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## FINITE ELEMENT ERROR ESTIMATES FOR ONE-DIMENSIONAL ELLIPTIC OPTIMAL CONTROL BY BV FUNCTIONS

**ABSTRACT.** We consider an optimal control problem governed by a one-dimensional elliptic equation that involves univariate functions of bounded variation as controls. For the discretization of the state equation we use linear finite elements and for the control discretization we analyze two strategies. First, we use variational discretization of the control and show that the  $L^2$ - and  $L^\infty$ -error for the state and the adjoint state are of order  $\mathcal{O}(h^2)$  and that the  $L^1$ -error of the control behaves like  $\mathcal{O}(h^2)$ , too. These results rely on a structural assumption that implies that the optimal control of the original problem is piecewise constant and that the adjoint state has nonvanishing first derivative at the jump points of the control. If, second, piecewise constant control discretization is used, we obtain  $L^2$ -error estimates of order  $\mathcal{O}(h)$  for the state and  $W^{1,\infty}$ -error estimates of order  $\mathcal{O}(h)$  for the adjoint state. Under the same structural assumption as before we derive an  $L^1$ -error estimate of order  $\mathcal{O}(h)$  for the control. We discuss optimization algorithms and provide numerical results for both discretization schemes indicating that the error estimates are optimal.

**1. Introduction.** In this paper we derive a priori error estimates for two finite element discretizations of the optimal control problem governed by a one-dimensional elliptic equation

$$\min_{(u,q)} \frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2 + \alpha \|q'\|_{\mathcal{M}(\Omega)} \quad \text{s.t.} \quad Au = q.$$

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Here,  $u \in V := H_0^1(\Omega)$  is the state and  $q \in Q := BV(\Omega)$  is the control, where  $BV(\Omega)$  denotes the space of functions of bounded variation (BV) on the interval  $\Omega := (0, 1)$ . The operator  $A$  is elliptic and  $\alpha$  is a positive real number. The two finite element schemes that will be analyzed are identical in regard to the discretization of state and adjoint state, but they differ in the treatment of the control. In the *variational discretization* the control is not discretized, while in the second scheme the control is discretized by piecewise constant functions.

The significance of the above control problem is given by the use of the BV-seminorm  $\|q'\|_{\mathcal{M}(\Omega)}$  in the objective. This favors piecewise constant controls with only a limited number of jumps, which makes this problem type interesting in many practical applications. The precise functional analytic setting will be provided in the next section.

Optimal control problems with BV-controls defined in one space dimension are strongly related to control problems with measures as controls. Both BV optimal control problems and optimal control problems with measures have attracted significant research interest in the recent past, see, e.g. [8, 13, 14, 17, 22, 23] for the former and [11, 12, 15, 16, 29, 30] for the latter.

Error estimates for PDE-constrained optimal control problems involving measures have been presented in [11, 30, 31, 34, 35]. For error estimates of further sparsity promoting optimal control problems with PDEs see for example [19, 30]. The literature on error estimates for optimal control problems with controls in BV is rather limited. We are only aware of [14, 18]. Error estimates and numerical analysis for inverse problems involving BV-functions are studied in [5, 6]. Related discussion of ODE-constrained control problems involving discontinuous functions and their numerical analysis can be found in, e.g., [1, 2, 10, 21, 25, 26, 37, 38].

The main difficulty in deriving error estimates for the above problem is given by the fact that it lacks certain coercivity properties that are usually employed to obtain error estimates for the controls, for instance by suitably testing the first order necessary optimality conditions. Hence, only error estimates for the state and the adjoint state can be proven in a rather direct manner; these are, however, suboptimal. To obtain an error estimate for the control and also to improve the error estimates for state and adjoint state, we make use of a structural assumption on the Lagrange multiplier  $\bar{\Phi}$  arising from the convex subdifferential of the term  $\|q'\|_{\mathcal{M}(\Omega)}$ . Specifically, we assume that  $\bar{\Phi}$ , which is a  $C^2$  function in  $\bar{\Omega}$ , has only finitely many global extreme points and that it exhibits quadratic growth near those points (i.e.,  $\bar{\Phi}'' \neq 0$  near those points; see Assumption 4.4 and Assumption 4.5). Since the jump set of the optimal control is contained in the set of global extreme points of  $\bar{\Phi}$ , see Corollary 1, this assumption implies that the optimal control admits only finitely many jumps, which is a rather typical situation in practice. In addition, it ensures that the adjoint state has nonvanishing first derivative near the global extreme points of  $\bar{\Phi}$ , which is closely related to assumptions used to derive error estimates for bang-bang control problems, see, e.g., [7, 20, 24].

Starting from possibly suboptimal error estimates for the state and adjoint state and incorporating the structural assumption, we are able to derive an error estimate for the controls in  $L^1$  for both variational control discretization, where the order of the error is  $\mathcal{O}(h^2)$ , and piecewise constant control discretization, where we obtain  $\mathcal{O}(h)$ . Moreover, we provide numerical experiments which indicate that the established error estimates are optimal. To further substantiate the use of the  $L^1$ -norm in the error estimates for the control, we include numerical results for the order of

convergence of the controls with respect to the  $L^2$ -norm. These results clearly show that in both discretization schemes the order of convergence in  $L^1$  is higher than the one in  $L^2$ .

Let us stress that the essential structural assumption on  $\bar{\Phi}$  cannot be transferred to settings in which the control domain is of dimension greater than one. This is due to the fact that in such settings the Lagrange multiplier  $\bar{\Phi}$  does not characterize the jump set of the optimal control. While this implies that the *control* domain is limited to an interval, this is not the case for the domain of the *state*. We expect that the analysis presented in this paper can be extended to problems where the state lives on a domain of dimension larger than one.

This paper is structured as follows. In Section 2 we provide the precise problem setting and discuss existence of optimal solutions as well as first order optimality conditions for the continuous problem. Section 3 is concerned with the same aspects, but for the two discretization schemes. In Section 4 we derive both the basic and the improved error estimates, which is why this section also contains the structural assumption. The numerical experiments are presented in Section 5.

**2. The continuous problem.** We will consider the following model problem in the one-dimensional spatial domain  $\Omega := (0, 1)$ . Given the parameter  $\alpha > 0$ , a desired state  $u_d \in L^\infty(\Omega)$ , and functions  $a \in C^{0,1}(\bar{\Omega})$  and  $d_0 \in L^\infty(\Omega)$  satisfying  $a(x) \geq \nu > 0$  with a constant  $\nu > 0$  for all  $x \in \bar{\Omega}$  and  $d_0(x) \geq 0$  for a.e.  $x \in \Omega$ , we are looking for a control  $q \in Q := BV(\Omega)$  and an associated state  $u \in V := H_0^1(\Omega)$  solving the optimal control problem

$$\min_{(u,q) \in V \times Q} \underbrace{\frac{1}{2} \|u - u_d\|_{L^2(\Omega)}^2 + \alpha \|q'\|_{\mathcal{M}(\Omega)}}_{=: J(u,q)} \quad \text{s.t.} \quad \mathbf{a}(u, w) = (q, w)_{L^2(\Omega)} \quad \forall w \in V,$$

where the bilinear form  $\mathbf{a}$  is given by

$$\mathbf{a}: V \times V \rightarrow \mathbb{R}, \quad \mathbf{a}(v, w) := (av', w')_{L^2(\Omega)} + (d_0v, w)_{L^2(\Omega)}.$$

**2.1. The state equation.** Recall from, e.g., [3, 28, 41] that the space  $BV(\Omega)$  is given by those functions  $v \in L^1(\Omega)$  for which the distributional derivative  $v'$  is a Radon measure, i.e.,

$$BV(\Omega) = \left\{ v \in L^1(\Omega) : \|v'\|_{\mathcal{M}(\Omega)} < \infty \right\},$$

where  $\mathcal{M}(\Omega)$  denotes the space of Radon measures. The space  $BV(\Omega)$  is a Banach space if equipped with the norm

$$\|v\|_{BV(\Omega)} := \|v\|_{L^1(\Omega)} + \|v'\|_{\mathcal{M}(\Omega)},$$

see, e.g., [4, Thm. 10.1.1]. Moreover,  $BV(\Omega)$  embeds continuously into  $L^p(\Omega)$  for  $p \in [1, \infty]$  and compactly into  $L^p(\Omega)$  for  $p \in [1, \infty)$ , see, e.g., [3, Cor. 3.49 together with Prop. 3.21]. As  $BV(\Omega)$  embeds into  $L^2(\Omega)$  we note that for every  $q \in BV(\Omega)$  the Lax-Milgram theorem readily guarantees existence of a unique associated state  $u = u(q) \in V$ . Thus, the use of the solution or control-to-state operator

$$S: Q \subset V^* \rightarrow V$$

is justified. We note in passing that  $S: V^* \rightarrow V$  is a self-adjoint isomorphism. In fact, because we are working in dimension one, the following strong regularity result can be proven by standard arguments.

**Lemma 2.1.** *Let  $p \in (1, \infty]$ . For all  $v \in L^p(\Omega)$  there holds  $Sv \in W^{2,p}(\Omega) \cap V$ , and the estimate*

$$\|Sv\|_{W^{2,p}(\Omega)} \leq C \|v\|_{L^p(\Omega)}$$

is satisfied, where the constant  $C > 0$  is independent of  $v$  and  $p$ .

Introducing the reduced objective  $j: Q \rightarrow \mathbb{R}$ ,  $j(q) := J(S(q), q)$ , we can now analyze the reduced version of the original problem, given by

$$\min_{q \in Q} j(q). \quad (\text{P})$$

We will demonstrate that (P) admits a unique solution, characterize this solution by means of optimality conditions, and draw some conclusions from the optimality conditions regarding the structure of the optimal solution. Due to convexity we need not distinguish between local and global solutions, and first order necessary conditions are also sufficient.

## 2.2. Existence of optimal controls.

**Theorem 2.2.** *Problem (P) admits a unique optimal control  $\bar{q} \in Q$  with associated optimal state  $\bar{u} \in V$ .*

*Proof.* The injectivity of  $S$  implies that  $j$  is strictly convex, so (P) has at most one solution. To establish existence of  $\bar{q}$ , let us consider a minimizing sequence  $(q_n)_{n \in \mathbb{N}}$  of  $j$  with  $j(q_n) \leq j(0)$  for all  $n \in \mathbb{N}$ . Our goal is to bound the BV-norm of that sequence. Since there holds

$$\alpha \|q'_n\|_{\mathcal{M}(\Omega)} \leq j(q_n) \leq j(0), \quad (1)$$

it only remains to establish that  $(\|q_n\|_{L^1(\Omega)})_{n \in \mathbb{N}}$  is bounded. From [3, Thm. 3.44] it follows that

$$\|q_n - \hat{q}_n\|_{L^1(\Omega)} \leq C_{iso} \|q'_n\|_{\mathcal{M}(\Omega)} \leq \frac{C_{iso} j(0)}{\alpha}, \quad (2)$$

where  $\hat{q}_n := \frac{1}{|\Omega|} \int_{\Omega} q_n \, dx$  and  $C_{iso}$  depends only on  $\Omega$ . Estimate (2) implies via the inverse triangle inequality that for all  $n \in \mathbb{N}$  there holds

$$\|q_n\|_{L^1(\Omega)} \leq \frac{C_{iso} j(0)}{\alpha} + |\hat{q}_n|, \quad (3)$$

where we have used that  $|\Omega| = 1$ . Moreover, we have

$$\begin{aligned} |\hat{q}_n| \|S1\|_{L^2(\Omega)} &\leq \|S(\hat{q}_n - q_n)\|_{L^2(\Omega)} + \|Sq_n\|_{L^2(\Omega)} \\ &\leq \|S\|_{\mathcal{L}(V^*, L^2(\Omega))} \|\hat{q}_n - q_n\|_{V^*} + \|Sq_n\|_{L^2(\Omega)}. \end{aligned}$$

Making use of the embedding  $L^1(\Omega) \hookrightarrow V^*$  with constant  $C_{emb}$  we infer that the first term on the right-hand side can be bounded using (2), and the second term can be bounded by (1). Together, this yields

$$\begin{aligned} |\hat{q}_n| \|S1\|_{L^2(\Omega)} &\leq \frac{C_{iso} C_{emb} j(0)}{\alpha} \|S\|_{\mathcal{L}(V^*, L^2(\Omega))} + \|Sq_n - u_d\|_{L^2(\Omega)} + \|u_d\|_{L^2(\Omega)} \\ &\leq \frac{C_{iso} C_{emb} j(0)}{\alpha} \|S\|_{\mathcal{L}(V^*, L^2(\Omega))} + \sqrt{2j(0)} + \|u_d\|_{L^2(\Omega)}. \end{aligned}$$

This and (3) imply

$$\begin{aligned} \|q_n\|_{L^1(\Omega)} &\leq \frac{C_{iso} j(0)}{\alpha} \\ &\quad + \|S1\|_{L^2(\Omega)}^{-1} \left( \frac{C_{iso} C_{emb} j(0)}{\alpha} \|S\|_{\mathcal{L}(V^*, L^2(\Omega))} + 2\sqrt{2j(0)} \right), \end{aligned}$$

where we have used that  $S1 \neq 0$  and that  $\|u_d\|_{L^2(\Omega)} \leq \sqrt{2j(0)}$ . In view of (1) we have thus found for all  $n \in \mathbb{N}$  that

$$\begin{aligned} \|q_n\|_{BV(\Omega)} &\leq \frac{(C_{iso} + 1)j(0)}{\alpha} \\ &\quad + \|S1\|_{L^2(\Omega)}^{-1} \left( \frac{C_{iso}C_{emb}j(0)}{\alpha} \|S\|_{\mathcal{L}(V^*, L^2(\Omega))} + 2\sqrt{2j(0)} \right). \end{aligned} \quad (4)$$

Since  $BV(\Omega)$  is compactly embedded in  $L^1(\Omega)$ , there is a subsequence  $(q_{n_k})_{k \in \mathbb{N}}$  of  $(q_n)$  and a  $\bar{q} \in L^1(\Omega)$  such that  $q_{n_k} \rightarrow \bar{q}$  in  $L^1(\Omega)$  for  $k \rightarrow \infty$ . By continuity of the mapping  $L^2(\Omega) \ni q \mapsto \frac{1}{2}\|Sq - u_d\|_{L^2(\Omega)}^2$  and lower semicontinuity of  $q \mapsto \|q'\|_{\mathcal{M}(\Omega)}$  with respect to the  $L^1(\Omega)$  topology, cf. [41, Thm. 5.2.1], we deduce that  $j(\bar{q}) = \inf_{q \in Q} j(q)$ .  $\square$

**2.3. Optimality conditions.** Next, we provide necessary and sufficient optimality conditions for the optimal solution.

