DE GRUYTER Depend. Model. 2020; 8:107–118

Research Article Open Access

Henrik Sloot*

The deFinetti representation of generalised Marshall–Olkin sequences

https://doi.org/10.1515/demo-2020-0006
Received January 28, 2020; accepted May 22, 2020

Abstract: We show that each infinite exchangeable sequence $\tau_1, \tau_2, \ldots$ of random variables of the generalised Marshall–Olkin kind can be uniquely linked to an additive subordinator via its deFinetti representation. This is useful for simulation, model estimation, and model building.

Keywords: Shock model, Stochastic processes, Survival analysis, Marshall–Olkin, de Finetti

MSC: 60G09, 60G51, 62H05

1 Introduction

The Marshall–Olkin distribution was introduced by eponymous authors in [21] as the multivariate exponential distribution satisfying a strong multivariate lack-of-memory property. A random vector $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)'$ has a *Marshall–Olkin distribution* if non-negative parameters λ_I, $\emptyset \neq I \subseteq \{1, \ldots, d\}$, exist such that $\boldsymbol{\tau}$ has the survival-function

$$\bar{F}(\boldsymbol{t}) = \exp\left\{ - \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} \lambda_I \max_{i \in I} t_i \right\}, \quad \forall \boldsymbol{t} = (t_1, \ldots, t_d)' \geq 0, \tag{1}$$

and the parameters λ_I, $\emptyset \neq I \subseteq \{1, \ldots, d\}$, fulfil the condition

$$\sum_{I \ni i} \lambda_I > 0, \quad \forall i \in \{1, \ldots, d\}. \tag{2}$$

A simple calculation shows that the sums in Eq. (2) correspond to the rates of the exponentially distributed univariate margins τ_i, $i \in \{1, \ldots, d\}$, respectively. Hence, the condition in Eq. (2) ensures that $\tau_i < \infty$ a.s. for all $i \in \{1, \ldots, d\}$.

In [21], the authors proposed the *exogenous shock model* as a natural stochastic model for the Marshall–Olkin distribution. This model is based on independent, exponentially distributed random times corresponding to the failure of multiple components of a system at once. In particular, for $\lambda_I \geq 0$, $\emptyset \neq I \subseteq \{1, \ldots, d\}$, fulfilling the condition in Eq. (2), let $E_I \sim \text{Exp}(\lambda_I)$ be independent exponentially distributed random variables with rates λ_I, $\emptyset \neq I \subseteq \{1, \ldots, d\}$, respectively, where we use the convention that an exponentially distributed random variable with rate zero is almost surely infinite. Define $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_d)'$ by

$$\tau_i := \min\left\{ E_I \, : \, I \ni i \right\}, \quad i \in \{1, \ldots, d\}. \tag{3}$$

Then $\boldsymbol{\tau}$ has a Marshall–Olkin distribution with parameters λ_I, $\emptyset \neq I \subseteq \{1, \ldots, d\}$.

We are interested in exchangeable random vectors and sequences of (generalised) Marshall–Olkin kind. These subclasses have been intensively studied in the last decade for the *classical* Marshall–Olkin distribution.

***Corresponding Author: Henrik Sloot:** Technical University of Munich, Parkring 11, 85748 Garching-Hochbrück,
E-mail: henrik.sloot@tum.de

- *The exchangeable subclass* is studied in [11, Chp. 3]. The author has proven that exchangeability corresponds to the property
$$\lambda_I = \lambda_J, \quad \forall \emptyset \neq I, J \subseteq \{1, \dots, d\} \ : \ |I| = |J|.$$
Furthermore, he has shown that the survival function in Eq. (1) of an exchangeable Marshall–Olkin distribution can be reparametrised as follows:
$$\bar{F}(\boldsymbol{t}) = \exp\left\{-\sum_{i=1}^{d} a_{i-1} t_{[i]}\right\}, \quad \forall \boldsymbol{t} = (t_1, \dots, t_d)' \geq 0, \tag{4}$$
where $t_{[1]} \geq \cdots \geq t_{[d]}$ is $\boldsymbol{t}$ in descending order. The sequence $a_0, a_1, \dots a_{d-1}$ is defined by
$$a_{i-1} = \sum_{j=0}^{d-i} \binom{d-i}{j} \lambda_{j+1}, \quad i \in \{1, \dots, d\},$$
where $\lambda_i = \lambda_I$ for $i = |I|$. Finally, he provides a characterisation theorem that states that a function $\bar{F}$ of the form of Eq. (4) is a survival function if, and only if, the sequence $a_0, a_1, \dots, a_{d-1}$ is *d-monotone*. A sequence $a_0, a_1, \dots, a_{d-1}$ is called *d-monotone* if $(-1)^{i-1} \Delta^{i-1} a_{d-i} \geq 0$ for $i = 1, \dots, d$. In this case, the author shows that $\lambda_i = (-1)^{i-1} \Delta^{i-1} a_{d-i}$, $i \in \{1, \dots, d\}$.
- The *extendible* subclass is studied in [11, Chp. 4]. A d-variate Marshall–Olkin distributed random vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)'$ is called *extendible (in the class of Marshall–Olkin distributions)* if an exchangeable sequence of random variables $\{\tilde{\tau}_i\}_{i \in \mathbb{N}}$ exists such that each d-variate subsequence is equal in law to $\boldsymbol{\tau}$ and each finite subsequence has a Marshall–Olkin distribution. The author of the aforementioned reference found a unique link between extendible Marshall–Olkin distributions and Lévy subordinators via a deFinetti representation. In particular, he has shown that an infinite exchangeable Marshall–Olkin sequence $\{\tau_i\}_{i \in \mathbb{N}}$ is conditionally iid and can be written as
$$\tau_i = \inf\{t > 0 \ : \ \Lambda_t \geq E_i\}, \quad i \in \mathbb{N}, \tag{5}$$
where E_i are iid unit exponential random variables independent of a Lévy subordinator $\{\Lambda_t\}_{t \geq 0}$ on $[0, \infty]$. This model is also called the *Lévy frailty model*.

A natural generalisation of the classical Marshall–Olkin distribution is achieved if we allow non-constant hazard rates in the exogenous shock model construction in Eq. (3), see [10]. This means that we replace $\lambda_I \cdot \max_{i \in I} t_i$ with a cumulative hazard rate function $H_I(\max_{i \in I} t_i)$ and the exponential shocks $E_I \sim \text{Exp}(\lambda_I)$ in Eq. (3) with $Z_I \sim 1 - \exp\{-H_I\}$, $\emptyset \neq I \subseteq \{1, \dots, d\}$, respectively. A *cumulative hazard rate function* is a non-negative, non-decreasing, and continuous function on the non-negative half-line that starts in zero. Previous works exist on special cases of this generalisation, e.g. [9], which discusses the bivariate case, and [22], which assumes that $H_I(t) \equiv \lambda_I H(t)$.

