



Effective Mass of the Polaron—Revisited

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Abstract. Properties of the energy–momentum relation for the Fröhlich polaron are of continuing interest, especially for large values of the coupling constant. By combining spectral theory with the available results on the central limit theorem for the polaron path measure, we prove that, except for an intermediate range of couplings, the inverse effective mass is strictly positive and coincides with the diffusion constant. Such a result is established also for polaron-type models with a suitable ultraviolet cut-off and for arbitrary values of the coupling constant. We point out a slightly stronger variant of the central limit theorem which would imply that the energy–momentum relation has a *unique* global minimum attained at zero momentum.

1. Introduction

Polaron refers to an electron interacting with the lattice vibrations of a polar crystal, see [1, 7, 23] as a guide to the physics literature. In the conventional approximations, the quantum Hamiltonian reads

$$H = \frac{1}{2}p^2 + \int_{\mathbb{R}^d} dk \omega(k) a^*(k) a(k) + \sqrt{\alpha} \int_{\mathbb{R}^d} dk \frac{\hat{v}(k)}{\sqrt{2\omega(k)}} (e^{ikx} a(k) + e^{-ikx} a^*(k)). \quad (1.1)$$

We use units in which the bare electron mass equals one. x, p are position and momentum of the electron in \mathbb{R}^d , $a^*(k), a(k)$ are the creation and annihilation operators of a free scalar Bose field over \mathbb{R}^d with commutation relations $[a(k), a^*(k')] = \delta(k - k')$, ω is the dispersion relation of the Bose field, $\omega \geq 0$, continuous and strictly positive almost everywhere, and $\omega(Rk) = \omega(k)$ for all rotations R . The form factor \hat{v} is assumed to be real, rotation invariant and

has the Fourier transform¹ \hat{v} . $v(x)$ physically describes the smearing of the interaction between the electron and the Bose field. It is a standard convention to call $g = \hat{v}/\sqrt{2\omega}$ the coupling function. Finally, α is the coupling constant, $\alpha \geq 0$. Formally, H acts on the Hilbert space $\mathcal{H} = L^2(\mathbb{R}^d) \otimes \mathcal{F}$ with \mathcal{F} the Fock space of the Bose field. The coupling between field and particle is translation invariant, and hence the total momentum

$$P = p + P_f, \quad P_f = \int_{\mathbb{R}^d} dk k a^*(k)a(k), \tag{1.2}$$

is conserved.

The *Fröhlich polaron* corresponds to the specific choice $d = 3$, $\omega(k) = 1$, and $g(k) = (\sqrt{2\pi}|k|)^{-1}$. In particular, $\|g\|_2 = \infty$ because of ultraviolet divergence. Since the strong coupling physics is dominated by the large $|k|$ behavior of the coupling function, no ultraviolet cut-off can be afforded² and a separate discussion is required, see Sect. 4. The acoustic polaron corresponds to $\omega(k) = |k|$ and other variations can be found in the physics literature. As common practice [15, 23], we thus keep d, ω, g general for a while and add further assumptions on the way. For the purpose of this introductory discussion $\|g\|_2 < \infty$ is assumed. Under precise conditions to be stated in Sect. 3, H is a self-adjoint operator and has the fiber decomposition

$$H = \Pi^* \left(\int_{\mathbb{R}^d}^{\oplus} dP H(P) \right) \Pi, \tag{1.3}$$

since P is conserved. (Here $\Pi := F e^{iP_t x}$, where F is the Fourier transform from x to P variable). The fiber Hamiltonian reads

$$H(P) = \frac{1}{2}(P - P_f)^2 + \int_{\mathbb{R}^d} dk \omega(k) a^*(k)a(k) + \sqrt{\alpha} \int_{\mathbb{R}^d} dk g(k)(a(k) + a^*(k)) \tag{1.4}$$

and acts on \mathcal{F} . The energy–momentum relation, $E(P)$, is the bottom of the spectrum of $H(P)$,

$$E(P) = \inf \text{spec}(H(P)). \tag{1.5}$$

By construction, $E(RP) = E(P)$ for all rotations R and hence $E(P) = E_r(|P|)$. Also, in generality,

$$E(P) \geq E(0). \tag{1.6}$$

As a widely accepted definition, the effective mass is the inverse of the curvature of $E(P)$ at $P = 0$, which by rotation invariance means

$$(m_{\text{eff}})^{-1} = E_r''(0). \tag{1.7}$$

In [13] an alternative definition of the effective mass has been proposed, which is based on the response of the ground state energy of the polaron to a weak confining potential. The agreement with (1.7) is proved.

¹We use the convention $\hat{v}(k) := \frac{1}{(2\pi)^{d/2}} \int_{\mathbb{R}^d} dx e^{-ikx} v(x)$ and occasionally write $(Fv)(k) := \hat{v}(k)$.

²More precisely, the cut-off should be sent to infinity before the coupling constant.

A long-standing open problem is to analyze the effective mass of the Fröhlich polaron in the strong coupling regime [20,23]. As the coupling is increased, more and more bosons are bound to the electron and one would expect the effective mass to increase with α , presumably to diverge in the limit. Recent progress has been achieved by Lieb and Seiringer [12], who by functional analytic methods prove that indeed $m_{\text{eff}}(\alpha)$ diverges as $\alpha \rightarrow \infty$. A mathematically orthogonal approach is to study the polaron path measure, as originally introduced by Feynman [4]; see also [5,22]. Mathematically this corresponds to a standard Brownian motion with a Gibbsian-like weight which depends only on the increments. Thus one expects to still observe diffusive behavior on large scales, however with an effective diffusion constant, σ^2 . In other words, one conjectures the validity of a central limit theorem (CLT) for such weighted Brownian motion. In fact, the CLT has become now available for a large class of polaron models, including the assertion that $\sigma^2 > 0$. We refer to Sect. 2 for more details.

As argued in [23], and presumably before, effective mass and diffusion constant should be related as

$$(m_{\text{eff}})^{-1} = \sigma^2. \quad (1.8)$$

However, at the time, the reasoning was based on considering second moments for the position of the weighted Brownian motion, a piece of information which is not so easily available from current CLT proofs. For us, this by itself is a convincingly enough reason to reconsider the case. As an extra bonus, apparently not noted before, the conventional CLT also yields spectral information about properties apparently unaccessible by current functional analytic techniques. To explain this point in more detail, we assume the lower bound $\omega(k) \geq c_0 > 0$ and further conditions as stated in Sects. 3 and 4. Then $H(P)$ has a unique ground state with energy $E(P)$ for $|P| < \gamma$ for some $\gamma > 0$. Furthermore, the continuum edge of $H(P)$ is strictly larger than $E(P)$ in this ball. Possible eigenvalues have a finite multiplicity and can accumulate only at the continuum edge [6,15]. In particular, E_r is real analytic in P and $\sqrt{\alpha}$. From perturbation theory, requiring α to be sufficiently small, one then infers that

$$E_r''(0) > 0. \quad (1.9)$$

But for larger α , it is difficult to exclude $E(P)$ to have vanishing curvature at $P = 0$. Under to be stated conditions we will establish the identity (1.8). Hence from $\sigma^2 > 0$ one concludes that $m_{\text{eff}} < \infty$ for all α .

A related issue is the long-standing (physically obvious) conjecture

$$E(P) > E(0), \quad P \neq 0. \quad (1.10)$$

The weaker property (1.6) follows from a Kato inequality for H [6]. So the real issue is to exclude points $P^* \neq 0$ at which $E(P^*) = E(0)$. Property (1.10) is claimed in [7, Statement 2]. In the proof on p. 78, the authors argue with the degeneracy of the ground state of H . But, under the common assumptions, H has no ground state at all. To have a ground state would require the set $\{P \in \mathbb{R}^d \mid E(P) = E(0)\}$ to have nonzero Lebesgue measure. In Sect. 5, we will explain how (1.10) follows from a CLT with yet to be studied boundary

conditions. Alternatively, one might invoke a suitable large deviation result in the context of the available boundary conditions.

Our paper is organized as follows. In Sect. 2, we explain the connection to the probabilistic CLT and discuss recent results of interest in our context. In Sects. 3 and 4, we show the relations (1.8), (1.9) for polaron-type models with the UV cut-off $\|g\|_2 < \infty$ and for the Fröhlich polaron, respectively. In Sect. 5, we study the functional analytic side of a CLT with general two-sided pinning.