**Theorem 2.3.** *The control  $\bar{q} \in Q$  with associated state  $\bar{u} \in V$  is optimal for Problem (P) if and only if there exists a unique adjoint state  $\bar{z} \in W^{2,\infty}(\Omega) \cap V$  such that  $(\bar{u}, \bar{q}, \bar{z})$  and the  $W^{3,\infty}(\Omega)$  function  $\bar{\Phi}: [0, 1] \rightarrow \mathbb{R}$ ,  $\bar{\Phi}(x) := \int_0^x \bar{z}(s) ds$  satisfy  $\bar{\Phi}(1) = 0$  as well as*

$$\int_{\Omega} \bar{\Phi} d\bar{q}' = \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} \quad \text{and} \quad \|\bar{\Phi}\|_{\infty} \leq \alpha,$$

$$\begin{aligned} \mathbf{a}(\bar{u}, w) &= (\bar{q}, w)_{L^2(\Omega)} \quad \forall w \in V, \\ \mathbf{a}(w, \bar{z}) &= (w, \bar{u} - u_d)_{L^2(\Omega)} \quad \forall w \in V, \end{aligned}$$

and

$$-(\bar{z}, q - \bar{q})_{L^2(\Omega)} \leq \alpha \left[ \|q'\|_{\mathcal{M}(\Omega)} - \|\bar{q}'\|_{\mathcal{M}(\Omega)} \right] \quad \forall q \in Q.$$

*Proof.* Using convex analysis, e.g. [32], the optimality of  $\bar{q}$  is equivalent to

$$0 \in \partial j(\bar{q}),$$

where  $\partial j(\bar{q})$  denotes the subdifferential of  $j$  at the point  $\bar{q}$ . By the chain rule and the sum rule, e.g. [32, Proposition 3.28] and [32, Thm. 3.30], this is equivalent to

$$-S^*(S\bar{q} - u_d) \in \partial \left( \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} \right). \quad (5)$$

Note that the sum rule is applicable since both summands of  $j$  are continuous on  $Q$ . Defining  $\bar{z} := S^*(S\bar{q} - u_d)$  and recalling  $\bar{u} = S\bar{q}$  we obtain

$$\begin{aligned} \mathbf{a}(\bar{u}, w) &= (\bar{q}, w)_{L^2(\Omega)} \quad \forall w \in V, \\ \mathbf{a}(w, \bar{z}) &= (w, \bar{u} - u_d)_{L^2(\Omega)} \quad \forall w \in V. \end{aligned}$$

In particular, the asserted regularity of  $\bar{z}$  follows from Lemma 2.1, which in turn implies  $\bar{\Phi} \in W^{3,\infty}(\Omega)$ . Furthermore, the definition of the subdifferential implies that (5) can be equivalently expressed as

$$-(\bar{z}, q - \bar{q})_{L^2(\Omega)} \leq \alpha \left[ \|q'\|_{\mathcal{M}(\Omega)} - \|\bar{q}'\|_{\mathcal{M}(\Omega)} \right] \quad \forall q \in Q.$$

Testing with  $q = 2\bar{q}$ ,  $q = 0$  and  $q = \tilde{q} + \bar{q}$  for any  $\tilde{q} \in Q$  yields the equivalent system

$$\begin{aligned} -(\bar{z}, \bar{q})_{L^2(\Omega)} &= \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)}, \\ |(\bar{z}, q)_{L^2(\Omega)}| &\leq \alpha \|q'\|_{\mathcal{M}(\Omega)} \quad \forall q \in Q. \end{aligned} \quad (6)$$

Inserting  $q = 1$  into (6) supplies  $\bar{\Phi}(1) = \int_{\Omega} \bar{z} ds = 0$ . By the definition of the distributional derivative of BV functions, (6) is equivalent to

$$\begin{aligned} \int_{\Omega} \bar{\Phi} d\bar{q}' &= \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)}, \\ \left| \int_{\Omega} \bar{\Phi} dq' \right| &\leq \alpha \|q'\|_{\mathcal{M}(\Omega)} \quad \forall q \in Q. \end{aligned} \quad (7)$$

For  $x \in \Omega$  let  $q := 1_{(x,1)} \in Q$  be the characteristic function of the interval  $(x, 1)$ . We have  $q' = \delta_x$  and hence (7) yields  $|\bar{\Phi}(x)| \leq \alpha$ .  $\square$

**Structural conclusions.** With the optimality conditions of Theorem 2.3 at hand, we can now derive helpful structural properties that hold without additional assumptions.

**Corollary 1.** *If  $\bar{q}$  is optimal for (P), then there hold*

$$\begin{aligned} \text{supp}(\bar{q}'_+) &\subset \{x \in \Omega: \bar{\Phi}(x) = \alpha\}, \\ \text{supp}(\bar{q}'_-) &\subset \{x \in \Omega: \bar{\Phi}(x) = -\alpha\}, \end{aligned}$$

where  $\bar{q}'_+$  and  $\bar{q}'_-$  denote the positive and the negative part of the Jordan decomposition of the measure  $\bar{q}'$ . Moreover, we have

$$\text{supp}(\bar{q}') \subset \{x \in \Omega: |\bar{\Phi}(x)| = \alpha\} \cup \{x \in \Omega: \bar{z}(x) = 0\}. \quad (8)$$

*Proof.* Let  $\hat{x} \in \Omega$  with  $\bar{\Phi}(\hat{x}) < \alpha$ . By the continuity of  $\bar{\Phi}$  there is an open neighborhood  $U \subset \Omega$  of  $\hat{x}$  and  $\delta > 0$  such that  $\bar{\Phi} \leq \alpha - \delta$  on  $U$ . Then we have

$$\begin{aligned} \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} &= \int_{\Omega} \bar{\Phi} d\bar{q}'_+ - \int_{\Omega} \bar{\Phi} d\bar{q}'_- \leq \int_{\Omega \setminus U} \alpha d\bar{q}'_+ + \int_U (\alpha - \delta) d\bar{q}'_+ + \int_{\Omega} \alpha d\bar{q}'_- \\ &= \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} - \delta \bar{q}'_+(U). \end{aligned}$$

Thus  $\bar{q}'_+(U) = 0$  and  $\hat{x} \notin \text{supp}(\bar{q}'_+)$ . The claim for  $\bar{q}'_-$  follows analogously. The first inclusion in (8) follows from

$$\text{supp}(\bar{q}') = \text{supp}(\bar{q}'_+) \cup \text{supp}(\bar{q}'_-) \subset \{x \in \Omega: |\bar{\Phi}(x)| = \alpha\}.$$

Theorem 2.3 implies that every  $x$  with  $|\bar{\Phi}(x)| = \alpha$  is either a global maximum or minimum of the  $C^1$  function  $\bar{\Phi}$  and hence satisfies  $0 = \bar{\Phi}'(x) = \bar{z}(x)$ , establishing the second inclusion in (8).  $\square$

**3. Finite element discretization.** For the discretization of (P) we divide  $\bar{\Omega} = [0, 1]$  into  $1 < l$  subintervals  $T_i = (x_{i-1}, x_i)$  of size  $h_i$  defined by the spatial nodes

$$0 = x_0 < x_1 < \dots < x_l = 1, \quad \mathcal{N}_h := \{x_0, x_1, \dots, x_l\}.$$

We obtain  $\Omega = \bigcup_{1 \leq i \leq l} T_i$  and set  $\mathcal{T}_h := \bigcup_{1 \leq i \leq l} \{T_i\}$ , where  $h := \max_{1 \leq i \leq l} h_i$  denotes the mesh width.

**3.1. Discretization of the state equation.** To discretize the state equation we use linear finite elements, i.e., the discrete state space  $V_h$  is given by

$$V_h := \{v_h \in V \cap C_0(\bar{\Omega}) : v_h|_T \text{ is linear for all } T \in \mathcal{T}_h\},$$

where  $C_0(\bar{\Omega})$  denotes the continuous functions on  $\bar{\Omega}$  that vanish on  $\partial\Omega$ .

For further reference we recall that the *Ritz projection* associated to the bilinear form  $\mathbf{a}$ , denoted  $R_h: V \rightarrow V_h$ , satisfies

$$\mathbf{a}(R_h v, w_h) = \mathbf{a}(v, w_h) \quad \forall w_h \in V_h.$$

It is well known that for each  $v \in V$  this variational equality has a unique solution. Moreover, the discrete solution operator is denoted by  $S_h: V^* \rightarrow V_h$  and satisfies, with  $u_h := S_h v$ ,

$$\mathbf{a}(u_h, w_h) = (v, w_h)_{L^2(\Omega)} \quad \forall w_h \in V_h.$$

Since these identities, in fact, uniquely determine  $R_h$  and  $S_h$ , it follows that  $S_h = R_h S$  on  $V^*$ .

Concerning the approximation quality of  $S_h$  we cite the following well-known results.

**Lemma 3.1.** *There exist  $C > 0$  and  $h_0 > 0$  such that for every  $h \in (0, h_0]$  and all  $v \in L^2(\Omega)$  there hold*

$$\|Sv - S_h v\|_{L^2(\Omega)} \leq Ch^2 \|v\|_{L^2(\Omega)} \quad \text{and} \quad \|Sv - S_h v\|_V \leq Ch \|v\|_{L^2(\Omega)}.$$

*Proof.* Cf., e.g., [27, Section 3.2].  $\square$

**Lemma 3.2.** *There exist  $C > 0$  and  $h_0 > 0$  such that for every  $h \in (0, h_0]$  and all  $v \in L^\infty(\Omega)$  there holds*

$$\|Sv - S_h v\|_{L^\infty(\Omega)} \leq Ch^2 \|v\|_{L^\infty(\Omega)}.$$

*Proof.* This is the main theorem of [40], keeping the regularity from Lemma 2.1 in mind.  $\square$

The next lemma shows that  $S_h$  is stable from  $L^2(\Omega)$  to  $W^{1,\infty}(\Omega)$ .

**Lemma 3.3.** *There exist  $C > 0$  and  $h_0 > 0$  such that for every  $h \in (0, h_0]$  and all  $v \in L^2(\Omega)$  there holds*

$$\|S_h v\|_{W^{1,\infty}(\Omega)} \leq C \|Sv\|_{W^{1,\infty}(\Omega)} \leq C \|v\|_{L^2(\Omega)}.$$

*Proof.* This is a consequence of the stability result from [9, Thm. 8.1.11], the embedding  $H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$ , and Lemma 2.1.  $\square$

**Lemma 3.4.** *Let  $w \in H^2(\Omega) \cap V$  and  $R_h w$  its Ritz projection. Then there are  $C, h_0 > 0$  such that for each  $h \in (0, h_0]$  we have*

$$\|(R_h w - w)'\|_{L^\infty(\Omega)} \leq Ch^{\frac{1}{2}} \|w\|_{H^2(\Omega)}.$$

*If  $w \in W^{2,\infty}(\Omega) \cap V$  we even have*

$$\|(R_h w - w)'\|_{L^\infty(\Omega)} \leq Ch \|w\|_{W^{2,\infty}(\Omega)}.$$

*In both cases, the constant  $C > 0$  is independent of  $w$  and  $h$ .*

*Proof.* Lemma 3.3 implies that the Ritz projection is stable in  $W^{1,\infty}(\Omega)$  and thus

$$\begin{aligned} \|R_h w - w\|_{W^{1,\infty}(\Omega)} &\leq \|R_h(w - I_h w)\|_{W^{1,\infty}(\Omega)} + \|I_h w - w\|_{W^{1,\infty}(\Omega)} \\ &\leq C \|I_h w - w\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

Here,  $I_h w$  is the usual nodal interpolant of  $w$ . The two estimates now follow from [9, Thm. 4.4.20].  $\square$

**3.2. Variational control discretization.** In this section we discuss the variational discretization of problem (P), in which the controls are not discretized explicitly. We show that the resulting semi-discrete problem admits a unique solution, characterize this solution by means of optimality conditions, and draw conclusions from the optimality conditions regarding the structure of the optimal solution.

The variationally discretized version of (P) is given by

$$\begin{aligned} \min_{(u_h, q) \in V_h \times Q} & \frac{1}{2} \|u_h - u_d\|_{L^2(\Omega)}^2 + \alpha \|q'\|_{\mathcal{M}(\Omega)} \\ \text{s.t.} & \quad \mathbf{a}(u_h, w_h) = (q, w_h)_{L^2(\Omega)} \quad \forall w_h \in V_h. \end{aligned}$$

Defining  $j_h: Q \rightarrow \mathbb{R}$  by  $j_h(q) := J(S_h(q), q)$ , its reduced formulation reads

$$\min_{q \in Q} j_h(q). \quad (\text{P}_{\text{vd}})$$

Theorem 2.2 has the following discrete counterpart.

**Theorem 3.5.** *Problem (P<sub>vd</sub>) admits a unique optimal control  $\bar{q}_h \in Q$  with associated optimal state  $\bar{u}_h \in V_h$ . There exist  $C > 0$  and  $h_0 > 0$  such that for all  $h \in (0, h_0]$  the controls satisfy  $\|\bar{q}_h\|_{BV(\Omega)} \leq C$ .*

*Proof.* The proof of Theorem 2.2 can be used verbatim as there holds  $S_h 1 \neq 0$ . It remains to establish the estimate for the controls. As in the proof of Theorem 2.2 we can derive (4) for  $\bar{q}_h$  instead of  $\bar{q}$ . Passing to the limit in this version of (4) yields

$$\begin{aligned} \|\bar{q}_h\|_{BV(\Omega)} & \leq \frac{(C_{iso} + 1)j_h(0)}{\alpha} \\ & + \|S_h 1\|_{L^2(\Omega)}^{-1} \left[ \frac{C_{iso} C_{emb} j_h(0)}{\alpha} \|S_h\|_{\mathcal{L}(V^*, L^2(\Omega))} + \sqrt{8j_h(0)} \right]. \end{aligned} \quad (9)$$

From Lemma 3.1 we obtain

$$\|S1\|_{L^2(\Omega)} - \|S_h 1\|_{L^2(\Omega)} \leq \|S1 - S_h 1\|_{L^2(\Omega)} \leq Ch^2 \|1\|_{L^2(\Omega)} \leq \frac{1}{2} \|S1\|_{L^2(\Omega)} \quad (10)$$

for  $h$  sufficiently small, thus  $\|S_h 1\|_{L^2(\Omega)} \geq \frac{1}{2} \|S1\|_{L^2(\Omega)}$ . Furthermore,  $S_h = R_h S$  on  $V^*$  and the  $H^1(\Omega)$ -stability of the Ritz projection imply

$$\|S_h\|_{\mathcal{L}(V^*, L^2(\Omega))} \leq C \|S\|_{\mathcal{L}(V^*, V)}. \quad (11)$$

Inequalities (10) and (11) in conjunction with (9) and  $j_h(0) = j(0)$  yield the desired boundedness of  $\|\bar{q}_h\|_{BV(\Omega)}$  independent of  $h$ .  $\square$

We point out that the control space  $Q$  is not discretized, hence the optimal control  $\bar{q}_h$  belongs to  $BV(\Omega)$ . We prefer the notation  $\bar{q}_h$  nonetheless, because the variationally discretized problem depends on  $h$ .

We collect without proof optimality conditions and structural properties analogous to the continuous setting.

**Theorem 3.6.** *The control  $\bar{q}_h \in Q$  with associated state  $\bar{u}_h \in V_h$  is optimal for Problem (P<sub>vd</sub>) if and only if there exists a unique adjoint state  $\bar{z}_h \in V_h$  such that  $(\bar{u}_h, \bar{q}_h, \bar{z}_h)$  and the  $C^1$  function  $\bar{\Phi}_h: [0, 1] \rightarrow \mathbb{R}$ ,  $\bar{\Phi}_h(x) := \int_0^x \bar{z}_h(s) ds$  satisfy  $\bar{\Phi}_h(1) =$*



0 as well as

$$\int_{\Omega} \bar{\Phi}_h d\bar{q}'_h = \alpha \|\bar{q}'_h\|_{\mathcal{M}(\Omega)} \quad \text{and} \quad \|\bar{\Phi}_h\|_{L^\infty(\Omega)} \leq \alpha,$$

$$\begin{aligned} \mathbf{a}(\bar{u}_h, w_h) &= (\bar{q}_h, w_h)_{L^2(\Omega)} & \forall w_h \in V_h, \\ \mathbf{a}(w_h, \bar{z}_h) &= (w_h, \bar{u}_h - u_d)_{L^2(\Omega)} & \forall w_h \in V_h, \end{aligned}$$

and

$$-(\bar{z}_h, q - \bar{q}_h)_{L^2(\Omega)} \leq \alpha \left[ \|q'\|_{\mathcal{M}(\Omega)} - \|\bar{q}'_h\|_{\mathcal{M}(\Omega)} \right] \quad \forall q \in Q.$$

**Corollary 2.** *If  $\bar{q}_h$  is optimal for  $(\mathbf{P}_{\text{vd}})$ , then there hold*

$$\begin{aligned} \text{supp}((\bar{q}'_h)_+) &\subset \{x \in \Omega : \bar{\Phi}_h(x) = \alpha\}, \\ \text{supp}((\bar{q}'_h)_-) &\subset \{x \in \Omega : \bar{\Phi}_h(x) = -\alpha\}, \end{aligned}$$

where  $(\bar{q}'_h)_+$  and  $(\bar{q}'_h)_-$  denote the positive and the negative part of the Jordan decomposition of the measure  $\bar{q}'_h$ . Moreover, we have

$$\text{supp}(\bar{q}'_h) \subset \{x \in \Omega : |\bar{\Phi}_h(x)| = \alpha\} \cup \{x \in \Omega : \bar{z}_h(x) = 0\}.$$

**3.3. Piecewise constant control discretization.** In this section we present a discretization for  $(\mathbf{P})$  in which the controls  $q_h$  are piecewise constant. We denote the space of piecewise constant functions on  $\mathcal{T}_h$  by

$$Q_h := \{q_h \in BV(\Omega) : q_h|_T = \text{const. for all } T \in \mathcal{T}_h\}.$$

Now the discretization of  $(\mathbf{P})$  is given by

$$\begin{aligned} \min_{(u_h, q_h) \in V_h \times Q_h} & \frac{1}{2} \|u_h - u_d\|_{L^2(\Omega)}^2 + \alpha \|q'_h\|_{\mathcal{M}(\Omega)} \\ \text{s.t.} & \quad \mathbf{a}(u_h, w_h) = (q_h, w_h)_{L^2(\Omega)} \quad \forall w_h \in V_h. \end{aligned}$$

With  $j_h(q_h) := J(S_h(q_h), q_h)$  its reduced formulation reads

$$\min_{q_h \in Q_h} j_h(q_h). \tag{P_{cd}}$$

Note that in contrast to  $(\mathbf{P}_{\text{vd}})$  the control  $q_h$  is now discretized and has the form

$$q_h = a_h + \sum_{j=1}^{l-1} c_h^j 1_{(x_j, 1)}, \quad \text{hence} \quad q'_h = \sum_{j=1}^{l-1} c_h^j \delta_{x_j} \tag{12}$$

for some  $a_h, c_h^j \in \mathbb{R}$ ,  $1 \leq j \leq l-1$ .

