

Optimal Information Control in Cyber-Physical Systems

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Abstract: In this paper, we address optimal information control in cyber-physical systems. In particular, we study the optimal closed-loop policy for transmission of measurements of a stochastic dynamical system through a communication channel given estimation and communication costs. We develop a framework for optimizing an aggregate cost function that incorporates the estimation and the communication costs over a finite time horizon. We obtain the optimal closed-loop policy, and show that it can be expressed directly in terms of the value of information. In addition, we propose an approximation algorithm that yields a suboptimal closed-loop policy. Numerical and simulation results are presented for a simple system.

Keywords: Closed-Loop Policy, Estimation, Event-Driven Sampling, Value of Information

1. INTRODUCTION

Cyber-physical systems are tomorrow's systems in which cyber and physical components interact in all scales and levels. They deeply integrate computation, communication, and control into physical systems. In this paper, we address optimal information control in cyber-physical systems. Consider two agents Alice and Bob. Alice who has access to measurements of a stochastic dynamical system in the environment desires to inform optimally Bob whose task is to estimate the state of the system. However, the directed communication of Alice to Bob has a price due to the associated energy consumption, communication constraints, computational limits, and security issues. Therefore, Alice should devise a policy that transmits only data that is of *valuable information* to Bob. From an information theoretic perspective, Alice and Bob require a *source encoder* and a *source decoder*, respectively. Then, a challenging problem is to design an optimal encoding policy given estimation cost at the decoder and cost of the communication. This study has a broad range of applications including planetary exploration, environmental monitoring, wearable sensing, teleoperation, and many other examples of cyber-physical systems (Lee (2008)).

In this study, the encoder employs a sampler to control the information flow in the communication channel. *Nonuniform sampling* (Mark and Todd (1981)) and its important subclass *event-driven sampling* (Åström and Bernhardsson (2002)) in the context of the estimation problem have received early attention in the literature. Meier et al. (1967) extend the work of Kushner (1964) by looking at the measurement control problem subject to measurement cost and constraints, and by proposing dynamic programming (DP) and the gradient method as computational procedures. Åström and Bernhardsson (2002) show that

event-driven sampling can outperform periodic sampling with respect to the estimation error of a scalar linear system under a sampling rate constraint. Rabi et al. (2012) study optimal event-driven sampling as a stopping time problem for a scalar system under a finite transmission budget constraint. Molin and Hirche (2012) investigate the optimal design for event-driven sampling in a scalar system with communication cost by considering a two-player problem. Sijs and Lazar (2012) study the estimation problem in event-driven sampling taking into account the implied knowledge when no measurement is transmitted. Furthermore, in the last few years several sampling policies have been proposed including ones based on the error between the current measurement and the last transmitted measurement (Otanez et al. (2002); Miskowicz (2006)), on the measurement innovation (Wu et al. (2013)), on the covariance of the estimation error (Trimpe and D'Andrea (2014); Soleymani et al. (2016b); Soleymani et al. (2016a)), and on the Kullback-Leibler divergence between the prior and the posterior conditional distributions (Marck and Sijs (2010)).

In the sense of Witsenhausen (1971), and Bar-Shalom and Tse (1974), we classify sampling policies based on the information pattern of the problem under study into: *open-loop policies*, *feedback policies*, and *closed-loop policies*. All previous works on nonuniform sampling for estimation are based on either open-loop policies (Kushner (1964); Meier et al. (1967); Trimpe and D'Andrea (2014); Soleymani et al. (2016b); Soleymani et al. (2016a)) or feedback policies (Otanez et al. (2002); Miskowicz (2006); Rabi et al. (2012); Molin and Hirche (2012); Sijs and Lazar (2012); Wu et al. (2013)). In this paper, for the first time, we study the optimal closed-loop sampling policy. The information pattern in our problem is characterized as follows:

- (1) The encoder has access to imperfect information, i.e., the encoder cannot access the state of the process,
- (2) The estimator used at the decoder is causal, i.e., the estimator depends on past and present transmitted measurements,
- (3) The estimator used at the decoder neglects the implied knowledge when no measurement is transmitted,
- (4) The sampling policy is closed-loop, i.e., the policy takes into account past and present measurements and the statistics of measurements in the future.

We develop a framework for optimizing an aggregate cost function that incorporates the estimation and the communication costs over a finite time horizon given the aforementioned information pattern. We define the *value of information* (VOI) as “the maximum value that the encoder would be willing to pay for the transmission of a measurement”. We show that the optimal closed-loop policy can be expressed directly in terms of the value of information. In addition, we propose an approximation algorithm that yields a suboptimal closed-loop policy.

