

Constant mean curvature surfaces in hyperbolic 3-space via loop groups

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Dedicated to the memory of Hongyou Wu

Abstract. In hyperbolic 3-space \mathbb{H}^3 surfaces of constant mean curvature H come in three types, corresponding to the cases $0 \leq H < 1$, $H = 1$, $H > 1$. Via the Lawson correspondence the latter two cases correspond to constant mean curvature surfaces in Euclidean 3-space \mathbb{E}^3 with $H = 0$ and $H \neq 0$, respectively. These surface classes have been investigated intensively in the literature. For the case $0 \leq H < 1$ there is no Lawson correspondence in Euclidean space and there are relatively few publications. Examples have been difficult to construct. In this paper we present a generalized Weierstrass type representation for surfaces of constant mean curvature in \mathbb{H}^3 with particular emphasis on the case of mean curvature $0 \leq H < 1$. In particular, the generalized Weierstrass type representation presented in this paper enables us to construct simultaneously minimal surfaces ($H = 0$) and non-minimal constant mean curvature surfaces ($0 < H < 1$).

Introduction

Harmonic maps into (semi-simple) Riemannian symmetric spaces can be studied by infinite dimensional Lie group theory (*loop groups*). The dressing action of the loop group enables one to construct “non-trivial solutions” from a trivial solution and, meromorphically, new solutions from old/known ones. This is based on a Weierstrass type representation for harmonic maps which was established by the first named author of the present paper, F. Pedit and H. Wu [22] in terms of loop group decompositions.

As an application of the loop group method, constant mean curvature surfaces in Euclidean 3-space \mathbb{E}^3 , pseudo-spherical surfaces in \mathbb{E}^3 and their indefinite analogues (spacelike or timelike surfaces in Minkowski 3-space) are studied extensively [19, 28]. The loop group method to construct such *integrable surfaces* is frequently called *DPW method* or *generalized Weierstrass type representation* for surface geometry.

The second named author is partially supported by Kakenhi 18540068, 21546067. The third named author is partially supported by Kakenhi 20740045.

In Euclidean geometry, the starting point of the DPW method is that integrable surfaces are naturally associated with harmonic maps into the 2-sphere \mathbb{S}^2 with respect to appropriate conformal structures. More precisely, a surface in \mathbb{E}^3 is of constant mean curvature if and only if its Gauss map is harmonic. This characterization of constancy of mean curvature is referred to as the *Ruh–Vilms property*. On the other hand, let $f : M \rightarrow \mathbb{E}^3$ be a surface in \mathbb{E}^3 with negative or positive Gaussian curvature. Then the Gaussian curvature is constant if and only if its Gauss map is harmonic with respect to the conformal structure determined by the second fundamental form. This is another instance of the Ruh–Vilms property. Analogously, in Minkowski 3-space the corresponding integrable surfaces are associated with harmonic maps into hyperbolic 2-space \mathbb{H}^2 or de Sitter 2-space $\mathbb{S}^{1,1}$.

In Möbius geometry, a conformally immersed surface in Möbius 3-space \mathcal{M}^3 is Möbius minimal (Willmore) if and only if its conformal Gauss map (central sphere congruence) is a harmonic map into the moduli space of oriented 2-spheres. Note that via the projective light-cone model of the Möbius 3-space, the moduli space of all oriented 2-spheres (including point spheres) is identified with the de Sitter 4-space $\mathbb{S}^{1,3}$, a semi-simple Lorentzian symmetric space [6].

F. E. Burstall and U. Hertrich-Jeromin [10] proved the Ruh–Vilms property for certain surfaces in Lie sphere geometry as well as projective differential geometry. More precisely, let $f : M \rightarrow \mathcal{M}^3$ be a surface in Möbius space. Then M is said to be a *Lie minimal surface* if M is a critical point of the Lie-area functional. Note that the Lie-area functional is invariant under the Möbius group as well as the Lie sphere transformation group. On the other hand, for every surface in real projective 3-space $\mathbb{R}P^3$, a projectively invariant area functional (projective area) has been introduced. A critical point of the projective area is called a *projective minimal surface*. Both the Lie-minimality and projective minimality are characterized by the harmonicity of appropriate Gauss maps taking values in a certain non-compact Grassmann manifold equipped with an invariant indefinite semi-Riemannian metric.

In addition, E. Musso and L. Nicolodi [37] showed that in Laguerre geometry, Laguerre-minimality of surfaces is characterized by the harmonicity of the Laguerre Gauss map (*middle sphere congruence*). Thus all these surfaces classes are defined by the Ruh–Vilms property.

Here we should remark that all the target manifolds of *harmonic Gauss maps* in Lie-sphere, Möbius, projective and Laguerre geometries are semi-simple semi-Riemannian *symmetric spaces* equipped with invariant indefinite metrics.

It would be interesting to characterize all surface geometries to which the DPW method is applicable. So far only partial results are known: See, e.g., the survey [16] by the first named author and the recent classification, carried out by the third named author, of surfaces in 3-dimensional space forms which are *real forms* of complex constant mean curvature surfaces [33].

In this article, we give a *new* example of such a surface geometry. More precisely we shall show that the DPW method is also applicable to surfaces in hyperbolic 3-space \mathbb{H}^3 of constant mean curvature $0 \leq H < 1$.

The geometry of a CMC surface in \mathbb{H}^3 depends on the range of the mean curvature. In case $H > 1$, there exist compact CMC surfaces in \mathbb{H}^3 . It is well known that there exist natural, locally bijective correspondences (so-called *Lawson correspondences*) between CMC surfaces in 3-dimensional space forms. In particular, CMC surfaces in \mathbb{H}^3 with $H \geq 1$ have corresponding CMC or minimal surfaces (*Lawson correspondents*) in Euclidean 3-space or the 3-sphere. Based on the Lawson correspondences, M. Kilian, W. Rossman, N. Schmitt and the third named author [42] studied CMC surfaces in \mathbb{H}^3 with $H > 1$ via the DPW method. In

case $H = 1$, the Lawson correspondent in \mathbb{E}^3 is a minimal surface, and R. Bryant [7] gave the Weierstrass type representation in terms of a holomorphic differential equation.

On the other hand, by the maximum principle, there are no compact CMC surfaces in \mathbb{H}^3 (without boundary) such that $0 \leq H \leq 1$. Moreover, CMC surfaces in \mathbb{H}^3 with $0 \leq H < 1$ have neither Lawson correspondents in Euclidean 3-space \mathbb{E}^3 nor in the 3-sphere \mathbb{S}^3 . This means that CMC surface geometry with $0 \leq H < 1$ in hyperbolic space has special and unusual features and hence it is of interest.

There is another motivation for the study of CMC surfaces with $0 < H < 1$. D. Chopp and J. Velling [15] found numerical evidence that \mathbb{H}^3 can be foliated by CMC discs with $0 < H < 1$ that share a common Jordan curve boundary in the ideal boundary of \mathbb{H}^3 .

In contrast to the case $H \geq 1$, there are relatively few papers on CMC surfaces in \mathbb{H}^3 with $0 \leq H < 1$. Initial studies on the existence and regularity of CMC surfaces in \mathbb{H}^3 with $0 \leq H < 1$ can be found in M. Anderson [1], K. Uhlenbeck [45], K. Polthier [40], Y. Tonegawa [44] and J. Velling [47].

In [3], M. Babich and A. Bobenko considered minimal surfaces of finite type in \mathbb{H}^3 and described them explicitly in terms of theta functions.

One of the remarkable points of the present study is that – as in the already known cases – we associate a certain kind of *Gauss map* to constant mean curvature surfaces in \mathbb{H}^3 . And the constancy of the mean curvature is equivalent to the harmonicity of the Gauss map. Thus the Ruh–Vilms property holds. However, unlike all surface geometries discussed by the DPW method so far, our target space is *not* a symmetric space but a real 5-dimensional 4-symmetric space.

In general, the loop group approach can not be applied to harmonic maps into non-symmetric homogeneous spaces. Thus it is not obvious how to apply the loop group method to our case. More precisely, the loop group method has two main ingredients. One is a zero curvature representation and the other one is a loop group decomposition. The zero curvature representation is equivalent with the existence of a flat connections and this representation enables us to use loop groups. A loop group decomposition is at the heart of the DPW method and recovers surfaces from holomorphic potentials.

In this paper, the Gauss map of CMC surfaces in \mathbb{H}^3 has as target space a 4-symmetric space. Thus the DPW method is, *a priori*, not applicable. However it turns out that these Gauss maps are Legendre maps. In the situation under consideration this suffices to obtain a zero curvature representation for all Gauss maps of CMC surfaces in \mathbb{H}^3 (and in particular for mean curvature H satisfying $0 \leq H < 1$).

Applying the DPW method in this setting, we obtain a potential in a 4-graded Lie algebra. However, not every potential in this 4-graded Lie algebra will admit the application of the DPW method, since the admissible condition will not be satisfied in general [12].

It is a fortunate accident that the Maurer–Cartan form of the moving frame of every CMC immersion into \mathbb{H}^3 with $0 \leq H < 1$ is also contained in same real form of $\Lambda\mathrm{SL}_2\mathbb{C}_\sigma \times \Lambda\mathrm{SL}_2\mathbb{C}_\sigma$ and this is part of some “symmetric space setting” (which actually is not induced from any finite dimensional symmetric space). This enables us to apply the DPW method via the standard way and therefore two main features of the loop group method can be established and the loop group method is applicable.

The reason for the applicability of the DPW method in our case can also be explained by using the Lawson correspondence which transforms our original harmonic Gauss map into a *primitive* map, to which the DPW method is applicable. By the Lawson correspondence for a CMC surface with mean curvature $0 \leq H < 1$ in \mathbb{H}^3 , the CMC surface can be considered as

a minimal surface in $\mathbb{H}^3(c)$ for some $c < 0$, where $\mathbb{H}^3(c)$ denotes the hyperbolic 3-space of sectional curvature c . At this point it is important to note that the Gauss map of a CMC surface in $\mathbb{H}^3(c)$ is primitive if and only if the surface is minimal. Finally we recall that primitive maps into k -symmetric ($k > 2$) spaces form a special class of harmonic maps and it is known that the DPW method is applicable to primitive maps [21]. Moreover, we interpret our original Gauss map of a CMC surface in \mathbb{H}^3 , which is only a Legendre harmonic map, as a primitive Gauss map of some CMC surface in $\mathbb{H}^3(c)$, and thus the loop group method becomes applicable.

We would like to point out that the DPW method for CMC surfaces in \mathbb{H}^3 presented in [42] is based on the Lawson correspondence, so the authors of [42] can treat CMC surfaces in \mathbb{H}^3 with mean curvature $H > 1$. In the present study, we give a unified approach to both cases: $H > 1$ and $0 \leq H < 1$. The case $H = 1$ has been studied extensively already, e.g., in [7, 46], and will not be considered in this paper.

This paper is organized as follows. After establishing the requisite knowledge on harmonic maps into normal semi-Riemannian homogeneous spaces and homogeneous geometry of hyperbolic space in Sections 1 and 2, we shall devote Sections 3 and 4 to surface geometry in \mathbb{H}^3 in terms of $\text{SL}_2 \mathbb{C}$ -valued functions. In Section 5, we shall give a loop group formulation of CMC surfaces with $H > 1$. This formulation is different from the one used in [42]. We shall clarify the Lawson correspondences between CMC surfaces in \mathbb{H}^3 with $H > 1$ and CMC surfaces in \mathbb{E}^3 in terms of loop groups.

In Section 6, we shall give a loop group formulation for CMC surfaces with $0 \leq H < 1$. The two cases discussed in Sections 5 and 6 are distinguished by the automorphism of the Kac–Moody Lie algebra which characterizes the Lax equations of corresponding types of CMC surfaces.

The key tool of the present study is a “contact geometric characterization” of CMC surfaces. For every (oriented) surface $f : M \rightarrow \mathbb{H}^3$, there exists a smooth map F into the unit tangent sphere bundle $U\mathbb{H}^3$ of \mathbb{H}^3 . The map F is referred to as the *Gauss map* of M . One can see that the Gauss map F of a surface M is a Legendre map in the sense of V. I. Arnold [2], that is, it is tangent to the canonical contact structure. Moreover, the constancy of the mean curvature is equivalent to the harmonicity of that Gauss map. Thus the Ruh–Vilms property holds.

In Section 7, we shall characterize harmonic Gauss maps in a way which is different from the Ruh–Vilms property. The Legendre property of the Gauss map will be characterized in terms of 4-symmetric structure of $U\mathbb{H}^3$. These characterizations in terms of contact geometry and 4-symmetric structure yield a zero-curvature representation for Legendre harmonic maps. Based on these results, in Sections 8 and 9, we shall give a DPW method for Legendre harmonic maps (and hence for CMC surfaces with $0 \leq H < 1$). In the final Section 10, we shall exhibit some examples of CMC surfaces with $0 \leq H < 1$ via the DPW method established in this paper.

Acknowledgement. We would like to thank Idrisse Khemar and the referee for helpful comments on a preliminary version of this paper. Part of this work was carried out during the workshop “Surface Theory: Research in Pairs” at Kloster Schöntal, March 2008, funded by DFG Grant DO776. This work was started when the second named author visited the University of Kansas in 2000. He would like to express his sincere thanks to the Department of Mathematics. Some of the results of this article were reported at the workshop “Progress in Surface Theory” held at Mathematisches Forschungsinstitut Oberwolfach, May 2010.

1. Harmonic maps into normal semi-Riemannian homogeneous spaces

1.1. Let G/H be a reductive homogeneous space with semi-simple Lie group G . We equip G/H with a G -invariant semi-Riemannian metric which is derived from (a constant multiple of) the Killing form of G . Assume that the Lie algebra \mathfrak{h} of H is non-degenerate with respect to the induced scalar product. Then the orthogonal complement \mathfrak{p} of \mathfrak{h} is non-degenerate and can be identified with the tangent space of G/H at the origin $o = H$. The Lie algebra \mathfrak{g} is decomposed into the direct sum

$$\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$$

of linear subspaces. The resulting homogeneous semi-Riemannian manifold is a *normal semi-Riemannian homogeneous space* [14].

1.2. A smooth map $\psi : M \rightarrow N$ of a Riemann surface M into a semi-Riemannian manifold N is said to be a *harmonic map* if its tension field $\text{tr}(\nabla d\psi)$ vanishes. When the target space N is a normal semi-Riemannian homogeneous space G/H , the harmonic map equation for ψ has a particularly simple form. Here we recall a zero curvature representation for harmonic maps into normal semi-Riemannian homogeneous spaces due to Burstall and Pedit [12].

Now let $\psi : \mathbb{D} \rightarrow G/H$ be a smooth map from a simply connected domain $\mathbb{D} \subset \mathbb{C}$ into a normal semi-Riemannian homogeneous space. Take a frame $\Psi : \mathbb{D} \rightarrow G$ of ψ and put $\alpha := \Psi^{-1}d\Psi$. Then we have the identity (*Maurer–Cartan equation*)

$$d\alpha + \frac{1}{2}[\alpha \wedge \alpha] = 0.$$

Decompose α along the Lie algebra decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{p}$ as

$$\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{p}}, \quad \alpha_{\mathfrak{h}} \in \mathfrak{h}, \quad \alpha_{\mathfrak{p}} \in \mathfrak{p}.$$

We decompose $\alpha_{\mathfrak{p}}$ with respect to the conformal structure of \mathbb{D} as

$$\alpha_{\mathfrak{p}} = \alpha'_{\mathfrak{p}} + \alpha''_{\mathfrak{p}}.$$

Here $\alpha'_{\mathfrak{p}}$ and $\alpha''_{\mathfrak{p}}$ are the $(1, 0)$ and $(0, 1)$ part of $\alpha_{\mathfrak{p}}$, respectively.

The harmonicity of ψ is equivalent to

$$(1.1) \quad d(*\alpha_{\mathfrak{p}}) + [\alpha \wedge *\alpha_{\mathfrak{p}}] = 0.$$

Here $*$ denotes the Hodge star operator of \mathbb{D} .

Assume that

$$(1.2) \quad [\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = 0.$$

Then one can check that the harmonic map equation together with the Maurer–Cartan equation is equivalent to the following *zero-curvature representation*:

$$d\alpha_{\lambda} + \frac{1}{2}[\alpha_{\lambda} \wedge \alpha_{\lambda}] = 0,$$

where $\alpha_{\lambda} := \alpha_{\mathfrak{h}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_{\mathfrak{p}}$ with $\lambda \in \mathbb{S}^1$.

Proposition 1.1 ([12]). *Let \mathbb{D} be a region in \mathbb{C} and $\psi : \mathbb{D} \rightarrow G/H$ a harmonic map which satisfies the admissible condition (1.2). Then the loop of connections $d + \alpha_\lambda$ is flat for all λ . Namely*

$$(1.3) \quad d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all λ . Conversely assume that \mathbb{D} is simply connected. Let $\alpha_\lambda = \alpha_{\mathfrak{h}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_{\mathfrak{p}}$ be an \mathbb{S}^1 -family of \mathfrak{g} -valued 1-forms satisfying (1.3) for all $\lambda \in \mathbb{S}^1$. Then there exists a 1-parameter family of maps $\Psi_\lambda : \mathbb{D} \rightarrow G$ such that

$$\Psi_\lambda^{-1}d\Psi_\lambda = \alpha_\lambda \quad \text{and} \quad \psi_\lambda = \Psi_\lambda \bmod H : \mathbb{D} \rightarrow G/H$$

is harmonic for all λ .