We now address existence of optimal solutions and optimality conditions for Problem  $(\mathbf{P}_{\text{cd}})$ .

**Theorem 3.7.** *Problem  $(\mathbf{P}_{\text{cd}})$  admits a unique optimal control  $\hat{q}_h \in Q_h$  with associated optimal state  $\hat{u}_h \in V_h$ . There exist  $C > 0$  and  $h_0 > 0$  such that for all  $h \in (0, h_0]$  we have  $\|\hat{q}_h\|_{BV(\Omega)} \leq C$ .*

*Proof.* The proof is the same as for Theorem 3.5.  $\square$

**Theorem 3.8.** *The control  $\hat{q}_h \in Q_h$  with associated state  $\hat{u}_h \in V_h$  is optimal for Problem (P<sub>cd</sub>) if and only if there exists a unique adjoint state  $\hat{z}_h \in V_h$  such that  $(\hat{u}_h, \hat{q}_h, \hat{z}_h)$  and the  $C^1$  function  $\hat{\Phi}_h: [0, 1] \rightarrow \mathbb{R}$ ,  $\hat{\Phi}_h(x) := \int_0^x \hat{z}_h(s) ds$  satisfy  $\hat{\Phi}_h(1) = 0$  as well as*

$$\int_{\Omega} \hat{\Phi}_h d\hat{q}'_h = \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} \quad \text{and} \quad \max_{0 \leq j \leq l} |\hat{\Phi}_h(x_j)| \leq \alpha,$$

$$\begin{aligned} \mathbf{a}(\hat{u}_h, w_h) &= (\hat{q}_h, w_h)_{L^2(\Omega)} & \forall w_h \in V_h, \\ \mathbf{a}(w_h, \hat{z}_h) &= (w_h, \hat{u}_h - u_d)_{L^2(\Omega)} & \forall w_h \in V_h, \end{aligned}$$

and

$$-(\hat{z}_h, q_h - \hat{q}_h)_{L^2(\Omega)} \leq \alpha \left[ \|q'_h\|_{\mathcal{M}(\Omega)} - \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} \right] \quad \forall q_h \in Q_h.$$

*Proof.* As in the proof of Theorem 2.3 the optimality of  $\hat{q}_h \in Q_h$  is equivalent to

$$-\hat{z}_h := -S_h^*(S_h \hat{q}_h - u_d) \in \partial(\alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)}).$$

Also as in the proof of Theorem 2.3, in particular (6), this is equivalent to

$$\begin{aligned} -(\hat{z}_h, \hat{q}_h)_{L^2(\Omega)} &= \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)}, \\ |(\hat{z}_h, q_h)_{L^2(\Omega)}| &\leq \alpha \|q'_h\|_{\mathcal{M}(\Omega)} \quad \forall q_h \in Q_h. \end{aligned} \tag{13}$$

It remains to establish the statements for  $\hat{\Phi}_h$ . Testing with  $q_h := 1 \in Q_h$  in (13) shows  $\int_{\Omega} \hat{z}_h(s) ds = 0$  and thus  $\hat{\Phi}_h(1) = 0$ . Moreover, (13) can be expressed as

$$\begin{aligned} \int_{\Omega} \hat{\Phi}_h d\hat{q}'_h &= \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)}, \\ \left| \int_{\Omega} \hat{\Phi}_h dq'_h \right| &\leq \alpha \|q'_h\|_{\mathcal{M}(\Omega)} \quad \forall q_h \in Q_h. \end{aligned} \tag{14}$$

Because  $1_{(x_j, 1)} \in Q_h$  and  $(1_{(x_j, 1)})' = \delta_{x_j}$  for  $j = 0, 1, \dots, l$ , we infer from the inequality in (14) that

$$|\hat{\Phi}_h(x_j)| = \left| \int_{\Omega} \hat{\Phi}_h d(1_{(x_j, 1)})' \right| \leq \alpha \|\delta_{x_j}\|_{\mathcal{M}(\Omega)} = \alpha. \quad \square$$

**Remark 1.** The information on  $\hat{\Phi}_h$  in Theorem 3.8 concerns only the gridpoints. It is therefore not ensured (and in general not true) that  $\|\hat{\Phi}_h\|_{\infty} \leq \alpha$ .

**Corollary 3.** *If  $\hat{q}_h \in Q_h$  is optimal for (P<sub>cd</sub>), then there holds*

$$\begin{aligned} \text{supp}((\hat{q}'_h)_+) &\subset \{x_j \in \mathcal{N}_h : \hat{\Phi}_h(x_j) = \alpha\}, \\ \text{supp}((\hat{q}'_h)_-) &\subset \{x_j \in \mathcal{N}_h : \hat{\Phi}_h(x_j) = -\alpha\}, \end{aligned}$$

where  $(\hat{q}'_h)_+$  and  $(\hat{q}'_h)_-$  denote the positive and the negative part of the Jordan decomposition of the measure  $\hat{q}'_h$ .

*Proof.* Recall that

$$\hat{q}_h = a_h + \sum_{j=1}^{l-1} c_h^j 1_{(x_j, 1)}, \quad \text{hence} \quad \hat{q}'_h = \sum_{j=1}^{l-1} c_h^j \delta_{x_j}$$

for real numbers  $a_h, c_h^1, c_h^2, \dots, c_h^{l-1}$ . Let  $x_{j^*} \in \text{supp}((\hat{q}'_h)_+)$  for some  $j^* \in \{1, \dots, l-1\}$ . Note that this is equivalent to saying that  $c_h^{j^*} > 0$ . Assume that  $\hat{\Phi}_h(x_{j^*}) < \alpha$ . By (14) we have that

$$\alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} = \int_{\Omega} \hat{\Phi}_h d\hat{q}'_h = \sum_{i=1, i \neq j^*}^{l-1} c_h^i \hat{\Phi}_h(x_i) + c_h^{j^*} \hat{\Phi}_h(x_{j^*}).$$

By Theorem 3.8 we thus find

$$\alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} < \sum_{i=1, i \neq j^*}^{l-1} |c_h^i| \alpha + c_h^{j^*} \alpha = \alpha \sum_{i=1}^{l-1} |c_h^i| = \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)},$$

a contradiction that implies  $\hat{\Phi}_h(x_{j^*}) = \alpha$  and hence the statement for  $\text{supp}((\hat{q}'_h)_+)$ . Analogously, we obtain the assertion for  $\text{supp}((\hat{q}'_h)_-)$ .  $\square$

**Remark 2.** Note that at non-gridpoints,  $|\hat{\Phi}_h|$  may assume larger values than  $\alpha$ . This implies that  $x_{j^*}$  with  $|\hat{\Phi}_h(x_{j^*})| = \alpha$  is not necessarily an extreme point of  $\hat{\Phi}_h$ . It is therefore not ensured that  $\hat{\Phi}'_h(x_{j^*}) = \hat{z}_h(x_{j^*}) = 0$ . This stands in stark contrast to both the continuous and the variationally discretized problems, where every point at which  $|\bar{\Phi}|$ , respectively,  $|\bar{\Phi}_h|$  attains the value  $\alpha$  is necessarily an extreme point and thus a root of  $\bar{z}$ , respectively,  $\bar{z}_h$ . However, if  $|\hat{\Phi}_h(x_{j^*})| = \alpha$  for some  $j^* \in \{1, \dots, l-1\}$ , then Rolle's theorem yields the existence of  $\xi \in (x_{j^*-1}, x_{j^*+1})$  with  $\hat{\Phi}'_h(\xi) = \hat{z}_h(\xi) = 0$ . That is, there is a root of  $\hat{z}_h$  whose distance to  $x_{j^*}$  is no more than  $h$ . This will suffice to prove error estimates of order  $\mathcal{O}(h)$ .

For later use let us define an  $L^2$ -projection operator onto the space of piecewise constant functions and collect useful properties of this operator.

**Definition 3.9.** For  $i = 0, 1, \dots, l-1$  we introduce

$$\Pi_h : BV(\Omega) \rightarrow Q_h, \quad \Pi_h q|_{(x_i, x_{i+1})} := (x_{i+1} - x_i)^{-1} \int_{x_i}^{x_{i+1}} q(s) ds.$$

It is easy to check that for any  $v_h \in Q_h$  and  $q \in BV(\Omega)$  we have

$$(\Pi_h q - q, v_h)_{L^2(\Omega)} = 0. \quad (15)$$

We have the following estimates.

**Lemma 3.10.** For any  $q \in BV(\Omega)$  there hold

- $\|\Pi_h q - q\|_{L^1(\Omega)} \leq h \|q'\|_{\mathcal{M}(\Omega)},$
- $\|(\Pi_h q)'\|_{\mathcal{M}(\Omega)} \leq \|q'\|_{\mathcal{M}(\Omega)},$
- $\|q - \Pi_h q\|_{L^\infty(\Omega)} \leq h \|q'\|_{L^\infty(\Omega)}$  provided  $q \in W^{1,\infty}(\Omega).$

*Proof.* The first two estimates are taken from [14, Proposition 16].

By Rademacher's theorem (e.g. [3, Thm. 2.14])  $q$  is Lipschitz continuous with Lipschitz constant  $\|q'\|_{L^\infty(\Omega)}$ . Thus, a straightforward estimate shows for any  $i = 0, 1, \dots, l-1$  and  $x \in (x_i, x_{i+1})$

$$q(x) - \frac{1}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} q(s) ds \leq \frac{\|q'\|_{L^\infty(\Omega)}}{x_{i+1} - x_i} \int_{x_i}^{x_{i+1}} |x-s| ds \leq (x_{i+1} - x_i) \|q'\|_{L^\infty(\Omega)}.$$

The definition of  $h$  yields the desired last inequality.  $\square$

## 4. Finite element error estimates.

### 4.1. Error estimates for variational control discretization.

4.1.1. *Basic error estimates for state and adjoint state.* We begin this section by proving a priori estimates for the errors in the optimal state and the adjoint state.

**Lemma 4.1.** *There exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 \leq C(h^4 - (R_h\bar{z} - \bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)}).$$

*Proof.* The optimality conditions for  $\bar{q}$  and  $\bar{q}_h$  from Theorems 2.3 and 3.6 provide

$$\begin{aligned} -(\bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} &\leq \alpha \|\bar{q}'_h\|_{\mathcal{M}(\Omega)} - \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)}, \\ (\bar{z}_h, \bar{q}_h - \bar{q})_{L^2(\Omega)} &\leq \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} - \alpha \|\bar{q}'_h\|_{\mathcal{M}(\Omega)}. \end{aligned}$$

Adding these two inequalities and inserting  $R_h\bar{z}$  yields

$$(\bar{z}_h - R_h\bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} + (R_h\bar{z} - \bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} = (\bar{z}_h - \bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} \leq 0. \quad (16)$$

We can rearrange the first term by first using the state equations, cf. Theorems 2.3 and 3.6, and then using the definition of the Ritz projection. This demonstrates

$$(\bar{z}_h - R_h\bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} = \mathbf{a}(\bar{u}_h - \bar{u}, \bar{z}_h - R_h\bar{z}) = \mathbf{a}(\bar{u}_h - R_h\bar{u}, \bar{z}_h - \bar{z}).$$

Invoking the definition of the adjoint equations, cf. Theorems 2.3 and 3.6, and  $R_hS = S_h$  this reads

$$\begin{aligned} (\bar{z}_h - R_h\bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} &= (\bar{u}_h - \bar{u}, \bar{u}_h - R_h\bar{u})_{L^2(\Omega)} \\ &= \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 - (\bar{u}_h - \bar{u}, \bar{u} - R_h\bar{u})_{L^2(\Omega)}. \end{aligned}$$

Inserting this into (16) yields

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 \leq -(R_h\bar{z} - \bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} + (\bar{u}_h - \bar{u}, \bar{u} - R_h\bar{u})_{L^2(\Omega)}.$$

Hölder's inequality and Young's inequality supply

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 \leq -(R_h\bar{z} - \bar{z}, \bar{q}_h - \bar{q})_{L^2(\Omega)} + \frac{1}{2}\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 + \frac{1}{2}\|\bar{u} - R_h\bar{u}\|_{L^2(\Omega)}^2.$$

Recalling that  $\|\bar{u} - R_h\bar{u}\|_{L^2(\Omega)}^2 = \|S\bar{q} - S_h\bar{q}\|_{L^2(\Omega)}^2 \leq Ch^4$  by Lemma 3.1, the assertion follows after subtraction of  $\frac{1}{2}\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2$ .  $\square$

**Lemma 4.2.** *There exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \leq Ch.$$

*Proof.* By Lemma 4.1 we have that

$$\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}^2 \leq C(h^4 + \|R_h\bar{z} - \bar{z}\|_{L^2(\Omega)}\|\bar{q}_h - \bar{q}\|_{L^2(\Omega)}).$$

By Lemma 3.1 and Theorem 3.7 the first term is of order  $Ch^2$ . Taking the root yields the desired estimate.  $\square$

We readily deduce an error estimate for the adjoint state.

**Lemma 4.3.** *There exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{z}_h - \bar{z}\|_{W^{1,\infty}(\Omega)} \leq Ch.$$

*Proof.* We have

$$\begin{aligned} \|\bar{z}_h - \bar{z}\|_{W^{1,\infty}(\Omega)} &= \|S_h^*(\bar{u}_h - u_d) - S^*(\bar{u} - u_d)\|_{W^{1,\infty}(\Omega)} \\ &\leq \|S_h^*(\bar{u}_h - u_d) - S^*(\bar{u}_h - u_d)\|_{W^{1,\infty}(\Omega)} + \|S^*(\bar{u}_h - \bar{u})\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

Lemma 3.2 together with Lemma 3.4 and Lemma 2.1 show that

$$\|S_h^*(\bar{u}_h - u_d) - S^*(\bar{u}_h - u_d)\|_{W^{1,\infty}(\Omega)} \leq C(h^2 + h)\|\bar{u}_h - u_d\|_{L^\infty(\Omega)} \leq Ch,$$

where we used that  $u_d \in L^\infty(\Omega)$  and that  $\|\bar{u}_h\|_{L^\infty(\Omega)} \leq C$ , the latter being a consequence of Lemma 3.2. Moreover, by means of  $H^2(\Omega) \hookrightarrow W^{1,\infty}(\Omega)$  and Lemma 2.1 we obtain

$$\|S^*(\bar{u}_h - \bar{u})\|_{W^{1,\infty}(\Omega)} \leq C\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \leq Ch,$$

where the last inequality is due to Lemma 4.2.  $\square$

**4.1.2. Improved error estimates under structural assumptions.** We now improve the  $L^2(\Omega)$  convergence order for the state to  $\mathcal{O}(h^2)$  and deduce from this that the controls have  $L^1(\Omega)$  convergence order  $\mathcal{O}(h^2)$ , and that the adjoint state has  $L^\infty(\Omega)$  convergence order  $\mathcal{O}(h^2)$ . To achieve this, we work with a structural assumption: We consider situations where the continuous optimal control admits finitely many jumps. More precisely, we assume that the number of minima and maxima of the function  $\bar{\Phi}$  is finite. This number bounds the number of jumps of the optimal control. Since these maxima and minima are in fact roots of the continuous adjoint state, regularity and convergence results for the discrete adjoint state allow to prove that the discrete problem admits a similar structure. In the following we will frequently use the regularity  $\bar{z} \in W^{2,\infty}(\Omega)$  from Theorem 2.3.

