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OPTIMAL ESTIMATION OF A SUBSET OF INTEGERS WITH APPLICATION TO GNSS

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ABSTRACT. The problem of integer or mixed integer/real valued parameter estimation in linear models is considered. It is a well-known result that for zero-mean additive Gaussian measurement noise the integer least-squares estimator is optimal in the sense of maximizing the probability of correctly estimating the full vector of integer parameters. In applications such as global navigation satellite system ambiguity resolution, it can be beneficial to resolve only a subset of all integer parameters. We derive the estimator that leads to the highest possible success rate for a given integer subset and compare its performance to suboptimal integer mappings via numerical studies. Implementation aspects of the optimal estimator as well as subset selection criteria are discussed.

Keywords: Integer estimation, GNSS ambiguity resolution, Partial fixing, Success rate

1. INTRODUCTION

We study a system of linear or linearized observation equations in the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{a} + \boldsymbol{\eta}, \tag{1}$$

with $\boldsymbol{a} \in \mathbb{Z}^n$ the vector of unknown integer parameters, which are linked to the vector of observations $\boldsymbol{y} \in \mathbb{R}^q$ via the full-rank matrix $\boldsymbol{A} \in \mathbb{R}^{q \times n}$. The noise vector $\boldsymbol{\eta} \in \mathbb{R}^q$ is assumed to follow a zero mean Gaussian distribution with covariance matrix $\boldsymbol{Q}$. The estimation of $\boldsymbol{a}$ is usually decomposed into two steps (Teunissen, 1993). In the first step the integer property of $\boldsymbol{a}$ is simply disregarded and the so called float solution $\hat{\boldsymbol{a}}$ is computed via a standard linear least-squares estimation

$$\hat{\boldsymbol{a}} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{A}\right)^{-1}\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{y}. \tag{2}$$

This float solution $\hat{\boldsymbol{a}}$ is a minimal sufficient statistic, i.e., it contains the same information about the unknown $\boldsymbol{a}$ as the observation $\boldsymbol{y}$ itself. The second step, which takes into account the integer property of $\boldsymbol{a}$, can therefore be based on $\hat{\boldsymbol{a}}$ instead of $\boldsymbol{y}$. From the law of error propagation we know that $\hat{\boldsymbol{a}}$ follows a Gaussian distribution:

$$\hat{\boldsymbol{a}} \sim \mathcal{N}\left(\boldsymbol{a}, \boldsymbol{Q}_{\hat{a}}\right) \quad \text{with} \quad \boldsymbol{Q}_{\hat{a}} = \left(\boldsymbol{A}^{\mathrm{T}}\boldsymbol{Q}^{-1}\boldsymbol{A}\right)^{-1}. \tag{3}$$

In global navigation satellite system (GNSS) applications, only some of the unknown parameters are integer valued, i.e., the observation model is of the form

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{a} + \boldsymbol{B}\boldsymbol{b} + \boldsymbol{\eta}, \tag{4}$$

with $\boldsymbol{b} \in \mathbb{R}^p$ the vector of real valued unknowns such as incremental receiver coordinates or instrumental and atmospheric delays and $\boldsymbol{B} \in \mathbb{R}^{q\times p}$. The measurement vector $\boldsymbol{y}$ usually contains code and carrier phase measurements, and $\boldsymbol{a}$ are the carrier phase integer ambiguities. The estimation of $\boldsymbol{a}$ and $\boldsymbol{b}$ can now be decomposed into three steps, where the first one is again the float solution $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ from a linear least-squares estimation

$$\begin{bmatrix} \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{b}} \end{bmatrix} = \left(\begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{B}^{\mathrm{T}} \end{bmatrix} \boldsymbol{Q}^{-1} \begin{bmatrix} \boldsymbol{A} & \boldsymbol{B} \end{bmatrix} \right)^{-1} \begin{bmatrix} \boldsymbol{A}^{\mathrm{T}} \\ \boldsymbol{B}^{\mathrm{T}} \end{bmatrix} \boldsymbol{Q}^{-1}\boldsymbol{y}. \tag{5}$$

The float solution is Gaussian distributed as

$$\begin{bmatrix} \hat{\boldsymbol{a}} \\ \hat{\boldsymbol{b}} \end{bmatrix} \sim \mathcal{N}\left(\begin{bmatrix} \boldsymbol{a} \\ \boldsymbol{b} \end{bmatrix}, \begin{bmatrix} \boldsymbol{Q}_{\hat{a}} & \boldsymbol{Q}_{\hat{a}\hat{b}} \\ \boldsymbol{Q}_{\hat{b}\hat{a}} & \boldsymbol{Q}_{\hat{b}} \end{bmatrix} \right) \tag{6}$$

and delivers a minimal sufficient statistic. In the second step the float solution $\hat{\boldsymbol{a}}$ is used to resolve some or all of the elements of $\boldsymbol{a}$ as integer values. The motivation to do so is usually to improve the precision of the estimate of $\boldsymbol{b}$ compared to $\hat{\boldsymbol{b}}$. Thus, the correlation between $\hat{\boldsymbol{a}}$ and $\hat{\boldsymbol{b}}$ is utilized in the third and final step to correct $\hat{\boldsymbol{b}}$ for the resolved integers. In GNSS positioning with short observation time spans, ambiguity resolution can improve the position accuracy by roughly two orders of magnitude compared to the float solution, depending on the chosen subset of ambiguities and given that the estimated integers are correct. The performance analyses of partial ambiguity resolution (PAR) techniques in, e.g., Verhagen et al. (2011), Odijk et al. (2014), Nardo et al. (2016), and Brack (2016) show that PAR can be very beneficial for obtaining both fast and reliable positioning results.

