

Renormalization group evolution of flavour invariants

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ABSTRACT: The fermion spectrum in the Standard Model (SM) exhibits hierarchical structures between the eigenvalues of the Yukawa matrices which determine the fermion masses, as well as certain hierarchical patterns in the mixing matrix that describes weak transitions between different fermion generations. A basis-independent description of the SM flavour structure can be given in terms of a complete set of flavour invariants. In this paper, we construct a convenient set of such invariants, and discuss the general form of the renormalization-group equations. We also discuss the simplifications that arise from exploiting hierarchies in Yukawa couplings and mixings which are present in the SM or its minimal-flavour violating extensions.

KEYWORDS: Quark Masses and SM Parameters, Renormalization Group, CP violation

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1 Introduction

In the Standard Model (SM) of particle physics, the Yukawa couplings of quarks and charged leptons to the Higgs field are the only sources of flavour structure. The singular values of the Yukawa matrices, together with the vacuum expectation value (VEV) of the Higgs field determine the fermion masses, and the relative orientation between the up- and down-quark Yukawa matrices results in the Cabibbo-Kobayashi-Maskawa (CKM) matrix, responsible for charged flavour transitions in weak interactions. In the quantum-field theoretical formulation of the SM, the Yukawa matrices enter as coupling parameters in the Lagrange density. In the following, we will focus on the quark sector, where one has

$$-\mathcal{L}_{\text{Yukawa}} = Y_U^{ij} \bar{Q}_L^i \tilde{H} U_R^j + Y_D^{ij} \bar{Q}_L^i H D_R^j + \text{h.c.} \quad (1.1)$$

Here and in the following Y_U and Y_D denote the Yukawa matrices for up- and down-type quarks, Q_L, U_R, D_R are the left-handed quark doublet and right-handed singlets, respectively, and H, \tilde{H} is the Higgs field and its SU(2) conjugate. The indices $i, j = 1 \dots n_g$ denote the quark generations/families ($n_g = 3$ in the SM).

As all other couplings in the SM, after renormalization of ultraviolet divergencies, the Yukawa matrices in (1.1) are to be interpreted as effective parameters with the scale-dependence controlled by renormalization-group (RG) equations [1–5]. The structure of the RG equations and their solutions have been extensively studied in the past. In [6], the resulting one-loop RG evolution of the CKM matrix elements (in a given parametrization) has been studied, and approximate analytic solutions have been derived on the basis of the observed hierarchies in quark masses and mixing angles in the SM. Generalizations to particular new physics (NP) frameworks have also been derived, notably for 2-Higgs-doublet models or supersymmetric extensions of the SM, see, for instance, [7, 8]. Recently, the effect of possible NP contributions has been studied in a model-independent way, by considering the RG effects from dimension-six operators in an effective field theory (SM-EFT) approach [9]. Finally, Bednyakov et al. [10] have recently computed the three-loop RG coefficients for the SM Yukawa matrices.

The RG equations are usually formulated in matrix form, i.e. the scale-variation of the Yukawa matrices is given by a matrix polynomial of Y_U and Y_D . Since the gauge sector of the SM is invariant under unitary field redefinitions for the individual quark multiplets, the RG equations have to transform covariantly under such changes of flavour basis (see below). This also implies a certain degree of redundancy in the RG equations, because from the 18 complex matrix entries in Y_U and Y_D only 10 physical parameters are observables.

In this paper, we will therefore reformulate the RG equations in terms of *flavour invariants*, i.e. objects constructed from Y_U and Y_D which are independent of the choice of flavour basis. As has been shown in [11] from the basic algebraic principle of Hilbert series, one can define *eleven* polynomially independent flavour invariants for three quark generations. These fix the six quark masses, the three mixing angles and the sine and cosine of the CP-violating phase in the CKM matrix (see also [12]). As a corollary, using Cayley-Hamilton identities for matrix products (cf. [13]), this also implies that any flavour-covariant product of Yukawa matrices that appears on the right-hand side of the RG equations for Y_U or Y_D can be reduced to a finite basis of flavour matrices with coefficients given as polynomials of flavour invariants. It is then a straightforward, though tedious, task to derive the RG equations for the set of flavour invariants.

Although the RG equations for the flavour invariants contain the same information as the original flavour-covariant equations, the formulation in terms of flavour invariants, under certain circumstances, may be considered advantageous. For instance, the form of the RG equations is universal, not only for the SM, but also for all extensions that obey the principle of minimal flavour violation (MFV) in the technical sense of [14]. Furthermore, the hierarchical pattern of masses and mixing directly translates into a well-defined power counting for (suitably chosen) flavour invariants, which can be exploited to simplify the RG equations. An attractive physical picture arises if one assumes these hierarchies to be associated to some dynamical NP mechanism that can be traced back to an effective poten-

tial which determines the flavour structures at low energies. Within the MFV framework, the potential itself will have to be formulated in terms of flavour invariants, and the minimization of the potential should generate VEVs for the flavour invariants that reflect the particular pattern of (sequential) flavour-symmetry breaking in the SM (see [15]). Recent studies along these lines can be found, for instance, in [16–19]. Here, it turned out that in order to completely reproduce the SM flavour patterns from the minimum of an effective potential, one would generally have to introduce additional degrees of freedom apart from those related to the SM Yukawa matrices. Within the MFV framework these would be encoded as new spurion fields transforming according to some tensor representation of the SM flavour symmetry group [20, 21]. Such flavour structures also show up, for instance, as coupling constants in front of higher-dimensional operators in SM-EFT [22, 23].

The remaining paper is organized as follows: in the next section, we will first discuss a toy scenario with only two generations (2G) of SM quarks. The simplifications in the 2G case (no CP violation, closure of $SU(2)$ matrices under multiplication, small number of polynomially independent invariants) allow us to introduce our approach in a very transparent way, perform all calculations analytically and illustrate the RG equations for the flavour invariants in a graphical way. To this end, we will first give convenient definitions for flavour invariants and basic flavour matrices. In terms of these, the general form of the RG equations for flavour invariants will be derived. We also present analytical and numerical solutions for the RG equations that can be obtained from exploiting SM-like flavour hierarchies in the one-loop approximation. In section 3 we generalize our framework to the realistic case of three quark generations (3G). To keep the discussion transparent, we restrict ourselves to the one-loop approximation from the very beginning. Again, we derive the general form of the (one-loop) RG equations for the eleven flavour invariants, and discuss their approximate solutions in the SM. We close the paper with a short summary and outlook in section 4. Some technical details about the use of Cayley-Hamilton identities, the explicit form of the 3G flavour invariants, and the general form of the two-loop RG equations in the 3G case can be found in the appendices.

2 Two quark generations

As mentioned above, in this section, we restrict ourselves to two generations of left-handed quark doublets and right-handed up- and down-quark singlets in the SM. The gauge-kinetic terms of the SM Lagrangian are flavour-blind, and therefore independent unitary rotations of the quark multiplets define a flavour symmetry,

$$\mathcal{G}_{\text{quark}} = U(2)^3/U(1)_B \sim SU(2)_{Q_L} \times U(2)_{U_R} \times U(2)_{D_R}, \quad (2.1)$$

which is only broken by the Yukawa couplings in (1.1). Here we factored out a $U(1)_B$ symmetry for baryon number conservation, which is unaffected by the SM Yukawa interactions [15]. (More precisely, we find it convenient to factor out a $U(1)_{Q_L}$ transformation acting on the left-handed doublets only). In a particular flavour basis, the Yukawa matrices

for up- and down-type quarks read

$$Y_U = \begin{pmatrix} y_u & 0 \\ 0 & y_c \end{pmatrix}, \quad Y_D = V_{\text{Cabibbo}} \begin{pmatrix} y_d & 0 \\ 0 & y_s \end{pmatrix} = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix} \begin{pmatrix} y_d & 0 \\ 0 & y_s \end{pmatrix}. \quad (2.2)$$

Under a change of flavour basis, the Yukawa matrices transform as

$$Y_U \rightarrow V_{Q_L} Y_U V_{U_R}^\dagger, \quad Y_D \rightarrow V_{Q_L} Y_D V_{D_R}^\dagger, \quad (2.3)$$

where $V_{Q_L} \in \text{SU}(2)_{Q_L}$ etc.

2.1 Flavour invariants

In the 2G case, one can construct five flavour invariants that are polynomially independent (see e.g. [11] and references therein for the mathematical background). In the following, to set the stage for the 3G case to be discussed in section 3, we will discuss the construction and properties of these flavour invariants step by step.

From the Yukawa matrices Y_U and Y_D , one can construct flavour invariants in terms of traces or determinants of matrix products constructed from the non-negative hermitian matrices

$$U = Y_U Y_U^\dagger, \quad D = Y_D Y_D^\dagger. \quad (2.4)$$

These transform as $V_{Q_L} U V_{Q_L}^\dagger$ and $V_{Q_L} D V_{Q_L}^\dagger$ under basis transformations for the left-handed quark doublets. A convenient choice for non-negative invariants is

$$\begin{aligned} I_1 \equiv \text{tr}[U] &= y_u^2 + y_c^2 \geq 0, & I_2 \equiv \text{tr}[D] &= y_d^2 + y_s^2 \geq 0, \\ \widehat{I}_3 \equiv \det[U] &= y_u^2 y_c^2 \geq 0 & \widehat{I}_4 \equiv \det[D] &= y_d^2 y_s^2 \geq 0, \end{aligned} \quad (2.5)$$

and

$$\widehat{I}_5 \equiv \text{tr}[UD] = (y_c^2 y_s^2 + y_d^2 y_u^2) \cos^2 \theta + (y_c^2 y_d^2 + y_s^2 y_u^2) \sin^2 \theta \geq 0. \quad (2.6)$$

