

# Ionosphere electron density modeling using the regularized constraint optimization approach

Ganesh Lalgudi Gopalakrishnan and Michael Schmidt

Deutsches Geodätisches Forschungsinstitut der Technischen Universität München (DGFI-TUM)

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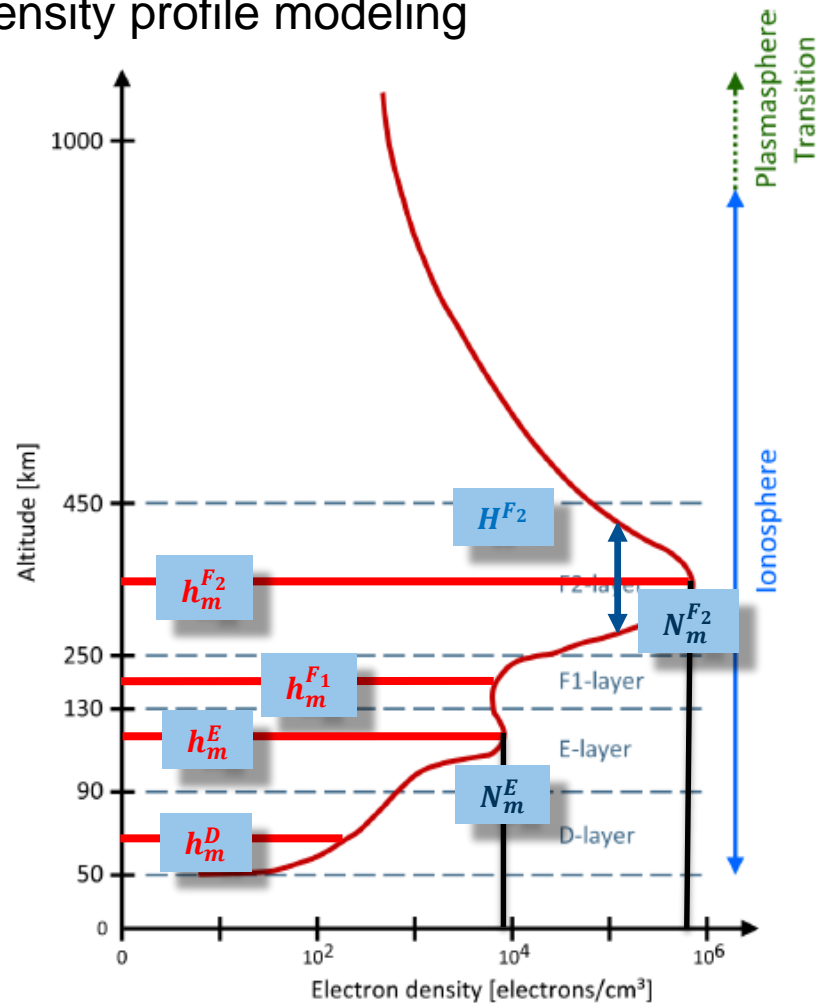
Contact : ganesh.lalgudi-gopalakrishnan@tum.de

Vienna, 10 April 2019



# Electron density profile modeling

**Multi-Chapman layer** approach for vertical electron density profile modeling



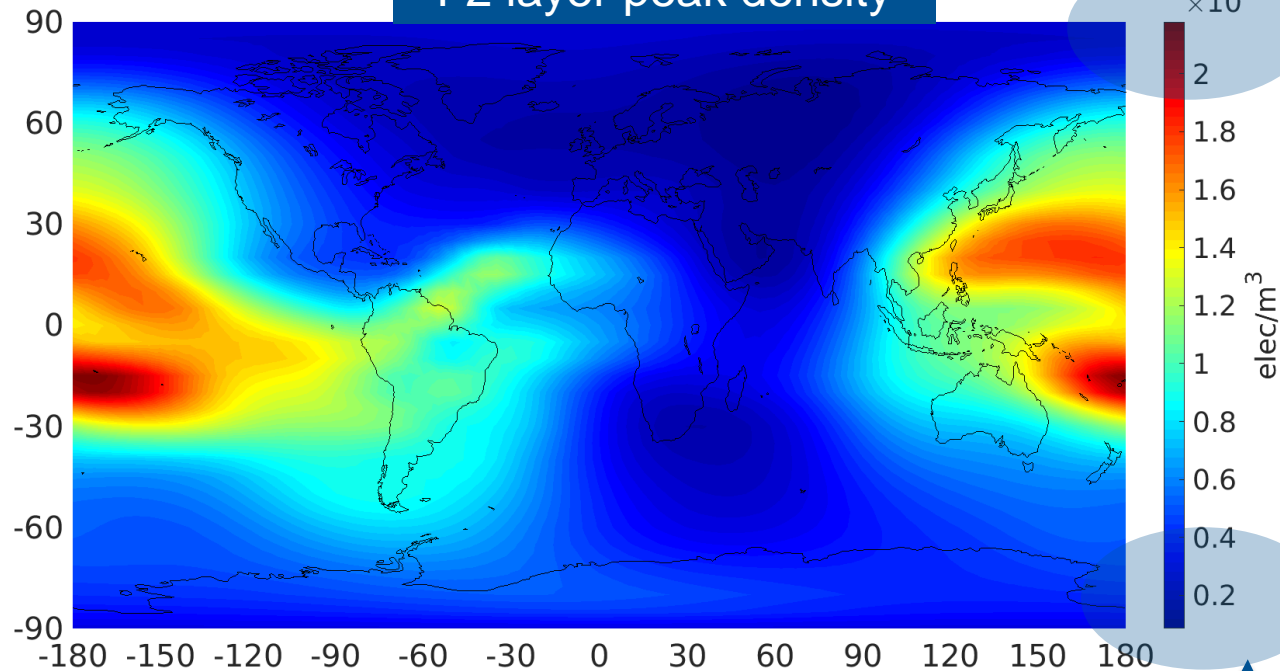
- Ionosphere could be divided into **discrete layers** (D, E, F1, F2 layers and plasmasphere)
- Electron density of each layer varies with height in a deterministic manner, e.g. following the **Chapman profile function**
- The Chapman profile function is controlled by **three key parameters**, namely the **peak density** (maximum), the corresponding **peak height** and the **scale height**
- In this work, our aim is the **modelling** of the ionosphere key parameters