The essential structural assumption reads as follows.

**Assumption 4.4.** Suppose that

$$\{x \in \Omega: |\bar{\Phi}(x)| = \alpha\}$$

is finite. The elements of this set are denoted by  $\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m$ , i.e.,

$$\{x \in \Omega: |\bar{\Phi}(x)| = \alpha\} = \{\bar{x}^1, \bar{x}^2, \dots, \bar{x}^m\},$$

with  $m = 0$  indicating that these sets are empty.

To interpret this assumption recall from Corollary 1 that

$$\text{supp}(\bar{q}') \subset \{x \in \Omega: |\bar{\Phi}(x)| = \alpha\},$$

hence  $\text{supp}(\bar{q}')$  is also finite. Thus, there exist real numbers  $\bar{a}$  and  $\bar{c}^i$ ,  $1 \leq i \leq m$ , such that

$$\bar{q} = \bar{a} + \sum_{i=1}^m \bar{c}^i 1_{(\bar{x}^i, 1)}, \quad \bar{q}' = \sum_{i=1}^m \bar{c}^i \delta_{\bar{x}^i}, \quad (17)$$

where some of the coefficients may be zero. In addition, (8) yields  $\bar{z}(\bar{x}^i) = 0$ ,  $1 \leq i \leq m$ , i.e., the  $\bar{x}^i$  are roots of the continuous adjoint state. Under a mild additional assumption it is possible to prove that the discrete adjoint state  $\bar{z}_h$  admits roots  $\bar{x}_h^i$  close to the  $\bar{x}^i$ . Specifically, the distance  $|\bar{x}^i - \bar{x}_h^i|$  is of order  $\mathcal{O}(h)$ .

The additional assumption reads as follows.

**Assumption 4.5.** Let Assumption 4.4 be fulfilled and suppose  $\bar{z}'(\bar{x}^i) \neq 0$  for  $i = 1, 2, \dots, m$ .

We point out that Assumption 4.5 is equivalent to the existence of numbers  $\kappa > 0$  and  $R > 0$  such that  $|\bar{\Phi}(x)| \leq \alpha - \kappa|x - \bar{x}^i|^2$  for all  $x \in B_R(\bar{x}^i)$ ,  $1 \leq i \leq m$ . That is, Assumption 4.5 imposes a *quadratic growth condition* on  $\bar{\Phi}$  near its extreme points  $\bar{x}^i$ . Also note that the discrete counterparts  $\bar{\Phi}_h$  and  $\hat{\Phi}_h$  of  $\bar{\Phi}$  are piecewise quadratic functions.

Let us now prove the existence of unique roots of the discrete adjoint state in small neighborhoods of the points  $\bar{x}^i$ .

**Lemma 4.6.** *If Assumption 4.5 is fulfilled, then there exist  $R, \delta, h_0 > 0$  such that the following holds for all  $h \in (0, h_0]$  and all  $i = 1, 2, \dots, m$ .  $|\bar{z}'| \geq \delta$  on  $B_R(\bar{x}^i)$  and  $\bar{z}_h$  has a unique root  $\bar{x}_h^i$  in  $B_R(\bar{x}^i)$ . In addition, there hold  $B_R(\bar{x}^i) \cap \partial\Omega = \emptyset$ , the  $B_R(\bar{x}^i)$  are pairwise disjoint, and the roots  $\bar{x}_h^i$  satisfy  $|\bar{x}^i - \bar{x}_h^i| \leq Ch$  for a constant  $C > 0$  that does not depend on  $h$ .*

*Proof.* We first note that  $\bar{x}^i \in \Omega$  is satisfied for  $i = 1, 2, \dots, m$  since  $\bar{\Phi}(x) = 0$  for  $x \in \partial\Omega$ , whereas  $|\bar{\Phi}(\bar{x}^i)| = \alpha > 0$  for  $i = 1, 2, \dots, m$ . Hence, we can assume without loss of generality that  $R > 0$  is chosen so small that  $B_R(\bar{x}^i) \subset \Omega$  for  $i = 1, 2, \dots, m$ . Moreover, we can choose  $R > 0$  so small that all  $B_R(\bar{x}^i)$  are pairwise disjoint. Thus, it is sufficient to argue for one  $i \in \{1, 2, \dots, m\}$ . We write  $\bar{x} := \bar{x}^i$  for this  $i$ .

Since  $\bar{z} \in H^2(\Omega)$ , we have  $\bar{z}' \in C(\bar{\Omega})$ . Thus, Assumption 4.5 implies the existence of  $R > 0$  and  $\delta > 0$  such that  $\bar{x}$  is the only solution of  $\bar{z}(x) = 0$  in  $B_R(\bar{x})$  and such that  $|\bar{z}'(x)| \geq \delta > 0$  for all  $x \in B_R(\bar{x})$ . Since  $\bar{z}'$  is continuous, this inequality implies that  $\bar{z}'$  does not change sign in  $B_R(\bar{x})$ , hence  $\bar{z}$  is strictly monotone in  $B_R(\bar{x})$ . In view of Lemma 4.3 we can also achieve that  $\bar{z}'_h$  has for all sufficiently small  $h$  the same sign as  $\bar{z}'$  a.e. in  $B_R(\bar{x})$ . Hence,  $\bar{z}'_h$  is either positive or negative almost everywhere in  $B_R(\bar{x})$ .

Evidently, the strict monotonicity of  $\bar{z}$  implies that  $\bar{z}$  assumes both negative and positive values in  $B_R(\bar{x})$ . Fix  $x_-, x_+ \in B_R(\bar{x})$  with  $\bar{z}(x_-) < 0$  and  $\bar{z}(x_+) > 0$ . Using Lemma 4.3 it follows that for  $h_0 > 0$  sufficiently small  $\bar{z}_h(x_-) < 0$  and  $\bar{z}_h(x_+) > 0$  for all  $h \in (0, h_0]$ . Thus, the intermediate value theorem implies for every  $h \in (0, h_0]$  the existence of  $\bar{x}_h \in (x_-, x_+)$  with  $\bar{z}_h(\bar{x}_h) = 0$  as claimed.

Suppose that there were an additional root  $\hat{x}_h$  of  $\bar{z}_h$  in  $B_R(\bar{x})$ . Then, by the fundamental theorem of calculus for Sobolev functions, we obtain  $0 = \bar{z}_h(\hat{x}_h) - \bar{z}_h(\bar{x}_h) = \int_{\bar{x}_h}^{\hat{x}_h} \bar{z}'_h(x) dx$ . However, since  $\bar{z}'_h$  is either positive or negative almost everywhere in  $B_R(\bar{x})$ , this cannot be true. Hence,  $\bar{x}_h$  is indeed the only root of  $\bar{z}_h$  in  $B_R(\bar{x})$ .

It remains to establish the estimate  $|\bar{x} - \bar{x}_h| \leq Ch$ . Using  $0 = \bar{z}(\bar{x}) = \bar{z}_h(\bar{x}_h)$  and the mean value theorem yields

$$\bar{z}'(\xi)(\bar{x} - \bar{x}_h) = \bar{z}(\bar{x}) - \bar{z}(\bar{x}_h) = \bar{z}_h(\bar{x}_h) - \bar{z}(\bar{x}_h)$$

for a  $\xi \in B_R(\bar{x})$ . Taking absolute values and using  $1/|\bar{z}'(\xi)| \leq 1/\delta$  this implies  $|\bar{x} - \bar{x}_h| \leq |\bar{z}_h(\bar{x}_h) - \bar{z}(\bar{x}_h)|/\delta \leq Ch/\delta$ , where we applied Lemma 4.3 again.  $\square$

In the next lemma we conclude that in the neighborhoods  $B_R(\bar{x}^i)$  only the  $\bar{x}_h^i$  can satisfy  $|\bar{\Phi}_h(x)| = \alpha$  and that there cannot be any points outside these neighborhoods where  $|\bar{\Phi}_h(x)| = \alpha$  holds.

**Lemma 4.7.** *Suppose that Assumption 4.5 is valid and let  $R > 0$  and  $\bar{x}_h^i$ ,  $1 \leq i \leq m$ , be as in Lemma 4.6. Then there is  $h_0 > 0$  such that for all  $h \in (0, h_0]$  and all  $x \in \bar{\Omega}$  we have*

$$|\bar{\Phi}_h(x)| = \alpha \implies x = \bar{x}_h^i \text{ for some } i \in \{1, 2, \dots, m\}.$$

*Proof.* Let  $h_0 > 0$  be from Lemma 4.6 and let  $h \in (0, h_0]$  and  $x \in \bar{\Omega}$  be such that  $|\bar{\Phi}_h(x)| = \alpha$ . From Corollary 2 we know that  $\bar{z}_h(x) = 0$ . We distinguish two cases.

**Case 1:**  $x \in B_R(\bar{x}^i)$  for some  $i \in \{1, 2, \dots, m\}$

In this case the claim follows from Lemma 4.6.

**Case 2:**  $x \in \bar{\Omega} \setminus \bigcup_{i=1}^m B_R(\bar{x}^i)$

It is sufficient to show that in this case,  $|\bar{\Phi}_h(x)| = \alpha$  cannot be satisfied. To this end, we will demonstrate that there is  $\epsilon > 0$  such that  $|\bar{\Phi}(x)| \leq \alpha - \epsilon$  for all

$x \in \bar{\Omega} \setminus \bigcup_{i=1}^m B_R(\bar{x}^i)$ . Granted this claim, we infer from the definitions of  $\bar{\Phi}$  and  $\bar{\Phi}_h$  together with Lemma 4.3 and  $|\Omega| = 1$  that

$$\|\bar{\Phi} - \bar{\Phi}_h\|_{L^\infty(\Omega)} \leq \|\bar{z} - \bar{z}_h\|_{L^1(\Omega)} \leq \|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \leq Ch.$$

Thus we obtain, for  $h$  sufficiently small, that  $|\bar{\Phi}_h(x)| \leq \alpha - \frac{\epsilon}{2}$  for all  $x \in \bar{\Omega} \setminus \bigcup_{i=1}^m B_R(\bar{x}^i)$  proving that  $|\bar{\Phi}_h(x)| \neq \alpha$  for all these  $x$ , as desired.

To establish the existence of said  $\epsilon$ , note that  $|\bar{\Phi}|$  is continuous on the compact set  $\bar{\Omega} \setminus \bigcup_{i=1}^m B_R(\bar{x}^i)$ . Hence, it attains a maximum on this set, and from Assumption 4.4 and  $\|\bar{\Phi}\|_{L^\infty(\Omega)} \leq \alpha$ , cf. Theorem 2.3, it is evident that this maximum is smaller than  $\alpha$ , which shows that the desired  $\epsilon$  exists, thereby concluding the proof.  $\square$

Lemmas 4.6 and 4.7 guarantee the existence of  $m$  well-defined pairs  $(\bar{x}^i, \bar{x}_h^i)$  that are roots of the continuous and discrete adjoint state, respectively. By Corollary 1 and Corollary 2 we have

$$\begin{aligned} \text{supp}(\bar{q}') &\subset \{x \in \Omega: |\bar{\Phi}(x)| = \alpha\} \subset \{x \in \Omega: \bar{z}(x) = 0\}, \\ \text{supp}(\bar{q}'_h) &\subset \{x \in \Omega: |\bar{\Phi}_h(x)| = \alpha\} \subset \{x \in \Omega: \bar{z}_h(x) = 0\}. \end{aligned}$$

Therefore, Lemmas 4.6 and 4.7 together with Assumption 4.4 imply that the number of points of the support of  $\bar{q}'$  and  $\bar{q}'_h$  are both bounded by  $m$ . Using Lemma 4.6 we observe for the cardinality of the involved sets that

$$\#\left\{x \in \bigcup_{i=1}^m B_R(\bar{x}^i): \bar{z}_h(x) = 0\right\} = \#\left\{x \in \bigcup_{i=1}^m B_R(\bar{x}^i): \bar{z}(x) = 0\right\} = m.$$

Yet, by virtue of Lemma 4.7 this implies

$$\#\{x \in \Omega: |\bar{\Phi}_h(x)| = \alpha\} \leq \#\{x \in \Omega: |\bar{\Phi}(x)| = \alpha\} = m,$$

but it can happen, at least for large  $h$ , that

$$\#\text{supp}(\bar{q}') < \#\text{supp}(\bar{q}'_h).$$

Since we know from Corollary 2 and Lemma 4.7 that

$$\text{supp}(\bar{q}'_h) \subset \{x \in \Omega: |\bar{\Phi}_h(x)| = \alpha\} \quad \text{and} \quad \#\{x \in \Omega: |\bar{\Phi}_h(x)| = \alpha\} \leq m,$$

we find the following discrete analogue to the continuous representation (17): There exist real numbers  $\bar{a}_h$  and  $\bar{c}_h^i$ ,  $1 \leq i \leq m$ , such that

$$\bar{q}_h = \bar{a}_h + \sum_{i=1}^m \bar{c}_h^i 1_{(\bar{x}_h^i, 1)}, \quad \bar{q}'_h = \sum_{i=1}^m \bar{c}_h^i \delta_{\bar{x}_h^i}. \quad (18)$$

Note that some of the coefficients may be zero. In addition, we recall that  $\bar{z}_h(\bar{x}_h^i) = 0$  for  $i = 1, \dots, m$  by definition, cf. Lemma 4.6.

Next we estimate the difference between the jump heights of the optimal control  $\bar{q}$  and its counterpart  $\bar{q}_h$ .

**Lemma 4.8.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  the optimal controls  $\bar{q} = \bar{a} + \sum_{i=1}^m \bar{c}^i 1_{(\bar{x}^i, 1)}$  of (P) and  $\bar{q}_h = \bar{a}_h + \sum_{i=1}^m \bar{c}_h^i 1_{(\bar{x}_h^i, 1)}$  of (P<sub>vd</sub>) satisfy*

$$\sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| \leq C (h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}).$$

*Proof.* Let  $R, h_0 > 0$  be the quantities from Lemma 4.6. Then for all  $h \in (0, h_0]$  the balls  $B_{\frac{3}{4}R}(\bar{x}^i)$  are contained in  $\Omega$  and are pairwise disjoint for  $i = 1, \dots, m$ . For any  $1 \leq i \leq m$  we can thus choose a function  $g \in C_c^\infty(\Omega)$  such that  $g = 1$  on  $B_{\frac{R}{2}}(\bar{x}^i)$  and  $g = 0$  on  $\bar{\Omega} \setminus B_{\frac{3}{4}R}(\bar{x}^i)$ . For  $h$  small enough we have  $\bar{x}_h^i \in B_{\frac{R}{2}}(\bar{x}^i)$  for all  $i \in \{1, 2, \dots, m\}$  by Lemma 4.6.