2. Probabilistic Approach and Central Limit Theorem

We choose the boundary states $\phi_{\pm} = \varphi_{\pm} \otimes \Omega \in \mathcal{H}$ with Ω the Fock vacuum and φ_{\pm} nonzero and real-valued, and define

$$\tilde{G}_{T_-, T_+}(k, t) = \langle \phi_-, e^{-T_- H} e^{ikx} e^{-tH} e^{-ikx} e^{-T_+ H} \phi_+ \rangle, \quad t, T_-, T_+ \geq 0. \tag{2.1}$$

Using the direct integral decomposition (1.3), (1.4), see also formula (B.1), one obtains the identity

$$\tilde{G}_{T_-, T_+}(k, t) = \int_{\mathbb{R}^d} dP \overline{\hat{\varphi}_-(P)} \hat{\varphi}_+(P) \langle \Omega, e^{-T_- H(P)} e^{-tH(P+k)} e^{-T_+ H(P)} \Omega \rangle. \tag{2.2}$$

In spirit of the Feynman–Kac formula for a Schrödinger operator, the semi-group e^{-tH} , $t \geq 0$, can be written as a weighted average with respect to a Gaussian measure. For the particle trajectories, we introduce the Wiener process \mathbb{P}^W , i.e., standard Brownian motion starting with Lebesgue measure on \mathbb{R}^d , with expectation \mathbb{E}^W . The continuous paths of the Wiener process are denoted by $q(t)$. The Bose field maps to the Gaussian process $u(x, t)$ whose path measure is denoted by \mathbb{P}^G , with expectation \mathbb{E}^G . The Gaussian process has mean zero, is stationary in space time and is uniquely defined through its covariance

$$\mathbb{E}^G(u(x, t)u(x', t')) = \int_{\mathbb{R}^d} dk \frac{1}{(2\pi)^d} \frac{1}{2\omega(k)} e^{ik(x-x')} e^{-\omega(k)|t-t'|}. \tag{2.3}$$

Then

$$\begin{aligned} \tilde{G}_{T_-, T_+}(k, t) &= \mathbb{E}^W \times \mathbb{E}^G \left(\varphi_-(q(-T_-)) \varphi_+(q(T_+ + t)) e^{-ik(q(t) - q(0))} \right. \\ &\quad \left. \times \exp \left[\sqrt{\alpha} \int_{-T_-}^{T_+ + t} ds \int_{\mathbb{R}^d} dx v(x) u(q(s) - x, s) \right] \right), \end{aligned} \tag{2.4}$$

which makes more explicit how $v(x)$ smears the field u relative to the position of the particle. If $\|g\|_2 < \infty$, the term in the square brackets is a well-defined Gaussian random variable with respect to \mathbb{P}^G . The Gaussian average \mathbb{E}^G can

be carried out explicitly leading to

$$\begin{aligned} \tilde{G}_{T_-, T_+}(k, t) &= \mathbb{E}^W \left(\varphi_-(q(-T_-)) \varphi_+(q(T_+ + t)) e^{-ik(q(t) - q(0))} \right. \\ &\quad \left. \times \exp \left[\frac{1}{2} \alpha \int_{-T_-}^{T_+ + t} ds \int_{-T_-}^{T_+ + t} ds' W(q(s) - q(s'), s - s') \right] \right) \end{aligned} \tag{2.5}$$

with

$$W(x, t) = \int_{\mathbb{R}^d} dk |g(k)|^2 e^{ikx} e^{-\omega(k)|t|}. \tag{2.6}$$

Note that W is real, continuous, rotation invariant in x , and $|W(x, t)| \leq \|g\|_2^2$. In particular, the integrand under the double time integral appearing in (2.5) is pathwise bounded and continuous.

The Fröhlich polaron is the special case $d = 3$, $\omega = 1$, and $g(k) = (\sqrt{2\pi}|k|)^{-1}$, thereby defining the Hamiltonian H^{Fr} , for which self-adjointness is established in [6, 8]. The kernel W of the Fröhlich polaron is given by

$$W^{\text{Fr}}(x, t) = |x|^{-1} e^{-|t|}, \tag{2.7}$$

which is no longer bounded. Still the factor $\exp[\cdot]$ in (2.5) is integrable [3]. To establish the validity of the basic identity (2.5) for H^{Fr} , one introduces the cut-off coupling $g_\kappa(k) = (\sqrt{2\pi}|k|)^{-1} e^{-\frac{1}{\kappa}|k|}$, thereby defining the Hamiltonian H_κ . The strong limit $\lim_{\kappa \rightarrow \infty} e^{-tH_\kappa} = e^{-tH^{\text{Fr}}}$ is established in [6, 14], which controls the left side of (2.5). On the right side, W is replaced by

$$W_\kappa(x, t) = |x|^{-1} \frac{2}{\pi} \arctan(\kappa|x|) e^{-|t|}, \tag{2.8}$$

which increases monotonously to $W^{\text{Fr}}(x, t)$. Thus by monotonicity, the right-hand side of (2.5) converges to the corresponding expression with kernel given by $W^{\text{Fr}}(x, t)$ and hence (2.5) remains valid for the Fröhlich polaron.

In (2.5), the reference process is a standard Brownian motion over the time interval $[-T_-, t + T_+]$. The Brownian motion is pinned by the function φ_- at the left border and by φ_+ at the right one. The Brownian path is weighted by the exponential of the double time integral involving W . Note that the weight depends only on the increments. To have a probability measure, we have to normalize by the partition function $\tilde{G}_{T_-, T_+}(0, t)$. The difference $q(t) - q(0)$ is the Brownian motion increment over the time interval $[0, t]$. Of interest is its characteristic function, i.e., the Fourier transform of the corresponding probability density function. Altogether, this leads to the normalized characteristic function

$$G_{T_-, T_+}(k, t) = \tilde{G}_{T_-, T_+}(k, t) / \tilde{G}_{T_-, T_+}(0, t). \tag{2.9}$$

Depending on the precise setup, one then has to establish the limits $T_-, T_+ \rightarrow \infty$ followed by the CLT which requires $t \rightarrow \infty$.

In the probabilistic literature, two distinct boundary conditions have been studied and we discuss them one by one. In both cases $T_- = 0$, $\varphi_-(x) = \delta(x)$, and $\varphi_+(x) = 1$, of which the latter two have to be approximated by a suitable sequence of L^2 functions. We set $T_+ = T$ in the sequel.

In [2] and the follow-up by Gubinelli [10], the authors require the conditions

$$\int_{\mathbb{R}^d} dk |g(k)|^2 \left(\sum_{j=1,2,3} \omega(k)^{-j} \right) < \infty, \quad \int_{\mathbb{R}^d} dk |g(k)|^2 |k|^2 \left(\sum_{j=2,4} \omega(k)^{-j} \right) < \infty. \tag{2.10}$$

They consider $\tilde{G}_{0,T}$ of the form

$$\begin{aligned} &\tilde{G}_{0,T}(k, t) \\ &= \mathbb{E}^W \left(\delta(q(0)) e^{-ikq(t)} \exp \left[\frac{1}{2} \alpha \int_0^{T+t} ds \int_0^{T+t} ds' W(q(s) - q(s'), s - s') \right] \right) \end{aligned} \tag{2.11}$$

and establish the limit

$$\lim_{T \rightarrow \infty} G_{0,T}(k, t) = G_{0,\infty}(k, t). \tag{2.12}$$

The CLT is proved, thus ensuring the limit

$$\lim_{\epsilon \rightarrow 0} G_{0,\infty}(\epsilon k, \epsilon^{-2} t) = e^{-\frac{1}{2} \sigma^2 k^2 t} \tag{2.13}$$

for some $\sigma > 0$. In fact, the stronger functional CLT is established, see [2, Theorem 1.1].

It is instructive to rewrite the expectation values from above in the language of operators as in (2.1), (2.2), with the result

$$\begin{aligned} \tilde{G}_{0,T}(k, t) &= \langle \phi_-, e^{ikx} e^{-tH} e^{-ikx} e^{-TH} \phi_+ \rangle \\ &= \int_{\mathbb{R}^d} dP \delta(P) \langle \Omega, e^{-tH(P+k)} e^{-TH(P)} \Omega \rangle = \langle \Omega, e^{-tH(k)} e^{-TH(0)} \Omega \rangle, \end{aligned} \tag{2.14}$$

where we used $\hat{\varphi}_-(P) = (2\pi)^{-d/2}$, and $\hat{\varphi}_+(P) = (2\pi)^{d/2} \delta(P)$. For polaron-type models treated in Sect. 3, $H(0)$ has a spectral gap and a unique ground state ψ_0 , thus by the spectral theorem

$$G_{0,\infty}(k, t) = \langle \Omega, e^{-t(H(k) - E(0))} \psi_0 \rangle / \langle \Omega, \psi_0 \rangle. \tag{2.15}$$

More recently, Mukerjee and Varadhan studied the CLT under weaker conditions than imposed in [2, 10]. Their starting formula is

$$\tilde{G}_{0,0}(k, t) = \mathbb{E}^W \left(\delta(q(0)) e^{-ikq(t)} \exp \left[\frac{1}{2} \alpha \int_0^t ds \int_0^t ds' W(q(s) - q(s'), s - s') \right] \right), \tag{2.16}$$

hence $T = 0$, which one recognizes as a particular case of (2.14) and

$$G_{0,0}(k, t) = \langle \Omega, e^{-tH(k)} \Omega \rangle / \langle \Omega, e^{-tH(0)} \Omega \rangle. \tag{2.17}$$

In [17, Theorem 4.2], the CLT of the following form is established for the Fröhlich polaron,

$$\lim_{\epsilon \rightarrow 0} G_{0,0}(\epsilon k, \epsilon^{-2} t) = e^{-\frac{1}{2} \sigma^2 k^2 t}, \tag{2.18}$$

for some $\sigma > 0$, with the restriction $\alpha \in [0, \alpha_0) \cup (\alpha_1, \infty)$ for some $0 < \alpha_0 < \alpha_1 < \infty$. The functional CLT is not touched upon.

