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On the index of Siegel grids and its application to the tomography of quasicrystals

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Abstract

We give a characterization of when the index of Siegel grids is finite. As a main application, we solve a basic decomposition problem for the discrete tomography of quasicrystals that live on finitely generated $\mathbb{Z}$-modules in some $\mathbb{R}^s$.

1. Introduction

The present paper is motivated by a basic problem from the discrete tomography of mathematical quasicrystals, the so-called model sets. In general, the discrete tomography of model sets in $\mathbb{R}^s$ is concerned with the reconstruction of a finite quasicrystalline structure F that is only accessible through certain X-ray images i.e., through the cardinalities of their intersection with all affine subspaces that are translates of a given small number m of linear subspaces $S_1, \dots, S_m$ of $\mathbb{R}^s$ spanned by model set vectors; see Section 3 for precise definitions, references and background information.

For the purpose of the present paper, model sets are special discrete subsets of some translate of some finitely generated $\mathbb{Z}$-module Z in $\mathbb{R}^s$; see e.g. [1] for a comprehensive treatment of modules. The module Z itself can be thought of as a projection of some lattice L in $\mathbb{R}^d$ on $\mathbb{R}^s$ parallel to some linear subspace U of $\mathbb{R}^d$. For $i = 1, \dots, m$ let $\mathcal{T}_i$ denote the set of all translates $t + S_i$ that intersect F. Then, of course, F has to be contained in its tomographic grid

$$H_F := \bigcap_{i=1}^{m} \bigcup_{T \in \mathcal{T}_i} T.$$

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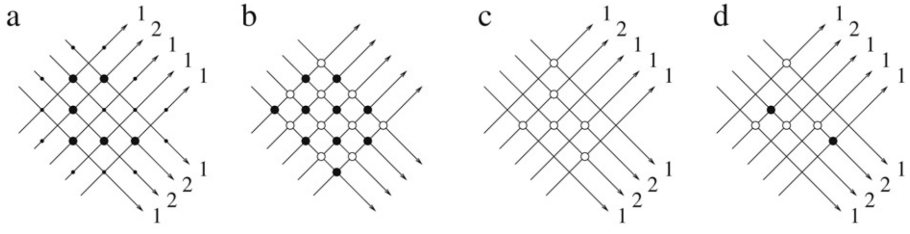

Fig. 1. (a) A subset F of $\mathbb{Z}^2$, X-ray lines for $S_1 = \text{lin}\{(1,1)^{\mathrm{T}}\}$, $S_2 = \text{lin}\{(1,-1)^{\mathrm{T}}\}$. (b) The tomographic grid divides into two equivalence classes of copies of $\mathbb{Z}^2$. (c) A set F' with the same X-rays as F contained in the 'white' $\mathbb{Z}^2$. (d) Another set F'' with the same X-rays. The points of F'' are scattered over both copies of $\mathbb{Z}^2$; hence F'' is not admissible.

In general, however, the X-ray information does not suffice to determine the underlying set F precisely. Moreover, in general H_F will not be contained in a single translate of Z, but in a union of several translates of Z; see Fig. 1 for an example. Any solution of the underlying reconstruction problem must, however, completely belong to just one set $t + Z$.

Hence a basic algorithmic task in the discrete tomography of model sets is to compute a partition of the tomographic grid into subsets that are contained in a single translate of the underlying module Z. This, of course, leads immediately to the structural problem of whether there exists a uniform bound on the cardinality of translates needed. This decomposition problem was introduced in [6] and solved for cyclotomic model sets i.e., planar model sets that are contained in some (unknown) translate of the smallest subring $\mathbb{Z}[\zeta_N]$ of $\mathbb{C}$ that contains $\mathbb{Z}$ and the primitive Nth root of unity $\zeta_N := \mathrm{e}^{\frac{2\pi\mathrm{i}}{N}}$. As is well known, $\mathbb{Z}[\zeta_N]$ is a finitely generated $\mathbb{Z}$-module of rank $\phi(N)$, where ϕ denotes Euler's totient function i.e., $\phi(N)$ is the number of integers j with $1 \leq j \leq N$ that are coprime to N; see e.g. [38,24] for more information on cyclotomic fields. Using the specific algebraic structure in this situation, [6] shows that for two lines S_1 and S_2 that are spanned by a vector from $\mathbb{Z}[\zeta_N]$, respectively, already the complete tomographic grid

$$H := \bigcap_{i=1}^{m} \bigcup_{z \in \mathbb{Z}[\zeta_N]} (z + S_i)$$

(with $m = 2$) decomposes into finitely many equivalence classes $t + \mathbb{Z}[\zeta_N]$ with $t \in \mathbb{Q}[\zeta_N]$ (or, which is the same, $t \in \mathbb{Q}(\zeta_N)$), a result that is fundamental for a subsequent polynomial-time reconstruction algorithm; see [6].

In the present paper, we study this decomposition problem for general finitely generated $\mathbb{Z}$-modules in some $\mathbb{R}^s$. We will give a complete characterization when the number of translational equivalence classes is finite. As a simple corollary, we obtain the mentioned result for cyclotomic model sets; see Corollary 3.3. However, our results apply to more general model sets in arbitrary dimensions and do not rely on specific algebraic properties, hence allowing us to handle even structures that are generated by non-algebraic reals. As a matter of fact, our approach is rooted in the geometry of numbers rather than in algebra, and uses the concept of Siegel grids as introduced in Section 2. The question of when the index of Siegel grids is finite can be seen to be equivalent to the existence of a finite lattice refinement that hosts simultaneous 'pseudodiophantine' solutions to given systems of linear equations with real coefficients.

The paper is organized as follows. Section 2 introduces the basic notion of Siegel grids that allows us to formulate and study the underlying problem within the geometry of numbers and states our main characterization of when the index of Siegel grids is finite. Section 3 gives a brief account of some relevant notions from the discrete tomography of quasicrystals and states the main consequences of the previous characterization to this field of application. Sections 4 and 5 provide all corresponding proofs and derive further results and corollaries.