When the target space G/H is a semi-Riemannian symmetric space, then the admissible condition is fulfilled automatically for any ψ , since $[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{h}$.

1.3. Primitive maps. Let G be a semi-simple Lie group with automorphism τ of order $k > 2$. A normal semi-Riemannian homogeneous space G/H is said to be a (regular) *semi-Riemannian k-symmetric space* if $G_\tau^\circ \subset H \subset G_\tau$. Here G_τ is the Lie subgroup of all fixed points of τ and G_τ° the identity component of it.

We denote the induced Lie algebra automorphism of \mathfrak{g} by the same letter τ . Now we have the eigenspace decomposition of the complexified Lie algebra $\mathfrak{g}^{\mathbb{C}}$,

$$\mathfrak{g}^{\mathbb{C}} = \sum_{j \in \mathbb{Z}_k} \mathfrak{g}_j^{\mathbb{C}}.$$

Here $\mathfrak{g}_j^{\mathbb{C}}$ is the eigenspace of τ with eigenvalue ζ^j . Here ζ is the primitive k -th root of unity. In particular, $\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{h}$ and $\overline{\mathfrak{g}_{-1}^{\mathbb{C}}} = \mathfrak{g}_1^{\mathbb{C}}$.

Definition ([12]). Let $\psi : M \rightarrow G/H$ be a smooth map of a Riemann surface into a regular semi-Riemannian k -symmetric space. Then ψ is said to be a *primitive map* if any frame Ψ has $\alpha'_{\mathfrak{p}}$ taking value in $\mathfrak{g}_{-1}^{\mathbb{C}}$.

One can see that every primitive map satisfies the admissible condition (1.2). In fact,

$$[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}] = [\alpha_{-1} \wedge \alpha_1] \in \mathfrak{g}_0^{\mathbb{C}} = \mathfrak{h}.$$

For more information on primitive maps, we refer to [12] and [21].

2. Hyperbolic 3-space

2.1. Let $\mathbb{E}^{1,3}$ be the 4-dimensional Minkowski space with scalar product:

$$\langle \mathbf{x}, \mathbf{y} \rangle = -x_0y_0 + x_1y_1 + x_2y_2 + x_3y_3.$$

We denote by $e_0 = (1, 0, 0, 0)$, $e_1 = (0, 1, 0, 0)$, $e_2 = (0, 0, 1, 0)$, $e_3 = (0, 0, 0, 1)$ the natural basis of $\mathbb{E}^{1,3}$. We will identify Euclidean 3-space $\mathbb{E}^3 = \{\mathbf{x} \in \mathbb{E}^{1,3} \mid x_0 = 0\}$. Let ϵ denote the *signature matrix* defined by $\epsilon = \text{diag}(-1, 1, 1, 1)$. Then we define the *Lorentz group* $O_{1,3}$ by

$$O_{1,3} = \{A \in \text{GL}_4 \mathbb{R} \mid A^t \epsilon A = \epsilon\}.$$

Note that $O_{1,3}$ acts isometrically on $\mathbb{E}^{1,3}$, and has four connected components. We denote by $SO_{1,3}^+$ the identity component of $O_{1,3}$. We equip the Lie algebra $\mathfrak{so}_{1,3}$ of $SO_{1,3}^+$ with an invariant scalar product

$$(2.1) \quad \langle X, Y \rangle = \frac{1}{2} \text{tr}(XY), \quad X, Y \in \mathfrak{so}_{1,3}.$$

Since $O_{1,3}$ is non-compact, this scalar product is indefinite. More precisely it has signature $(3, 3)$.

Remark. The Killing form φ of $\mathfrak{so}_{1,3}$ is

$$\varphi(X, Y) = -2 \text{tr}(XY), \quad X, Y \in \mathfrak{so}_{1,3}.$$

We equip the tangent space $T_{e_0} \mathbb{H}^3 \subset \mathfrak{so}_{1,3}$ with the inner product $\langle \cdot, \cdot \rangle$ given by

$$\langle X, Y \rangle = -\frac{1}{4} \varphi(X, Y) = \frac{1}{2} \text{tr}(XY).$$

2.2. The Minkowski 4-space $\mathbb{E}^{1,3}$ is identified with the space $\text{Her}_2 \mathbb{C}$ of all complex Hermitian 2×2 -matrices:

$$(2.2) \quad \mathbb{E}^{1,3} \cong \text{Her}_2 \mathbb{C} = \left\{ \mathbf{x} = \begin{pmatrix} x_0 + x_1 & x_3 - ix_2 \\ x_3 + ix_2 & x_0 - x_1 \end{pmatrix} \mid x_0, x_1, x_2, x_3 \in \mathbb{R} \right\}.$$

The space $\text{Her}_2 \mathbb{C}$ is spanned by the orthonormal basis

$$e_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad e_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad e_3 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is easy to see that $-\det \mathbf{x} = -x_0^2 + x_1^2 + x_2^2 + x_3^2$ for $\mathbf{x} \in \text{Her}_2 \mathbb{C}$. Thus the Lorentzian metric of $\text{Her}_2 \mathbb{C}$ is described as

$$(2.3) \quad \langle \mathbf{x}, \mathbf{y} \rangle = -\frac{1}{2} \text{tr}(\mathbf{x} e_2 \mathbf{y}^t e_2), \quad \mathbf{x}, \mathbf{y} \in \text{Her}_2 \mathbb{C}.$$

In particular, we have

$$\langle \mathbf{x}, \mathbf{x} \rangle = -\det \mathbf{x}, \quad \mathbf{x} \in \text{Her}_2 \mathbb{C}.$$

2.3. The hyperbolic 3-space \mathbb{H}^3 of curvature -1 is realized as a quadric

$$\mathbb{H}^3 = \{\mathbf{x} \in \mathbb{E}^{1,3} \mid \langle \mathbf{x}, \mathbf{x} \rangle = -1, x_0 > 0\}.$$

Under the identification (2.2), we have

$$\mathbb{H}^3 = \{\mathbf{x} \in \text{Her}_2 \mathbb{C} \mid \det \mathbf{x} = 1, \text{tr } \mathbf{x} > 0\}.$$

With respect to the metric induced from (2.3), \mathbb{H}^3 is of constant curvature -1 .

The special linear group $G = \mathrm{SL}_2 \mathbb{C}$ acts isometrically and transitively on the hyperbolic 3-space via the action

$$(2.4) \quad A : \mathrm{SL}_2 \mathbb{C} \times \mathbb{H}^3 \rightarrow \mathbb{H}^3, \quad (g, x) \mapsto gxg^*,$$

where g^* denotes \bar{g}^t . The isotropy subgroup of this action at e_0 is the special unitary group SU_2 . Hence \mathbb{H}^3 is represented by $\mathbb{H}^3 = G/K = \mathrm{SL}_2 \mathbb{C}/\mathrm{SU}_2$ as a Riemannian homogeneous space. The natural projection $\pi : G \rightarrow \mathbb{H}^3$ is given explicitly by $\pi(g) = gg^*$, $g \in G$. In other words, \mathbb{H}^3 is represented as

$$\mathbb{H}^3 = \{gg^* \mid g \in G\}.$$

Remark. It is important to note that in this context the simple Lie group $\mathrm{SL}_2 \mathbb{C}$ is regarded as a simple real Lie group and as a double covering of the special Lorentz group $\mathrm{SO}_{1,3}^+$. The real Lie algebra $\mathfrak{sl}_2 \mathbb{C}$ is spanned by the basis

$$(2.5) \quad \{ie_1, ie_2, ie_3, e_1, e_2, e_3\}.$$

The bi-invariant semi-Riemannian metric $\langle \cdot, \cdot \rangle$ on G corresponding to the scalar product (2.1) via the isomorphism (2.2) has the signature $(-, -, -, +, +, +)$. The tangent space $T_{e_0} \mathbb{H}^3$ is identified with

$$\mathfrak{m} = \mathfrak{sl}_2 \mathbb{C} \cap \mathrm{Her}_2 \mathbb{C} = \left\{ \begin{pmatrix} a & \bar{b} \\ b & -a \end{pmatrix} \mid a \in \mathbb{R}, b \in \mathbb{C} \right\} = \mathbb{R}e_1 \oplus \mathbb{R}e_2 \oplus \mathbb{R}e_3.$$

On $\mathfrak{m} = T_{e_0} \mathbb{H}^3$, the scalar product (2.1) is positive definite and coincides with the restriction of the scalar product (2.3) on \mathfrak{m} . Hence $\mathbb{H}^3 = \mathrm{SL}_2 \mathbb{C}/\mathrm{SU}_2$ is a normal Riemannian homogeneous space, especially, Riemannian symmetric space.

2.4. Let us denote by $\mathrm{U}\mathbb{H}^3$ the *unit tangent sphere bundle* of \mathbb{H}^3 . Namely, $\mathrm{U}\mathbb{H}^3$ is the manifold of all unit tangent vectors of \mathbb{H}^3 . Then $\mathrm{U}\mathbb{H}^3$ is identified with the submanifold

$$\{(x, v) \mid \langle x, x \rangle = -1, \langle v, v \rangle = 1, \langle x, v \rangle = 0, x_0 > 0\}$$

of $\mathbb{E}^{1,3} \times \mathbb{E}^{1,3}$. The tangent space $T_{(x,v)} \mathrm{U}\mathbb{H}^3$ at a point (x, v) is expressed as

$$T_{(x,v)} \mathrm{U}\mathbb{H}^3 = \{(X, V) \in \mathbb{E}^{1,3} \times \mathbb{E}^{1,3} \mid \langle x, X \rangle = 0, \langle v, V \rangle = 0, \langle x, V \rangle + \langle v, X \rangle = 0\}.$$

Define a 1-form ω on $\mathrm{U}\mathbb{H}^3$ by

$$\omega_{(x,v)}(X, V) = \langle X, v \rangle = -\langle x, V \rangle.$$

Then one can see that ω is a *contact form* on $\mathrm{U}\mathbb{H}^3$, i.e., $(d\omega)^2 \wedge \omega \neq 0$. The distribution

$$\mathcal{D}_{(x,v)} := \{(X, V) \in T_{(x,v)} \mathrm{U}\mathbb{H}^3 \mid \omega_{(x,v)}(X, V) = 0\}$$

is called the *canonical contact structure* of $\mathrm{U}\mathbb{H}^3$, see Section 7.2. Under the identification (2.2), $\mathrm{U}\mathbb{H}^3$ is represented as

$$\mathrm{U}\mathbb{H}^3 = \{(x, v) \in \mathrm{Her}_2 \mathbb{C} \times \mathrm{Her}_2 \mathbb{C} \mid \det x = 1, \mathrm{tr} x > 0, \det v = -1, \langle x, v \rangle = 0\}.$$

The special liner group $G = \mathrm{SL}_2 \mathbb{C}$ acts isometrically and transitively on $\mathrm{U}\mathbb{H}^3$ via the action

$$g \cdot (\mathbf{x}, \mathbf{v}) = (g\mathbf{x}g^*, g\mathbf{v}g^*).$$

The isotropy subgroup of G at $(\mathbf{e}_0, \mathbf{e}_1)$ is

$$H = \left\{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \right\} = \mathrm{U}_1.$$

Thus the unit tangent sphere bundle $\mathrm{U}\mathbb{H}^3$ is as homogeneous space $G/H = \mathrm{SL}_2 \mathbb{C}/\mathrm{U}_1$. The Lie algebra \mathfrak{h} of H is

$$(2.6) \quad \mathfrak{h} = \left\{ \begin{pmatrix} ia_2 & 0 \\ 0 & -ia_2 \end{pmatrix} \mid a_2 \in \mathbb{R} \right\} = \mathfrak{u}_1 = \mathbb{R}(i\mathbf{e}_1).$$

The tangent space $T_{(\mathbf{e}_0, \mathbf{e}_1)} \mathrm{U}\mathbb{H}^3$ is identified with the complement

$$(2.7) \quad \mathfrak{p} = \left\{ \begin{pmatrix} a_1 & b_1 + ib_2 \\ c_1 + ic_2 & -a_1 \end{pmatrix} \mid a_1, b_1, b_2, c_1, c_2 \in \mathbb{R} \right\}$$

of the isotropy algebra in $\mathfrak{sl}_2 \mathbb{C}$. The semi-Riemannian metric induced on the homogeneous space $\mathrm{U}\mathbb{H}^3 \cong \mathrm{SL}_2 \mathbb{C}/\mathrm{U}_1$ has signature $(2, 3)$. One can see that $\mathrm{U}\mathbb{H}^3$ is a normal semi-Riemannian homogeneous space (hence it is naturally reductive) but not a semi-Riemannian symmetric space.

2.5. Let $\pi_2 : \mathrm{U}\mathbb{H}^3 \rightarrow \mathbb{H}^3$ denote the natural projection. Then, we can see from the relations

$$\mathfrak{p} = \{\mathbb{R}\mathbf{e}_1 \oplus \mathbb{R}\mathbf{e}_2 \oplus \mathbb{R}\mathbf{e}_3\} \oplus \{\mathbb{R}(i\mathbf{e}_2) \oplus \mathbb{R}(i\mathbf{e}_3)\} = \mathfrak{m} \oplus \{\mathbb{R}(i\mathbf{e}_2) \oplus \mathbb{R}(i\mathbf{e}_3)\}$$

that the horizontal distribution \mathcal{H}_2 and the vertical distribution \mathcal{V}_2 with respect to the fibering $\pi_2 : \mathrm{U}\mathbb{H}^3 \rightarrow \mathbb{H}^3$ are generated by

$$\mathcal{H}_2(\mathbf{e}_0, \mathbf{e}_1) = \mathfrak{m}, \quad \mathcal{V}_2(\mathbf{e}_0, \mathbf{e}_1) = \mathbb{R}(i\mathbf{e}_2) \oplus \mathbb{R}(i\mathbf{e}_3).$$

Since the restriction of the semi-Riemannian metric of $\mathrm{U}\mathbb{H}^3$ to \mathcal{V}_2 is non-degenerate, the horizontal subspace of \mathcal{H}_2 can be defined by $\mathcal{H}_2 = \mathcal{V}_2^\perp$. Moreover, it is easy to see that $d\pi_2$ restricted to \mathcal{H}_2 preserves the scalar product. Thus π_2 is a semi-Riemannian submersion.

2.6. Next we consider $\mathrm{Geo}(\mathbb{H}^3)$ the space of all oriented geodesics in \mathbb{H}^3 . Let us denote by $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$ the Grassmann manifold of all oriented *timelike* planes in $\mathbb{E}^{1,3}$. Take a point $\gamma \in \mathrm{Geo}(\mathbb{H}^3)$, then γ is given by the intersection of \mathbb{H}^3 with a timelike plane $W \in \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$. By identifying γ with W , the space $\mathrm{Geo}(\mathbb{H}^3)$ is identified with the Grassmann manifold $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$. Hereafter we identify $\mathrm{Geo}(\mathbb{H}^3)$ with $\mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$ in this way.

There exists a natural projection $\pi_1 : \mathrm{U}\mathbb{H}^3 \rightarrow \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3})$:

$$\pi_1(\mathbf{x}, \mathbf{v}) = \mathbf{x} \wedge \mathbf{v}.$$

The Grassmann manifold $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ is a homogeneous space of $G = \text{SL}_2 \mathbb{C}$. The isotropy subgroup of G at $e_0 \wedge e_1$ is

$$D = \text{GL}_1 \mathbb{C} = \left\{ \begin{pmatrix} w & 0 \\ 0 & \frac{1}{w} \end{pmatrix} \mid w \in \mathbb{C}^\times \right\} \cong \mathbb{C}^\times.$$

The Lie algebra \mathfrak{d} of D is

$$(2.8) \quad \mathfrak{d} = \left\{ \begin{pmatrix} v & 0 \\ 0 & -v \end{pmatrix} \mid v \in \mathbb{C} \right\} = \mathbb{R}e_1 \oplus \mathbb{R}(ie_1).$$

The tangent space $T_{e_0 \wedge e_1} \text{Gr}_{1,1}(\mathbb{E}^{1,3})$ of $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ at $e_0 \wedge e_1$ is identified with

$$(2.9) \quad \mathfrak{q} = \mathbb{R}e_2 \oplus \mathbb{R}(ie_2) \oplus \mathbb{R}e_3 \oplus \mathbb{R}(ie_3).$$

The induced semi-Riemannian metric on $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ has signature $(2, 2)$. One can see that the resulting normal semi-Riemannian homogeneous space $\text{SL}_2 \mathbb{C} / \text{GL}_1 \mathbb{C}$ is a semi-Riemannian symmetric space.

The horizontal and vertical distributions with respect to π_1 are generated by

$$\mathcal{H}_1(e_0, e_1) = \mathfrak{q}, \quad \mathcal{V}_1(e_0, e_1) = \mathbb{R}e_1.$$

Since the restriction of the semi-Riemannian metric of $U\mathbb{H}^3$ to \mathcal{V}_1 is non-degenerate, the horizontal subspace of \mathcal{H}_1 can be defined by $\mathcal{H}_1 = \mathcal{V}_1^\perp$. Moreover, it is easy to see that $d\pi_1$ restricted to \mathcal{H}_1 preserves the scalar product. Thus π_1 is a semi-Riemannian submersion.

The contact form ω on $U\mathbb{H}^n$ induces a symplectic form Ω on the Grassmann manifold $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ so that $\pi_1^*\Omega = d\omega$.

