is computed.

④ The set of abstraction errors $\mathscr{L}$ is computed based on the reachable set $\mathscr{R}^{abstract}(\tau_k)$.

⑤ When $\mathscr{L} \not\subseteq \bar{\mathscr{L}}$, the abstraction error is not admissible, requiring the assumption $\bar{\mathscr{L}}$ to be enlarged. If several enlargements are not successful, one has to split the reachable set and continue with one more partial reachable set from then on.

⑥ If $\mathscr{L} \subseteq \bar{\mathscr{L}}$, the abstraction error is accepted and the reachable set is obtained by using the tighter abstraction error: $\dot{x}(t) \in f^{abstract}(x(t),u(t)) \oplus \mathscr{L}$.

⑦ It remains to increase the time step ($k := k+1$) and cancel redundant reachable sets that are already covered by previously computed reachable sets. This decreases the number of reachable sets that have to be considered in the next time interval.

The necessity of splitting reachable sets is indicated in the workspace outputs using the keyword `split`.

6.2. Quadrotor Example

We study the dynamics of a quadrotor as derived in [7, eq. (16) - (19)]. Let us first introduce the variables required to describe the model: The inertial (north) position x_1, the inertial (east) position x_2, the altitude x_3, the longitudinal velocity x_4, the lateral velocity x_5, the vertical velocity x_6, the roll angle x_7, the pitch angle x_8, the yaw angle x_9, the roll rate x_{10}, the pitch rate x_{11}, and the yaw rate x_{12}. We further require the following parameters: gravity constant $g = 9.81$ [m/s^2], radius of center mass $R = 0.1$ [m], distance of motors to center mass $l = 0.5$ [m], motor mass $M_{rotor} = 0.1$ [kg], center mass $M = 1$ [kg], and total mass $m = M + 4M_{rotor}$.

From the above parameters we can compute the moments of inertia as

$$\begin{aligned} J_x &= \frac{2}{5} M R^2 + 2 l^2 M_{rotor}, \\ J_y &= J_x, \\ J_z &= \frac{2}{5} M R^2 + 4 l^2 M_{rotor}. \end{aligned}$$

Finally, we can write the set of ordinary differential equations for the quadrotor according to [7, eq. (16) - (19)]:

$$\begin{cases} \dot{x}_1 &= \cos(x_8)\cos(x_9)x_4 + \Big(\sin(x_7)\sin(x_8)\cos(x_9) - \cos(x_7)\sin(x_9)\Big)x_5 \\ &\quad + \Big(\cos(x_7)\sin(x_8)\cos(x_9) + \sin(x_7)\sin(x_9)\Big)x_6 \\ \dot{x}_2 &= \cos(x_8)\sin(x_9)x_4 + \Big(\sin(x_7)\sin(x_8)\sin(x_9) + \cos(x_7)\cos(x_9)\Big)x_5 \\ &\quad + \Big(\cos(x_7)\sin(x_8)\sin(x_9) - \sin(x_7)\cos(x_9)\Big)x_6 \\ \dot{x}_3 &= \sin(x_8)x_4 - \sin(x_7)\cos(x_8)x_5 - \cos(x_7)\cos(x_8)x_6 \\ \dot{x}_4 &= x_{12}x_5 - x_{11}x_6 - g\sin(x_8) \\ \dot{x}_5 &= x_{10}x_6 - x_{12}x_4 + g\cos(x_8)\sin(x_7) \\ \dot{x}_6 &= x_{11}x_4 - x_{10}x_5 + g\cos(x_8)\cos(x_7) - \frac{F}{m} \\ \dot{x}_7 &= x_{10} + \sin(x_7)\tan(x_8)x_{11} + \cos(x_7)\tan(x_8)x_{12} \\ \dot{x}_8 &= \cos(x_7)x_{11} - \sin(x_7)x_{12} \\ \dot{x}_9 &= \frac{\sin(x_7)}{\cos(x_8)}x_{11} + \frac{\cos(x_7)}{\cos(x_8)}x_{12} \\ \dot{x}_{10} &= \frac{J_y - J_z}{J_x}x_{11}x_{12} + \frac{1}{J_x}\tau_\phi \\ \dot{x}_{11} &= \frac{J_z - J_x}{J_y}x_{10}x_{12} + \frac{1}{J_y}\tau_\theta \\ \dot{x}_{12} &= \frac{J_x - J_y}{J_z}x_{10}x_{11} + \frac{1}{J_z}\tau_\psi \end{cases}$$

To check interesting control specifications, we stabilize the quadrotor using simple PD controllers for height, roll, and pitch. The inputs to the controller are the desired values for height, roll, and pitch u_1, u_2, and u_3, respectively. The equations of the controllers are

$$\begin{aligned} F &= mg - 10(x_3 - u_1) + 3x_6 && \text{(height control)}, \\ \tau_\phi &= -(x_7 - u_2) - x_{10} && \text{(roll control)}, \\ \tau_\theta &= -(x_8 - u_3) - x_{11} && \text{(pitch control)}. \end{aligned}$$

We leave the heading uncontrolled so that we set $\tau_\psi = 0$.

