Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

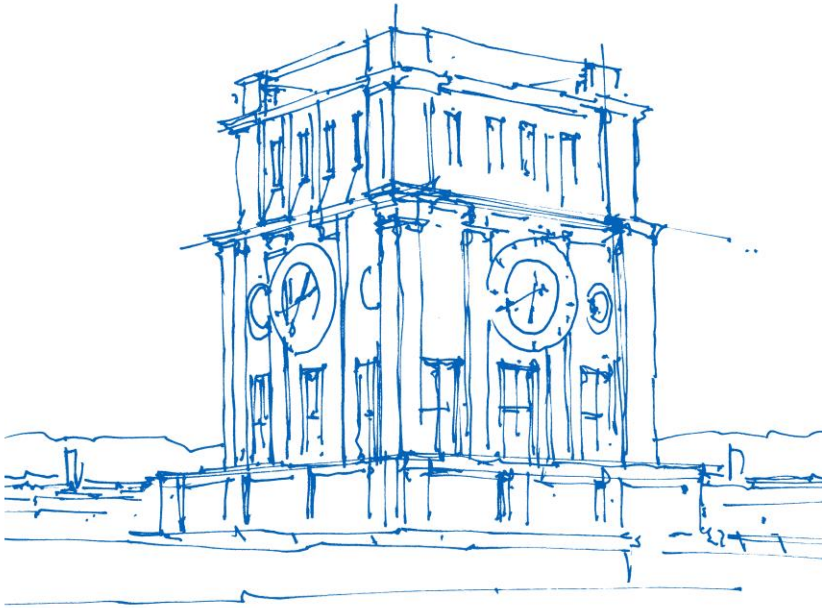
On the $\mathcal{H}_2$-pseudo-optimal bilinear model reduction

Maria Cruz Varona

Applied Numerical Analysis Seminar

Virginia Tech, Math Department

Blacksburg, April 24th 2017



Brief personal introduction

Maria Cruz Varona
M.Sc. Electrical Engineering

Research assistant (since 08/14)

Chair of Automatic Control

Department of Mechanical Engineering

Technical University of Munich

maria.cruz@tum.de

www.rt.mw.tum.de

Research interests:

System theory, model order reduction of nonlinear dynamical systems by Krylov-subspace methods

At VT:

Visiting Serkan Gugercin until June 30th (office 465)

Motivation

Large-scale full order models (FOMs)

$$\mathbf{E}\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}\mathbf{u}$$

$$\mathbf{y} = \mathbf{C}\mathbf{x}$$

$$\det(\mathbf{E}) \neq 0$$

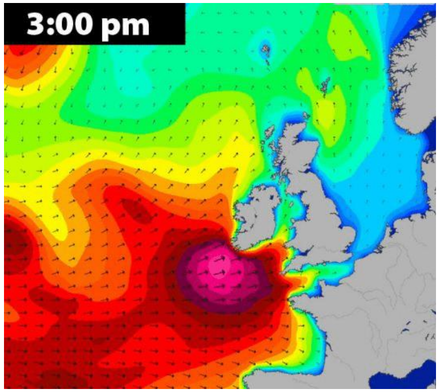
MOR

Reduced order models (ROMs)

$$\mathbf{E}_r\dot{\mathbf{x}}_r = \mathbf{A}_r\mathbf{x}_r + \mathbf{B}_r\mathbf{u}$$

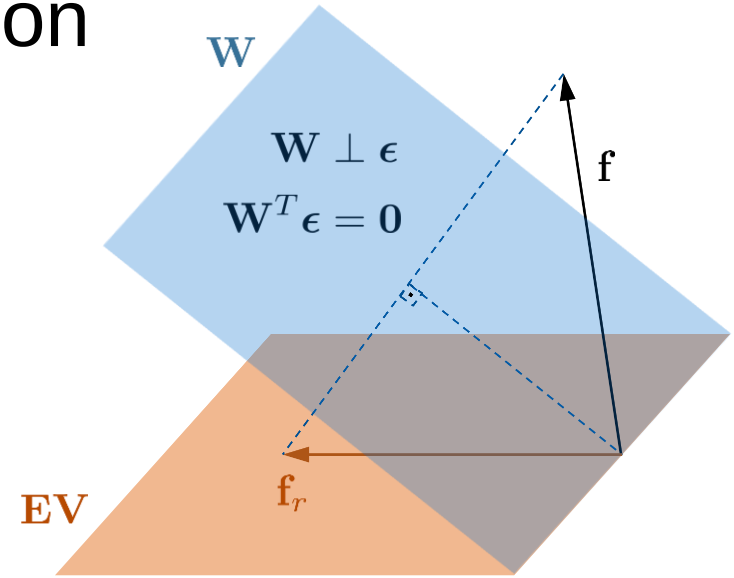
$$\mathbf{y}_r = \mathbf{C}_r\mathbf{x}_r$$

$$\mathbf{x}_r \in \mathbb{R}^r, \; r \ll n$$



10^3 — 10^4 — 10^5 — 10^6 → n

Projective Model Order Reduction



Assumption: Dynamical system does not transit all regions of the state-space equally often, but mainly stays and evolves in a subspace of lower dimension

Approximation of the state vector:

$$\mathbf{x} = \mathbf{V} \; \mathbf{x}_r + \mathbf{e} \;, \quad \mathbf{V} \in \mathbb{R}^{n \times r}$$

Projection procedure in subspace $\mathcal{V} = \mathrm{Im}(\mathbf{EV})$**:**

1. Replace $\mathbf{x}$ by its approximation
2. Reduce the number of equations (via projection with $\mathbf{\Pi} = \mathbf{EV}(\mathbf{W}^T\mathbf{EV})^{-1}\mathbf{W}^T$)
3. Petrov-Galerkin condition

$$\overbrace{\mathbf{W}^T \; \mathbf{E} \; \mathbf{V}}^{\mathbf{E}_r} \; \dot{\mathbf{x}}_r = \overbrace{\mathbf{W}^T \; \mathbf{A} \; \mathbf{V}}^{\mathbf{A}_r} \; \mathbf{x}_r + \overbrace{\mathbf{W}^T \; \mathbf{B}}^{\mathbf{B}_r} \; \mathbf{u}$$

$$\mathbf{y}_r = \underbrace{\mathbf{C} \; \mathbf{V}}_{\mathbf{C}_r} \; \mathbf{x}_r$$