The Grassmann manifold $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ admits an invariant product structure P , i.e., an endomorphism field P satisfying $P^2 = \text{Id}$, compatible with the metric. Moreover, P is parallel with respect to the Levi-Civita connection. The resulting homogeneous space $\text{SL}_2 \mathbb{C} / \text{GL}_1 \mathbb{C}$ is an indefinite para-Kähler symmetric space [31]. The fundamental 2-form of this para-Kähler symmetric space is compatible with the canonical contact form ω . More precisely the fundamental 2-form of $\text{Gr}_{1,1}(\mathbb{E}^{1,3})$ is -2Ω .

3. Surfaces in \mathbb{H}^3

3.1. Let us denote by $\mathcal{M}^3(c)$ the 3-dimensional simply connected Riemannian space form of curvature c .

In this subsection we recall two properties of CMC surfaces in space forms which play a fundamental role in this paper. The first one is the existence of 1-parameter isometric deformations of CMC surfaces. The second one is the so-called Lawson correspondences between CMC surfaces.

Let $f : M \rightarrow \mathcal{M}^3(c)$ be a conformal immersion of a Riemann surface M into the space form $\mathcal{M}^3(c)$ with unit normal vector field n . Clearly, by replacing, if necessary, the unit normal n of an immersion by $-n$, we can assume that the mean curvature satisfies $H \geq 0$.

Take a simply connected coordinate region $(\mathbb{D}, z) \subset M$, then the first fundamental form I is written as

$$(3.1) \quad I = e^u dz d\bar{z}.$$

The Gauss–Codazzi equations of f are given by

$$u_{z\bar{z}} + \frac{1}{2}(H^2 + c)e^u - 2|Q|^2e^{-u} = 0, \quad Q_{\bar{z}} = \frac{1}{2}H_z e^u.$$

The *Hopf differential* of f is defined by $Q dz^2$ with $Q = \langle f_{zz}, n \rangle$. The constancy of the mean curvature is characterized as follows.

Proposition 3.1. *Let M be a Riemann surface. A conformal immersion $f : M \rightarrow \mathbb{H}^3$ is of constant mean curvature if and only if its Hopf differential is holomorphic.*

Assume that H is constant, then the Gauss–Codazzi equations are invariant under the deformation

$$Q \mapsto \lambda^{-1}Q, \quad \lambda \in \mathbb{S}^1.$$

Hence, on the simply connected region \mathbb{D} , there exists a 1-parameter deformation family of conformal constant mean curvature immersions $\{f_\lambda\}$ through $f_1 = f$. All these immersions have the same induced metric and mean curvature. The family $\{f_\lambda\}$ is referred to as the *associated family* of the original immersion f .

The Gauss–Codazzi equations show the following fact:

Proposition 3.2. *Let $f : \mathbb{D} \rightarrow \mathcal{M}^3(c)$ be a simply connected open surface of constant mean curvature H . Take a pair (\tilde{c}, \tilde{H}) of real numbers such that $H^2 + c = \tilde{H}^2 + \tilde{c}$. Then there exists a conformal immersion $\tilde{f} : \mathbb{D} \rightarrow \mathcal{M}^3(\tilde{c})$ with constant mean curvature \tilde{H} whose induced metric is the original metric of (\mathbb{D}, f) .*

The correspondence $f \mapsto \tilde{f}$ is frequently called the *Lawson correspondence*. In particular, surfaces of constant mean curvature $H = \pm 1$ in \mathbb{H}^3 correspond to minimal surfaces in \mathbb{E}^3 .

Via the Lawson correspondences, one can construct constant mean curvature surfaces in \mathbb{S}^3 or \mathbb{H}^3 by the generalized Weierstrass type representation (DPW method) of CMC surfaces in \mathbb{E}^3 . For instance, in [42], some CMC surfaces in \mathbb{S}^3 or \mathbb{H}^3 are constructed.

However, surfaces of constant mean curvature H in \mathbb{H}^3 such that $0 \leq H < 1$ have no corresponding surfaces in \mathbb{E}^3 or \mathbb{S}^3 . This has motivated us to establish a DPW method for CMC surfaces in \mathbb{H}^3 without any restrictions on the range of the mean curvature.

3.2. Let again $f : M \rightarrow \mathbb{H}^3$ be a conformal immersion with unit normal n .

For the purposes of this paper it will be important to consider the map

$$(3.2) \quad F := (f, n) : M \rightarrow \mathrm{U}\mathbb{H}^3.$$

In our paper, the smooth map F will be called the *Gauss map* of f . The Gauss map satisfies $\langle df, n \rangle = 0$. In Section 2.4 we have introduced the canonical contact form ω of $\mathrm{U}\mathbb{H}^3$. By the definition of ω , we have

$$F^*\omega = \langle df, n \rangle.$$

Hence the Gauss map F satisfies the *Legendre condition*:

$$F^*\omega = 0.$$

Note that the Gauss map F is also called the *Legendre lift* of f as in [2]. The Legendre property will be discussed in Section 7.2. The following result which is very important for this paper is due to T. Ishihara [29].

Proposition 3.3 (T. Ishihara). *Let M be a Riemann surface. A conformal immersion $f : M \rightarrow \mathbb{H}^3$ has constant mean curvature if and only if its Gauss map is harmonic with respect to the metric induced from (2.1).*

In Section 4.2, we will give a proof of this result in terms of frames and the Sym formula.

4. The 2×2 -matrix model for immersions into \mathbb{H}^3

4.1. Now let again $f : M \rightarrow \mathbb{H}^3$ be a CMC surface with unit normal n . Let \mathbb{D} denote the universal cover of M with complex coordinate z . Since $\mathbb{D} \cong \mathbb{S}^2$ can only occur for totally umbilic CMC immersions with $H > 1$ and cannot occur for CMC immersions with $0 \leq H \leq 1$ (see [43]), we can assume that $\mathbb{D} \subset \mathbb{C}$ is open. (Usually we will assume $\mathbb{D} = \mathbb{C}$ or \mathbb{D} being an open unit disk.) Hence, without loss of generality, \mathbb{D} is open in \mathbb{C} .

Denote by (x, y) the associated isothermal coordinates, i.e., $z = x + iy$, where the induced metric is expressed as $I = e^u(dx^2 + dy^2)$.

The *coordinate frame* Ψ of f with respect to (x, y) is a map from \mathbb{D} into the Lorentz group $\mathrm{SO}_{1,3}^+$ defined by

$$\Psi = (f, n, e^{-u/2} f_y, e^{-u/2} f_x).$$

As mentioned above, the action (2.4) induces a double covering $\tilde{\pi} : \mathrm{SL}_2 \mathbb{C} \rightarrow \mathrm{SO}_{1,3}^+$. Since \mathbb{D} is simply connected, the lift $\hat{\Phi}$ of the coordinate frame Ψ to $\mathrm{SL}_2 \mathbb{C}$ is determined uniquely by

$$(4.1) \quad e_j \mapsto \hat{\Phi} e_j \hat{\Phi}^*$$

up to sign. The lift $\hat{\Phi}$ satisfies the following Gauss–Weingarten formulas:

$$(4.2) \quad \begin{aligned} \hat{\Phi}^{-1} \hat{\Phi}_z &= \begin{pmatrix} \frac{u_z}{4} & \frac{1}{2}(H+1)e^{u/2} \\ -Qe^{-u/2} & -\frac{u_z}{4} \end{pmatrix}, \\ \hat{\Phi}^{-1} \hat{\Phi}_{\bar{z}} &= \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & \bar{Q}e^{-u/2} \\ -\frac{1}{2}(H-1)e^{u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}. \end{aligned}$$

Let us consider the associated family $\{f_\lambda\}_{\lambda \in \mathbb{S}^1}$ of $f = f_1$. The immersion f_λ has the Hopf differential $\lambda^{-1} Q dz^2$. The corresponding $\mathrm{SL}_2 \mathbb{C}$ -valued frame is denoted by $\hat{\Phi}_\lambda$. For our purposes, it will be useful to perform the following gauge transformation:

$$(4.3) \quad \tilde{\Phi} := \hat{\Phi}_{\lambda^2} \begin{pmatrix} \sqrt{\lambda} & 0 \\ 0 & \frac{1}{\sqrt{\lambda}} \end{pmatrix}.$$

Then we obtain $\tilde{U} = \tilde{\Phi}^{-1}\tilde{\Phi}_z$ and $\tilde{V} = \tilde{\Phi}^{-1}\tilde{\Phi}_{\bar{z}}$ with

$$(4.4) \quad \begin{aligned} \tilde{U} &= \begin{pmatrix} \frac{u_z}{4} & \frac{\lambda^{-1}}{2}(H+1)e^{u/2} \\ -\lambda^{-1}Qe^{-u/2} & -\frac{u_{\bar{z}}}{4} \end{pmatrix}, \\ \tilde{V} &= \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & \lambda\bar{Q}e^{-u/2} \\ -\frac{\lambda}{2}(H-1)e^{u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}. \end{aligned}$$

From (4.4), \tilde{U} and \tilde{V} are elements of the *loop algebra* of $\mathfrak{sl}_2\mathbb{C}$:

$$\Lambda\mathfrak{sl}_2\mathbb{C}_\sigma = \{g : \mathbb{S}^1 \rightarrow \mathfrak{sl}_2\mathbb{C} \mid g(-\lambda) = \sigma g(\lambda)\},$$

where

$$(4.5) \quad \sigma = \text{Ad}(\mathbf{e}_1).$$

It will turn out to be useful to consider the analytic loops in $\Lambda\mathfrak{sl}_2\mathbb{C}_\sigma$. Those loops will be denoted by $\tilde{\Lambda}\mathfrak{sl}_2\mathbb{C}_\sigma$. We will use a similar notation for loop groups. It is easy to see that $\tilde{U}, \tilde{V} \in \tilde{\Lambda}\mathfrak{sl}_2\mathbb{C}_\sigma$ and $\tilde{\Phi} \in \tilde{\Lambda}\text{SL}_2\mathbb{C}_\sigma$. Then we obtain the *Sym formula* for CMC surfaces in \mathbb{H}^3 which is easily seen to be equivalent to the corresponding formula used in [3, 5].

Proposition 4.1. *Let $\tilde{\Phi}$ be a solution to (4.4). Then*

$$\tilde{f}_\lambda = \tilde{\Phi}\tilde{\Phi}^*$$

is a loop of immersions of constant mean curvature H with unit normal vector field

$$\tilde{n}_\lambda = \tilde{\Phi}\mathbf{e}_1\tilde{\Phi}^*.$$

For $\lambda = 1$ we obtain $\tilde{f}_{\lambda=1} = f$.

Proof. A direct computation shows

$$\tilde{f}_z = \frac{\lambda^{-1}}{2}e^{u/2}\tilde{\Phi}(\mathbf{e}_3 + i\mathbf{e}_2)\tilde{\Phi}^*, \quad \tilde{f}_{\bar{z}} = \frac{\lambda}{2}e^{u/2}\tilde{\Phi}(\mathbf{e}_3 - i\mathbf{e}_2)\tilde{\Phi}^*.$$

From these equations we obtain

$$\mathbf{I}_\lambda = e^u dz d\bar{z}.$$

Hence \tilde{f}_λ is an immersion for all $\lambda \in \mathbb{S}^1$. The unit normal for \tilde{f}_λ is given by $\tilde{n}_\lambda = \tilde{\Phi}\mathbf{e}_1\tilde{\Phi}^*$. Thus

$$\tilde{n}_z = \tilde{\Phi}(U\mathbf{e}_1 + \mathbf{e}_1 V^*)\tilde{\Phi}^* = -\frac{\lambda^{-1}}{2}\{He^{u/2}\tilde{\Phi}(\mathbf{e}_3 + \mathbf{e}_2)\tilde{\Phi}^* + 2Qe^{-u/2}\tilde{\Phi}(\mathbf{e}_3 - \mathbf{e}_2)\tilde{\Phi}^*\}.$$

From these equations one can check that each \tilde{f}_λ has constant mean curvature H and Hopf differential $\lambda^{-2}Qdz^2$. \square

Remark. The construction above shows that all one needs for the construction of constant mean curvature surfaces are a real number H , a holomorphic function Q and a real valued function u such that with \tilde{U} and \tilde{V} as in (4.4) the 1-form $\tilde{\alpha} = \tilde{U}dz + \tilde{V}d\bar{z}$ is integrable.

4.2. In this section, we discuss the harmonicity of the Gauss map associated with \tilde{f}_λ . Let $f : M \rightarrow \mathbb{H}^3$ be a conformal immersion as above. Let $\hat{\Phi}$ denote the $\text{SL}_2 \mathbb{C}$ -valued frame which is a lift of the coordinate frame Ψ and put $\hat{\alpha} := \hat{\Phi}^{-1} d\hat{\Phi}$. Then we decompose $\hat{\alpha}$ as

$$\hat{\alpha} = \hat{\alpha}_{\mathfrak{h}} + \hat{\alpha}'_{\mathfrak{p}} + \hat{\alpha}''_{\mathfrak{p}}$$

according to the Lie algebra decomposition $\mathfrak{sl}_2 \mathbb{C} = \mathfrak{h} \oplus \mathfrak{p}$ as in (2.6) and (2.7), respectively. A direct computation shows

$$[\hat{\alpha}'_{\mathfrak{p}} \wedge \hat{\alpha}''_{\mathfrak{p}}] = -\frac{1}{4} \{(H^2 - 1)e^u - 4|Q|^2 e^{-u}\} e_1 dz \wedge d\bar{z}.$$

This is contained in \mathfrak{h} . Therefore the \mathfrak{p} -part of $[\hat{\alpha}'_{\mathfrak{p}} \wedge \hat{\alpha}''_{\mathfrak{p}}]$ vanishes and $\hat{\alpha}$ satisfies the admissible condition (1.2). Moreover, it is easy to check that

$$d(*\hat{\alpha}_{\mathfrak{p}}) + [\hat{\alpha}' \wedge * \hat{\alpha}_{\mathfrak{p}}] = -ie^{u/2} \begin{pmatrix} 0 & H_{\bar{z}} \\ -H_z & 0 \end{pmatrix} dz \wedge d\bar{z}$$

holds. Since (1.1) describes the harmonicity of the Gauss map, this formula implies Proposition 3.3.

Remark. On the unit tangent sphere bundle $U\mathbb{H}^3$, we can define Riemannian metrics so that the natural projection π_2 is a Riemannian submersion. One of such metrics is the *Sasaki lift metric*. It is not difficult to see that the Gauss map F of a non-minimal CMC surface is *never* harmonic with respect to the Sasaki lift metric; see [30, p. 271, proof of Corollary].

5. CMC surfaces with $H > 1$

As pointed out in the introduction, the case $H = 1$ is special and has been investigated already intensively. Therefore this case will not be considered in this paper. In this section, we study CMC surfaces with mean curvature H such that $H > 1$. In this case, we may write $H = \coth q$, $q \in \mathbb{R}_{>0}$.

We perform a gauge transformation:

$$(5.1) \quad \Phi := \tilde{\Phi} \begin{pmatrix} e^{q/4} & 0 \\ 0 & e^{-q/4} \end{pmatrix}.$$

We call Φ the *extended frame* of a CMC immersion f with $H > 1$. Moreover we put

$$\mathcal{H} = e^{-q}(H + 1) \in \mathbb{R}, \quad \nu = -e^{-q/2}\lambda.$$

Then $H = \coth q$ implies $\mathcal{H} = e^{-q}(H + 1) = e^q(H - 1)$. Moreover, the Lax pair

$$U = \Phi^{-1} \Phi_z, \quad V = \Phi^{-1} \Phi_{\bar{z}}$$

is given by

$$(5.2) \quad U = \begin{pmatrix} \frac{u_z}{4} & -\frac{1}{2}\nu^{-1}\mathcal{H}e^{u/2} \\ \nu^{-1}Qe^{-u/2} & -\frac{u_z}{4} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{u_{\bar{z}}}{4} & -\nu\bar{Q}e^{-u/2} \\ \frac{1}{2}\nu\mathcal{H}e^{u/2} & \frac{u_{\bar{z}}}{4} \end{pmatrix}.$$

Clearly, the matrices U and V are holomorphic in the parameter $v \in \mathbb{C}^\times$. In particular, the gauged frame Φ defined in (5.1) above can be considered to be a holomorphic function in v , where v is restricted to the circle of radius $r = e^{-q/2}$. Noting that everything is holomorphic in $v \in \mathbb{C}^\times$, it is straightforward to check that the 1-form $\alpha = Udz + Vd\bar{z}$ is fixed by the following loop algebra automorphism:

$$(5.3) \quad \tau_3 : g(v) \mapsto -g^*\left(\frac{1}{v}\right).$$

This automorphism is said to be of type C_3 (almost compact automorphism of the third kind), [33]. The Maurer–Cartan form $\alpha = \Phi^{-1}d\Phi$ has the decomposition

$$(5.4) \quad \alpha = v^{-1}\alpha_{-1} + \alpha_0 + v\alpha_1.$$

$\tau_3(\alpha) = \alpha$ translates into

$$\overline{\alpha_0} = -\alpha_0, \quad \alpha_{-1} = -\alpha_1^*.$$

The mapping Φ takes values in the twisted loop group

$$\Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_3} := \{g : \mathbb{S}^1 \rightarrow \mathrm{SL}_2 \mathbb{C} \mid g(-v) = \sigma g(v), \tau_3(g)(v) = g(v)\},$$

where σ is defined in (4.5) and $\tau_3(g)(v) = g(1/\bar{v})^{*-1}$.