# Estimated key parameters – F2 layer peak density and peak height

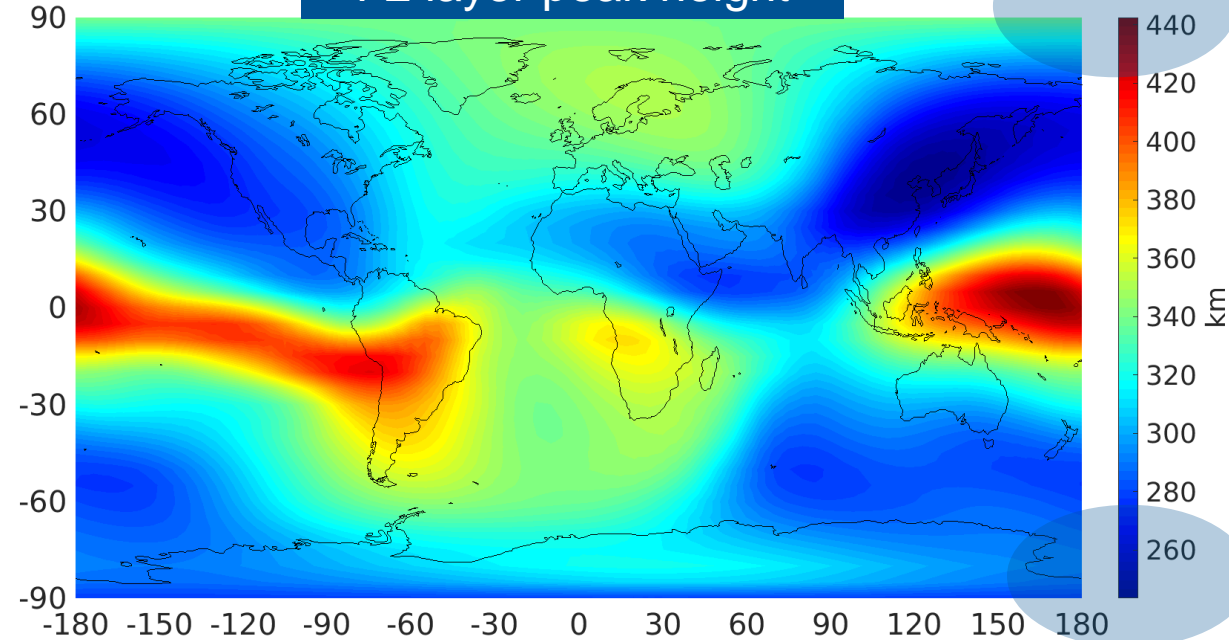
In this work, the ionospheric key parameters are estimated from observations or functionals of the electron density.

F2 layer peak density

Inequality for upper bound



F2 layer peak height



Estimation of key parameters involves application of constraints relating to the bounds of key parameters and using regularization.

Inequality for upper and lower bound

# Electron density profile modeling

- **Multi-layer model** is defined as the sum of several **Chapman profile functions** [Feltens, 2007], one for each layer, i.e.

$$\begin{aligned}
 N_e(h) &= N_e^D(h) + N_e^E(h) + N_e^{F_1}(h) + N_e^{F_2}(h) + N_e^P(h) \\
 &= \sum_{Q=1}^4 N_m^Q p^Q(h) + N_0^P p^P(h)
 \end{aligned}$$

with  $Q \in \{D, E, F_1, F_2\}$  and  $P = \text{plasmasphere}$ .

- The **profile function**  $p^Q(h)$  represents the electron density distribution of the layer  $Q$ .
- It is characterized by the **peak height**  $h_m^Q$ , the **peak density** value  $N_m^Q$  and the **scale height**  $H^Q$ .
- The **plasmasphere profile function**  $p^P(h)$  is characterized by the **plasmasphere scale height**  $H^P$ .

# Electron density profile modeling

- Multi-layer Chapman model characterized by a **set**  $\mathcal{K}$  of **key parameters**, i.e.

$$\begin{aligned}\mathcal{K} &= \{N_m^D, h_m^D, H^D, N_m^E, h_m^E, H^E, N_m^{F_1}, h_m^{F_1}, H^{F_1}, N_m^{F_2}, h_m^{F_2}, H^{F_2}, N_0^P, H^P\} \\ &= \{\kappa_1, \kappa_2, \dots, \kappa_r, \dots, \kappa_R\}\end{aligned}$$

- For each key parameter  $\kappa_r$  we may set up a series expansion in **2-D tensor products** of (endpoint-interpolating) **Uniform polynomial B-Spline (UBS) functions**  $N_{J,k}^2(x)$  with level value  $J \in \{J_1, J_2\}$ , shift  $k \in \{k_1, k_2\}$  and variable  $x \in \{\lambda, \varphi\}$ .