Apart from discrete ambiguities related to renaming the original quark fields in the flavour eigenbasis, they determine the four eigenvalues for the Yukawa couplings and the Cabibbo mixing angle. Invariants built from traces of higher powers of U and D are related to the above via Cayley-Hamilton identities (see appendix A and e.g. the discussion in [13]). Still, for the following discussion, we further define the polynomially *dependent* invariants

$$I_3 \equiv \frac{I_1^2}{2} - 2\widehat{I}_3 = \frac{1}{2}(y_c^2 - y_u^2)^2, \quad I_4 \equiv \frac{I_2^2}{2} - 2\widehat{I}_4 = \frac{1}{2}(y_s^2 - y_d^2)^2, \quad (2.7)$$

and

$$\begin{aligned} I_5 \equiv \widehat{I}_5 - \frac{I_1 I_2}{2} &= \frac{1}{2}(y_c^2 - y_u^2)(y_s^2 - y_d^2) \cos(2\theta), \\ \widetilde{I}_5 \equiv I_3 I_4 - I_5^2 &= \frac{1}{4}(y_c^2 - y_u^2)^2 (y_s^2 - y_d^2)^2 \sin^2(2\theta). \end{aligned} \quad (2.8)$$

Triplet matrices and triplet invariants. It is further convenient to divide the matrices U and D into singlet and triplet components with respect to the flavour group factor $SU(2)_{Q_L}$,

$$U = \frac{1}{2} \text{tr}[U] \mathbb{1} + U_3, \quad D = \frac{1}{2} \text{tr}[D] \mathbb{1} + D_3, \quad (2.9)$$

A third independent triplet matrix can be defined as

$$A_3 = \frac{i}{2} [U_3, D_3]. \quad (2.10)$$

Polynomial basis. For generic Yukawa entries, any 2×2 matrix M that transforms as $V_{Q_L} M V_{Q_L}^\dagger$ under $SU(2)_{Q_L}$ can be written as a finite polynomial of matrices from the set $\{\mathbb{1}, U_3, D_3, A_3\}$. For those matrices, the following multiplication tables for symmetric and anti-symmetric products of matrices holds.

$\{, \}$	U_3	D_3	A_3	$i[,]$	U_3	D_3	A_3
U_3	$I_3 \mathbb{1}$	$I_5 \mathbb{1}$	0	U_3	0	$2\bar{A}_3$	$-2\bar{D}_3$
D_3	$I_5 \mathbb{1}$	$I_4 \mathbb{1}$	0	D_3	$-2\bar{A}_3$	0	$2\bar{U}_3$
A_3	0	0	$\frac{1}{2} \tilde{I}_5 \mathbb{1}$	A_3	$2\bar{D}_3$	$-2\bar{U}_3$	0

This explicitly shows that the set $\{\mathbb{1}, U_3, D_3, A_3\}$ closes under matrix multiplication with prefactors that are polynomials of the flavour invariants. Here we defined the *dual* matrices

$$\bar{U}_3 \equiv \frac{i}{2} [D_3, A_3] = \frac{I_4}{2} U_3 - \frac{I_5}{2} D_3, \quad \bar{D}_3 \equiv \frac{i}{2} [A_3, U_3] = \frac{I_3}{2} D_3 - \frac{I_5}{2} U_3, \quad (2.11)$$

and

$$\bar{A}_3 \equiv \frac{i}{2} [U_3, D_3] = A_3, \quad (2.12)$$

which can be obtained from the inverse of the metric

$$G_{XY} = \text{tr}[X_3 Y_3] = \begin{pmatrix} I_3 & I_5 & 0 \\ I_5 & I_4 & 0 \\ 0 & 0 & \frac{\tilde{I}_5}{2} \end{pmatrix} \quad (X, Y = U, D, A), \quad (2.13)$$

as

$$\begin{pmatrix} \bar{U}_3 \\ \bar{D}_3 \\ \bar{A}_3 \end{pmatrix} = \frac{\tilde{I}_5}{2} G^{-1} \begin{pmatrix} U_3 \\ D_3 \\ A_3 \end{pmatrix}. \quad (2.14)$$

From this we can read off the orthogonality relations between triplet matrices and their dual,

$$\text{tr}[X_3 \bar{Y}_3] = \frac{\tilde{I}_5}{2} \delta_{XY} \quad (X, Y = U, D, A). \quad (2.15)$$

This can be used, for instance, to decompose a generic $SU(2)_{Q_L}$ triplet matrix as

$$M_3 = \sum_{X=U,D,A} \frac{2 \text{tr}[\bar{X}_3 M_3]}{\tilde{I}_5} X_3. \quad (2.16)$$

Similarly, a matrix M_U that transforms as a bi-doublet under $SU(2)_{Q_L} \times SU(2)_{U_R}$ can be decomposed as

$$M_U = \frac{\text{tr}[M_U Y_U^{-1}]}{2} \mathbb{1} + \sum_{X=U,D,A} \frac{2 \text{tr}[\tilde{X}_3 M_U Y_U^{-1}]}{\tilde{I}_5} X_3, \quad (2.17)$$

and analogously for $U_R \leftrightarrow D_R$. Higher tensor representations of the flavour symmetry group and their expansion can be constructed from $M_{3,U,D}$. Notice that for *generic* matrices $M_{3,U,D}$, the coefficients in these expansions are enhanced by $(\tilde{I}_5)^{-1}$ and $Y_{U,D}^{-1}$, respectively. In contrast, the MFV hypothesis assumes these coefficients to be of order 1 or smaller (see again [13]).

2.2 Renormalization-group equations

2.2.1 General form

The Yukawa matrices are subject to renormalization-group (RG) evolution. The generic form for the RG-running of the Yukawa matrices $Y_{U,D}$ can be written in manifestly flavour-symmetric form (see e.g. [2, 3]). Using the generic decomposition into basis matrices as discussed above, we thus write

$$\begin{aligned} \frac{dY_U(\mu)}{d \ln \mu} &= (a_0(I_i, \mu) \mathbb{1} + a_1(I_i, \mu) U_3 + a_2(I_i, \mu) D_3 + i a_3(I_i, \mu) A_3) Y_U(\mu), \\ \frac{dY_D(\mu)}{d \ln \mu} &= (b_0(I_i, \mu) \mathbb{1} + b_1(I_i, \mu) D_3 + b_2(I_i, \mu) U_3 - i b_3(I_i, \mu) A_3) Y_D(\mu). \end{aligned} \quad (2.18)$$

Each of the coefficients a_i, b_i depends on flavour invariants which arise from loop diagrams including additional Higgs-Yukawa couplings. (At one-loop accuracy, only terms at most quadratic in the Yukawa couplings can appear within the round brackets etc.) In the SM (or, in general, in constrained MFV models without additional sources of CP violation), the coefficients will be *real* polynomials of the flavour invariants.¹ This immediately translates into RG equations for the matrices U and D , and from this we obtain

$$\frac{dI_1}{d \ln \mu} = \text{tr} \left[\frac{dU}{d \ln \mu} \right] = 2 (a_0 I_1 + a_1 I_3 + a_2 I_5), \quad (2.19)$$

$$\frac{dI_2}{d \ln \mu} = \text{tr} \left[\frac{dD}{d \ln \mu} \right] = 2 (b_0 I_2 + b_1 I_4 + b_2 I_5). \quad (2.20)$$

In a similar way, one obtains the RG equations for the remaining invariants in a straightforward manner. The RG equations for the invariants \hat{I}_3, \hat{I}_4 take a particularly simple form

$$\frac{d\hat{I}_3}{d \ln \mu} = 4a_0 \hat{I}_3, \quad \frac{d\hat{I}_4}{d \ln \mu} = 4b_0 \hat{I}_4. \quad (2.21)$$

For the invariant I_5 , we obtain

$$\frac{dI_5}{d \ln \mu} = (2a_0 + 2b_0 + a_1 I_1 + b_1 I_2) I_5 + a_2 I_1 I_4 + b_2 I_2 I_3 + (a_3 + b_3) \tilde{I}_5, \quad (2.22)$$

¹Furthermore, if weak isospin-violating corrections are neglected, the coefficients a_i and b_i will be related, see e.g. [4, 24].

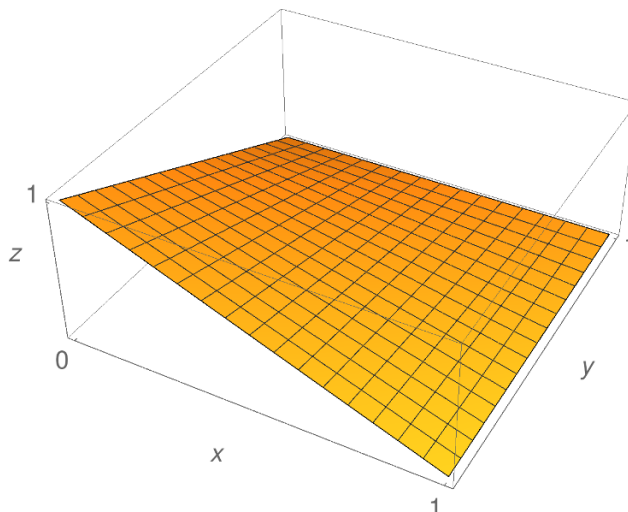


Figure 1. Illustration of the “phase space” (the region below the shaded area) for the normalized invariants x, y, z defined in the text.

and for the invariant \tilde{I}_5 , we get

$$\frac{d\tilde{I}_5}{d\ln\mu} = (4a_0 + 4b_0 + 2a_1I_1 + 2b_1I_2 - 2(a_3 + b_3)I_5) \tilde{I}_5, \quad (2.23)$$

Discussion. From (2.21) and (2.23) we observe that some limiting cases in the phase space of flavour invariants are stable under RG evolution:

- The case $\hat{I}_3 = 0$: in terms of physical parameters, this corresponds to one vanishing eigenvalue in the up-quark sector, $y_u = 0$, and otherwise generic values for $y_{d,s,c}$ and θ .
- The case $\hat{I}_4 = 0$: this corresponds to one vanishing eigenvalue in the down-quark sector, $y_d = 0$, and otherwise generic values for $y_{u,s,c}$ and θ .
- The case $\tilde{I}_5 = 0$: this corresponds to $\sin 2\theta = 0$, i.e. no mixing and otherwise generic Y_U and Y_D ; or degenerate eigenvalues in the up-quark sector ($y_u = y_c$) or in the down-quark sector ($y_d = y_s$), respectively.