- The exchangeable generalised Marshall–Olkin distribution and the exchangeable exogenous shock model are studied in [25]. Similar to the classical Marshall–Olkin case, the author has proven that exchangeability corresponds to the property
$$H_I(t) = H_J(t), \quad \forall t > 0, \ \forall \emptyset \neq I, J \subseteq \{1, \dots, d\} \ : \ |I| = |J|.$$
Furthermore, he has shown that a reparametrisation is possible, similar to the classical Marshall–Olkin case, by replacing $a_{i-1} \cdot t_{[i]}$ by $A_{i-1}(t_{[i]})$, $i \in \{1, \dots, d\}$, in Eq. (4). He also provides an analytical characterisation, which is discussed in Section 3.
- The extendible subclass of generalised Marshall–Olkin distributions is studied in [14] and [25, Sec. 3]. In [14, Prop. 3.1], it is shown that if the subordinator Λ in Eq. (5) is assumed to be an additive subordinator in $[0, \infty]$, then each finite margin of $\{\tau_i\}_{i \in \mathbb{N}}$ has an extendible generalised Marshall–Olkin distribution. We call this stochastic model the *additive-frailty model*.

Contribution:
This article provides the following novel result, which was posed as an open problem for further research in [25, p. 147 sq.]: every exchangeable sequence $\tau_1, \tau_2, \ldots$ with finite margins of generalised Marshall–Olkin type has an implicit representation as an additive frailty model. In particular, an additive subordinator Λ and an iid sequence of unit exponential random variables $E_1, E_2, \ldots$, independent of Λ, exist such that Eq. (5) is fulfilled almost surely. Recall that the converse of this statement was proven in [14, Prop. 3.1]. Consequently, we complete this result and establish a novel one-to-one connection between sequences of generalised Marshall–Olkin type and additive subordinators.

The article is structured as follows: we introduce the mathematical background and notation in Section 2, we summarise existing results on exchangeable generalised Marshall–Olkin distributions in Section 3, and we present the main result in Section 4. In Section 5, we conclude the article. The main proof requires some technical results involving exchangeable sequences and Bernstein functions. For the interested reader, we summarise the required background in Appendices A and B.

2 Mathematical background and notation

In this section, we give a short overview of the required mathematical background and the used notation.

We assume basic knowledge of the theory on multivariate distribution functions and probability theory. Furthermore, we assume that the reader is familiar with the Lévy–Khintchine characterisation of additive subordinators. *Additive processes* are real-valued, stochastic processes, which are defined on the non-negative half-line, start at zero, have independent increments, and have càdlàg path. An *additive subordinator* is a non-decreasing additive process which tends almost surely to infinity. Excellent books on additive processes and Lévy processes in particular are [2, 24]. We deviate slightly from the standard theory by allowing the additive subordinator to jump to an absorbing point associated with ∞ at a random time, which is independent from the subordinator. The corresponding (cumulative) hazard rate is called *(cumulative) killing hazard rate* and is equal to the zero function if almost surely no killing occurs. The Lévy–Khintchine characterisation states that each additive subordinator is uniquely determined in law by its family of Laplace exponents. These Laplace exponents are from the family of *Bernstein functions*, hereafter denoted by $\mathcal{BF}$. A function $\psi : (0, \infty) \to (0, \infty)$ is a Bernstein function if it is infinitely often differentiable and has the following property

$$(-1)^{n-1} \psi^{(n)}(x) \geq 0, \quad \forall x > 0, \; n \in \mathbb{N}.$$

One can show, see e.g. [3, Prop. 6.12] and [26, Thm. 3.2], that a function $\psi : (0, \infty) \to (0, \infty)$ is a Bernstein function if, and only if,

$$(-1)^{n-1} \Delta^n \psi(x) \geq 0, \quad \forall x > 0, \; n \in \mathbb{N}.$$

Here, Δ is the forward iterated difference operator. A Bernstein function ψ is assumed to be extended to the domain $[0, \infty)$ by the convention $\psi(0) = 0$. Excellent books on Bernstein functions are [3, 26].

We denote random variables with capital or Greek letters, e.g. X or τ, and (random) vectors with bold letters, e.g. $\boldsymbol{X}$, $\boldsymbol{\tau}$, or $\boldsymbol{t}$. We write $X \sim F$ if X has the distribution function F. We assume that operators are applied component-wise to vectors. That means $\boldsymbol{\tau} > \boldsymbol{t}$ is equivalent to $\tau_i > t_i$ for all $i \in \{1, \ldots, d\}$. Finally, we denote the descending order of a vector $\boldsymbol{t} \in [0, \infty)^d$ by $t_{[1]} \geq \cdots \geq t_{[d]}$.

We denote the class of continuous, real functions by $\mathcal{C}^{(0)}$, we write $\Delta f \geq 0$ if the function f is non-decreasing everywhere, and we use the notation $f(x-) := \lim_{y \nearrow x} f(y)$ as well as $f(x+) := \lim_{y \searrow x} f(y)$. Finally, for a real number x, we denote the smallest integer i with $i \geq x$ by $\lceil x \rceil$.

3 Exchangeable generalised Marshall–Olkin distributions

In this section, we give a short introduction into exchangeable generalised Marshall–Olkin distributions. For a more detailed treatment of the exchangeable subclass, see [14, 25].