Rational Interpolation by Krylov-subspace methods

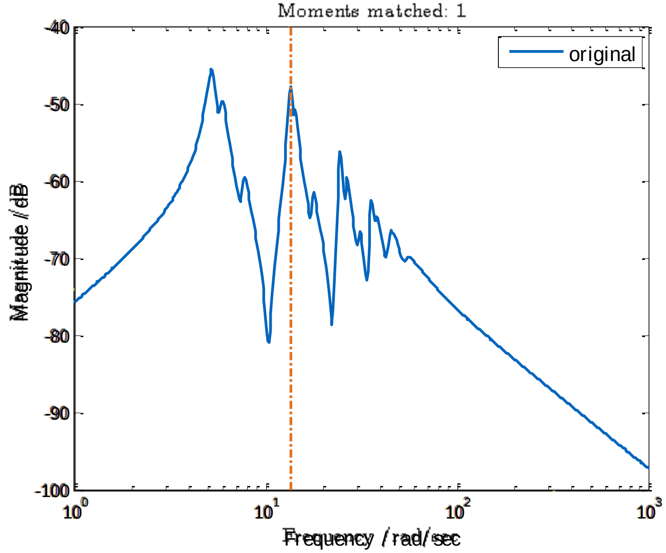
Moments of a transfer function

$$\mathbf{G}(s) = \mathbf{C}(s\mathbf{E} - \mathbf{A})^{-1}\mathbf{B}$$

$$= \mathbf{G}(\Delta s + s_0) = -\sum_{i=0}^{\infty} \mathbf{M}_i(s_0)(s - s_0)^i$$

s_0 : interpolation point (shift)

$\mathbf{M}_i(s_0)$: i-th moment around s_0



Moment Matching by Rational Krylov-subspaces

Bases for input and output Krylov-subspaces:

$$\mathrm{Im}(\mathbf{V}) = \left[(\mathbf{A} - \sigma_1\mathbf{E})^{-1}\mathbf{B}\,\mathbf{r}_1, \ldots, (\mathbf{A} - \sigma_r\mathbf{E})^{-1}\mathbf{B}\,\mathbf{r}_r\right]$$

$$\mathrm{Im}(\mathbf{W}) = \left[(\mathbf{A} - \mu_1\mathbf{E})^{-T}\mathbf{C}^T\,\mathbf{l}_1, \ldots, (\mathbf{A} - \mu_r\mathbf{E})^{-T}\mathbf{C}^T\,\mathbf{l}_r\right]$$

$\Leftrightarrow$

$$\mathbf{A}\,\mathbf{V} - \mathbf{E}\,\mathbf{V}\,\mathbf{S}_V = \mathbf{B}\,\mathbf{R}$$

$$\mathbf{A}^T\,\mathbf{W} - \mathbf{E}^T\,\mathbf{W}\,\mathbf{S}_W^T = \mathbf{C}^T\,\mathbf{L}$$

$\lambda_i(\mathbf{S}_V) = \sigma_i$: shifts

$\mathbf{R}, \mathbf{L}$: tangential dir.

Moments from full and reduced order model around certain shifts match!

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

$\mathcal{H}_2$-optimal model reduction of linear systems

Goal: Find ROM that minimizes the $\mathcal{H}_2$-error: $\|G - G_r\|_{\mathcal{H}_2} = \min_{\dim(\widetilde{G}_r)=r} \left\| G - \widetilde{G}_r \right\|_{\mathcal{H}_2}$

$\mathcal{H}_2$-norm for linear systems

$$\|G\|_{\mathcal{H}_2} = \sqrt{\langle G, G \rangle_{\mathcal{H}_2}} = \left(\frac{1}{2\pi} \int_{-\infty}^{\infty} G(-\imath\omega)\, G(\imath\omega)\, \mathrm{d}\omega \right)^{\frac{1}{2}}$$

$\mathcal{H}_2$-inner product for linear systems

$$\langle G, G_r \rangle_{\mathcal{H}_2} = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(-\imath\omega)\, G_r(\imath\omega)\, \mathrm{d}\omega = \sum_{k=1}^{r} \phi_{r,k}\, G(-\overline{\lambda}_{r,k})$$

ROM in pole-residue form

$$G_r(s) = \sum_{k=1}^{r} \frac{\phi_{r,k}}{s - \lambda_{r,k}}$$

$$\phi_{r,k} = \hat{c}_{r,k} \cdot \hat{b}_{r,k}$$

$\mathcal{H}_2$-error in pole-residue form

$$\mathcal{J} = \|G - G_r\|^2_{\mathcal{H}_2} = \langle G - G_r, G - G_r \rangle_{\mathcal{H}_2} = \|G\|^2_{\mathcal{H}_2} - 2\, \langle G, G_r \rangle_{\mathcal{H}_2} + \|G_r\|^2_{\mathcal{H}_2}$$

$$= \|G\|^2_{\mathcal{H}_2} - 2 \sum_{k=1}^{r} \phi_{r,k}\, G(-\overline{\lambda}_{r,k}) + \sum_{k=1}^{r} \sum_{\ell=1}^{r} \frac{\phi_{r,k}\, \phi_{r,\ell}}{-\overline{\lambda}_{r,k} - \lambda_{r,\ell}}$$

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

$\mathcal{H}_2$-optimal model reduction of linear systems

First-order necessary optimality conditions: $\mathcal{J} = \|\boldsymbol{\Sigma} - \boldsymbol{\Sigma}_r\|^2_{\mathcal{H}_2} = f(\mathbf{A}, \boxed{\boldsymbol{\Lambda}_r}, \mathbf{B}, \boxed{\hat{\mathbf{B}}_r}, \mathbf{C}, \boxed{\hat{\mathbf{C}}_r}) \rightarrow \min$

I) $\mathbf{G}(-\overline{\lambda}_{r,i})\hat{\mathbf{b}}_{r,i} = \mathbf{G}_r(-\overline{\lambda}_{r,i})\hat{\mathbf{b}}_{r,i}$

II) $\hat{\mathbf{c}}^T_{r,i}\mathbf{G}(-\overline{\lambda}_{r,i}) = \hat{\mathbf{c}}^T_{r,i}\mathbf{G}_r(-\overline{\lambda}_{r,i})$

III) $\hat{\mathbf{c}}^T_{r,i}\mathbf{G}'(-\overline{\lambda}_{r,i})\hat{\mathbf{b}}_{r,i} = \hat{\mathbf{c}}^T_{r,i}\mathbf{G}'_r(-\overline{\lambda}_{r,i})\hat{\mathbf{b}}_{r,i}$

I)+II) $\dfrac{\partial \mathcal{J}}{\partial \phi_{r,i}} = 0 \Rightarrow G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$