The task is to change the height from 0 [m] to 1 [m] within 5 [s]. A goal region $[0.98, 1.02]$ of the height x_3 has to be reached within 5 [s] and the height has to stay below 1.4 for all times. After 1 [s] the height should stay above 0.9 [m]. The initial position of the quadrotor is uncertain in all directions within $[-0.4, 0.4]$ [m] and also the velocity is uncertain within $[-0.4, 0.4]$ [m/s] for all directions. All other values are initialized as 0.

The MATLAB code that implements the simulation and reachability analysis of the quadrotor example is:

```
%% set options
options.tStart=0; %start time
options.tFinal=5; %final time
options.x0=zeros(12,1); %initial state for simulation
options.R0=zonotope([options.x0,0.4*diag([ones(6,1); zeros(6,1)])]);
options.timeStep=0.1; %time step size for reachable set computation
options.taylorTerms=4; %number of taylor terms for reachable sets
options.zonotopeOrder=50; %zonotope order
options.intermediateOrder=5;
options.reductionTechnique='girard';
options.errorOrder=1;
options.polytopeOrder=2; %polytope order
options.reductionInterval=1e3;
options.maxError = 1*ones(12,1);
options.originContained = 0;
options.advancedLinErrorComp = 0;
options.tensorOrder = 2;
options.uTrans = [1;0;0];
options.U = zonotope([0;0;0]); %input for reachability analysis

%% specify continuous dynamics
quadrocopter = nonlinearSys(12,3,@quadrocopterControlledEq,options);

%% compute reachable set
tic
Rcont = reach(quadrocopter, options);
tComp = toc;
disp(['computation time of reachable set: ',num2str(tComp)]);

%% create random simulations
runs = 60;
fractionVertices = 0.5;
fractionInputVertices = 0.5;
inputChanges = 6;
simRes = simulate_random(quadrocopter, options, runs, ...
    fractionVertices, fractionInputVertices, inputChanges);

%% plot results
figure;
hold on

%plot time elapse
for i=1:length(Rcont)
    %get Uout
    Uout1 = interval(project(Rcont{i}{1},3));
```

```
    %obtain times
    t1 = (i-1)*options.timeStep;
    t2 = i*options.timeStep;
    %generate plot areas as interval hulls
    IH = interval([t1; infimum(Uout1)], [t2; supremum(Uout1)]);

    plotFilled(IH,[1 2],[.75 .75 .75],'EdgeColor','none');
end

%plot simulation results
for i=1:(length(simRes.t))
    plot(simRes.t{i},simRes.x{i}(:,3),'Color',0*[1 1 1]);
end

%label plot
box on
xlabel('t');
ylabel('altitude x_3');
axis([0,5,-0.5,1.5]);
```

The reachable set and the simulation are plotted in Fig. 5 for a time horizon of $t_f = 5$.

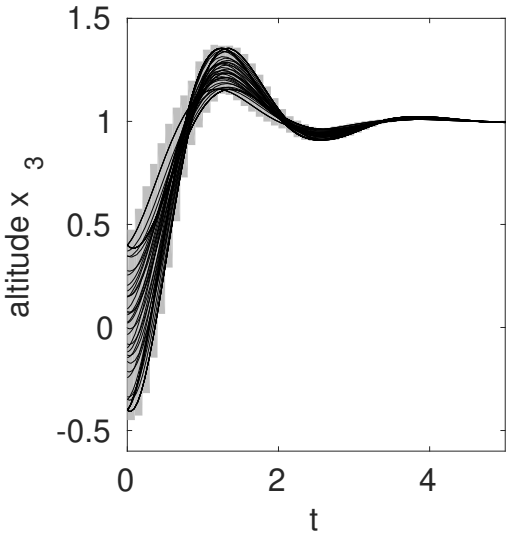


Figure 5. Illustration of the reachable set of the quadrotor. The black box shows the initial set and the black line shows the simulated trajectory.