2. The index of Siegel grids: Concept, notation, and main results

Let Z be a finitely generated $\mathbb{Z}$-module in some real space $\mathbb{R}^s$ and let $S_1, \dots, S_m$ be linear subspaces of $\mathbb{R}^s$. Then the set

$$\begin{aligned} G &:= G(Z; S_1, \dots, S_m) := \bigcap_{i=1}^{m} \bigcup_{z \in Z} (z + S_i) \\ &= \left\{ g \in \mathbb{R}^s : [\forall (i = 1, \dots, m)\ \exists (z_i \in Z \wedge x_i \in S_i) : g = z_i + x_i] \right\} \end{aligned}$$

is called the *Siegel grid* of $(Z; S_1, \dots, S_m)$. Note that every Siegel grid is a $\mathbb{Z}$-module; hence Siegel grids 'interpolate' the extremal cases $S_1 = \cdots = S_m = \{0\}$ and $S_1 = \cdots = S_m = \mathbb{R}^s$ where we have

$$G(Z; \{0\}, \dots, \{0\}) = Z \quad \wedge \quad G(Z; \mathbb{R}^s, \dots, \mathbb{R}^s) = \mathbb{R}^s.$$

In his famous *Lectures on the Geometry of Numbers* [36], C.L. Siegel gave a beautiful proof that the closure of $\mathbb{Z}$-modules in $\mathbb{R}^s$ or, as he called them, *vector groups*, is a Siegel grid of the form $G = G(L; W)$, where L is a lattice and W is a linear subspace [36, Lect. VI, Section 2] and applied it to obtain Kronecker's theorem [23, Ch. IV] on the approximate solution of a system of linear diophantine equations with real coefficients [36, Lect. VI, Section 6].

Now, let S be a subspace of $S_1 \cap \cdots \cap S_m$, and let the relation

$$\sim := \sim_S \subset G \times G$$

be defined by

$$g_1 \sim g_2 :\Leftrightarrow g_1 - g_2 \in Z + S.$$

Obviously, $\sim$ is an equivalence relation. The number of equivalence classes $|G/_\sim|$ is called the *index* of G with respect to S. We are interested in the question of when exactly $|G/_\sim|$ is finite.

Note that the finiteness of the index is invariant under linear transformations. Hence, we may assume that $\mathrm{lin}_{\mathbb{R}}(Z) = \mathbb{R}^s$ and that Z contains $\mathbb{Z}^s$. We will do this whenever we want to explicitly reveal the geometric flavor of our arguments as, under the latter assumption, the relevant linear mappings become projections parallel to their kernel.

Let $p_1, \dots, p_d \in \mathbb{R}^s$ be generators of the $\mathbb{Z}$-module Z (with $p_1, \dots, p_s$ being the standard unit vectors of $\mathbb{R}^s$) and let $P := [p_1, \dots, p_d] \in \mathbb{R}^{s \times d}$. Then, of course, $Z = P\mathbb{Z}^d$. Hence Z is the projection of $\mathbb{Z}^d$ on $\mathbb{R}^s$ parallel to the space $U := \ker(P)$. Therefore we may equivalently consider the index of $G(\mathbb{Z}^d; S_1 + U, \dots, S_m + U)$ with respect to $S + U$, where S resp. S_i is embedded in $\mathbb{R}^d$ via $S \times \{0\}^{d-s}$ resp. $S_i \times \{0\}^{d-s}$. Since the index will never be finite if $S + U$ is a proper subspace of $(S_1 + U) \cap \cdots \cap (S_m + U)$, we will in the following (without loss of generality) deal with the standard situation of

$$G := G(V_1, \dots, V_m) := G(\mathbb{Z}^d; V_1, \dots, V_m) \quad \wedge \quad \sim := \sim_V,$$

where $V_1, \dots, V_m$ are linear subspaces of $\mathbb{R}^d$ and

$$V := V_1 \cap \cdots \cap V_m.$$

Again, without loss of generality, we will assume that all the V_i are non-trivial subspaces of $\mathbb{R}^d$ for, otherwise, G coincides with $\mathbb{Z}^d$ or some V_i is redundant. We will frequently use the notation

$$\iota(V_1, \ldots, V_m) := |G/_\sim|$$

rather than

$$|G(V_1, \ldots, V_m)/_{\sim_V}|$$

to explicitly signify the involved subspaces.

A linear subspace of $\mathbb{R}^d$ is called *rational* if it admits a basis of integer vectors. As is well known, the index of a Siegel grid G is finite whenever all involved subspaces are rational; cf. [36, Lect. V, Section 6]. The problem becomes, however, much more intricate if the spaces are not rational.

As it turns out, the Siegel grids are intimately related to questions involving 'nearly diophantine' simultaneous solutions of systems of linear equations with real coefficients. To be more precise, let for $i = 1, \ldots, m$

$$n_i \in \mathbb{N} \quad \wedge \quad A_i \in \mathbb{R}^{n_i \times d} \quad \wedge \quad b_i \in \mathbb{R}^{n_i},$$

and set

$$A := \begin{bmatrix} A_1 \\ \vdots \\ A_m \end{bmatrix} \quad \wedge \quad b := \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix} \quad \wedge \quad n := \sum_{i=1}^{m} n_i.$$

Now, let $\mathscr{B} := \mathscr{B}(A_1, \ldots, A_m)$ denote the set of all vectors $b \in \mathbb{R}^n$ such that the full system $Ax = b$ is feasible over $\mathbb{R}^d$, while the m partial systems $A_1 z_1 = b_1, \ldots, A_m z_m = b_m$ individually admit solutions in $\mathbb{Z}^d$. Observe that the set $\mathscr{B}$ is a finitely generated submodule of $D\mathbb{Z}^{dm}$, where

$$D := \begin{bmatrix} A_1 & 0 & \ldots & 0 \\ 0 & A_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \ldots & 0 & A_m \end{bmatrix}.$$

Hence there are an $r \in \mathbb{N}_0$ and a matrix $B \in \mathbb{R}^{n \times r}$ such that $\mathscr{B} = B\mathbb{Z}^r$.

We are interested in the question whether there exists a finite lattice refinement L of $\mathbb{Z}^d$ such that $Az = b$ is solvable over L for each $b \in \mathscr{B}$. If this is the case, then we will call the solutions *pseudodiophantine*. Now, let for $i = 1, \ldots, m$

$$V_i := \ker(A_i).$$

If there exist $x \in \mathbb{R}^d$ and $z_1, \ldots, z_m \in \mathbb{Z}^d$ such that

$$A_1 x = b_1, \ldots, A_m x = b_m \quad \wedge \quad A_1 z_1 = b_1, \ldots, A_m z_m = b_m,$$

then the spaces $z_1 + V_1, \ldots, z_m + V_m$ intersect in x i.e.,

$$x \in G(V_1, \ldots, V_m).$$

In fact, there exist pseudodiophantine solutions for each right hand side $b \in \mathscr{B}$ if and only if the index $\iota(V_1, \ldots, V_m)$ is finite.

Throughout the paper, the above notation

$$A_1, \ldots, A_m, A, \mathscr{B}, D, V_1, \ldots, V_m, V, n_1, \ldots, n_m, n, r, B$$

will be fixed. Further, to avoid trivialities, we assume

$$d \geq 2 \quad \wedge \quad m \geq 2.$$

Our first theorem gives a characterization in terms of the inherent rational dependencies. It shows, in particular, that $\iota(V_1, \ldots, V_m) < \infty$ if and only if the equivalence classes of $G(V_1, \ldots, V_m)$ have *rational* representations, a property that is especially interesting from an algorithmic viewpoint, since it allows a finite precision encoding.