The outline of the paper is as follows. After an introduction on notations, the information control problem is formulated in Section 2. In Section 3, we obtain the optimal information control and propose an approximation algorithm. We illustrate numerical and simulation results in Section 4. Finally, concluding remarks are made in Section 5.

1.1 Notations

In this paper, we represent an n dimensional vector with $x = [x_1, \dots, x_n]^T$ where x_i is its i th component. We write x^T to denote the transpose of the vector x . The identity matrix with dimension n is denoted by I_n . We use C^\dagger to denote the Moore-Penrose inverse of the matrix C . We write $\delta_{kk'}$ to denote the Kronecker delta function. We write $p(x)$ to denote the probability distribution of the stochastic variable x . The expected value and the covariance of x are denoted by $\mathbb{E}[x]$ and $\text{Cov}[x]$, respectively. The normal distribution with mean μ and covariance Σ is denoted by $N(\mu, \Sigma)$. For matrices A and B , we write $A \succ 0$ and $B \succeq 0$ to mean that A and B are positive definite and positive semi-definite, respectively.

2. INFORMATION CONTROL PROBLEM

2.1 Dynamical System and Information Control

Consider a discrete-time dynamical system generated by the following linear state equation:

$$x_k = Fx_{k-1} + w_{k-1}, \quad (1)$$

$$y_k = Hx_k + v_k, \quad (2)$$

for $k = 1, 2, \dots$ where $x_k \in \mathbb{R}^n$ is the state of the system at time k , F is the state matrix, $w_k \in \mathbb{R}^n$ is a white noise sequence with zero mean and covariance $Q\delta_{kk'}$ where $Q \succ 0$, $y_k \in \mathbb{R}^p$ is the output of the system at time k , H is the output matrix, and $v_k \in \mathbb{R}^p$ is a white noise sequence with zero mean and covariance $R\delta_{kk'}$ where $R \succ 0$. It is assumed that the initial state x_0 is a Gaussian vector with zero mean and covariance P_0 , and that x_0 , w_k , and v_k are mutually independent.

Measurements of the system are available to a source encoder that samples measurements at times k_s for $s = 1, \dots, M$ where M is unknown. Samples are transmitted through a communication channel, and received by a source decoder. Through this study, the decoder assumes that measurements are never compromised.

Definition 1. (information control). The *information control* δ_k at time k is

$$\delta_k = \begin{cases} 1, & \text{if } \exists s : k = k_s, \\ 0, & \text{otherwise,} \end{cases} \quad (3)$$

where $\delta_0 = 0$.

A set of information controls $\pi = \{\delta_1, \dots, \delta_N\}$ is called an information control policy (or a sampling policy). In addition, a policy is closed-loop if it takes into account past and present measurements and the statistics of measurements in the future.

Definition 2. (encoder’s information set). The *encoder’s information set* is the σ -algebra generated by past and present measurements and past information controls, i.e.,

$$\mathcal{J}_k = \sigma\{y_l, \delta_{l-1} \mid l \leq k\}. \quad (4)$$

Remark 1. The information set \mathcal{J}_k is available at time k to the encoder for decision making, i.e., $\delta_k = \delta_k(\mathcal{J}_k)$.

Definition 3. (decoder’s information set). The *decoder’s information set* is the σ -algebra generated by measurements transmitted to the decoder, i.e.,

$$\mathcal{I}_k = \sigma\{y_l \mid l \leq k, \delta_l = 1\}. \quad (5)$$

Remark 2. The information set \mathcal{I}_k specifies the set of measurements available at time k for filtration at the decoder. Notice that the encoder can reconstruct the decoder’s information set \mathcal{I}_k from its information set \mathcal{J}_k , i.e., $\mathcal{I}_k \subset \mathcal{J}_k$.

The encoder’s information set is autonomous, i.e.,

$$\mathcal{J}_k = \sigma\{\mathcal{J}_{k-1}, y_k, \delta_{k-1}\}. \quad (6)$$

However, the decoder’s information set is a function of the information control, i.e., $\mathcal{I}_k = \mathcal{I}_k(\delta_k)$. In particular, we can write

$$\mathcal{I}_k(\delta_k) = \begin{cases} \sigma\{\mathcal{I}_{k-1}, y_k\}, & \text{if } \delta_k = 1, \\ \mathcal{I}_{k-1}, & \text{otherwise.} \end{cases} \quad (7)$$

2.2 Estimate Dynamics

Filtration at the decoder is based on the decoder’s information set $\mathcal{I}_k(\delta_k)$. We assume that the estimator neglects the implied knowledge when no measurement is transmitted. Therefore, the conditional distribution $p(x_k | \mathcal{I}_k(\delta_k))$ is a Gaussian distribution. Define

$$\hat{x}_k = \mathbb{E}[x_k | \mathcal{I}_k(\delta_k)], \quad (8)$$

$$P_k = \text{Cov}[x_k | \mathcal{I}_k(\delta_k)]. \quad (9)$$

The conditional distribution $N(\hat{x}_k, P_k)$ evolves in time due to the system dynamics, and is updated at times k_s due to measurements.