[Limberger, 2015], [Liang, 2017]

# Challenges in electron density modelling

- We provide here a **closed-loop simulation** in order to demonstrate that the model parameters can be estimated by considering appropriate equality and inequality constraints
- Typically in electron density modelling problem (depending on the B-Spline levels) we may have **~300 - 1700 parameters** to estimate (per key parameter).
- Simulated observations are sampled from a  $5^\circ \times 5^\circ$  grid using the **IRI model** with a non-uniform altitude step size sampling (between 80 to 1000 km). The F2 layer is sampled with smaller intervals than the remaining layers
- Altogether we have around **250000 observations** electron density observations.
- When multiple key parameters are to be estimated, their the B-spline coefficients of the corresponding parameters are highly **correlated**.
- The **sensitivity** of the coefficients of the different parameters are different.
- Furthermore, several **constraints** need to be imposed on the bounds of the different key parameters.
- Therefore, classical least squares based solution is not optimal for modelling. We use the technique of **inequality constrained convex optimization**, see Röse-Körner (2015).

# Convex optimization approach

- In order to solve the aforementioned challenges, we apply constraints on the model parameters in the optimization problem.
- Constraints could be imposed in the form of **equality** or **inequality** constraints.

## Example of equality constraints

- Set one (or more) unknown parameter (coefficients) to a fixed value; the ionospheric D layer peak density could, e.g. be set to zero

## Example of inequality constraints

- Set one (or more) unknown parameters to a larger (or smaller) value than a given bound
  - Peak electron density of F2 layer shall be always positive
  - Peak height of F2 layer shall always be larger than that of F1 layer

# Convex optimization approach

The approach for estimation of B-Spline coefficients (using optimization approach) is as follows:

- The **cost function** (shown in previous slide) and its **Lagrangian** are expressed in “**standard form**” (quadratic form).

A system of equation in the form  $A x = l + \varepsilon$  has a quadratic standard form  $x^T Q x - 2x^T q + l^T P l$  which is derived from the residual sum of squares  $\frac{1}{2} (A x - l)^T P (A x - l)$ ; ( $Q = A^T P A$ ) and ( $q = A^T P l$ )

$x$  is the unknown parameter vector,  $l$  is the observation vector,  $\varepsilon$  is the observation noise and  $P$  is the inverse covariance matrix

- The constraints are transformed to their **convex** representation ( $B x \leq b$ ), where  $B$  is the **constraint matrix** (consisting of 0, 1 or -1),  $x$  is the **vector of B-Spline coefficients** (for the chosen key parameters) and  $b$  is the **vector of bounds** on the optimization variable ( $x$ ).



# Convex optimization approach with constraints

How do we solve the system of (electron density observation) equations subject to **inequality constraints** ?

We use **convex optimization** approach.

Representing an inequality constraint in convex form  
(for one key parameter – F2 layer peak density  $N_m^{F2}$  )

This signifies if or not a constraint is imposed

$$Bx \leq b$$

Each element on LHS of inequality is less than or equal to the corresponding element on RHS



0 signifies NO CONSTRAINT  
+1 or -1 signifies CONSTRAINT

0	• $[M]_d^{N_m^{F2}}$	$[d^{N_m^{F2}}] \leq 0$	← No constraint on $N_m^{F2}$
-1	• $[M]_d^{N_m^{F2}}$	$[d^{N_m^{F2}}] \leq 0$	← Lower bound constraint on $N_m^{F2}$

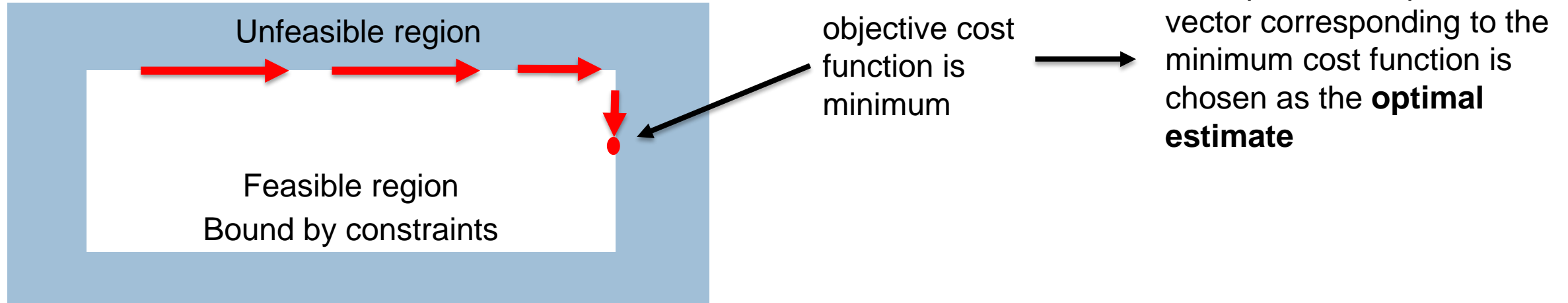
Jacobian matrix for  
 $N_m^{F2}$  coefficients

B-spline  $N_m^{F2}$   
coefficients

Constraint bounds

# Convex optimization approach

- The constraints define an **feasibility region** (white box in the figure below) over which the cost function is evaluated using **Active set approach**. The main idea is to **traverse** (red arrows) the feasibility region and determine the point where the objective cost function is minimum with respect to the optimization variable.