We generalise the *classical* Marshall–Olkin distribution by allowing arbitrary continuous, cumulative hazard rate functions in the exogenous shock model. This is equivalent to having continuous, non-negative, and unbounded shock-times. For this, we define the class of continuous, cumulative hazard rate functions $\mathcal{H}$ and its unbounded subclass $\mathcal{H}_0$ by

$$\mathcal{H} := \left\{ H : [0,\infty) \to [0,\infty) \ : \ H \in \mathcal{C}^{(0)}, \ \Delta H \geq 0, \ H(0) = 0 \right\}$$

and

$$\mathcal{H}_0 := \left\{ H \in \mathcal{H} \ : \ H(\infty-) = \infty \right\}.$$

We say that a random vector $\boldsymbol{\tau} \in [0,\infty)^d$ has a *generalised Marshall–Olkin distribution* if functions $H_I \in \mathcal{H}$, $\emptyset \neq I \subseteq \{1,\ldots,d\}$, exist such that $\boldsymbol{\tau}$ has survival function

$$\bar{F}(\boldsymbol{t}) = \exp\left\{ - \sum_{\emptyset \neq I \subseteq \{1,\ldots,d\}} H_I\left(\max_{i \in I} t_i \right) \right\}, \quad \forall \boldsymbol{t} \geq 0,$$

and the hazard rate functions fulfil the condition

$$\sum_{I \ni i} H_I \in \mathcal{H}_0, \quad \forall i \in \{1,\ldots,d\}.$$

This condition, which generalises the condition in Eq. (2), is equivalent to the margins being almost surely finite, since $\sum_{I \ni i} H_I$ are the marginal cumulative hazard rates. With a simple calculation, we can establish a generalised version of the exogenous shock model in Eq. (3) for generalised Marshall–Olkin distributions by replacing $\lambda_I \cdot \max_{i \in I} t_i$ with $H_I(\max_{i \in I} t_i)$ and E_I with $Z_I \sim 1 - \exp\{-H_I\}$, $\emptyset \neq I \subseteq \{1,\ldots,d\}$, see [14, Proof of Thm. 1.1 (iv)⇒ (i)].

Below, we present a characterisation of exchangeable generalised Marshall–Olkin distributions. We know from [14, Prop. 2.1] that, similar to the classical Marshall–Olkin case, exchangeability is equivalent to the property $H_I = H_J$ for all $\emptyset \neq I, J \subseteq \{1,\ldots,d\}$ with $|I| = |J|$. Furthermore, the following characterisation result has been proven in [14]:

Lemma 1 ([14, Thm. 1.1]). *Let $\bar{F} : [0,\infty)^d \to [0,1]$ be a function such that functions $A_0,\ldots,A_{d-1} \in \mathcal{H}$ with $A_0 \in \mathcal{H}_0$ and $A_i(0) = 0$ with*

$$\bar{F}(\boldsymbol{t}) = \exp\left\{ - \sum_{i=1}^{d} A_{i-1}\left(t_{[i]}\right) \right\}, \quad \forall \boldsymbol{t} \geq 0,$$

exist, where $t_{[1]} \geq \cdots \geq t_{[d]}$ is $\boldsymbol{t}$ in descending order. Then the following statements are equivalent:

1. *$\bar{F}$ is the survival function of a random vector on $[0,\infty)^d$.*
2. *It holds that $H_i : [0,\infty) \to [0,\infty)$, $t \mapsto (-1)^{i-1} \Delta^{i-1} A_{d-i}(t) \in \mathcal{H}$ for all $i \in \{1,\ldots,d\}$, where the difference operator is understood to be applied to the (finite) sequence $A_0(t),\ldots,A_{d-1}(t)$ for fixed $t \geq 0$.*

Finally, we can construct a random vector $\boldsymbol{\tau}$ with survival function $\bar{F}$ via an exogenous shock model with $H_I := H_i$ if $|I| = i$, $\emptyset \neq I \subseteq \{1,\ldots,d\}$.

Proof of Lemma 1. This is a direct corollary of [14, Thm. 1.1]. However, since we changed the notation, we will give a short explanation: if we take the standardisation of the margins into account, the aforementioned result affirms that the first statement of this lemma is equivalent to $H_i \in \mathcal{H}$ for all $i \in \{1,\ldots,d\}$, where

$$H_i(t) := \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} A_{d-i+j}(t), \quad \forall t \geq 0.$$

Now, we obtain the claim as a corollary from [14, Thm. 1.1] by using [11, Lem. 2.5.2] which implies that

$$(-1)^{i-1} \Delta^{i-1} A_{d-i}(t) = \sum_{j=0}^{i-1} (-1)^j \binom{i-1}{j} A_{d-i+j}(t), \quad \forall t \geq 0.$$

□

4 The deFinetti representation of GMO sequences

In this section, we characterise the deFinetti representation of exchangeable generalised Marshall–Olkin sequences. We begin with an overview of general deFinetti representations.

We know from *deFinetti's theorem*, see [1, Thm. 3.1], that an almost surely unique random distribution function F exists for each exchangeable sequence $\tau_1, \tau_2, \ldots$ such that almost surely

$$\mathbb{P}\left(\tau_1 \leq x_1, \ldots, \tau_d \leq x_d \middle| F\right) = \prod_{i=1}^{d} F_{x_i}, \quad \forall x_1, \ldots, x_d \in \mathbb{R},\ d \in \mathbb{N}. \tag{6}$$

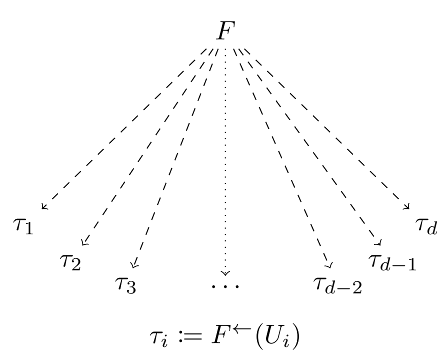
For a non-decreasing function h, we define its *generalised (right) inverse* $h^{\leftarrow}$ by $h^{\leftarrow}(y) := \inf\{x : h(x) \geq y\}$ with $\inf \emptyset = 0$, see [7] for a detailed discussion of generalised inverses. If the random distribution function F has almost surely no jumps, we have that almost surely

$$\tau_i = F^{\leftarrow}(U_i) = \inf\left\{x \in \mathbb{R} \ : \ F_x \geq U_i\right\}, \quad i \in \mathbb{N}, \tag{7}$$