For illustration, we thus define normalized invariants (for $I_{1,2} \neq 0$),

$$x = 1 - \bar{x} \equiv \frac{4\hat{I}_3}{I_1^2}, \quad y = 1 - \bar{y} \equiv \frac{4\hat{I}_4}{I_2^2}, \quad z = 1 - \bar{z} \equiv \frac{4\tilde{I}_5}{I_1^2 I_2^2}, \quad (2.24)$$

which take values in the unit interval $[0, 1]$, with the additional constraints

$$z \leq \bar{x} \bar{y}. \quad (2.25)$$

This is illustrated in figure 1. As a consequence of the above observations, there will be now RG flow from the “phase-space” edges, defined by $x = 0$, $y = 0$, or $z = 0$, into the

bulk. This can be understood as a consequence of a residual flavour symmetry. In contrast, the case $z = \bar{x}\bar{y}$ is not protected by symmetry. A more detailed discussion of the residual flavour symmetries associated with this situation will be given in [25] (see also [26]).

2.2.2 Exploiting flavour hierarchies

The RG equations simplify when one exploits flavour hierarchies in the Yukawa matrices. For instance, in a SM-like scenario, we can consider the limit where all but one Yukawa coupling, say y_c in the 2G toy case, are small. In this case, the basis of triplet matrices in (2.18) can be reduced to U_3 , and consequently only the coefficients a_0, b_0, a_1, b_2 are relevant to first approximation.² The RG equations for the invariants in this approximation read (also using $I_3 \simeq I_1^2/2$)

$$\begin{aligned} \frac{dI_1}{d \ln \mu} &\simeq (2a_0 + a_1 I_1) I_1, & \frac{d\hat{I}_3}{d \ln \mu} &= 4a_0 \hat{I}_3, \\ \frac{dI_2}{d \ln \mu} &\simeq 2b_0 I_2 + 2b_2 I_5, & \frac{d\hat{I}_4}{d \ln \mu} &= 4b_0 \hat{I}_4, \\ \frac{d\tilde{I}_5}{d \ln \mu} &\simeq (4a_0 + 4b_0 + 2a_1 I_1) \tilde{I}_5. \end{aligned} \tag{2.26}$$

Solving for the four coefficients, leaves one general relation between the five invariants and their derivatives which can be written as

$$\frac{d\tilde{I}_5}{\tilde{I}_5} \simeq 2 \frac{dI_1}{I_1} + \frac{d\hat{I}_4}{\hat{I}_4} \quad (\text{for } y_c \gg y_{u,d,s}). \tag{2.27}$$

This implies

$$\frac{\tilde{I}_5(\mu)}{\tilde{I}_5(\mu_0)} \simeq \frac{I_1^2(\mu) \hat{I}_4(\mu)}{I_1^2(\mu_0) \hat{I}_4(\mu_0)} \Leftrightarrow \frac{y}{z} \simeq \text{const.}, \tag{2.28}$$

or, in terms of Yukawa eigenvalues and the Cabibbo angle,³

$$(y_s/y_d - y_d/y_s)^2 \sin^2(2\theta) \simeq \text{const.} \quad (\text{for } y_c \gg y_{u,d,s}). \tag{2.29}$$

Putting in experimental values for the quark-mass ratio and the Cabibbo angle, the constant on the r.h.s. ranges between 60 and 90.

2.2.3 One-loop solutions in the SM

To illustrate the numerical effect of the RG equations, we consider the one-loop RG coefficients in the SM. The system of RG equations further simplifies if we neglect electroweak corrections, leading to the values summarized in table 1. For the starting values of the invariants in the 2G case, we consider a toy model where we neglect the first generation

²A complementary approach would perform the limit $y_c \gg y_{u,d,s}$ from the very beginning and consider invariants under the *reduced* flavour symmetry only, see [27, 28].

³In models with texture zeros one typically relates the Cabibbo angle to the *square root* of y_d/y_s , see e.g. [29]. Therefore, such relations — in general — are not scale invariant in the limit of hierarchical Yukawa couplings.

a_0	b_0	$a_1 = b_1$	$a_2 = b_2$
$\frac{3}{8\pi^2} \left(I_1 + I_2 + \frac{I_1 - I_2}{2n_g} \right) - \frac{2\alpha_s}{\pi}$	$\frac{3}{8\pi^2} \left(I_1 + I_2 + \frac{I_2 - I_1}{2n_g} \right) - \frac{2\alpha_s}{\pi}$	$\frac{3}{16\pi^2}$	$-\frac{3}{16\pi^2}$

Table 1. Non-vanishing RG coefficients in the SM as defined in the text, following from [6]. Here n_g denotes the number of generations, and electroweak contributions have been neglected for simplicity.

in the SM, such that the large Yukawa couplings from the third generation lead to non-trivial effects on the r.h.s. of the RG equations. Exploiting again the hierarchies in the SM Yukawa entries, we then find

$$\frac{dI_1}{d \ln \mu} \simeq \frac{9}{8\pi^2} I_1^2 - \frac{4\alpha_s}{\pi} I_1. \quad (2.30)$$

Using the one-loop expression for the QCD β -function,

$$\frac{d\alpha_s}{d \ln \mu} \simeq -\frac{\beta_0}{2\pi} \alpha_s^2, \quad (2.31)$$

one obtains the explicit solution

$$I_1(\mu) \simeq \eta^{8/\beta_0} I_1(\mu_0) G(\mu, \mu_0), \quad \eta = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)} \quad (2.32)$$

where we defined the RG-evolution function

$$G(\mu, \mu_0) := \exp \left[\frac{9}{8\pi^2} \int_{\mu_0}^{\mu} \frac{d\mu}{\mu} I_1(\mu) \right] \simeq \left(1 + I_1(\mu_0) \frac{9}{4\pi} \frac{\eta - \eta^{8/\beta_0}}{(\beta_0 - 8) \alpha_s(\mu)} \right)^{-1}. \quad (2.33)$$

This coincides with [6], where the approximate RG flow of the top Yukawa coupling has been derived (with $I_1(\mu) \simeq (y_t(\mu))^2$ and $\beta^{(1)}$ in [6] is defined as $\beta^{(0)}/4$ in our convention.)

For the remaining invariants, using $y_c \gg y_s \gg y_{u,d}$ and $|\sin 2\theta| \ll 1$, we have

$$\frac{dI_2}{d \ln \mu} \simeq \frac{3}{8\pi^2} I_1 I_2 - \frac{4\alpha_s}{\pi} I_2 \quad \Leftrightarrow \quad \frac{d(I_2/I_1)}{d \ln \mu} \simeq -\frac{3}{4\pi^2} I_1 \left(\frac{I_2}{I_1} \right), \quad (2.34)$$

and

$$\begin{aligned} \frac{d\hat{I}_3}{d \ln \mu} &\simeq \frac{15}{8\pi^2} I_1 \hat{I}_3 - \frac{8\alpha_s}{\pi} \hat{I}_3 &\Leftrightarrow \quad \frac{dx}{d \ln \mu} &\simeq -\frac{3}{8\pi^2} I_1 x, \\ \frac{d\hat{I}_4}{d \ln \mu} &\simeq \frac{9}{8\pi^2} I_1 \hat{I}_4 - \frac{8\alpha_s}{\pi} \hat{I}_4 &\Leftrightarrow \quad \frac{dy}{d \ln \mu} &\simeq \frac{3}{8\pi^2} I_1 y, \\ \frac{d\tilde{I}_5}{d \ln \mu} &\simeq \frac{27}{8\pi^2} I_1 \tilde{I}_5 - \frac{16\alpha_s}{\pi} \tilde{I}_5 &\Leftrightarrow \quad \frac{dz}{d \ln \mu} &\simeq \frac{3}{8\pi^2} I_1 z. \end{aligned} \quad (2.35)$$

We see that once the RG-solution for $I_1(\mu)$ has been constructed, the RG equations for the remaining invariants can be easily solved by separation of variables. Using the RG function $G(\mu, \mu_0)$ defined in (2.33), we have

$$I_2(\mu) \simeq \eta^{8/\beta_0} I_2(\mu_0) [G(\mu, \mu_0)]^{1/3}, \quad (2.36)$$

and

$$x(\mu) \simeq [G(\mu, \mu_0)]^{-1/3} x(\mu_0), \quad y(\mu) \simeq [G(\mu, \mu_0)]^{1/3} y(\mu_0), \quad z(\mu) \simeq [G(\mu, \mu_0)]^{1/3} z(\mu_0). \quad (2.37)$$

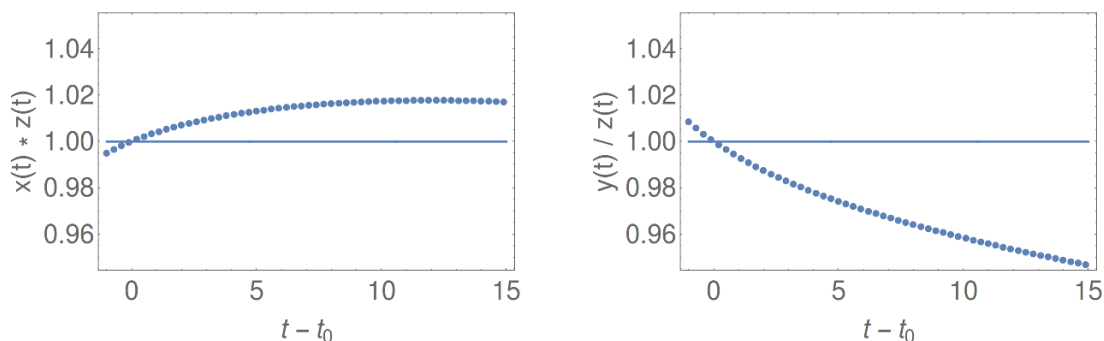


Figure 2. Comparison of numerical [dots] and approximate analytical (2.37) [solid line] solution for combinations of flavour invariants, $x(t)z(t)$ and $y(t)/z(t)$, normalized to the values at $t = \ln \mu/\mu_0 \rightarrow t_0 = 0$. The following starting values have been used: $I_1(t_0) = 1.0$, $I_2(t_0) = 0.1$, $x(t_0) = 0.005$, $y(t_0) = 0.048$, $z(t_0) = 0.105$.

2.2.4 Numerical illustration

In figure 2 we provide illustrations for the one-loop RG flow of the combinations of flavour invariants x, y, z in the SM, and compare the exact numerical solutions with the approximation in (2.37). We observe that — for the chosen numerical starting values — even for values as large as $t = \ln \mu/\mu_0 = 15$, the differences between the exact and approximate solutions are always below 5%.