The focus of this contribution is on how to find an integer estimate for a certain subset of integer parameters based on $\boldsymbol{Q}_{\hat{a}}$ and the realization of $\hat{\boldsymbol{a}}$, which applies to both discussed observation models. The integer fixing takes place either in the original n dimensional parameter space of $\boldsymbol{a}$ or after the reparameterization

$$\hat{\boldsymbol{a}}' = \boldsymbol{Z}\hat{\boldsymbol{a}}, \quad \boldsymbol{Q}_{\hat{a}'} = \boldsymbol{Z}\boldsymbol{Q}_{\hat{a}}\boldsymbol{Z}^{\mathrm{T}}, \tag{7}$$

where $\boldsymbol{Z} \in \mathbb{Z}^{n\times n}$ and $\operatorname{abs}(\det \boldsymbol{Z}) = 1$, such that $\boldsymbol{Z}^{-1}$ is also an integer unimodular matrix. Applying such a transformation does not affect the outcome of a search for closest n dimensional integer grid points in the metric defined by $\boldsymbol{Q}_{\hat{a}}$ (Agrell et al., 2002), thus leaving the result of integer least-squares (ILS) unchanged. It does, however, fundamentally change the problem of partial integer fixing, irrespective of the integer mapping that is used. Common choices of $\boldsymbol{Z}$ for PAR in the context of GNSS are widelaning techniques for ambiguities on multiple frequencies (Hatch, 1982) or decorrelation/reduction transformations (e.g., Jazaeri et al., 2014). Let the set of indexes that correspond to the integer parameters that are resolved as integers and kept as float values be given by $\mathcal{I}$ and $\bar{\mathcal{I}}$, respectively (with $\mathcal{I} \cup \bar{\mathcal{I}} = \{1, \ldots, n\}$, $\mathcal{I} \cap \bar{\mathcal{I}} = \emptyset$). All parameters with an index in $\mathcal{I}$ are fixed to integers via the mapping $\mathcal{S}(\cdot)$:

$$\check{\boldsymbol{a}}' = \mathcal{S}\left(\hat{\boldsymbol{a}}'\right), \quad \mathcal{S}(\cdot)\colon \mathbb{R}^n \mapsto \mathbb{Z}^{|\mathcal{I}|}. \tag{8}$$

Note that generally only a subset of the integer parameters is resolved, i.e., the integer solution $\check{\boldsymbol{a}}'$ is of dimension $|\mathcal{I}|$ instead of n, where $|\mathcal{I}|$ denotes the cardinality of the set $\mathcal{I}$. The goal is now to design the mapping function $\mathcal{S}(\cdot)$ such that the probability of a correct integer estimate $P\left(\check{\boldsymbol{a}}' = \boldsymbol{Z}_{\mathcal{I}}\boldsymbol{a}\right)$ is as large as possible for the given index set (subscripts $\mathcal{I}$ are used to select all entries of a vector/rows of a matrix with index $i \in \mathcal{I}$). If system model (4) is employed, the float solution $\hat{\boldsymbol{b}}$ is then corrected for the fixed integers

$$\check{\boldsymbol{b}} = \hat{\boldsymbol{b}} - \boldsymbol{Q}_{\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}'_{\mathcal{I}}}\boldsymbol{Q}_{\hat{\boldsymbol{a}}'_{\mathcal{I}}}^{-1}\left(\hat{\boldsymbol{a}}'_{\mathcal{I}} - \check{\boldsymbol{a}}'\right), \tag{9}$$

with $\boldsymbol{Q}_{\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}'_{\mathcal{I}}} = \boldsymbol{Q}_{\hat{\boldsymbol{b}}\hat{\boldsymbol{a}}}\boldsymbol{Z}_{\mathcal{I}}^{\mathrm{T}}$ and $\boldsymbol{Q}_{\hat{\boldsymbol{a}}'_{\mathcal{I}}} = \boldsymbol{Z}_{\mathcal{I}}\boldsymbol{Q}_{\hat{\boldsymbol{a}}}\boldsymbol{Z}_{\mathcal{I}}^{\mathrm{T}}$.

The remainder of this paper is organized as follows. In Section 2, we generalize the class of integer estimators as introduced in Teunissen (1999) such that it is capable of partial integer estimation. Two simple but suboptimal examples of partial integer estimators are given in Section 3. This raises the question of which estimator to prefer. We introduce the estimator that leads to the highest probability of correct integer estimates in Section 4 and prove its optimality. In Section 5, some computational aspects of the optimal estimator are studied, and criteria for selecting the integer subset in GNSS applications are given in Section 6. A comparison between the optimal and suboptimal estimators is presented in Section 7. For the sake of notational simplicity, $\hat{\boldsymbol{a}}$ and $\check{\boldsymbol{a}}$ will be used instead of $\hat{\boldsymbol{a}}'$ and $\check{\boldsymbol{a}}'$ in Sections 2–5 and 7.

2. PARTIAL INTEGER ESTIMATION

The problem is to determine an integer estimate of the parameters $\boldsymbol{a}_{\mathcal{I}}$ with the a-priori defined index subset $\mathcal{I}$ based on the realization of the float solution $\hat{\boldsymbol{a}}$, i.e., to define a mapping $\mathcal{S}(\cdot)\colon \mathbb{R}^n \mapsto \mathbb{Z}^{|\mathcal{I}|}$. The probably most intuitive approach is to only consider the part $\hat{\boldsymbol{a}}_{\mathcal{I}}$ of the float solution $\hat{\boldsymbol{a}}$, which lies in the subspace that corresponds to the parameters that are to be resolved as integers, and to apply an integer estimator such as ILS to $\hat{\boldsymbol{a}}_{\mathcal{I}}$. This implies that only the information $\hat{\boldsymbol{a}}_{\mathcal{I}}$ is used for computing the integer estimate whereas $\hat{\boldsymbol{a}}_{\bar{\mathcal{I}}}$ is not, meaning that the parameters $\boldsymbol{a}_{\bar{\mathcal{I}}}$ that are not resolved as integers are treated as if they were conceptually real valued. According to the system models (1) and (4), the parameters $\boldsymbol{a}_{\bar{\mathcal{I}}}$ are, however, integers, which should not be ignored when defining the partial integer mapping.

The most general approach is to assign a subset $S_{\boldsymbol{z}} \subset \mathbb{R}^n$ to each integer vector $\boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}$, which implicitly defines the integer mapping $\mathcal{S}(\cdot)$:

$$S_{\boldsymbol{z}} = \left\{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{z} = \mathcal{S}\left(\boldsymbol{x}\right)\right\}, \quad \forall \boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}. \tag{10}$$

With these regions the integer estimator (8) can be explicitly written as

$$\check{\boldsymbol{a}} = \sum_{\boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}} s_{\boldsymbol{z}}\left(\hat{\boldsymbol{a}}\right)\boldsymbol{z}, \quad \text{with} \quad s_{\boldsymbol{z}}\left(\hat{\boldsymbol{a}}\right) = \begin{cases} 1 & \text{if } \hat{\boldsymbol{a}} \in S_{\boldsymbol{z}} \\ 0 & \text{else.} \end{cases} \tag{11}$$