III) $\dfrac{\partial \mathcal{J}}{\partial \lambda_{r,i}} = 0 \Rightarrow G'(-\overline{\lambda}_{r,i}) = G'_r(-\overline{\lambda}_{r,i})$

Algorithm 1 Iterative Rational Krylov Algorithm (IRKA)

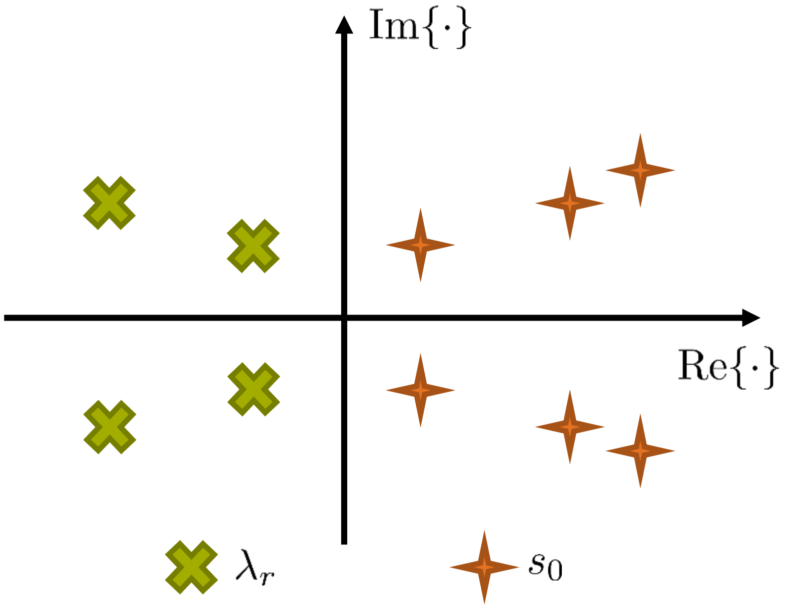
Input: $\boldsymbol{\Sigma} := (\mathbf{E}, \mathbf{A}, \mathbf{B}, \mathbf{C})$, $\boldsymbol{\Sigma}_r := (\mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$
Output: locally $\mathcal{H}_2$-optimal ROM $\boldsymbol{\Sigma}_r^{\mathrm{opt}}$

while (change in $\boldsymbol{\Lambda}_r > \epsilon$) **do**
    $\mathbf{E}_r^{-1}\mathbf{A}_r = \mathbf{R}\boldsymbol{\Lambda}_r\mathbf{R}^{-1}$, $\hat{\mathbf{B}}_r = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{B}_r$, $\hat{\mathbf{C}}_r = \mathbf{C}_r\mathbf{R}$
    Solve

$$\mathbf{A}\,\mathbf{V} - \mathbf{E}\,\mathbf{V}\,(-\overline{\boldsymbol{\Lambda}}_r) = \mathbf{B}\,\hat{\mathbf{B}}_r^T$$

$$\mathbf{A}^T\,\mathbf{W} - \mathbf{E}^T\,\mathbf{W}\,(-\overline{\boldsymbol{\Lambda}}_r^T) = \mathbf{C}^T\,\hat{\mathbf{C}}_r$$

    $\mathbf{V} = \mathrm{orth}(\mathbf{V})$, $\mathbf{W} = \mathrm{orth}(\mathbf{W})$
    $\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}$
end while
$\mathbf{E}_r^{\mathrm{opt}} = \mathbf{E}_r$, $\mathbf{A}_r^{\mathrm{opt}} = \mathbf{A}_r$, $\mathbf{B}_r^{\mathrm{opt}} = \mathbf{B}_r$, $\mathbf{C}_r^{\mathrm{opt}} = \mathbf{C}_r$



Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

$\mathcal{H}_2$-optimal vs. $\mathcal{H}_2$-pseudo-optimal model reduction

$\mathcal{H}_2$-optimality

- Problem:

$$\|G - G_r\|_{\mathcal{H}_2} = \min_{\dim(\widetilde{G}_r)=r} \left\|G - \widetilde{G}_r\right\|_{\mathcal{H}_2}$$

- Necessary conditions for local $\mathcal{H}_2$-optimality (SISO): (Meier-Luenberger)

$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$$

$$G'(-\overline{\lambda}_{r,i}) = G_r'(-\overline{\lambda}_{r,i})$$

- G_r minimizes the $\mathcal{H}_2$ error locally within the set of all ROMs of order r

$\mathcal{H}_2$-pseudo-optimality [Wolf '14]

- Problem: $\Lambda = \{\lambda_{r,1}, \ldots, \lambda_{r,r}\},\ \lambda_i \in \mathbb{C}^-$

$$\|G - G_r\|_{\mathcal{H}_2} = \min_{\widetilde{G}_r \in \mathcal{G}(\Lambda)} \left\|G - \widetilde{G}_r\right\|_{\mathcal{H}_2}$$

- Necessary **and** sufficient condition for global $\mathcal{H}_2$-pseudo-optimality:

$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$$

- Pseudo-optimal means optimal in a certain subspace
- G_r minimizes the $\mathcal{H}_2$ error globally within the subset of all ROMs of order r with poles Λ

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Advantages of $\mathcal{H}_2$-pseudo-optimal model reduction

[Wolf '14]

✓ **Convexity of the $\mathcal{H}_2$-pseudo-optimal problem**

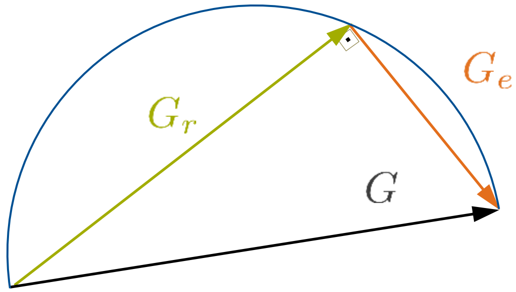
Objective function is quadratic w.r.t. the residues → globally optimal ROM

$$\mathcal{J} = \|G\|_{\mathcal{H}_2}^2 - 2\sum_{k=1}^{r} \phi_{r,k}\, G(-\overline{\lambda}_{r,k}) + \sum_{k=1}^{r}\sum_{\ell=1}^{r} \frac{\phi_{r,k}\,\phi_{r,\ell}}{-\overline{\lambda}_{r,k} - \lambda_{r,\ell}}$$

✓ **Stability can be guaranteed by construction**

- Poles of ROM are the mirror images of the chosen shifts: $\Lambda(\mathbf{S}_V) = \Lambda(-\mathbf{E}_r^{-1}\mathbf{A}_r)$
  → choice of the shifts is twice as important!