Consider the transformation $I_k = P_k^{-1}$ where I_k is the Fisher information matrix (FIM) (Guo et al. (2012)). Following the Kolmogorov forward equation (Åström (2006)), the estimate and the FIM in the interval $(k_{s-1}, k_s^-]$ are propagated as

$$\hat{x}_k = F\hat{x}_{k-1}, \quad k \in (k_{s-1}, k_s^-], \quad (10)$$

$$I_k = (FI_{k-1}^{-1}F^T + Q)^{-1}, \quad k \in (k_{s-1}, k_s^-], \quad (11)$$

where k_s^- denotes time k_s before the estimate and the FIM are updated. Following Bayes' rule (Åström (2006)), the estimate and the FIM at time k_s are updated as

$$\hat{x}_k = \hat{x}_{k^-} + K_k(y_k - H\hat{x}_{k^-}), \quad k = k_s, \quad (12)$$

$$I_k = I_{k^-} + H^T R^{-1} H, \quad k = k_s, \quad (13)$$

where $K_k = I_k^{-1} H^T R^{-1}$ is the gain of the filter.

We write the discrete-time switched dynamics of the estimate and the FIM in terms of the information control:

$$\hat{x}_k = F\hat{x}_{k-1} + K_k(y_k - HF\hat{x}_{k-1})\delta_k, \quad (14)$$

$$I_k = (FI_{k-1}^{-1}F^T + Q)^{-1} + H^T R^{-1} H \delta_k, \quad (15)$$

which are shortly expressed by $\hat{x}_k = \psi_k(\hat{x}_{k-1}, I_{k-1}, \delta_k)$ and $I_k = \phi_k(I_{k-1}, \delta_k)$.

2.3 Estimation Criterion and Communication Price

We use mean square error (MSE) to measure the distortion between the state of the system and its estimate at the decoder over the time horizon N :

$$J_\pi^e = \mathbb{E} \left[\sum_{k=0}^N (x_k - \hat{x}_k)^T (x_k - \hat{x}_k) \right]. \quad (16)$$

We assume that the communication price per measurement λ_k is a random variable given by a Gaussian sequence with mean μ_λ and variance σ_λ^2 . However, its value at the current time is known to the encoder. Then, the communication cost over the time horizon N is

$$J_\pi^c = \mathbb{E} \left[\sum_{k=1}^N \lambda_k \delta_k \right]. \quad (17)$$

We define the aggregate cost function as a convex combination of the estimation and the communication cost functions defined in (16), (17):

$$J_\pi = \mathbb{E} \left[\sum_{k=0}^N g_k(\hat{x}_k, \delta_k) \right] \quad (18)$$

where the stage cost $g_k(\hat{x}_k, \delta_k)$ at time k is

$$g_k(\hat{x}_k, \delta_k) = \theta e_k^T e_k + (1 - \theta) \lambda_k \delta_k \quad (19)$$

where $\theta \in [0, 1]$ and $e_k = x_k - \hat{x}_k$ is the estimation error at time k . Notice that the expectation in (18) is with respect to x_k, y_k, λ_k , and δ_k for all $k = 0, 1, \dots, N$.

Problem 1. Given the dynamical system defined by (1), (2), the information control defined by (3), and the filtration given by (14), (15), find the optimal closed-loop information control policy $\pi^* = \{\delta_1^*, \dots, \delta_N^*\}$ that minimizes the aggregate cost function J_π .

3. OPTIMAL INFORMATION CONTROL

3.1 Optimal Closed-Loop Policy

The optimal closed-loop information control policy π^* is the solution of the following finite horizon stochastic optimization problem:

$$\text{minimize} \quad \mathbb{E} \left[\sum_{k=0}^N g_k(\hat{x}_k, \delta_k) \right] \quad (20)$$

$$\begin{aligned} \text{subject to} \quad & x_k = Fx_{k-1} + w_{k-1}, \\ & y_k = Hx_k + v_k, \\ & \hat{x}_k = \psi_k(\hat{x}_{k-1}, I_{k-1}, \delta_k), \\ & I_k = \phi_k(I_{k-1}, \delta_k), \end{aligned}$$

with variables $x_k \in \mathbb{R}^n$, $y_k \in \mathbb{R}^p$, $\hat{x}_k \in \mathbb{R}^n$, $I_k \succ 0$, and $\delta_k \in \{0, 1\}$ for all $k = 1, 2, \dots, N$, and with initial conditions x_0, \hat{x}_0 , and I_0 .