The constraints that have to be imposed on the construction of the regions $S_{\boldsymbol{z}}$ have been formulated in Teunissen (1999) for $|\mathcal{I}| = n$, i.e., for the case that the full set of integer parameters is resolved. For the general case in which an arbitrary set $\mathcal{I}$ of integer parameters is resolved, the criteria that have to met when defining an integer mapping scheme are slightly different. The following three properties seem reasonable to be possessed by the regions $S_{\boldsymbol{z}}$, as will be explained thereafter:

$$\begin{array}{rl} (i) & \bigcup_{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}} S_{\boldsymbol{z}} = \mathbb{R}^n \\ (ii) & \mathrm{Int}\,(S_{\boldsymbol{z}}) \cap \mathrm{Int}\,(S_{\boldsymbol{u}}) = \emptyset, \quad \forall \boldsymbol{z}, \boldsymbol{u} \in \mathbb{Z}^{|\mathcal{I}|}, \boldsymbol{z} \neq \boldsymbol{u} \\ (iii) & S_{\boldsymbol{v}_{\mathcal{I}}} = S_{\boldsymbol{0}} + \boldsymbol{v}, \quad \forall \boldsymbol{v} \in \mathbb{Z}^n. \end{array} \tag{12}$$

The first two conditions state that the regions $S_{\boldsymbol{z}}$ have to cover the space $\mathbb{R}^n$ without gaps and overlaps, meaning that the (partial) integer mapping $\mathcal{S}(\cdot)$ leads to a unique $|\mathcal{I}|$ dimensional integer estimate for any float solution $\hat{\boldsymbol{a}} \in \mathbb{R}^n$. The third condition states that if the float solution $\hat{\boldsymbol{a}}$ is shifted by an arbitrary integer vector $\boldsymbol{v} \in \mathbb{Z}^n$, then the integer estimate for the subset $\mathcal{I}$ has to be shifted by $\boldsymbol{v}_{\mathcal{I}}$. This implies that the regions $S_{\boldsymbol{z}}$ are translated copies of each other for any integer shift in the directions that correspond to the subset $\mathcal{I}$, and that the regions $S_{\boldsymbol{z}}$ are invariant for any integer shift in the directions that correspond to the subset $\bar{\mathcal{I}}$. In other words, if any entry of the float solution $\hat{\boldsymbol{a}}$ that is to be resolved is shifted by an arbitrary integer, then the integer solution is shifted by the same amount. On the other hand, if any entry of the float solution $\hat{\boldsymbol{a}}$ that is not resolved is shifted by an arbitrary integer, this must not affect the integer solution of the subset to be resolved.

For any non-empty set $\mathcal{I}$ there are two possible outcomes of a partial integer estimator: Either the integer mapping was correct or incorrect. The probability of a correct integer estimate $\check{\boldsymbol{a}}$ is given by

$$P\left(\check{\boldsymbol{a}} = \boldsymbol{a}_{\mathcal{I}}\right) = \int_{S_{\boldsymbol{a}_{\mathcal{I}}}} p_{\boldsymbol{a}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}, \tag{13}$$

with $p_{\boldsymbol{a}}(\boldsymbol{x})$ the probability density function (pdf) of $\hat{\boldsymbol{a}}$. The failure rate follows as

$$P\left(\check{\boldsymbol{a}} \neq \boldsymbol{a}_{\mathcal{I}}\right) = 1 - P\left(\check{\boldsymbol{a}} = \boldsymbol{a}_{\mathcal{I}}\right). \tag{14}$$

3. EXAMPLES OF PARTIAL INTEGER ESTIMATORS

As already mentioned in Section 2, one possible approach to estimate the subset $\mathcal{I}$ of integers from $\hat{\boldsymbol{a}}$ is to apply ILS to the partial float solution $\hat{\boldsymbol{a}}_{\mathcal{I}}$, i.e.,

$$\check{\boldsymbol{a}}_{\mathrm{parILS}} = \underset{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}}{\operatorname{argmin}} \left\| \hat{\boldsymbol{a}}_{\mathcal{I}} - \boldsymbol{z} \right\|^2_{\boldsymbol{Q}_{\hat{a}_{\mathcal{I}}}}, \tag{15}$$

where $\boldsymbol{Q}_{\hat{a}_{\mathcal{I}}}$ follows from selecting the entries that correspond to $\hat{\boldsymbol{a}}_{\mathcal{I}}$ from $\boldsymbol{Q}_{\hat{\boldsymbol{a}}}$. The pull-in regions $S_{\boldsymbol{z}}$ of this estimator are given by

$$S_{\boldsymbol{z}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \left\| \boldsymbol{x}_{\mathcal{I}} - \boldsymbol{z} \right\|^2_{\boldsymbol{Q}_{\hat{a}_{\mathcal{I}}}} \leq \left\| \boldsymbol{x}_{\mathcal{I}} - \boldsymbol{u} \right\|^2_{\boldsymbol{Q}_{\hat{a}_{\mathcal{I}}}}, \forall \boldsymbol{u} \in \mathbb{Z}^{|\mathcal{I}|} \right\}, \quad \forall \boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}. \tag{16}$$

Essentially, this estimator considers a *reduced* $|\mathcal{I}|$ *dimensional space* and resolves the *full vector of integers* (which is then of dimension $|\mathcal{I}|$) in that space. As ILS is optimal in the sense of maximizing the success rate for estimating the full vector of integer parameters (Teunissen, 1999), this estimator seems to be a reasonable choice. It does, however, completely neglect the fact that the parameters $\boldsymbol{a}_{\bar{\mathcal{I}}}$ are integers as well, since the computation of the integer solution (15) is independent of $\hat{\boldsymbol{a}}_{\bar{\mathcal{I}}}$.

This leads us to a different approach, which is to compute the full n dimensional ILS solution $\boldsymbol{a}^*$ and to select those entries from $\boldsymbol{a}^*$, which correspond to the subset $\mathcal{I}$:

$$\check{\boldsymbol{a}}_{\mathrm{fullILS}} = \boldsymbol{a}^*_{\mathcal{I}}, \quad \text{with} \quad \boldsymbol{a}^* = \underset{\boldsymbol{v}\in\mathbb{Z}^n}{\operatorname{argmin}} \left\| \hat{\boldsymbol{a}} - \boldsymbol{v} \right\|^2_{\boldsymbol{Q}_{\hat{\boldsymbol{a}}}}. \tag{17}$$

Accordingly, the regions $S_{\boldsymbol{z}}$ of this estimator are given by the union of all pull-in regions of the full n dimensional ILS estimator that correspond to an integer $\boldsymbol{v} \in \mathbb{Z}^n$ with $\boldsymbol{v}_{\mathcal{I}} = \boldsymbol{z}$:

$$S_{\boldsymbol{z}} = \bigcup_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{z}} \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \|\boldsymbol{x} - \boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}} \leq \|\boldsymbol{x} - \boldsymbol{u}\|^2_{\boldsymbol{Q}_{\hat{a}}}, \forall \boldsymbol{u} \in \mathbb{Z}^n \right\}, \quad \forall \boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}. \tag{18}$$

This estimator takes the integer property of $\boldsymbol{a}_{\bar{\mathcal{I}}}$ into account, but ILS in combination with a selection of entries of the resulting n dimensional integer vector is no longer optimal.