✓ **Structured orthogonality condition:** $\langle G - G_r, G_r\rangle_{\mathcal{H}_2} = 0 \iff \langle G, G_r\rangle_{\mathcal{H}_2} = \langle G_r, G_r\rangle_{\mathcal{H}_2}$



- Makes the optimization easier:

$$\mathcal{J} = \|G\|_{\mathcal{H}_2}^2 - 2\langle G, G_r\rangle_{\mathcal{H}_2} + \|G_r\|_{\mathcal{H}_2}^2 = \|G\|_{\mathcal{H}_2}^2 - \|G_r\|_{\mathcal{H}_2}^2$$

- Change of paradigm:

$$\min \|G - G_r\|_{\mathcal{H}_2} \overset{\mathcal{H}_2\text{-pseudo-opt.}}{\Longrightarrow} \max \|G_r\|_{\mathcal{H}_2}$$

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Applications of $\mathcal{H}_2$-pseudo-optimality in linear MOR

[Beattie/Gugercin '12], [Wolf '14]

1.) Explicit computation of pseudo-optimal ROMs

Example: Fixing reduced poles and right directions, optimizing left directions

Algorithm 1 Pseudo-optimal rational Krylov (PORK)

Input: $\mathbf{V}$, $\mathbf{S}_V$, $\mathbf{R}$, $\mathbf{C}$, such that $\mathbf{AV}-\mathbf{EVS}_V=\mathbf{BR}$

Output: $\mathcal{H}_2$-pseudo-optimal ROM $\boldsymbol{\Sigma}_r := (\mathbf{E}_r, \mathbf{A}_r, \mathbf{B}_r, \mathbf{C}_r)$

1: $\mathbf{P}_r^{-1} = \mathrm{lyap}\left(-\mathbf{S}_V^T, \mathbf{R}^T\mathbf{R}\right)$

2: $\mathbf{B}_r = -\left(\mathbf{P}_r^{-1}\right)^{-1}\mathbf{R}^T$

3: $\mathbf{A}_r = \mathbf{S}_V + \mathbf{B}_r\mathbf{R}$, $\mathbf{E}_r = \mathbf{I}$, $\mathbf{C}_r = \mathbf{CV}$

$$\Longleftrightarrow \quad \overbrace{\left[\hat{\mathbf{c}}_{r,1}, \dots, \hat{\mathbf{c}}_{r,r}\right]}^{\hat{\mathbf{C}}_r} \overbrace{\begin{bmatrix} \frac{\hat{\mathbf{b}}_{r,1}^T\hat{\mathbf{b}}_{r,1}}{-\overline{\lambda}_{r,1}-\lambda_{r,1}} & \cdots & \frac{\hat{\mathbf{b}}_{r,1}^T\hat{\mathbf{b}}_{r,r}}{-\overline{\lambda}_{r,1}-\lambda_{r,r}} \\ \vdots & \ddots & \vdots \\ \frac{\hat{\mathbf{b}}_{r,r}^T\hat{\mathbf{b}}_{r,1}}{-\overline{\lambda}_{r,r}-\lambda_{r,1}} & \cdots & \frac{\hat{\mathbf{b}}_{r,r}^T\hat{\mathbf{b}}_{r,r}}{-\overline{\lambda}_{r,r}-\lambda_{r,r}} \end{bmatrix}}^{\mathbf{M}}$$

$$= \underbrace{\left[\mathbf{G}(-\overline{\lambda}_{r,1})\hat{\mathbf{b}}_{r,1}, \dots, \mathbf{G}(-\overline{\lambda}_{r,r})\hat{\mathbf{b}}_{r,r}\right]}_{\mathbf{Y}}$$

$$\Longrightarrow \quad \begin{aligned} [\mathbf{A}_r, \mathbf{B}_r, \mathbf{E}_r] &= f(\mathbf{S}_V, \mathbf{R}) \\ \mathbf{C}_r &= \mathbf{CV} \end{aligned}$$

$$\Longrightarrow \quad \hat{\mathbf{C}}_r = \mathbf{Y}\mathbf{M}^{-1}$$

2.) Residue correction: inner loop within IRKA for optimized residue directions

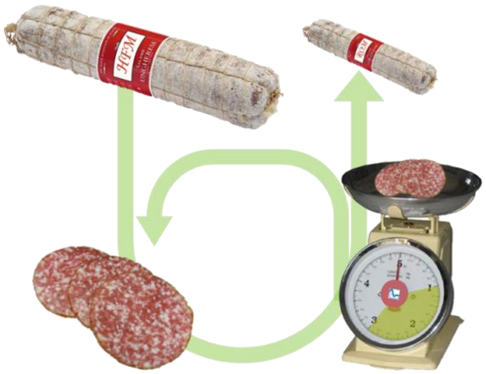
- Fix reduced poles and right residue directions → optimize left residue directions
- Fix reduced poles and left residue directions → optimize right residue directions

→ Better convergence of IRKA (with less outer iterations) due to residue correction

Applications of $\mathcal{H}_2$-pseudo-optimality in linear MOR

3.) Cumulative Reduction (CuRe) and Stability Preserving Adaptive Rational Krylov (SPARK)

- Make use of H2-pseudo-optimality in each CuRe iteration for adaptive choice of shifts
- Restriction to H2-pseudo-optimal ROMs simplifies the cost function



$$G(-\overline{\lambda}_{r,i}) = G_r(-\overline{\lambda}_{r,i})$$
$$G'(-\overline{\lambda}_{r,i}) = G_r'(-\overline{\lambda}_{r,i})$$

$$\min \|G - G_r\|_{\mathcal{H}_2}$$
$$\Downarrow$$
$$\max \|G_r\|_{\mathcal{H}_2}$$
$$\Downarrow$$
$$\mathcal{J}(a,b) = -\|G_r\|_{\mathcal{H}_2}^2 = -C_r P_r C_r^T$$

4.) Lyapunov equations: link between ADI & Krylov-subspaces with H2-pseudo-optimal shifts

ADI = RKSM + H2-pseudo-optimal shifts

Bilinear Systems

Nonlinear state equation

$$\mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{f}(\mathbf{x}(t)) + \mathbf{B}\mathbf{u}(t)$$
$$\mathbf{y}(t) = \mathbf{C}\mathbf{x}(t)$$

$$\det(\mathbf{E}) \neq 0$$

Carleman bilinearization

[Rugh '81]