Define the *value function* $V_k(\mathcal{J}_k)$ at time k as

$$V_k(\mathcal{J}_k) = \min_{\delta_k, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k}^N g_l(\hat{x}_l, \delta_l) \middle| \mathcal{J}_k \right]. \quad (21)$$

Lemma 1. The optimal closed-loop information control policy π^* in (20) is obtained by

$$\begin{aligned} \delta_k^*(\mathcal{J}_k) = \operatorname{argmin}_{\delta_k} \mathbb{E} \left[& g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, \delta_k), \delta_k) \right. \\ & \left. + V_{k+1}(\sigma\{\mathcal{J}_k, y_{k+1}, \delta_k\}) \middle| \mathcal{J}_k \right], \quad (22) \end{aligned}$$

subject to (1), (2), (14), (15), for $k = 1, 2, \dots, N$ with condition $V_{N+1} = 0$.

Proof. From the definition of the value function and expectation properties, we have

$$\begin{aligned} \mathbb{E} [V_k(\mathcal{J}_k)] &= \min_{\delta_k, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k}^N g_l(\hat{x}_l, \delta_l) \right] \\ &= \mathbb{E} \left[\min_{\delta_k, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k}^N g_l(\hat{x}_l, \delta_l) \middle| \mathcal{J}_k \right] \right]. \end{aligned}$$

Besides, $J_\pi = \mathbb{E}[V_0]$. Therefore, the policy minimizing the value function is the optimal closed-loop information control policy. Moreover, we can write the aggregate cost function as

$$J_\pi = \mathbb{E} \left[\sum_{l=0}^{k-1} g_l(\hat{x}_l, \delta_l) \right] + \mathbb{E} \left[\sum_{l=k}^N g_l(\hat{x}_l, \delta_l) \right]$$

where only the second term depends on the information control δ_k . Following the principle of optimality for imperfect information, we have

$$\begin{aligned} V_k(\mathcal{J}_k) &= \min_{\delta_k, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k}^N g_l(\hat{x}_l, \delta_l) \middle| \mathcal{J}_k \right] \\ &= \min_{\delta_k} \mathbb{E} \left[g_k(\hat{x}_k, \delta_k) + V_{k+1}(\mathcal{J}_{k+1}) \middle| \mathcal{J}_k \right] \end{aligned}$$

where $\hat{x}_k = \psi_k(\hat{x}_{k-1}, I_{k-1}, \delta_k)$ and $\mathcal{J}_{k+1} = \sigma\{\mathcal{J}_k, y_{k+1}, \delta_k\}$. Hence, the optimal information control $\delta_k^*(\mathcal{J}_k)$ at time k is the argument of the above minimum. \blacksquare

Transmission of a measurement can decrease the estimation error e_k . However, there is a price λ_k associated with a transmission. Therefore, the encoder should at each time k decide whether it is worthwhile to transmit a measurement or not. This leads us to the following definition.

Definition 4. (value of information). The *value of information* (VOI) is the maximum value that the encoder would be willing to pay for the transmission of a measurement y_k at time k , i.e.,

$$\alpha_k(\mathcal{J}_k) = \mathbb{E} \left[g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 0), 0) - g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 1), 0) + V_{k+1}(\sigma\{\mathcal{J}_k, y_{k+1}, 0\}) - V_{k+1}(\sigma\{\mathcal{J}_k, y_{k+1}, 1\}) \middle| \mathcal{J}_k \right]. \quad (23)$$

Proposition 1. Let $\alpha_k(\mathcal{J}_k)$ be the value of information at time k . The information control policy

$$\delta_k^*(\mathcal{J}_k) = \begin{cases} 1, & \text{if } \alpha_k(\mathcal{J}_k) \geq (1-\theta)\lambda_k, \\ 0, & \text{otherwise,} \end{cases} \quad (24)$$

is optimal and closed-loop.

Proof. In the view of Lemma 1, the optimal information control $\delta_k^*(\mathcal{J}_k)$ at time k is 1 if

$$\mathbb{E} \left[g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 0), 0) + V_{k+1}(\sigma\{\mathcal{J}_k, y_{k+1}, 0\}) \middle| \mathcal{J}_k \right] - \mathbb{E} \left[g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 1), 1) + V_{k+1}(\sigma\{\mathcal{J}_k, y_{k+1}, 1\}) \middle| \mathcal{J}_k \right] \geq 0,$$

and is 0 otherwise. We can write $g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 1), 1) = g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 1), 0) + (1-\theta)\lambda_k$. Substituting this in the above equation, we obtain (24). ■

In order to use the optimal closed-loop information control policy π^* proposed in (24), we need to compute at time k the expected value of the stage cost and the expected value of the optimal cost-to-go conditioned on the encoder's information set \mathcal{J}_k .