Using the structure of the optimal controls, the definition of the distributional derivative, and the definition of the state equation, we infer for all  $h \in (0, h_0]$  that

$$\begin{aligned} |\bar{c}^i - \bar{c}_h^i| &= |\langle \bar{q}' - \bar{q}_h', g \rangle_{\mathcal{M}(\Omega), C(\bar{\Omega})}| = |-(\bar{q} - \bar{q}_h, g')_{L^2(\Omega)}| \\ &\leq \left| (\bar{q} - \bar{q}_h, R_h(g'))_{L^2(\Omega)} \right| + \left| (\bar{q} - \bar{q}_h, g' - R_h(g'))_{L^2(\Omega)} \right|. \end{aligned} \quad (19)$$

For the second term on the right-hand side we observe

$$\left| (\bar{q} - \bar{q}_h, g' - R_h(g'))_{L^2(\Omega)} \right| \leq \left( \|\bar{q}\|_{L^2(\Omega)} + \|\bar{q}_h\|_{L^2(\Omega)} \right) \|g' - R_h(g')\|_{L^2(\Omega)} \leq Ch^2$$

due to Lemma 3.1 and the boundedness of  $\bar{q}_h$  independent of  $h$  (after decreasing  $h_0$  if necessary), cf. Theorem 3.5. Using the state equation for the first term we obtain

$$\begin{aligned} \left| (\bar{q} - \bar{q}_h, R_h(g'))_{L^2(\Omega)} \right| &= \left| \mathbf{a}(\bar{u} - \bar{u}_h, R_h(g')) \right| \\ &\leq \left| \mathbf{a}(\bar{u}, R_h(g')) - g' \right| + \left| \mathbf{a}(\bar{u}_h, g') \right| \leq \left| \mathbf{a}(\bar{u}, R_h(g')) - g' \right| + \left| \mathbf{a}(\bar{u} - \bar{u}_h, g') \right| \\ &= \left| (\bar{q}, R_h(g'))_{L^2(\Omega)} - g' \right| + \left| \int_{\Omega} (\bar{u} - \bar{u}_h)((ag'')' + d_0g') dx \right| \\ &\leq Ch^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)} \|(ag'')' + d_0g'\|_{L^2(\Omega)}, \end{aligned}$$

where the second inequality is obtained by virtue of Lemma 3.1 and integration by parts. Inserting the two obtained estimates into (19) yields the assertion after summation.  $\square$

From the previous lemma we derive an estimate for the difference between the offsets and the jump positions of  $\bar{q}$  and  $\bar{q}_h$ .

**Lemma 4.9.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  the optimal controls  $\bar{q} = \bar{a} + \sum_{i=1}^m \bar{c}^i 1_{(\bar{x}^i, 1)}$  and  $\bar{q}_h = \bar{a}_h + \sum_{i=1}^m \bar{c}_h^i 1_{(\bar{x}_h^i, 1)}$  satisfy*

$$|\bar{a} - \bar{a}_h| + \sum_{i=1}^m |\bar{x}^i - \bar{x}_h^i| \leq C(h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}).$$

*Proof.* Lemma 4.6 and Corollary 1 imply

$$0 = \bar{z}(\bar{x}^i) - \bar{z}(\bar{x}_h^i) + \bar{z}(\bar{x}_h^i) - \bar{z}_h(\bar{x}_h^i) = \bar{z}'(\xi^i)(\bar{x}^i - \bar{x}_h^i) + \bar{z}(\bar{x}_h^i) - \bar{z}_h(\bar{x}_h^i)$$

for some  $\xi^i$  between  $\bar{x}^i$  and  $\bar{x}_h^i$ . By Lemma 4.6 we also have  $|\bar{z}'| \geq \delta > 0$  in a neighborhood of  $\bar{x}^i$  containing  $\bar{x}_h^i$  for  $i = 1, 2, \dots, m$  for  $h$  sufficiently small. Thus, by Lemmas 3.1, 3.2 and 3.3 we find

$$\begin{aligned} |\bar{x}^i - \bar{x}_h^i| &\leq C\|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \\ &\leq C\|\bar{z} - R_h\bar{z}\|_{L^\infty(\Omega)} + C\|S_h\bar{u} - \bar{z}_h\|_{L^\infty(\Omega)} \\ &\leq C(h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}). \end{aligned} \quad (20)$$



It remains to estimate the difference in the offsets. To this end, we denote  $\mathcal{S} := S^* S$  and  $\mathcal{S}_h := S_h^* S_h$  and observe that

$$\begin{aligned} \bar{z} - \bar{z}_h &= S^* S \left( \bar{a} + \sum_{i=1}^m \bar{c}^i 1_{(\bar{x}^i, 1)} \right) - S_h^* S_h \left( \bar{a}_h + \sum_{i=1}^m \bar{c}_h^i 1_{(\bar{x}_h^i, 1)} \right) - (S^* - S_h^*) u_d \\ &= (\bar{a} - \bar{a}_h) \mathcal{S} 1 + \bar{a}_h (\mathcal{S} - \mathcal{S}_h) 1 + \sum_{i=1}^m (\bar{c}^i - \bar{c}_h^i) \mathcal{S} 1_{(\bar{x}^i, 1)} \\ &\quad + \sum_{i=1}^m \bar{c}_h^i (\mathcal{S} - \mathcal{S}_h) 1_{(\bar{x}^i, 1)} + \sum_{i=1}^m \bar{c}_h^i \mathcal{S}_h (1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)}) - (S^* - S_h^*) u_d. \end{aligned}$$

By Theorems 2.3 and 3.6 the means of  $\bar{z}$  and  $\bar{z}_h$  vanish. Integration hence shows

$$\begin{aligned} 0 &= (\bar{a} - \bar{a}_h) \int_{\Omega} \mathcal{S} 1 \, dx + \bar{a}_h \int_{\Omega} (\mathcal{S} - \mathcal{S}_h) 1 \, dx + \sum_{i=1}^m (\bar{c}^i - \bar{c}_h^i) \int_{\Omega} \mathcal{S} 1_{(\bar{x}^i, 1)} \, dx \\ &\quad + \sum_{i=1}^m \bar{c}_h^i \int_{\Omega} (\mathcal{S} - \mathcal{S}_h) 1_{(\bar{x}^i, 1)} \, dx + \sum_{i=1}^m \bar{c}_h^i \int_{\Omega} \mathcal{S}_h (1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)}) \, dx \\ &\quad - \int_{\Omega} (S^* - S_h^*) u_d \, dx. \end{aligned}$$

As  $S$  is an isomorphism, we have

$$\int_{\Omega} \mathcal{S} 1 \, dx = \int_{\Omega} S^* S 1 \, dx = \|S 1\|_{L^2(\Omega)}^2 \neq 0$$

and therefore

$$\begin{aligned} |\bar{a} - \bar{a}_h| &\leq \|S 1\|_{L^2(\Omega)}^{-2} \left( |\bar{a}_h| \|(\mathcal{S} - \mathcal{S}_h) 1\|_{L^1(\Omega)} + \sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| \|S 1_{(\bar{x}^i, 1)}\|_{L^1(\Omega)} \right. \\ &\quad + \sum_{i=1}^m |\bar{c}_h^i| \|(\mathcal{S} - \mathcal{S}_h) 1_{(\bar{x}^i, 1)}\|_{L^1(\Omega)} + \sum_{i=1}^m |\bar{c}_h^i| \|\mathcal{S}_h (1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)})\|_{L^1(\Omega)} \\ &\quad \left. + \|(S^* - S_h^*) u_d\|_{L^1(\Omega)} \right). \end{aligned} \tag{21}$$

From Lemma 3.1 and  $S = S^*$  we deduce

$$\begin{aligned} \|\mathcal{S} - \mathcal{S}_h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} &\leq \|S^*\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \|S - S_h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \\ &\quad + \|S^* - S_h^*\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \|S_h\|_{\mathcal{L}(L^2(\Omega), L^2(\Omega))} \leq Ch^2. \end{aligned}$$

This, Lemma 3.1 and (21) yield, with  $|\bar{c}_h|_1 := \sum_{i=1}^m |\bar{c}_h^i|$ ,

$$\begin{aligned} |\bar{a} - \bar{a}_h| &\leq Ch^2 (|\bar{a}_h| + |\bar{c}_h|_1 + 1) \\ &\quad + C \sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| + C \sum_{i=1}^m |\bar{c}_h^i| \|\mathcal{S}_h (1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)})\|_{L^1(\Omega)}. \end{aligned}$$

We have that  $\|\mathcal{S}_h\|_{\mathcal{L}(L^1(\Omega), L^1(\Omega))} = \|S_h^* S_h\|_{\mathcal{L}(L^1(\Omega), L^1(\Omega))} \leq C$ , since  $S_h^* = S_h$  and, by standard energy norm estimates,

$$\|S_h v\|_{L^1(\Omega)} \leq \|S_h v\|_{H_0^1(\Omega)} \leq C \|v\|_{H^{-1}(\Omega)} \leq C \|v\|_{L^1(\Omega)}$$

in one space dimension. We can therefore continue the estimate by

$$|\bar{a} - \bar{a}_h| \leq Ch^2 (|\bar{a}_h| + |\bar{c}_h|_1 + 1) + C \sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| + C \sum_{i=1}^m |\bar{c}_h^i| \|1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)}\|_{L^1(\Omega)}. \quad (22)$$

From the definition of  $\bar{q}_h$  we obtain

$$\frac{1}{2} \|\bar{a}_h S1 + \sum_{i=1}^m \bar{c}_h^i S1_{(\bar{x}_h^i, 1)} - u_d\|_{L^2(\Omega)}^2 + \alpha \left\| \sum_{i=1}^m \bar{c}_h^i \delta_{\bar{x}_h^i} \right\|_{\mathcal{M}(\Omega)} = j_h(\bar{q}_h) \leq j_h(0) = j(0).$$

This implies  $|\bar{c}_h|_1 = \left\| \sum_{i=1}^m \bar{c}_h^i \delta_{\bar{x}_h^i} \right\|_{\mathcal{M}(\Omega)} \leq C$  and because of  $S1 \neq 0$  it also yields  $|\bar{a}_h| \leq C$  with constants independent of  $h$ . By Lemma 4.8 we have  $\sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| \leq C(h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)})$ . Obviously, it also holds that  $\|1_{(\bar{x}^i, 1)} - 1_{(\bar{x}_h^i, 1)}\|_{L^1(\Omega)} = |\bar{x}^i - \bar{x}_h^i|$ . Thus, (20) and (22) show

$$|\bar{a} - \bar{a}_h| \leq Ch^2 + C\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}. \quad \square$$

The previous two results have the following consequence.

**Corollary 4.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{q} - \bar{q}_h\|_{L^1(\Omega)} \leq C(h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}).$$

*Proof.* As

$$\|\bar{q}_h - \bar{q}\|_{L^1(\Omega)} \leq |\bar{a}_h - \bar{a}| |\Omega| + \sum_{i=1}^m |\bar{c}_h^i - \bar{c}_i| \|1_{(\bar{x}_h^i, 1)}\|_{L^1(\Omega)} + \sum_{i=1}^m |\bar{c}^i| \|1_{(\bar{x}_h^i, 1)} - 1_{(\bar{x}^i, 1)}\|_{L^1(\Omega)},$$

the result follows from Lemma 4.8 together with Lemma 4.9.  $\square$

In view of Corollary 4 it remains to estimate  $\|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}$ . We are now able to establish convergence order  $h^2$  for the optimal state.

**Theorem 4.10.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)} \leq Ch^2.$$

*Proof.* Combining Lemma 4.1 with Hölder's inequality and Corollary 4 leads to

$$\|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 \leq C \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} (h^2 + \|\bar{u} - \bar{u}_h\|_{L^2(\Omega)}) + Ch^4.$$

By Young's inequality this yields

$$\frac{1}{2} \|\bar{u}_h - \bar{u}\|_{L^2(\Omega)}^2 \leq C \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} h^2 + C \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)}^2 + Ch^4.$$

Since  $\bar{z} \in W^{2, \infty}(\Omega)$ , the error estimate of the Ritz projection from Lemma 3.2 thus implies the assertion.  $\square$

Finally, we obtain convergence of order  $h^2$  also for the optimal control and the optimal adjoint state, but with respect to the  $L^1(\Omega)$ -norm and the  $L^\infty(\Omega)$ -norm, respectively.

**Corollary 5.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have the following estimates of the structural differences of  $\bar{q}$  and  $\bar{q}_h$*

$$\sum_{i=1}^m |\bar{x}^i - \bar{x}_h^i| \leq Ch^2, \quad \sum_{i=1}^m |\bar{c}^i - \bar{c}_h^i| \leq Ch^2 \quad \text{and} \quad |\bar{a} - \bar{a}_h| \leq Ch^2.$$

We also have the error estimates

$$\|\bar{q} - \bar{q}_h\|_{L^1(\Omega)} \leq Ch^2 \quad \text{and} \quad \|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \leq Ch^2.$$

*Proof.* For the first four claims combine Lemma 4.8, Lemma 4.9 and Corollary 4 with Theorem 4.10. For the last claim note that Lemma 3.1 and Lemma 2.1 imply, due to the embedding  $H_0^1(\Omega) \hookrightarrow L^\infty(\Omega)$ ,

$$\begin{aligned} \|\bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} &\leq \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} + \|R_h \bar{z} - \bar{z}_h\|_{L^\infty(\Omega)} \\ &= \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} + \|S_h \bar{u} - S_h \bar{u}_h\|_{L^\infty(\Omega)} \\ &\leq \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} \\ &\quad + C \|(S_h - S)(\bar{u} - \bar{u}_h)\|_{H_0^1(\Omega)} + C \|S(\bar{u} - \bar{u}_h)\|_{H_0^1(\Omega)} \\ &\leq \|\bar{z} - R_h \bar{z}\|_{L^\infty(\Omega)} + Ch^2, \end{aligned}$$

where we have also used Theorem 4.10 to deduce the last inequality. The claim follows by taking into account the Ritz projection error from Lemma 3.2.  $\square$

**4.2. Error estimates for piecewise constant control discretization.** In this section we prove convergence rates for  $(\mathbf{P}_{\text{cd}})$ . Let us stress that we can only expect  $\|\hat{q}_h - \bar{q}\|_{L^1(\Omega)} = \mathcal{O}(h)$  because for  $\bar{q} = 1_{(\bar{x},1)}$ ,  $\bar{x} \in \Omega$ , we have  $\|1_{(x_j,1)} - 1_{(\bar{x},1)}\|_{L^1(\Omega)} = |x_j - \bar{x}| = \mathcal{O}(h)$  for any node  $x_j$ . We will establish precisely this order of convergence and emphasize that the numerical experiments in Sections 5.3 and 5.4 indicate that this order is indeed optimal.

As in the variationally discrete case we begin by establishing an error estimate for the state and the adjoint state that holds without any structural assumption on the optimal controls. In fact, we are not able to improve this further. Still, in a second step we can derive an error estimate for the control relying on the same structural assumptions as in the variationally discretized setting.

4.2.1. *Basic error estimates for state and adjoint equation.*

**Lemma 4.11.** *Let  $h_0 > 0$  be as in Theorem 3.7. For any  $h \in (0, h_0]$  the optimal state  $\hat{u}_h$  associated with the optimal control  $\hat{q}_h$  to  $(\mathbf{P}_{\text{cd}})$  satisfies*

$$\|\hat{u}_h - \bar{u}\|_{L^2(\Omega)} \leq Ch$$

with a constant  $C$  independent of  $h$ .