Bilinear model

$$\zeta : \begin{cases} \mathbf{E}\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{j=1}^{m} \mathbf{N}_j \mathbf{x}(t) u_j(t) + \mathbf{B}\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}\mathbf{x}(t), \;\; \mathbf{x}(0) = \mathbf{x}_0 \end{cases}$$

$\mathbf{E}, \mathbf{A}, \mathbf{N}_j \in \mathbb{R}^{n \times n}$

$\mathbf{B} \in \mathbb{R}^{n \times m}; \;\; \mathbf{C} \in \mathbb{R}^{p \times n}$

Input-output representation (SISO)

$$y(t) = \sum_{k=1}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \underbrace{\mathbf{c}^T e^{\mathbf{E}^{-1}\mathbf{A}\tau_k} \mathbf{E}^{-1}\mathbf{N} \cdots \mathbf{E}^{-1}\mathbf{N} e^{\mathbf{E}^{-1}\mathbf{A}\tau_2} \mathbf{E}^{-1}\mathbf{N} e^{\mathbf{E}^{-1}\mathbf{A}\tau_1} \mathbf{E}^{-1}\mathbf{b}}_{g_k^{\mathrm{reg}}(\tau_1,\ldots,\tau_k)}$$
$$\times u(t-\tau_k) \cdots u(t-\tau_k - \ldots - \tau_1)\, \mathrm{d}\tau_k \cdots \mathrm{d}\tau_1$$

$g_k^{\mathrm{reg}}(\tau_1, \ldots, \tau_k)$: regular Volterra kernels

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Bilinear System‘s Theory (SISO)

Generalized transfer functions:

$$G_k(s_1,\ldots,s_k) = \mathbf{c}^T(s_k\mathbf{E}-\mathbf{A})^{-1}\mathbf{N}\cdots\mathbf{N}\,(s_2\mathbf{E}-\mathbf{A})^{-1}\,\mathbf{N}\,(s_1\mathbf{E}-\mathbf{A})^{-1}\mathbf{b}$$

$\mathcal{H}_2$-norm for bilinear systems [Flagg ’12]

$$\|\zeta\|^2_{\mathcal{H}_2} := \sum_{k=1}^{\infty}\frac{1}{(2\pi)^k}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} G_k(-\imath\omega_1,\ldots,-\imath\omega_k)\,G_k(\imath\omega_1,\ldots,\imath\omega_k)\,\mathrm{d}\omega_1\cdots\mathrm{d}\omega_k$$

Transfer functions in pole-residue form

$$G_k(s_1,\ldots,s_k) = \sum_{l_1=1}^{n}\cdots\sum_{l_k=1}^{n}\frac{\phi_{l_1,\ldots,l_k}}{\prod_{i=1}^{k}(s_i-\lambda_{l_i})}$$

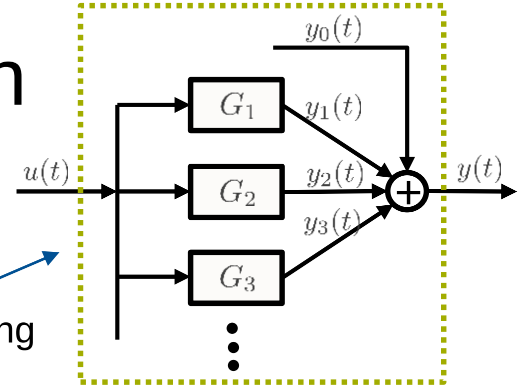
$$\phi(l_1,\ldots,l_k) = \hat{c}_{l_k}\cdot\hat{n}_{l_k,l_{k-1}}\cdot\ldots\cdot\hat{n}_{l_2,l_1}\cdot\hat{b}_{l_1}$$

$\mathcal{H}_2$-norm for bilinear systems in pole-residue form

$$\|\zeta\|^2_{\mathcal{H}_2} = \sum_{k=1}^{\infty}\sum_{l_1=1}^{n}\cdots\sum_{l_k=1}^{n}\phi_{l_1,\ldots,l_k}\,G_k(-\overline{\lambda}_{l_1},\ldots,-\overline{\lambda}_{l_k})$$

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Multipoint Volterra Series Interpolation



Goal: Enforcing multipoint interpolation of the underlying Volterra series

Moment matching

Multipoint Volterra series interpolation [Flagg/Gugercin '15]

Set of interpolation points: $S = \{\sigma_1, \ldots, \sigma_r\}, \quad i = 1, \ldots, r$

$$\sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\ldots,l_{k-1},i} \, G_k(\sigma_{l_1}, \ldots, \sigma_{l_{k-1}}, \sigma_i) = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\ldots,l_{k-1},i} \, G_{k,r}(\sigma_{l_1}, \ldots, \sigma_{l_{k-1}}, \sigma_i)$$

This approach interpolates the weighted series at the interpolation points $\sigma_1, \ldots, \sigma_r$

Weighting matrices: $\mathbf{U}_V = \{u_{ij}^v\}, \; \mathbf{U}_W = \{u_{ij}^w\} \in \mathbb{R}^{r \times r}$

$$\eta_{l_1,\ldots,l_{k-1},i} = u_{i,l_{k-1}}^v u_{l_{k-1},l_{k-2}}^v \ldots u_{l_2,l_1}^v \text{ for } k \geq 2 \text{ and } \eta_{l_1} = 1 \text{ for } l_1 = 1, \ldots, r$$

Weights and shifts are defined by the user

Example: $\eta_{1,2,3} = u_{3,2}^v \cdot u_{2,1}^v$

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Multipoint Volterra Series Interpolation

Explicit computation of Volterra series-based interpolation [Flagg/Gugercin '15]

$$\mathbf{v}_i = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \eta_{l_1,\ldots,l_{k-1},i} (\mathbf{A} - \sigma_i \mathbf{E})^{-1} \mathbf{N} (\mathbf{A} - \sigma_{l_{k-1}} \mathbf{E})^{-1} \mathbf{N} \ldots \mathbf{N} (\mathbf{A} - \sigma_1 \mathbf{E})^{-1} \mathbf{b}$$

$$\mathbf{w}_i = \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_{k-1}=1}^{r} \vartheta_{l_1,\ldots,l_{k-1},i} (\mathbf{A} - \mu_i \mathbf{E})^{-T} \mathbf{N}^T (\mathbf{A} - \mu_{l_{k-1}} \mathbf{E})^{-T} \mathbf{N}^T \ldots \mathbf{N}^T (\mathbf{A} - \mu_1 \mathbf{E})^{-T} \mathbf{c}$$