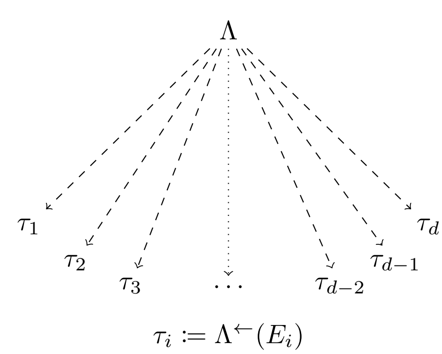
for an iid uniform sequence $U_1, U_2, \ldots$, independent of F, which is defined by $U_i := F(\tau_i)$. If $\operatorname{supp}(F) \subseteq [0, \infty]$, we can rewrite Eq. (7) as

$$\tau_i = \Lambda^{\leftarrow}(E_i) = \inf\left\{t > 0 \ : \ \Lambda_t \geq E_i\right\}, \quad i \in \mathbb{N}, \tag{8}$$

for a (càdlàg) subordinator Λ and a sequence $E_1, E_2, \ldots$ of iid unit exponential random variables, independent of Λ. For this, we define $\Lambda = -\log(1 - F)$ and $E_i = -\log(1 - U_i)$, $i \in \{1, \ldots, d\}$. Note that we define a *subordinator* as a $[0, \infty]$-valued, non-decreasing, càdlàg process on $[0, \infty)$ that starts at zero and tends to infinity for $t \to \infty$. If the random distribution function F may possibly have jumps, then Eqs. (7) and (8) still hold if there is an additional iid uniform sequence $W_1, W_2, \ldots$, which is independent of $\tau_1, \tau_2, \ldots$, defined on the probability space. The sequence $W_1, W_2, \ldots$ is required to modify $F(\tau_i)$ to a uniform random variable by a random interpolation at its (random) atoms, see [23].



(a) With random distribution function



(b) With random hazard rate

Figure 1: A visualisation of both deFinetti representations: (1) Draw F (resp. Λ) (2) For each component i, draw U_i (resp. E_i) and transform with generalised inverse of F (resp. Λ). We have $F = 1 - \exp\{-\Lambda\}$ and $U_i = 1 - \exp\{-E_i\}$.

Before moving on to the main result of this article, we want to outline three applications of the deFinetti representation:

1. We can use the deFinetti representation to sample from certain distributions efficiently in high-dimensions as illustrated in Fig. 1. See, e.g., [11, 16, 19] for applications of this technique.
2. We can build low-parametric, dimensionless families of multivariate distributions from parametrised subordinators, see, e.g., [4, 15, 17] for examples. We call these families *dimensionless*, since a random vector from such a model can be defined as the margin of an infinite sequence. Consequently, these families are not inherently linked to a specific dimension.

3. We can use the deFinetti representation for exchangeable sequences to build hierarchical models for non-exchangeable sequences. We refer the interested reader to [12, 18, 20].

Below, we state the main result of this article and investigate the subordinator, which is implied by the deFinetti representation of generalised Marshall–Olkin sequences. We already know from [11] that Marshall–Olkin sequences are uniquely linked to Lévy subordinators via Eq. (8). That remains true if we generalise the Marshall–Olkin definition as in Section 3 and generalise the Lévy subordinator to an additive subordinator.

Theorem 1 (Main result). *Let $\tau_1, \tau_2, \dots$ be an infinite exchangeable sequence of generalised Marshall–Olkin kind, i.e. a sequence of functions $A_0, A_1, \dots \in \mathcal{H}$ with $A_0 \in \mathcal{H}_0$ and $A_i(0) = 0$ exists such that for $d \geq 2$*

$$\mathbb{P}(\tau_1 > t_1, \dots, \tau_d > t_d) = \exp\left\{ -\sum_{i=1}^{d} A_{i-1}\left(t_{[i]}\right) \right\}, \quad \forall \boldsymbol{t} \geq 0. \tag{9}$$

Furthermore, assume that an iid uniform sequence $W_1, W_2, \dots$ which is independent of $\tau_1, \tau_2, \dots$ is defined on the probability space. Then, an additive subordinator Λ and iid unit exponentially distributed random variables $E_1, E_2, \dots$, independent of Λ, exist such that almost surely

$$\tau_i = \inf\left\{ t > 0 \ : \ \Lambda_t \geq E_i \right\}, \quad \forall i \in \mathbb{N}. \tag{10}$$

Conversely, if Λ is an additive subordinator with Laplace exponents $\{\psi_t\}_{t \geq 0}$, $E_1, E_2, \dots$ are iid unit exponentially distributed random variables, independent of Λ, and $\tau_1, \tau_2, \dots$ are constructed according to Eq. (10)*, then for all $d \geq 2$ the random vector $\boldsymbol{\tau}_d = (\tau_1, \dots, \tau_d)$ has an exchangeable generalised Marshall–Olkin distribution with*

$$A_i(t) = \psi_t(i+1) - \psi_t(i), \quad \forall t \geq 0, \ i \in \mathbb{N}_0. \tag{11}$$

Proof. Firstly, note that the backward direction is a corollary of [14, Prop. 3.1] by considering marginal transformations.

For the forward direction, which is the main contribution of this article, we use deFinetti's theorem, see [1, Thm. 3.1], to obtain the existence of a random distribution function F such that the sequence is conditionally iid given F and Eq. (6) holds. We define

$$U_i := F_{\tau_i} + W_i\left(F_{\tau_i} - F_{\tau_i -}\right), \quad i \in \mathbb{N}.$$

We use [23, Sec. 2] to obtain that, conditioned on F, $U_1, U_2, \dots$ are uniform and fulfil almost surely Eq. (6). In particular, we have that almost surely

$$\mathbb{P}\left(U_1 \leq u_1, \dots, U_d \leq u_d \middle| F\right) = \prod_{i=1}^{d} u_i, \quad \boldsymbol{u} \in [0,1]^d.$$

In summary, the sequence $U_1, U_2, \dots$ is iid uniform and independent of F. We use the transformations $\Lambda = -\log(1 - F)$ and $E_i = -\log(1 - U_i)$, $i \in \mathbb{N}$ and obtain a subordinator Λ and an iid unit exponential sequence $E_1, E_2, \dots$, independent of Λ, such that Eq. (10) holds almost surely.