Theorem 2.1. *The following statements are equivalent:*

(i) *The index $\iota(V_1, \ldots, V_m)$ is finite.*
(ii) *There exists a matrix $Q \in \mathbb{Q}^{d \times r}$ such that*

$$B = AQ.$$

(iii) *Each equivalence class in $G(V_1, \ldots, V_m)$ is of the form $q + V + \mathbb{Z}^d$ for some $q \in \mathbb{Q}^d$.*

Moreover, if (ii) *holds and $\delta > 0$ is a common denominator of all coefficients of Q, then*

$$\iota(V_1, \ldots, V_m) \leq \delta^r.$$

Note that if A is totally unimodular, the equality $B = AQ$ implies that Q can be chosen to have integer entries and thus $\iota(V_1, \ldots, V_m) = \delta = \delta^r = 1$; hence in this case the bound in (iii) is tight.

In Theorem 2.1, the matrix B encodes the structure of $\mathscr{B}$ as a submodule of $D\mathbb{Z}^{dm}$ or, more intuitively, the linear dependencies between the A_i's. Geometrically, if all matrices A_i have full row rank, the special case $\mathscr{B} = D\mathbb{Z}^{dm}$ corresponds to the fact that the m affine spaces $z_1 + V_1, \ldots, z_m + V_m$ intersect for each arbitrary choice of vectors $z_1, \ldots, z_m \in \mathbb{Z}^d$. Hence $\mathscr{B} = D\mathbb{Z}^{dm}$ if and only if A has full row rank. The following corollary shows that in this special setting Q will reflect the underlying 'decoupled' structure.

Corollary 2.2. *If $\mathscr{B} = D\mathbb{Z}^{dm}$, then the following statements are equivalent:*

(i) *$\iota(V_1, \ldots, V_m)$ is finite.*
(ii) *For each $i \in \{1, \ldots, m\}$ there exists $Q_i \in \mathbb{Q}^{d \times d}$ such that*

$$A_i Q_i = A_i \quad \wedge \quad A_j Q_i = 0 \in \mathbb{R}^{n_j \times d} \quad \textit{for } j \in \{1, \ldots, m\} \setminus \{i\}.$$

In particular, Q_m can be chosen as $I_d - \sum_{l=1}^{m-1} Q_l$, where I_d denotes the $d \times d$ unit matrix.

The next theorem indicates that the finiteness of $\iota(V_1, \ldots, V_m)$ is closely related to the 'degree of (ir)rationality' of $V_1, \ldots, V_m$. For $i = 1, \ldots, m$ let

$$\operatorname{rat}(V_i) := \operatorname{lin}_{\mathbb{R}}(V_i \cap \mathbb{Q}^d).$$

Theorem 2.3. *Suppose that for each $i \in \{1, \ldots, m\}$ there exists $Q_i \in \mathbb{Q}^{d \times d}$ such that*

$$A_i Q_i = A_i \quad \wedge \quad A_j Q_i = 0 \in \mathbb{R}^{n_j \times d} \quad \textit{for } j \in \{1, \ldots, m\} \setminus \{i\}.$$

Further, for $i = 1, \ldots, m$, let A_i contain at least k_i $\mathbb{Q}$-linearly independent columns. Then, for $i \in \{1, \ldots, m\}$,

$$k_i \leq \min \left\{ \dim \left(\operatorname{rat}(V_j) \right) : j \in \{1, \ldots, m\} \setminus \{i\} \right\}.$$

If, additionally, $V \cap \mathbb{Z}^d = \{0\}$*, then*

$$\sum_{i=1}^{m} k_i \leq \sum_{i=1}^{m} \dim(\mathrm{rat}(V_i)) \leq (m-1)d.$$

Note that the requirement $V \cap \mathbb{Z}^d = \{0\}$ is a natural condition. In fact, we could essentially assume it without loss of generality, since a rational subspace of V can be projected out to reduce the dimension.

The following theorem contains a statement that is somewhat converse to Theorem 2.3.

Theorem 2.4. *Let* $V \cap \mathbb{Z}^d = \{0\}$ *and*

$$\sum_{i=1}^{m} \dim(\mathrm{rat}(V_i)) = (m-1)d.$$

Then $\iota(V_1, \ldots, V_m) < \infty$.

3. The decomposition problem in the discrete tomography of quasicrystals: Notation and main results

We will use our results on Siegel grids to solve the decomposition problem in the discrete tomography of mathematical quasicrystals that live on some finitely generated $\mathbb{Z}$-module Z in some $\mathbb{R}^s$ i.e, lie in Z up to translation. As a service to the reader we will begin with a short description of model sets, the standard mathematical model for quasicrystals, and will then briefly introduce the concept of discrete tomography. We are not aiming at the most general descriptions, but will concentrate on those facts that will enable us to state the basic decomposition problem in a self-contained way. The main part of this section will show how our characterization of when the index of Siegel grids is finite will translate into a solution of this problem. As a simple corollary we will derive a result on certain planar model sets, an example being the cyclotomic model sets studied in [6].

In their most general form, model sets are defined via some *cut-and-project scheme* that involves locally compact Abelian groups $\mathscr{G}$ and $\mathscr{H}$, a discrete co-compact additive subgroup $\mathscr{L}$ of $\mathscr{G} \oplus \mathscr{H}$ and a subset of $\mathscr{G}$, the so-called window; see [25,33,9]. Since it is not quasicrystals and their remarkable properties but rather $\mathbb{Z}$-modules in some $\mathbb{R}^s$ that are in the focus of the present paper, we will, for the sake of the intuitiveness of the exposition, not introduce general model sets in 'their natural habitat' but restrict ourselves to a description that shows the main geometric flavor of cut-and-project schemes. Note, however, that our results are much more general than they might seem at first glance to those familiar with general model sets, since all we need is that the structures of interest live on some arbitrary finitely generated $\mathbb{Z}$-module Z in some $\mathbb{R}^s$ which is the case for all quasicrystals of practical relevance; see, in particular, [33, Sec. 5] for some explanatory theoretical results.