Depending on the order, the two approaches (15) and (17) may be identical when using sequential conditional rounding, also referred to as integer bootstrapping (BS) (Babai, 1986; Blewitt, 1989; Teunissen, 1998), instead of ILS. Both approaches are identical, if ILS is replaced by simple component wise rounding.

4. MAXIMIZING THE SUCCESS RATE

The estimator that resolves the a-priori defined subset $\mathcal{I}$ of $\boldsymbol{a}$ with the highest possible probability of correct integer estimates $P(\check{\boldsymbol{a}} = \boldsymbol{a}_{\mathcal{I}})$ (13) within the class of partial integer estimators (12) is given in the following theorem.

Theorem: Let $\mathcal{I}$ be the subset of integer parameters to be resolved and let the partial integer estimator $\check{\boldsymbol{a}}_{\text{opt}}$ for $\boldsymbol{a}_{\mathcal{I}}$ be given by

$$\check{\boldsymbol{a}}_{\text{opt}} = \underset{\boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}}{\text{argmax}} \sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{z}} \exp\left(-\frac{1}{2}\|\hat{\boldsymbol{a}} - \boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}}\right), \tag{19}$$

where the float solution $\hat{\boldsymbol{a}}$ follows the Gaussian distribution $\mathcal{N}(\boldsymbol{a}, \boldsymbol{Q}_{\hat{a}})$, then

$$P(\check{\boldsymbol{a}}_{\text{opt}} = \boldsymbol{a}_{\mathcal{I}}) \geq P(\check{\boldsymbol{a}} = \boldsymbol{a}_{\mathcal{I}}) \tag{20}$$

with $\check{\boldsymbol{a}}$ any partial integer estimator as defined in (11), which meets the criteria (12).

The proof is as follows. Let the regions $\bar{S}_{\boldsymbol{u}}, \forall \boldsymbol{u} \in \mathbb{Z}^{|\bar{\mathcal{I}}|}$, be arbitrary regions that fulfill the properties of the pull-in regions $S_{\boldsymbol{z}}$ (12) for the index set $\bar{\mathcal{I}}$ instead of $\mathcal{I}$, and let the regions $R_{\boldsymbol{v}}, \forall \boldsymbol{v} \in \mathbb{Z}^n$, be defined as $R_{\boldsymbol{v}} = S_{\boldsymbol{v}_{\mathcal{I}}} \cap \bar{S}_{\boldsymbol{v}_{\bar{\mathcal{I}}}}$ (this implies that $R_{\boldsymbol{v}}, \forall \boldsymbol{v} \in \mathbb{Z}^n$, cover $\mathbb{R}^n$ without gaps and overlaps and are integer translated copies of each other for any integer $\in \mathbb{Z}^n$). The pull-in regions $S_{\boldsymbol{z}}$ can be written as $S_{\boldsymbol{z}} = \bigcup_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{z}} R_{\boldsymbol{v}}$. With $p_{\boldsymbol{a}}(\boldsymbol{x})$ the Gaussian pdf of the float solution, the pull-in regions $S^*_{\boldsymbol{z}}$ of the optimal estimator (19) for the index set $\mathcal{I}$ are given by

$$S^*_{\boldsymbol{z}} = \left\{ \boldsymbol{x} \in \mathbb{R}^n \mid \sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{z}} p_{\boldsymbol{v}}(\boldsymbol{x}) \geq \sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{u}} p_{\boldsymbol{v}}(\boldsymbol{x}), \forall \boldsymbol{u} \in \mathbb{Z}^{|\mathcal{I}|} \right\}. \tag{21}$$

Then,

$$\sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{a}_{\mathcal{I}}} p_{\boldsymbol{v}}(\boldsymbol{x}) \geq \sum_{\boldsymbol{u} \in \mathbb{Z}^{|\mathcal{I}|}} \sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{u}} p_{\boldsymbol{v}}(\boldsymbol{x}) s_{\boldsymbol{u}}(\boldsymbol{x}), \quad \forall \boldsymbol{x} \in S^*_{\boldsymbol{a}_{\mathcal{I}}}, \tag{22}$$

with $s_{\boldsymbol{u}}(\cdot)$ the indicator function of an arbitrary partial integer estimator for subset $\mathcal{I}$ with pull-in regions $S_{\boldsymbol{u}}$. We integrate both sides of (22) over the subset $R^*_{\boldsymbol{a}} = S^*_{\boldsymbol{a}_{\mathcal{I}}} \cap \bar{S}_{\boldsymbol{a}_{\bar{\mathcal{I}}}}$ of $S^*_{\boldsymbol{a}_{\mathcal{I}}}$, where $\bar{S}_{\boldsymbol{a}_{\bar{\mathcal{I}}}}$ is arbitrary within the above mentioned constraints:

$$\sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{a}_{\mathcal{I}}} \int_{R^*_{\boldsymbol{a}}} p_{\boldsymbol{v}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x} \geq \sum_{\boldsymbol{u} \in \mathbb{Z}^{|\mathcal{I}|}} \sum_{\boldsymbol{v} \in \mathbb{Z}^n | \boldsymbol{v}_{\mathcal{I}} = \boldsymbol{u}} \int_{R^*_{\boldsymbol{a}} \cap S_{\boldsymbol{u}}} p_{\boldsymbol{v}}(\boldsymbol{x}) \mathrm{d}\boldsymbol{x}. \tag{23}$$

The coordinate transform $\boldsymbol{y} = \boldsymbol{x} + \boldsymbol{a} - \boldsymbol{v}$ leads to (we make use of $p_{\boldsymbol{v}}(\boldsymbol{x} + \boldsymbol{c}) = p_{\boldsymbol{v}-\boldsymbol{c}}(\boldsymbol{x})$ and (iii) from (12))

$$\sum_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{a}_{\mathcal{I}}} \int_{R^*_{2\boldsymbol{a}-\boldsymbol{v}}} p_{\boldsymbol{a}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \geq \sum_{\boldsymbol{u}\in\mathbb{Z}^{|\mathcal{I}|}} \sum_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{u}} \int_{R^*_{2\boldsymbol{a}-\boldsymbol{v}}\cap S_{\boldsymbol{u}+\boldsymbol{a}_{\mathcal{I}}-\boldsymbol{v}_{\mathcal{I}}}} p_{\boldsymbol{a}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{24}$$