In the related study [18], the strong coupling limit and its relation to the Pekar process are investigated. Mukherjee [19] also starts from (2.16) and considers a general weight function W , for which he requires $|W(x, t)| \leq C(1 + |t|)^{-(2+\delta)}$ for some $C, \delta > 0$. In particular, this condition covers the polaron whenever $g \in L^2$. [In the currently posted version, in addition $W \geq 0$ is required. As communicated to us by the author this condition can be dropped.] In [19, Theorem 2.1] the conventional CLT of the form (2.18) is proved for arbitrary $\alpha \geq 0$.

Physically one is also interested in the behavior of $E(P)$ away from the origin. Starting from (2.17), instead of $k = \mathcal{O}(\epsilon)$ one would have to consider $k = P = \mathcal{O}(1)$, which probabilistically is a problem of large deviations. In Sect. 5, we explore a different approach by starting from (2.1) with general square-integrable boundary functions φ_{\pm} in the limit $T_{\pm} \rightarrow \infty$, but still invoking a CLT.

3. Polaron-Type Models with a UV Cut-off

In this section, we show that $\sigma > 0$ appearing in the CLT (2.18) coincides with the square root of the inverse effective mass for a large class of polaron-type Hamiltonians with a UV cut-off. It is convenient to start from a family of the fiber Hamiltonians of the form

$$H(P) = \frac{1}{2}(P - P_f)^2 + H_f + \sqrt{\alpha} \int_{\mathbb{R}^d} dk g(k)(a(k) + a^*(k)), \quad (3.1)$$

where $H_f = \int_{\mathbb{R}^d} dk \omega(k)a^*(k)a(k)$, $P_f = \int_{\mathbb{R}^d} dk k a^*(k)a(k)$. Further assumptions are listed in

Condition C (i) $g \in L^2(\mathbb{R}^d)$ is real and rotation invariant. The coupling constant $\alpha \geq 0$ is arbitrary.

(ii) $\omega(k) \geq c_0 > 0$, ω is continuous, rotation invariant, and sub-additive in the sense that

$$\omega(k_1 + k_2) \leq \omega(k_1) + \omega(k_2). \quad (3.2)$$

Then, by the Kato–Rellich theorem, $H(P)$ are self-adjoint, semi-bounded operators on the domain $D(P_f^2 + H_f)$ which is independent of P . By the direct integral formula (1.3), one obtains a Hamiltonian of the form (1.1). Under the above assumptions, the HVZ theorem for these models was shown in [6, 15]. All the properties below can be found in [15] except for part 0 for which we refer to [9] or [25, Section 15.2], and part 6 which can be found in [16]. We refer to [15] for a discussion of the literature.

Lemma 3.1 [6, 15]. *Assume Condition C and define $E(P) = \inf \text{spec}(H(P))$, $E_{\text{ess}}(P) = \inf \text{spec}_{\text{ess}}(H(P))$. Then the following statements hold true:*

0. $E(0) \leq E(P)$ for all $P \in \mathbb{R}^d$.

1. $E_{\text{ess}}(P) = \inf_{k \in \mathbb{R}^d} (E(P - k) + \omega(k))$.
2. The set $\mathcal{I}_0 := \{P \in \mathbb{R}^d \mid E(P) < E_{\text{ess}}(P)\}$ is non-empty and contains a neighborhood of any global minimum of $|P| \mapsto E_r(|P|)$.
3. $E(P)$ is an isolated, simple eigenvalue for $P \in \mathcal{I}_0$.
4. Suppose that ω is bounded along some sequence $\{k_n\}_{n \in \mathbb{N}}$ s.t. $\lim_{n \rightarrow \infty} |k_n| = \infty$. Then, for any $\bar{E} \in \mathbb{R}$, and any sequence $\{P_n\}_{n \in \mathbb{N}}$ s.t. $\lim_{n \rightarrow \infty} |P_n| = \infty$, $E(P_n) \leq \bar{E}$ we have $\lim_{|P_n| \rightarrow \infty} (E_{\text{ess}}(P_n) - E(P_n)) = 0$.
For general ω , $E(P) \geq c_1 \omega(P) + c_2$ for some $c_1 > 0$.
5. For $P \in \mathcal{I}_0$ we have $|\langle \Omega, \psi_P \rangle| > 0$, where ψ_P is the ground state of $H(P)$.
6. $\mathcal{I}_0 \ni P \mapsto E(P)$ and $P \mapsto |\psi_P\rangle\langle\psi_P|$ are real analytic functions.

Let us comment briefly on the proof of properties 1–5 and the role of various assumptions on ω . We consider the thresholds

$$E^{(n)}(P) := \inf_{k_1, \dots, k_n \in \mathbb{R}^d} (E(P - k_1 - \dots - k_n) + \omega(k_1) + \dots + \omega(k_n)). \tag{3.3}$$

Assuming only that ω is continuous, bounded and massive (i.e. $\omega(k) \geq c_0 > 0$), Theorem 2.1 of [15] gives $E_{\text{ess}}(P) = \inf_{n \geq 1} E^{(n)}(P)$. As sub-additivity of ω clearly gives monotonicity of thresholds, with this additional assumption one obtains property 1 of Lemma 3.1 above. As pointed out in [15], it is clear from this relation, and from the fact that ω is massive, that if $E(P) < \inf_{P'} E(P') + \inf_{k'} \omega(k')$ then $E(P) < E_{\text{ess}}(P)$, which gives property 2 of Lemma 3.1. Clearly, the spectrum below $E_{\text{ess}}(P)$ consists at most of eigenvalues of finite multiplicity with $E_{\text{ess}}(P)$ as the only possible accumulation point. Thus $E(P)$ is an isolated eigenvalue, which is simple by Theorem 2.3 of [15]. Thus we obtain property 3 of Lemma 3.1. For the first part of property 4 and property 5, we refer to Theorems 2.3 and 2.4 of [15]. (The assumption $\lim_{|k| \rightarrow \infty} \omega(k)/|k|^2 = 0$ from Theorem 2.3 of [15] holds in our case by the sub-additivity and Fekete’s lemma). The second part of property 4 can be found in [6] (see also [25, Section 15.2, property (v)]).

As for part 6, we note that for ξ in the resolvent set of $H(P_0)$ the function $P \mapsto (H(P) - \xi)^{-1}$ can be expanded around any $P_0 \in \mathbb{R}^d$ as in formula (A.2). The real analyticity of the eigenprojections $\mathcal{I}_0 \ni P \mapsto |\psi_P\rangle\langle\psi_P|$ follows immediately via the Cauchy formula. (We note that by a suitable choice of the phase, we can ensure that $\mathcal{I}_0 \ni P \mapsto \psi_P$ is norm-continuous, which is the property we will need below). Since $|P| \mapsto H(\hat{P}|P|)$ is a real analytic family of self-adjoint operators in the sense of [11, Chapter VII, §1, §3] and $P \mapsto E(P)$ is a rotation invariant function, we obtain by [11, Chapter VII, §3] that $\mathcal{I}_0 \ni P \mapsto E(P)$ is real analytic.

Now we are ready to state and prove our main result concerning polaron-type models with a UV cut-off.