*Proof.* By Theorem 3.7 we have that for any  $h \in (0, h_0]$  there exists a unique optimal control  $\hat{q}_h$  to  $(\mathbf{P}_{\text{cd}})$  with associated state  $\hat{u}_h$  and adjoint state  $\hat{z}_h$ . We test the variational inequality from Theorem 3.8 with  $q_h = \Pi_h \bar{q} \in Q_h$  and the variational inequality from Theorem 2.3 with  $q = \hat{q}_h$  and obtain

$$\begin{aligned} -(\hat{z}_h, \Pi_h \bar{q} - \hat{q}_h)_{L^2(\Omega)} + \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} &\leq \alpha \|(\Pi_h \bar{q})'\|_{\mathcal{M}(\Omega)}, \\ -(\hat{z}_h, \hat{q}_h - \bar{q})_{L^2(\Omega)} + \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} &\leq \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)}. \end{aligned}$$

Adding those two lines and using Lemma 3.10 we find

$$\begin{aligned} -(\hat{z}_h, \Pi_h \bar{q} - \hat{q}_h)_{L^2(\Omega)} - (\hat{z}_h, \hat{q}_h - \bar{q})_{L^2(\Omega)} + \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} + \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} \\ \leq \alpha \|(\Pi_h \bar{q})'\|_{\mathcal{M}(\Omega)} + \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} \leq \alpha \|\bar{q}'\|_{\mathcal{M}(\Omega)} + \alpha \|\hat{q}'_h\|_{\mathcal{M}(\Omega)}. \end{aligned}$$

Rearranging terms and using (15) leads to

$$(\bar{z} - \hat{z}_h, \bar{q} - \hat{q}_h)_{L^2(\Omega)} \leq (\hat{z}_h, \Pi_h \bar{q} - \bar{q})_{L^2(\Omega)} = (\hat{z}_h - \Pi_h \hat{z}_h, \Pi_h \bar{q} - \bar{q})_{L^2(\Omega)}.$$

By Lemma 3.10 we obtain

$$(\bar{z} - \hat{z}_h, \bar{q} - \hat{q}_h)_{L^2(\Omega)} \leq \|H_h \bar{q} - \bar{q}\|_{L^1(\Omega)} \|H_h \hat{z}_h - \hat{z}_h\|_{L^\infty(\Omega)} \leq h^2 \|\hat{q}'_h\|_{\mathcal{M}(\Omega)} \|\hat{z}'_h\|_{L^\infty(\Omega)}.$$

Using Theorem 3.5, Lemma 3.3 and the boundedness of  $\|\hat{u}_h - u_d\|_{L^2(\Omega)}$ , which is due to Theorem 3.7, we find  $\|\hat{q}'_h\|_{\mathcal{M}(\Omega)}, \|\hat{z}'_h\|_{L^\infty(\Omega)} \leq C$  and thus

$$(\bar{z} - \hat{z}_h, \bar{q} - \hat{q}_h)_{L^2(\Omega)} \leq Ch^2. \quad (23)$$

We introduce the auxiliary state  $\tilde{u}_h := S_h \bar{q}$  and observe with the boundedness results from Theorem 2.2 and Theorem 3.7 together with Lemma 3.1 that

$$\begin{aligned} \|\tilde{u}_h - \hat{u}_h\|_{L^2(\Omega)}^2 &= (S_h(\bar{q} - \hat{q}_h), S_h(\bar{q} - \hat{q}_h))_{L^2(\Omega)} \\ &= (S_h^*(S_h \bar{q} - S_h \hat{q}_h), \bar{q} - \hat{q}_h)_{L^2(\Omega)} \\ &= (S^*(S \bar{q} - u_d), \bar{q} - \hat{q}_h)_{L^2(\Omega)} - (S_h^*(S_h \hat{q}_h - u_d), \bar{q} - \hat{q}_h)_{L^2(\Omega)} \\ &\quad - (S^*(S - S_h)\bar{q}, \bar{q} - \hat{q}_h)_{L^2(\Omega)} - ((S_h^* - S^*)u_d, \bar{q} - \hat{q}_h)_{L^2(\Omega)} \\ &\quad - ((S^* - S_h^*)S_h \bar{q}, \bar{q} - \hat{q}_h)_{L^2(\Omega)} \\ &\leq (\bar{z} - \hat{z}_h, \bar{q} - \hat{q}_h)_{L^2(\Omega)} + Ch^2, \end{aligned}$$

pointing out that due to  $S^* = S$  and  $S_h^* = S_h$  the same finite element discretization error estimates as for the state equation apply to the adjoint states. Combining this with (23) leads to  $\|\tilde{u}_h - \hat{u}_h\|_{L^2(\Omega)} \leq Ch$ . Therefore, the assertion follows from

$$\|\bar{u} - \hat{u}_h\|_{L^2(\Omega)} \leq \|\bar{u} - \tilde{u}_h\|_{L^2(\Omega)} + \|\tilde{u}_h - \hat{u}_h\|_{L^2(\Omega)} \leq Ch,$$

where the first summand is of order  $h^2$  by Lemma 3.1.  $\square$

The preceding lemma has the following consequence.

**Corollary 6.** *Let  $h_0 > 0$  be from Theorem 3.7 and  $h \in (0, h_0]$ . Let  $(\hat{u}_h, \hat{q}_h, \hat{z}_h)$  be the optimal triple of  $(\mathbf{P}_{\text{cd}})$  and  $(\bar{u}, \bar{q}, \bar{z})$  the optimal triple of  $(\mathbf{P})$ . Then there holds*

$$\|\bar{z} - \hat{z}_h\|_{W^{1,\infty}(\Omega)} \leq Ch$$

with a constant  $C > 0$  independent of  $h$ .

*Proof.* The proof is essentially the same as for Lemma 4.3, with Lemma 4.11 replacing Lemma 4.2.  $\square$

**4.2.2. Improved error estimates under structural assumptions.** Similarly as in the variationally discrete setting we will now use the structural Assumptions 4.4 and 4.5 to derive an  $L^1(\Omega)$ -error estimate for the control. We recall that Assumption 4.4 ensures that  $\bar{\Phi}$  has only finitely many minima and maxima, which in turn implies that the optimal control exhibits only finitely many jumps. The main idea underlying the proof of the error estimate is to examine the distance between jump points and jump heights of the continuous and the discrete optimal control. Note that the discrete optimal control  $\hat{q}_h$  is piecewise constant and can only admit jumps at the gridpoints  $x_j$  with  $|\hat{\Phi}_h(x_j)| = \alpha$ . These jumps can only occur close to points where  $|\bar{\Phi}| = \alpha$ , i.e., in the vicinity of the  $\bar{x}^i$ ,  $i = 1, 2, \dots, m$ , as the following result shows.

**Lemma 4.12.** *Suppose that Assumption 4.4 is valid and let  $R > 0$  be as in Lemma 4.6. Then there is  $h_0 > 0$  such that for all  $h \in (0, h_0]$  and all  $x \in \bar{\Omega} \setminus \bigcup_{i=1}^m B_{\frac{R}{2}}(\bar{x}_i)$  we have*

$$|\hat{\Phi}_h(x)| < \alpha.$$

*Proof.* The proof follows along the lines of Case 2 in Lemma 4.7.  $\square$

Next we investigate the behavior of  $\hat{\Phi}_h$  inside the balls  $B_R(\bar{x}^i)$ . Note that if  $|\hat{\Phi}_h| < \alpha$  in  $B_R(\bar{x}^i)$ , then  $\hat{q}_h$  will not admit a jump in  $B_R(\bar{x}^i)$ , hence  $\hat{c}_h^j = 0$  in (12) for all  $j$  with  $x_j \in B_R(\bar{x}^i)$ . We therefore consider points where  $|\hat{\Phi}_h| \geq \alpha$  and remark that points with  $|\hat{\Phi}_h| > \alpha$  can actually exist because  $\hat{\Phi}_h$  is piecewise quadratic.

**Lemma 4.13.** *Let Assumption 4.5 hold and let  $R > 0$  be as in Lemma 4.6. There exists an  $h_0 > 0$  such that the following holds for all  $h \in (0, h_0]$ . If  $|\hat{\Phi}_h(\hat{x})| \geq \alpha$  for some  $\hat{x} \in B_R(\bar{x}^i)$  and some  $i \in \{1, 2, \dots, m\}$ , then  $\hat{z}_h$  has a unique root  $\hat{x}_h^i$  in  $B_R(\bar{x}^i)$  and there holds  $\hat{x}_h^i \in B_{\frac{R}{2}}(\bar{x}^i)$ . Moreover, the point  $\hat{x}_h^i$  is the unique local maximizer of  $|\hat{\Phi}_h|$  in  $B_R(\bar{x}^i)$  and satisfies  $|\hat{\Phi}_h(\hat{x}_h^i)| \geq \alpha$  and  $|\bar{x}^i - \hat{x}_h^i| \leq Ch$  with a constant  $C$  not depending on  $h$ .*

*Proof.* Without loss of generality let us assume that  $h_0 \leq R/2$ . We argue for the case  $\hat{\Phi}_h(\hat{x}) \geq \alpha$  for some  $\hat{x} \in B_R(\bar{x}^i)$  and an  $i \in \{1, 2, \dots, m\}$ . The case  $\hat{\Phi}_h(\hat{x}) \leq -\alpha$  can be handled analogously. Due to  $\hat{\Phi}_h(\hat{x}) \geq \alpha$  we infer from Lemma 4.12 that  $\hat{x} \in B_{\frac{R}{2}}(\bar{x}^i)$ . Since  $h_0 \leq R/2$ , we find gridpoints  $\hat{x}_{l,h}^i$  and  $\hat{x}_{r,h}^i$  that satisfy  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i \in B_R(\bar{x}^i)$  and  $\hat{x}_{l,h}^i < \hat{x} < \hat{x}_{r,h}^i$ . Since  $\hat{\Phi}_h$  satisfies  $|\hat{\Phi}_h(x_j)| \leq \alpha$  for all  $0 \leq j \leq l$ , cf. Theorem 3.8, we have

$$\hat{\Phi}_h(\hat{x}_{l,h}^i), \hat{\Phi}_h(\hat{x}_{r,h}^i) \leq \alpha \leq \hat{\Phi}_h(\hat{x}).$$

Hence, the continuous function  $\hat{\Phi}_h$  attains a local maximum at some  $\hat{x}_h^i \in (\hat{x}_{l,h}^i, \hat{x}_{r,h}^i)$ . Clearly, there hold  $0 = \hat{\Phi}'_h(\hat{x}_h^i) = \hat{z}_h(\hat{x}_h^i)$  and  $\hat{\Phi}_h(\hat{x}_h^i) \geq \hat{\Phi}_h(\hat{x}) \geq \alpha$ , with the latter implying  $\hat{x}_h^i \in B_{\frac{R}{2}}(\bar{x}^i)$  by Lemma 4.12. The uniqueness of the root  $\hat{x}_h^i$  in  $B_R(\bar{x}^i)$  can be established as in the proof of Lemma 4.6. The estimate for  $|\bar{x}^i - \hat{x}_h^i|$  also follows as in the proof of Lemma 4.6.  $\square$

In the gridpoints we have  $|\hat{\Phi}_h| \leq \alpha$ . Next we show that  $|\hat{\Phi}_h(x_j)| = \alpha$  for a gridpoint  $x_j$  can only hold if  $x_j = \hat{x}_h^i$  for some  $i \in \{1, \dots, m\}$  or if  $|\hat{\Phi}_h(\hat{x}_h^i)| > \alpha$  and  $x_j$  is close to  $\hat{x}_h^i$ .

**Corollary 7.** *Let Assumption 4.5 hold and let  $R$  be as in Lemma 4.13. There exists  $h_0 > 0$  such that the following holds for all  $h \in (0, h_0]$ . If  $|\hat{\Phi}_h(\hat{x})| \geq \alpha$  for some  $\hat{x} \in B_R(\bar{x}^i)$  and some  $i \in \{1, 2, \dots, m\}$ , then the point  $\hat{x}_h^i \in B_{\frac{R}{2}}(\bar{x}^i)$  from Lemma 4.13 satisfies exactly one of the following two statements:*

1.  $|\hat{\Phi}_h(\hat{x}_h^i)| = \alpha$  and  $|\hat{\Phi}_h(y)| < \alpha$  for all  $y \in B_R(\bar{x}^i) \setminus \{\hat{x}_h^i\}$ .
2.  $|\hat{\Phi}_h(\hat{x}_h^i)| > \alpha$  and there exist exactly two points  $y_l^i, y_r^i \in B_R(\bar{x}^i)$  such that  $|\hat{\Phi}_h(y_l^i)| = |\hat{\Phi}_h(y_r^i)| = \alpha$  and  $\hat{x}_h^i \in (y_l^i, y_r^i)$ . In addition,  $y_l^i, y_r^i \in [\hat{x}_{l,h}^i, \hat{x}_{r,h}^i]$ , where  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i \in B_R(\bar{x}^i)$  are the gridpoints closest to  $\hat{x}_h^i$  that satisfy  $\hat{x}_{l,h}^i < \hat{x}_h^i < \hat{x}_{r,h}^i$ . Furthermore, there holds  $|\hat{\Phi}_h(y)| < \alpha$  for all  $y \in B_R(\bar{x}^i) \setminus [\hat{x}_{l,h}^i, \hat{x}_{r,h}^i]$ .

Moreover, we have

$$\text{supp}(\hat{q}'_h) \subset \left\{ x \in \mathcal{N}_h : \left| \hat{\Phi}_h(x) \right| = \alpha \right\} \quad \text{and} \quad \# \left\{ x \in \mathcal{N}_h : \left| \hat{\Phi}_h(x) \right| = \alpha \right\} \leq 2m. \quad (24)$$

*Proof.* The first part of (24) is just a restatement of Corollary 3 and the second part of (24) follows from the main statement in combination with Lemma 4.12.

We assume  $\hat{\Phi}_h(\hat{x}) \geq \alpha$ ; the case  $\hat{\Phi}_h(\hat{x}) \leq -\alpha$  is treated analogously.

We first consider statement 1. Assume  $\hat{\Phi}(\hat{x}_h^i) = \alpha$ . Since  $\hat{x}_h^i$  is the unique maximizer of  $\hat{\Phi}$  on  $B_R(\bar{x}^i)$  by Lemma 4.13 the statement readily follows.

To establish 2, let us assume that  $\hat{\Phi}_h(\hat{x}_h^i) > \alpha$ . Let  $x_l, x_r \in B_R(\bar{x}^i)$  be the two gridpoints closest to  $\hat{x}_h^i$  that satisfy  $\hat{x}_{l,h}^i < \hat{x}_h^i < \hat{x}_{r,h}^i$ . The existence of such  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i$  is ensured if  $h_0 < R/2$ . Since  $\hat{\Phi}_h(\hat{x}_{l,h}^i), \hat{\Phi}_h(\hat{x}_{r,h}^i) \leq \alpha$ , the intermediate value theorem implies that there exist  $y_l^i \in [\hat{x}_{l,h}^i, \hat{x}_h^i]$  and  $y_r^i \in (\hat{x}_h^i, \hat{x}_{r,h}^i]$  such that  $\hat{\Phi}_h(y_l^i) = \hat{\Phi}_h(y_r^i) = \alpha$ . In particular, we have  $\hat{x}_h^i \in (y_l^i, y_r^i)$ .

To demonstrate uniqueness of  $y_l^i, y_r^i$  in  $B_R(\bar{x}^i)$ , assume there were an  $x \in B_R(\bar{x}^i) \setminus \{y_l^i, y_r^i\}$  with  $\hat{\Phi}_h(x) = \alpha$ . If  $x < \hat{x}_h^i$ , then Rolle's theorem yields a  $\xi$  between  $x$  and  $y_l^i$  with  $\hat{z}_h(\xi) = 0$ , which contradicts the uniqueness of the root  $\hat{x}_h^i$  of  $\hat{z}_h$  in  $B_R(\bar{x}^i)$  proven in Lemma 4.13. If  $x > \hat{x}_h^i$ , then we readily obtain a similar contradiction. Since  $x = \hat{x}_h^i$  is excluded due to  $\hat{\Phi}_h(\hat{x}_h^i) > \alpha$ , we conclude that  $\hat{\Phi}_h(x) = \alpha$  for  $x \in B_R(\bar{x}^i)$  if and only if  $x \in \{y_l^i, y_r^i\}$ .