Link Krylov-Sylvester

$$\mathbf{V} = [\mathbf{v}_1, \ldots, \mathbf{v}_r] \in \mathbb{R}^{n \times r}$$

$$\mathbf{W} = [\mathbf{w}_1, \ldots, \mathbf{w}_r] \in \mathbb{R}^{n \times r}$$

Volterra Sylvester equations

$$\mathbf{A}\mathbf{V} - \mathbf{E}\mathbf{V}\mathbf{S}_V - \mathbf{N}\mathbf{V}\mathbf{U}_V^T = \mathbf{b}\mathbf{o}^T$$

$$\mathbf{A}^T\mathbf{W} - \mathbf{E}^T\mathbf{W}\mathbf{S}_W^T - \mathbf{N}^T\mathbf{W}\mathbf{U}_W^T = \mathbf{c}\mathbf{o}^T$$

$\mathcal{H}_2$-optimal model reduction of bilinear systems

Goal: Find ROM that minimizes the $\mathcal{H}_2$-error: $\mathcal{J} = \|\zeta - \zeta_r\|_{\mathcal{H}_2}^2 = f(\mathbf{A}, \boxed{\mathbf{\Lambda}_r}, \mathbf{N}_j, \boxed{\hat{\mathbf{N}}_{j,r}}, \mathbf{B}, \boxed{\hat{\mathbf{B}}_r}, \mathbf{C}, \boxed{\hat{\mathbf{C}}_r}) \rightarrow \min$

Necessary conditions for $\mathcal{H}_2$-optimality [Benner/Breiten '12]

I) $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{C}}_{r,ij}} = 0$ **II)** $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{B}}_{r,ij}} = 0$ **III)** $\dfrac{\partial \mathcal{J}}{\partial \lambda_{r,i}} = 0$ **IV)** $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{N}}_{r,ij}} = 0$

Algorithm 1 Bilinear Iterative Rational Krylov Algorithm (B-IRKA)

Input: $\zeta := (\mathbf{E}, \mathbf{A}, \mathbf{N}_1, \ldots, \mathbf{N}_m, \mathbf{B}, \mathbf{C})$, $\zeta_r := (\mathbf{E}_r, \mathbf{A}_r, \mathbf{N}_{1,r}, \ldots, \mathbf{N}_{m,r}, \mathbf{B}_r, \mathbf{C}_r)$
Output: locally $\mathcal{H}_2$-optimal ROM ζ_r^{opt}

while (change in $\mathbf{\Lambda}_r > \epsilon$) **do**

$\mathbf{E}_r^{-1}\mathbf{A}_r = \mathbf{R}\mathbf{\Lambda}_r\mathbf{R}^{-1}$, $\hat{\mathbf{N}}_{j,r} = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{N}_{j,r}\mathbf{R}$, $\hat{\mathbf{B}}_r = \mathbf{R}^{-1}\mathbf{E}_r^{-1}\mathbf{B}_r$, $\hat{\mathbf{C}}_r = \mathbf{C}_r\mathbf{R}$

Solve

$$\mathbf{A}\,\mathbf{V} - \mathbf{E}\,\mathbf{V}\,(-\overline{\mathbf{\Lambda}}_r) - \sum_{j=1}^{m} \mathbf{N}_j\,\mathbf{V}\,\hat{\mathbf{N}}_{j,r}^T = \mathbf{B}\,\hat{\mathbf{B}}_r^T$$

$$\mathbf{A}^T\,\mathbf{W} - \mathbf{E}^T\,\mathbf{W}\,(-\overline{\mathbf{\Lambda}}_r^T) - \sum_{j=1}^{m} \mathbf{N}_j^T\,\mathbf{W}\,\hat{\mathbf{N}}_{j,r} = \mathbf{C}^T\,\hat{\mathbf{C}}_r$$

$\mathbf{V} = \text{orth}(\mathbf{V})$, $\mathbf{W} = \text{orth}(\mathbf{W})$

$\mathbf{E}_r = \mathbf{W}^T\mathbf{E}\mathbf{V}$, $\mathbf{A}_r = \mathbf{W}^T\mathbf{A}\mathbf{V}$, $\mathbf{N}_{j,r} = \mathbf{W}^T\mathbf{N}_j\mathbf{V}$, $\mathbf{B}_r = \mathbf{W}^T\mathbf{B}$, $\mathbf{C}_r = \mathbf{C}\mathbf{V}$

end while

$\mathbf{E}_r^{\text{opt}} = \mathbf{E}_r$, $\mathbf{A}_r^{\text{opt}} = \mathbf{A}_r$, $\mathbf{N}_{j,r}^{\text{opt}} = \mathbf{N}_{j,r}$, $\mathbf{B}_r^{\text{opt}} = \mathbf{B}_r$, $\mathbf{C}_r^{\text{opt}} = \mathbf{C}_r$

$\mathcal{H}_2$-optimality and multipoint interpolation

Theorem: Connection of $\mathcal{H}_2$-optimality to Volterra series interpolation [Flagg/Gugercin '15]

Let ζ be a bilinear SISO system of dimension n and let ζ_r be an $\mathcal{H}_2$-optimal reduced model of dimension r. Then ζ_r satisfies the following multipoint Volterra series interpolation conditions:

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\phi_{r,l_1,\ldots,l_{k-1},i}G_k(-\overline{\lambda}_{r,l_1},\cdots,-\overline{\lambda}_{r,i}) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\phi_{r,l_1,\ldots,l_{k-1},i}G_{k,r}(-\overline{\lambda}_{r,l_1},\cdots,-\overline{\lambda}_{r,i})$$

$$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\phi_{r,l_1,\ldots,l_{k-1},i}\left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j}G_k(-\overline{\lambda}_{r,l_1},\cdots,-\overline{\lambda}_{r,i})\right) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r}\phi_{r,l_1,\ldots,l_{k-1},i}\left(\sum_{j=1}^{k}\frac{\partial}{\partial s_j}G_{k,r}(-\overline{\lambda}_{r,l_1},\cdots,-\overline{\lambda}_{r,i})\right)$$

$\phi_{r,l_1,\ldots,l_{k-1},i}$: reduced order residues

λ_{r,l_i} : reduced order poles

$i = 1,\ldots,r$

Thus, the $\mathcal{H}_2$-optimality conditions imply multipoint Volterra series interpolation conditions, with weights given by the reduced order residues, and interpolation points (shifts) given by the mirror images of the reduced order poles.