In figure 3 we illustrate the RG flow for the boundaries of the “phase-space” of flavour invariants, defined by $z \equiv 0$, $y \equiv 0$, $x \equiv 0$, respectively, see the discussion in section 2.2.1. Again, we have chosen a hierarchical scenario with $I_2(\mu_0) \ll I_1(\mu_0)$. We observe that

- The relation $y(t)/z(t) \simeq \text{const.}$ holds on the whole plane $x = 0$, which is in line with our derivation of (2.28) which only required $I_1 \gg I_2$.
- In contrast, $x(t)y(t) \simeq \text{const.}$ only holds in the vicinity of $x \sim y \sim 0$ (where $y_u \ll y_c$ and $y_d \ll y_s$) and for θ near zero (which requires the solution with $I_5(\mu_0) = -\sqrt{I_3 I_4 - \tilde{I}_5(\mu_0) I_5}$ shown on the left-hand side).
- The same is true for $x(t)z(t) \simeq \text{const.}$

3 Three quark generations

For three quark generations in the SM, the flavour symmetry group to consider now is

$$\mathcal{G}_{\text{quark}} = \text{U}(3)^3/\text{U}(1)_B \sim \text{SU}(3)_{Q_L} \times \text{U}(3)_{U_R} \times \text{U}(3)_{D_R}. \quad (3.1)$$

The corresponding Yukawa matrices again transform as bi-fundamentals under a change of flavour basis,

$$Y_U \rightarrow V_{Q_L} Y_U V_{U_R}^\dagger, \quad Y_D \rightarrow V_{Q_L} Y_D V_{D_R}^\dagger.$$

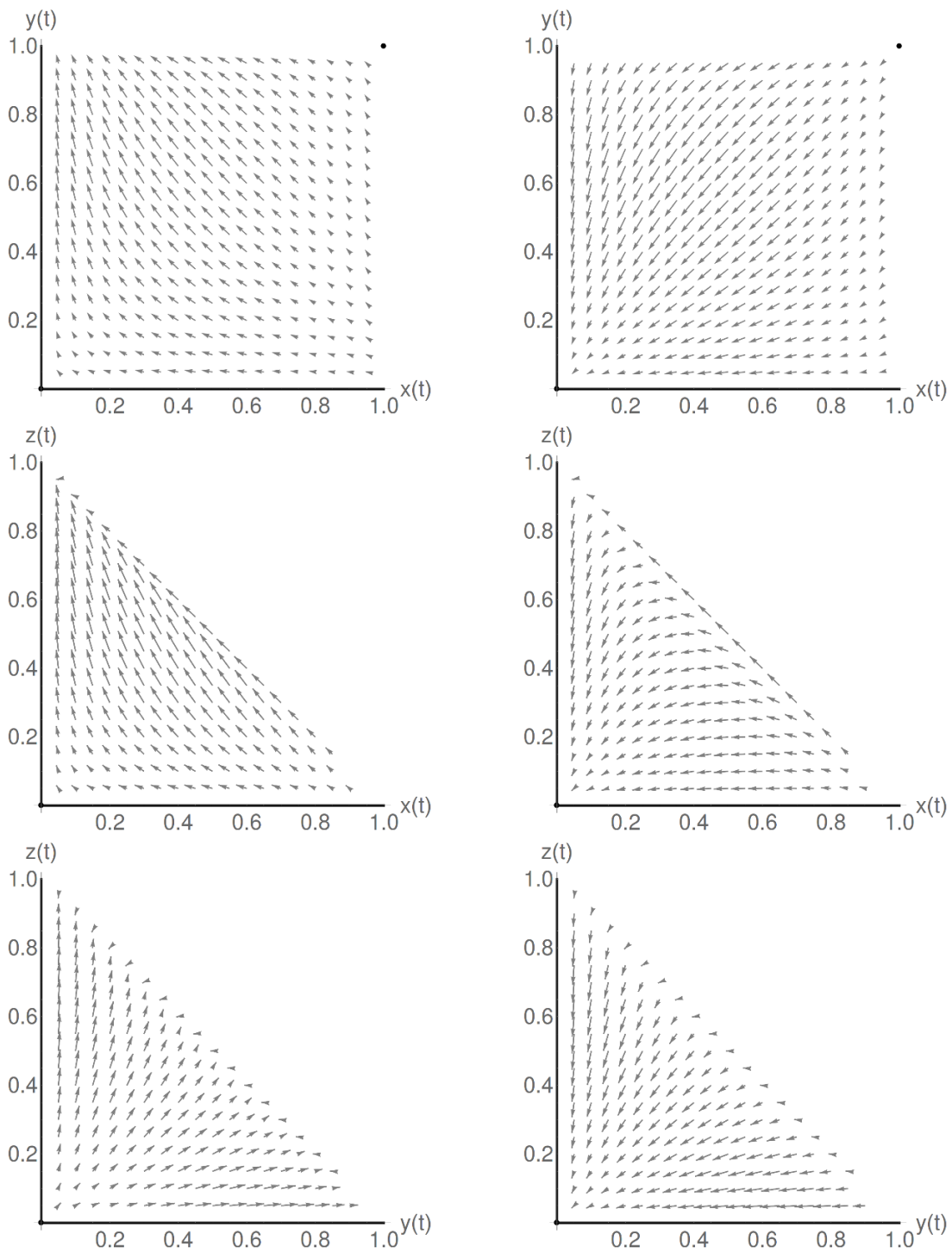


Figure 3. Numerical illustration of the RG flow at the “phase-space boundaries” for flavour invariants in the SM (2G, one-loop accuracy, neglecting electroweak gauge couplings). Top: $z \equiv 0$; center: $y \equiv 0$; bottom $x \equiv 0$. Each arrow indicates the RG flow from $t_0 = 0$ to $t = 5$; the starting values are again chosen as $I_1(\mu_0) = 1.0$, $I_2(\mu_0) = 0.1$, $\alpha_s(\mu_0) = 0.2$. The plots on the left (right) are generated assuming $I_5(\mu_0) = \mp \sqrt{I_3 I_4 - \tilde{I}_5(\mu_0)}$.

In a particular flavour basis, they are given by

$$Y_U = \begin{pmatrix} y_u & 0 & 0 \\ 0 & y_c & 0 \\ 0 & 0 & y_t \end{pmatrix}, \quad Y_D = V_{\text{CKM}} \begin{pmatrix} y_d & 0 & 0 \\ 0 & y_s & 0 \\ 0 & 0 & y_b \end{pmatrix}. \quad (3.2)$$

In the subsequent analysis, it turns out to be more convenient to discuss the flavour invariants as a function of the CKM elements V_{ij} without choosing a particular parametrization in terms of mixing angles which would directly reflect the unitarity of the CKM matrix V_{CKM} .

3.1 Flavour invariants

As discussed in [11], the SM quark sector in the 3G case can be described in terms of 10+1 polynomially independent invariants, which determine 6 Yukawa eigenvalues, 3 mixing angles and the cosine and sine of the CP-violating phase (in a given parametrization of the CKM matrix). With a similar procedure as in the 2G case, we will now explicitly construct a convenient set for these 11 invariants from the non-negative hermitian matrices,

$$U \equiv Y_U Y_U^\dagger, \quad D \equiv Y_D Y_D^\dagger,$$

which now transform under the $SU(3)_{Q_L}$ flavour symmetry. For later use we also define the adjoint matrices, satisfying

$$U \text{ adj } U = \det U \quad \text{etc.}$$

With this, we can easily construct a complete set of polynomially independent positive semi-definite invariants. For the unmixed invariants, we define

$$\begin{aligned} I_1 &\equiv \text{tr}(U) \geq 0, & I_2 &\equiv \text{tr}(D) \geq 0, \\ \widehat{I}_3 &\equiv \text{tr}(\text{adj } U) \geq 0, & \widehat{I}_4 &\equiv \text{tr}(\text{adj } D) \geq 0, \\ \widehat{I}_6 &\equiv \det(U) \geq 0, & \widehat{I}_8 &\equiv \det(D) \geq 0, \end{aligned} \quad (3.3)$$

These determine the six singular values of the Yukawa matrices. The CKM elements are then determined by mixed invariants which we define in a similar way. CP-even invariants can be chosen as

$$\widehat{I}_5 \equiv \text{tr}(UD) \geq 0, \quad \widehat{I}_7 \equiv \text{tr}(D \text{ adj } U) \geq 0, \quad \widehat{I}_9 \equiv \text{tr}(U \text{ adj } D) \geq 0, \quad (3.4)$$

and

$$\widehat{I}_{10} \equiv \text{tr}(\text{adj}(UD)) \geq 0, \quad (3.5)$$

According to the discussion in [11], there is an eleventh, CP-odd, invariant that cannot be expressed as a polynomial of the other ten invariants, as defined above. It is related to the Jarlskog determinant [30] and can be chosen as

$$I_{11}^- = -\frac{3i}{8} \det[U, D]. \quad (3.6)$$

Explicit expression in terms of Yukawa couplings and CKM elements can be found in appendix B.

tr []	U_8	D_8	adj U_8	adj D_8	S_8	A_8	A_U	A_D
U_8	I_3	I_5	I_6	I_9	\tilde{I}_7	0	0	0
D_8	+	I_4	I_7	I_8	\tilde{I}_9	0	0	0
adj U_8	+	+	P_1	P_2	P_3	0	0	0
adj D_8	+	+	+	P_4	P_5	0	0	0
S_8	+	+	+	+	I_{10}	0	0	0
A_8	0	0	0	0	0	\tilde{I}_{10}	Q_1	Q_2
A_U	0	0	0	0	0	+	Q_3	Q_4
A_D	0	0	0	0	0	+	+	Q_5

Table 2. Traces of basic octet matrices constructed from $Y_U Y_U^\dagger$ and $Y_D Y_D^\dagger$ as defined in the text.