Since the regions $R^*_{\boldsymbol{v}}$ do not overlap, the sum of integrals on the left side of (24) can be written as a single integral over the region $\bigcup_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{a}_{\mathcal{I}}} R^*_{2\boldsymbol{a}-\boldsymbol{v}} = \bigcup_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{a}_{\mathcal{I}}} R^*_{\boldsymbol{v}} = S^*_{\boldsymbol{a}_{\mathcal{I}}}$. On the right side of (24) we notice that all terms have in common that $\boldsymbol{v}_{\mathcal{I}} = \boldsymbol{u}$, i.e., $S_{\boldsymbol{u}+\boldsymbol{a}_{\mathcal{I}}-\boldsymbol{v}_{\mathcal{I}}} = S_{\boldsymbol{a}_{\mathcal{I}}}$ and constant for all terms of the sums. The only quantity that depends on the argument of the sums are the non-overlapping $R^*_{2\boldsymbol{a}-\boldsymbol{v}}$. We can therefore again replace the sum of integrals with a single integral. Since the two sums simply represent a sum over all n dimensional integer vectors, the region of integration is $\bigcup_{\boldsymbol{v}\in\mathbb{Z}^n} R^*_{2\boldsymbol{a}-\boldsymbol{v}}\cap S_{\boldsymbol{a}_{\mathcal{I}}} = \mathbb{R}^n \cap S_{\boldsymbol{a}_{\mathcal{I}}} = S_{\boldsymbol{a}_{\mathcal{I}}}$, and we finally have

$$\int_{S^*_{\boldsymbol{a}_{\mathcal{I}}}} p_{\boldsymbol{a}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y} \geq \int_{S_{\boldsymbol{a}_{\mathcal{I}}}} p_{\boldsymbol{a}}(\boldsymbol{y})\mathrm{d}\boldsymbol{y}. \tag{25}$$

Comparing (25) to (13) shows that the left and right side of (25) are the probabilities of correctly resolving $\boldsymbol{a}_{\mathcal{I}}$ for the two estimators defined by $S^*_{\boldsymbol{a}_{\mathcal{I}}}$ and $S_{\boldsymbol{a}_{\mathcal{I}}}$, which concludes the proof.

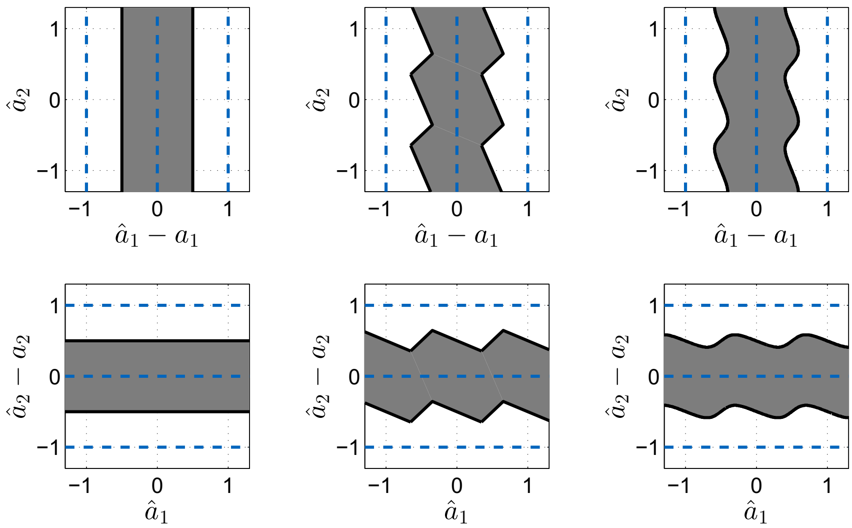

Fig. 1. Regions $S_{\boldsymbol{z}}$ for a two dimensional example; first row: $\mathcal{I} = \{1\}$, second row: $\mathcal{I} = \{2\}$; left: partial ILS, center: full ILS + selection, right: optimal; the corresponding integer solutions are indicated in blue; the regions of successful ambiguity resolution are marked in gray.

If $\mathcal{I}$ corresponds to the full set of integers, the Theorem reduces to the optimality property of ILS, since the sum then only contains one term. Figure 1 shows a two dimensional example with $\mathcal{I} = \{1\}$ in the first row and $\mathcal{I} = \{2\}$ in the second row, i.e., only the first/second integer is resolved. The boundaries of the regions $S_{\boldsymbol{z}}$, which lead to different integer values according to (11), are shown for the two suboptimal strategies from Section 3 and the optimal integer mapping. The first column corresponds to partial

ILS, i.e., ILS is applied to $\hat{\boldsymbol{a}}_{\mathcal{I}}$ (15), the second column to full ILS followed by a selection of entries (17), and the third column to the optimal strategy (19). Each region leads to a different but unique one dimensional fixed solution $\check{\boldsymbol{a}}$, indicated by the blue dashed lines. The gray regions lead to correct integer estimates $\check{\boldsymbol{a}} = \boldsymbol{a}_{\mathcal{I}}$.

5. COMPUTATION OF THE OPTIMAL ESTIMATOR

In this section we discuss the problem of how to implement the optimal partial integer estimator (19) for any index set $\mathcal{I}$ that is not the full set (for $|\mathcal{I}| = n$, (19) reduces to a standard ILS problem). An equivalent formulation of the optimization problem (19) is given by

$$\check{\boldsymbol{a}}_{\text{opt}} = \underset{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}}{\operatorname{argmin}} \sum_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}\neq\boldsymbol{z}} \exp\left(-\frac{1}{2}\|\hat{\boldsymbol{a}}-\boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}}\right) = \underset{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}}{\operatorname{argmin}}\, C(\boldsymbol{z}), \tag{26}$$

with $C(\boldsymbol{z})$ the cost function as defined in (26). The minimization of this cost function over the set $\mathbb{Z}^{|\mathcal{I}|}$ can be solved via search (and shrink) as follows. The cost function $C(\boldsymbol{z})$ is initialized with, e.g., $\boldsymbol{z} = \boldsymbol{a}^*_{\mathcal{I}}$ as $C(\boldsymbol{a}^*_{\mathcal{I}}) = d$, where $\boldsymbol{a}^*$ is the n dimensional ILS solution. The goal is now to find all integer candidates $\boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}$ with $C(\boldsymbol{z}) \leq d$. The integer candidate with the minimum value of the cost function $C(\cdot)$ is the optimizer of (26). This search strategy can be combined with a *shrinking* of the search space, i.e., whenever a valid integer candidate $\boldsymbol{z}$ is found, the value of d is reduced to $C(\boldsymbol{z})$. The search for integer candidates $\boldsymbol{z} \in \mathbb{Z}^{|\mathcal{I}|}$ can be formulated as a tree-search problem, as the cost function can be written as