Moreover, the first formula in Proposition 4.1 can now be reinterpreted as

$$(5.5) \quad f_v := \Phi \begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2} \end{pmatrix} \Phi^*.$$

This CMC immersion into \mathbb{H}^3 has mean curvature $H = \coth q$ and the unit normal

$$(5.6) \quad n_v = \Phi \begin{pmatrix} e^{-q/2} & 0 \\ 0 & -e^{q/2} \end{pmatrix} \Phi^*.$$

Note that the matrix $\mathrm{diag}(e^{-q/2}, e^{q/2})$ corresponds under the isomorphism (2.2) to the point $(\cosh(q/2), -\sinh(q/2), 0, 0) \in \mathbb{H}^3$. Conversely, the following result holds:

Proposition 5.1. *Let \mathcal{H} be a positive real number, u a real valued function and Q a holomorphic function on the simply connected domain $\mathbb{D} \subset \mathbb{C}$. Let v be a complex parameter. Assume that the differential 1-form $\alpha = Udz + Vd\bar{z}$ is integrable and let Φ denote a solution to $\Phi^{-1}d\Phi = \alpha$. Then (5.5) defines for v of absolute value $e^{-q/2}$ a CMC immersion into \mathbb{H}^3 with mean curvature $H = \coth q$ and the unit normal is defined in (5.6).*

The Lawson correspondence (see Section 3.1 for more details) between CMC surfaces in \mathbb{H}^3 with mean curvature $H > 1$ and CMC surfaces in \mathbb{E}^3 with mean curvature \mathcal{H} has now in our setting the following simple explanation: Consider a CMC surface in \mathbb{H}^3 and let U and V denote the associated matrices, but now consider these matrices as functions of $v \in \mathbb{S}^1$. Then it is straightforward to check that the 1-form $\alpha = Udz + Vd\bar{z}$ has the decomposition as in (5.4) and satisfies all the conditions for being the Maurer–Cartan form of the extended frame of some CMC surface in \mathbb{E}^3 with Hopf differential Q and mean curvature \mathcal{H} .

The converse construction, starting from some CMC surface in \mathbb{E}^3 and ending up with some CMC surface in \mathbb{H}^3 works out analogously.

Thus for the case $H > 1$, constructions of CMC surfaces in \mathbb{H}^3 are reduced to those for CMC surfaces in Euclidean 3-space [22]. In other words, we can construct CMC surfaces in \mathbb{H}^3 with $H > 1$ via the generalized Weierstrass type representation (DPW method) for CMC surfaces in Euclidean 3-space. See Section 3.1 for another explanation for this fact.

Remark. One can make this relation into a 1–1 relation by fixing the initial conditions of the extended frames at some base point.

Let us identify \mathfrak{su}_2 with Euclidean 3-space via the correspondence

$$x_1(i\mathbf{e}_1) + x_2(i\mathbf{e}_2) + x_3(i\mathbf{e}_3) \longleftrightarrow (x_1, x_2, x_3).$$

The Euclidean inner product $dx_1^2 + dx_2^2 + dx_3^2$ corresponds to the inner product

$$\langle X, Y \rangle = -\frac{1}{2} \operatorname{tr}(XY), \quad X, Y \in \mathfrak{su}_2.$$

Then one can see that

$$\varphi_v = \operatorname{Ad}(\Phi)(i\mathbf{e}_1) : \mathbb{D} \times C_r \rightarrow \mathbb{S}^2 \subset \mathfrak{su}_2$$

is a loop of harmonic maps, where C_r is a radius r circle.

6. CMC surfaces with $0 \leq H < 1$

6.1. Now we start our study of CMC surfaces with mean curvature H such that $0 \leq H < 1$. In this case, we may write $H = \tanh q$, $q \in \mathbb{R}_{\geq 0}$.

We perform the gauge transformation:

$$(6.1) \quad \Phi := \tilde{\Phi} \begin{pmatrix} e^{(q+\pi i)/4} & 0 \\ 0 & e^{-(q+\pi i)/4} \end{pmatrix}.$$

We call Φ the *extended frame* of a CMC immersion f with $0 \leq H < 1$. Moreover we put

$$\mathcal{H} = ie^{-q}(H+1) \in i\mathbb{R}, \quad v = e^{-q/2}\lambda, \quad \mathcal{Q} = -iQ.$$

Note that $H = \tanh q$ implies, $\mathcal{H} = ie^{-q}(H+1) = -ie^q(H-1)$, and the Lax pair

$$U = \Phi^{-1}\Phi_z, \quad V = \Phi^{-1}\Phi_{\bar{z}}$$

is given explicitly by the matrices

$$(6.2) \quad U = \begin{pmatrix} \frac{u_z}{4} & -\frac{1}{2}v^{-1}\mathcal{H}e^{u/2} \\ v^{-1}\mathcal{Q}e^{-u/2} & -\frac{u_z}{4} \end{pmatrix}, \quad V = \begin{pmatrix} -\frac{u_z}{4} & -v\bar{\mathcal{Q}}e^{-u/2} \\ \frac{1}{2}v\mathcal{H}e^{u/2} & \frac{u_z}{4} \end{pmatrix}.$$

Considering, as in Section 5, everything as holomorphic expressions in $v \in \mathbb{C}^\times$, it is straightforward to check that the 1-form $\alpha = U dz + V d\bar{z}$ is fixed by the following automorphism of the loop algebra:

$$(6.3) \quad \tau_4 : g(v) \mapsto -\operatorname{Ad}(\mathcal{R}) \left\{ g\left(\frac{i}{\bar{v}}\right) \right\}^*, \quad \mathcal{R} = \begin{pmatrix} \frac{1}{\sqrt{i}} & 0 \\ 0 & \sqrt{i} \end{pmatrix}.$$

This automorphism is said to be of type C_4 (almost compact automorphism of the fourth kind), [33]. The Maurer–Cartan form $\alpha = \Phi^{-1}d\Phi$ has the decomposition

$$(6.4) \quad \alpha = v^{-1}\alpha_{-1} + \alpha_0 + v\alpha_1.$$

Moreover, $\tau_4(\alpha) = \alpha$ translates into

$$\overline{\alpha_0} = -\alpha_0, \quad \alpha_{-1} = i \operatorname{Ad}(\mathcal{R})(\alpha_1)^*.$$

The mapping Φ takes values in the twisted loop group

$$\Lambda \operatorname{SL}_2 \mathbb{C}_{\sigma, \tau_4} := \{g : \mathbb{S}^1 \rightarrow \operatorname{SL}_2 \mathbb{C} \mid g(-v) = \sigma g(v), \tau_4(g)(v) = g(v)\},$$

where σ is defined in (4.5) and

$$(6.5) \quad \tau_4(g)(v) = \operatorname{Ad}(\mathcal{R})\left\{g\left(\frac{i}{v}\right)\right\}^{*-1}.$$

Moreover, the first formula in Proposition 4.1 can be reinterpreted as

$$(6.6) \quad f_v := \Phi \begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2} \end{pmatrix} \Phi^*,$$

thus reproducing the given CMC immersion in \mathbb{H}^3 with $0 \leq H = \tanh q < 1$. Its unit normal can be written in the form

$$(6.7) \quad n_v = \Phi \begin{pmatrix} e^{-q/2} & 0 \\ 0 & -e^{q/2} \end{pmatrix} \Phi^*.$$

Conversely, the following result holds:

Proposition 6.1. *Let \mathcal{H} be a purely imaginary constant, u a real function and Q a holomorphic function on the simply connected domain $\mathbb{D} \subset \mathbb{C}$. Let v be a complex parameter. Assume that the differential 1-form $\alpha = Udz + Vd\bar{z}$ is integrable and let Φ denote a solution to $\Phi^{-1}d\Phi = \alpha$. Then (6.6) defines for v of absolute value $e^{-q/2}$ a CMC immersion into \mathbb{H}^3 with mean curvature $H = \tanh q$ and normal as defined in (6.7).*

Remark. As pointed out above, the Maurer–Cartan form $\alpha = \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1$ is a type C_4 real form of the complex CMC surface equation [33]. But it does not correspond naturally to a CMC surface in \mathbb{E}^3 , since \mathcal{H} is not real.

6.2. As indicated in the introduction the extended frame (6.1) of a CMC surface f with mean curvature $H = \tanh q$ can also be considered as the extended frame of a minimal surface in the hyperbolic 3-space of sectional curvature $-1/\cosh^2 q$. For simplicity, this paper primarily considers surfaces in $\mathbb{H}^3(-1)$. However, the formalism can easily be adjusted to fit surfaces in $\mathbb{H}^3(c)$ with $c < 0$. It suffices to “scale” a given surface, and thus $\mathbb{H}^3(-1)$ inside $\operatorname{Her}_2 \mathbb{C}$, by the factor $1/\sqrt{|c|}$. The radial deformation of the loop parameter λ on unit circle to a radius r circle changes the mean curvature for a CMC surface, which is given by conjugation of a diagonal matrix to the extended frame. Combining the scaling and the radial deformation, the Lawson correspondence for CMC surfaces in $\mathbb{H}^3(c)$ is obtained.

More precisely, let $\tilde{\Phi} = \tilde{\Phi}_\lambda$ be the $\Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$ -valued map defined by (4.3) which frames the associated family of a CMC surface $f : \mathbb{D} \rightarrow \mathbb{H}^3(-1)$ with mean curvature H . Then for any real number q ,

$$(6.8) \quad f_\lambda = \frac{1}{\cosh q - H \sinh q} \tilde{\Phi}_\lambda \begin{pmatrix} e^{q/2} & 0 \\ 0 & e^{-q/2} \end{pmatrix} \tilde{\Phi}_\lambda^* \Big|_{\lambda=e^{q/2}}$$

defines a CMC surface of mean curvature $H_\lambda = H \cosh q - \sinh q$ in the hyperbolic space $\mathbb{H}^3(K_\lambda)$ of sectional curvature $K_\lambda = -(\cosh q - H \sinh q)^2$. The surface f_λ has the same metric and the same Hopf differential as f . By definition, $H_\lambda^2 + K_\lambda = H^2 - 1$. Thus f_λ is a Lawson correspondent of $f : \mathbb{D} \rightarrow \mathbb{H}^3(-1)$ in $\mathbb{H}^3(K_\lambda)$.

Now we consider a CMC surface f with mean curvature $H = \tanh q$, then the Lawson correspondent f_λ has the mean curvature $H_\lambda = 0$ and thus is a minimal surface. Note that the sectional curvature of the ambient space is $K_\lambda = -1/\cosh^2 q$ and the extended frame of f_λ is given as in (6.1). In Section 7.5 we will show that the minimality of the surface in $\mathbb{H}^3(c)$, $c < 0$, and the primitivity of the Gauss map of the surface are equivalent.

7. 4-symmetric structure of the unit tangent sphere bundle

7.1. As we have seen in the preceding section, every CMC surface with $0 \leq H < 1$ admits a loop group valued map Φ which is fixed under the type C_4 automorphism τ_4 .

In this section we study the automorphism τ of $\mathfrak{sl}_2 \mathbb{C}$ which is obtained by first extending τ_4 to the untwisted loop algebra $\Lambda \mathfrak{sl}_2 \mathbb{C}$ using the formula (6.3) and then restricting it to $\mathfrak{sl}_2 \mathbb{C} \subset \Lambda \mathfrak{sl}_2 \mathbb{C}$. The automorphism τ is given on $\mathfrak{g} = \mathfrak{sl}_2 \mathbb{C}$ by the formula

$$(7.1) \quad \tau(X) = -\mathrm{Ad}(\mathcal{R})X^*,$$

where \mathcal{R} is defined in (6.3). More explicitly,

$$(7.2) \quad \tau \begin{pmatrix} a & b \\ c & -a \end{pmatrix} = \begin{pmatrix} -\bar{a} & i\bar{c} \\ -i\bar{b} & \bar{a} \end{pmatrix}.$$

It is easy to see that τ is of order 4. The eigenspace decomposition of the *complexified Lie algebra* $\mathfrak{g}^\mathbb{C}$ with respect to τ is given by

$$\mathfrak{g}^\mathbb{C} = \mathfrak{g}_0^\mathbb{C} \oplus \mathfrak{g}_1^\mathbb{C} \oplus \mathfrak{g}_2^\mathbb{C} \oplus \mathfrak{g}_3^\mathbb{C},$$

where $\mathfrak{g}_k^\mathbb{C}$ is the eigenspace corresponding to the eigenvalue i^k . Note that the complexified Lie algebra $\mathfrak{g}^\mathbb{C} = (\mathfrak{sl}_2 \mathbb{C})^\mathbb{C}$ is realized as $\mathfrak{g} \times \mathfrak{g}$. This construction can be described as follows:

Consider the map

$$\iota : X \in \mathfrak{sl}_2 \mathbb{C} \mapsto (X, \bar{X}) \in \mathfrak{sl}_2 \mathbb{C} \times \mathfrak{sl}_2 \mathbb{C}.$$

This is an injective homomorphism of the real Lie algebra $\mathfrak{sl}_2 \mathbb{C}$ into $\mathfrak{sl}_2 \mathbb{C} \times \mathfrak{sl}_2 \mathbb{C}$. Since $\iota(\mathfrak{sl}_2 \mathbb{C}) \cap i\iota(\mathfrak{sl}_2 \mathbb{C}) = \{0\}$, the image $\iota(\mathfrak{sl}_2 \mathbb{C})$ is a real form of $\mathfrak{sl}_2 \mathbb{C} \times \mathfrak{sl}_2 \mathbb{C}$. The latter Lie algebra carries the natural complex structure given by multiplying a complex number to each of the two factors. By transporting τ via ι we obtain

- (i) τ acts on $\iota(\mathfrak{sl}_2 \mathbb{C})$ as $\tau(X, \bar{X}) = (\tau(X), \overline{\tau(X)})$.

We now define the complex linear extension $\hat{\tau}$ to $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$ as follows:

$$(ii) \quad \hat{\tau}(X, Y) = (\tau(\bar{Y}), \overline{\tau(X)}).$$

It is straightforward to show that, indeed, $\hat{\tau}$ is complex linear relative to i acting on the first and the second factor equally by multiplication.

Moreover, on $\iota(X)$ the new $\hat{\tau}$ acts like (i). Thus (ii) is the complex linear extension of the original τ on $\mathfrak{sl}_2\mathbb{C}$ to the product $\mathfrak{sl}_2\mathbb{C} \times \mathfrak{sl}_2\mathbb{C}$.

Since τ is an automorphism of order 4, also $\hat{\tau}$ is an automorphism of order 4 and we deduce the following result:

Lemma 7.1. $[\mathfrak{g}_k^\mathbb{C}, \mathfrak{g}_l^\mathbb{C}] \subset \mathfrak{g}_{k+l}^\mathbb{C} \pmod{4}$.

The automorphism has two *real eigenvalues* $i^0 = 1$ and $i^2 = -1$. The corresponding eigenspaces $\mathfrak{g}_0^\mathbb{C}$ and $\mathfrak{g}_2^\mathbb{C}$ of $\mathfrak{g}^\mathbb{C} \cong \mathfrak{g} \times \mathfrak{g}$ are computed explicitly as

$$\mathfrak{g}_0^\mathbb{C} = \left\{ \left(\begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix}, \begin{pmatrix} -x_1 & 0 \\ 0 & x_1 \end{pmatrix} \right) \right\}, \quad \mathfrak{g}_2^\mathbb{C} = \left\{ \left(\begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix}, \begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix} \right) \right\},$$

with $x_1 \in \mathbb{C}$. The corresponding real subspaces of $\mathfrak{sl}_2\mathbb{C}$ are

$$\mathfrak{g} \cap \mathfrak{g}_0^\mathbb{C} = \mathfrak{h} = \mathbb{R}(ie_1) \quad \text{and} \quad \mathfrak{g} \cap \mathfrak{g}_2^\mathbb{C} = \mathbb{R}(e_1).$$

Moreover, the eigenspaces $\mathfrak{g}_1^\mathbb{C}$ and $\mathfrak{g}_3^\mathbb{C}$ for eigenvalues i and $-i$ are computed explicitly as

$$\mathfrak{g}_1^\mathbb{C} = \left\{ \left(\begin{pmatrix} 0 & x_2 \\ x_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_3 \\ x_2 & 0 \end{pmatrix} \right) \right\}, \quad \mathfrak{g}_3^\mathbb{C} = \left\{ \left(\begin{pmatrix} 0 & x_2 \\ x_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_3 \\ -x_2 & 0 \end{pmatrix} \right) \right\},$$

with $x_2, x_3 \in \mathbb{C}$. The automorphism τ defines the semi-Riemannian 4-symmetric space $\mathrm{SL}_2\mathbb{C}/U_1$. The space is isomorphic with the unit tangent sphere bundle $U\mathbb{H}^3$ as shown in Section 2.4. The complexified tangent space $\mathfrak{p}^\mathbb{C} = (T_{(e_0, e_1)}U\mathbb{H}^3)^\mathbb{C}$ is given by

$$\mathfrak{p}^\mathbb{C} = \mathfrak{g}_1^\mathbb{C} \oplus \mathfrak{g}_2^\mathbb{C} \oplus \mathfrak{g}_3^\mathbb{C}.$$

Comparing this with the fibration $\pi_1 : U\mathbb{H}^3 \rightarrow \mathrm{Geo}(\mathbb{H}^3)$ discussed in Section 2.4, we have

$$\mathcal{H}_1^\mathbb{C} = \mathfrak{g}_1^\mathbb{C} \oplus \mathfrak{g}_3^\mathbb{C}, \quad \mathcal{V}_1^\mathbb{C} = \mathfrak{g}_2^\mathbb{C}.$$

7.2. Here we recall the notion of a contact manifold.

Definition. A 1-form ω on a manifold L of dimension $2n - 1$ is said to be a *contact form* if $(d\omega)^{n-1} \wedge \omega \neq 0$ on M . A hyperplane field $\mathcal{D} \subset TL$ on L is called a *contact structure* if for any point $p \in L$, there exists a contact form ω defined on a neighborhood U_p of p such that $\mathrm{Ker} \omega = \mathcal{D}$ on U_p .