Theorem 3.2. *Consider polaron-type models satisfying Condition C. Then, for all $\alpha \geq 0$,*

$$(m_{\text{eff}})^{-1} = \sigma^2 > 0. \tag{3.4}$$

Proof. The proof relies on the CLT as stated in (2.18). We consider the expression

$$G_{0,0}(\epsilon k, \epsilon^{-2}t) = \frac{\langle \Omega, e^{-\frac{t}{\epsilon^2}(H(\epsilon k) - E(0))} \Omega \rangle}{\langle \Omega, e^{-\frac{t}{\epsilon^2}(H(0) - E(0))} \Omega \rangle}. \tag{3.5}$$

The spectral calculus gives $\lim_{\epsilon \rightarrow 0} \langle \Omega, e^{-\frac{t}{\epsilon^2}(H(0) - E(0))} \Omega \rangle = \langle \Omega, \psi_0 \rangle \langle \psi_0, \Omega \rangle$. Concerning the numerator in (3.5), we obtain

$$\langle \Omega, e^{-\frac{t}{\epsilon^2}(H(\epsilon k) - E(0))} \Omega \rangle = \langle \Omega, e^{-\frac{t}{\epsilon^2}(H(\epsilon k) - E(0))} \psi_{\epsilon k} \rangle \langle \psi_{\epsilon k}, \Omega \rangle \tag{3.6}$$

$$+ \langle \Omega, e^{-\frac{t}{\epsilon^2}(H(\epsilon k) - E(0))} (|\psi_{\epsilon k}\rangle \langle \psi_{\epsilon k}|)^\perp \Omega \rangle \tag{3.7}$$

$$\xrightarrow{\epsilon \rightarrow 0} \langle \Omega, \psi_0 \rangle \langle \psi_0, \Omega \rangle e^{-\frac{1}{2} t k^2 (\partial_{|P|}^2 E_r)(0)}, \tag{3.8}$$

where in the leading term (3.6) we used the analyticity of $P \mapsto E(P)$ near zero (see Lemma 3.1) and we noted that the expression in (3.7) tends to zero as $\epsilon \rightarrow 0$ by the spectral calculus. \square

We remark that a priori the diffusion constant obtained from the CLT of the characteristic function (2.17) could differ from the one of (2.13). Our analysis implies that they agree under Condition C and assumption (2.10). Indeed, making use of the CLT in (2.13) and following the steps of the proof of Theorem 3.2, we obtain $\sigma^2 = m_{\text{eff}}^{-1}$ also for the diffusion constant from (2.13). This modification of the proof amounts to replacing $G_{0,0}$ with $G_{0,\infty}$, that is replacing Ω with ψ_0 in (3.5)–(3.8).

4. The Fröhlich Polaron

Let $H_\kappa(P)$ be the polaron Hamiltonian (3.1) with $d = 3$, $\omega \equiv 1$ and $g(k) = \frac{\chi_{[0,\kappa]}(|k|)}{\sqrt{2\pi|k|}}$, where κ is the UV cut-off.³ Explicitly, it has the form

$$H_\kappa(P) = \frac{1}{2}(P - P_f)^2 + N_f + \sqrt{\alpha} \int_{|k| \leq \kappa} dk \frac{1}{\sqrt{2\pi|k|}} (a(k) + a^*(k)), \tag{4.1}$$

where N_f is the number operator. It is well known, that this sequence of Hamiltonians converges in the norm-resolvent sense as $\kappa \rightarrow \infty$ to the limiting Fröhlich Hamiltonian $H^{\text{Fr}}(P)$. Also, the sequence of the full Hamiltonians $H_\kappa = \Pi^* \int^\oplus dP H_\kappa(P) \Pi$ converges in the norm-resolvent sense to $H^{\text{Fr}} = \Pi^* \int^\oplus dP H^{\text{Fr}}(P) \Pi$, cf. [8] and references therein. Making use of these approximation properties, Lemma 3.1 and further results from [14], it is easy to establish the following:

Lemma 4.1. *Let $E(P) = \inf \text{spec}(H^{\text{Fr}}(P))$ and $E_{\text{ess}}(P) = \inf \text{spec}_{\text{ess}}(H^{\text{Fr}}(P))$. Then the following statements hold true:*

- 0. $E(0) \leq E(P)$ for all $P \in \mathbb{R}^3$.

³We use here a different UV cut-off than in the discussion in Sect. 2. However, the limiting Fröhlich Hamiltonians $H^{\text{Fr}}(P)$ are the same, as one can infer from [14, Proposition A.4] and the strong convergence of the Gross transform.

1. $E_{\text{ess}}(P) = E(0) + 1$.
2. The set $\mathcal{I}_0 := \{P \in \mathbb{R}^d \mid E(P) < E_{\text{ess}}(P)\}$ contains a neighborhood of any global minimum of $|P| \mapsto E_r(|P|)$.
3. $E(P)$ is an isolated, simple eigenvalue for $P \in \mathcal{I}_0$.
4. All global minima of $|P| \mapsto E_r(|P|)$ are contained in a compact set.
5. For $P \in \mathcal{I}_0$, we have $|\langle \Omega, \psi_P \rangle| > 0$, where ψ_P is the ground state of $H(P)$.
6. $\mathcal{I}_0 \ni P \mapsto E(P)$ and $P \mapsto |\psi_P\rangle\langle\psi_P|$ are real analytic functions.

Let us comment on the proofs of the above properties. It is a general consequence of the strong resolvent convergence that for any eigenvalue λ of $H(P)$, there exists an approximating sequence $\lambda_\kappa \rightarrow \lambda$ of eigenvalues of $H_\kappa(P)$ [21, Theorem VIII.24]. Therefore, part 0 of Lemma 4.1 follows from part 0 of Lemma 3.1. Next, by [14, Proposition A.4], $\lim_{\kappa \rightarrow \infty} E_{\text{ess},\kappa}(P) = E_{\text{ess}}(P)$, where $E_{\text{ess},\kappa}(P)$ is the bottom of the essential spectrum of $H_\kappa(P)$. Now part 1 of Lemma 4.1 follows from part 1 of Lemma 3.1 applied to the case of $\omega \equiv 1$. (Alternatively, one can refer to [24, Section IV]). Parts 2 and 3 of Lemma 4.1 follow from parts 1 and 2 of the same lemma, considering that the proof of [14, Theorem 6.4] gives the uniqueness of the ground state whenever it exists, also outside of the ball $|P| < \sqrt{2}$ from the statement of the theorem. Concerning part 4, suppose by contradiction that there is a sequence P_ℓ , $\ell \in \mathbb{N}$, s.t. $E(P_\ell) = E(0)$ and $|P_\ell| \rightarrow \infty$. We pick a function $f \in C_0^\infty(\mathbb{R})$ supported in a ball around $E(0)$ of radius strictly smaller than 1 and s.t. $0 \leq f \leq 1$ and $f(E(0)) = 1$. Then, by the norm-resolvent convergence of H_κ and [21, Theorem VIII.20], we have

$$\begin{aligned} 0 &= \lim_{\kappa \rightarrow \infty} \|f(H_\kappa) - f(H^{\text{Fr}})\| = \lim_{\kappa \rightarrow \infty} \sup_{P \in \mathbb{R}^3} \|f(H_\kappa(P)) - f(H^{\text{Fr}}(P))\| \\ &\geq \lim_{\kappa \rightarrow \infty} \sup_{\ell \geq \ell_\kappa} \|f(H_\kappa(P_\ell)) - f(H^{\text{Fr}}(P_\ell))\| = 1, \end{aligned} \tag{4.2}$$

which is a contradiction. Here in the third step, we choose ℓ_κ so large that the spectrum of $H_\kappa(P_\ell)$ is outside of the support of f , which is possible by Lemma 3.1, part 4. Part 5 of Lemma 4.1 is a consequence of the strict positivity statement in [14, Theorem 6.4], where again we can disregard the restriction $|P| < \sqrt{2}$, considering the structure of the proof. Part 6 is proven analogously as the corresponding part of Lemma 3.1, given the input from ‘‘Appendix A.’’

Now we come to our main result concerning the Fröhlich polaron.

Theorem 4.2. *Consider the Fröhlich polaron. Then, for all $\alpha \in [0, \alpha_0) \cup (\alpha_1, \infty)$ for some $0 < \alpha_0 < \alpha_1 < \infty$,*

$$(m_{\text{eff}})^{-1} = \sigma^2 > 0. \tag{4.3}$$

Proof. The claim follows from the CLT stated in (2.18) by the same steps as in the proof of Theorem 3.2. Instead of Lemma 3.1, Lemma 4.1 is used. \square

5. A CLT for Two-Sided Pinning

We return to the setup of Eq. (2.1) with square-integrable boundary functions $\varphi_\pm = \varphi$, $T_\pm = T$, and $T \rightarrow \infty$. A probabilistic study of this variant does not

seem to be available in the literature, and we focus on the functional analytic side. It will be more transparent to work in a general framework, which includes the polaron models discussed so far, but many more, e.g., systems with a non-quadratic energy momentum relation for the electron.