To prove that  $|\hat{\Phi}_h(x)| < \alpha$  for all  $x \in B_R(\bar{x}^i) \setminus [\hat{x}_{l,h}^i, \hat{x}_{r,h}^i]$ , we assume without loss of generality that  $h_0 \leq R/4$  so that we are able to find gridpoints  $\tilde{x}_{l,h}^i, \tilde{x}_{r,h}^i \in B_R(\bar{x}^i)$  with  $\tilde{x}_{l,h}^i < \hat{x}_{l,h}^i \leq y_l^i$  and  $y_r^i \leq \hat{x}_{r,h}^i < \tilde{x}_{r,h}^i$ . Because we have established that  $\hat{\Phi}_h(x) = \alpha$  for  $x \in B_R(\bar{x}^i)$  if and only if  $x \in \{y_l^i, y_r^i\}$ , there holds  $\hat{\Phi}_h(\tilde{x}_{l,h}^i), \hat{\Phi}_h(\tilde{x}_{r,h}^i) < \alpha$ . Thus, a continuity argument supplies  $\hat{\Phi}_h(x) < \alpha$  for all  $x \in B_R(\bar{x}^i) \setminus [\hat{y}_l^i, \hat{y}_r^i]$ . The claim follows since  $[\hat{y}_l^i, \hat{y}_r^i] \subset [\hat{x}_{l,h}^i, \hat{x}_{r,h}^i]$ .  $\square$

Summarizing we now know that  $\hat{q}_h$  cannot jump outside of any  $B_R(\bar{x}^i)$ ,  $1 \leq i \leq m$ , and that inside every  $B_R(\bar{x}^i)$  jumps can only occur at  $\hat{x}_h^i$  (Case 1) or at any of the two points  $y_l^i$  and  $y_r^i$  (Case 2),  $1 \leq i \leq m$ . In addition, such a jump can only occur if the respective point is a gridpoint. In contrast, in the variational discrete setting the jumps of  $\bar{q}_h$  are not restricted to gridpoints. For clarification we point out that there might well be situations, for large  $h$ , where the continuous optimal control  $\bar{q}$  jumps at  $\bar{x}^i$ , but the discrete optimal control  $\hat{q}_h$  does not admit a jump in  $B_R(\bar{x}^i)$ . Vice versa, for large  $h$  it may happen that  $\hat{q}_h$  exhibits one or two jumps in  $B_R(\bar{x}^i)$ , but  $\bar{q}$  does not jump in  $B_R(\bar{x}^i)$ .

To obtain a convergence result, we need to estimate the difference in the jump points and the corresponding coefficients. In the remainder of this section we use the following notation. We write  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i$  according to Corollary 7 if the second case of Corollary 7 applies. If the first case of Corollary 7 applies, then  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i$  denote the left and right neighbor of  $\hat{x}_h^i$ , provided that  $\hat{x}_h^i$  itself is not a gridpoint. If it is a gridpoint, then we denote by  $\hat{x}_{l,h}^i$  its left neighbor and set  $\hat{x}_{r,h}^i := \hat{x}_h^i$ . If neither case applies, then we have  $|\hat{\Phi}_h| < \alpha$  in  $B_R(\bar{x}^i)$ . In this case,  $\hat{x}_{l,h}^i, \hat{x}_{r,h}^i$  are taken to be the gridpoints adjacent to each other and satisfying  $\bar{x}^i \in [\hat{x}_{l,h}^i, \hat{x}_{r,h}^i]$ . We observe that  $|\hat{x}_{l,h}^i - \hat{x}_h^i|, |\hat{x}_{r,h}^i - \hat{x}_h^i| \leq h$  whenever  $\hat{x}_h^i$  exists, and  $|\hat{x}_{l,h}^i - \bar{x}^i|, |\hat{x}_{r,h}^i - \bar{x}^i| \leq h$  otherwise. In view of Lemma 4.13 this immediately implies the following result, that does not require a proof.

**Lemma 4.14.** *Let Assumption 4.5 hold. There exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  and  $i = 1, 2, \dots, m$  we have  $|\bar{x}^i - \hat{x}_{l,h}^i|, |\bar{x}^i - \hat{x}_{r,h}^i| \leq Ch$ .*

By virtue of the inclusion in (24) the preceding discussion furthermore shows that  $\hat{q}_h$  can be represented as follows. There exist real numbers  $\hat{a}_h, \hat{c}_{l,h}^i, \hat{c}_{r,h}^i$ ,  $1 \leq i \leq m$ ,

such that

$$\hat{q}_h = \hat{a}_h + \sum_{i=1}^m (\hat{c}_{l,h}^i \mathbf{1}_{(\hat{x}_{l,h}^i, 1)} + \hat{c}_{r,h}^i \mathbf{1}_{(\hat{x}_{r,h}^i, 1)}), \quad \hat{q}'_h = \sum_{i=1}^m (\hat{c}_{l,h}^i \delta_{\hat{x}_{l,h}^i} + \hat{c}_{r,h}^i \delta_{\hat{x}_{r,h}^i}), \quad (25)$$

where some of the coefficients may be zero.

We estimate the difference between the jump heights of the optimal control  $\bar{q}$  and its discrete counterpart  $\hat{q}_h$ .

**Lemma 4.15.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  the optimal controls  $\hat{q}_h = \hat{a}_h + \sum_{i=1}^m (\hat{c}_{l,h}^i \mathbf{1}_{(\hat{x}_{l,h}^i, 1)} + \hat{c}_{r,h}^i \mathbf{1}_{(\hat{x}_{r,h}^i, 1)})$  and  $\bar{q} = \bar{a} + \sum_{i=1}^m \bar{c}^i \mathbf{1}_{(\bar{x}^i, 1)}$  satisfy*

$$\sum_{i=1}^m |\bar{c}^i - (\hat{c}_{l,h}^i + \hat{c}_{r,h}^i)| \leq Ch.$$

*Proof.* The proof of Lemma 4.8 remains valid for  $\hat{q}_h, \hat{u}_h, \hat{z}_h$  and yields

$$\sum_{i=1}^m |\bar{c}^i - (\hat{c}_{l,h}^i + \hat{c}_{r,h}^i)| \leq C (h^2 + \|\bar{u} - \hat{u}_h\|_{L^2(\Omega)}).$$

Applying Lemma 4.11 establishes the desired estimate.  $\square$

The difference between the offsets and the jump positions of  $\bar{q}$  and  $\hat{q}_h$  can be estimated as follows.

**Lemma 4.16.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  the optimal controls  $\hat{q}_h = \hat{a}_h + \sum_{i=1}^m (\hat{c}_{l,h}^i \mathbf{1}_{(\hat{x}_{l,h}^i, 1)} + \hat{c}_{r,h}^i \mathbf{1}_{(\hat{x}_{r,h}^i, 1)})$  and  $\bar{q} = \bar{a} + \sum_{i=1}^m \bar{c}^i \mathbf{1}_{(\bar{x}^i, 1)}$  satisfy*

$$|\bar{a} - \hat{a}_h| + \sum_{i=1}^m (|\bar{x}^i - \hat{x}_{l,h}^i| + |\bar{x}^i - \hat{x}_{r,h}^i|) \leq Ch.$$

*Proof.* In view of Lemma 4.14 it only remains to estimate the difference  $|\bar{a} - \hat{a}_h|$ . This can be accomplished almost verbatim as in Lemma 4.9.  $\square$

We obtain the following error estimate for the control in  $L^1(\Omega)$ .

**Corollary 8.** *Suppose that Assumption 4.5 is valid. Then there exist  $C, h_0 > 0$  such that for all  $h \in (0, h_0]$  we have*

$$\|\bar{q} - \hat{q}_h\|_{L^1(\Omega)} \leq Ch.$$

*Proof.* The desired estimate follows by combining Lemma 4.15 and Lemma 4.16.  $\square$

**5. Numerical Experiments.** In this section we introduce an algorithm to solve the optimization problems  $(P_{vd})$  and  $(P_{cd})$  based on the PDAP method described for example in [33, 39]. Moreover, we discuss the error estimates for both discretization schemes on two numerical examples.

**5.1. Optimization algorithm for variational discretization.** We recall from (18) that there is a number  $m \in \mathbb{N}$  such that the optimal control  $\bar{q}_h$  for  $(\mathbf{P}_{\text{vd}})$  and its derivative can be expressed as  $\bar{q}_h = \bar{a}_h + \sum_{i=1}^m \bar{c}_h^i \mathbf{1}_{(\bar{x}_h^i, 1)}$ , respectively,  $\bar{q}'_h = \sum_{i=1}^m \bar{c}_h^i \delta_{\bar{x}_h^i}$  for suitable coefficients  $\bar{a}_h, \bar{c}_h := (\bar{c}_h^1, \dots, \bar{c}_h^m)^T \in \mathbb{R}^m$  and points  $\bar{x}_h^1, \dots, \bar{x}_h^m \in \Omega = (0, 1)$  that satisfy  $\bar{z}_h(\bar{x}_h^i) = 0$  for  $0 \leq i \leq m$ . Let us assume for a moment that we know  $\{\bar{x}_h^i\}_{i=1}^m$ . We can then determine the coefficients  $\bar{a}_h$  and  $\bar{c}_h$  by solving the finite-dimensional, convex optimization problem

$$\min_{a_h \in \mathbb{R}, c_h \in \mathbb{R}^m} \frac{1}{2} \|u_h - u_d\|_{L^2(\Omega)}^2 + \alpha \sum_{i=1}^m |c_h^i| \quad \text{s.t.} \quad u_h = S_h \left( a_h + \sum_{i=1}^m c_h^i \mathbf{1}_{(\bar{x}_h^i, 1)} \right). \quad (26)$$

Since we do not know  $\{\bar{x}_h^i\}_{i=1}^m$  beforehand, the algorithmic idea is to work with approximations of this set. We start with an approximation  $\{t_{(0)}^i\}_{i=0}^{m_{(0)}}$  that satisfies  $0 < t_{(0)}^0 < t_{(0)}^1 < t_{(0)}^2 < \dots < t_{(0)}^{m_{(0)}} < 1$ . Next we solve (26) using  $\{t_{(0)}^i\}_{i=1}^{m_{(0)}}$  instead of  $\{\bar{x}_h^i\}_{i=1}^m$  by a semi-smooth Newton method, cf. [36]. Note that (26) is a finite-dimensional problem of dimension  $m_{(0)} + 1$ , independently of  $h$ . This yields  $(q_h^{(0)}, u_h^{(0)}, z_h^{(0)})$ . We compute the roots  $\{t_{(1)}^i\}_{i=1}^{m_{(1)}} \subset (0, 1)$  of  $z_h^{(0)}$  and solve (26) using  $\{t_{(1)}^i\}_{i=1}^{m_{(1)}}$  instead of  $\{\bar{x}_h^i\}_{i=1}^m$  to obtain  $(q_h^{(1)}, u_h^{(1)}, z_h^{(1)})$ . This process is iterated. We call the step of the algorithm where the new estimate  $\{t_{(k+1)}^i\}_{i=1}^{m_{(k+1)}}$  of  $\{\bar{x}_h^i\}_{i=1}^m$  is obtained, the *outer* iteration. The *inner* iteration consists of solving (26). The outer iteration and thereby the overall algorithm are terminated if an approximation  $t_{(k)} := (t_{(k)}^1, \dots, t_{(k)}^{m_{(k)}})^T$ ,  $k \geq 1$ , is obtained that satisfies

$$m_{(k)} = m_{(k-1)} \quad \text{and} \quad \|t_{(k)} - t_{(k-1)}\|_2 \leq \epsilon_{\text{out}}, \quad (\text{T})$$

where  $\epsilon_{\text{out}} > 0$  is some small tolerance, e.g.,  $\epsilon_{\text{out}} = 10^{-10}$ .

All in all, these considerations give rise to the following algorithm.

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**Algorithm 1:** Solving the semi-discrete problem  $(\mathbf{P}_{\text{vd}})$

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**Input:**  $m_{(0)} \in \mathbb{N} \cup \{0\}$ ,  $t_{(0)} \in \mathbb{R}^{m_{(0)}}$  and  $\epsilon_{\text{in}}, \epsilon_{\text{out}} > 0$

1 **for**  $k = 0, 1, 2, \dots$  **do** // outer iteration

2     **if** (T) **holds then** let  $m := m_{(k)}$ ,  $\bar{x}_h := t_{(k)}$ , and extract  $(\bar{a}_h, \bar{c}_h)$  from  $q_h^{(k)}$ ;  
       **STOP** // check termination criterion

3     Obtain  $(q_h^{(k)}, u_h^{(k)}, z_h^{(k)})$  by solving (26) to tolerance  $\epsilon_{\text{in}}$  // inner iter.

4     Compute the roots  $t_{(k+1)} \in \mathbb{R}^{m_{(k+1)}}$  of  $z_h^{(k)}$  // next approximation

5 **end**

**Output:**  $\bar{x}_h \in \mathbb{R}^m$ ,  $(\bar{a}_h, \bar{c}_h) \in \mathbb{R}^{m+1}$

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While it is theoretically possible that the inner iteration does not converge, we did not observe divergence in the numerical experiments that we carried out. However, we did sometimes observe cycling of the outer iteration, e.g.,  $t_{(2k+2)} = t_{(2k)}$  and  $t_{(2k+3)} = t_{(2k+1)}$  for all  $k$  sufficiently large. Since this did only occur for iterates with an equal number of roots of the adjoint state, the following modification of line 4 was possible and turned out to be sufficient: Compute the roots  $t_{(k+1)}$  in line 4, and if  $\|t_{(k+1)} - t_{(k)}\|_2 \geq \|t_{(k)} - t_{(k-1)}\|_2$ , then use  $0.5t_{(k)} + 0.5t_{(k+1)}$  instead of  $t_{(k+1)}$  as new approximation of  $\{\bar{x}_h^i\}_{i=1}^m$ .

For the numerical computations we use

$$a_h^{(0)} := 0, \quad m_{(0)} := 0, \quad t_{(0)} := \{\}, \quad \epsilon_{\text{out}} := 10^{-10}, \quad \text{and} \quad \epsilon_{\text{in}} := 10^{-12}. \quad (27)$$



We stress that our intent is to display the order of convergence, hence the parameter choices are made in such a way that the computed solutions are highly accurate.

**5.2. Optimization algorithm for full discretization.** The algorithm that we use to solve  $(\mathbf{P}_{\text{cd}})$  is very similar to Algorithm 1. In fact, there are only two differences: The approximating points  $\{t_{(k)}^i\}_{i=1}^{m(k)}$  have to be gridpoints and, in view of our theoretical findings from Corollary 7, we may add two gridpoints for every root of  $z_h^{(k)}$ . To meet these demands we first compute the roots of  $z_h^{(k)}$  in the same way as in Algorithm 1. Subsequently, every root is replaced by the two gridpoints adjacent to that root, except if a root happens to be on a gridpoint, in which case only that gridpoint is used. This is in agreement with Corollary 7. Indeed, if a gridpoint is added at which no jump occurs, then the inner iteration accounts for this by yielding zero for the corresponding coefficient (recall the representation (25)). Since these are the only changes in Algorithm 1, we do not state the resulting algorithm. In the numerical experiments we use the same set of parameters as for Algorithm 1, cf. (27).

**5.3. Example 1: Known Solution.** We construct an example by defining the following quantities:

- $c := 12 - 4\sqrt{8}$ ,  $x_c := \frac{1}{2\pi} \arccos(\frac{c}{4})$ ;
- $\alpha := 10^{-5}$ ;
- $\bar{q} := 0.5 + 1_{(x_c,1)} - 2 \cdot 1_{(0.5,1)} + 1.5 \cdot 1_{(1-x_c,1)}$ ;
- $\bar{u} := S(\bar{q})$ ,  $a(x) := 1$ ,  $d_0(x) := 0$ ;
- $\bar{\Phi}(x) := \frac{\alpha}{2c} [(1 - \cos(4\pi x)) - c(1 - \cos(2\pi x))]$  (a linear combination of a wave with two positive peaks and one negative peak; the peaks are not equidistant throughout  $\Omega$ , but symmetrical to 0.5);
- $\bar{z} := \bar{\Phi}'$ ;
- $u_d := \bar{u} + \bar{z}''$ .

It is straightforward to check that these quantities satisfy the conditions from Theorem 2.3. In particular, given this  $\alpha$  and this  $u_d$  the exact solution to  $(\mathbf{P})$  is  $\bar{q}$ . The approximated solutions to this problem are depicted in Figure 1.

Figure 2 displays the errors between solutions to the original problem  $(\mathbf{P})$  and solutions to the variationally discretized problem  $(\mathbf{P}_{\text{vd}})$ . We observe that the error estimates of Theorem 4.10 and Corollary 5 are indeed sharp. In addition, the  $L^2(\Omega)$ -error of the controls is *not* of order  $h^2$ , showing that the derived error estimates for the control are not satisfied for the  $L^2(\Omega)$ -norm. We remark that an error estimate of order  $\mathcal{O}(h)$  for the controls with respect to the  $L^2(\Omega)$ -norm follows easily from Corollary 5.