7. Hybrid Systems

In CORA, hybrid systems are modeled by hybrid automata. Besides a continuous state x, there also exists a discrete state v for hybrid systems. The continuous initial state may take values within continuous sets while only a single initial discrete state is assumed without loss of generality[4]. The switching of the continuous dynamics is triggered by *guard sets*. Jumps in the continuous state are considered after the discrete state has changed. One of the most intuitive examples where jumps in the continuous state can occur is the bouncing ball example (see Sec. 7.2), where the velocity of the ball changes instantaneously when hitting the ground.

[4] In the case of several initial discrete states, the reachability analysis can be performed for each discrete state separately.

7.1. Hybrid Automata

The formal definition of the hybrid automaton is similarly defined as in [12]. The main difference is the consideration of uncertain parameters and the restrictions on jumps and guard sets. A hybrid automaton $HA = (\mathscr{V}, v^0, \mathscr{X},\ \mathscr{X}^0, \mathscr{U}, \mathscr{P}, \mathtt{inv}, \mathtt{T}, \mathtt{g}, \mathtt{h}, \mathtt{f})$, as it is considered in CORA, consists of:

- the finite set of locations $\mathscr{V} = \{v_1, \ldots, v_\xi\}$ with an initial location $v^0 \in \mathscr{V}$.
- the continuous state space $\mathscr{X} \subseteq \mathbb{R}^n$ and the set of initial continuous states $\mathscr{X}^0$ such that $\mathscr{X}^0 \subseteq \mathtt{inv}(v^0)$.
- the continuous input space $\mathscr{U} \subseteq \mathbb{R}^m$.
- the parameter space $\mathscr{P} \subseteq \mathbb{R}^p$.
- the mapping[5] $\mathtt{inv}$: $\mathscr{V} \to 2^{\mathscr{X}}$, which assigns an invariant $\mathtt{inv}(v) \subseteq \mathscr{X}$ to each location v.
- the set of discrete transitions $\mathtt{T} \subseteq \mathscr{V} \times \mathscr{V}$. A transition from $v_i \in \mathscr{V}$ to $v_j \in \mathscr{V}$ is denoted by (v_i, v_j).
- the guard function $\mathtt{g} : \mathtt{T} \to 2^{\mathscr{X}}$, which associates a guard set $\mathtt{g}((v_i, v_j))$ for each transition from v_i to v_j, where $\mathtt{g}((v_i, v_j)) \cap \mathtt{inv}(v_i) \neq \emptyset$.
- the jump function $\mathtt{h} : \mathtt{T} \times \mathscr{X} \to \mathscr{X}$, which returns the next continuous state when a transition is taken.
- the flow function $\mathtt{f} : \mathscr{V} \times \mathscr{X} \times \mathscr{U} \times \mathscr{P} \to \mathbb{R}^n$, which defines a continuous vector field for the time derivative of x: $\dot{x} = \mathtt{f}(v, x, u, p)$.

The invariants $\mathtt{inv}(v)$ and the guard sets $\mathtt{g}((v_i, v_j))$ are modeled by polytopes. The jump function is restricted to a linear map

$$x' = K_{(v_i,v_j)}\, x + l_{(v_i,v_j)}, \tag{6}$$

where x' denotes the state after the transition is taken and $K_{(v_i,v_j)} \in \mathbb{R}^{n \times n}$, $l_{(v_i,v_j)} \in \mathbb{R}^n$ are specific for a transition (v_i, v_j). The input sets $\mathscr{U}_v$ are modeled by zonotopes and are also dependent on the location v. Note that in order to use the results from reachability analysis of nonlinear systems, the input $u(t)$ is assumed to be locally Lipschitz continuous. The set of parameters $\mathscr{P}_v$ can also be chosen differently for each location v.

The evolution of the hybrid automaton is described informally as follows. Starting from an initial location $v(0) = v^0$ and an initial state $x(0) \in \mathscr{X}^0$, the continuous state evolves according to the flow function that is assigned to each location v. If the continuous state is within a guard set, the corresponding transition can be taken and has to be taken if the state would otherwise leave the invariant $\mathtt{inv}(v)$. When the transition from the previous location v_i to the next location v_j is taken, the system state is updated according to the jump function and the continuous evolution within the next invariant.

Because the reachability of discrete states is simply a question of determining if the continuous reachable set hits certain guard sets, the focus of CORA is on the continuous reachable sets. Clearly, as for the continuous systems, the reachable set of the hybrid system has to be over-approximated in order to verify the safety of the system. An illustration of a reachable set of a hybrid automaton is given in Fig. 6.