A $(2n - 1)$ -manifold L with a contact structure \mathcal{D} is called a *contact manifold*. If a contact manifold (L, \mathcal{D}) admits a globally defined contact form ω which annihilates \mathcal{D} , i.e., $\mathrm{Ker} \omega = \mathcal{D}$, then (L, \mathcal{D}) is said to be a contact manifold in the *strict sense*.

Definition. Let M^n be an n -manifold and $F : M \rightarrow L$ a smooth map into a contact $(2n - 1)$ -manifold. Then F is said to be *Legendre* if $dF(TM) \subset \mathcal{D}$.

In particular, if L admits a global contact form ω , then F is Legendre if and only if $F^*\omega = 0$.

Now let (N^n, g) be a Riemannian n -manifold. Then its unit tangent sphere bundle UN admits a canonical contact structure. In case $N = \mathbb{H}^3$, one can check that the canonical contact structure of \mathbb{H}^3 is given by

$$\mathcal{D} = \mathcal{H}_1 = (\mathfrak{g}_1^\mathbb{C} \oplus \mathfrak{g}_3^\mathbb{C}) \cap \mathfrak{g}.$$

By using this fact and results of Section 3.2 we obtain:

Proposition 7.2. *Let \mathbb{D} be a simply connected domain in \mathbb{C} and $F : \mathbb{D} \rightarrow U\mathbb{H}^3$ a smooth map with frame $\Phi : \mathbb{D} \rightarrow \mathrm{SL}_2 \mathbb{C}$, i.e., Φ is a map satisfying*

$$F = (\Phi\Phi^*, \Phi e_1\Phi^*).$$

Denote by $\alpha = \Phi^{-1}d\Phi$ the pull-back of the Maurer–Cartan form by Φ and decompose $(\alpha, \bar{\alpha})$ as

$$(\alpha, \bar{\alpha}) = \alpha_0^\mathbb{C} + \alpha_1^\mathbb{C} + \alpha_2^\mathbb{C} + \alpha_3^\mathbb{C}$$

according to the eigenspace decomposition with respect to τ . Then the component $\alpha_2^\mathbb{C}$ is given by $\alpha_2^\mathbb{C} = (\alpha_2, \bar{\alpha}_2)$ with

$$\alpha_2 = \frac{1}{2}F^*\omega e_1.$$

Thus F is Legendre if and only if $\alpha_2 = 0$. In particular, if $F = (f, n)$ is the Gauss map of a conformal immersion $f : \mathbb{D} \rightarrow \mathbb{H}^3$, then

$$\alpha_2 = \frac{1}{2}F^*\omega e_1 = \frac{1}{2}\langle df, n \rangle e_1 = 0.$$

7.3. Let $F : \mathbb{D} \rightarrow G/H = \mathrm{SL}_2 \mathbb{C}/U_1$ be a Legendre map with frame Φ . Then we have the eigenspace decomposition of $\alpha = \Phi^{-1}d\Phi$,

$$(\alpha, \bar{\alpha}) = \alpha_0^\mathbb{C} + \alpha_1^\mathbb{C} + \alpha_3^\mathbb{C}, \quad \alpha_0^\mathbb{C} = (\alpha_0, \bar{\alpha}_0).$$

On the other hand, we have the decomposition $\alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{p}}$. We denote the first component of $\alpha_j^\mathbb{C}$ by α_j' . Comparing these decompositions, we get

$$(7.3) \quad \alpha = \alpha_{\mathfrak{h}} + \alpha_{\mathfrak{p}}, \quad \alpha_{\mathfrak{h}} = \alpha_0, \quad \alpha_{\mathfrak{p}} = \alpha'_p + \alpha''_p = \alpha_1 + \alpha_3.$$

We express the type-decompositions of α_1 and α_3 with respect to the conformal structure of \mathbb{D} as

$$\alpha_1 = \alpha'_1 + \alpha''_1, \quad \alpha_3 = \alpha'_3 + \alpha''_3.$$

Then from Lemma 7.1 and the integrability of α , we derive

$$[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = [(\alpha'_1 + \alpha'_3) \wedge (\alpha''_1 + \alpha''_3)]_{\mathfrak{p}} = [\alpha'_1 \wedge \alpha''_1] + [\alpha'_3 \wedge \alpha''_3]$$

and

$$[\alpha_1 \wedge \alpha_1] + [\alpha_3 \wedge \alpha_3] = 0.$$

Noting $[\alpha_1 \wedge \alpha_1] = 2[\alpha'_1 \wedge \alpha'_1]$ and $[\alpha_3 \wedge \alpha_3] = 2[\alpha''_3 \wedge \alpha'_3]$, we conclude that

$$[\alpha'_{\mathfrak{p}} \wedge \alpha''_{\mathfrak{p}}]_{\mathfrak{p}} = [\alpha'_1 \wedge \alpha''_1] + [\alpha'_3 \wedge \alpha''_3] = \frac{1}{2}[\alpha_1 \wedge \alpha_1] + \frac{1}{2}[\alpha_3 \wedge \alpha_3] = 0.$$

Now we arrive at the following *zero curvature representation* for Legendre harmonic maps.

Proposition 7.3. *Let $F : \mathbb{D} \rightarrow \mathrm{SL}_2 \mathbb{C}/\mathrm{U}_1$ be a Legendre harmonic map with frame $\Phi : \mathbb{D} \rightarrow \mathrm{SL}_2 \mathbb{C}$. Then $\alpha_\lambda = \Phi^{-1}d\Phi = \alpha_{\mathfrak{h}} + \lambda^{-1}\alpha'_{\mathfrak{p}} + \lambda\alpha''_{\mathfrak{p}}$ satisfies*

$$d\alpha_\lambda + \frac{1}{2}[\alpha_\lambda \wedge \alpha_\lambda] = 0$$

for all $\lambda \in \mathbb{C}^\times$. Here $\alpha_{\mathfrak{h}}$ and $\alpha_{\mathfrak{p}}$ are defined in (7.3) and $\alpha'_{\mathfrak{p}}$ (resp. $\alpha''_{\mathfrak{p}}$) is the $(1, 0)$ -part (resp. $(0, 1)$ -part) of $\alpha_{\mathfrak{p}}$.

7.4. The square τ^2 of τ is an involutive automorphism of \mathfrak{g} . The 1-eigenspace and (-1) -eigenspace of τ^2 on \mathfrak{g} are

$$\mathfrak{g} \cap (\mathfrak{g}_0^\mathbb{C} \oplus \mathfrak{g}_2^\mathbb{C}) = \mathfrak{d} \quad \text{and} \quad \mathfrak{g} \cap (\mathfrak{g}_1^\mathbb{C} \oplus \mathfrak{g}_3^\mathbb{C}) = \mathfrak{q},$$

where \mathfrak{d} and \mathfrak{q} are defined in (2.8) and (2.9), respectively. Hence (G, τ^2) defines the semi-Riemannian symmetric space

$$G/D = \mathrm{SL}_2 \mathbb{C}/\mathbb{C}^\times = \mathrm{Gr}_{1,1}(\mathbb{E}^{1,3}).$$

The 4-symmetric space $G/H = \mathrm{SL}_2 \mathbb{C}/\mathrm{U}_1$ is a fiber bundle over G/D with standard fiber

$$D/H = \mathbb{C}^\times/\mathrm{U}_1 = \mathbb{R}_{>0}.$$

Now let $F : \mathbb{D} \rightarrow G/H$ be a Legendre harmonic map with frame $\Phi : \mathbb{D} \rightarrow G$. Since F is Legendre, integrating $\Phi_\lambda^{-1}d\Phi_\lambda = \alpha_\lambda$, we get the associated family $\{F_\lambda\}$ of F .

Decompose α according to the Lie algebra decomposition $\mathfrak{g} = \mathfrak{d} \oplus \mathfrak{q}$:

$$\alpha_{\mathfrak{d}} = \alpha_0 + \alpha_2 = \alpha_0, \quad \alpha_{\mathfrak{q}} = \alpha_1 + \alpha_3.$$

Then the decomposition above can be rephrased as

$$\alpha_\lambda = \alpha_{\mathfrak{d}} + \lambda^{-1}\alpha'_{\mathfrak{q}} + \lambda\alpha''_{\mathfrak{q}}.$$

This formula implies that the projected map $\mathcal{G} = \pi_1 \circ F$ is harmonic and satisfies $\mathcal{G}^*\Omega = 0$. Moreover $\mathcal{G}_\lambda := \pi_1 \circ F_\lambda$ gives the associated family $\{\mathcal{G}_\lambda\}$ of \mathcal{G} , where π_1 is the natural projection $\pi_1 : G/H \rightarrow G/D$.

Remark. Proposition 7.3 is valid for any horizontal harmonic maps into semi-Riemannian 4-symmetric spaces. More precisely, let $(G/H, \tau)$ be a semi-Riemannian 4-symmetric space with semi-Riemannian homogeneous projection $\pi_1 : G/H \rightarrow G/D$ onto the semi-Riemannian symmetric space $(G/D, \tau^2)$. Take a map $F : \mathbb{D} \rightarrow G/H$ which is horizontal with respect to π_1 and let $\Phi : \mathbb{D} \rightarrow G$ be its frame. Decompose $\alpha = \Phi^{-1}d\Phi$ as $\alpha = \sum_{j=0}^3 \alpha_j$ according to the eigenspace decomposition of τ . Then one can see that F is horizontal if and only if $\alpha_2 = 0$. In addition, the harmonicity of F is equivalent to the flatness of the connections $d + \alpha_\lambda$, where α_λ is defined as in Proposition 7.3 (see [11]).

7.5. In this section, we prove the following characterization of minimal surfaces.

Proposition 7.4. *Let $\mathbb{H}^3(c)$ be the hyperbolic 3-space of sectional curvature $c < 0$. Then the unit tangent sphere bundle $U\mathbb{H}^3(c)$ is a 4-symmetric space. Moreover, a surface $f : \mathbb{D} \rightarrow \mathbb{H}^3(c)$ is minimal if and only if its Gauss map is a primitive map with respect to the 4-symmetric structure of $U\mathbb{H}^3(c)$.*

Proof. It is clear that $U\mathbb{H}^3(c)$ is a 4-symmetric space with respect to τ defined in (7.2). Let $f : M \rightarrow \mathbb{H}^3(c)$ be a conformal immersion with unit normal n . Take a simply connected coordinate domain $(\mathbb{D}, z) \subset M$ and denote by Ψ the coordinate frame defined on \mathbb{D} . Let $\hat{\Phi}$ be a lift of Ψ to $SL_2 \mathbb{C}$ as in (4.1). Then the Maurer–Cartan form $\hat{\alpha} = \hat{\Phi}^{-1} d\hat{\Phi}$ is given by (4.2) with $H + 1$ and $H - 1$ replaced by $H + \sqrt{|c|}$ and $H - \sqrt{|c|}$, respectively. Now we decompose $(\hat{\alpha}, \bar{\hat{\alpha}})$ according to the eigenspace decomposition of τ . Then we have

$$(\hat{\alpha}, \bar{\hat{\alpha}}) = \hat{\alpha}_0^\mathbb{C} + \hat{\alpha}_1^\mathbb{C} + \hat{\alpha}_2^\mathbb{C} + \hat{\alpha}_3^\mathbb{C},$$

where

$$\begin{aligned} \hat{\alpha}_0^\mathbb{C} &= \left(\begin{pmatrix} x_1 & 0 \\ 0 & -x_1 \end{pmatrix}, \begin{pmatrix} -x_1 & 0 \\ 0 & x_1 \end{pmatrix} \right), \quad \hat{\alpha}_2^\mathbb{C} = (\mathbf{0}, \mathbf{0}), \\ \hat{\alpha}_1^\mathbb{C} &= \left(\begin{pmatrix} 0 & x_2 \\ x_3 & 0 \end{pmatrix}, \begin{pmatrix} 0 & -x_3 \\ x_2 & 0 \end{pmatrix} \right), \quad \hat{\alpha}_3^\mathbb{C} = \left(\begin{pmatrix} 0 & x_4 \\ x_5 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x_5 \\ -x_4 & 0 \end{pmatrix} \right), \end{aligned}$$

with

$$\begin{aligned} x_1 &= \frac{1}{4}(u_z dz - u_{\bar{z}} d\bar{z}), \quad x_2 = \frac{\sqrt{|c|}}{2} e^{u/2} dz, \quad x_3 = -Q e^{-u/2} dz - \frac{H}{2} e^{u/2} d\bar{z}, \\ x_4 &= \frac{H}{2} e^{u/2} dz + \bar{Q} e^{-u/2} d\bar{z}, \quad x_5 = \frac{\sqrt{|c|}}{2} e^{u/2} d\bar{z}. \end{aligned}$$

From these equations, we deduce that f is minimal if and only if its Gauss map F is primitive. \square

Remark. Setting $\tau(g) = \text{Ad } \mathcal{R}(g^*)^{-1}$ we obtain an automorphism of $G = SL_2 \mathbb{C}$, the differential of which coincides with τ as given in (7.1). By abuse of language we will use the same notation for the group level and for the Lie algebra level as well as for the corresponding complexified objects.

In [21], the original loop group approach [22] was extended to include primitive harmonic maps into compact Riemannian k -symmetric spaces. In our case the 4-symmetric space under consideration is *non-compact* and *indefinite*. This has far-reaching consequences. We have therefore included in Sections 8 and 9 a brief description of the corresponding technical details. In particular, the Iwasawa decomposition has not only one, but two open cells (the union of which is dense). Implications of this can already be seen in the examples presented in Section 10.

8. Potentials

8.1. We recall loop groups and the Birkhoff decomposition. The twisted loop group is defined as

$$(8.1) \quad \Lambda \text{SL}_2 \mathbb{C}_\sigma = \{g : \mathbb{S}^1 \rightarrow \text{SL}_2 \mathbb{C} \mid g \text{ is continuous and } g(-\lambda) = \sigma g(\lambda)\},$$

where σ is defined in (4.5). More strictly, we assume that the coefficients of all $g \in \Lambda \text{SL}_2 \mathbb{C}_\sigma$ are in the *Wiener algebra*

$$\mathcal{A} = \left\{ f(\lambda) = \sum_{n \in \mathbb{Z}} f_n \lambda^n : \mathbb{S}^1 \rightarrow \mathbb{C} \mid \sum_{n \in \mathbb{Z}} |f_n| < \infty \right\}.$$

The Wiener algebra is a Banach algebra relative to the norm $\|f\| = \sum |f_n|$, and \mathcal{A} consists of continuous functions. Thus $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ is a Banach Lie group. We denote the Lie algebra of $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ by $\Lambda \mathfrak{sl}_2 \mathbb{C}_\sigma$, which consists of maps $g : \mathbb{S}^1 \rightarrow \mathfrak{sl}_2 \mathbb{C}$.

We will need to consider two subgroups of $\Lambda \text{SL}_2 \mathbb{C}_\sigma$, the *twisted plus loop group* and the *minus loop group* as follows: Let \mathbf{B} a subgroup of $\text{SL}_2 \mathbb{C}$. Let $\Lambda_B^+ \text{SL}_2 \mathbb{C}_\sigma$ be the group of maps into $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ which can be extended holomorphically to $D = \{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ and which take values in \mathbf{B} at $\lambda = 0$. Similarly, let $\Lambda_B^- \text{SL}_2 \mathbb{C}_\sigma$ be the group of maps into $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ which can be extended holomorphically to $E = \{\lambda \in \mathbb{C} \mid 1 < |\lambda|\} \cup \{\infty\}$ and take values in \mathbf{B} at $\lambda = \infty$. If $\mathbf{B} = \{\text{Id}\}$ we write the subscript $*$ instead of \mathbf{B} , if $\mathbf{B} = \text{SL}_2 \mathbb{C}$ we abbreviate $\Lambda_B^+ \text{SL}_2 \mathbb{C}_\sigma$ and $\Lambda_B^- \text{SL}_2 \mathbb{C}_\sigma$ by $\Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ and $\Lambda^- \text{SL}_2 \mathbb{C}_\sigma$, respectively.

It is clear that the loop groups $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, defined in Sections 5 and 6 are also subgroups of $\Lambda \text{SL}_2 \mathbb{C}_\sigma$. The Lie algebras of $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_j}$ are denoted by $\Lambda \mathfrak{sl}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, and those are subalgebras of $\Lambda \mathfrak{sl}_2 \mathbb{C}_\sigma$.

Theorem 8.1 (Birkhoff decomposition [41]). *The maps*

$$\Lambda_*^- \text{SL}_2 \mathbb{C}_\sigma \times \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma \rightarrow \Lambda \text{SL}_2 \mathbb{C}_\sigma \quad \text{and} \quad \Lambda_*^+ \text{SL}_2 \mathbb{C}_\sigma \times \Lambda^- \text{SL}_2 \mathbb{C}_\sigma \rightarrow \Lambda \text{SL}_2 \mathbb{C}_\sigma$$

are analytic diffeomorphisms onto the open dense subsets $\Lambda_*^- \text{SL}_2 \mathbb{C}_\sigma \cdot \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ and $\Lambda_*^+ \text{SL}_2 \mathbb{C}_\sigma \cdot \Lambda^- \text{SL}_2 \mathbb{C}_\sigma$ of $\Lambda \text{SL}_2 \mathbb{C}_\sigma$, respectively. The open dense subsets will be called the left big cell and the right big cell, respectively.