$$C(\boldsymbol{z}) = \sum_{i=1}^{|\mathcal{I}|} c_i(z_1,\ldots,z_i), \tag{27}$$

where $c_i(z_1,\ldots,z_i)$ are non-negative additive increments that only depend on all entries of $\boldsymbol{z}$ up to level i. They are given by

$$c_i(z_1,\ldots,z_i) = \sum_{\substack{\boldsymbol{v}\in\mathbb{Z}^n|_{v_{\mathcal{I}_1}=z_1,\ldots,v_{\mathcal{I}_{i-1}}=z_{i-1}} \\ v_{\mathcal{I}_i}\neq z_i}} \exp\left(-\frac{1}{2}\|\hat{\boldsymbol{a}}-\boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}}\right), \tag{28}$$

with $\mathcal{I}_i$ the ith element of the index set. That is, the sum in (26) for computing $C(\boldsymbol{z})$ can be built up by adding new terms in each level. The possible values at level i follow from $c_i(z_1,\ldots,z_i) \leq d - \sum_{j=1}^{i-1} c_j(z_1,\ldots,z_j)$.

Since the evaluation of $C(\boldsymbol{z})$ in (26) comprises a sum over infinitely many integer vectors $\boldsymbol{v}$, it cannot be computed exactly but has to be approximated, i.e., the integer vectors $\boldsymbol{v}$ have to be chosen from a finite set instead of $\mathbb{Z}^n$. This finite set is defined such that all integers, whose contribution to the sum is too small, are omitted. This results in the set $\Theta^\lambda_{\hat{\boldsymbol{a}}}$ of integers, which lie within an ellipsoidal region centered around $\hat{\boldsymbol{a}}$, i.e.,

$$\Theta^\lambda_{\boldsymbol{x}} = \left\{\boldsymbol{v}\in\mathbb{Z}^n \mid \|\boldsymbol{x}-\boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}} < \lambda^2\right\}. \tag{29}$$

The larger the value of the size parameter λ of $\Theta^\lambda_{\hat{\boldsymbol{a}}}$ is chosen, the closer the approximation will be to the exact value of $C(\boldsymbol{z})$. The same problem arises in the computation of the best integer-equivariant estimator, where a good choice for λ was found from the condition $P(\|\hat{\boldsymbol{a}}-\boldsymbol{a}\|^2_{\boldsymbol{Q}_{\hat{a}}} < \lambda^2) = 1-\alpha$ with a small value of α (Teunissen, 2005). Since

$\|\hat{\boldsymbol{a}} - \boldsymbol{a}\|^2_{\boldsymbol{Q}_{\hat{a}}}$ follows a central χ^2 distribution with n degrees of freedom, the value of λ can be determined.

The evaluation of (28) includes a search for integer candidates in $\Theta^\lambda_{\hat{a}}$ in each level $i = 1, \ldots, |\mathcal{I}|$. In order to save computational complexity, one should first search for *all* integers $\boldsymbol{v} \in \Theta^\lambda_{\hat{a}}$ and store them together with $\exp(-\frac{1}{2}\|\hat{\boldsymbol{a}} - \boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}})$. Irrespective of the chosen parameterization in (7), this first search should be carried out after applying a decorrelation transformation, which makes the search more efficient. The search for the optimal solution $\check{\boldsymbol{a}}_{\text{opt}} \in \mathbb{Z}^{|\mathcal{I}|}$ is then performed by simply selecting candidates from that list when evaluating (28).

6. CRITERIA FOR SELECTING THE INTEGER SUBSET

In GNSS applications, there are two criteria that the user is interested in when selecting the subset $\mathcal{I}$ of integer parameters to be resolved (Verhagen et al., 2011): The *precision* of the estimate $\check{\boldsymbol{b}}$ (9) that he can expect after resolving $\boldsymbol{a}'_{\mathcal{I}}$, and the *reliability* of the integer estimate $\check{\boldsymbol{a}}'$ (8).

Under the assumption that all $\check{\boldsymbol{a}}'$ in (9) were resolved to their correct integers, $\check{\boldsymbol{b}}$ is Gaussian distributed with mean value $\boldsymbol{b}$ and the conditional covariance matrix

$$\boldsymbol{Q}_{\hat{b}|\hat{a}'_{\mathcal{I}}} = \boldsymbol{Q}_{\hat{b}} - \boldsymbol{Q}_{\hat{b}\hat{a}'_{\mathcal{I}}} \boldsymbol{Q}^{-1}_{\hat{a}'_{\mathcal{I}}} \boldsymbol{Q}_{\hat{a}'_{\mathcal{I}}\hat{b}} \tag{30}$$

for the given subset $\mathcal{I}$. The probability of an incorrect ambiguity estimate $\check{\boldsymbol{a}}'$ directly tells us how much confidence we can put in the presented statistics of $\check{\boldsymbol{b}}$. If this failure rate is sufficiently small, the user can employ $\boldsymbol{Q}_{\hat{b}|\hat{a}'_{\mathcal{I}}}$ from (30) to evaluate which precision he can expect from the estimate $\check{\boldsymbol{b}}$ after the algorithm that he chose determined the transformation $\boldsymbol{Z}$ and index set $\mathcal{I}$. If the user application requires a certain minimum precision of $\check{\boldsymbol{b}}$, (30) can be used to decide, whether a specific subset $\mathcal{I}$ is useful for the application or not. Since

$$\boldsymbol{Q}_{\hat{b}|\hat{a}'_{\mathcal{I}}} \preccurlyeq \boldsymbol{Q}_{\hat{b}|\hat{a}'_{\mathcal{J}}}, \quad \forall \mathcal{J} \subseteq \mathcal{I}, \tag{31}$$

where $\preccurlyeq$ is defined in terms of positive-definiteness, we know that once we have found an index set $\mathcal{I}$ that does not lead to the required precision of $\check{\boldsymbol{b}}$, all subsets of $\mathcal{I}$ cannot reach that precision as well and do not have to be considered.