Let $\mathcal{H}, \mathfrak{H}$ be Hilbert spaces and $\Pi: \mathcal{H} \rightarrow L^2(\mathbb{R}^d, \mathfrak{H})$ a unitary. For any $\phi \in \mathcal{H}$, we have the corresponding representation $\phi = \Pi^* \int_{\mathbb{R}^d}^{\oplus} dP \phi_P$. For any $k \in \mathbb{R}^d$, we define the unitary $U(k)$ by its action on such vectors ϕ

$$U(k)\phi = \Pi^* \int_{\mathbb{R}^d}^{\oplus} dP \phi_{P+k}. \tag{5.1}$$

Furthermore, we are interested in self-adjoint operators H on a domain $D(H) \subset \mathcal{H}$ which have the representation

$$H = \Pi^* \left(\int_{\mathbb{R}^d}^{\oplus} dP H(P) \right) \Pi. \tag{5.2}$$

Here $\mathbb{R}^d \ni P \mapsto H(P)$ is a real analytic family of positive operators with domains $D(H(P)) \subset \mathfrak{H}$, as stated more precisely in the standing assumption 0 below. Furthermore, we note that for any bounded Borel function f

$$U(k)f(H)U(k)^* = \Pi^* \left(\int_{\mathbb{R}^d}^{\oplus} dP f(H(P+k)) \right) \Pi. \tag{5.3}$$

In this section, we impose the following *standing assumptions*:

0. The family $P \mapsto H(P)$ is real analytic in the sense that for any $P_0 \in \mathbb{R}^d$ and any $\xi \notin \text{spec}(H(P_0))$ there exists a real neighborhood N_{P_0} of P_0 s.t. $\xi \notin \text{spec}(H(P))$ for any $P \in N_{P_0}$ and $N_{P_0} \ni P \mapsto (H(P) - \xi)^{-1}$ is real analytic. As a consequence, for any \hat{P} on the unit sphere $|P| \mapsto H(\hat{P}|P|)$ is a real analytic family of unbounded operators in the sense of [11, Chapter VII, §1]. Another consequence of this property and of the Helffer–Sjöstrand method of almost analytic extensions is the strong continuity of $\mathbb{R}^d \ni P \mapsto e^{-tH(P)}$, which will be used in the proofs below.
1. The function $P \mapsto E(P) := \text{inf spec}(H(P))$ is rotation invariant and we write as before $E(P) = E_r(|P|)$. E attains its global minima in the sets $M_\ell = \{P \in \mathbb{R}^d \mid |P| = Q_\ell\}$, $\ell = 0, 1, 2, \dots, L$, where $0 \leq Q_0 < Q_1 < \dots < Q_L$ and L finite. Also, we assume $E_r(Q_\ell) = 0$.
2. E is analytic in sets $\tilde{M}_\ell = \{P \in \mathbb{R}^d \mid |P| \in \Delta_\ell\}$, where Δ_ℓ is a neighborhood of Q_ℓ . Then we have $E_r(Q_\ell + R) \sim R^{n_\ell}$ for small R and some $n_\ell \in \mathbb{N}$, $n_\ell \geq 2$.
3. For $P \in \tilde{M}_\ell$, $E(P)$ are simple eigenvalues and the corresponding family of projections $\tilde{M}_\ell \ni P \mapsto |\psi_P\rangle\langle\psi_P|$ is strongly continuous. (It easily follows that $P \mapsto \psi_P$ can be chosen strongly continuous by a suitable choice of the phases, possibly at a cost of shrinking \tilde{M}_ℓ . We assume that such a choice has been made).
4. There exist vectors $\phi \in \mathcal{H}$ such that $\tilde{M}_\ell \ni P \mapsto |\langle\phi_P, \psi_P\rangle|$ are continuous and nonzero on M_ℓ .

The above assumptions hold, in particular, for models of Sect. 3 and 4 as shown in the following proposition. The proof is postponed to “Appendix B.”

Proposition 5.1. *For the Fröhlich polaron (4.1) and the polaron-type models (3.1) satisfying Condition C, the following properties hold true:*

- (a) *The models satisfy the standing assumptions 0, 1, 2, 3 above.*
- (b) *Let $\phi = \varphi \otimes \Omega \in \mathcal{H}$ be s.t. $\varphi \in L^2(\mathbb{R}^d)$, $\hat{\varphi} \in C(\mathbb{R}^d)$ and $\hat{\varphi}(p) > 0$ for all $p \in \mathcal{I}_0$. For such ϕ assumption 4 above holds.*

Coming back to the general framework, we note that assumptions 2, 3 follow from 0,1 and analytic perturbation theory [11, Chapter VII, §3] if $E(P)$ are simple, isolated eigenvalues for $P \in \tilde{M}_\ell$. However, our discussion in this section does not require spectral gaps above $E(P)$. Also the standard relation $E(0) \leq E(P)$, $P \in \mathbb{R}^d$, for polaron-type models, cf. Lemmas 3.1, 4.1, does not follow from the standing assumptions above. However, with additional input which we now explain, we will obtain not only this relation, but even $E(0) < E(P)$, $P \neq 0$, for $d \geq 2$.

For the two-sided boundary condition, the properly normalized characteristic function reads

$$\tilde{G}_T(k, t) := \langle \phi, e^{-TH}U(k)e^{-tH}U(k)^*e^{-TH}\phi \rangle, \quad G_T(k, t) := \tilde{G}_T(k, t)/\tilde{G}_T(0, t) \tag{5.4}$$

for $\phi \in \mathcal{H}$ and $t, T \geq 0$. By the spectral theorem, the denominator above is different from zero for any finite T . Furthermore, for ϕ as in assumption 4, the limits

$$G_\infty(k, t) := \lim_{T \rightarrow \infty} G_T(k, t) \quad \text{and} \quad \lim_{\epsilon \rightarrow 0} G_\infty(\epsilon k, \epsilon^{-2}t) \tag{5.5}$$

exist under our standing assumptions. The explicit expressions are provided in Proposition 5.4 and Lemma 5.5 below. We expect that the latter limit has the form suggested by the CLT.

Conjecture 5.2. *There exists $\phi \in \mathcal{H}$ as in assumption 4 above, such that the CLT of the form*

$$\lim_{\epsilon \rightarrow 0} G_\infty(\epsilon k, \epsilon^{-2}t) = e^{-\frac{1}{2}\sigma^2 k^2 t} \tag{5.6}$$

holds true for some $\sigma > 0$.

The consequences of this conjecture for models satisfying the above standing assumptions are collected in the following theorem.

Theorem 5.3. *Suppose that Conjecture 5.2 holds true for some $\phi \in \mathcal{H}$ as in assumption 4 and $\sigma > 0$. Then, for $d \geq 2$, there is a global minimum at zero (i.e., $Q_0 = 0$). Furthermore,*

- (a) $\sigma^2 = (\partial_{|P|}^2 E_r)(0)$,
- (b) $E(P) > E(0)$ for $P \neq 0$.

For $d = 1$ we obtain that $\sigma^2 = (\partial_{|P|}^2 E_r)(Q_\ell)$ for $\ell = 0, 1, 2, \dots, L$ and $0 \leq Q_0 < Q_1 < \dots < Q_L$.

We stated a minimal conjecture as required for Theorem 5.3 to hold. In fact, the CLT should be in force for a large set of boundary functions, e.g., those satisfying 4. of the standing assumptions. Our theorem then asserts that the diffusion constant is always given by $\sigma^2 = (\partial_{|P|}^2 E_{\text{r}})(0)$.

The observation behind Theorem 5.3 is fairly elementary and can be grasped most easily for the polaron models underlying (2.2). We note that the P -integral has the weight $|\hat{\varphi}(P)|^2 > 0$ for φ as in Proposition 5.1(b). Thus in the limit $T \rightarrow \infty$ the P -integral concentrates on the set of global minima $\{P \in \mathbb{R}^d \mid E(P) = E(0)\}$. If the CLT would hold, the limit expression must have come only from $P = 0$ and hence $E(P) > E(0)$ for $P \neq 0$.

The actual proof is more involved and the remaining part of this section is devoted to proving Theorem 5.3. We start with two auxiliary results, which do not rely on Conjecture 5.2. In the following proposition, $d\Omega$ denotes the spherical measure on S^{d-1} normalized to $|S^{d-1}|$ and \hat{P} denotes an element of S^{d-1} .

Proposition 5.4. *The following statements hold:*

1. *If $Q_0 = 0$ and it is the only global minimum of E , then*

$$\lim_{T \rightarrow \infty} G_T(k, t) = \langle \psi_0, e^{-tH(k)} \psi_0 \rangle. \tag{5.7}$$

2. *If $Q_0 = 0$ and there are other global minima at $Q_\ell > 0$, $\ell = 1, 2, \dots, L$, we set $n := \max_{\ell \neq 0} (n_\ell)$ (see assumption 2) and distinguish the following cases:*

- (a) *For $n > \frac{n_0}{d}$*

$$\lim_{T \rightarrow \infty} G_T(k, t) = \frac{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2 \langle \psi_{Q_{\ell}\hat{P}}, e^{-tH(Q_{\ell}\hat{P}+k)} \psi_{Q_{\ell}\hat{P}} \rangle}{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2}, \tag{5.8}$$

where $C_{\ell} > 0$ and the sums extend only over $\ell > 0$ s.t. $n_{\ell} = n$.