With this perspective, let us now give an elementary indication of the geometric genesis of model sets. In their basic geometric form, model set in some s-dimensional real vector space Y are commonly defined via a linear cut-and-project scheme. So, let

$$s \in \{1, \ldots, d-1\} \quad \wedge \quad X = \{0\}^s \times \mathbb{R}^{d-s} \quad \wedge \quad Y = \mathbb{R}^s \times \{0\}^{d-s}.$$

Further, let

$$\Pi_X : \mathbb{R}^d \to Y \quad \wedge \quad \Pi_Y : \mathbb{R}^d \to X$$

denote the projection parallel to X, Y, respectively. Let $L \subset \mathbb{R}^d$ be a lattice of rank d. Note that, in particular,

$$Z_Y := \Pi_X(L) \quad \wedge \quad Z_X := \Pi_Y(L)$$

are finitely generated $\mathbb{Z}$-modules. We will frequently identify X, Y with $\mathbb{R}^{d-s}$, $\mathbb{R}^s$, respectively, and hence particularly regard Z_Y as a subset of $\mathbb{R}^s$. As a standard assumption in the theory of quasicrystals, let the restriction $\Pi_X|_L$ on L be injective. Of course, this is equivalent to

$$X \cap L = \{0\}$$

and implies that Z_Y is not discrete. The space Y is called the *physical space* since Z_Y hosts the quasicrystals. Naturally, for direct applications to real physical structures, the dimension of Y could be restricted to three or, if layered objects are considered, to two. However, we will deal with the decomposition problem in general. The mathematical quasicrystals are now selected from Z_Y by the so-called *star map*

$$\cdot^\star := \Pi_Y \circ \Pi_X |_L^{-1} : Z_Y \to Z_X$$

together with a so-called *window*, an appropriate bounded subset W of X. More precisely, let

$$\Lambda(W) := \{z \in Y : z^\star \in W\} \quad \wedge \quad \mathscr{M}(W) := \{y + \Lambda(W + x) : x \in X \wedge y \in Y\}.$$

Each element of $\mathscr{M}(W)$ is called a *model set* (with respect to the cut-and-project scheme $(X, Y; W)$). The fact that translations are allowed within X and Y reflects the problem that in physical applications a natural choice of the translational origin is not possible while the rotational orientation of a probe in an electron microscope can be determined in the diffraction mode prior to taking images in the high resolution mode. For more information on quasicrystals and aperiodic tilings see [25,37,11,12,32,13,22,14,30,26,27,33,34,29,5,4] and other papers quoted there.

Using the high resolution mode in electron microscopy and an image analysis technique developed in [35] and [21], one can in principle reach a tomographic resolution at the atomic scale. Hence the problem of the reconstruction of a crystalline or quasicrystalline atomic structure that is only accessible through a (small) number of its images under high resolution transmission electron microscopy can be modeled in terms of a finite point set F whose cardinalities of intersections with query sets parallel to the imaging directions are known. More precisely and more generally, let F be a finite subset of some linear space Y that lives on some $\mathbb{Z}$-module Z in Y. Further, let S be a proper subspace of Y, and let $\mathcal{T}$ denote the family of all affine spaces $t + S$. Then the (discrete) *X-ray* of F parallel to S is the function

$$X_S F : \mathcal{T} \to \mathbb{N}_0$$

defined by

$$X_S F(t + S) := |F \cap (t + S)| .$$

Now suppose that X-ray information on the otherwise unknown set F is available for m different subspaces $S_1, \ldots, S_m$ that are spanned by vectors of the same $\mathbb{Z}$-module Z. The basic *inverse problem of discrete tomography* is to reconstruct (all, an appropriate) such set(s) from the given X-ray information. See [16,18,17,19] for surveys on discrete tomography, [2] for related stability issues and [6,7,20] for other results on the discrete tomography of quasicrystals.

It is clear that one can directly restrict the set of all possible solutions. In fact, let $\mathcal{T}_1, \ldots, \mathcal{T}_m$ denote the corresponding supports i.e., $\mathcal{T}_i$ is the family of all translates $t + S_i$ that intersect F. Then, the 'unknown' set F is contained in the *tomographic grid*

$$H_F := \bigcap_{i=1}^{m} \bigcup_{T \in \mathcal{T}_i} T$$

of F. It is, however, not clear how to (efficiently) determine F in H_F. The 'classical' crystalline case with a fixed origin corresponds to the situation that Z is a lattice and F is contained in Z; hence one can further restrict the reconstruction to $H_F \cap Z$. In general, however, H_F will not be contained in a single translate of Z, but in a union of several translates of Z. Any feasible solution of the underlying reconstruction problem must, however, entirely belong to just one such class; see Fig. 1 for an example.

This requirement leads directly to the so-called *decomposition problem of discrete tomography*, of whether there is a uniform bound, independent of F, on the number of elements of a partition of the tomographic grid into subsets that are contained in a single translate of the underlying module. Equivalently, this is a question of whether the *complete tomographic grid*

$$H := \bigcap_{i=1}^{m} \bigcup_{z \in Z} (z + S_i)$$

decomposes into finitely many equivalence classes $q + Z$. In the lattice case, this is simple and well known. The general problem is exactly that of the finiteness of the index of the Siegel grid $G(Z; S_1, \ldots, S_m)$.

In order to transform H to the standard situation of Section 2, let

$$p_1, \ldots, p_d \in \mathbb{R}^s$$

be generators of the $\mathbb{Z}$-module Z, and let

$$P := [p_1, \ldots, p_d] \in \mathbb{R}^{s \times d} \quad \wedge \quad U := \ker(P).$$

Again, we may assume that $[p_1, \ldots, p_s]$ is the standard unit matrix in $\mathbb{R}^s$. Then Z is the projection of $\mathbb{Z}^d$ on $\mathbb{R}^s$ parallel to the space U. Of course, with $S := S_1 \cap \cdots \cap S_m$ and

$$V_i := S_i + U \quad (i = 1, \ldots, m) \quad \wedge \quad V := V_1 \cap \cdots \cap V_m,$$

we have

$$\left| G(Z; S_1, \ldots, S_m) /_{\sim_S} \right| < \infty \Leftrightarrow \iota(V_1, \ldots, V_m) < \infty.$$

The following two theorems are motivated by the classical lattice setting in the plane, where $\iota(V_1, V_2) < \infty$ for *each* pair of non-parallel lines $V_i := S_i := z_i\mathbb{R}$ with $z_i \in \mathbb{Z}^2$ and $i = 1, 2$.