Octet matrices and octet invariants. As in the 2G case, we can also construct basic flavour matrices as octet representations of the flavour group factor $SU(3)_{Q_L}$. First, there are two polynomially independent octet matrices that are quadratic in the Yukawas, namely the traceless part of the matrices U and D (defined analogously to the 2G case),

$$U_8 \equiv U - \frac{1}{3} \text{tr}[U] \mathbf{1}, \quad D_8 \equiv D - \frac{1}{3} \text{tr}[D] \mathbf{1}. \quad (3.7)$$

For terms quartic in the Yukawas, we may define the octet part of the adjoint matrices,

$$\text{adj } U_8 \equiv \text{adj } U - \frac{1}{3} \text{tr}[\text{adj } U] \mathbf{1}, \quad \text{adj } D_8 \equiv \text{adj } D - \frac{1}{3} \text{tr}[\text{adj } D] \mathbf{1},$$

together with

$$S_8 \equiv \frac{1}{2} \{U, D\} - \frac{1}{3} \text{tr}[UD] \mathbf{1}, \quad A_8 \equiv \frac{i}{2} [U, D]. \quad (3.8)$$

Similarly, we define

$$A_U \equiv \frac{i}{2} [\text{adj } U, D], \quad A_D \equiv \frac{i}{2} [\text{adj } D, U]. \quad (3.9)$$

For generic Yukawa entries, the eight hermitian matrices as defined above provide a basis for octet matrices in $SU(3)_{Q_L}$. The symmetric (but non-orthogonal) metric defined by the traces of matrix products contains flavour invariants for the 3×3 case. It is summarized in table 2. Here, the unhatted invariants are related to the hatted ones via

$$I_3 \equiv \text{tr}[U_8^2] = \frac{2}{3} I_1^2 - 2\hat{I}_3, \quad I_4 \equiv \text{tr}[D_8^2] = \frac{2}{3} I_2^2 - 2\hat{I}_4,$$

$$I_6 \equiv \text{tr}[U_8 \text{adj} U_8] = 3\widehat{I}_6 - \frac{I_1 \widehat{I}_3}{3}, \quad I_8 \equiv \text{tr}[D_8 \text{adj} D_8] = 3\widehat{I}_8 - \frac{I_2 \widehat{I}_4}{3}, \quad (3.10)$$

and

$$I_5 \equiv \text{tr}[U_8 D_8] = \widehat{I}_5 - \frac{I_1 I_2}{3}, \quad (3.11)$$

and

$$I_7 \equiv \text{tr}[D_8 \text{adj} U_8] = \widehat{I}_7 - \frac{I_2 \widehat{I}_3}{3}, \quad I_9 \equiv \text{tr}[U_8 \text{adj} D_8] = \widehat{I}_9 - \frac{I_1 \widehat{I}_4}{3}, \quad (3.12)$$

and

$$\begin{aligned} \widetilde{I}_7 &\equiv \text{tr}[U_8 S_8] = \widehat{I}_7 - I_2 \widehat{I}_3 + \frac{2I_1 \widehat{I}_5}{3}, \\ \widetilde{I}_9 &\equiv \text{tr}[D_8 S_8] = \widehat{I}_9 - I_1 \widehat{I}_4 + \frac{2I_2 \widehat{I}_5}{3}, \end{aligned} \quad (3.13)$$

Finally, one has

$$I_{10} \equiv \text{tr}[S_8 S_8] = \frac{\widehat{I}_3 \widehat{I}_4 - \widehat{I}_{10}}{2} + \frac{(\widehat{I}_5 - I_1 I_2) \widehat{I}_5}{6} + \frac{I_1 \widetilde{I}_9 + I_2 \widetilde{I}_7}{2}. \quad (3.14)$$

Further polynomially dependent invariants that appear in table 2 are given by

$$\widetilde{I}_{10} \equiv \text{tr}[A_8^2] = I_{10} + 2\widehat{I}_{10} - \frac{2\widehat{I}_5^2}{3}, \quad (3.15)$$

and

$$P_1 \equiv \text{tr}[\text{adj} U_8 \text{adj} U_8] = -2I_1 \widehat{I}_6 + \frac{2\widehat{I}_3^2}{3}, \quad P_4 \equiv \text{tr}[\text{adj} D_8 \text{adj} D_8] = -2I_2 \widehat{I}_8 + \frac{2\widehat{I}_4^2}{3}, \quad (3.16)$$

and

$$P_2 \equiv \text{tr}[\text{adj} U_8 \text{adj} D_8] = \widehat{I}_{10} - \frac{\widehat{I}_3 \widehat{I}_4}{3}, \quad (3.17)$$

and

$$P_3 \equiv \text{tr}[\text{adj} U_8 S_8] = I_2 \widehat{I}_6 - \frac{\widehat{I}_3 \widehat{I}_5}{3}, \quad P_5 \equiv \text{tr}[\text{adj} D_8 S_8] = I_1 \widehat{I}_8 - \frac{\widehat{I}_4 \widehat{I}_5}{3}. \quad (3.18)$$

Furthermore,

$$\begin{aligned} Q_1 &\equiv \text{tr}[A_8 A_U] = -\frac{I_1 (\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{2} + \frac{3I_4 \widehat{I}_6}{4} - \frac{\widehat{I}_5 \widehat{I}_7}{2} - \frac{\widehat{I}_3 \widehat{I}_9}{2}, \\ Q_2 &\equiv \text{tr}[A_8 A_D] = +\frac{I_2 (\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{2} - \frac{3I_3 \widehat{I}_8}{4} + \frac{\widehat{I}_5 \widehat{I}_9}{2} + \frac{\widehat{I}_4 \widehat{I}_7}{2}, \end{aligned} \quad (3.19)$$

and

$$\begin{aligned} Q_3 &\equiv \text{tr}[A_U A_U] = \frac{\widehat{I}_3 (\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{2} + \frac{\widehat{I}_6 (3I_9 - I_1 I_4 + I_2 I_5)}{2} + \frac{(I_2 \widehat{I}_3 - \widehat{I}_7) \widehat{I}_7}{2}, \\ Q_5 &\equiv \text{tr}[A_D A_D] = \frac{\widehat{I}_4 (\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{2} + \frac{\widehat{I}_8 (3I_7 - I_2 I_3 + I_1 I_5)}{2} + \frac{(I_1 \widehat{I}_4 - \widehat{I}_9) \widehat{I}_9}{2} \end{aligned} \quad (3.20)$$

and

$$Q_4 \equiv \text{tr}[A_U A_D] = \frac{(\widehat{I}_5 - I_1 I_2)(\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{4} + \frac{(\widehat{I}_7 - I_2 \widehat{I}_3)(\widehat{I}_9 - I_1 \widehat{I}_4)}{4} - \frac{(\widehat{I}_6 - I_1 \widehat{I}_3)(\widehat{I}_8 - I_2 \widehat{I}_4)}{4} - 2\widehat{I}_6 \widehat{I}_8. \quad (3.21)$$

In order to project onto the eight basis matrices one needs the inverse of the metric in table 2. The explicit result is rather lengthy and not very instructive, and we therefore refrain from quoting it here. We checked however that the metric is not singular for generic Yukawa entries.

3.2 One-loop RG equations

In the 3G case, again, any flavour matrix that arises as a flavour-covariant product of SM Yukawa matrices Y_U and Y_D can be written as a linear combination of a finite set of basic matrices (constructed from Y_U and Y_D) with coefficients given as *polynomials* of a finite number of flavour invariants (as a corollary to the discussion in [11]). The most general form of the RG equations then can be written as

$$\begin{aligned} \frac{dY_U}{d \ln \mu} &= (a_0 \mathbb{1} + a_1 U_8 + a_2 D_8 + a_3 \text{adj } U_8 + a_4 \text{adj } D_8 + a_5 S_8 + ia_6 A_8 \\ &\quad + ia_7 A_U + ia_8 A_D) Y_U(\mu), \\ \frac{dY_D}{d \ln \mu} &= (b_0 \mathbb{1} + b_1 D_8 + b_2 U_8 + b_3 \text{adj } D_8 + b_4 \text{adj } U_8 + b_5 S_8 - ib_6 A_8 \\ &\quad - ib_7 A_D - ib_8 A_U) Y_D(\mu), \end{aligned} \quad (3.22)$$

As compared to the 2G case, the expressions for the RG equations of the flavour invariants derived from this general parametrization become rather lengthy. (The explicit structure of the two-loop expressions can be found in (C.1) in the appendix.)

3.2.1 General form

If we restrict ourselves to the RG equations at one-loop accuracy, we can write

$$\begin{aligned} \frac{dY_U}{d \ln \mu} &= (a_0 \mathbb{1} + a_1 U_8 + a_2 D_8 + \dots) Y_U(\mu), \\ \frac{dY_D}{d \ln \mu} &= (b_0 \mathbb{1} + b_1 D_8 + b_2 U_8 + \dots) Y_D(\mu), \end{aligned} \quad (3.23)$$

where the coefficients a_0, b_0 are first-order polynomials of flavour invariants, and $a_{1,2}, b_{1,2}$ are constant, see again table 1. The RG equations for the quadratic invariants then take the same form as in the 2G-case,

$$\frac{dI_1}{dt} \simeq 2a_0 I_1 + 2a_1 I_3 + 2a_2 I_5, \quad \frac{dI_2}{dt} \simeq 2b_0 I_2 + 2b_1 I_4 + 2b_2 I_5. \quad (3.24)$$

We remind the reader of the difference between the hatted and unhatted invariants, as defined in section 3.1. For the remaining unmixed invariants, we also find simple expressions

$$\frac{d\widehat{I}_3}{dt} \simeq 4a_0 \widehat{I}_3 - 2a_1 I_6 - 2a_2 I_7, \quad \frac{d\widehat{I}_4}{dt} \simeq 4b_0 \widehat{I}_4 - 2b_1 I_8 - 2b_2 I_9, \quad (3.25)$$

and

$$\frac{d\widehat{I}_6}{dt} = 6a_0 \widehat{I}_6, \quad \frac{d\widehat{I}_8}{dt} = 6b_0 \widehat{I}_8. \quad (3.26)$$