Now, we have to prove that Λ is an additive subordinator. By a simple uniqueness-in-distribution argument and [25, p. 41], we determine that this is equivalent to the existence of a family of Bernstein functions $\{\psi_t\}_{t \geq 0} \subseteq \mathcal{BF}$, fulfilling the conditions

$$\psi_0 \equiv 0, \tag{12a}$$

$$\psi_s - \psi_t \in \mathcal{BF} \quad \forall s > t \geq 0, \tag{12b}$$

$$t \mapsto \psi_t(x) \in \mathcal{C}^{(0)}, \quad \forall x \geq 0, \tag{12c}$$

such that Eq. (11) holds. Below, we show that a family of Bernstein functions with these properties exists. With Lemma 1, we have

$$H_i^{(d)} := (-1)^{i-1} \Delta^{i-1} A_{d-i} \in \mathcal{H}, \quad i \in \{1, \dots, d\}.$$

Fix $s > t \geq 0$. Then, we have for arbitrary $d \in \mathbb{N}$

$$H_i^{(d)}(s) - H_i^{(d)}(t) \geq 0, \quad \forall i \in \{1, \ldots, d\}$$
$$\Leftrightarrow \quad (-1)^{i-1} \Delta^{i-1}\big(A_{d-i}(s) - A_{d-i}(t)\big) \geq 0, \quad \forall i \in \{1, \ldots, d\}.$$

This implies that the sequence $A_0(s) - A_0(t), A_1(s) - A_1(t), \ldots$ is completely monotone, see [11, Lem. 3.3.2]. A completely monotone sequence can be uniquely represented by the series of first-order iterated differences of a Bernstein function on $\mathbb{N}_0$, see [11, Sec. 4.1], [8, Cor. 4.2], and [3, Prop. 6.12]. Therefore, we obtain the existence of a unique Bernstein function $\psi_{s,t}$ with $\psi_{s,t}(0) = 0$ such that

$$A_i(s) - A_i(t) = \psi_{s,t}(i+1) - \psi_{s,t}(i) \quad \forall i \in \mathbb{N}.$$

This implies

$$\begin{aligned}\psi_{s,t}(i+1) - \psi_{s,t}(i) &= A_i(s) - A_i(t)\\ &= \big[A_i(s) - A_i(0)\big] - \big[A_i(t) - A_i(0)\big]\\ &= \big[\psi_{s,0}(i+1) - \psi_{s,0}(i)\big] - \big[\psi_{t,0}(i+1) - \psi_{t,0}(i)\big], \quad \forall i \in \mathbb{N}.\end{aligned}$$

Thus, if we set $\psi_u = \psi_{u,0}$ for $u \in \{t, s\}$, we obtain $\psi_0 \equiv 0$ and

$$\psi_t(i) + \psi_{s,t}(i) = \psi_s(i), \quad \forall i \in \mathbb{N}.$$

We use the fact that Bernstein functions are determined by their values on $\mathbb{N}_0$, see [3, Prop. 6.12] and [26, Thm. 3.2], and we get $\psi_s - \psi_t = \psi_{s,t} \in \mathcal{BF}$. Finally, we use that a Bernstein function is non-negative and monotone increasing to obtain the following formula for $s > t \geq 0$ and $x \geq 0$ that

$$0 \leq \psi_s(x) - \psi_t(x) = \psi_{s,t}(x) \leq \psi_{s,t}(\lceil x \rceil) = \sum_{j=1}^{\lceil x \rceil} A_{j-1}(s) - A_{j-1}(t).$$

Hence, the continuity of $A_0, A_1, \ldots$ implies $\lim_{t_k \to t} \psi_{t_k}(x) = \psi_t(x)$ for all $t, x \geq 0$. □

Recovery of the subordinator

Theorem 1 motivates the following questions: firstly, what are non-trivial examples of how the forward direction of this theorem can be used and secondly, how can we use the theorem to learn more about the implied subordinator. A non-trivial example is an exchangeable, but not comonotone or independent, generalised Marshall–Olkin sequence, which is not directly generated by a deFinetti model. Given such a sequence, the theorem only guarantees the existence of a deFinetti representation, but does not explicitly state the law of the subordinator or how it can be explicitly recovered. In the following, we use an example adapted from [13, Expl. 6.3] to demonstrate how the subordinator can be identified and recovered.

We consider an exogenous shock model in which each component can fail due to independent individual shocks or a common global shock. For this, let $H, H^G \in \mathcal{H}$ with $H + H^G \in \mathcal{H}_0$ be cumulative hazard rate functions and define $A = H$ and $A_0 = H + H^G$. Furthermore, let $Z_G \sim 1 - \exp\{-H^G\}$ and let $Z_1, Z_2, \ldots$ be an iid sequence with distribution function $1 - \exp\{-H\}$ that is independent of Z_G. We define the random sequence $\tau_1, \tau_2, \ldots$ by

$$\tau_i := \min\{Z_i, Z_G\}, \quad i \in \mathbb{N}.$$

Recovery of the subordinator law

In the first step, we use the generalised version of the exogenous shock model representation from Eq. (3) and the novel result from Theorem 1 to determine that the subordinator, implied by the deFinetti representation, is an additive subordinator with cumulative killing hazard rate H^G and deterministic part H. Since the sequence $\tau_1, \tau_2, \dots$ is exchangeable and of generalised Marshall–Olkin kind, we know that the random vector $\boldsymbol{\tau} = (\tau_1, \dots, \tau_d)$ has an exchangeable generalised Marshall–Olkin distribution for each $d \in \mathbb{N}$. We use Lemma 1 and determine that the corresponding survival function is

$$\bar{F}(t) = \exp\left\{-A_0\left(t_{[1]}\right) - \sum_{i=2}^{d} A\left(t_{[i]}\right)\right\}, \quad \boldsymbol{t} = (\tau_1, \dots, \tau_d) \geq 0.$$

Then, we conclude with Theorem 1 that an additive subordinator Λ with the characterising family of Bernstein functions $\{\psi_t\}_{t\geq 0}$ exists such that

$$A_i(t) = \psi_t(i+1) - \psi_t(i), \quad i \in \mathbb{N}_0 \quad \forall t \geq 0.$$

This implies for $t \geq 0$ that

$$\begin{aligned} A_0(t) &= \psi_t(1), \\ A(t) &= \psi_t(i+1) - \psi_t(i), \quad i \in \mathbb{N}, \\ \psi_t(i) &= \begin{cases} A_0(t) + (i-1)A(t) & i \in \mathbb{N}, \\ 0 & i = 0. \end{cases} \end{aligned}$$

As Bernstein functions are uniquely defined by their values on $\mathbb{N}_0$, we verify that

$$\psi_t(x) = \underbrace{(A_0(t) - A(t))}_{=H^G(t)} 1_{\{x>0\}} + x \underbrace{A(t)}_{=H(t)}, \quad t, x \geq 0.$$

This family of Bernstein functions can be identified with an additive subordinator with (inhomogeneous) cumulative killing hazard rate $H^G(t)$ and deterministic part $H(t)$. In particular, a random variable $Z \sim 1 - \exp\{-H^G\}$ exists such that

$$\Lambda_t = H(t) + \infty \cdot 1_{\{Z\leq t\}} = \begin{cases} H(t) & t < Z, \\ \infty & t \geq Z. \end{cases}$$

Note that so far, we only know that some random variable Z exists such that this equation holds. A natural conjecture is that $Z = Z^G$, which is proven in the following.