$\mathcal{H}_2$-pseudo-optimal bilinear model reduction

$\mathcal{H}_2$-optimality

$$\|\zeta - \zeta_r\|_{\mathcal{H}_2} = \min_{\dim(\tilde{\zeta}_r)=r} \left\|\zeta - \tilde{\zeta}_r\right\|_{\mathcal{H}_2}$$

ζ_r satisfies

I) $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{C}}_{r,ij}} = 0$ **II)** $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{B}}_{r,ij}} = 0$

III) $\dfrac{\partial \mathcal{J}}{\partial \lambda_{r,i}} = 0$ **IV)** $\dfrac{\partial \mathcal{J}}{\partial \hat{\mathbf{N}}_{r,ij}} = 0$

$\mathcal{H}_2$-pseudo-optimality

$\Lambda = \{\lambda_{r,1}, \ldots, \lambda_{r,r}\}$: fixed reduced poles

$\mathcal{G}(\Lambda)$: Subset of reduced models

ζ_r satisfies $\|\zeta - \zeta_r\|_{\mathcal{H}_2} = \min_{\tilde{\zeta}_r \in \mathcal{G}(\Lambda)} \left\|\zeta - \tilde{\zeta}_r\right\|_{\mathcal{H}_2}$

iff

$$\zeta(-\overline{\lambda}_{r,i}) = \zeta_r(-\overline{\lambda}_{r,i})$$
$$\zeta'(-\overline{\lambda}_{r,i}) = \zeta_r'(-\overline{\lambda}_{r,i}) \qquad i = 1, \ldots, r$$

I)+II)+IV) $$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r} \phi_{r,l_1,\ldots,l_{k-1},i} G_k(-\overline{\lambda}_{r,l_1}, \cdots, -\overline{\lambda}_{r,i}) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r} \phi_{r,l_1,\ldots,l_{k-1},i} G_{k,r}(-\overline{\lambda}_{r,l_1}, \cdots, -\overline{\lambda}_{r,i})$$

III) $$\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r} \phi_{r,l_1,\ldots,l_{k-1},i} \left(\sum_{j=1}^{k} \frac{\partial}{\partial s_j} G_k(-\overline{\lambda}_{r,l_1}, \cdots, -\overline{\lambda}_{r,i})\right) = \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_{k-1}=1}^{r} \phi_{r,l_1,\ldots,l_{k-1},i} \left(\sum_{j=1}^{k} \frac{\partial}{\partial s_j} G_{k,r}(-\overline{\lambda}_{r,l_1}, \cdots, -\overline{\lambda}_{r,i})\right)$$

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

$\mathcal{H}_2$-pseudo-optimal bilinear model reduction

$\mathcal{H}_2$-norm for bilinear systems [Flagg '12]

$$\|\zeta\|^2_{\mathcal{H}_2} := \sum_{k=1}^{\infty} \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_k(-\imath\omega_1, \ldots, -\imath\omega_k)\, G_k(\imath\omega_1, \ldots, \imath\omega_k)\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_k$$

Is the H2-norm being induced by an H2-inner product?: $\|\zeta\|_{\mathcal{H}_2} = \sqrt{\langle \zeta, \zeta \rangle_{\mathcal{H}_2}}$?

$\mathcal{H}_2$-inner product for bilinear systems (ongoing work!)

$$\langle \zeta, \zeta_r \rangle_{\mathcal{H}_2} = \sum_{k=1}^{\infty} \frac{1}{(2\pi)^k} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} G_k(-\imath\omega_1, \ldots, -\imath\omega_k)\, G_{k,r}(\imath\omega_1, \ldots, \imath\omega_k)\, \mathrm{d}\omega_1 \cdots \mathrm{d}\omega_k$$

$$\vdots$$

$$= \sum_{k=1}^{\infty} \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \phi_{r,l_1,\ldots,l_k}\, G_k(-\overline{\lambda}_{r,l_1}, \ldots, -\overline{\lambda}_{r,l_k})$$

Bilinear ROM in pole-residue form

$$G_{k,r}(s_1, \ldots, s_k) = \sum_{l_1=1}^{r} \cdots \sum_{l_k=1}^{r} \frac{\phi_{r,l_1,\ldots,l_k}}{\prod_{i=1}^{k} (s_i - \lambda_{r,l_i})}$$

$$\phi_r(l_1, \ldots, l_k) = \hat{c}_{r,l_k} \cdot \hat{n}_{r,l_k,l_{k-1}} \cdot \ldots \cdot \hat{n}_{r,l_2,l_1} \cdot \hat{b}_{r,l_1}$$

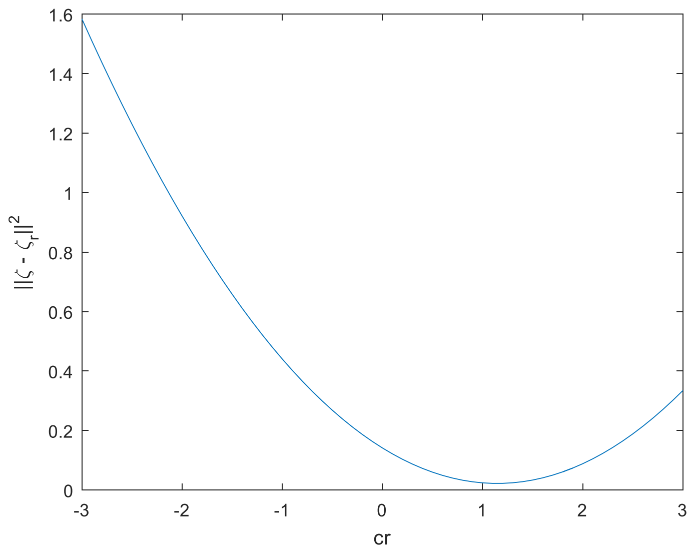
Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

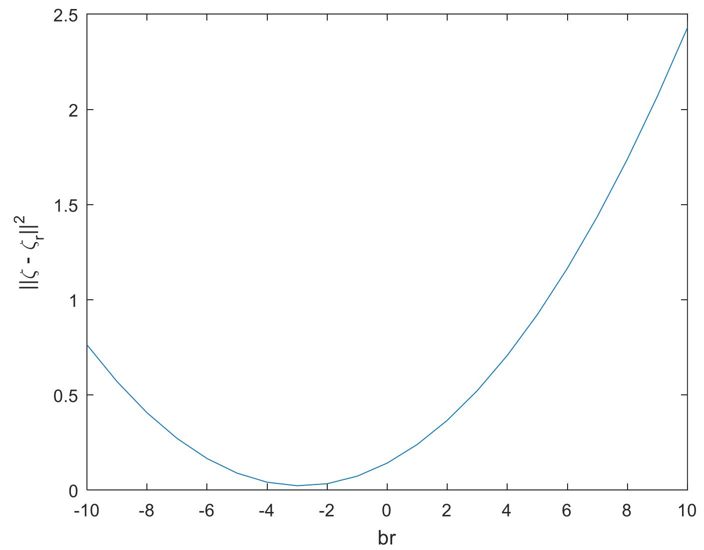
Convexity of the problem: MATLAB example