- (b) *For $n = \frac{n_0}{d}$*

$$\begin{aligned} \lim_{T \rightarrow \infty} G_T(k, t) &= \frac{c_0 |\langle \phi_0, \psi_0 \rangle|^2 \langle \psi_0, e^{-tH(k)} \psi_0 \rangle + \sum_{\ell} c_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2 \langle \psi_{Q_{\ell}\hat{P}}, e^{-tH(Q_{\ell}\hat{P}+k)} \psi_{Q_{\ell}\hat{P}} \rangle}{c_0 |\langle \phi_0, \psi_0 \rangle|^2 + \sum_{\ell} c_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2}, \end{aligned} \tag{5.9}$$

where $c_0, c_{\ell} > 0$ and the sums extend only over $\ell > 0$ s.t. $n_{\ell} = n$.

- (c) *For $n < \frac{n_0}{d}$*

$$\lim_{T \rightarrow \infty} G_T(k, t) = \langle \psi_0, e^{-tH(k)} \psi_0 \rangle. \tag{5.10}$$

3. *If $0 < Q_0 < Q_1 < \dots < Q_L$, for $L \geq 0$, we obtain*

$$\lim_{T \rightarrow \infty} G_T(k, t) = \frac{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2 \langle \psi_{Q_{\ell}\hat{P}}, e^{-tH(Q_{\ell}\hat{P}+k)} \psi_{Q_{\ell}\hat{P}} \rangle}{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2}, \tag{5.11}$$

where $C_{\ell} > 0$ and the sum extends over ℓ s.t. $n_{\ell} = \bar{n} := \max_{\ell'} n_{\ell'}$.

For $d = 1$, the angular integrations above amount to summations over $\hat{P} = \pm 1$.

Proof. In the fiber representation, expression (5.4) has the following form

$$\tilde{G}_T(k, t) = \int_{\mathbb{R}^d} dP \langle \phi_P, e^{-TH(P)} e^{-tH(P+k)} e^{-TH(P)} \phi_P \rangle. \tag{5.12}$$

We denote the spectral measure of $H(P)$ by $M_P(\cdot)$ and choose $\delta > 0$ s.t. $E(P) \leq \delta$ implies that $P \in \tilde{M} := \bigcup_{\ell=0}^L \tilde{M}_\ell$. By spectral calculus, we have $n\text{-}\lim_{T \rightarrow \infty} e^{-TH(P)} M_P((\delta, \infty)) = 0$, $s\text{-}\lim_{T \rightarrow \infty} e^{-TH(P)} M_P((E(P), \infty)) = 0$. (5.13)

Therefore, it suffices to study

$$\begin{aligned} \tilde{G}_T^{(1)}(k, t) &= \int_{\tilde{M}} dP \langle \phi_P, \psi_P \rangle \langle \psi_P, e^{-TH(P)} e^{-tH(P+k)} e^{-TH(P)} \psi_P \rangle \langle \psi_P, \phi_P \rangle \\ &= \int_{\tilde{M}} dP |\langle \phi_P, \psi_P \rangle|^2 e^{-2TE(P)} \langle \psi_P, e^{-tH(P+k)} \psi_P \rangle \\ &= \int_{\tilde{M}} dP |\langle \phi_P, \psi_P \rangle|^2 e^{-(2T+t)E(P)} \langle \psi_P, e^{-tH(P+k)} e^{tH(P)} \psi_P \rangle. \end{aligned} \tag{5.14}$$

Hence, setting $G_T^{(1)}(k, t) := \tilde{G}_T^{(1)}(k, t) / \tilde{G}_T^{(1)}(0, t)$,

$$\begin{aligned} G_T^{(1)}(k, t) &= \int_{\tilde{M}} dP \left\{ \frac{|\langle \phi_P, \psi_P \rangle|^2 e^{-(2T+t)E(P)}}{\int_{\tilde{M}} dP' |\langle \phi_{P'}, \psi_{P'} \rangle|^2 e^{-(2T+t)E(P')}} \right\} \langle \psi_P, e^{-tH(P+k)} e^{tH(P)} \psi_P \rangle. \end{aligned} \tag{5.15}$$

We write $\hat{P} := P/|P|$ and move on to polar coordinates in P and P' integrations:

$$\begin{aligned} G_T^{(1)}(k, t) &= \sum_\ell \int d\Omega(\hat{P}) \int_{\Delta_\ell} d|P| |P|^{d-1} \langle \psi_{|P|\hat{P}}, e^{-tH(|P|\hat{P}+k)} e^{tH(|P|\hat{P})} \psi_{|P|\hat{P}} \rangle \\ &\quad \times \left\{ \frac{|\langle \phi_{|P|\hat{P}}, \psi_{|P|\hat{P}} \rangle|^2 e^{-(2T+t)E_r(|P|)}}{\sum_\ell \int d\Omega(\hat{P}') \int_{\Delta_\ell} d|P'| |P'|^{d-1} |\langle \phi_{|P'|\hat{P}'}, \psi_{|P'|\hat{P}'} \rangle|^2 e^{-(2T+t)E_r(|P'|)}} \right\}, \end{aligned} \tag{5.16}$$

where we also used that $P \mapsto E(P)$ is rotation invariant.

Let us first consider a possible global minimum at zero. Since E is analytic near zero, we have that $E_r(|P|) \sim |P|^{n_0}$ in this region for some $n_0 \in \mathbb{N}_0$, $n_0 \geq 2$. Thus an elementary analysis gives for the numerator in (5.16)

$$\begin{aligned} \lim_{T \rightarrow \infty} (2T+t)^{d/n_0} \int d\Omega(\hat{P}) \int_{\Delta_0} d|P| |P|^{d-1} |\langle \phi_{|P|\hat{P}}, \psi_{|P|\hat{P}} \rangle|^2 e^{-(2T+t)E_r(|P|)} \\ \times \langle \psi_{|P|\hat{P}}, e^{-tH(|P|\hat{P}+k)} e^{tH(|P|\hat{P})} \psi_{|P|\hat{P}} \rangle \\ = C_0 |\langle \phi_0, \psi_0 \rangle|^2 \langle \psi_0, e^{-tH(k)} \psi_0 \rangle \int_0^\infty dU e^{-U} U^{(d/n_0)-1}, \end{aligned} \tag{5.17}$$

for some $C_0 > 0$. An analogous formula holds for the denominator in (5.16)

$$\begin{aligned} \lim_{T \rightarrow \infty} (2T + t)^{d/n_0} \int d\Omega(\hat{P}) \int_{\Delta_0} d|P| |P|^{d-1} |\langle \phi_{|P|\hat{P}}, \psi_{|P|\hat{P}} \rangle|^2 e^{-(2T+t)E_r(|P|)} \\ = C_0 |\langle \phi_0, \psi_0 \rangle|^2 \int_0^\infty dU e^{-U} U^{(d/n_0)-1}. \end{aligned} \tag{5.18}$$

Let us now analyze a global minimum at $Q_\ell \neq 0$. By analyticity, we have that $E_r(Q_\ell + R) \sim R^{n_\ell}$ near $R = 0$ for some $n_\ell \in \mathbb{N}_0$, $n_\ell \geq 2$. In this case, we obtain for the numerator in (5.16)

$$\begin{aligned} \lim_{T \rightarrow \infty} (2T + t)^{(1/n_\ell)} \int d\Omega(\hat{P}) \int_{\Delta_\ell} d|P| |P|^{d-1} |\langle \phi_{|P|\hat{P}}, \psi_{|P|\hat{P}} \rangle|^2 e^{-(2T+t)E_r(|P|)} \\ \times \langle \psi_{|P|\hat{P}}, e^{-tH(|P|\hat{P}+k)} e^{tH(|P|\hat{P})} \psi_{|P|\hat{P}} \rangle \\ = C_\ell \int d\Omega(\hat{P}) |\langle \phi_{Q_\ell \hat{P}}, \psi_{Q_\ell \hat{P}} \rangle|^2 \langle \psi_{Q_\ell \hat{P}}, e^{-tH(Q_\ell \hat{P}+k)} \psi_{Q_\ell \hat{P}} \rangle \int_0^\infty dU e^{-U} U^{(1/n)-1}, \end{aligned} \tag{5.19}$$

for some $C_\ell > 0$. For the denominator in (5.16), we get in this case

$$\begin{aligned} \lim_{T \rightarrow \infty} (2T + t)^{(1/n_\ell)} \int d\Omega(\hat{P}) \int_{\Delta_\ell} d|P| |P|^{d-1} |\langle \phi_{|P|\hat{P}}, \psi_{|P|\hat{P}} \rangle|^2 e^{-(2T+t)E_r(|P|)} \\ = C_\ell \int d\Omega(\hat{P}) |\langle \phi_{Q_\ell \hat{P}}, \psi_{Q_\ell \hat{P}} \rangle|^2 \int_0^\infty dU e^{-U} U^{(1/n)-1}. \end{aligned} \tag{5.20}$$

By substituting (5.17)–(5.20) back to formula (5.16) and considering the different cases from the statement of the proposition, we complete the proof. \square