Let us define the innovation at the encoder at time k as

$$\nu_k = y_k - HF\hat{x}_{k-1}. \quad (25)$$

We will show that the conditional expected value of the stage cost at time k is a function of the FIM I_{k-1} , the innovation ν_k , and the information control δ_k .

Theorem 1. Assume that the output matrix H has full rank. The expected value of the stage cost at time k conditioned on \mathcal{J}_k is

$$\begin{aligned} & \mathbb{E} [g(\hat{x}_k, \delta_k) | \mathcal{J}_k] \\ &= (1-\theta)\lambda_k\delta_k + \theta \left(\bar{\nu}_k^T \bar{H} \bar{\nu}_k - 2\bar{\nu}_k^T \bar{H} R \Gamma_k^{-1} \nu_k + \text{tr}(\bar{H}R - \bar{H}R\Gamma_k^{-1}R + \bar{H}R\Gamma_k^{-1}\nu_k\nu_k^T\Gamma_k^{-1}R) \right) \end{aligned} \quad (26)$$

where

$$\begin{aligned} \bar{\nu}_k &= (I_p - HK_k\delta_k)\nu_k, \\ \bar{H} &= H^\dagger{}^T H^\dagger, \\ \Gamma_k &= H^T \phi(I_{k-1}, 0)^{-1} H + R. \end{aligned}$$

Proof. The expected value of the stage cost at time k conditioned on \mathcal{J}_k can be written as

$$\begin{aligned} \mathbb{E} [g(\hat{x}_k, \delta_k) | \mathcal{J}_k] &= \mathbb{E} [\theta e_k^T e_k + (1-\theta)\lambda_k\delta_k | \mathcal{J}_k] \\ &= \theta \mathbb{E} [e_k^T e_k | \mathcal{J}_k] + (1-\theta)\lambda_k\delta_k \end{aligned} \quad (27)$$

where e_k is a function of δ_k . For the error e_k , we can write

$$\begin{aligned} He_k &= Hx_k - H\hat{x}_k = y_k - H\hat{x}_k - v_k \\ &= y_k - HF\hat{x}_{k-1} - HK_k(y_k - HF\hat{x}_{k-1})\delta_k - v_k \\ &= (I_p - HK_k\delta_k)\nu_k - v_k = \bar{\nu}_k - v_k. \end{aligned} \quad (28)$$

Employing the Moore-Penrose inverse, we have

$$e_k = H^\dagger(\bar{\nu}_k - v_k). \quad (29)$$

Since H has full rank, H^\dagger exists. The inner product of the error by itself using the definition of \bar{H} yields

$$\begin{aligned} e_k^T e_k &= (\bar{\nu}_k - v_k)^T \bar{H}(\bar{\nu}_k - v_k) \\ &= \bar{\nu}_k^T \bar{H} \bar{\nu}_k - 2\bar{\nu}_k^T \bar{H} v_k + v_k^T \bar{H} v_k. \end{aligned} \quad (30)$$

For the measurement noise v_k and the measurement y_k at time k , it is easy to show that

$$\mathbb{E} [v_k, y_k | \mathcal{J}_{k-1}] = \begin{bmatrix} 0 \\ HF\hat{x}_{k-1} \end{bmatrix}, \quad (31)$$

$$\text{Cov} [v_k, y_k | \mathcal{J}_{k-1}] = \begin{bmatrix} R & R \\ R & H^T \phi_k(I_{k-1}, 0)^{-1} H + R \end{bmatrix}. \quad (32)$$

Then, we can obtain the mean and the covariance of the measurement noise v_k conditioned on $\mathcal{J}_k = \sigma\{\mathcal{J}_{k-1}, y_k, \delta_{k-1}\}$, i.e., we have

$$\begin{aligned} \mathbb{E}[v_k | \mathcal{J}_k] &= R\Gamma_k^{-1}(y_k - HF\hat{x}_{k-1}) \\ &= R\Gamma_k^{-1}\nu_k, \end{aligned} \quad (33)$$

$$\text{Cov}[v_k | \mathcal{J}_k] = R - R\Gamma_k^{-1}R. \quad (34)$$

Furthermore, from trace and expectation properties, we can write

$$\begin{aligned} \mathbb{E}[v_k^T \bar{H} v_k | \mathcal{J}_k] &= \mathbb{E} [\text{tr}(v_k^T \bar{H} v_k) | \mathcal{J}_k] \\ &= \mathbb{E} [\text{tr}(\bar{H} v_k v_k^T) | \mathcal{J}_k] = \text{tr}(\bar{H} \mathbb{E}[v_k v_k^T | \mathcal{J}_k]) \\ &= \text{tr}(\bar{H} \text{Cov}[v_k | \mathcal{J}_k] + \bar{H} \mathbb{E}[v_k | \mathcal{J}_k] \mathbb{E}[v_k | \mathcal{J}_k]^T). \end{aligned} \quad (35)$$