Explicit recovery of the subordinator

In the second step, to derive the subordinator explicitly, we use that

$$F_t(\omega) = \mathbb{E}\left[1_{\{\tau_1\leq t\}} \middle| \mathcal{T}\right](\omega), \quad \forall \omega \in \Omega \setminus N$$

for the tail-σ-algebra $\mathcal{T}$ of the sequence $\tau_1, \tau_2, \dots$ and a $\mathbb{P}$-nullset N, see [1, Lem. 2.15 and 2.19]. Furthermore, we use that $Z_j > t$ for infinitely many j and therefore

$$\left\{Z^G > t\right\} = \bigcap_{i\geq 1} \bigcup_{j\geq i} \left\{\min\{Z_j, Z_G\} > t\right\}, \quad t \geq 0.$$

Consequently, Z_G is measurable with respect to $\mathcal{T}$. Moreover, we have for $\omega \in \Omega \setminus N$

$$\begin{aligned} \Lambda_t(\omega) &= -\log\left(1 - F_t(\omega)\right) = -\log\left(\mathbb{E}\left[1_{\{\tau_1>t\}} \middle| \mathcal{T}\right](\omega)\right) \\ &= -\log\left(\mathbb{E}\left[1_{\{Z_1>t\}}\right] 1_{\{Z^G(\omega)>t\}}\right) \\ &= -\log\left(\mathbb{E}\left[1_{\{Z_1>t\}}\right]\right) + \infty 1_{\{Z^G(\omega)\leq t\}} \end{aligned}$$

$$= \begin{cases} H(t) & t < Z_G(\omega), \\ \infty & t \geq Z_G(\omega), \end{cases}$$

where we use the convention that $0 \cdot \infty = 0$. Finally, given an iid uniform sequence $W_1, W_2, \ldots$, independent of $\tau_1, \tau_2, \ldots$, we can construct the sequence $E_1, E_2, \ldots$ by

$$U_i := \begin{cases} \left(1 - e^{-H(Z_i)}\right) & Z_i < Z_G, \\ 1 - (1 - W_i)\, e^{-H(Z_G)} & Z_i \geq Z_G. \end{cases} \qquad i \in \mathbb{N}$$

and

$$E_i := -\log(1 - U_i) = \begin{cases} H(Z_i) & Z_i < Z_G \\ H(Z_G) - \log(1 - W_i) & Z_i \geq Z_G. \end{cases} \qquad i \in \mathbb{N}.$$

Now, Theorem 1 implies that the sequence $E_1, E_2, \ldots$ is iid unit exponential, independent of F, and we conclude that almost surely

$$\tau_i = \inf\left\{t > 0 \ : \ \Lambda_t \geq E_i\right\}, \quad i \in \mathbb{N}.$$

5 Conclusion

We have shown that exchangeable sequences $\tau_1, \tau_2, \ldots$ of a generalised Marshall–Olkin kind are uniquely linked to additive subordinators via a deFinetti representation. In particular, in a suitably extended probability space, we have almost surely that

$$\tau_i = \inf\{t > 0 \ : \ \Lambda_t \geq E_i\}, \quad i \in \mathbb{N},$$

where Λ is an additive subordinator and the sequence $E_1, E_2, \ldots$ is iid unit exponential and independent of Λ.

Acknowledgements: Many thanks to Lexuri Fernández, Florian Brück, Matthias Scherer, and the two anonymous reviewers for their feedback on this article.

A Exchangeable sequences and DeFinetti's theorem

In this section, we summarise the background on exchangeable sequences and deFinetti representations. An extensive reference on the deFinetti representation of exchangeable sequences and exchangeability in general, which contains all results that are presented in this section, is [1].

We call a sequence $\tau_1, \tau_2, \ldots$ *exchangeable* if

$$(\tau_1, \ldots, \tau_d) \stackrel{d}{=} (\tau_{\pi(1)}, \ldots, \tau_{\pi(d)}),$$

for each $d \in \mathbb{N}$ and permutation π on $\{1, \ldots, d\}$. A well-known result, first established by Bruno deFinetti in [6], states that sequences $\tau_1, \tau_2, \ldots$ are exchangeable if, and only if, they are conditionally iid. While this statement is clear and simple, there are some technical details hidden in the expression *conditionally iid*. In our case, since generalised Marshall–Olkin distributions have singular components and additive subordinators have jumps, these details become very important. This is explained in more detail with an example at the end of this section. For this reason, we outline below how an exchangeable sequence can be represented by a random distribution function and an iid uniform sequence.

DeFinetti Representation (See [1, Thm. 3.1]). *A sequence $\tau_1, \tau_2, \ldots$ is exchangeable if, and only if, a random measure α exists such that the product measure α^∞ is a regular conditional distribution of $\tau_1, \tau_2, \ldots$ given $\sigma(\alpha)$.*

In the following, we show how the *directing measure* α can be calculated from the sequence $\tau_1, \tau_2, \ldots$. For this, assume that the sequence $\tau_1, \tau_2, \ldots$ is defined in the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and let $\mathcal{T}$ be its tail-σ-algebra. On existence, α is a.s. unique, $\mathcal{T}$ measurable, and a regular conditional distribution for τ_1 given $\mathcal{T}$, see [1, Lem. 2.15 and 2.19]. Thus, we have

$$\alpha(\omega, A) = \mathbb{P}\big(\tau_1 \in A | \mathcal{T}\big)(\omega), \quad \forall \omega \in \Omega \setminus N, \ A \in \mathcal{B},$$

where N is a $\mathbb{P}$-nullset. In the following, we assume w.l.o.g. that $\alpha(\omega, A) = 0$ for all $\omega \in N$, $A \in \mathcal{B}$. Finally, since $\alpha(\omega, \cdot)$ is a (random) probability measure on $\mathbb{R}$, we may identify $\alpha(\omega, \cdot)$ with a random distribution function $F(\omega)$ via

$$F_t(\omega) := \alpha(\omega, (-\infty, t]), \quad \forall t \in \mathbb{R}, \ \omega \in \Omega.$$

If another sequence of iid uniform random variables $W_1, W_2, \ldots$, which is independent of $\tau_1, \tau_2, \ldots$ is defined on the probability space, we can refine deFinetti's theorem:

Corollary 1 (Cf. [1, Thm. 3.1]). *Let $W_1, W_2, \ldots$ be an iid uniform sequence and let $\tau_1, \tau_2, \ldots$ be independent thereof. The sequence $\tau_1, \tau_2, \ldots$ is exchangeable if, and only if, a random distribution function F and an iid uniform sequence $U_1, U_2, \ldots$, independent of F, exist such that*

$$\tau_i = \inf \left\{ t \in \mathbb{R} \ : \ F_t \geq U_i \right\} \text{ a.s.}, \quad \forall i \in \mathbb{N}. \tag{13}$$

Corollary 2 (Cf. [1, Thm. 3.1]). *Let $W_1, W_2, \ldots$ be an iid uniform sequence and let $\tau_1, \tau_2, \ldots \geq 0$ be independent thereof. The sequence $\tau_1, \tau_2, \ldots$ is exchangeable if, and only if, a random subordinator Λ and an iid unit exponential sequence $E_1, E_2, \ldots$, independent of Λ, exist such that*

$$\tau_i = \inf \left\{ t \geq 0 \ : \ \Lambda_t \geq E_i \right\} \text{ a.s.}, \quad \forall i \in \mathbb{N}.$$

Proof of Corollaries 1 and 2. Firstly, the claim from Corollary 2 follows directly from Corollary 1 with the transformations $\Lambda = -\log(1 - F)$ and $E_i = -\log(1 - U_i)$, $i \in \mathbb{N}$. Secondly, we use [23, Prop. 2.1] to ascertain that $U_1, U_2, \ldots$ are iid uniform conditioned on $\mathcal{T}$ and that Eq. (13) holds, where we define

$$U_i := F_{\tau_i -} + W_i \left(F_{\tau_i} - F_{\tau_i -} \right), \quad \forall i \in \mathbb{N}.$$

Finally, with the definition of the regular conditional distribution, we establish that $U_1, U_2, \ldots$ is an iid uniform sequence that is independent of $\mathcal{T}$, hence also independent of F. □

We conclude this section with an example that explains the need for additional randomness, in form of an iid uniform sequence $W_1, W_2, \ldots$, in the two preceding theorems. This example also highlights that not every *conditionally independent* sequence has a representation as in Eq. (13) when only the original probability space is considered. For this, let $(\Omega, \mathcal{F}, \mathbb{P})$ be the Lebesgue probability space on the interval $[0, 1]$ and define

$$U_i(\omega) := \omega, \quad \omega \in [0, 1], \ i \in \mathbb{N}.$$

Clearly, the sequence $U_1, U_2, \ldots$ is exchangeable and U_1 is measurable with respect to the sequences tail-σ-algebra $\mathcal{T}$. Hence, we can calculate the random distribution function F, corresponding to the sequences directing measure α, for all ω excluding a Lebesgue-nullset and $u \in [0, 1]$ by

$$F_u(\omega) = \mathbb{E}\left[1_{\{U_1 \in [0,u]\}} | \mathcal{T} \right](\omega) = 1_{\{U_1(\omega) \in [0,u]\}}.$$

Since $\sigma(F) = \sigma(U_1) = \mathcal{F}$, there is no additional iid sequence independent of F defined on this probability space. If we now consider the enclosing probability product space, on which U_1 as well as an iid uniform sequence $W_1, W_2, \ldots$, independent of U_1, are defined, we have

$$\begin{aligned} U_i = U_1 &= \inf \left\{ u \in [0, 1] \ : \ 1_{\{U_1 \in [0,u]\}} \geq W_i \right\} \\ &= \inf \left\{ u \in [0, 1] \ : \ F_u \geq W_i \right\}, \quad i \in \mathbb{N}. \end{aligned}$$

B Bernstein functions and completely monotone sequences

The proof of the main theorem relies heavily on the connection between additive and Lévy subordinators, so-called *Bernstein functions*, and *completely monotone sequences*. As the topic cannot be treated in detail without using deep results of functional analysis and measure theory, we will limit ourselves to presenting the main results. Extensive references on this topic are [3, 26]. Another excellent reference is [11, Chp. 3 and 4].

A *Bernstein function* is a function $\psi : (0, \infty) \to [0, \infty)$ that has derivatives of arbitrary order $\psi^{(i)}$, $i \in \mathbb{N}$, and fulfils

$$(-1)^{i-1}\psi^{(i)}(x) \geq 0, \quad \forall x > 0,\ i \in \mathbb{N}.$$

We denote the set of all Bernstein functions by $\mathcal{BF}$ and use the convention that a Bernstein function may be extended to $[0, \infty)$ by setting $\psi(0) := 0$. It is well-known, see, e.g. [26, Thm. 3.2], that a function $\psi : (0, \infty) \to [0, \infty)$ is a Bernstein function if, and only if, real numbers $a, b \geq 0$ and a Lévy-measure ν on $(0, \infty)$ exist such that

$$\psi(x) = a + bx + \int_{(0,\infty)} \left(1 - \mathrm{e}^{-xu}\right) \nu(\mathrm{d}u), \quad \forall x > 0,$$

where we call a measure ν on $(0, \infty)$ a *Lévy measure* if $\int_{(0,\infty)} (1 \wedge x)\, \nu(\mathrm{d}x) < \infty$. In that case (a, b, ν) is uniquely determined by ψ and is called the *Lévy triplet*.

Bernstein functions ψ with $\psi(0) = 0$ can be uniquely linked to so-called completely monotone sequences. For a (countably infinite) sequence $a_0, a_1, \dots$, let Δ be the discrete difference operator defined by $\Delta a_i := a_{i+1} - a_i$ and define recursively $\Delta^n a_i := \Delta(\Delta^{n-1} a_i)$. We call the sequence $a_0, a_1, \dots$ *completely monotone* if

$$(-1)^i \Delta^i a_k \geq 0, \quad \forall i, k \in \mathbb{N}_0.$$

We call a finite sequence $a_0, \dots, a_{d-1}$ *d-monotone* if

$$(-1)^i \Delta^i a_k \geq 0, \quad \forall i, k \in \mathbb{N}_0\ :\ i + k < d. \tag{14}$$

In particular, a sequence $a_0, a_1, \dots$ is completely monotone if, and only if, the sequences $a_0, a_1, \dots, a_{d-1}$ are d-monotone for all $d \in \mathbb{N}$. Furthermore, a sequence $a_0, a_1, \dots, a_{d-1}$ is d-monotone if, and only if, Eq. (14) is fulfilled for $i, k \in \mathbb{N}_0$ with $i + k = d - 1$, see [11, Lem. 3.3.2]. Moreover, a sequence $a_0, a_1, \dots$ is completely monotone if, and only if, a Bernstein function ψ exists with $a_i = \psi(i + 1) - \psi(i)$ for all $i \in \mathbb{N}_0$, see [11, Sec. 4.1] and cf. [8, Cor. 4.2] or [3, Prop. 6.12]. Note, that this implies that Bernstein function are uniquely determined by their values on the natural numbers.