7.2. Bouncing Ball Example

We demonstrate the syntax of CORA for the well-known bouncing ball example, see e.g., [13, Section 2.2.3]. Given is a ball in Fig. 7 with dynamics $\ddot{s} = -g$, where s is the vertical position and g is the gravity constant. After impact with the ground at $s = 0$, the velocity

[5] $2^{\mathscr{X}}$ is the power set of $\mathscr{X}$.

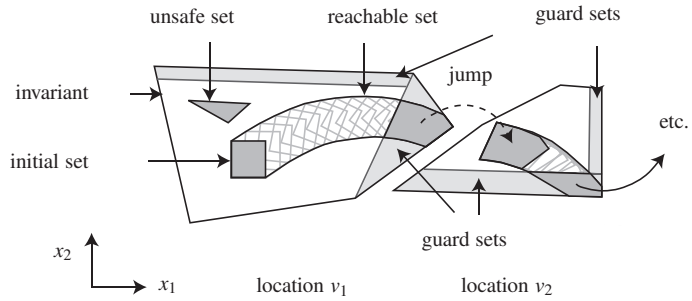
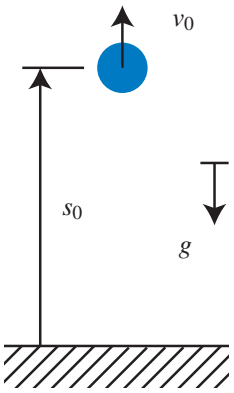


Figure 6. Illustration of the reachable set of a hybrid automaton.

changes to $v' = -\alpha v$ ($v = \dot{s}$) with $\alpha \in [0,1]$. The corresponding hybrid automaton can be formalized as presented in Fig. 7.



$$
\begin{aligned}
\mathscr{V} &= \{v_1\} \\
\mathscr{X} &= \mathbb{R}^+ \times \mathbb{R} \quad \text{(ball is above ground)} \\
\mathscr{U} = \mathscr{Y}_c &= \{\} \\
\mathtt{T} &= \{(z_1, z_1)\} \\
\mathtt{inv}(z_1) &= \{[x_1, x_2]^T | x_1 \in \mathbb{R}_0^+, x_2 \in \mathbb{R}\} \\
\mathtt{g}((z_1, z_1)) &= \{[x_1, x_2]^T | x_1 = 0, x_2 \in \mathbb{R}_0^-\} \\
\mathtt{h}((z_1, z_1), x) &= \begin{bmatrix} x_1 \\ -\alpha x_2 \end{bmatrix} \\
\mathtt{f}(z_1, x) &= \begin{bmatrix} x_2 \\ -g \end{bmatrix}
\end{aligned}
$$

Figure 7.: Bouncing ball.

The MATLAB code that implements the simulation and reachability analysis of the bouncing ball example is:

```
%%set reachability options
options.R0 = zonotope([[1; 0], diag([0.05, 0.05])]); %initial set
options.startLoc = 1; %initial location
options.finalLoc = 0; %0: no final location
options.tStart = 0; %start time
options.tFinal = 1.7; %final time
options.timeStepLoc{1} = 0.05; %time step size in location 1
options.taylorTerms = 10; %Taylor terms for exponential matrix
options.zonotopeOrder = 20; %maximum zonotope order
options.polytopeOrder = 10; %maximum order for conversion to polytopes
options.reductionTechnique = 'girard'; %reduction technique
options.enclosureEnables = 5; %choose enclosure method(s)
options.originContained = 0; %origin contained in input set?
options.isHyperplaneMap = 0; %hyperplane maps are not used
options.guardIntersect = 'polytope'; %convert guards to polytopes
options.uLoc{1} = 0; % no inputs
options.Uloc{1} = zonotope(0); % no inputs


%% set simulation options
simOptions.x0 = [1; 0]; %initial state for simulation
simOptions.startLoc = 1; %initial location
simOptions.finalLoc = 0; %0: no final location
simOptions.tStart = 0; %start time
simOptions.tFinal = 1.7; %final time
simOptions.uLoc{1} = 0; % no inputs
simOptions.Uloc{1} = zonotope(0); % no inputs


```

```
%% specify hybrid automaton
% converetd hybrid automaton model of the bouncing ball obtained from
% "spaceex2cora(bball.xml);"
HA = bball;

%simulate hybrid automaton
HA = simulate(HA,simOptions);

%compute reachable set
[HA] = reach(HA,options);

%% choose projection and plot
figure
hold on
options.projectedDimensions = [1 2];
options.plotType = 'b';
plot(HA,'reachableSet',options); %plot reachable set
plotFilled(options.R0,options.projectedDimensions,'w','EdgeColor','k');
plot(HA,'simulation',options); %plot simulation
axis([0,1.2,-6,4]);
```

The reachable set and the simulation are plotted in Fig. 8 for a time horizon of $t_f = 1.7$.