By taking the expectation of (30) conditioned on \mathcal{J}_k and using (33), (34), (35), and the fact that $\bar{\nu}_k$ is \mathcal{J}_k -measurable, we have

$$\begin{aligned} & \mathbb{E}[e_k^T e_k | \mathcal{J}_k] \\ &= \bar{\nu}_k^T \bar{H} \bar{\nu}_k - 2\bar{\nu}_k^T \bar{H} \mathbb{E}[v_k | \mathcal{J}_k] + \mathbb{E}[v_k^T \bar{H} v_k | \mathcal{J}_k] \\ &= \bar{\nu}_k^T \bar{H} \bar{\nu}_k - 2\bar{\nu}_k^T \bar{H} \mathbb{E}[v_k | \mathcal{J}_k] \\ &\quad + \text{tr}(\bar{H} \text{Cov}[v_k | \mathcal{J}_k] + \bar{H} \mathbb{E}[v_k | \mathcal{J}_k] \mathbb{E}[v_k | \mathcal{J}_k]^T) \\ &= \bar{\nu}_k^T \bar{H} \bar{\nu}_k - 2\bar{\nu}_k^T \bar{H} R \Gamma_k^{-1} \nu_k \\ &\quad + \text{tr}(\bar{H}R - \bar{H}R\Gamma_k^{-1}R + \bar{H}R\Gamma_k^{-1}\nu_k\nu_k^T\Gamma_k^{-1}R). \end{aligned} \quad (36)$$

This completes the proof. ■

The expected value of the cost-to-go at time k is a function of future measurements averaged out conditioned on the information set \mathcal{J}_k . Next, we will show that the expected value of the optimal cost-to-go at each time can be obtained by a deterministic optimization problem.

Theorem 2. Assume that past and present measurements do not affect the cost-to-go in the future. The expected value of the optimal cost-to-go starting from $k+1$ conditioned on \mathcal{J}_k is

$$\begin{aligned} & \mathbb{E} [V_{k+1}(\mathcal{J}_{k+1}) | \mathcal{J}_k] \\ &= \min_{\delta_{k+1}, \dots, \delta_N} \sum_{l=k+1}^N \theta \text{tr} \phi_l^{-1}(I_{l-1}, \delta_l) + (1-\theta)\mu\lambda\delta_l. \end{aligned} \quad (37)$$

Proof. From the definition of the value function, expectation properties, and assumption that past and present measurements do not affect the cost-to-go, we have

$$\begin{aligned} & \mathbb{E} [V_{k+1}(\mathcal{J}_{k+1}) | \mathcal{J}_k] \\ &= \mathbb{E} \left[\min_{\delta_{k+1}, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k+1}^N g_l(\hat{x}_l, \delta_l) \middle| \mathcal{J}_{k+1} \right] \middle| \mathcal{J}_k \right] \\ &= \mathbb{E} \left[\min_{\delta_{k+1}, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k+1}^N g_l(\hat{x}_l, \delta_l) \middle| \mathcal{J}_{k+1} \right] \right] \\ &= \min_{\delta_{k+1}, \dots, \delta_N} \mathbb{E} \left[\sum_{l=k+1}^N g_l(\hat{x}_l, \delta_l) \right]. \end{aligned} \quad (38)$$

Next, we will find the first term of the summation:

$$\begin{aligned} & \mathbb{E} [g_{k+1}(\hat{x}_{k+1}, \delta_{k+1})] \\ &= \mathbb{E} [\theta e_{k+1}^T e_{k+1} + (1 - \theta) \lambda_{k+1} \delta_{k+1}] \\ &= \theta \mathbb{E} [e_{k+1}^T e_{k+1}] + (1 - \theta) \mu_\lambda \delta_{k+1}. \end{aligned} \quad (39)$$