Notice that the last two relations — with our convention in (3.22) where the coefficients $a_{i>0}, b_{i>0}$ always multiply *traceless* matrices — are exact. The one-loop RG equations for the mixed invariants are determined as

$$\frac{d\widehat{I}_5}{dt} \simeq (2a_0 + 2b_0) \widehat{I}_5 + (2a_1 + 2b_2) \widetilde{I}_7 + (2a_2 + 2b_1) \widetilde{I}_9, \quad (3.27)$$

together with

$$\begin{aligned} \frac{d\widehat{I}_7}{dt} &= (4a_0 + 2b_0) \widehat{I}_7 + (2a_1 - 2b_2) \left(\frac{I_1 \widehat{I}_7}{3} - I_2 \widehat{I}_6 \right) \\ &\quad - (2a_2 - 2b_1) \left(\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4 + \frac{2I_2 \widehat{I}_7}{3} \right), \end{aligned} \quad (3.28)$$

$$\begin{aligned} \frac{d\widehat{I}_9}{dt} &= (4b_0 + 2a_0) \widehat{I}_9 + (2b_1 - 2a_2) \left(\frac{I_2 \widehat{I}_9}{3} - I_1 \widehat{I}_8 \right) \\ &\quad - (2b_2 - 2a_1) \left(\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4 + \frac{2I_1 \widehat{I}_9}{3} \right), \end{aligned} \quad (3.29)$$

and

$$\frac{d\widehat{I}_{10}}{dt} = (4a_0 + 4b_0) \widehat{I}_{10} + (2a_1 + 2b_2) \left(\frac{I_1 \widehat{I}_{10}}{3} - \widehat{I}_4 \widehat{I}_6 \right) + (2a_2 + 2b_1) \left(\frac{I_2 \widehat{I}_{10}}{3} - \widehat{I}_3 \widehat{I}_8 \right). \quad (3.30)$$

The RG equations for the Jarlskog invariant is simple, and to one-loop accuracy reads

$$\frac{dI_{11}^-}{dt} = (6a_0 + 6b_0 + 2a_1 I_1 + 2b_1 I_2) I_{11}^-. \quad (3.31)$$

3.2.2 Exploiting flavour hierarchies

The RG equations again simplify when one exploits flavour hierarchies in the SM Yukawa matrices, which are also applicable to MFV extensions of the SM. For concreteness, we relate the scaling of the quark Yukawa couplings to the Wolfenstein parameter λ in the CKM matrix, as it can be realized in Froggatt-Nielsen models [31] (see also [15, 20]), assuming

$$V_{12} \sim \lambda, \quad V_{23} \sim \lambda^2, \quad V_{13} \sim \lambda^3, \quad (3.32)$$

and

$$y_t \sim \lambda^0, \quad y_b \sim \lambda^2, \quad y_c \sim \lambda^3, \quad y_s \sim \lambda^6, \quad y_{u,d} \sim \lambda^8. \quad (3.33)$$

Defining $\epsilon = \lambda^2$, the individual invariants scale as (see appendix B)

$$\begin{aligned} I_1 &\simeq y_t^2 \sim \epsilon^0, & \widehat{I}_3 &\simeq y_t^2 y_c^2 \sim \epsilon^3, & \widehat{I}_6 &= y_t^2 y_c^2 y_u^2 \sim \epsilon^{11}, \\ I_2 &\simeq y_b^2 \sim \epsilon^2, & \widehat{I}_4 &\simeq y_b^2 y_s^2 \sim \epsilon^8, & \widehat{I}_8 &= y_b^2 y_s^2 y_d^2 \sim \epsilon^{16}, \end{aligned} \quad (3.34)$$

and

$$\begin{aligned} \widehat{I}_5 &\simeq y_t^2 y_b^2 |V_{tb}|^2 = I_1 I_2 + \mathcal{O}(\epsilon^4), & \widehat{I}_7 &\simeq y_t^2 y_b^2 y_c^2 |V_{ub}|^2 \sim \epsilon^8, \\ \widehat{I}_{10} &\simeq y_t^2 y_b^2 y_c^2 y_s^2 |V_{ud}|^2 = \widehat{I}_3 \widehat{I}_4 + \mathcal{O}(\epsilon^{12}), & \widehat{I}_9 &\simeq y_t^2 y_b^2 y_s^2 |V_{td}|^2 \sim \epsilon^{11}, \end{aligned} \quad (3.35)$$

and

$$I_{11}^- \sim \epsilon^{16}. \quad (3.36)$$

The leading terms in the (one-loop) RG equations are then identified as⁴

$$\begin{aligned} \frac{dI_1}{dt} &\simeq \left(2a_0 + \frac{4a_1}{3} I_1\right) I_1, & \frac{dI_2}{dt} &\simeq \left(2b_0 + \frac{4b_2}{3} I_1\right) I_2, \\ \frac{d\widehat{I}_3}{dt} &\simeq \left(4a_0 + \frac{2a_1}{3} I_1\right) \widehat{I}_3, & \frac{d\widehat{I}_4}{dt} &\simeq \left(4b_0 + \frac{2b_2}{3} I_1\right) \widehat{I}_4, \end{aligned} \quad (3.37)$$

and

$$\frac{d(\widehat{I}_5 - I_1 I_2)}{dt} \simeq \left(2a_0 + 2b_0 + \frac{4a_1}{3} I_1 - \frac{2b_2}{3} I_1\right) (\widehat{I}_5 - I_1 I_2), \quad (3.38)$$

and

$$\begin{aligned} \frac{d\widehat{I}_6}{dt} &= 6a_0 \widehat{I}_6, & \frac{d\widehat{I}_7}{dt} &\simeq \left(4a_0 + 2b_0 + \frac{2a_1 - 2b_2}{3} I_1\right) \widehat{I}_7, \\ \frac{d\widehat{I}_8}{dt} &= 6b_0 \widehat{I}_8, & \frac{d\widehat{I}_9}{dt} &\simeq \left(4b_0 + 2a_0 + \frac{4a_1 - 4b_2}{3} I_1\right) \widehat{I}_9, \end{aligned} \quad (3.39)$$

and

$$\frac{d(\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{dt} \simeq \left(4a_0 + 4b_0 + \frac{2a_1 + 2b_2}{3} I_1\right) (\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4). \quad (3.40)$$

together with

$$\frac{d\widehat{I}_{11}^-}{dt} \simeq (6a_0 + 6b_0 + 2a_1 I_1) I_{11}^-. \quad (3.41)$$

As in the 2G example, only the coefficients a_0, b_0, a_1, b_2 in (3.23) are needed in this approximation. Solving for the latter, one obtains

$$\begin{aligned} a_0 dt &= -\frac{dI_1}{6I_1} + \frac{d\widehat{I}_3}{3\widehat{I}_3}, & a_1 dt &\simeq \frac{1}{2I_1} \left(2 \frac{dI_1}{I_1} - \frac{d\widehat{I}_3}{\widehat{I}_3}\right), \\ b_0 dt &= -\frac{dI_2}{6I_2} + \frac{d\widehat{I}_4}{3\widehat{I}_4}, & b_2 dt &\simeq \frac{1}{2I_1} \left(2 \frac{dI_2}{I_2} - \frac{d\widehat{I}_4}{\widehat{I}_4}\right). \end{aligned} \quad (3.42)$$

This leaves 7 relations that can be used to identify RG-invariant combinations of flavour invariants,

$$\frac{d\widehat{I}_6}{\widehat{I}_6} + \frac{dI_1}{I_1} = 2 \frac{d\widehat{I}_3}{\widehat{I}_3}, \quad \frac{d\widehat{I}_8}{\widehat{I}_8} + \frac{dI_2}{I_2} \simeq 2 \frac{d\widehat{I}_4}{\widehat{I}_4}, \quad (3.43)$$

⁴We do not include the power-counting for the gauge-coupling constants here as has been advocated in [24].

and

$$\frac{d(\widehat{I}_5 - I_1 I_2)}{\widehat{I}_5 - I_1 I_2} + \frac{dI_2}{I_2} \simeq \frac{dI_1}{I_1} + \frac{d\widehat{I}_4}{\widehat{I}_4}, \quad (3.44)$$

and

$$\frac{d\widehat{I}_7}{\widehat{I}_7} + \frac{dI_2}{I_2} \simeq \frac{d\widehat{I}_3}{\widehat{I}_3} + \frac{d\widehat{I}_4}{\widehat{I}_4}, \quad \frac{d\widehat{I}_9}{\widehat{I}_9} + 2 \frac{dI_2}{I_2} \simeq \frac{dI_1}{I_1} + 2 \frac{d\widehat{I}_4}{\widehat{I}_4}, \quad (3.45)$$

and

$$\frac{d(\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4)}{\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4} \simeq \frac{d\widehat{I}_3}{\widehat{I}_3} + \frac{d\widehat{I}_4}{\widehat{I}_4}, \quad \frac{dI_{11}^-}{I_{11}^-} + \frac{dI_2}{I_2} \simeq \frac{dI_1}{I_1} + \frac{d\widehat{I}_3}{\widehat{I}_3} + 2 \frac{d\widehat{I}_4}{\widehat{I}_4}. \quad (3.46)$$

Here each of the invariants is to be read as a function of $(I_1, I_2, \widehat{I}_3, \widehat{I}_4)$. As in the 2G case, the relations can be easily integrated, resulting in

$$\frac{I_1 \widehat{I}_6}{(\widehat{I}_3)^2} \simeq \text{const.} \simeq \frac{y_u^2}{y_c^2} \sim \epsilon^5, \quad \frac{I_2 \widehat{I}_8}{(\widehat{I}_4)^2} \simeq \text{const.} \simeq \frac{y_d^2}{y_s^2} \sim \epsilon^2, \quad (3.47)$$

and

$$\frac{I_1 \widehat{I}_4}{I_2 (\widehat{I}_5 - I_1 I_2)} \simeq \text{const.} \simeq \frac{y_s^2}{y_b^2 |V_{cb}|^2} \sim \epsilon^2, \quad (3.48)$$

and

$$\frac{\widehat{I}_3 \widehat{I}_4}{I_2 \widehat{I}_7} \simeq \text{const.} \simeq \frac{y_s^2}{y_b^2 |V_{ub}|^2} \sim \epsilon, \quad \frac{I_1 \widehat{I}_4^2}{I_2 \widehat{I}_9} \simeq \text{const.} \simeq \frac{y_s^2}{y_b^2 |V_{td}|^2} \sim \epsilon, \quad (3.49)$$

and

$$\frac{\widehat{I}_{10} - \widehat{I}_3 \widehat{I}_4}{\widehat{I}_3 \widehat{I}_4} \simeq \text{const.} \simeq |V_{us}|^2 \sim \epsilon, \quad \frac{I_1 \widehat{I}_3 \widehat{I}_4}{I_2 I_{11}^-} \simeq \text{const.} \simeq \frac{4y_s^2}{3y_b^2 \text{Im}[V_{ud} V_{ub}^* V_{td}^* V_{tb}]} \sim \epsilon. \quad (3.50)$$

This explicitly shows, how the known simplifications for the RG solutions of quark masses and mixing angles that arise in the limit of large top-quark Yukawa coupling (see also [32]) can be translated to the set of flavour invariants in a straightforward manner.