Theorem 3.1. *Let $U \cap \mathbb{Z}^d = \{0\}$. Let $\mathscr{S}$ denote a set of at least $2m - 1$ non-trivial subspaces of $\mathbb{R}^s$ which have the property that, for each m element subset $\{S_1, \ldots, S_m\}$ and $z_1, \ldots, z_m \in Z$,*

$$(z_1 + S_1) \cap \cdots \cap (z_m + S_m) \neq \emptyset \quad \wedge \quad S_1 \cap \cdots \cap S_m = \{0\},$$

and that $G(Z; S_1, \ldots, S_m)$ has finite index. Then

$$d \le m \left\lfloor \frac{d}{2} \right\rfloor.$$

Theorem 3.1 implies, in particular, that in each planar model set whose internal space is of odd dimension, there must exist two module lines whose complete tomographic grid does not decompose into finitely many translational equivalence classes. So, a necessary condition for the index in the planar case to be always finite is that the underlying dimension d is even. The next result gives a partial converse. It proves finiteness in the 'classical non-discrete' planar cases involving a 2-dimensional vector space $\mathbb{V}$ over a proper finite real field extension $\Bbbk$ of $\mathbb{Q}$ i.e., $\Bbbk$ is a field, $\mathbb{Q} \subset \Bbbk \subset \mathbb{R}$ and, viewed as a $\mathbb{Q}$-vector space, $1 < \dim_{\mathbb{Q}} \Bbbk < \infty$. As Example 5.2 shows, the 'product structure' is indeed relevant.

Theorem 3.2. *Let $\Bbbk$ be a proper finite real field extension of $\mathbb{Q}$, $\mathbb{V}$ a $\Bbbk$-vector space of dimension 2, and $d := 2 \cdot \dim_{\mathbb{Q}}(\Bbbk)$. Further, let $p_1, \ldots, p_d$ be a $\mathbb{Q}$-basis of $\mathbb{V}$, and let Z be the $\mathbb{Z}$-module in $\mathbb{R}^2$ generated by $p_1, \ldots, p_d$. Then for each linearly independent pair $z_1, z_2 \in Z$, the Siegel grid $G(Z; z_1\mathbb{R}, z_2\mathbb{R})$ decomposes into finitely many equivalence classes.*

Note that, under the assumptions of Theorem 3.2, $\mathrm{cl}(Z) = \mathbb{R}^2$. Let us further point out that for each $j \in \mathbb{N}$, there exists a real field extension $\Bbbk$ of $\mathbb{Q}$ of degree j. In fact, noting that by Eisenstein's irreducibility criterion (see, e.g., [10, Sec. 3.10]) the polynomial $x^j - 2$ is irreducible over $\mathbb{Q}$, we may, for instance, choose $\mathbb{Q}(\sqrt[j]{2})$. Hence for each even d there are dense $\mathbb{Z}$-modules of rank d in the plane whose tomographic or Siegel grids have a finite index no matter which module lines S_1, S_2 are chosen.

Since the cyclotomic rings (regarded as subsets of $\mathbb{R}^2$) are also covered by Theorem 3.2, we obtain the following result of [6] as a corollary.

Corollary 3.3. *Let $N \in \mathbb{N}$. Then for each linearly independent pair $z_1, z_2 \in \mathbb{Z}[\zeta_N]$, the Siegel grid $G\,(\mathbb{Z}[\zeta_N]; z_1\mathbb{R}, z_2\mathbb{R})$ decomposes into finitely many equivalence classes modulo $\mathbb{Z}[\zeta_N]$.*

4. Siegel grids: Proofs and further results

We begin with the proof of Theorem 2.1 and Corollary 2.2. Here, and in the following, for $l \in \mathbb{N}$, the standard unit vectors of $\mathbb{R}^l$ will be denoted by $u_1, \ldots, u_l$, and I_l is the $l \times l$ identity matrix.

Proof of Theorem 2.1. "(i) $\Rightarrow$ (ii)": We prove that for each $i \in \{1, \ldots, r\}$ the i-th column of B is a $\mathbb{Q}$-linear combination of the columns of A. So, let $i \in \{1, \ldots, r\}$. Since $\iota(V_1, \ldots, V_m) < \infty$, there are only finitely many different sets of the form

$$\{x \in \mathbb{R}^d : Ax = jBu_i\} + \mathbb{Z}^d$$

for $j \in \mathbb{N}$. Therefore there exist $j_1, j_2 \in \mathbb{N}$ with $j_1 < j_2$, such that

$$\{x \in \mathbb{R}^d : Ax = j_1 Bu_i\} + \mathbb{Z}^d = \{x \in \mathbb{R}^d : Ax = j_2 Bu_i\} + \mathbb{Z}^d.$$

By the definition of $\mathscr{B}$ the sets in question are non-empty. So, let $y_1 \in \{x \in \mathbb{R}^d : Ax = j_1 Bu_i\}$, $y_2 \in \{x \in \mathbb{R}^d : Ax = j_2 Bu_i\}$, and $z \in \mathbb{Z}^d$ such that $y_1 = y_2 + z$. Then

$$Ay_1 = j_1 Bu_i = A(y_2 + z) = Ay_2 + Az = j_2 Bu_i + Az,$$

and hence

$$Bu_i = A\left(\frac{1}{j_1 - j_2} z\right).$$

Thus Bu_1 is indeed a $\mathbb{Q}$-linear combination of the columns of A.

"(ii) ⇒ (iii)": If $B = AQ$ for some $Q \in \mathbb{Q}^{d\times r}$, then for each $w \in \mathbb{Z}^r$ the equation $Ax = Bw$ is equivalent to $x - Qw \in \ker(A)$. Hence,

$$\{x \in \mathbb{R}^d : Ax = Bw\} + \mathbb{Z}^d = Qw + V + \mathbb{Z}^d$$

i.e., each equivalence classes has a rational representative.

"(iii) ⇒ (ii)": The assumption (iii) implies, in particular, that the system $Ax = Bu_i$ has a rational solution q_i for each $i \in \{1, \dots, r\}$. With $Q := [q_1 \dots, q_r]$ we obtain $AQ = BI_r = B$.

"(ii) ⇒ (i)": Let $Q \in \mathbb{Q}^{d\times r}$ with $B = AQ$, and let $\delta > 0$ be a common denominator of the entries of Q. Since

$$G(V_1, \dots, V_m)/_\sim = \left\{\{x \in \mathbb{R}^d : Ax = b\} + \mathbb{Z}^d : b \in \mathscr{B}\right\}$$

it suffices to show that for each $b \in \mathscr{B}$, there exists a vector $t \in \{0, 1, \dots, \delta - 1\}^r$ such that

$$\{x \in \mathbb{R}^d : Ax = b\} + \mathbb{Z}^d = \{x \in \mathbb{R}^d : Ax = Bt\} + \mathbb{Z}^d.$$

This will also prove the final assertion of the theorem.