However,

$$e_{k+1} = x_{k+1} - \hat{x}_{k+1}. \quad (40)$$

Hence, the mean and the covariance of e_{k+1} are obtained as

$$\mathbb{E}[e_{k+1}] = 0, \quad (41)$$

$$\text{Cov}[e_{k+1}] = \phi_{k+1}^{-1}(I_k, \delta_{k+1}). \quad (42)$$

Therefore,

$$\begin{aligned} & \mathbb{E} [e_{k+1}^T e_{k+1}] \\ &= \text{tr} (\text{Cov}[e_{k+1}] + \mathbb{E}[e_{k+1}] \mathbb{E}[e_{k+1}]^T) \\ &= \text{tr} \phi_{k+1}^{-1}(I_k, \delta_{k+1}). \end{aligned} \quad (43)$$

The rest of the terms can be derived analogously. This completes the proof. \blacksquare

Remark 3. One can calculate the optimal cost-to-go in (37) using a deterministic DP algorithm.

3.2 Suboptimal Closed-Loop Policy

The DP calculation becomes intractable when the time space and the state space are large (known as curse of dimensionality). However, we can make a trade-off between the convenient implementation and adequate performance. Let $L_{k+1}(I_{k-1}, \delta_k)$ be an approximation of the conditional expected value of the optimal cost-to-go starting from $k+1$ which is in fact a function of the FIM I_{k-1} and the information control δ_k . The approximate value of information is defined as

$$\begin{aligned} \tilde{\alpha}_k(\mathcal{J}_k) &= L_{k+1}(I_{k-1}, 0) - L_{k+1}(I_{k-1}, 1) \\ &+ \mathbb{E} \left[g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 0), 0) - g_k(\psi_k(\hat{x}_{k-1}, I_{k-1}, 1), 0) \middle| \mathcal{J}_k \right]. \end{aligned} \quad (44)$$

Then, a suboptimal closed-loop information control policy $\pi^+ = \{\delta_1^+, \dots, \delta_N^+\}$ is obtained by using (44) in (24).

Now, consider the discrete-time switched dynamics of the FIM using the Euler method (Soleymani et al. (2016b)):

$$\begin{aligned} I_k &= -\ln(F)^T I_{k-1} - I_{k-1} \ln(F) + I_{k-1} \\ &- \epsilon I_{k-1} \bar{Q} I_{k-1} + H^T R^{-1} H \delta_k + \mathcal{O}(\epsilon^2), \end{aligned} \quad (45)$$

where $\ln(F)$ is the matrix logarithm of F , \bar{Q} is the covariance of the continuous-time white noise process calculable using the Van Loan method (Van Loan (1977)), and ϵ is the time step-size. An approximation of the FIM dynamics is $I_k = \tilde{\phi}_k(I_{k-1}, \delta_k)$ where $\tilde{\phi}_k(I_{k-1}, \delta_k)$ is obtained when $\mathcal{O}(\epsilon^2)$ is discarded in (45).

In the following, we use a semi-definite programming (SDP) relaxation (Vandenberghe and Boyd (1996)) to find a lower bound which can be used as $L_{k+1}(I_{k-1}, \delta_k)$.

Theorem 3. A lower bound on

$$\min_{\delta_{k+1}, \dots, \delta_N} \sum_{l=k+1}^N \theta \text{tr} I_l^{-1} + (1 - \theta) \mu_\lambda \delta_l \quad (46)$$

subject to $I_l = \tilde{\phi}_l(I_{l-1}, \delta_l)$ is given by the optimal value of the following SDP:

$$\begin{aligned} & \text{minimize} \quad \sum_{l=1}^{N-k} \theta \text{tr} I_l^{-1} + (1 - \theta) \mu_\lambda \delta_l \\ & \text{subject to} \quad I_l = -\ln(F)^T I_{l-1} - I_{l-1} \ln(F) + I_{l-1} \\ & \quad \quad \quad - \epsilon D_{l-1} + H^T R^{-1} H \delta_l, \\ & \quad \quad \quad \begin{bmatrix} D_l & I_l \\ I_l & \bar{Q}^{-1} \end{bmatrix} \succeq 0, \end{aligned} \quad (47)$$

with variables $I_l \succ 0$, $D_l \succ 0$, and $\delta_l \in [0, 1]$ for all $l = 1, 2, \dots, N - k$, and with initial conditions $I_0 = I_k$ and $D_0 = I_k \bar{Q} I_k$.