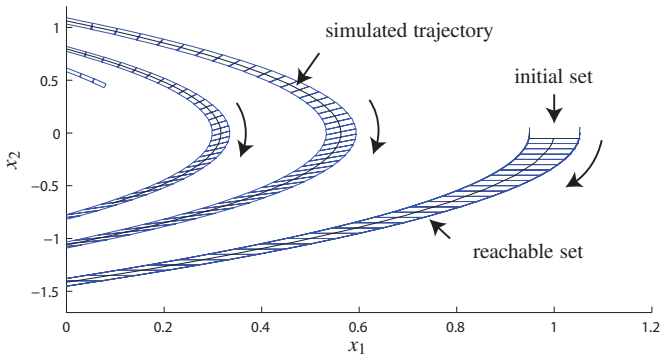


Figure 8. Illustration of the reachable set of the bouncing ball. The black box shows the initial set and the black line shows the simulated trajectory.

8. Further Continuous Dynamics

These system classes are also implemented in CORA:

- Linear systems with uncertain parameters (constant parameters)
- Linear systems with uncertain parameters (time-varying parameters)
- Nonlinear systems with uncertain parameters (constant parameters)
- Nonlinear systems with uncertain parameters
- Nonlinear discrete time systems
- Differential-algebraic systems
- Parallel hybrid automata

9. Conclusions

In this tutorial, we have demonstrated how one can easily build a reachability analyzer using CORA. Compared to other tools, one can more easily realize a reachability ana-

lyzer due to the use of MATLAB—a script-based language. CORA also contains basic operations on set and the set-based evaluation of nonlinear functions, which is also useful for other applications, such as set-based observers, conformance checking, controller synthesis, and fault detection, among others. The next release of CORA will contain automatic tuning of algorithmic parameters, such as the time step size, so that it becomes even more user-friendly.

References

[1] M. Althoff. *Reachability Analysis and its Application to the Safety Assessment of Autonomous Cars*. Dissertation, Technische Universität München, 2010. http://nbn-resolving.de/urn/resolver.pl?urn:nbn:de:bvb:91-diss-20100715-963752-1-4.

[2] M. Althoff. Reachability analysis of nonlinear systems using conservative polynomialization and non-convex sets. In *Hybrid Systems: Computation and Control*, pages 173–182, 2013.

[3] M. Althoff. An introduction to CORA 2015. In *Proc. of the Workshop on Applied Verification for Continuous and Hybrid Systems*, pages 120–151, 2015.

[4] M. Althoff and D. Grebenyuk. Implementation of interval arithmetic in CORA 2016. In *Proc. of the 3rd International Workshop on Applied Verification for Continuous and Hybrid Systems*, pages 91–105, 2016.

[5] M. Althoff, D. Grebenyuk, and N. Kochdumper. Implementation of Taylor models in CORA 2018. In *Proc. of the 5th International Workshop on Applied Verification for Continuous and Hybrid Systems*, pages 145–173, 2018.

[6] M. Althoff and B. H. Krogh. Zonotope bundles for the efficient computation of reachable sets. In *Proc. of the 50th IEEE Conference on Decision and Control*, pages 6814–6821, 2011.

[7] Randal Beard. Quadrotor dynamics and control rev 0.1. Technical report, Brigham Young University, 2008.

[8] A. Girard. Reachability of uncertain linear systems using zonotopes. In *Hybrid Systems: Computation and Control*, LNCS 3414, pages 291–305. Springer, 2005.

[9] J. Hoefkens, M. Berz, and K. Makino. *Scientific Computing, Validated Numerics, Interval Methods*, chapter Verified High-Order Integration of DAEs and Higher-Order ODEs, pages 281–292. Springer, 2001.

[10] G. Lafferriere, G. J. Pappas, and S. Yovine. Symbolic reachability computation for families of linear vector fields. *Symbolic Computation*, 32:231–253, 2001.

[11] J. K. Scott, D. M. Raimondo, G. R. Marseglia, and R. D. Braatz. Constrained zonotopes: A new tool for set-based estimation and fault detection. *Automatica*, 69:126–136, 2016.

[12] O. Stursberg and B. H. Krogh. Efficient representation and computation of reachable sets for hybrid systems. In *Hybrid Systems: Computation and Control*, LNCS 2623, pages 482–497. Springer, 2003.

[13] A. van der Schaft and H. Schumacher. *An Introduction to Hybrid Dynamical Systems*. Springer, 2000.