So, let $b \in \mathscr{B}$ and $w \in \mathbb{Z}^r$ such that $b = Bw$. Decomposing w by component-wise division modulo δ, we obtain $z \in \mathbb{Z}^r$ and $t \in \{0, 1, \dots, \delta - 1\}^r$ such that $w = \delta z + t$. Then $Bw = \delta Bz + Bt = \delta AQz + Bt$; hence $Ax = Bw$ is equivalent to $A(x - \delta Qz) = Bt$. It follows that

$$\{x \in \mathbb{R}^d : Ax = b\} = \{y + \delta Qz \in \mathbb{R}^d : Ay = Bt\}.$$

Since $\delta Qz \in \mathbb{Z}^d$, we conclude

$$\begin{aligned}\{x \in \mathbb{R}^d : Ax = b\} + \mathbb{Z}^d &= \{y \in \mathbb{R}^d : Ay = Bt\} + \delta Qz + \mathbb{Z}^d \\ &= \{x \in \mathbb{R}^d : Ax = Bt\} + \mathbb{Z}^d,\end{aligned}$$

which finishes the proof. □

Next we show that Corollary 2.2 is an immediate consequence of Theorem 2.1.

Proof of Corollary 2.2. Since $\mathscr{B} = D\mathbb{Z}^{dm}$, we apply Theorem 2.1 with $r = dm$ and $B = D$ to obtain a matrix $Q \in \mathbb{Q}^{d\times dm}$ with $D = AQ$. For $i = 1, \dots, m$ let Q_i denote its $d \times d$ submatix of the columns with index $(i - 1)d + 1, \dots, id$. Then $Q_1, \dots, Q_m$ have the asserted properties. The converse follows similarly.

Now, set $Q'_m := I_d - \sum_{l=1}^{m-1} Q_l$. Then, of course,

$$A_m Q'_m = A_m \left(I_d - \sum_{l=1}^{m-1} Q_l\right) = A_m$$

and for $j \in \{1, \dots, m - 1\}$

$$A_j Q'_m = A_j \left(I_d - \sum_{l=1}^{m-1} Q_l\right) = A_j - A_j = 0. \quad \square$$

One may wonder whether the finiteness of the index of a Siegel grid that is built with the aid of m spaces $V_1, \dots, V_m$ implies already the one obtained with one additional space V_{m+1}. The answer is not immediately obvious, since with $V = V_1 \cap \cdots \cap V_m$ and $V' := V \cap V_{m+1}$, in general, the relations $\sim_V$ and $\sim_{V'}$ are different. Suppose first that $V \subset V_{m+1}$; hence $\sim = \sim_V = \sim_{V'}$. Now, let

$$g_1, g_2 \in G(V_1, \ldots, V_m, V_{m+1}) \quad \wedge \quad g_1 \sim g_2.$$

Then, of course, $g_1, g_2 \in G(V_1, \ldots, V_m)$ and $g_1 - g_2 \in \mathbb{Z}^d + V$. Therefore, in this case,

$$\iota(V_1, \ldots, V_m) < \infty \Rightarrow \iota(V_1, \ldots, V_m, V_{m+1}) < \infty.$$

In general, however, the situation is more complicated.

Example 4.1. Let $\omega \in \mathbb{R} \setminus \mathbb{Q}$,

$$A_1 := [\omega, 1, 1], \qquad A_2 := [0, 1, 0], \qquad A_3 := [0, 0, 1] \in \mathbb{R}^{1\times 3},$$

and $V_i := \ker(A_i)$ for $i = 1, 2, 3$. Then V_1, V_2, V_3 are 2-dimensional subspaces of $\mathbb{R}^3$. Further, let

$$Q_{1,2} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 1 \end{bmatrix} \quad \wedge \quad Q_{1,3} := \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \quad \wedge \quad Q_{2,3} := \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} Q_{1,2} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \quad \wedge \quad \begin{bmatrix} A_1 \\ A_3 \end{bmatrix} Q_{1,3} = \begin{bmatrix} A_1 \\ 0 \end{bmatrix} \quad \wedge \quad \begin{bmatrix} A_2 \\ A_3 \end{bmatrix} Q_{2,3} = \begin{bmatrix} A_2 \\ 0 \end{bmatrix}.$$

Hence, by Corollary 2.2

$$\iota(V_1, V_2), \iota(V_1, V_3), \iota(V_2, V_3) < \infty.$$

Now, suppose $\iota(V_1, V_2, V_3) < \infty$. Then, again by Corollary 2.2, there exists a matrix $Q_1 \in \mathbb{Q}^{3\times 3}$ with

$$\begin{bmatrix} \omega & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} Q_1 = \begin{bmatrix} \omega & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence

$$Q_1 = \begin{bmatrix} 1 & \frac{1}{\omega} & \frac{1}{\omega} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in \mathbb{Q}^{3\times 3},$$

contradicting the choice of $\omega \notin \mathbb{Q}$.

Example 4.1 shows that even if the dimensions of the involved spaces V_1, V_2, V_3 are such that arbitrary translates will always intersect, the finiteness of $\iota(V_i, V_j)$ for each pair $(i, j) \in \{1, 2, 3\}^2$ does not imply the finiteness of $\iota(V_1, V_2, V_3)$. The following corollary shows, however, that the converse is indeed true.

Corollary 4.2. *Let $\mathscr{B} = D\mathbb{Z}^{dm}$ and $\iota(V_1, \ldots, V_m) < \infty$. Then, for each $l \in \{1, \ldots, m\}$ and $V_{i_1}, \ldots, V_{i_l} \subset \{V_1, \ldots, V_m\}$,*

$$\iota(V_{i_1}, \ldots, V_{i_l}) < \infty.$$

Proof. By Corollary 2.2 there exist $Q_1, \ldots, Q_m \in \mathbb{Q}^{d\times d}$ such that

$$A_i Q_i = A_i \quad \wedge \quad A_j Q_i = 0 \in \mathbb{R}^{n_j \times d}$$

for $i, j \in \{1, \ldots, m\}$ with $i \neq j$. Now, let $l \in \{1, \ldots, m\}$, $V_{i_1}, \ldots, V_{i_l} \subset \{V_1, \ldots, V_m\}$ and suppose, without loss of generality, that $V_{i_1}, \ldots, V_{i_l}$ are all different. Then, of course,

$$A_i Q_i = A_i \quad \wedge \quad A_j Q_i = 0$$

for $i, j \in \{i_1, \ldots, i_l\}$ with $i \neq j$, and the assertion follows again from Corollary 2.2. $\square$

The following lemma is needed in the proofs of Theorems 2.3 and 2.4.

Lemma 4.3. *Suppose that* $V \cap \mathbb{Z}^d = \{0\}$ *and, for* $i = 1, \ldots, m$, *let* $r_i := \dim(\mathrm{rat}(V_i))$. *Then*

$$\sum_{i=1}^{m} r_i \leq (m-1)d,$$

and equality implies that

$$\mathbb{Q}^d \cap \bigcap_{i=1}^{m} (z_i + V_i) \neq \emptyset$$

for any choice of $z_1, \ldots, z_m \in \mathbb{Z}^d$.