The reliability can be measured with the probability $P(\check{\boldsymbol{a}}' = \boldsymbol{a}'_{\mathcal{I}})$ of a correct integer estimate $\check{\boldsymbol{a}}'$, cf. (13). As mentioned above, this success rate is important, since the precision as given by (30) is only valid given that $\check{\boldsymbol{a}}' = \boldsymbol{a}'_{\mathcal{I}}$, otherwise $\check{\boldsymbol{b}}$ can have large errors. Unfortunately, the success rate for both the optimal partial integer estimator and the two presented suboptimal schemes cannot be computed in closed form. One can, however, evaluate lower bounds on the success rate for a given index set $\mathcal{I}$ and fixing scheme and decide on whether or not to resolve a certain subset of integers based on these bounds. For the partial ILS strategy (15), which applies ILS to the reduced float vector $\hat{\boldsymbol{a}}_{\mathcal{I}}$, we can use the standard success rate bounds of ILS, but in the reduced space. It is clear, that the lower bounds are also valid for the optimal partial integer estimator. The analysis in Verhagen et al. (2013) shows that the success rates of integer BS are a tight lower bound for the ILS success rates if a prior decorrelation function is used. The idea of using BS success rates for selecting the subset $\mathcal{I}$ with a given reliability constraint was introduced in Teunissen et al. (1999) and used for PAR in, e.g., Khanafseh and Pervan (2010), Verhagen et al. (2011), Odijk et al. (2014), and Nardo et al. (2016). Generally, the subset selection process includes the evaluation of success rate and/or precision (30) for different subsets $\mathcal{I}$, before one of them is selected. A very simple method to choose $\mathcal{I}$ is to use a truncated version of the sequentially computed BS success rate, which keeps adding ambiguities to the set $\mathcal{I}$ until the resulting success rate drops below a user defined threshold.

If the difference between the easy-to-compute BS success rates and the success rates of the presented partial integer estimators is small, we can use the BS success rates to decide whether or not a certain subset $\mathcal{I}$ of integers can be reliably resolved, without being too conservative. In the following a numerical comparison of this difference is presented for simulated single epoch dual frequency L1/L2 single baseline GPS positioning examples, where the satellite constellation is used as seen during GPS week 1,815 with an elevation cutoff angle of 10°. We consider 145 different epochs in the area of Munich, Germany. The standard deviation of the undifferenced measurements in zenith direction is assumed as 25 cm for code and 3 mm for phase observations, to which the elevation dependent exponential weighting function from Euler and Goad (1991) is applied. The between receiver (residual) differential ionospheric delays are modeled as zero mean additive Gaussian random variables with standard deviations of $0 - 3$ cm in zenith direction.

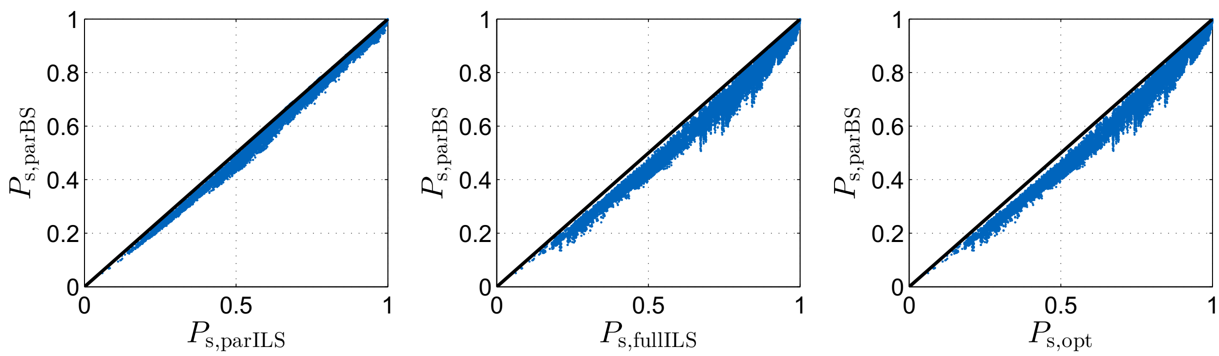

Fig. 2. Success rates of partial BS versus the actual success rates of partial ILS (left), full ILS + selection (center), and the optimal scheme (right); each figure shows 68,000 data points.

Figure 2 shows the analytic success rates $P_{\mathrm{s,parBS}}$ of partial BS against the success rates $P_{\mathrm{s,parILS}}$ of partial ILS, cf. (15), $P_{\mathrm{s,fullILS}}$ of full ILS + selection, cf. (17), and $P_{\mathrm{s,opt}}$ of optimal partial integer fixing, cf. (19), which are computed via Monte Carlo integration. For each positioning case, the success rates are evaluated for all subsets $\mathcal{I}$ except $\mathcal{I} = \{1, \ldots, n\}$, which lead to a precision of

$$\begin{bmatrix} \sigma_{\mathrm{E}} \\ \sigma_{\mathrm{N}} \\ \sigma_{\mathrm{U}} \end{bmatrix} \leq \begin{bmatrix} 2\,\mathrm{cm} \\ 2\,\mathrm{cm} \\ 6\,\mathrm{cm} \end{bmatrix}, \tag{32}$$

where σ_{E}, σ_{N}, and σ_{U} are the standard deviations of the coordinate estimates in the local east, north, and up frame that follow from $\boldsymbol{Q}_{\hat{\boldsymbol{b}}|\hat{\boldsymbol{a}}'_{\mathcal{I}}}$. That is, each data point in the figures corresponds to a different combination of measurement epoch, ionospheric uncertainty, and ambiguity subset. All partial integer fixing schemes are applied in the LAMBDA decorrelated space (Teunissen, 1995). Figure 2 shows that the success rates of partial BS are a tighter lower bound for partial ILS than for the optimal estimator, which implies that $P_{\mathrm{s,opt}}$ is in general noticeably larger than $P_{\mathrm{s,parILS}}$. Also, the BS success rates seem to work well as a lower bound for full ILS + selection, although they cannot be guaranteed to be a lower bound.

7. OPTIMAL VS SUBOPTIMAL PARTIAL INTEGER ESTIMATION

In Section 5 it was explained, how the optimal partial integer estimator as derived in Section 4 can be implemented. One might, however, be interested in using one of the simpler partial integer estimators (15) or (17) from Section 3, which only require an ILS solution, such as it is provided by, e.g., the LAMBDA method (Teunissen, 1995). If the sum in the computation of the optimal estimator (19) is simply approximated by its maximum term, (19) becomes equivalent to selecting entries of the full ILS solution, i.e., to (17). Furthermore, we can analyze (19) for the case that $\hat{\boldsymbol{a}}$ is of very high precision. A constant factor does not change the result of (19), thus

$$\check{\boldsymbol{a}}_{\mathrm{opt}} = \underset{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}}{\operatorname{argmax}} \sum_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{z}} \frac{\exp\left(-\frac{1}{2}\|\hat{\boldsymbol{a}}-\boldsymbol{v}\|^2_{\boldsymbol{Q}_{\hat{a}}}\right)}{\sum_{\boldsymbol{u}\in\mathbb{Z}^n}\exp\left(-\frac{1}{2}\|\hat{\boldsymbol{a}}-\boldsymbol{u}\|^2_{\boldsymbol{Q}_{\hat{a}}}\right)} = \underset{\boldsymbol{z}\in\mathbb{Z}^{|\mathcal{I}|}}{\operatorname{argmax}} \sum_{\boldsymbol{v}\in\mathbb{Z}^n|\boldsymbol{v}_{\mathcal{I}}=\boldsymbol{z}} w_{\boldsymbol{v}}(\hat{\boldsymbol{a}}). \quad (33)$$