Proof. First note that the functions $\text{tr} I_l^{-1}$ and $\mu_\lambda \delta_l$ are convex in I_l and δ_l given that $I_l \succ 0$. We relax the integrality constraint to $\delta_l \in [0, 1]$. Then, we aim at relaxing the quadratic term in the approximate FIM dynamics. Since this equality constraint does not influence the set of admissible control δ_l , such a relaxation does not violate the feasibility of the problem. By defining $D_l = I_l \bar{Q} I_l$, we have

$$\begin{aligned} I_l &= -\ln(F)^T I_{l-1} - I_{l-1} \ln(F) + I_{l-1} \\ &- \epsilon D_{l-1} + H^T R^{-1} H \delta_l \end{aligned} \quad (48)$$

which is a linear equation. We substitute the new constraint $D_l = I_l \bar{Q} I_l$ with its convex relaxation, i.e., $D_l - I_l \bar{Q} I_l \succeq 0$. This nonlinear convex inequality can be converted into an LMI by using the Schur complement:

$$\begin{bmatrix} D_l & I_l \\ I_l & \bar{Q}^{-1} \end{bmatrix} \succeq 0. \quad (49)$$

Using the new constraints (48), (49), and $\delta_l \in [0, 1]$, and following the convexity of the function in (46) we obtain the relaxed problem (47) which yields a lower bound on the function in (46). \blacksquare

Remark 4. Our approach produces a tractable approximation. Evaluating how accurate it is is a problem we will investigate elsewhere.

4. ILLUSTRATIVE EXAMPLES

In this section, we present numerical and simulation results for a system with the following discrete-time dynamics:

$$\begin{aligned} x_k &= 0.99x_{k-1} + w_{k-1} \\ y_k &= x_k + v_k \end{aligned}$$

with initial conditions $x_0 = 0$, $\hat{x}_0 = 0$, and $I_0 = 1$, and covariances $Q = 0.0495$, and $R = 1$. The time horizon is $N = 100$, the communication price per measurement is $\lambda_k = 0.2$, $\theta = 0.5$, and $\epsilon = 0.1$. We used the proposed VOI-based policy together with the approximation algorithm, and obtained a suboptimal closed-loop information control policy. The state, the estimate, and the covariance trajectories along with the sampling schedule for the VOI-based policy are illustrated in Fig. 1 and Fig. 2 for a realization of the system. The results are compared with a periodic policy that transmits every other time step. For the VOI-based and the periodic policies, the average stage costs are 0.1731 and 0.2718, and the total numbers of transmissions are $M = 51$ and $M = 50$, respectively.

5. CONCLUSION

In this work, we developed a framework for optimal information control in cyber-physical systems. We obtained

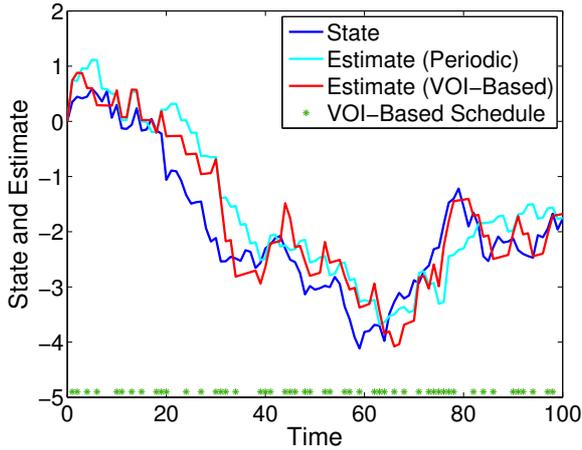


Fig. 1. Trajectories of the state and the estimates (periodic and VOI-based), and the VOI-based schedule.

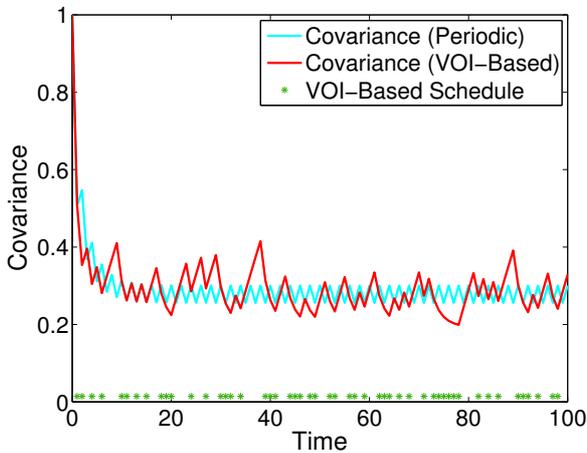


Fig. 2. Trajectories of the covariances (periodic and VOI-based), and the VOI-based schedule.

the optimal closed-loop policy, and showed that it can be expressed directly in terms of the value of information. Then, we showed that the value of information at each time consists of a feedback term and a feedforward term. The former is a function of the innovation, while the later can be calculated via DP. In addition, we propose an approximation algorithm based on an SDP relaxation that yields a suboptimal closed-loop policy.

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