In Figure 3 we compare the solutions of the fully discretized problem  $(\mathbf{P}_{\text{cd}})$  to the solutions of the original problem. Again we find the error estimates from Lemma 4.11, Corollary 6 and Corollary 8 to be sharp and the  $L^2(\Omega)$ -error of the controls to be of lower order than the  $L^1(\Omega)$ -error. Correspondingly, it is straightforward to deduce an error estimate in  $L^2(\Omega)$  of order  $\mathcal{O}(h^{\frac{1}{2}})$  for the controls. The slightly erratic behavior of the errors can be explained by the fact that on some grids the locations of the jumps of the continuous optimal control  $\bar{q}$  are better resolved by the gridpoints than on others; we stress that the grids are *not* nested.

**5.4. Example 2: Unknown Solution.** We consider  $\alpha := 10^{-5}$  and  $u_d(x) := 0.5\pi^{-2}(1 - \cos(2\pi x))$ . An approximate solution to  $(\mathbf{P})$  is shown in Figure 4.

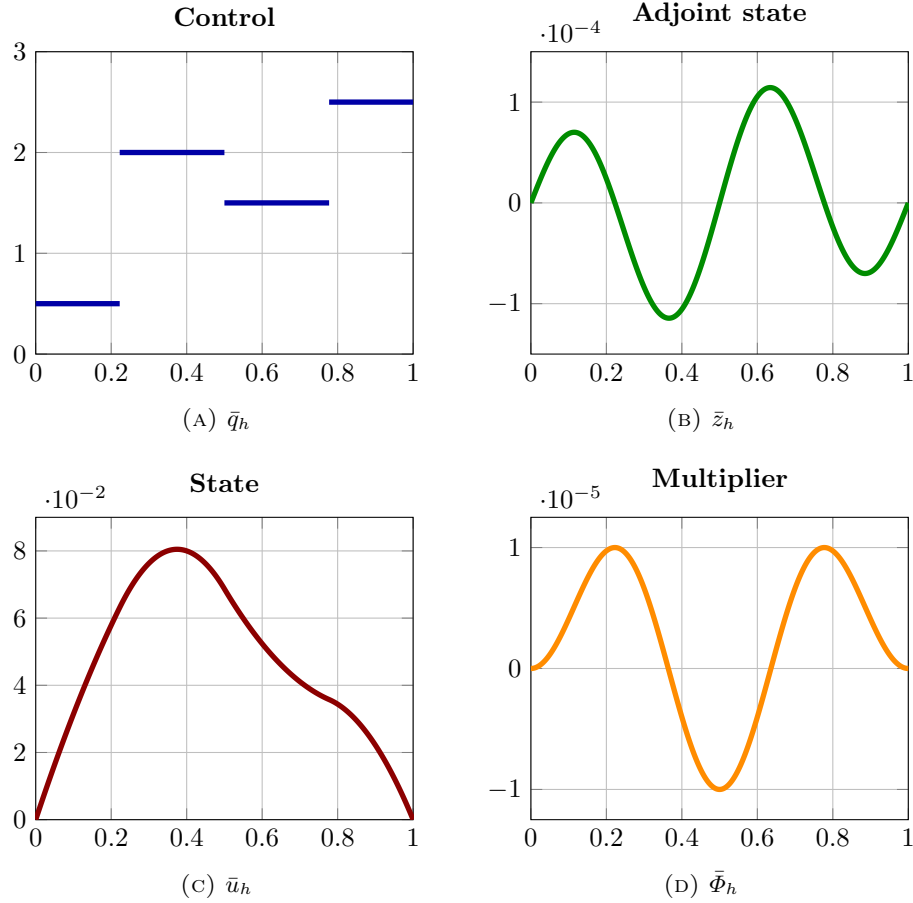


FIGURE 1. Example 1: The semi-discrete solution to the data from Section 5.3. The discretization parameter  $h$  is roughly  $3.8 \cdot 10^{-6}$ . The inclusions provided in Corollary 2 are clearly visible.

First we turn to the variationally discrete problem. As we do not have a known solution, we compute a reference solution  $(\bar{u}_{h_{\text{ref}}}, \bar{q}_{h_{\text{ref}}}, \bar{z}_{h_{\text{ref}}}, \bar{\Phi}_{h_{\text{ref}}})$  on a fine grid, more specifically  $h_{\text{ref}} \approx 9.5 \cdot 10^{-7}$ , and approximate the errors via  $\|\bar{q}_h - \bar{q}\|_{L^1(\Omega)} \approx \|\bar{q}_h - \bar{q}_{h_{\text{ref}}}\|_{L^1(\Omega)}$ . The same is done for the states and the adjoint states. Figure 5 displays the approximated errors. As in Example 1 we observe that the rates from Theorem 4.10 and Corollary 5 are sharp and that the  $L^2(\Omega)$ -error of the control is of lower order than the  $L^1(\Omega)$ -error.

The same procedure is applied to the fully discrete problem, and the results are depicted in Figure 6. Once again the proven rates turn out to be sharp and the  $L^2(\Omega)$ -rate is of lower order than the  $L^1(\Omega)$ -rate.

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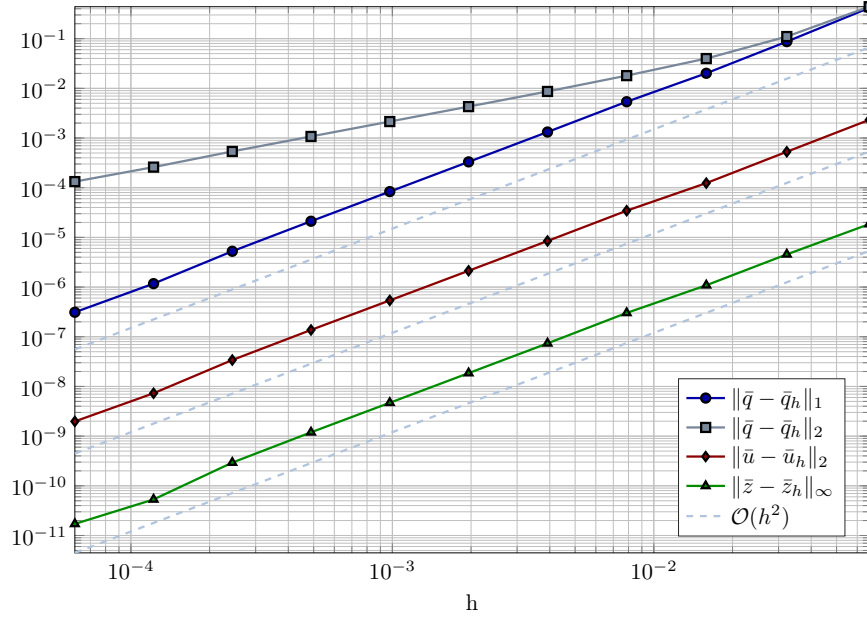


FIGURE 2. Example 1: Convergence plots of the errors of the solutions to the semi-discrete problem ( $P_{vd}$ ) compared to the exact solution. The exact solution is known.

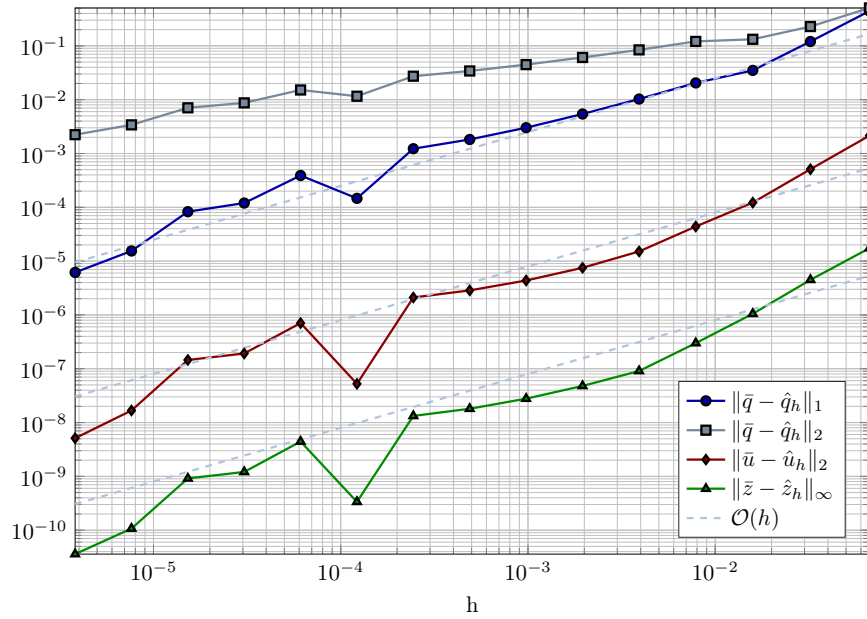


FIGURE 3. Example 1: Convergence plots of the errors of the solutions to the fully discrete problem ( $P_{cd}$ ) compared to the exact solution. The exact solution is known.

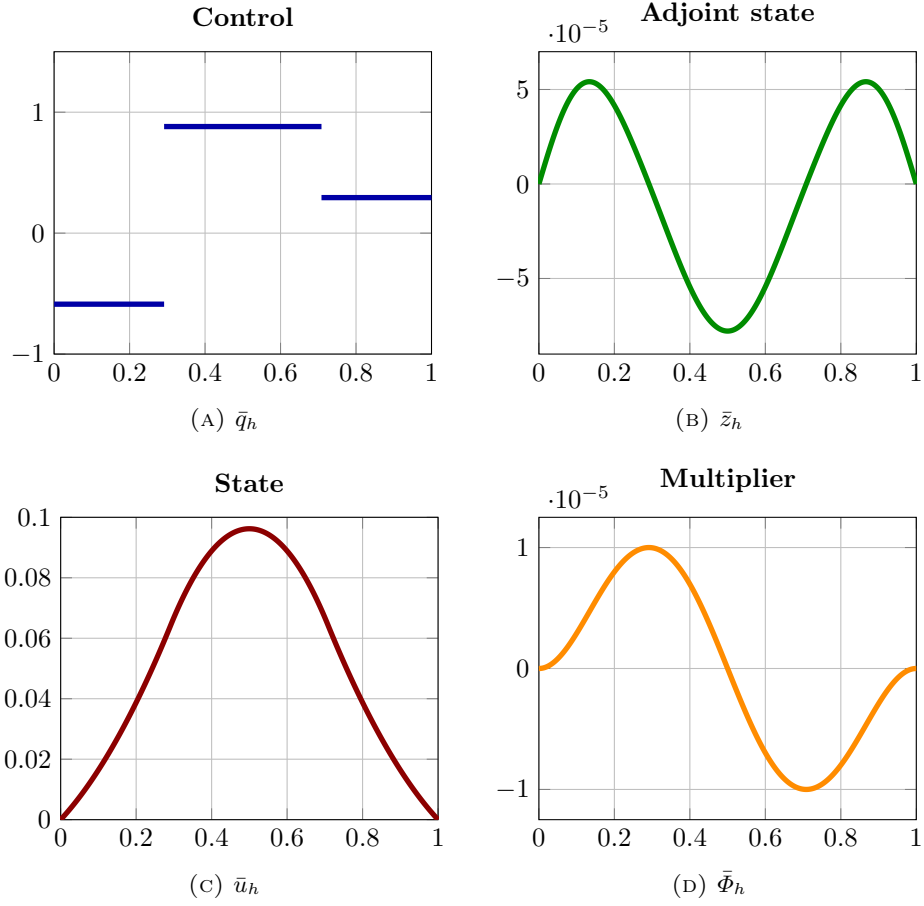


FIGURE 4. Example 2: The variationally discrete solution to the data from Section 5.4. The discretization parameter  $h$  is roughly  $3.8 \cdot 10^{-6}$ . The inclusions provided in Corollary 2 are clearly visible.

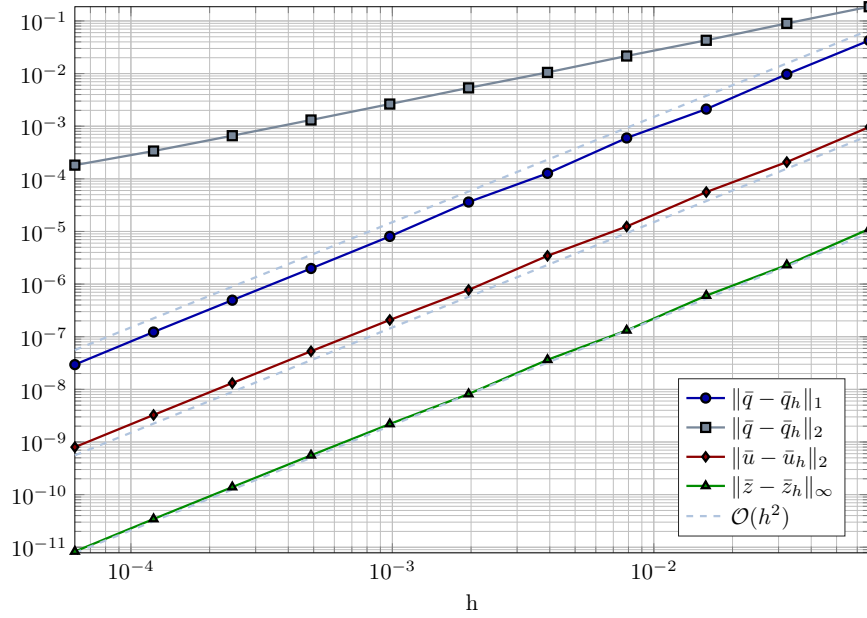


FIGURE 5. Example 2: Convergence plots of the errors of the solutions to the semi-discrete problem  $(P_{vd})$  compared to an approximation of the exact solution. The reference solution is computed as solution to  $(P_{vd})$  with  $h_{ref} \approx 3.8 \cdot 10^{-6}$ .

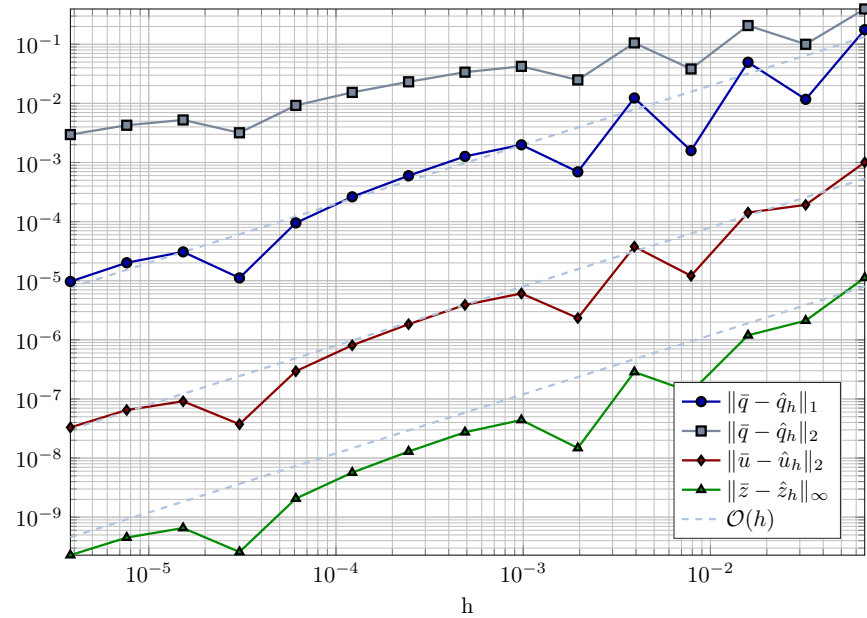


FIGURE 6. Example 2: Convergence plots of the errors of the solutions to the fully discrete problem  $(P_{cd})$  compared to an approximation of the exact solution. The reference solution is computed as solution to  $(P_{cd})$  with  $h_{ref} \approx 2.4 \cdot 10^{-7}$ .

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