Remark. In this paper the big cell always means the left big cell.

8.2. The holomorphic potential for a CMC surface f in \mathbb{H}^3 is a $\Lambda \mathfrak{sl}_2 \mathbb{C}_\sigma$ -valued holomorphic 1-form determined from the extended frame of f , which is an analogue of Weierstrass data for a minimal surface in \mathbb{R}^3 . The holomorphic potential reproduces the CMC surface f using the generalized Weierstrass type representation recalled in Section 9.

Let \mathbb{D} be a simply connected domain in \mathbb{C} , and let $\Phi : \mathbb{D} \rightarrow \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, the extended frame of a CMC surface f in \mathbb{H}^3 with $H \neq 1$ as defined in (5.1) or (6.1), respectively. The following proposition and corollary can be deduced from [22].

Proposition 8.2. *There exists a loop $g : \mathbb{D} \rightarrow \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$, holomorphic for $\lambda \in \mathbb{C}$, such that $\Phi g : \mathbb{D} \rightarrow \Lambda \text{SL}_2 \mathbb{C}_\sigma$ is holomorphic for $\lambda \in \mathbb{C}$ and $z \in \mathbb{D}$.*

Corollary 8.3 (Existence of a holomorphic potential). *Let $\Phi : \mathbb{D} \rightarrow \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, be the extended frame for some CMC surface f in \mathbb{H}^3 with $H \neq 1$ as defined in (5.1) or (6.1). Then there exist $g : \mathbb{D} \rightarrow \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ and a $\Lambda \mathfrak{sl}_2 \mathbb{C}$ -valued 1-form η such that*

$$(8.2) \quad \eta = (\Phi g)^{-1} d(\Phi g) = \sum_{j=-1}^{\infty} \lambda^j \eta_j,$$

where the η_j are λ -independent diagonal (resp. off-diagonal) holomorphic 1-forms if j is even (resp. odd). In particular, the 1-form $\lambda \eta$ is holomorphic in $z \in \mathbb{D}$ and $\lambda \in \mathbb{C}$ and the upper right entry of η_{-1} does not vanish on \mathbb{D} .

Remark. (i) The holomorphic 1-form η in (8.2) will be called the *holomorphic potential* of the immersion f .

- (ii) Holomorphic potentials are not unique, since the right multiplication of C by some holomorphic loop $C_+ : \mathbb{D} \rightarrow \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ gives another holomorphic potential, i.e., $\tilde{\eta} = \tilde{C}^{-1} d\tilde{C}$ is a new holomorphic potential, where $\tilde{C} = CC_+$ with C_+ holomorphic.
- (iii) The generalization of Grauert's theorem has so far only been proven for non-compact Riemann surfaces. Therefore, at this point we can only infer that holomorphic potentials exist for all CMC immersions defined on a non-compact Riemann surface. We would expect that the analogous result holds for a compact Riemann surface. But in this case the potential will probably only be meromorphic.

8.3. In Section 8.2, the holomorphic potential was derived from a CMC surface f . In this section, we give the normalized potential for f which is the $\Lambda \mathfrak{sl}_2 \mathbb{C}$ -valued meromorphic 1-form determined from the extended frame of f . Unlike the holomorphic potential, the normalized potential is not holomorphic, however, the Fourier expansion of it has only one coefficient. The following proposition again can be concluded from [22].

Proposition 8.4 (Existence of a normalized potential). *Let $\Phi : \mathbb{D} \rightarrow \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, be the extended frame for some CMC surface f in \mathbb{H}^3 with $H \neq 1$ as defined in (5.1) or (6.1). Then there exist $\Phi_+ : \mathbb{D} \rightarrow \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ and a $\Lambda \mathfrak{sl}_2 \mathbb{C}$ -valued 1-form ξ such that*

$$(8.3) \quad \xi = (\Phi \Phi_+)^{-1} d(\Phi \Phi_+) = \lambda^{-1} \xi_{-1},$$

where ξ_{-1} is an off-diagonal 1-form which is meromorphic on \mathbb{D} .

Remark. The meromorphic 1-form ξ in (8.3) will be called the *normalized potential* of the immersion f .

9. Generalized Weierstrass type representation

9.1. In Propositions 8.2 and 8.4 we have considered objects which are holomorphic in $\lambda \in \mathbb{C}^\times$. For the construction of CMC surfaces we need to obtain frames. Therefore we

consider the double loop groups $\mathcal{H} = \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma \times \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$. Then the subgroups \mathcal{H}_+ and \mathcal{H}_- of \mathcal{H} are defined as follows:

$$\mathcal{H}_+ = \Lambda^+ \mathrm{SL}_2 \mathbb{C}_\sigma \times \Lambda^- \mathrm{SL}_2 \mathbb{C}_\sigma, \quad \mathcal{H}_- = \{(g_1, g_2) \in \mathcal{H} \mid g_1 = g_2\}.$$

We quote [24, Theorem 2.6].

Theorem 9.1 (Generalized Iwasawa decomposition). *The map $\mathcal{H}_- \times \mathcal{H}_+ \rightarrow \mathcal{H}_- \mathcal{H}_+$ is an analytic diffeomorphism. The image is open and dense in \mathcal{H} . Moreover,*

$$\mathcal{H} = \bigcup_{n=0}^{\infty} \mathcal{H}_- w_n \mathcal{H}_+,$$

where $w_n = (\mathrm{Id}, (\begin{smallmatrix} \lambda^n & 0 \\ 0 & \lambda^{-n} \end{smallmatrix}))$ if $n = 2k$ and $(\mathrm{Id}, (\begin{smallmatrix} 0 & \lambda^n \\ -\lambda^{-n} & 0 \end{smallmatrix}))$ if $n = 2k + 1$.

9.2. We recall that $\Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_j}$, $j = 3, 4$, defined in Sections 5 and 6, are real forms of $\Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$ (see [33]). These real forms naturally induce Iwasawa decompositions of $\Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$:

Theorem 9.2 (Iwasawa decomposition for τ_3 , [41]). *The map*

$$(9.1) \quad \Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_3} \times \Lambda^+ \mathrm{SL}_2 \mathbb{C}_\sigma \rightarrow \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$$

is an analytic diffeomorphism.

The automorphism τ_3 in the theorem above is induced from some automorphism of $\mathrm{SL}_2 \mathbb{C}$. And this is used essentially in the proof of this theorem. For the automorphism τ_4 the situation is completely different, since τ_4 is not induced by some automorphism of the (finite dimensional) Lie group $\mathrm{SL}_2 \mathbb{C}$.

Theorem 9.3 (Iwasawa decomposition for τ_4). *Let $\omega_0 = (\begin{smallmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{smallmatrix})$. Then the map*

$$(9.2) \quad \Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_4} \times \{\mathrm{Id}, \omega_0\} \times \Lambda^+ \mathrm{SL}_2 \mathbb{C}_\sigma \rightarrow \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma, \quad (g, \delta, w_+) \mapsto g\delta w_+,$$

is an analytic diffeomorphism onto an open dense subset of $\Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$. The open dense subset will be called the Iwasawa core with two open cells.

Proof. Consider the map $g \in \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma \mapsto (g, \tau_4(g)) \in \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma \times \Lambda \mathrm{SL}_2 \mathbb{C}_\sigma$, where τ_4 is defined in (6.5). Applying the generalized Iwasawa decomposition of Theorem 9.1, we have

$$(g, \tau_4(g)) = (\Phi, \Phi)(\mathrm{Id}, W)(V_+, V_-).$$

Since $\Phi = gV_+^{-1}$ and $\Phi = \tau_4(g)V_-^{-1}W^{-1}$, we obtain

$$(9.3) \quad g^{-1}\tau_4(g) = V_+^{-1}WV_- \quad \text{on } \mathbb{S}^1.$$

Since $g^{-1}\tau_4(g)$ is in general not positive definite, the middle term W is not the identity element for the Birkhoff decomposition (9.3) in general. Consider the injective real analytic group homomorphism into

$$\Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_4} \rightarrow \mathrm{Gr}(H), \quad g \mapsto A_g \cdot H_+,$$

where $H = L^2(\mathbb{S}^1, \mathbb{C})$ with polarization $H = H_+ \oplus H_-$ (see [22, Chapter 2]). Let s be the non-trivial holomorphic section of the dual of the determinant bundle over $\text{Gr}(H)$. We now set $\ell(g) = s(A_{g^{-1}\tau_4(g)} \cdot H_+)$. It is known that $\ell(g) \neq 0$ if and only if $g^{-1}\tau_4(g)$ is in the big cell, i.e., $W = \text{Id}$ (see [22, Corollary 2.5]). The set given by $\ell(g) = 0$ in $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ can not contain any open subset and is closed, since $\ell(g)$ is a real analytic function and $\ell(\text{Id}) \neq 0$. Thus the set given by $\ell(g) \neq 0$ in $\Lambda \text{SL}_2 \mathbb{C}_\sigma$ is an open dense subset. Let us assume $\ell(g) \neq 0$ and $\ell(\text{Id}) \neq 0$, i.e., $W = \text{Id}$. Then taking τ_4 on both sides of (9.3), we obtain $\tau_4(V_+^{-1}V_-) = (V_+^{-1}V_-)^{-1}$ and hence

$$\tau_4(V_+) = k^{-1}V_- \quad \text{and} \quad \tau_4(V_-) = k^{-1}V_+,$$

where k is a λ -independent diagonal matrix with entries $k_0, k_0^{-1} \in \mathbb{R}$. Then from these symmetries, Φ acquires the symmetry $\tau_4(\Phi) = \Phi k$. If $k_0 > 0$, we let \tilde{k} be a real matrix such that $\tilde{k}^2 = k$ holds. Then with $\tilde{\Phi} = \Phi \tilde{k}$ and $\tilde{V}_+ = \tilde{k}^{-1}V_+$, we obtain

$$g = \tilde{\Phi} \tilde{V}_+, \quad \tilde{\Phi} \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}, \quad \tilde{V}_+ \in \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma.$$

If $k_0 < 0$, we set $\omega_0 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$ and choose a real matrix \check{k} such that $\check{k}^2 = -k$ holds. Then with $\check{\Phi} = \Phi \check{k} \omega_0$ and $\check{V}_+ = -\check{k}^{-1}V_+$, we have

$$g = \check{\Phi} \omega_0 \check{V}_+, \quad \check{\Phi} \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}, \quad \check{V}_+ \in \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma.$$

To show that the sets $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4} \cdot \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ and $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4} \omega_0 \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$ are open in $\Lambda \text{SL}_2 \mathbb{C}_\sigma$, it suffices to show that

$$(9.4) \quad \text{Lie}(\Lambda \text{SL}_2 \mathbb{C}_\sigma) = \text{Lie}(\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}) \oplus \text{Lie}(\Lambda^+ \text{SL}_2 \mathbb{C}_\sigma)$$

and

$$(9.5) \quad \text{Lie}(\Lambda \text{SL}_2 \mathbb{C}_\sigma) = \omega_0^{-1} \text{Lie}(\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}) \omega_0 \oplus \text{Lie}(\Lambda^+ \text{SL}_2 \mathbb{C}_\sigma),$$

where $\text{Lie}(G)$ denotes the Lie algebra of G (see [27, Chapter II, Lemma 2.4]). But these two equations can be proven by a straightforward computation. This completes the proof. \square

Remark. The Iwasawa decomposition for τ_4 in Theorem 9.3 can be rephrased as follows: Let $\hat{\tau}$ be an extension of τ in (7.1) to $G^\mathbb{C} = \text{SL}_2 \mathbb{C} \times \text{SL}_2 \mathbb{C}$ in Section 7.1 and

$$\begin{aligned} \Lambda G^\mathbb{C} &= \Lambda \text{SL}_2 \mathbb{C} \times \Lambda \text{SL}_2 \mathbb{C}, \\ \Lambda G_{\hat{\tau}}^\mathbb{C} &= \{(g, h) \in \Lambda G^\mathbb{C} \mid \hat{\tau}(g(\lambda), h(\lambda)) = (g(-i\lambda), h(-i\lambda))\}, \\ \Lambda G_{\hat{\tau}}^- &= \{(g, h) \in \Lambda G_{\hat{\tau}}^\mathbb{C} \mid h(\lambda) = \overline{g\left(\frac{1}{\bar{\lambda}}\right)}\}, \\ \Lambda^+ G_{\hat{\tau}}^\mathbb{C} &= \{(g, h) \in \Lambda G_{\hat{\tau}}^\mathbb{C} \mid g \in \Lambda^+ \text{SL}_2 \mathbb{C}, h \in \Lambda^- \text{SL}_2 \mathbb{C}\}. \end{aligned}$$

Since $\hat{\tau}^2$ is the involution $\hat{\sigma} = (\sigma, \sigma)$ with σ defined in (4.5), we have

$$\Lambda G_{\hat{\tau}}^\mathbb{C} \subset \Lambda G_{\hat{\sigma}}^\mathbb{C} = \Lambda \text{SL}_2 \mathbb{C}_\sigma \times \Lambda \text{SL}_2 \mathbb{C}_\sigma.$$

Then the Iwasawa decomposition Theorem 9.3 can be rephrased as follows: The multiplication map

$$(9.6) \quad \Lambda G_{\hat{\tau}} \times \{(\text{Id}, \text{Id}), (\omega_0, \omega_0)\} \times \Lambda^+ G_{\hat{\tau}}^\mathbb{C} \rightarrow \Lambda G_{\hat{\tau}}^\mathbb{C}, \quad (g, \delta, w_+) \mapsto g\delta w_+$$

is an analytic diffeomorphism onto an open dense subset of $\Lambda G_{\hat{\tau}}^\mathbb{C}$.

9.3. In Section 8, we discussed potentials associated with CMC surfaces in \mathbb{H}^3 with $H \neq 1$. In this section, we give conversely a construction of CMC surfaces from potentials, the *generalized Weierstrass type representation*:

The case $H > 1$.

Step 1. Take a holomorphic potential η as defined in (8.2).

Step 2. Solve the ordinary differential equation $dC = C\eta$ with $C(z_*, \lambda) = \text{Id}$, where z_* is some base point.

Step 3. Perform the Iwasawa decomposition for $C = \Phi V_+$ by Theorem 9.2, where $\Phi \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_3}$ and $V_+ \in \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$.

Theorem 9.4. *Up to a diagonal gauge $D \in U_1$ and a change of coordinates, the matrix Φ obtained in Step 3 is the extended frame of some CMC surface with $H > 1$.*

Step 4. Inserting Φ obtained in Step 3 into the Sym formula in (5.5), we obtain a CMC surface in \mathbb{H}^3 with $H = \coth q > 1$.

Proof of Theorem 9.4. Let us compute the Maurer–Cartan form of $\Phi = CV_+^{-1}$, where $\Phi \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_3}$. A direct computation shows that

$$(9.7) \quad \alpha = \Phi^{-1} d\Phi = V_+ C^{-1} dC V_+^{-1} - dV_+ V_+^{-1}.$$

Thus α has the form $\alpha = \sum_{j=-1}^{\infty} \lambda^j \alpha_j$. Since Φ is an element in $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_3}$, α satisfies $\tau_3(\alpha) = \alpha$, where τ_3 is defined in (5.3). Therefore

$$(9.8) \quad \alpha = \lambda^{-1} \alpha_{-1} + \alpha_0 + \lambda \alpha_1,$$

where $\overline{\alpha_0} = -\alpha_0$ and $\alpha_{-1} = \alpha_1^*$. It is easy to check that α has the form (5.2) up to a diagonal gauge $D = \text{diag}(e^{i\theta}, e^{-i\theta})$ and a change of coordinates [17, Section A.8]. Since $D \in U_1$, $C = \Phi D \cdot D^{-1} V_+$ is also an Iwasawa decomposition. Thus, by Proposition 5.1, ΦD is the extended frame of some CMC surface with $H > 1$. This completes the proof. \square

The case $0 \leq H < 1$.

Step 1. Take a holomorphic potential η as defined in (8.2).

Step 2. Solve the ordinary differential equation $dC = C\eta$ with $C(z_*, \lambda) = \text{Id}$, where z_* is some base point.

Step 3. Perform the Iwasawa decomposition for C , i.e., $C = \Phi V_+$ or $C = \Phi \omega_0 V_+$ for all z in \mathbb{D} such that $C(z, \lambda)$ is in $\mathbb{D}_1 \subset \mathbb{D}$ or $\mathbb{D}_2 \subset \mathbb{D}$, respectively, where $\Phi \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$, $\omega_0 = \begin{pmatrix} 0 & \lambda^{-1} \\ -\lambda & 0 \end{pmatrix}$, $V_+ \in \Lambda^+ \text{SL}_2 \mathbb{C}_\sigma$, and \mathbb{D}_1 and \mathbb{D}_2 are open subsets in \mathbb{D} such that the real valued functions k_0 defined in Theorem 9.3 are positive and negative on $z \in \mathbb{D}_1$ and $z \in \mathbb{D}_2$, respectively.