Lemma 5.5. *Suppose that $Q \geq 0$ is a global minimum of $|P| \mapsto E_r(|P|)$. Then the following relations hold:*

$$\lim_{\epsilon \rightarrow 0} \langle \psi_0, e^{-\frac{t}{\epsilon^2} H(\epsilon k)} \psi_0 \rangle = e^{-\frac{tk^2}{2} (\partial_{|P|}^2 E_r)(0)} \quad \text{for } Q = 0, \tag{5.21}$$

$$\lim_{\epsilon \rightarrow 0} \langle \psi_{Q\hat{P}}, e^{-\frac{t}{\epsilon^2} H(Q\hat{P}+\epsilon k)} \psi_{Q\hat{P}} \rangle = e^{-\frac{t}{2} (\partial_{|P|}^2 E_r)(Q)(\hat{P} \cdot k)^2} \quad \text{for } Q > 0. \tag{5.22}$$

Proof. Suppose that $Q = 0$. By shifting the vector $\psi_0 = \psi_{\epsilon k} + (\psi_0 - \psi_{\epsilon k})$, we obtain

$$\begin{aligned} \langle \psi_0, e^{-\frac{t}{\epsilon^2} H(\epsilon k)} \psi_0 \rangle &= \langle \psi_{\epsilon k}, e^{-\frac{t}{\epsilon^2} H(\epsilon k)} \psi_{\epsilon k} \rangle + R(\epsilon) \\ &= e^{-\frac{t}{\epsilon^2} E(\epsilon k)} + R(\epsilon) \xrightarrow{\epsilon \rightarrow 0} e^{-\frac{1}{2} t k^2 (\partial_{|P|}^2 E_r)(0)}, \end{aligned} \tag{5.23}$$

where $R(\epsilon)$ is an error term which tends to zero as $\epsilon \rightarrow 0$ by assumption 3.

Concerning the case $Q > 0$, we shift the vector as follows $\psi_{Q\hat{P}} = \psi_{Q\hat{P}+\epsilon k} + (\psi_{Q\hat{P}} - \psi_{Q\hat{P}+\epsilon k})$. This gives

$$\begin{aligned} \langle \psi_{Q\hat{P}}, e^{-\frac{t}{\epsilon^2} H(Q\hat{P}+\epsilon k)} \psi_{Q\hat{P}} \rangle &= \langle \psi_{Q\hat{P}+\epsilon k}, e^{-\frac{t}{\epsilon^2} H(Q\hat{P}+\epsilon k)} \psi_{Q\hat{P}+\epsilon k} \rangle + R(\epsilon) \\ &= e^{-\frac{t}{\epsilon^2} E(Q\hat{P}+\epsilon k)} + R(\epsilon) \xrightarrow{\epsilon \rightarrow 0} e^{-\frac{t}{2} (\partial_{|P|}^2 E_r(Q))(\hat{P} \cdot \hat{k})^2}, \end{aligned} \tag{5.24}$$

which completes the proof. □

Proof of Theorem 5.3. We start with the case $d \geq 2$. Suppose, by contradiction, that there is no global minimum at zero. Then, from the last part of Proposition 5.4 and Lemma 5.5 we obtain

$$e^{-\frac{tk^2\sigma^2}{2}} = \frac{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2 e^{-\frac{tk^2}{2} (\partial_{|P|}^2 E_r(Q_{\ell}))(\hat{P} \cdot \hat{k})^2}}{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2}. \tag{5.25}$$

We denote $x^2 := tk^2\sigma^2/2$, $f_{\ell}(\hat{P}) := C_{\ell} |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2$, $m_{\ell}^{-1} := \partial_{|P|}^2 E(Q_{\ell})$. This gives

$$\sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) = \sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) e^{x^2(1-\frac{m_{\ell}^{-1}}{\sigma^2}(\hat{P} \cdot \hat{k})^2)}. \tag{5.26}$$

As \hat{k} on the r.h.s. is arbitrary, we can replace it with $R\hat{k}$, where R is a rotation, and then average over the group of rotations. By a change of variables, this amounts to averaging f_{ℓ} on the r.h.s. w.r.t. rotations. We can therefore assume that the functions f_{ℓ} are constant and nonzero. Suppose first that all m_{ℓ}^{-1} are zero. Then we immediately obtain a contradiction by taking $x^2 \rightarrow \infty$. Now suppose that some⁴ $m_{\ell_1}^{-1} > 0$. Then we obtain from (5.26)

$$\sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) \geq \int d\Omega(\hat{P}) f_{\ell_1}(\hat{P}) e^{x^2(1-\frac{m_{\ell_1}^{-1}}{\sigma^2}(\hat{P} \cdot \hat{k})^2)} \chi\left(1 - \frac{m_{\ell_1}^{-1}}{\sigma^2}(\hat{P} \cdot \hat{k})^2 > 0\right). \tag{5.27}$$

As before, we obtain a contradiction by taking $x^2 \rightarrow \infty$, due to the fact that the functions f_{ℓ} are constant and nonzero.

Next, we prove part (b). Suppose, by contradiction, that there are several global minima in addition to the global minimum at zero. Let us assume first

⁴We note as an aside, that if some $m_{\ell}^{-1} > 0$ then all $m_{\ell}^{-1} > 0$ by definition of n . Indeed, $m_{\ell}^{-1} > 0$ means that $n_{\ell} = 2$. Then $n := \max_{\ell' \neq 0} (n_{\ell'}) = 2$, since the summation extends only over $n_{\ell} = n$.

that case (a) from Proposition 5.4 occurs, that is $n > \frac{n_0}{d}$. With the help of Lemma 5.5, we obtain from (5.8)

$$e^{-\frac{tk^2\sigma^2}{2}} = \frac{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2 e^{-\frac{tk^2}{2}(\partial_{|P|}^2 E_r(Q_{\ell}))(\hat{P} \cdot \hat{k})^2}}{\sum_{\ell} C_{\ell} \int d\Omega(\hat{P}) |\langle \phi_{Q_{\ell}\hat{P}}, \psi_{Q_{\ell}\hat{P}} \rangle|^2}. \tag{5.28}$$

We obtain a contradiction by repeating the steps (5.25)–(5.27) above.

Now let us assume that case (b) of Proposition 5.4 occurs, that is $n = \frac{n_0}{d}$. Since $d \geq 2$ and $n \geq 2$, we obtain that $n_0 > 2$ which implies $(\partial_{|P|}^2 E_r)(0) = 0$. With the help of Lemma 5.5, we obtain from (5.9) using the notation introduced above

$$\begin{aligned} c_0 |\langle \phi_0, \psi_0 \rangle|^2 + \sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) \\ = c_0 |\langle \phi_0, \psi_0 \rangle|^2 e^{x^2} + \sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) e^{x^2(1 - \frac{m_{\ell}^{-1}}{\sigma^2})(\hat{P} \cdot \hat{k})^2}. \end{aligned} \tag{5.29}$$

Due to the presence of the nonzero terms involving $c_0 |\langle \phi_0, \psi_0 \rangle|^2$, we immediately obtain a contradiction by taking $x^2 \rightarrow \infty$.

Finally, suppose that we are in case (c) of Proposition 5.4, that is $n < \frac{n_0}{d}$. Also in this case we have $n_0 > 2$ which implies $(\partial_{|P|}^2 E_r)(0) = 0$. By Eq. (5.10) and Lemma 5.5, we obtain $e^{-x^2} = 1$ which is immediately a contradiction. This concludes the proof of part (b) of the theorem.

Given that we have only one global minimum, we can apply formula (5.7). Together with Lemma 5.5, we obtain

$$e^{-\frac{tk^2\sigma^2}{2}} = e^{-\frac{tk^2(\partial_{|P|}^2 E_r)(0)}{2}}, \tag{5.30}$$

which gives part (a) of the theorem.

For $d = 1$, the reasoning above requires several modifications. From the assumption that there is no global minimum at zero, we obtain via (5.26) that $m_{\ell}^{-1} = \sigma^2$ for $\ell = 0, 1, \dots, L$, which is what we wanted to prove.

Now suppose that there is a global minimum at zero and possibly some nonzero global minima. In the case $n > \frac{n_0}{d}$ from the fact that $n_0 \geq 2$, we conclude that $n > 2$, hence $m_{\ell}^{-1} = 0$ for $\ell \neq 0$. In this situation, formula (5.26) gives directly a contradiction.

In the case $n = \frac{n_0}{d}$, we distinguish two sub-cases. First, for $n = n_0 > 2$ we have $m_{\ell}^{-1} = 0$ for all ℓ , including $\ell = 0$, and thus formula (5.29) gives a contradiction. Second, for $n = n_0 = 2$, we have $m_{\ell}^{-1} \neq 0$ and thus formula (5.29) has to be rewritten as follows

$$\begin{aligned} c_0 |\langle \phi_0, \psi_0 \rangle|^2 + \sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) \\ = c_0 |\langle \phi_0, \psi_0 \rangle|^2 e^{x^2(1 - \frac{m_0^{-1}}{\sigma^2})} + \sum_{\ell} \int d\Omega(\hat{P}) f_{\ell}(\hat{P}) e^{x^2(1 - \frac{m_{\ell}^{-1}}{\sigma^2})}, \end{aligned} \tag{5.31}$$

where angular integration denotes now summation over $\hat{P} = \pm 1$. Clearly, we avoid a contradiction iff $m_\ell^{-1} = \sigma^2$ for $\ell = 0, 1, \dots, L$. (It is important here that if $n = \max_{\ell \neq 0}(n_\ell) = 2$ then $n_\ell = 2$ for all $\ell \neq 0$).