$\mathcal{H}_2$-error for bilinear systems in pole-residue form (ongoing work!)

$$\mathcal{J} = \|\zeta - \zeta_r\|^2_{\mathcal{H}_2} = \|\zeta\|^2_{\mathcal{H}_2} - 2\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{r,l_1,\ldots,l_k}\,G_k(-\overline{\lambda}_{r,l_1},\ldots,-\overline{\lambda}_{r,l_k})$$

$$+\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\sum_{p_1=1}^{r}\cdots\sum_{p_k=1}^{r}\frac{\phi_{r,l_1,\ldots,l_k}\,\phi_{r,p_1,\ldots,p_k}}{\prod_{i=1}^{k}(-\overline{\lambda}_{r,l_i}-\lambda_{r,p_i})}$$





Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

Advantages of $\mathcal{H}_2$-pseudo-optimality (bi-MOR)

? **Convexity of the $\mathcal{H}_2$-pseudo-optimal problem**

Objective function seems to be quadratic w.r.t. the residues → globally optimal ROM?

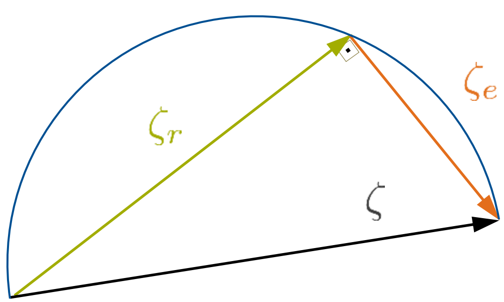
$$\mathcal{J} = \|\zeta\|_{\mathcal{H}_2}^2 - 2\sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\phi_{r,l_1,\ldots,l_k}\, G_k(-\overline{\lambda}_{r,l_1},\ldots,-\overline{\lambda}_{r,l_k}) + \sum_{k=1}^{\infty}\sum_{l_1=1}^{r}\cdots\sum_{l_k=1}^{r}\sum_{p_1=1}^{r}\cdots\sum_{p_k=1}^{r}\frac{\phi_{r,l_1,\ldots,l_k}\,\phi_{r,p_1,\ldots,p_k}}{\prod_{i=1}^{k}(-\overline{\lambda}_{r,l_i}-\lambda_{r,p_i})}$$

? **Stability can be guaranteed by construction**

- Poles of ROM are the mirror images of the chosen shifts.
- Weights (to be more precise $\hat{\mathbf{N}}_r$) are also being chosen

?

? **Structured orthogonality condition:** $\langle \zeta - \zeta_r, \zeta_r \rangle_{\mathcal{H}_2} = 0$



?

Applications of $\mathcal{H}_2$-pseudo-optimality in bi-MOR

1.) Explicit construction of bilinear H2-pseudo-optimal ROMs

Algorithm 1 Bilinear pseudo-optimal rational Krylov (BIPORK)

Input: $\mathbf{V}$, $\mathbf{S}_V$, $\mathbf{U}_V$, $\mathbf{R}$, $\mathbf{C}$, such that $\mathbf{AV} - \mathbf{EVS}_V - \mathbf{NVU}_V^T = \mathbf{BR}$
Output: $\mathcal{H}_2$ pseudo-optimal ROM ζ_r

1: $\mathbf{P}_r^{-1}$: solution of condition iii): $\mathbf{P}_r^{-1}\mathbf{S}_V + \mathbf{S}_V^T\mathbf{P}_r^{-1} - \mathbf{U}_V\mathbf{P}_r^{-1}\mathbf{U}_V^T - \mathbf{R}^T\mathbf{R} = \mathbf{0}$
2: $\mathbf{N}_r = -(\mathbf{P}_r^{-1})^{-1}\mathbf{U}_V\mathbf{P}_r^{-1}$ condition ii-2)
3: $\mathbf{B}_r = -\left(\mathbf{P}_r^{-1}\right)^{-1}\mathbf{R}^T$ condition ii-1)
4: $\mathbf{A}_r = \mathbf{S}_V + \mathbf{B}_r\mathbf{R} + \mathbf{N}_r\mathbf{U}_V^T$, $\mathbf{E}_r = \mathbf{I}_r$, $\mathbf{C}_r = \mathbf{CV}$

$\Leftrightarrow$?

$$\Longrightarrow \quad \begin{aligned} [\mathbf{A}_r, \mathbf{B}_r, \mathbf{N}_r, \mathbf{E}_r] &= f(\mathbf{S}_V, \mathbf{U}_V, \mathbf{R}) \\ \mathbf{C}_r &= \mathbf{CV} \end{aligned}$$

2.) Inner-loop in BIRKA with residue correction for better convergence of the algorithm

- Fix reduced poles, $\hat{\mathbf{N}}_r$ and right residue directions → optimize left residue directions
- Fix reduced poles, $\hat{\mathbf{N}}_r$ and left residue directions → optimize right residue directions

3.) Cumulative Reduction (CuRe) and SPARK for bilinear systems ?

4.) Bilinear Lyapunov equations: Bilinear low-rank ADI = RKSM + H2-pseudo-optimal shifts

Chair of Automatic Control
Department of Mechanical Engineering
Technical University of Munich

References

[Beattie/Gugercin '12] *Realization-independent H2-approximation*. In CDC proceedings.

[Benner/Breiten '12] *Interpolation-based H2-model reduction of bilinear control systems*. SIAM Journal on Matrix Analysis and Applications

[Flagg '12] *Interpolation Methods for the Model Reduction of Bilinear Systems*, PhD thesis, Virginia Tech

[Flagg/Gugercin '15] *Multipoint Volterra series interpolation and H2 optimal model reduction of bilinear systems*, SIAM Journal on Matrix…

[Gugercin et al '08] *H2 model reduction for large-scale linear dynamical systems*.

[Rugh '81] *Nonlinear system theory. The Volterra/Wiener Approach*

[Wolf '14] *H2 Pseudo-Optimal Model Order Reduction*, PhD thesis

Thank you for your attention!