Theorem 9.5. *Up to a diagonal gauge $D \in U_1$ and a change of coordinates, the matrices Φ and $\Phi \omega_0$ obtained in Step 3 are the extended frames of some CMC surface with $0 \leq H < 1$.*

Step 4. Inserting Φ or $\Phi \omega_0$ obtained in Step 3 into the Sym formula in (6.6), we obtain a CMC surface in \mathbb{H}^3 with $0 \leq H < 1$.

Proof of Theorem 9.5. Let us compute the Maurer–Cartan form of Φ if $C = \Phi V_+$ and for $\Phi\omega_0$ if $C = \Phi\omega_0 V_+$ with $\Phi \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$ and $V_+ \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$. Since Φ is an element in $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$, α satisfies $\tau_4(\alpha) = \alpha$, where τ_4 is defined in (6.3). Similar to the proof of Theorem 9.4, the Maurer–Cartan form of Φ and $\Phi\omega_0$, respectively, has the form

$$(9.9) \quad \alpha = \lambda^{-1}\alpha_{-1} + \alpha_0 + \lambda\alpha_1,$$

where $\overline{\alpha_0} = -\alpha_0$ and $\alpha_{-1} = i \text{Ad}(\mathcal{R})(\alpha_1)^*$. It is easy to check that, up to a diagonal gauge $D = \text{diag}(e^{i\theta}, e^{-i\theta})$ and a change of coordinates, α has the form (6.2) (see [17, Section A.8]). Since $D \in U_1$ and $D\omega_0 D = \omega_0$, we have that

$$C = \Phi D \cdot D^{-1} V_+ \quad \text{and} \quad C = \Phi\omega_0 D \cdot D^{-1} V_+ = \Phi D^{-1} \omega_0 \cdot D^{-1} V_+$$

are also Iwasawa decompositions. Thus, by Proposition 6.1, ΦD and $\Phi D^{-1} \omega_0$ are the extended frame of some CMC surface with $0 \leq H < 1$. \square

9.4. Let η be a potential, defined on \mathbb{D} , for a CMC immersion into \mathbb{H}^3 with mean curvature $0 \leq H < 1$. Let C be defined by $dC = C\eta$ and $C(z_*, \lambda) = \text{Id}$, where $z_* \in \mathbb{D}$ is some base point. In the proof of Theorem 9.3, we have seen that there exists a real analytic function $k_0 : \mathbb{D} \rightarrow \mathbb{R}$ such that the “singular set” $S_0 = \{z \in \mathbb{D} \mid k_0(z, \bar{z}) = 0\}$ divides \mathbb{D} into two open subsets \mathbb{D}_1 and \mathbb{D}_2 , where $C(z)$ is in the open Iwasawa cell containing Id for $z \in \mathbb{D}_1$ and in the other open Iwasawa cell for $z \in \mathbb{D}_2$. Let $C = \Phi V_+$ be an Iwasawa decomposition on the open Iwasawa cell containing Id for $z \in \mathbb{D}_1$. Examples like the one given in Section 10.1 show that the frame Φ associated with C will be generically singular along S_0 . Hence it does not seem to make sense to extend the immersion f associated with η in \mathbb{D}_1 across S_0 to \mathbb{D}_2 . However, in view of [20, Theorem 3.2], one can extend Φl meromorphically to $\mathbb{D} \times \bar{\mathbb{D}}$, if l is an appropriately chosen diagonal matrix independent of λ . Actually, with \tilde{k} as in the proof of Theorem 9.3 the matrices $\hat{\Phi} = \Phi \tilde{k}^{-1}$ and \tilde{k}^2 have meromorphic extension to $(z, w) \in \mathbb{D} \times \bar{\mathbb{D}}$. As a consequence, consider the CMC immersion of mean curvature $H = \tanh q$ given by the Sym formula $f = \Phi D_0 \Phi^*$ for $z \in \mathbb{D}_1$, where $D_0 = \text{diag}(e^{-q/2}, e^{q/2})$. Then we rephrase the Sym formula as

$$f = \Phi D_0 \Phi^* = (\Phi \tilde{k}^{-1}) \tilde{k}^2 D_0 (\Phi \tilde{k}^{-1})^*.$$

Then by the argument above, $\Phi \tilde{k}^{-1}$ and \tilde{k}^2 can be extended meromorphically to $(z, w) \in \mathbb{D} \times \bar{\mathbb{D}}$. Let us put $\hat{\Phi} = \Phi \tilde{k}^{-1}$ and $k = \tilde{k}^2$. Then $\hat{f} = \hat{\Phi} k D_0 \hat{\Phi}^*$ has a meromorphic extension to $(z, w) \in \mathbb{D} \times \bar{\mathbb{D}}$. We note that, on $w = \bar{z}$, $\tau_4(\hat{\Phi}) = \hat{\Phi} k$ and k has real diagonal entries, where τ_4 is defined in (6.5). We can now consider the analytic continuation from the one open Iwasawa cell for $z \in \mathbb{D}_1$, to the other open Iwasawa cell for $z \in \mathbb{D}_2$, i.e., $C = \tilde{\Phi} \omega_0 \tilde{V}_+$ is the Iwasawa decomposition for $z \in \mathbb{D}_2$. Then, it is easy to check that the entries of k are negative on $z \in \mathbb{D}_2$. Thus we can rephrase k as $k = -\check{k}^2$, where \check{k} is the diagonal matrix with positive entries which are independent of λ defined as in Theorem 9.3. Moreover $C = (\hat{\Phi} \check{k} \omega_0) \omega_0 (-\check{k}^{-1} \hat{V}_+)$ is the Iwasawa decomposition for $z \in \mathbb{D}_2$, where \hat{V}_+ is the meromorphic extension of $\check{k} V_+$. Using an obvious abbreviation $\tilde{\Phi} = \hat{\Phi} \check{k} \omega_0$, we can rephrase again the Sym formula as

$$\hat{f} = \tilde{\Phi} \omega_0 \check{k}^{-2} k D_0 \omega_0^* \tilde{\Phi}^* = -(\tilde{\Phi} \omega_0) D_0 (\tilde{\Phi} \omega_0)^*.$$

Therefore, it is natural to use for z in \mathbb{D}_2 the negative of the Sym formula (6.6) and to use this formula for $\tilde{\Phi} \omega_0$. Thus in the second open Iwasawa cell actually $\tilde{\Phi} \omega_0$ is the “frame” to use.

9.5. We have seen in Section 6 that coefficient matrices of the form (6.2) define CMC surfaces in \mathbb{H}^3 with mean curvature $0 \leq H < 1$.

In Section 9 we have shown that all such matrices can be obtained via a generalized Weierstrass type representation procedure from “potentials”. Thus we have a loop group procedure to construct all CMC surfaces in \mathbb{H}^3 with mean curvature $0 \leq H < 1$.

On the other hand, a simple minded approach would follow [22] more closely, since we know (Ishihara, Proposition 3.3) that the surfaces under consideration are characterized by the fact that their “Gauss map” (3.2) is a harmonic map into $U\mathbb{H}^3 = \text{SL}_2 \mathbb{C}/U_1$. Since $U\mathbb{H}^3$ is not a symmetric space, only a 4-symmetric space, [22] or [4] cannot be applied directly.

In Section 7 we have shown that one can indeed introduce a loop parameter into the Maurer–Cartan form α as in [12, Section 3.2], since the Gauss map is Legendre:

$$\alpha = \alpha_{\mathfrak{h}} + v^{-1}\alpha'_{\mathfrak{p}} + v\alpha''_{\mathfrak{p}}, \quad v \in \mathbb{C}^\times,$$

where $\alpha_{\mathfrak{h}}, \alpha_{\mathfrak{p}}$ are defined in (7.3). Integration of α yields an extended frame $\Phi = \Phi(z, \bar{z}, v)$.

There is no loop group method known for general harmonic maps into k -symmetric spaces if $k > 2$. However, since we were able to introduce a loop parameter one could decompose Φ anyway, say à la Birkhoff, see Theorem 8.1:

$$\Phi = \Phi_{1-}\Phi_{2+} = \Phi_{1+}\Phi_{2-}, \quad \Phi_{1\pm} \in \Lambda_*^\pm \text{SL}_2 \mathbb{C}_\sigma, \quad \Phi_{2\pm} \in \Lambda^\pm \text{SL}_2 \mathbb{C}_\sigma.$$

This would yield a potential $\xi = (\xi_-, \xi_+) = (\Phi_{1-}^{-1}d\Phi_{1-}, \Phi_{1+}^{-1}d\Phi_{1+})$.

A priori, the only information we have about ξ_\pm is that they are elements in $\Lambda^\pm \mathfrak{sl}_2 \mathbb{C}_\sigma$ with simple loop behavior.

If one would want to construct, conversely, extended frames from potentials of this type, one would run irreparably into the problem of violating the admissible condition (1.2). The simple observation that α is fixed under τ_4 , however, permits to fix this problem. Since $\tau_4(\Phi_{1-}) = \Phi_{1+}$, we have

$$\tau_4(\xi_-) = \xi_+.$$

Similar to [33], one can now show that for such potentials one can obtain an extended frame which is fixed under τ_4 .

It is not difficult to see that following [33] corresponds exactly to the approach (using an Iwasawa decomposition) explained in Sections 6 and 9.

9.6. Among the classes of surfaces that can be constructed by integrable system methods the surfaces of “finite type” have received special attention. This is due to the fact that these surfaces can be constructed fairly explicitly. Moreover, the algebro-geometric methods used for these surfaces are classical and beautiful. These methods have been highly successful for the construction of tori. On the other hand, since until now only surfaces without umbilical points can be constructed by algebro-geometric methods, it is necessary to develop more general methods which also work when umbilical points are present.

In the context of this paper the property of being of finite type can be expressed on all levels of our construction scheme. We sketch these constructions, but will not present any proofs.

The starting point for the construction of a CMC surface in \mathbb{H}^3 with $0 \leq H < 1$ in this paper are “potentials”. In the spirit of [12] and [23], we call a potential η to be of “finite type” if there exists a Laurent polynomial $\xi^o(\lambda)$ satisfying $\tau_4\xi^o = \xi^o$ such that $\eta = \lambda^{d-1}\xi^o(\lambda)dz$,

where d is odd and the maximal degree of λ occurring in ξ^o . In our construction scheme one obtains from η a CMC surface in \mathbb{H}^3 with $0 \leq H < 1$. Surfaces constructed from some ξ^o as above are usually called to be of *Symes finite type* in the sense of [23].

Our construction scheme produces from η an ODE solution $dC = C\eta$ satisfying $C(z_*, \lambda) = \text{Id}$ and from C via Iwasawa decomposition $C = \Phi V_+$, $\Phi(z_*, \bar{z}_*, \lambda) = \text{Id}$, the frame Φ of the CMC surface constructed from η . As in [12] (see also [23]) one can show the following theorem:

Theorem 9.6. *The Maurer–Cartan form $\alpha = \Phi^{-1}d\Phi$ can be represented in the form*

$$(9.10) \quad \alpha = \alpha_\xi = \left(\lambda^{-1} \xi_{-d} + \frac{1}{2} \xi_{-d+1} \right) dz + \tau_4 \left(\lambda^{-1} \xi_{-d} + \frac{1}{2} \xi_{-d+1} \right) d\bar{z},$$

where $\xi = \sum_{n \in \mathbb{Z}} \xi_n$ satisfies $\tau_4 \xi = \xi$ and the Lax equation

$$(9.11) \quad d\xi = [\xi, \alpha_\xi], \quad \xi(z_*) = \xi^o.$$

Moreover the ξ is polynomial in λ and satisfies

$$(9.12) \quad \xi = \Phi^{-1} \xi^o \Phi.$$

Such a ξ is called the polynomial Killing field.

Remark. In view of $\Phi = C V_+^{-1}$ one obtains that the degree of ξ is equal to the degree of ξ^o . Again following [12] (see also [23]) one calls a frame Φ to be of *finite type* if there exists some ξ^o such that the solution to (9.11) satisfies $\Phi^{-1}d\Phi = \alpha_\xi$. Moreover, a CMC surface in \mathbb{H}^3 with $0 \leq H < 1$ is called of *finite type* if its frame is of finite type. With this notation the theorem above states that finite type surfaces are equivalent to Symes finite type surfaces.

There are basically two ways to treat finite type surfaces algebro-geometrically. One way is to consider the spectral curve $Y = \{(\lambda, \mu) \in \mathbb{C}^\times \times \mathbb{C} \mid \det(\mu \cdot \text{Id} - \xi(z, \lambda)) = 0\}$ and the eigenline bundle \mathcal{L} of ξ over Y . This approach was carried out by I. McIntosh [35] for finite type harmonic maps into $\mathbb{C}P^n$ and it seems that his method should also apply to our case. The second approach is due to A. Bobenko, it produces algebro-geometric solutions to the complex sine-Gordon equation from line bundles over hyperelliptic curves via the corresponding theta functions. For the surfaces considered in this paper this was carried out in [3].

In the case of CMC surfaces in \mathbb{E}^3 , the relation between the loop group method as used in this paper and the approach of Bobenko was established in [18]. It would be interesting to see whether also for CMC surfaces of finite type in \mathbb{H}^3 a relation with Bobenko's approach can be realized in a similar fashion.

10. Examples of CMC immersions with $0 \leq H < 1$ in \mathbb{H}^3

In [42], examples of CMC immersions with $H > 1$ have been constructed from holomorphic potentials. In this section, we present some examples of CMC immersions with $0 \leq H < 1$ starting from holomorphic potentials. From now on we assume always $q \in \mathbb{R}_{\geq 0}$ and use the symbol λ instead of v . We recall that the resulting immersion by the Sym formula $f_{\lambda=e^{-q/2}}$ in (6.6) has the mean curvature $0 \leq H = \tanh q < 1$.

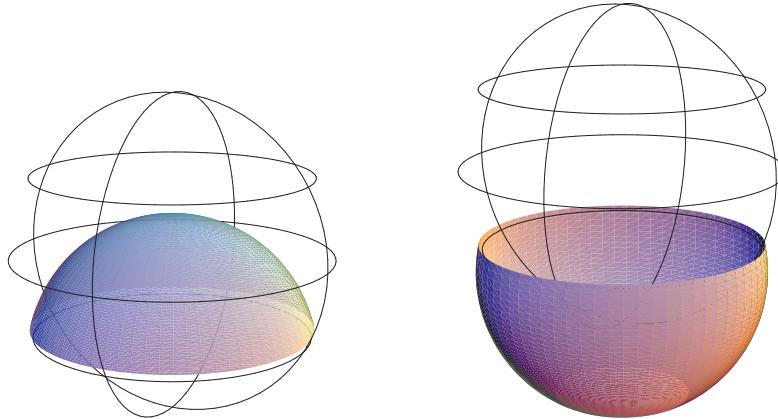


Figure 1. An equidistance surface with $H = \tanh(1)$, $|z| < 1$ (left) and an equidistance surface with $H = \tanh(1)$, $|z| > 1$ (right). Surfaces are shown in the Poincaré ball model and the outside of the Poincaré ball model, respectively.

All the examples given below have the feature that there exist some curves \mathcal{C} along which are immersions which tend to infinity. Each of the connected components of the immersion from $M \setminus \mathcal{C}$ is well defined and takes values in the forward \mathbb{H}^3 or the backward \mathbb{H}^3 (in the light cone picture of hyperbolic space). We hope to study this behavior in detail in a separate publication.

10.1. Umbilical surfaces (Figure 1). Assume an CMC immersion f with $0 \leq H < 1$ to be totally umbilical. It is easy to see that the Hopf differential Qdz^2 of an umbilic CMC immersion with $0 \leq H < 1$ vanishes, i.e., $Q = 0$. Then the corresponding normalized potential ξ as in (8.3) has the form

$$\xi = \lambda^{-1} \begin{pmatrix} 0 & h(z) \\ 0 & 0 \end{pmatrix} dz,$$

where $h(z)$ is a non-vanishing meromorphic function. Using the coordinate change $w = \int h dz$, ξ has the form defined in

$$(10.1) \quad \eta = \begin{pmatrix} 0 & \lambda^{-1} \\ 0 & 0 \end{pmatrix} dz.$$

Let us compute totally umbilic surfaces explicitly from holomorphic potentials η defined in (10.1). A solution C to the holomorphic differential equation $dC = C\eta$ with the above η and the Iwasawa decomposition for $C = \Phi V_+$ can be computed explicitly as follows:

$$C = \begin{pmatrix} 1 & \lambda^{-1}z \\ 0 & 1 \end{pmatrix} = \Phi V_+ = \begin{pmatrix} \frac{1}{\sqrt{1-|z|^2}} & \frac{\lambda^{-1}z}{\sqrt{1-|z|^2}} \\ \frac{\lambda\bar{z}}{\sqrt{1-|z|^2}} & \frac{1}{\sqrt{1-|z|^2}} \end{pmatrix} \begin{pmatrix} \frac{1}{\sqrt{1-|z|^2}} & 0 \\ \frac{-\lambda\bar{z}}{\sqrt{1-|z|^2}} & \sqrt{1-|z|^2} \end{pmatrix}.$$

The Sym formula (6.6) for the above Φ with $\lambda = e^{-q/2}$ gives

$$f_\lambda = \frac{1}{1-|z|^2} \begin{pmatrix} e^{-q/2} + e^{3q/2}|z|^2 & (e^{-q} + e^q)z \\ (e^{-q} + e^q)\bar{z} & e^{q/2} + e^{-3q/2}|z|^2 \end{pmatrix}.$$

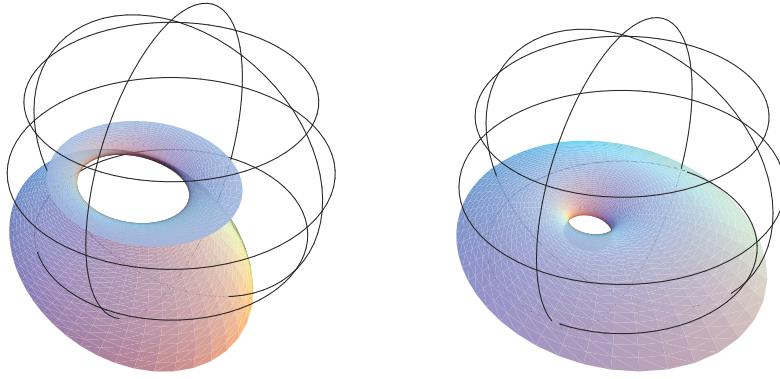


Figure 2. A portion of a surface of revolution with $H = \tanh(0.3)$ and $a = 0.3$ (left) and a portion of a minimal surface of revolution with $a = 0.2$ (right). Surfaces are shown in the Poincaré ball model and the outside of the Poincaré ball model, respectively.