In the case $n < \frac{n_0}{d}$, we obtain a contradiction as before. Thus in the case $d = 1$ the assumption that there are several global minima of E_r led us to the conclusion that the inverses of their effective masses $\partial_{|P|}^2 E_r(Q_\ell)$, $\ell = 0, 1, \dots, L$, must all be equal to σ^2 . □

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A. Analyticity of the Fröhlich Polaron in Total Momentum

In this appendix, we verify that $P \mapsto H^{\text{Fr}}(P)$ is a real analytic family as specified in the standing assumption 0 of Sect. 5.

Lemma A.1. *Suppose that $\xi \notin \text{spec}(H(P_0))$. Then $\xi \notin \text{spec}(H^{\text{Fr}}(P))$ for P in a neighborhood N_{P_0} of P_0 . The function $N_{P_0} \ni P \mapsto (H^{\text{Fr}}(P) - \xi)^{-1}$ is real analytic.*

Proof. First, suppose that $\xi \notin \text{spec}(H_\kappa(P))$ for κ sufficiently large and note that on $D(P_f^2 + N_f)$

$$H_\kappa(P) = H_\kappa(P_0) + \frac{1}{2}(P - P_0)^2 - (P - P_0) \cdot (P_f - P_0). \tag{A.1}$$

Consequently, for P in a small neighborhood N_{P_0} of P_0 , which a priori may depend on κ , the series on the r.h.s. below converges and defines the inverse of $(H_\kappa(P) - \xi)$:

$$\begin{aligned} & \frac{1}{H_\kappa(P) - \xi} \\ &= \frac{1}{H_\kappa(P_0) - \xi} \sum_{n=0}^\infty \left\{ - \left(\frac{1}{2}(P - P_0)^2 - (P - P_0) \cdot (P_f - P_0) \right) \frac{1}{H_\kappa(P_0) - \xi} \right\}^n. \end{aligned} \tag{A.2}$$

To eliminate the dependence of N_{P_0} on κ , we show in Lemma A.2 below, that

$$\|P_{f,j} \frac{1}{H_\kappa(P_0) - \xi}\| \leq c, \quad j = 1, 2, 3, \tag{A.3}$$

uniformly in κ .

Now we intend to take the limit $\kappa \rightarrow \infty$ on both sides of (A.2). By [21, Theorem VIII.23], if $\xi \notin \text{spec}(H^{\text{Fr}}(P))$, then $\xi \notin \text{spec}(H_\kappa(P))$ for κ sufficiently large and $(H_\kappa(P) - \xi)^{-1} \rightarrow (H^{\text{Fr}}(P) - \xi)^{-1}$ in norm. The same is true for P replaced with P_0 , and we can use (A.3) to exchange the limit $\kappa \rightarrow \infty$ with summation in (A.2). Thus the proof is complete. \square

Lemma A.2. *The following bounds hold uniformly in κ :*

$$\|P_{f,j}(H_\kappa(P) + i)^{-1}\| \leq c, \quad j = 1, 2, 3. \tag{A.4}$$

Proof. First, we recall some material from [14, Appendix A], referring there for more details. Let

$$\begin{aligned} T_{K,\kappa} &= \int_{\mathbb{R}^3} dk \beta_{K,\kappa} [a(k) - a^*(k)], \\ \beta_{K,\kappa}(k) &= -\sqrt{\alpha} \frac{1}{\sqrt{2\pi}} \frac{\chi_{[0,\kappa]}(|k|)}{|k|(1+k^2/2)} \chi_{[K,\infty)}(|k|), \end{aligned} \tag{A.5}$$

where K is chosen sufficiently large (depending on α but not on κ or P) as specified above Lemma A.3 of [14]. Let $H^{\text{free}}(P)$ denote the Hamiltonians (4.1) with $\alpha = 0$. Now the Gross-transformed Hamiltonians $\tilde{H}_\kappa(P) := e^{T_{K,\kappa}} H_\kappa(P) e^{-T_{K,\kappa}}$ are self-adjoint operators which converge in the norm-resolvent sense to a limiting Hamiltonian $\tilde{H}(P)$ [14, Proposition A.4]. As stated in the proof of this latter proposition, $H^{\text{free}}(P) \leq C'(\tilde{H}_\kappa(P) + C)$, with C, C' independent of κ . Hence,

$$\| |P - P_f| (\tilde{H}_\kappa(P) + C)^{-1/2} \|, \| N_f^{1/2} (\tilde{H}_\kappa(P) + C)^{-1/2} \| \leq C'. \tag{A.6}$$

Next, we can write on $D(H^{\text{free}}(P))$

$$e^{T_{K,\kappa}} P_f e^{-T_{K,\kappa}} = P_f + a(k\beta_{K,\kappa}) + a^*(k\beta_{K,\kappa}) + \int_{\mathbb{R}^3} dk k |\beta_{K,\kappa}(k)|^2, \tag{A.7}$$

where the last term is actually zero by symmetry. Noting the bound $\int dk |k\beta_{K,\kappa}(k)|^2 \leq c$, uniformly in κ , we have $\|a^{(*)}(k\beta_{K,\kappa})(1 + N_f)^{-1/2}\| \leq c$ uniformly in κ . Therefore, by estimates (A.6),

$$\begin{aligned} \|P_{f,j}(H_\kappa(P) + C)^{-1}\| &\leq C_0(\|P_{f,j}(\tilde{H}_\kappa(P) + C)^{-1}\| \\ &\quad + \|(1 + N_f)^{1/2}(\tilde{H}_\kappa(P) + C)^{-1}\|) \end{aligned} \tag{A.8}$$

is uniformly bounded in κ . This concludes the proof. \square

B. Proof of Proposition 5.1

Let us consider first polaron-type models satisfying Condition C. For the standing assumption 0 we refer to the discussion of Lemma 3.1, part 6. By part 2 of Lemma 3.1, \mathcal{I}_0 contains neighborhoods of all the global minima of $|P| \mapsto E_r(|P|)$. Thus, considering other items of this lemma, it suffices to show that there is a finite number of such minima to complete the proof of Proposition 5.1 (a). To this end, we first note that by parts 0 and 1 of Lemma 3.1, for any sequence $\{P_n\}_{n \in \mathbb{N}}$, $E_{\text{ess}}(P_n)$ is bounded if $\omega(P_n)$ is bounded. Furthermore, since $\omega(k) \geq c_0 > 0$, we have $E_{\text{ess}}(P) \geq E(0) + c_0$. Now by part 4 of the same lemma, all global minima must be localized in a compact set. (In particular, $P \mapsto E(P)$ cannot be a constant function in an open set, hence everywhere). Now suppose there is a finite accumulation point $(Q_*, E_r(Q_*))$ of the set of global minima. Since the spectrum is closed, the corresponding sphere belongs to the spectrum. By Lemma 3.1 parts 2 and 3, for $|P| = Q_*$, $E(P)$ is an isolated eigenvalue of $H(P)$. Thus, by analyticity, this eigenvalue can be continued to some neighborhood of Q_* in the radial direction. As any neighborhood contains infinitely many global minima, we conclude that $P \mapsto E(P)$ is constant near $|P| = Q_*$, hence everywhere, which is a contradiction.

Let us now move on to part (b). For the models in question we have $\Pi = F e^{iPt} x$, where F is the unitary Fourier transform from x to P variable. We decompose $\varphi \otimes \Omega = \Pi^* \int_{\mathbb{R}^d}^\oplus dP \phi_P$. Thus to find ϕ_P we compute

$$\Pi(\varphi \otimes \Omega) = \int_{\mathbb{R}^d}^\oplus dP \hat{\varphi}(P)\Omega, \tag{B.1}$$

which gives $\phi_P = \hat{\varphi}(P)\Omega$. The function $P \mapsto |\langle \phi_P, \psi_P \rangle| = |\hat{\varphi}(P)| |\langle \Omega, \psi_P \rangle|$ is continuous by continuity of $\hat{\varphi}$ and analyticity of $P \mapsto |\psi_P\rangle \langle \psi_P|$. It is nonzero by our assumption on $\hat{\varphi}$ and by part 5 of Lemma 3.1.

Concerning the Fröhlich polaron, standing assumption 0 is proven in ‘‘Appendix A.’’ The other claims are verified as above, using Lemma 4.1.

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