With $w_{\boldsymbol{v}}(\hat{\boldsymbol{a}}) = \frac{1}{1+\sum_{\boldsymbol{u}\in\mathbb{Z}^n|\boldsymbol{u}\neq\boldsymbol{v}}\exp\left(-\frac{1}{2\sigma^2}\left(\|\hat{\boldsymbol{a}}-\boldsymbol{u}\|^2_{\boldsymbol{G}}-\|\hat{\boldsymbol{a}}-\boldsymbol{v}\|^2_{\boldsymbol{G}}\right)\right)}$, where the factorization $\boldsymbol{Q}_{\hat{\boldsymbol{a}}} = \sigma^2\boldsymbol{G}$ is used, it follows that $\lim_{\sigma\to 0} w_{\boldsymbol{v}}(\hat{\boldsymbol{a}}) = 1$, if $\|\hat{\boldsymbol{a}}-\boldsymbol{v}\|^2_{\boldsymbol{G}} \leq \|\hat{\boldsymbol{a}}-\boldsymbol{u}\|^2_{\boldsymbol{G}}, \forall \boldsymbol{u}\in\mathbb{Z}^n$, and $\lim_{\sigma\to 0} w_{\boldsymbol{v}}(\hat{\boldsymbol{a}}) = 0$, else. Since the ILS solution does not depend on σ for the above factorization, this means that $w_{\boldsymbol{a}^*}(\hat{\boldsymbol{a}}) = 1$, with $\boldsymbol{a}^*$ the n dimensional ILS solution, and $w_{\boldsymbol{v}}(\hat{\boldsymbol{a}}) = 0, \forall \boldsymbol{v}\in\mathbb{Z}^n\setminus\boldsymbol{a}^*$. Accordingly, the sum in (33) will reach its maximum, if the full ILS solution is contained, and the optimal partial integer mapping becomes identical to selecting entries of the full ILS solution (17).

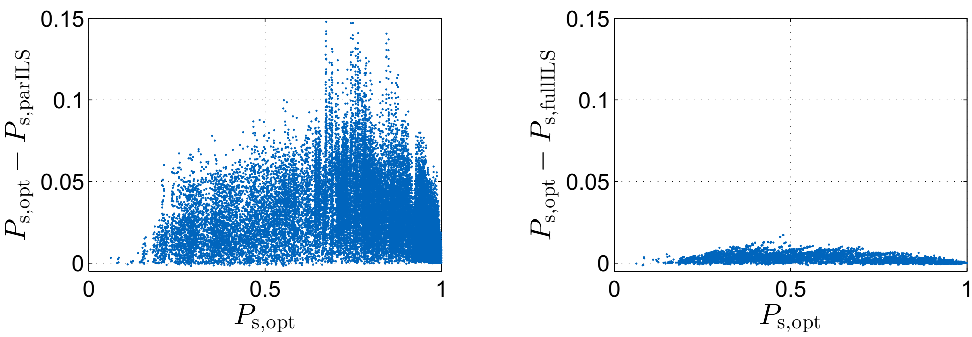

Fig. 3. Difference between success rates of the optimal partial integer estimator and partial ILS (left), and full ILS + selection (right); each figure shows 68,000 data points.

These arguments in favor of selecting entries of the full ILS solution over partial ILS are verified for the numerical examples from Section 6. Figure 3 shows the difference between the success rates of the optimal scheme and partial ILS/full ILS + selection. While partial ILS loses up to 15 % in terms of success rate compared to the optimal fixing scheme, full ILS + selection performs close to optimal for all examples.

Table 1. Average computation times of the search algorithms for partial ILS, full ILS + selection, and the optimal estimator.

	Partial ILS	Full ILS + selection	Optimal
Search for integer candidate(s)	0.056 ms	0.059 ms	0.446 ms
Search for integer solution	–	–	1.970 ms
Total search time	0.056 ms	0.059 ms	2.416 ms

In order for the optimal partial integer estimator to be used in real time GNSS applications, its computation time must be sufficiently small. Table 1 shows the average computation times of the search algorithms required to find the integer estimates for the above numerical examples. The timing was done using Matlab implementations on a standard desktop computer. Partial ILS and full ILS + selection only require the search for a closest integer candidate, which is slightly faster for partial ILS due to the smaller dimension of the search problem. For the optimal scheme, all integer candidates within $\Theta^{\lambda}_{\hat{\boldsymbol{a}}}$ (29) have to be found instead of only the best candidate, which causes a longer time for the candidate search (a value of $\alpha = 0.001$ was used to determine the size parameter λ of $\Theta^{\lambda}_{\hat{\boldsymbol{a}}}$). As explained in Section 5, a second search is required to determine the integer solution $\check{\boldsymbol{a}}_{\text{opt}}$ from these candidates. Although the average total search time of the optimal partial fixing scheme is clearly larger than the ones of the two suboptimal estimators, it is still well within the real time requirements for GNSS applications. It is noted that these timing results of course strongly depend on the employed hardware and implementation and should therefore only be seen as a rough indicator for the order of magnitude that can be expected for the computation time. Additional complexity resides in the LAMBDA decorrelation transformation that was applied before the search algorithms, which is identical for all three methods.

8. CONCLUSIONS

In this contribution a class of estimators for resolving a subset of integer parameters was defined. The optimal estimator for an arbitrary subset was derived for additive Gaussian noise, where optimal means that no other estimator of this class can reach a higher probability of correct integer estimates. This optimal estimator can be implemented by means of two consecutive searches. Numerical examples showed that for GNSS ambiguity resolution a very good suboptimal strategy is to compute the full dimensional ILS solution and to select those components thereof, which correspond to the given subset. The success rates of this strategy are close to optimal, while only the computation of an ILS solution is required. An application of this strategy is given in Brack and Günther (2014) and Brack (2015), where the subset of integers is not a-priori fixed but depends on the realization of the float solution. Usually one would then have to (iteratively) compute multiple integer estimates for different subsets and test them for acceptance. With this strategy, however, only a single integer solution – the full ILS solution – is required, which enables more efficient partial integer estimation schemes.

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