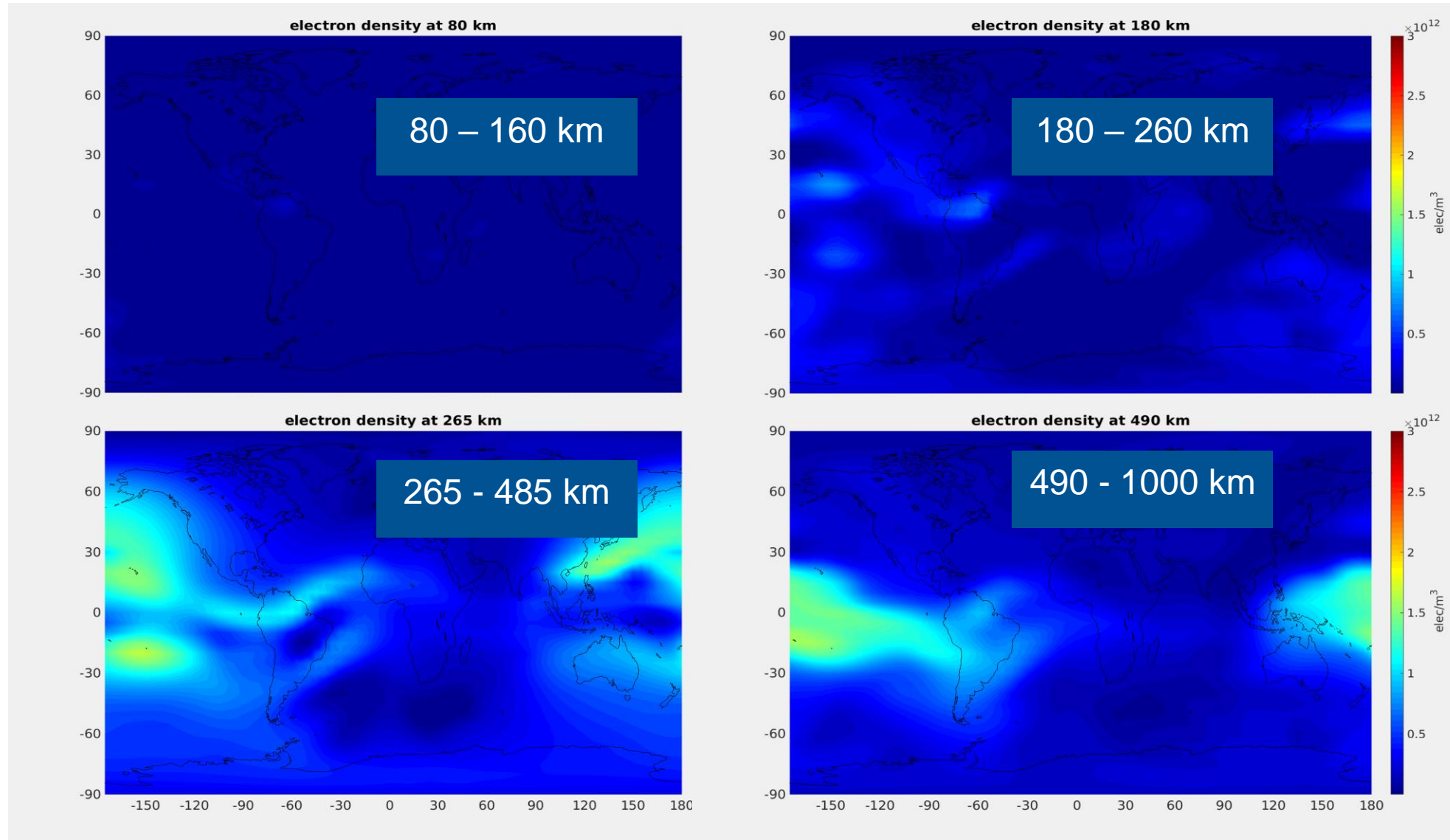


- The **traversal** along the feasibility region includes the determination of a general **direction** (of the red arrows) and a magnitude of **step** size to move from one point to another. The constraints are the boundary of the feasibility region. So the direction and step shall not violate the constraints (be always inside the feasibility region).

# Electron density observations (~250000 observations per epoch)

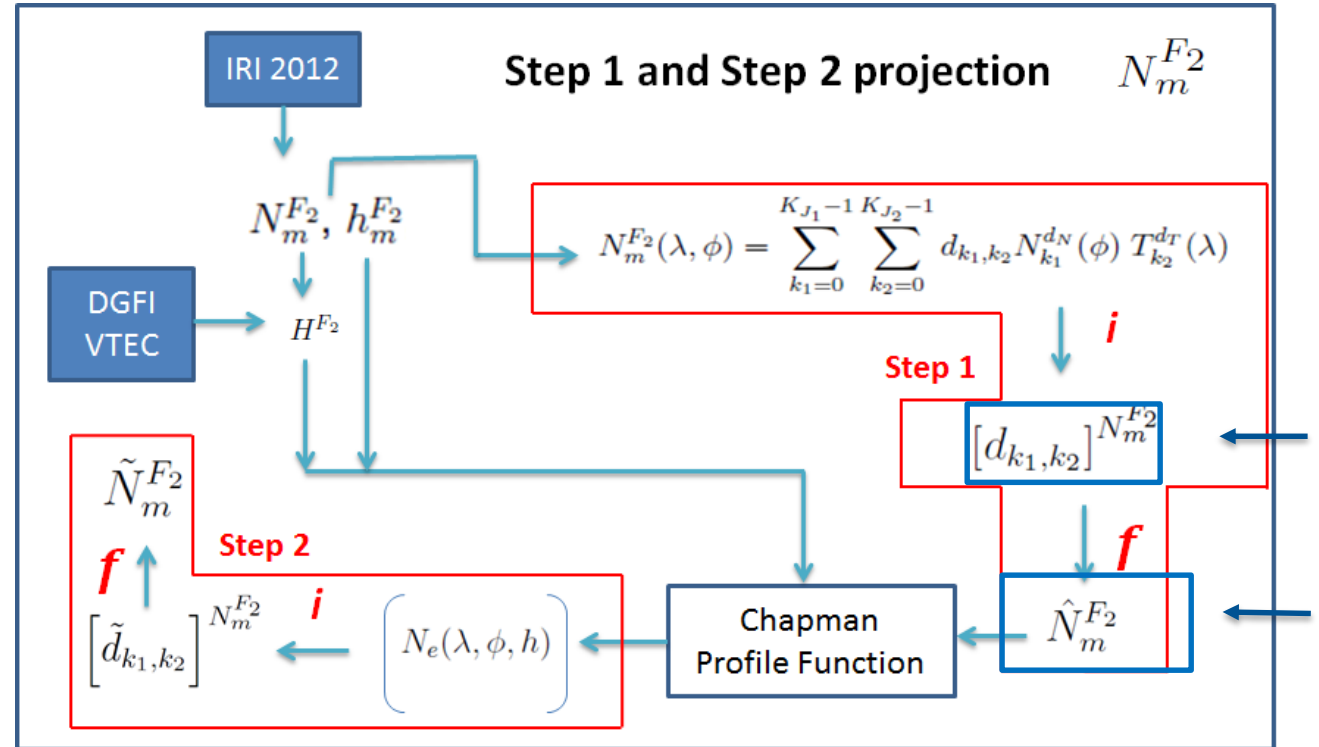
**Electron density observations** are generated from the IRI model