3.2.3 One-loop solutions in the SM

As in the 2G-case, we can derive explicit solutions to the RG equations, using the one-loop expressions for the coefficients in table 1 and the approximations from the Yukawa hierarchies discussed in the previous paragraph. With our definitions of flavour invariants, the approximate RG equations for the invariants $I_{1,2}$ and $\widehat{I}_{3,4}$ looks identical to the 2G case in (2.30), (2.34), (2.35). As a consequence, we can again express the running of the 11 invariants in terms of the RG function $G(\mu, \mu_0)$ defined in (2.33) from the evolution of the leading invariant

$$I_1(\mu) \simeq \eta^{8/\beta_0} I_1(\mu_0) G(\mu, \mu_0), \quad \eta = \frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}. \quad (3.51)$$

Defining normalized invariants as before (using a slightly different notation), we have

$$x_2(\mu) \equiv \frac{I_2(\mu)}{I_1(\mu)} \simeq [G(\mu, \mu_0)]^{-2/3} x_2(\mu_0),$$

$$\begin{aligned}
 x_3(\mu) &\equiv \frac{\widehat{I}_3(\mu)}{(I_1(\mu))^2} \simeq [G(\mu, \mu_0)]^{-1/3} x_3(\mu_0), \\
 x_4(\mu) &\equiv \frac{\widehat{I}_4(\mu)}{(I_2(\mu))^2} \simeq [G(\mu, \mu_0)]^{+1/3} x_4(\mu_0).
 \end{aligned}
 \tag{3.52}$$

and the remaining scaling relations follow from (3.47)–(3.50). In this way, we recover the results for the approximate RG running of CKM mixing angles as discussed in [6].

Comparison with Harrison et al. In a paper by Harrison et al. [33] it has been highlighted that, within the SM, the one-loop RG equations exhibit two combinations of flavour invariants that are stable with respect to RG flow,

$$\frac{d}{dt} \left(\frac{\text{tr}[UD]}{(\det[UD])^{1/3}} \right) = \frac{d}{dt} \frac{\widehat{I}_5}{(\widehat{I}_6 \widehat{I}_8)^{1/3}} = 0 \quad (\text{SM@1-loop}), \tag{3.53}$$

$$\frac{d}{dt} \left(\text{tr} [(UD)^{-1}] (\det[UD])^{1/3} \right) = \frac{d}{dt} \frac{\widehat{I}_{10}}{(\widehat{I}_6 \widehat{I}_8)^{2/3}} = 0 \quad (\text{SM@1-loop}). \tag{3.54}$$

In our notation, we have

$$\frac{d}{dt} \frac{\widehat{I}_5}{(\widehat{I}_6 \widehat{I}_8)^{1/3}} = \frac{2(a_1 + b_2) \widetilde{I}_7 + 2(a_2 + b_1) \widetilde{I}_9}{(\det[UD])^{1/3}}, \tag{3.55}$$

$$\frac{d}{dt} \frac{\widehat{I}_{10}}{(\widehat{I}_6 \widehat{I}_8)^{2/3}} = \frac{2(a_1 + b_2) (I_1 \widehat{I}_{10} - 3 \widehat{I}_4 \widehat{I}_6) + 2(a_2 + b_1) (I_2 \widehat{I}_{10} - 3 \widehat{I}_3 \widehat{I}_8)}{3 (\det[UD])^{2/3}}. \tag{3.56}$$

This indeed vanishes for $a_1 = -b_2 = -a_2 = b_1$ which holds within the SM, see table 1.

4 Summary and outlook

From the experimental as well as from the theoretical side (see e.g. the reviews in [34–38]), the quark flavour physics program is currently entering the precision era. The goal is to find hints to physics beyond the Standard Model (SM) from dedicated experiments, notably LHCb and BELLE II. Still, the answer to the flavour puzzle itself may reside at extremely high scales, possibly as high as the Planck scale. In any case, the determination of flavour observables occurs at low energies, and thus for any comparison with “new physics” models one needs to include the renormalization-group (RG) running of the flavour parameters in a given theoretical framework. In principle, there are various roads to discuss this. On the one hand, one can consider the entries of the 3×3 Yukawa matrices and study their RG evolution; but these depend on an arbitrary choice of basis in flavour space. Alternatively, one can use the physical parameters, i.e. the six quark masses together with four independent CKM parameters to describe quark mixing and CP violation in weak interactions; but these have rather complicated relations to the Yukawa couplings.

In this paper, we have chosen an intermediate point of view and considered simple combinations of Yukawa couplings that are independent of the orientation of the flavour basis. In terms of these flavour invariants we have formulated RG equations which are basis

independent and allow for a transparent implementation of flavour hierarchies as observed in the SM or its minimal-flavour-violating (MFV) extensions. Expanding systematically in small parameters, we have also constructed simple analytic solutions for the RG evolution of a set of polynomially independent flavour invariants.

Discussing the RG flow in terms of flavour invariants may be advantageous to discuss models with dynamical flavour symmetry breaking, where the Yukawa couplings emerge as vacuum expectation values (VEVs) of some scalar flavon fields. The scalar potential generating these VEVs will be constructed in terms of polynomials of flavour invariants of a given canonical mass dimension. In MFV-like constructions (see e.g. [39]), these can be reduced to the set of invariants discussed in this work. More complicated situations arise if one implements the spontaneous breaking of a gauged flavour symmetry on the level of renormalizable interactions. This leads to an “inverted-MFV” scenario, where the fundamental flavour invariants are approximately given as polynomials of the inverse Yukawa matrices [40]. Even more complicated relations can arise in a recently proposed model with dynamical flavour-symmetry breaking with a unification scheme according to Pati and Salam [41]. While the general form of the RG equations (3.22) will remain the same, the coefficients will have a more complicated dependence than in MFV scenarios. In any of these cases, the renormalization-group flow of the invariants is needed to constrain the theoretical NP parameters at a high scale from flavour observables at low scales, and eventually give us some clue on the solution of the flavour problem.

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A Cayley-Hamilton identities

A.1 Two-generation case

The Cayley-Hamilton identity for 2×2 matrices M reads

$$0 = M^2 - \text{tr}[M] M + \det M \mathbb{1} . \tag{A.1}$$

Taking the trace and solving for $\det M$, one obtains

$$\det M = \frac{1}{2} (\text{tr}^2[M] - \text{tr}[M^2]) . \tag{A.2}$$

Multiplying (A.1) with M^{-1} , and solving for $\text{adj } M = M^{-1} \det M$, one obtains

$$\text{adj } M = \text{tr}[M] \mathbb{1} - M \quad \Rightarrow \quad \text{tr}[\text{adj } M] = \text{tr}[M] . \tag{A.3}$$

Inserted back into (A.1) yields

$$M^2 = \text{tr}[\text{adj } M] M - \det[M] \mathbb{1} , \tag{A.4}$$

with $\det[M]$ given by (A.2). For traceless matrices, this further simplifies to

$$M^2 = \frac{1}{2} \text{tr}[M^2] \mathbb{1} \quad (\text{tr}[M] = 0). \quad (\text{A.5})$$

Therefore any power of 2×2 matrices M can be reduced to the basis $\{\mathbb{1}, M\}$ with coefficients built from polynomials of $\text{tr}[M]$, $\text{tr}[M^2]$ which are invariant under *unitary* basis transformations.

For matrices Y which transform under *bi-unitary* transformations, eq. (A.3) generalizes to

$$\text{adj } Y = \frac{\det[Y]}{\det[M]} Y^\dagger (\text{tr}[M] \mathbb{1} - M) \quad (M \equiv YY^\dagger). \quad (\text{A.6})$$

A.2 Three-generation case

The Cayley-Hamilton identity for 3×3 matrices M reads

$$0 = M^3 - \text{tr}[M] M^2 + \frac{1}{2} (\text{tr}^2[M] - \text{tr}[M^2]) M - \det M \mathbb{1}. \quad (\text{A.7})$$

Taking the trace and solving for $\det M$, one obtains

$$\det M = \frac{1}{3} \left(\text{tr}[M^3] - \frac{3}{2} \text{tr}[M] \text{tr}[M^2] + \frac{1}{2} \text{tr}^3[M] \right). \quad (\text{A.8})$$

Multiplying (A.7) with M^{-1} , and solving for $\text{adj } M = M^{-1} \det M$, one obtains

$$\begin{aligned} \text{adj } M &= M^2 - \text{tr}[M] M + \frac{1}{2} (\text{tr}^2[M] - \text{tr}[M^2]) \mathbb{1} \\ \Rightarrow \quad \text{tr}[\text{adj } M] &= \frac{1}{2} (\text{tr}^2[M] - \text{tr}[M^2]). \end{aligned} \quad (\text{A.9})$$

Inserted back into (A.7) yields

$$M^3 = \text{tr}[M] M^2 - \text{tr}[\text{adj } M] M + \det[M] \mathbb{1}, \quad (\text{A.10})$$

with $\det[M]$ given by (A.8). For traceless matrices, this further simplifies to

$$M^3 = \frac{1}{2} \text{tr}[M^2] M + \frac{1}{3} \text{tr}[M^3] \mathbb{1} \quad (\text{tr}[M] = 0). \quad (\text{A.11})$$

Therefore any power of 3×3 matrices M can be reduced to the basis $\{\mathbb{1}, M, M^2\}$ with coefficients built from invariants that are polynomials of $\text{tr}[M]$, $\text{tr}[M^2]$, $\text{tr}[M^3]$.