Proof. For $i = 1, \ldots, m$, let $R_i \in \mathbb{Q}^{(d-r_i)\times d}$ such that $\mathrm{rat}(V_i) = \ker(R_i)$, and set $R := [R_1^{\mathrm{T}}, \ldots, R_m^{\mathrm{T}}]^{\mathrm{T}}$. Of course,

$$\ker(R) = \bigcap_{i=1}^{m} \mathrm{rat}(V_i) \subset \bigcap_{i=1}^{m} V_i = V.$$

Suppose that $\sum_{i=1}^{m} r_i > (m-1)d$. Then $md - \sum_{i=1}^{m} r_i < d$ i.e., R has more columns than rows; hence V contains a non-zero integral vector, contradicting $V \cap \mathbb{Z}^d = \{0\}$.

Now, let $\sum_{i=1}^{m} r_i = (m-1)d$. Then R is quadratic and, again because of $V \cap \mathbb{Z}^d = \{0\}$, must have full rank. Thus for any choice of $z_1, \ldots, z_m \in \mathbb{Z}^d$, the system

$$Rx = \begin{bmatrix} R_1 z_1 \\ \vdots \\ R_m z_m \end{bmatrix}$$

of linear equations has a unique solution and this is rational. But then

$$\mathbb{Q}^d \cap \bigcap_{i=1}^{m} (z_i + \mathrm{rat}(V_i)) \subset \mathbb{Q}^d \cap \bigcap_{i=1}^{m} (z_i + V_i) \neq \emptyset. \quad \square$$

Now we give the proofs of Theorems 2.3 and 2.4.

Proof of Theorem 2.3. Let $i \in \{1, \ldots, m\}$ be fixed, let Q_i be as presumed, and suppose without loss of generality that the first k_i columns of A_i are $\mathbb{Q}$-linearly independent. Let $a_1, \ldots, a_{k_i}$ and $q_1, \ldots, q_{k_i}$ denote the first k_i columns of A_i and Q_i, respectively. Then we have for $l \in \{1, \ldots, k_i\}$ and $j \in \{1, \ldots, m\} \setminus \{i\}$

$$A_i q_l = a_l \quad \wedge \quad A_j q_l = 0.$$

Thus

$$q_1, \ldots, q_{k_i} \in \ker(A_j) \cap \mathbb{Q}^d.$$

Now, let

$$\lambda_1, \ldots, \lambda_{k_i} \in \mathbb{Q} \quad \wedge \quad \sum_{l=1}^{k_i} \lambda_l q_l = 0.$$

Then

$$A_i \left(\sum_{l=1}^{k_i} \lambda_l q_l \right) = \sum_{l=1}^{k_i} \lambda_l A_i q_l = \sum_{l=1}^{k_i} \lambda_l a_l = 0.$$

Since $a_1, \ldots, a_{k_i}$ are $\mathbb{Q}$-linearly independent, so are $q_1, \ldots, q_{k_i}$. But these vectors are rational; this implies that $q_1, \ldots, q_{k_i}$ are $\mathbb{R}$-linearly independent.

The final assertion follows now from Lemma 4.3. □

With the aid of Lemma 4.3, Theorem 2.4 is a direct consequence of Theorem 2.1.

5. The decomposition problem: Proofs

Beginning with Theorem 3.1, we give now the proofs of all assertions of Section 3.

Proof of Theorem 3.1. Let $S_1, \ldots, S_m \in \mathscr{S}$ be different. For $i = 1, \ldots, m$, let $A_i \in \mathbb{R}^{n_i \times d}$ have full row rank such that $S_i + U = \ker(A_i)$. Note that it follows from the assumption on $\mathscr{S}$ that $\mathscr{B} = D\mathbb{Z}^{dm}$ for each m element subset of $\mathscr{S}$.

Since the index of $G(Z; S_1, \ldots, S_m)$ is finite, $\iota(S_1 + U, \ldots, S_m + U) < \infty$, hence by Corollary 4.2, $\iota(S_1 + U, S_2 + U) < \infty$, and Corollary 2.2 yields a matrix $Q \in \mathbb{Q}^{d \times d}$ such that

$$\begin{bmatrix} A_1 \\ A_2 \end{bmatrix} Q = \begin{bmatrix} 0 \\ A_2 \end{bmatrix} \quad \wedge \quad \begin{bmatrix} A_1 \\ A_2 \end{bmatrix} (I_d - Q) = \begin{bmatrix} A_1 \\ 0 \end{bmatrix}.$$

Let $q_1, \ldots, q_d$ denote the columns of Q. Then, in particular,

$$q_1, \ldots, q_d \in S_1 + U \quad \wedge \quad u_1 - q_1, \ldots, u_d - q_d \in S_2 + U.$$

Since the rank of $[Q, I_d - Q]$ is d, at least one of the two matrices Q or $I_d - Q$ must contain at least $d/2$ linearly independent columns. Hence at least one of the spaces $\mathrm{rat}(S_1 + U)$ or $\mathrm{rat}(S_2 + U)$ has at least dimension $d/2$, say $S_1 + U$ i.e.,

$$\dim(\mathrm{rat}(S_1 + U)) \geq \left\lceil \frac{d}{2} \right\rceil.$$

Now, remove S_1 from $\mathscr{S}$ and apply the same argument successively. After m steps, we found m subspaces $S_1', \ldots, S_m' \in \mathscr{S}$ such that

$$\dim\left(\mathrm{rat}(S_i' + U)\right) \geq \left\lceil \frac{d}{2} \right\rceil$$

for $i = 1, \ldots, m$. Since $\bigcap_{i=1}^m (S_i' + U) \cap \mathbb{Z}^d = \{0\}$, we obtain with the aid of Theorem 2.3

$$(m-1)d \geq \sum_{i=1}^m \dim\left(\mathrm{rat}(S_i' + U)\right) \geq m \left\lceil \frac{d}{2} \right\rceil,$$

which yields the assertion. □

The following technical Lemma 5.1 will be used in the proof of Theorem 3.2.