Estimated Electron density values are related to the different key parameters of the **Chapman profile function**.



# Electron density modelling – peak density estimation

- **Step 1** projection refers to the estimation of “**true**” set of B-Spline coefficients.
- **Step 2** projection refers to the estimation of modelled B-Spline coefficients from electron density observations.
- This kind of architecture allows **closed loop validation** of the algorithms used.



“true” set of B-Spline coefficients

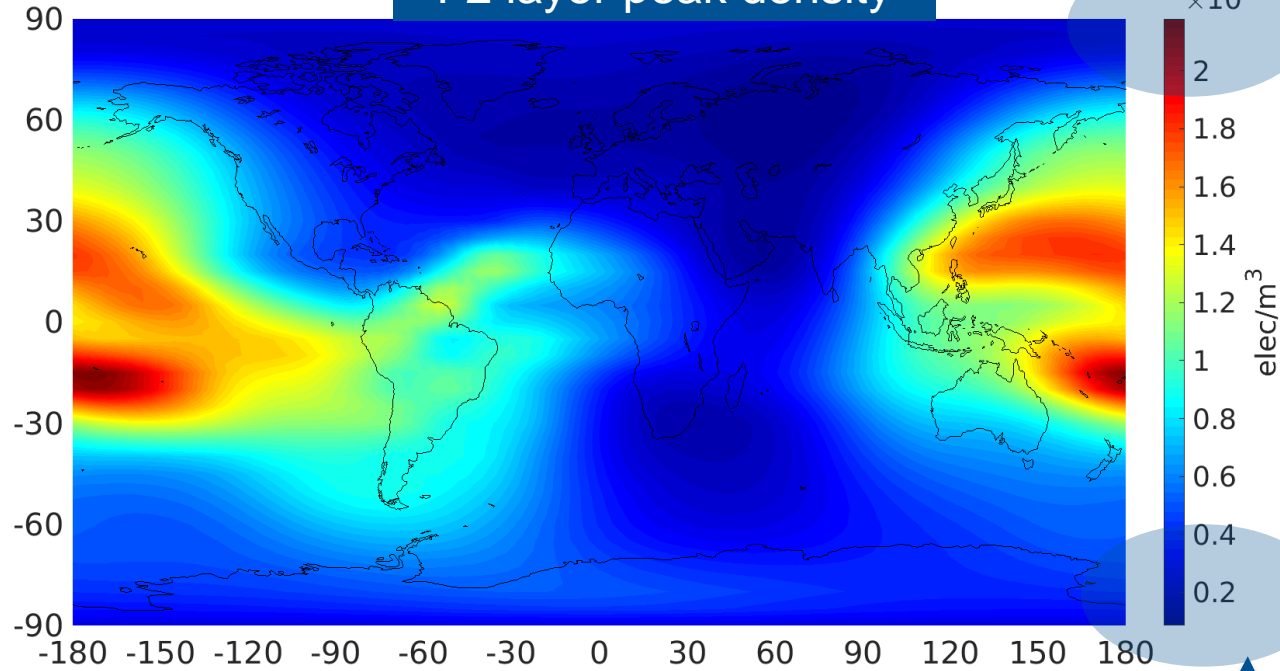
“true” set of peak density

- $i$  refers to an inverse projection
- $f$  refers to forward projection

# Estimated key parameters – F2 layer peak density and peak height

In this work, the ionospheric key parameters are estimated from observations or functionals of the electron density.