Similarly as before, for matrices Y which transform under *bi-unitary* transformations, eq. (A.9) generalizes to

$$\text{adj } Y = \frac{\det[Y]}{\det[M]} Y^\dagger (\text{tr}[\text{adj } M] \mathbb{1} - \text{tr}[M] M + M^2) \quad (M \equiv YY^\dagger). \quad (\text{A.12})$$

B 3G flavour invariants, Yukawa couplings and CKM elements

For our convention to define 10+1 polynomially independent flavour invariants, the explicit expressions in terms of Yukawa couplings and mixing angles read as follows. The quadratic invariants are

$$I_1 \equiv \text{tr}[U] = \sum_{i=u,c,t} y_i^2, \quad I_2 \equiv \text{tr}[D] = \sum_{j=d,s,b} y_j^2. \quad (\text{B.1})$$

Again, I_1 and I_2 quantify the overall size of flavour-symmetry breaking in the up- and down-quark sector, respectively. Quartic invariants appear as

$$\widehat{I}_3 \equiv \text{tr}[\text{adj} U] = \sum_{i=u,c,t} \widetilde{y}_i^2, \quad \widehat{I}_4 \equiv \text{tr}[\text{adj} D] = \sum_{j=d,s,b} \widetilde{y}_j^2, \quad (\text{B.2})$$

and

$$\widehat{I}_5 \equiv \text{tr}[UD] = \sum_{i=u,c,t} \sum_{j=d,s,b} y_i^2 y_j^2 |V_{ij}|^2, \quad (\text{B.3})$$

where we have defined $\widetilde{y}_u^2 = y_c^2 y_t^2$, $\widetilde{y}_d^2 = y_s^2 y_b^2$ etc. Continuing with the sixth-order invariants, we have

$$\widehat{I}_6 \equiv \text{tr}[U \text{adj} U] = 3 \det U = 3 y_u^2 y_c^2 y_t^2, \quad \widehat{I}_8 \equiv \text{tr}[D \text{adj} D] = 3 \det D = 3 y_d^2 y_s^2 y_b^2, \quad (\text{B.4})$$

and

$$\begin{aligned} \widehat{I}_7 \equiv \text{tr}[\text{adj} U D] &= \sum_{i=u,c,t} \sum_{j=d,s,b} \widetilde{y}_i^2 y_j^2 |V_{ij}|^2, \\ \widehat{I}_9 \equiv \text{tr}[U \text{adj} D] &= \sum_{i=u,c,t} \sum_{j=d,s,b} y_i^2 \widetilde{y}_j^2 |V_{ij}|^2. \end{aligned} \quad (\text{B.5})$$

The eight-order invariant reads

$$\widehat{I}_{10} \equiv \text{tr}[\text{adj} U \text{adj} D] = \sum_{i=u,c,t} \sum_{j=d,s,b} \widetilde{y}_i^2 \widetilde{y}_j^2 |V_{ij}|^2. \quad (\text{B.6})$$

Finally, the CP-odd invariant

$$I_{11}^- \equiv \text{tr}[A_8^3] = \frac{3}{4} (y_t^2 - y_c^2)(y_t^2 - y_u^2)(y_c^2 - y_u^2)(y_b^2 - y_s^2)(y_b^2 - y_d^2)(y_s^2 - y_d^2) \text{Im} [V_{ud} V_{ub}^* V_{td}^* V_{tb}], \quad (\text{B.7})$$

is proportional to the Jarlskog determinant [30].

C Two-loop RG equations for 3G flavour invariants

The two-loop approximation for RG equations of the quark Yukawa matrices Y_U and Y_D in (3.22) is obtained by keeping factors that are at most quartic in the Yukawa couplings, i.e. neglecting the contributions with the flavour matrices $A_{U,D}$,

$$\frac{dY_U}{d \ln \mu} = (a_0 \mathbf{1} + a_1 U_8 + a_2 D_8 + a_3 \text{adj} U_8 + a_4 \text{adj} D_8 + a_5 S_8 + ia_6 A_8 + \dots) Y_U(\mu),$$

$$\frac{dY_D}{d \ln \mu} = (b_0 \mathbb{1} + b_1 D_8 + b_2 U_8 + b_3 \text{adj } D_8 + b_4 \text{adj } U_8 + b_5 S_8 - ib_6 A_8 + \dots) Y_D(\mu). \quad (\text{C.1})$$

From this ansatz, it is straightforward — though tedious — to calculate the two-loop RG equations for the eleven flavour invariants. First, we have

$$\begin{aligned} \frac{dI_1}{dt} &\simeq 2a_0 I_1 + 2a_1 I_3 + 2a_2 I_5 + 2a_3 I_6 + 2a_4 I_9 + 2a_5 \tilde{I}_7, \\ \frac{dI_2}{dt} &\simeq 2b_0 I_2 + 2b_1 I_4 + 2b_2 I_5 + 2b_3 I_8 + 2b_4 I_7 + 2b_5 \tilde{I}_9, \end{aligned} \quad (\text{C.2})$$

where here and in the following the abbreviations for the combinations of flavour invariants are the same as in table 2. Then

$$\begin{aligned} \frac{d\hat{I}_3}{dt} &\simeq 4a_0 \hat{I}_3 - 2a_1 I_6 - 2a_2 I_7 - 2a_3 P_1 - 2a_4 P_2 - 2a_5 P_3, \\ \frac{d\hat{I}_4}{dt} &\simeq 4b_0 \hat{I}_4 - 2b_1 I_8 - 2b_2 I_9 - 2b_3 P_4 - 2b_4 P_2 - 2b_5 P_5 \end{aligned} \quad (\text{C.3})$$

and

$$\frac{d\hat{I}_6}{dt} = 6a_0 \hat{I}_6, \quad \frac{d\hat{I}_8}{dt} = 6b_0 \hat{I}_8. \quad (\text{C.4})$$

For the mixed invariants, one obtains

$$\begin{aligned} \frac{d\hat{I}_5}{dt} &\simeq (2a_0 + 2b_0) \hat{I}_5 + (2a_1 + 2b_2) \tilde{I}_7 + (2a_2 + 2b_1) \tilde{I}_9 \\ &\quad + (2a_3 + 2b_4) P_3 + (2a_4 + 2b_3) P_5 + (2a_5 + 2b_5) I_{10} + (2a_6 + 2b_6) \tilde{I}_{10}, \end{aligned} \quad (\text{C.5})$$

and

$$\begin{aligned} \frac{d\hat{I}_7}{dt} &\simeq (4a_0 + 2b_0) \hat{I}_7 + (2a_1 - 2b_2) \left(\frac{I_1 \hat{I}_7}{3} - I_2 \hat{I}_6 \right) - (2a_2 - 2b_1) \left(\hat{I}_{10} - \hat{I}_3 \hat{I}_4 + \frac{2I_2 \hat{I}_7}{3} \right) \\ &\quad + (2a_3 - 2b_4) \left(2\hat{I}_5 \hat{I}_6 - 3I_5 \hat{I}_6 - \frac{2\hat{I}_3 \hat{I}_7}{3} \right) + (2a_4 - 2b_3) \left(\frac{\hat{I}_4 \hat{I}_7}{3} - \hat{I}_3 \hat{I}_8 \right) \\ &\quad + (2a_5 - 2b_5) \left((2\hat{I}_4 - I_2^2) \hat{I}_6 + \frac{\hat{I}_5 \hat{I}_7}{3} + Q_1 \right) + (2a_6 + 2b_6) Q_1, \\ \frac{d\hat{I}_9}{dt} &\simeq (2a_0 + 4b_0) \hat{I}_9 - (2a_2 - 2b_1) \left(\frac{I_2 \hat{I}_9}{3} - I_1 \hat{I}_8 \right) - (2b_2 - 2a_1) \left(\hat{I}_{10} - \hat{I}_3 \hat{I}_4 + \frac{2I_1 \hat{I}_9}{3} \right) \\ &\quad - (2a_4 - 2b_3) \left(2\hat{I}_5 \hat{I}_8 - 3I_5 \hat{I}_8 - \frac{2\hat{I}_4 \hat{I}_9}{3} \right) - (2a_3 - 2b_4) \left(\frac{\hat{I}_3 \hat{I}_9}{3} - \hat{I}_4 \hat{I}_6 \right) \\ &\quad - (2a_5 - 2b_5) \left((2\hat{I}_3 - I_1^2) \hat{I}_8 + \frac{\hat{I}_5 \hat{I}_9}{3} - Q_2 \right) - (2a_6 + 2b_6) Q_2, \end{aligned} \quad (\text{C.6})$$

and

$$\frac{d\hat{I}_{10}}{dt} \simeq (4a_0 + 4b_0) \hat{I}_{10} + (2a_1 + 2b_2) \left(\frac{I_1 \hat{I}_{10}}{3} - \hat{I}_4 \hat{I}_6 \right) + (2a_2 + 2b_1) \left(\frac{I_2 \hat{I}_{10}}{3} - \hat{I}_3 \hat{I}_8 \right)$$

$$\begin{aligned}
& - (2a_3 + 2b_4) \left(\widehat{I}_6 (\widehat{I}_9 - I_1 \widehat{I}_4) + \frac{2\widehat{I}_3 \widehat{I}_{10}}{3} \right) - (2a_4 + 2b_3) \left((\widehat{I}_7 - I_2 \widehat{I}_3) \widehat{I}_8 + \frac{2\widehat{I}_4 \widehat{I}_{10}}{3} \right) \\
& + (2a_5 + 2b_5) \left(\frac{\widehat{I}_5 \widehat{I}_{10}}{3} - 3\widehat{I}_6 \widehat{I}_8 - Q_4 \right) - (2a_6 + 2b_6) Q_4.
\end{aligned} \tag{C.7}$$

and

$$\begin{aligned}
\frac{dI_{11}^-}{dt} & \simeq \left(6a_0 + 6b_0 + 2a_1 I_1 + 2b_2 I_2 - 2a_4 \widehat{I}_3 - 2b_4 \widehat{I}_4 \right) I_{11}^- \\
& + \left((a_5 + b_5) \frac{\widehat{I}_5 + I_1 I_2}{2} - (a_6 + b_6) I_5 \right) I_{11}^-.
\end{aligned} \tag{C.8}$$

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