A Bernstein function, and subsequently a completely monotone sequence, can be uniquely linked to the law of a Lévy subordinator, see [5, Thm. 1.2]. In particular, let ψ be a Bernstein function, then a Lévy subordinator Λ, uniquely determined in law, exists with Laplace exponent $x \mapsto t\psi(x)$ for all $t \geq 0$, i.e.

$$\mathbb{E}\left[\mathrm{e}^{-x\Lambda_t}\right] = \mathrm{e}^{-t\psi(x)}, \quad \forall t, x \geq 0. \tag{15}$$

Conversely, if Λ is a Lévy subordinator, then a Bernstein function ψ exists such that Eq. (15) holds.

This can be generalised, see, e.g. [25, p. 41]: let $\{\psi_t\}_{t\geq 0}$ be a family of Bernstein functions fulfilling

$$\psi_0 \equiv 0, \tag{12a rev.}$$

$$\psi_s - \psi_t \in \mathcal{BF} \quad \forall s > t \geq 0, \tag{12b rev.}$$

$$t \mapsto \psi_t(x) \in \mathcal{C}^{(0)}, \quad \forall x \geq 0. \tag{12c rev.}$$

Then, an additive subordinator, uniquely determined in law, exists with Laplace exponent $x \mapsto \psi_t(x)$ for all $t \geq 0$ and

$$\mathbb{E}\left[\mathrm{e}^{-x(\Lambda_s - \Lambda_t)}\right] = \mathrm{e}^{-(\psi_s - \psi_t)(x)}, \quad \forall s \geq t, x \geq 0. \tag{16}$$

Conversely, if Λ is an additive subordinator, then a family of Bernstein functions $\{\psi_t\}_{t\geq 0}$ exists fulfilling Eqs. (12a) to (12c) such that Eq. (16) holds.

References

[1] Aldous, D. J. (1985). *Exchangeability and Related Topics*. Springer, Berlin.

[2] Applebaum, D. (2009). *Lévy Processes and Stochastic Calculus.* Second edition. Cambridge University Press.

[3] Berg, C., J. P. R. Christensen, and P. Ressel (1984). *Harmonic Analysis on Semigroups*, Springer, New York.

[4] Bernhart, G., J.-F. Mai, and M. Scherer (2015). On the construction of low-parametric families of min-stable multivariate exponential distributions in large dimensions. *Depend. Model. 3*, 29–46.

[5] Bertoin, J. (1999). *Subordinators: Examples and Applications*. Springer, Berlin.

[6] de Finetti, B. (1937). La prévision: ses lois logiques, ses sources subjectives. *Ann. Inst. H. Poincaré 7*, 1–68.

[7] Embrechts, P. and M. Hofert (2013). A note on generalized inverses. *Math. Methods Oper. Res. 77*(3), 423–432.

[8] Gnedin, A. and J. Pitman (2008). Moments of convex distribution functions and completely alternating sequences. In D. Nolan and T. Speed (Eds.), *Probability and statistics: essays in honor of David A. Freedman.*, pp. 30–41, Inst. Math. Statist., Beachwood OH.

[9] Li, X. and F. Pellerey (2011). Generalized Marshall–Olkin distributions and related bivariate aging properties. *J. Multivariate Anal. 102*(10), 1399–1409.

[10] Lin, J. and X. Li (2014). Multivariate generalized Marshall–Olkin distributions and copulas. *Methodol. Comput. Appl. Probab. 16*(1), 53–78.

[11] Mai, J.-F. (2010). *Extendibility of Marshall–Olkin Distributions via Lévy Subordinators and an Application to Portfolio Credit Risk*. PhD thesis, Technische Universität München, Germany.

[12] Mai, J.-F. (2014). *Multivariate Exponential Distributions with Latent Factor Structure and Related Topics*. Habilitation thesis, Technische Universität München, Germany.

[13] Mai, J.-F. (2019). The infinite extendibility problem for exchangeable real-valued random vectors. Available at: https://arxiv.org/abs/1907.04054.

[14] Mai, J.-F., S. Schenk, and M. Scherer (2016). Exchangeable exogenous shock models. *Bernoulli 22*(2), 1278–1299.

[15] Mai, J.-F., S. Schenk, and M. Scherer (2017). Two novel characterizations of self-decomposability on the half-line. *J. Theoret. Probab. 30*(1), 365–383.

[16] Mai, J.-F. and M. Scherer (2009a). Efficiently sampling exchangeable Cuadras–Augé copulas in high dimensions. *Inform. Sci. 179*(17), 2872–2877.

[17] Mai, J.-F. and M. Scherer (2009b). Lévy-frailty copulas. *J. Multivariate Anal. 100*(7), 1567–1585.

[18] Mai, J.-F. and M. Scherer (2013). Sampling exchangeable and hierarchical Marshall-Olkin distributions. *Comm. Statist. Theory Methods 42*(4), 619–632.

[19] Mai, J.-F. and M. Scherer (2014). Characterization of extendible distributions with exponential minima via processes that are infinitely divisible with respect to time. *Extremes 17*(1), 77–95.

[20] Mai, J.-F. and M. Scherer (2017). *Simulating Copulas: Stochastic Models, Sampling Algorithms and Applications*. Second edition. World Scientific.

[21] Marshall, A. W. and I. Olkin (1967). A multivariate exponential distribution. *J. Amer. Statist. Assoc. 62*(317), 30–44.

[22] Muliere, P. and M. Scarsini (1987). Characterization of a Marshall–Olkin type class of distributions. *Ann. Inst. Statist. Math. 39*(2), 429–441.

[23] Rüschendorf, L. (2009). On the distributional transform, Sklar's theorem, and the empirical copula process. *J. Statist. Plann. Inference 139*(11), 3921–3927.

[24] Sato, K.-I. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

[25] Schenk, S. (2016). *Exchangeable Exogenous Shock Models*. PhD thesis, Technische Universität München, Germany.

[26] Schilling, R. L., R. Song, and Z. Vondracek (2012). *Bernstein Functions. Theory and Applications*. Second edition. De Gruyter, Berlin.