F2 layer peak density



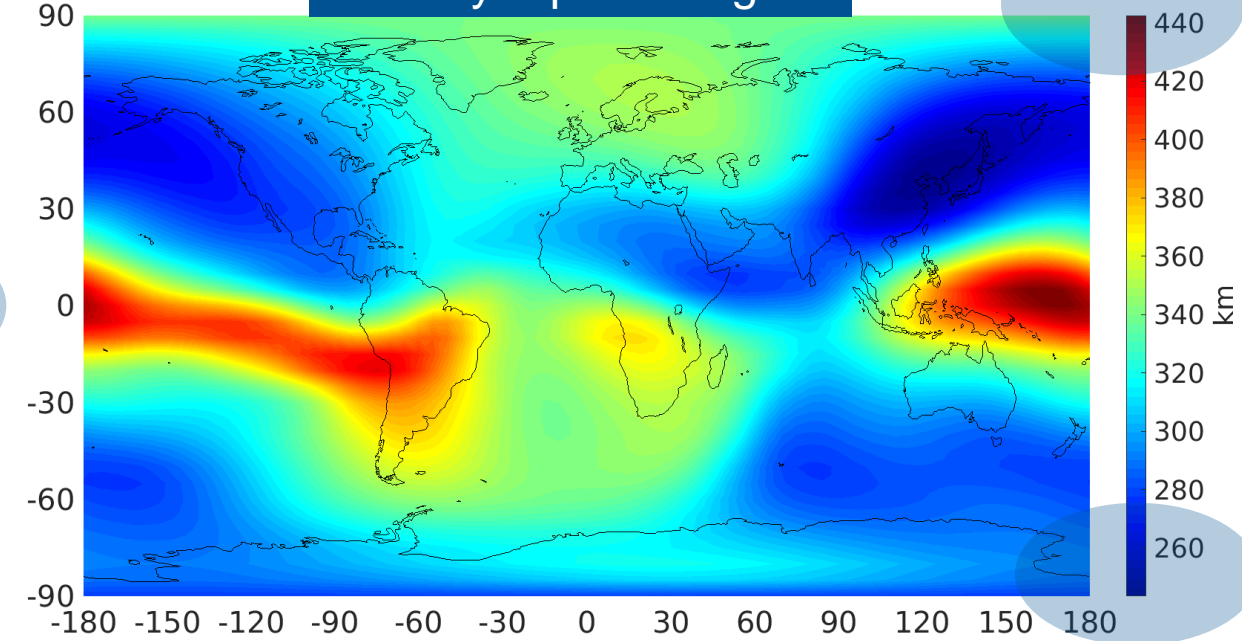
Inequality for upper bound



$\times 10^{12}$

elec/m<sup>3</sup>

F2 layer peak height



Inequality for upper and lower bound



Estimation of key parameters involves application of constraints relating to the bounds of key parameters and using regularization.

# Estimated key parameter with noisy observations

- Using the electron density observations, **several key parameters can be** modelled and their corresponding coefficients can be **estimated simultaneously**.
- In this presentation we show the key parameters modelled separately for clarity of analysis and clarity of the presented results.