Lemma 5.1. *Let $c_1, c_2 \in \mathbb{R}^d$ be linearly independent and $U = \ker[c_1, c_2]^{\mathrm{T}}$. Further, let $z_1, z_2 \in \mathbb{R}^d$ such that $U + z_1\mathbb{R} + z_2\mathbb{R} = \mathbb{R}^d$, and let a_1^{T} and a_2^{T} denote the rows of the $2 \times d$ matrix*

$$A_{c_1,c_2}(z_1, z_2) := \begin{bmatrix} z_1^{\mathrm{T}} \\ z_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} c_2, & -c_1 \end{bmatrix} \begin{bmatrix} c_1^{\mathrm{T}} \\ c_2^{\mathrm{T}} \end{bmatrix}.$$

Then we have, for $i = 1, 2$,

$$a_i^{\mathrm{T}} z_i = 0 \quad \wedge \quad a_i^{\perp} = U + z_i\mathbb{R} \quad \wedge \quad U = \ker\left(A_{c_1,c_2}(z_1, z_2)\right).$$

Proof. Since

$$\begin{bmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \end{bmatrix} = A_{c_1,c_2}(z_1, z_2) = \begin{bmatrix} z_1^{\mathrm{T}} c_2, & -z_1^{\mathrm{T}} c_1 \\ z_2^{\mathrm{T}} c_2, & -z_2^{\mathrm{T}} c_1 \end{bmatrix} \begin{bmatrix} c_1^{\mathrm{T}} \\ c_2^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} z_1^{\mathrm{T}} c_2 c_1^{\mathrm{T}} - z_1^{\mathrm{T}} c_1 c_2^{\mathrm{T}} \\ z_2^{\mathrm{T}} c_2 c_1^{\mathrm{T}} - z_2^{\mathrm{T}} c_1 c_2^{\mathrm{T}} \end{bmatrix}$$

we have $a_1^{\mathrm{T}} z_1 = a_2^{\mathrm{T}} z_2 = 0$. Also, a_1, a_2 are $\mathbb{R}$-linear combinations of c_1 and c_2, hence $U \subset a_i^{\perp}$. Therefore $a_i^{\perp} = U + z_i\mathbb{R}$ for $i = 1, 2$. Since $z_1, z_2 \in \mathbb{R}^d$ are $\mathbb{R}$-linearly independent, we finally conclude that

$$U \subset \ker[a_1, a_2]^{\mathrm{T}} \subset (U + z_1\mathbb{R}) \cap (U + z_2\mathbb{R}) \subset U. \quad \square$$

Note that the components of $A_{c_1,c_2}(z_1, z_2)$ are contained in the same field as the coefficients of c_1, c_2, z_1, z_2. This fact will be used in the following proof of Theorem 3.2.

Proof of Theorem 3.2. Due to the underlying invariance under linear transformations, we may assume that $\mathbb{V}$ is indeed $\mathbb{k}^2$.

Let $z_1, z_2 \in Z$ be linearly independent. In the following we use the standard identification of $\mathbb{R}^2$ with $\mathbb{R}^2 \times \{0\}^{d-2}$, the previous notation including

$$P = [p_1, \ldots, p_d] = \begin{bmatrix} c_1^{\mathrm{T}} \\ c_2^{\mathrm{T}} \end{bmatrix} \quad \wedge \quad U = \ker(P),$$

and we assume that $p_i = u_i$ for $i = 1, 2$. Since this assumption can always be satisfied by means of a linear transformation with entries in $\mathbb{k}$, this is again no restriction of generality. We show that $\iota(U + z_1\mathbb{R}, U + z_2\mathbb{R}) < \infty$.

By Lemma 5.1,

$$\ker\left(A_{c_1,c_2}(z_1, z_2)\right) = U \quad \wedge \quad U + z_i\mathbb{R} = a_i^{\perp} \quad (i = 1, 2).$$

Since

$$\begin{bmatrix} z_1^{\mathrm{T}} \\ z_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} c_2, & -c_1 \end{bmatrix} \in \mathbb{k}^{2\times 2}$$

and the columns of P are a $\mathbb{Q}$-basis of $\mathbb{k}^2$, so are the columns of $A_{c_1,c_2}(z_1, z_2)$. Hence for each $w_1, w_2 \in \mathbb{Z}^d$ the equation

$$\begin{bmatrix} a_1^{\mathrm{T}} \\ a_2^{\mathrm{T}} \end{bmatrix} x = \begin{bmatrix} a_1^{\mathrm{T}} w_1 \\ a_2^{\mathrm{T}} w_2 \end{bmatrix}$$

admits a rational solution, so the assertion follows from Theorem 2.1. $\square$

In the following example, Corollary 2.2 is used to show that the product structure in Theorem 3.2 cannot be abandoned.

Example 5.2. Let $\omega \in \mathbb{R}$ be such that 1, ω and ω^2 are $\mathbb{Q}$-linearly independent, and set

$$p_1 := \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \wedge \quad p_2 := \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad \wedge \quad p_3 := \begin{bmatrix} \omega \\ 0 \end{bmatrix} \quad \wedge \quad p_4 := \begin{bmatrix} \omega^2 \\ \omega \end{bmatrix},$$

and also

$$P := [p_1, p_2, p_3, p_4] \quad \wedge \quad Z := P\mathbb{Z}^4.$$

Further, let $c_1^{\mathrm{T}}, c_2^{\mathrm{T}}$ denote the rows of P, and let $z_i := u_i \in \mathbb{Z}^4$ for $i = 1, 2$. Then, according to Lemma 5.1,

$$A_{c_1,c_2}(z_1, z_2) = \begin{bmatrix} 0 & -1 & 0 & -\omega \\ 1 & 0 & \omega & \omega^2 \end{bmatrix}.$$

Now, suppose that the index of $G(Z; z_1\mathbb{R}, z_2\mathbb{R})$ is finite. Then, by Corollary 2.2, there exists a rational matrix $Q := (\kappa_{i,j})_{i,j=1,\ldots,4}$ such that

$$\begin{bmatrix} 0 & -1 & 0 & -\omega \\ 1 & 0 & \omega & \omega^2 \end{bmatrix} Q = \begin{bmatrix} 0 & -1 & 0 & -\omega \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence, in particular,

$$\kappa_{1,4} + \omega\kappa_{3,4} + \omega^2\kappa_{4,4} = 0 \quad \wedge \quad -\kappa_{2,4} - \omega\kappa_{4,4} = -\omega,$$

implying $0 = \kappa_{4,4} = 1$, a contradiction. Thus, $G(Z; z_1\mathbb{R}, z_2\mathbb{R})$ does not have a finite index.

Proof of Corollary 3.3. Since the assertion is trivial for $N \in \{1, 2\}$, we assume $N \geq 3$. Let $\mathbb{k} := \mathbb{Q}(\zeta_N) \cap \mathbb{R}$, and for $i = 1, \ldots, \phi(N)$, set $p_i := \zeta_N^{i-1}$. As it is standard fare in the theory of cyclotomic fields, $p_1, \ldots, p_{\phi(N)}$ form a $\mathbb{Z}$-basis of $\mathbb{Z}[\zeta_N]$ and also a $\mathbb{Q}$-basis of $\mathbb{Q}(\zeta_N)$, and $\mathbb{Q}(\zeta_N)$ is a $\mathbb{k}$-vector space of dimension 2 spanned by 1 and ζ_N; see e.g. [38,24]; cf. also [6]. Hence the assertion follows from Theorem 3.2. $\square$

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