- The **key parameter F2 layer peak density** is estimated with 1% measurement noise (1% of the peak magnitude of the electron density is added as additive white noise).
- 1% measurement noise can also be interpreted as adding noise up to 2 order of magnitude smaller than that of the electron density observations.
- The **accuracy** (projection error) and **relative accuracy** (relative projection error) under “additive noise” scenario are shown for the estimated **peak density** in subsequent slides (for comparison to “no noise” scenario).

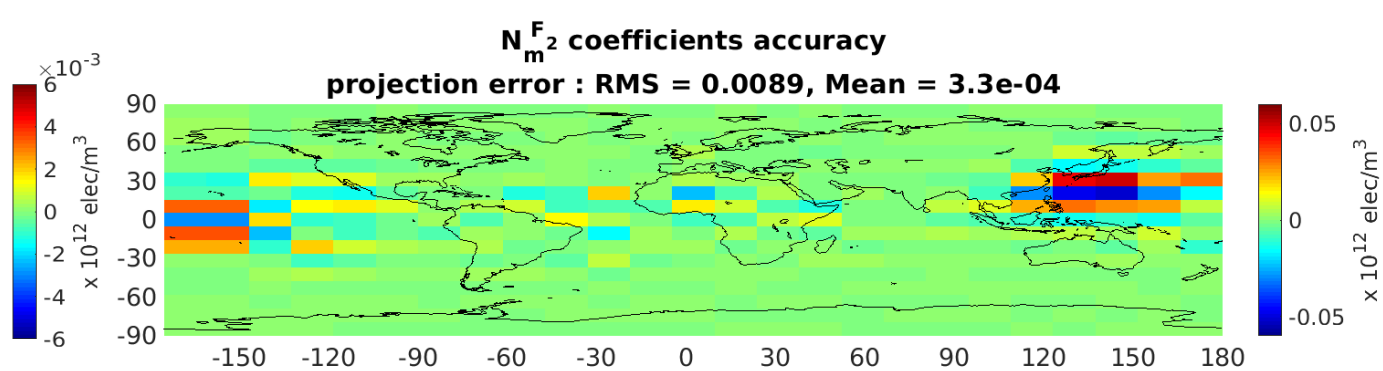
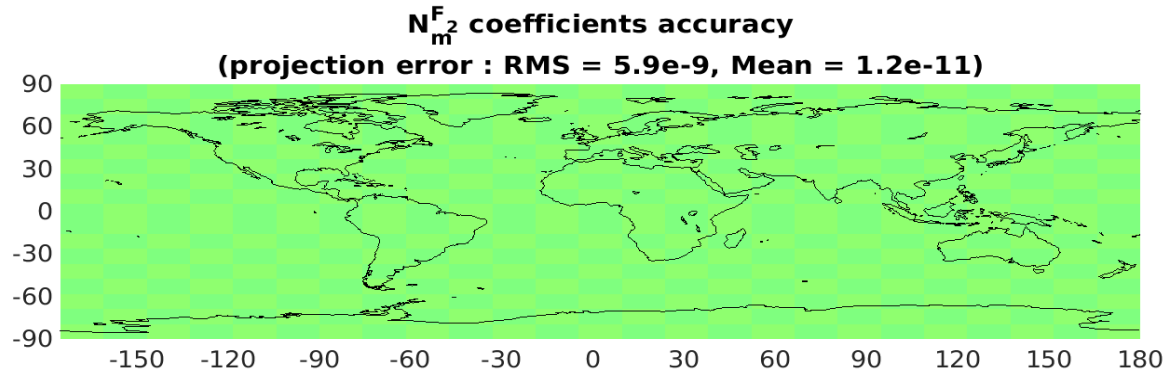
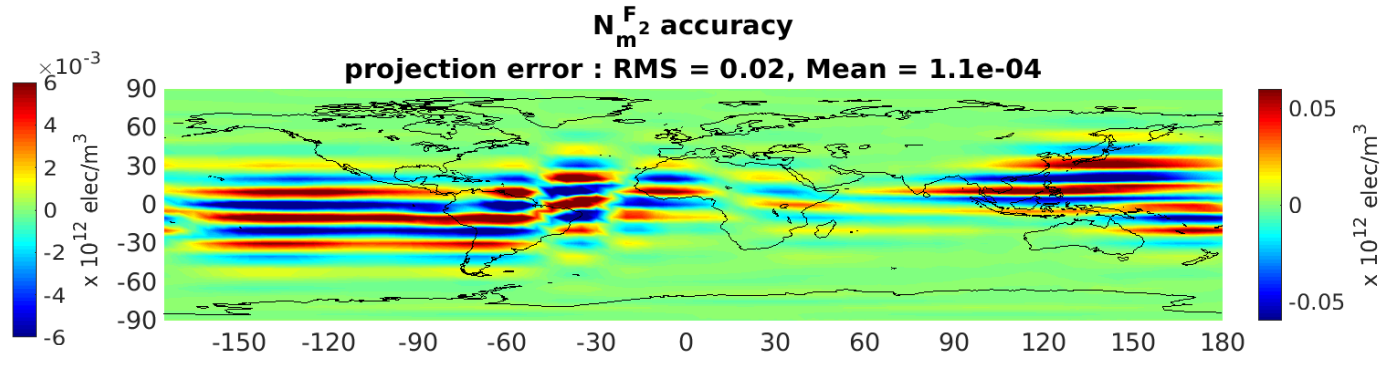
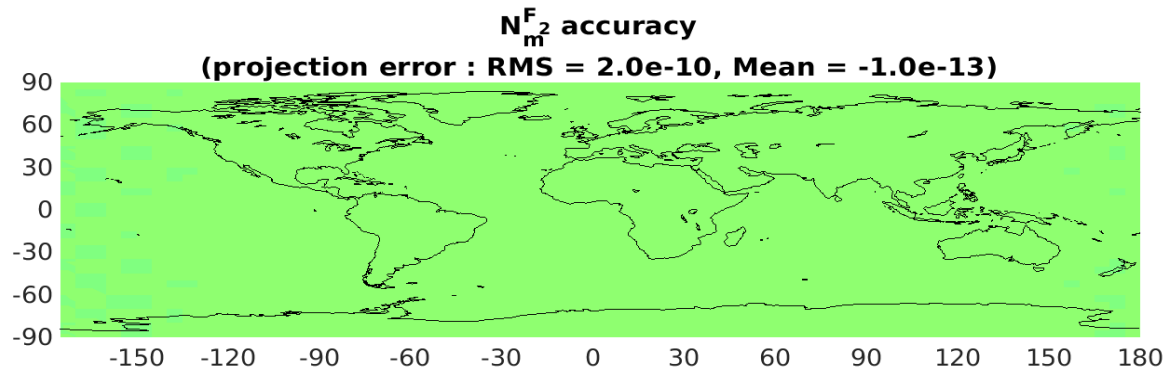


# F2 layer peak density accuracy

Peak density accuracy	RMS ( x 10 <sup>12</sup> elec/m <sup>3</sup> )	Mean ( x 10 <sup>12</sup> elec/m <sup>3</sup> )
No noise	2.0e-10	-1.0e-13%
1 % noise	0.02	0.05 %

The small values of accuracy and relative accuracy (next slide) for the peak density confirm the **validity of the algorithm** (for standalone estimation).

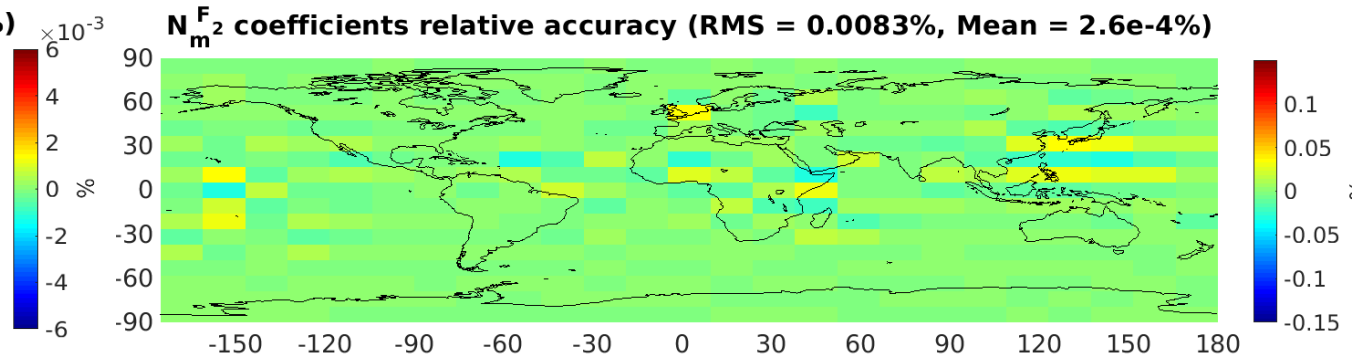
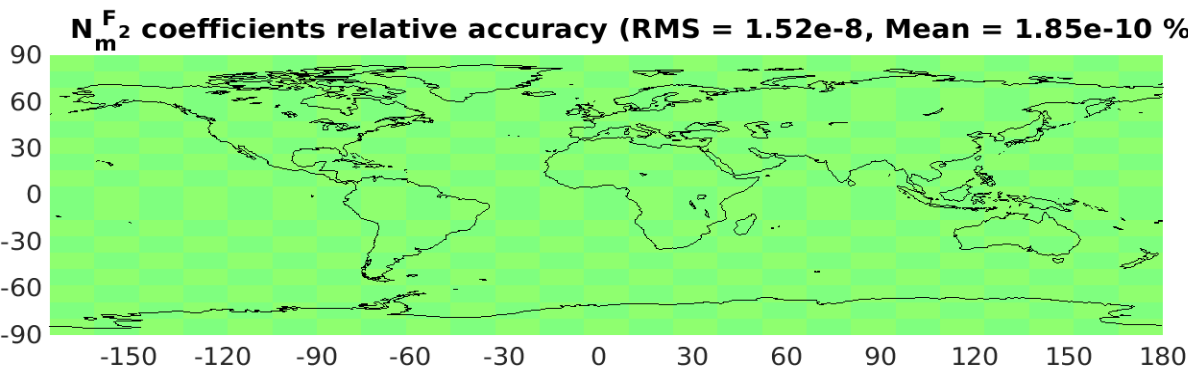
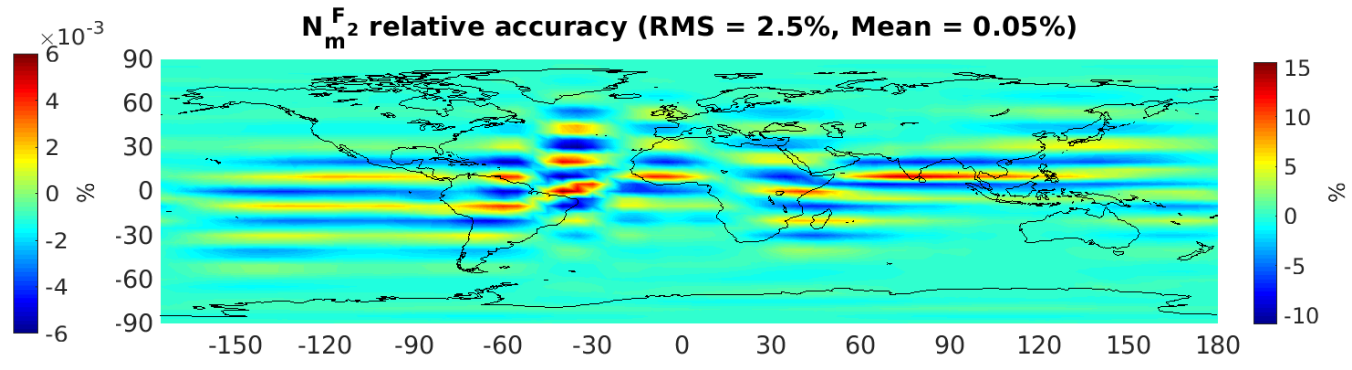
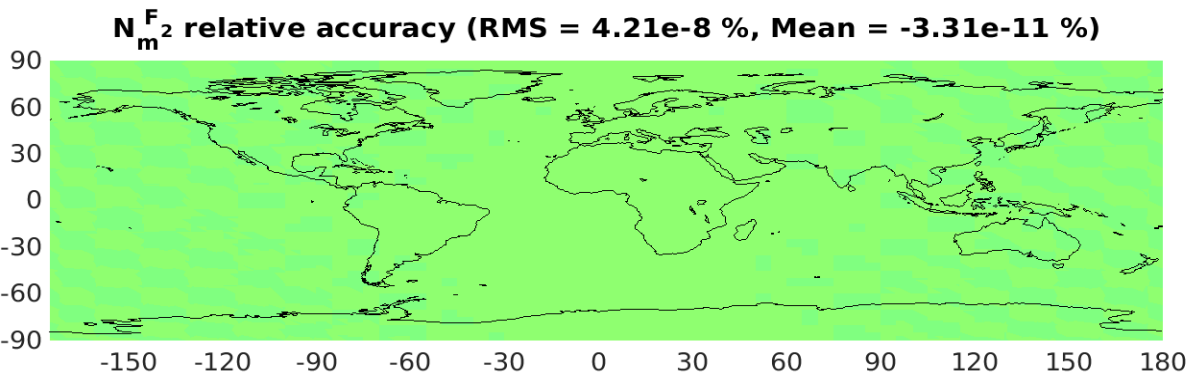
$$\Delta \tilde{N}_m^{F_2}(\lambda, \phi) = \tilde{N}_m^{F_2}(\lambda, \phi) - \hat{N}_m^{F_2}(\lambda, \phi)$$



# F2 layer peak density relative accuracy

Peak density relative accuracy	RMS (%)	Mean (%)
No noise	4.21e-8	-3.31e-11
1 % noise	2.5	0.05

$$N_m^{F_2} \text{ Relative accuracy} = \frac{\tilde{N}_m^{F_2} - \hat{N}_m^{F_2}}{\hat{N}_m^{F_2}}$$





# Summary

- We used **B-Splines** for ionosphere modelling from observations of electron density
- Electron density modelling is posed as a B-Spline coefficient estimation problem
- We have shown the example of F2 layer key parameters in this presentation
- Minimization of the objective cost function is performed **subject to inequality constraints and regularization**
- Inequality constraints and regularization are found to be necessary for simultaneous estimation of the coefficients of multiple key parameters

# References

- Erdogan, E., Schmidt, M., Seitz, F., & Durmaz, M. 2017. Near real-time estimation of ionosphere vertical total electron content from GNSS satellites using B-splines in a Kalman filter. *Annales Geophysicae*, 35(2), 263–277
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- Limberger, M. 2015. *Ionosphere modeling from GPS radio occultations and complementary data based on B-splines*, Deutsche Geodätische Kommission, Reihe C, 755, München.