Thus the resulting immersion is a totally umbilic surface, which is called an *equidistance surface* for $0 < H < 1$ and a *totally geodesic surface* for $H = 0$ (see [43, p. 77, Theorem 29]).

10.2. Surfaces of revolution (Figure 2).

We set

$$(10.2) \quad \eta = Adz = \begin{pmatrix} 0 & \lambda^{-1}a + \lambda b \\ \lambda^{-1}b - \lambda a & 0 \end{pmatrix} dz,$$

where $a, b \in \mathbb{R}$ and $b^2 - a^2 + ab(e^q - e^{-q}) = 1/4$. A solution to the holomorphic differential equation $dC = C\eta$ with the above potential η is

$$(10.3) \quad C = \exp(Az).$$

Let $\gamma : z \mapsto z + p$ be an automorphism of \mathbb{C} with $p \in \mathbb{C}^\times$. Noting that $\tau_4 A = -A$, it is easy to check that $\gamma^* C = C(p, \lambda)C$ and $C(p, \lambda) \in \Lambda \mathrm{SL}_2 \mathbb{C}_{\sigma, \tau_4}$ for all $p \in i\mathbb{R}$ and $|\lambda| = e^{-q/2}$.

Let p be purely imaginary and let $C = \Phi V_+$ denote the Iwasawa decomposition of C . Then the Iwasawa decomposition for $\gamma^* C = \hat{\Phi} \hat{V}_+$ can be computed as $\hat{\Phi} = C(p, \lambda)\Phi$ and $\hat{V}_+ = V_+$. Thus $C_q(p) = C(p, \lambda = e^{-q/2})$ acts on the resulting immersion f_λ at $\lambda = e^{-q/2}$, which is denoted by f_q , as a one parameter group of isometries with group parameter $p \in i\mathbb{R}$:

$$(10.4) \quad \gamma^* f_q = C_q(p) f_q C_q(p)^*.$$

Moreover, $C_q(p)$ has the eigenvalues

$$\exp\left(\pm p \sqrt{b^2 - a^2 + ab(\lambda^{-2} - \lambda^2)}\right) \Big|_{\lambda=e^{-q/2}} = \exp(\pm p/2).$$

Since $C_q(p) = \pm \mathrm{Id}$ for $p = 2k\pi i$ with $k \in \mathbb{Z}$, the resulting immersion closes up for $p = 2k\pi i$, i.e., $\gamma^* f_q = f_q$. The resulting immersion defines a surface of revolution with respect to the 1-parameter group (10.4). As its profile curve one can choose:

$$g(x) = \Phi \begin{pmatrix} e^{-q/2} & 0 \\ 0 & e^{q/2} \end{pmatrix} \Phi^* \Big|_{z=x, \lambda=e^{-q/2}},$$

where $x \in \mathbb{R}$. For the axis of the surface of revolution we obtain

$$\ell = \{X \in \text{Her}_2 \mathbb{C} \mid C_q(p)X - XC_q(p)^{*^{-1}} = 0, p \in i\mathbb{R}\} \cap \mathbb{H}^3.$$

Even though the axis ℓ could be an empty set, it is natural to call the resulting immersion a surface of revolution [36].

10.3. Radially symmetric surfaces.

We set

$$\eta = \begin{pmatrix} 0 & \lambda^{-1} \\ z^k \lambda & 0 \end{pmatrix} dz,$$

where $k \in \mathbb{N}$. Set $\epsilon : (z, \lambda) \mapsto (az, a^{(k+2)/2}\lambda)$. Then we have $\epsilon^*\eta = T\eta T^{-1}$, where $T = \text{diag}(a^{k/4}, a^{-k/4})$. Moreover, we assume that $a \in \mathbb{S}^1$. Let C denote the solution to $dC = C\eta$, $C(0, \lambda) = \text{Id}$. Let $C = \Phi V_+$ be the Iwasawa decomposition of C and assume that Φ is in the first open Iwasawa cell. Then the Iwasawa decomposition for ϵ^*C is $(\epsilon^*\Phi)(\epsilon^*V_+)$, thus $\epsilon^*\Phi$ is again in the first open Iwasawa cell. Moreover $\epsilon^*\Phi$ and ϵ^*V_+ can be computed as $\epsilon^*\Phi = T\Phi T^{-1}$ and $\epsilon^*V_+ = TV_+T^{-1}$. Since $V_+(\lambda = 0)$ is diagonal, the last relation shows that $V_+(\lambda = 0)$ only depends on $|z|$. In view of $C = \Phi V_+$, comparison of the potential considered in this section with (6.2) shows that the metric only depends on $|z|$. The resulting surface is a radially symmetric surface. The minimal surface of this type has been investigated in [38].

10.4. Cylinders.

We set

$$\eta = S dz = \begin{pmatrix} 0 & (a\lambda^{-1} + b\lambda)h(z) \\ (-b\lambda^{-1} + a\lambda)\bar{h}(\bar{z}) & 0 \end{pmatrix} dz,$$

where $h(z)$ is a periodic holomorphic function on \mathbb{C} with period $p \in \mathbb{R}^\times$ and $a, b \in \mathbb{R}^\times$. Moreover we assume $b = ae^{-q}$. It is easy to check that S has the following two properties: $\tau_4 S = S$ and $S(z, \lambda = e^{-q/2})$ is an upper triangular matrix.

Let $C(z, \lambda)$ denote the solution to $dC = C\eta$, $C(z = 0, \lambda) = \text{Id}$. By the Picard–Lindelöf iteration, C can be computed as

$$C = \text{Id} + \int_0^z C\eta dt + \int_0^z \left(\int_0^t \eta dt_1 \right) \eta dt + \dots$$

Set $\gamma : z \mapsto z + p$ with $p \in \mathbb{R}^*$. Then η is invariant under γ , and we obtain $\gamma^*C = C(p)C$. From the properties of $S(z, \lambda)$ stated above, we obtain $C(p) \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$ and

$$C(p)|_{\lambda=e^{-q/2}} = \begin{pmatrix} 1 & \int_0^p h(t)dt \\ 0 & 1 \end{pmatrix}.$$

Let $C = \Phi V_+$ be the Iwasawa decomposition of C . Then γ^*C has the Iwasawa decomposition $\gamma^*C = \hat{\Phi} \hat{V}_+$, where $\hat{\Phi} = C(p)F$ and $\hat{V}_+ = V_+$, since $C(p) \in \Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4}$. Therefore the immersion f_q obtained by the Sym formula closes up, i.e., $\gamma^*f_q = f_q$, if and only if $\int_0^p h(t)dt = 0$. It is not difficult to find periodic holomorphic functions $h(z)$ with period p satisfying this condition.

10.5. Totally symmetric surfaces.

We set

$$(10.5) \quad \eta = - \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda X^2 & 0 \end{pmatrix} \frac{dz}{z} + \begin{pmatrix} 0 & \lambda^{-1} \\ \lambda X^2 & 0 \end{pmatrix} \frac{dz}{z-1} + \begin{pmatrix} 0 & 0 \\ \lambda(X^2 - \frac{1}{4}) & 0 \end{pmatrix} dz,$$

where $a, b \in \mathbb{R}$, $X = \sqrt{b^2 - a^2 + ab(\lambda^{-2} - \lambda^2)}$ with $b^2 - a^2 + ab(e^q - e^{-q}) = 1/4$. We assume that

$$(10.6) \quad b^2 \not\equiv a^2 \pmod{S} \quad \text{and} \quad -1 + 4 \sin^2(\pi X) \neq 0 \quad \text{on } \lambda \in \mathbb{S}^1,$$

where $S = \{y \in \mathbb{Z} \mid x^2 = y \text{ for some } x \in \mathbb{Z}\}$. It is known that one can construct CMC immersions from $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$ into \mathbb{E}^3 from potentials η which have the form (10.5). However, in this case a and b need to be chosen differently, see [25] and [42]. It turns out that the conditions (10.6) are equivalent to the fact that monodromy matrices can be simultaneously conjugated into $\Lambda \text{SL}_2 \mathbb{C}_{\sigma, \tau_4} \subset \Lambda \text{SL}_2 \mathbb{C}_{\sigma}$. In our case, following an analogous procedure, it can be shown that the resulting CMC immersion with $H = \tanh q$ in \mathbb{H}^3 given by the holomorphic potential in (10.5) is well-defined on $\mathbb{C}P^1 \setminus \{0, 1, \infty\}$. The detailed computation has been discussed in [32].

References

- [1] M. T. Anderson, Complete minimal varieties in hyperbolic space, *Invent. Math.* **69** (1982), 477–494.
- [2] V. I. Arnol'd, S. M. Guseñ-Zade and A. N. Varchenko, Singularities of differentiable maps, vol. I, Monogr. Math. **82**, Birkhäuser, Boston 1985.
- [3] M. Babich and A. Bobenko, Willmore tori with umbilic lines and minimal surfaces in hyperbolic space, *Duke Math. J.* **72** (1993), 151–185.
- [4] V. Balan and J. Dorfmeister, Weierstrass-type representation for harmonic maps into general symmetric spaces via loop groups, *J. Math. Soc. Japan* **57** (2005), 69–94.
- [5] A. Bobenko, Surfaces of constant mean curvature and integrable equations, *Russian Math. Surveys* **46** (1991), 1–45.
- [6] R. L. Bryant, A duality theorem for Willmore surfaces, *J. Differential Geom.* **20** (1984), 23–53.
- [7] R. L. Bryant, Surfaces of mean curvature one in hyperbolic space, *Théorie des variétés minimales et applications* (Palaiseau 1983–1984), Astérisque **154-155** (1988), 321–347.
- [8] L. Bungart, On analytic fiber bundles, I: Holomorphic fiber bundles with infinite dimensional fibers, *Topology* **7** (1968), 55–68.
- [9] F. E. Burstall, Harmonic tori in spheres and complex projective spaces, *J. reine angew. Math.* **469** (1995), 149–177.
- [10] F. E. Burstall and U. Hertrich-Jeromin, Harmonic maps in unfashionable geometries, *Manuscripta Math.* **108** (2002), 171–189.
- [11] F. E. Burstall and I. Khemar, Twistors, 4-symmetric spaces and integrable systems, *Math. Ann.* **344** (2009), 451–461.
- [12] F. E. Burstall and F. Pedit, Harmonic maps via Adler-Kostant-Symes theory, in: *Harmonic maps and integrable systems*, Aspects Math. **E23**, Vieweg, Braunschweig (1994), 221–272.
- [13] F. E. Burstall and J. H. Rawnsley, Twistor theory for Riemannian symmetric spaces with applications to harmonic maps of Riemann surfaces, *Lecture Notes in Math.* **1424**, Springer, Berlin 1990.
- [14] J. Cheeger and D. G. Ebin, Comparison theorems in Riemannian geometry, AMS Chelsea Publishing, Providence 2008. Revised reprint of the 1975 original.
- [15] D. Chopp and J. A. Velling, Foliations of hyperbolic space by constant mean curvature surfaces sharing ideal boundary, *Experiment. Math.* **12** (2003), 339–350.
- [16] J. Dorfmeister, Generalized Weierstraß representation of surfaces, in: *Surveys on geometry and integrable systems*, Adv. Stud. Pure Math. **51**, Mathematical Society of Japan, Tokyo (2008), 55–111.
- [17] J. Dorfmeister and G. Haak, Meromorphic potentials and smooth surfaces of constant mean curvature, *Math. Z.* **224** (1997), 603–640.

- [18] J. Dorfmeister and G. Haak, On constant mean curvature surfaces with periodic metric, *Pacific J. Math.* **182** (1998), 229–287.
- [19] J. Dorfmeister, J. Inoguchi and M. Toda, Weierstraß-type representation of timelike surfaces with constant mean curvature, in: *Differential geometry and integrable systems (Tokyo 2000)*, Contemp. Math. **308**, American Mathematical Society, Providence (2002), 77–99.
- [20] J. Dorfmeister and S.-P. Kobayashi, Coarse classification of constant mean curvature cylinders, *Trans. Amer. Math. Soc.* **359** (2007), 2483–2500.
- [21] J. Dorfmeister, I. McIntosh, F. Pedit and H. Wu, On the meromorphic potentials for a harmonic surfaces in a k -symmetric space, *Manuscripta Math.* **92** (1997), 143–152.
- [22] J. Dorfmeister, F. Pedit and H. Wu, Weierstrass type representation of harmonic maps into symmetric spaces, *Comm. Anal. Geom.* **6** (1998), 633–668.
- [23] J. Dorfmeister and I. Sterling, Finite type Lorentz harmonic maps and the method of Symes, *Differential Geom. Appl.* **17** (2002), 43–53.
- [24] J. Dorfmeister and H. Wu, Constant mean curvature surfaces and loop groups, *J. reine angew. Math.* **440** (1993), 43–76.
- [25] J. Dorfmeister and H. Wu, Construction of constant mean curvature n -noids from holomorphic potentials, *Math. Z.* **258** (2008), 773–803.
- [26] C. L. Epstein, The hyperbolic Gauss map and quasiconformal reflections, *J. reine angew. Math.* **372** (1986), 96–135.
- [27] S. Helgason, Differential geometry, Lie groups, and symmetric spaces, *Grad. Stud. Math.* **34**, American Mathematical Society, Providence (2001). Corrected reprint of the 1978 original.
- [28] J. Inoguchi, Spacelike surfaces and harmonic maps, preprint 2008.
- [29] T. Ishihara, The harmonic Gauss maps in a generalized sense, *J. London Math. Soc. (2)* **26** (1982), 104–112.
- [30] G. R. Jensen and M. Rigoli, Harmonic Gauss maps, *Pacific J. Math.* **136** (1989), 261–282.
- [31] S. Kaneyuki and M. Kozai, Paracomplex structures and affine symmetric spaces, *Tokyo J. Math.* **8** (1985), 81–98.
- [32] S.-P. Kobayashi, Totally symmetric surfaces of constant mean curvature in hyperbolic 3-space, *Bull. Aust. Math. Soc.* **82** (2010), 240–253.
- [33] S.-P. Kobayashi, Real forms of complex surfaces of constant mean curvature, *Trans. Amer. Math. Soc.* **363** (2011), 1765–1788.
- [34] H. B. Lawson, Jr., Complete minimal surfaces in S^3 , *Ann. of Math. (2)* **92** (1970), 335–374.
- [35] I. McIntosh, A construction of all non-isotropic harmonic tori in complex projective space, *Internat. J. Math.* **6** (1995), 831–879.
- [36] H. Mori, Stable complete constant mean curvature surfaces in \mathbf{R}^3 and H^3 , *Trans. Amer. Math. Soc.* **278** (1983), 671–687.
- [37] E. Musso and L. Nicolodi, A variational problem for surfaces in Laguerre geometry, *Trans. Amer. Math. Soc.* **348** (1996), 4321–4337.
- [38] V. Y. Novokshenov, Minimal surfaces in the hyperbolic space and radial-symmetric solutions of the cosh-Laplace equation, in: *Algebraic and geometric methods in mathematical physics (Kaciveli 1993)*, Math. Phys. Stud. **19**, Kluwer Academic Publishers, Dordrecht (1996), 357–370.
- [39] B. O’Neill, Semi-Riemannian geometry with applications to relativity, Pure and Applied Mathematics **103**, Academic Press, New York 1983.
- [40] K. Polthier, Geometric a priori estimates for hyperbolic minimal surfaces, *Bonner Math. Schriften* **263**, Universität Bonn, Bonn 1994.
- [41] A. Pressley and G. Segal, Loop groups, Oxford Mathematical Monographs, Clarendon Press, Oxford 1986.
- [42] N. Schmitt, M. Kilian, S.-P. Kobayashi and W. Rossman, Unitarization of monodromy representations and constant mean curvature trinoids in 3-dimensional space forms, *J. London Math. Soc. (2)* **75** (2007), 563–581.
- [43] M. Spivak, A comprehensive introduction to differential geometry, vol. IV, 2nd ed., Publish or Perish, Wilmington 1979.
- [44] Y. Tonegawa, Existence and regularity of constant mean curvature hypersurfaces in hyperbolic space, *Math. Z.* **221** (1996), 591–615.
- [45] K. K. Uhlenbeck, Closed minimal surfaces in hyperbolic 3-manifolds, in: *Seminar on minimal submanifolds*, Ann. of Math. Stud. **103**, Princeton University Press, Princeton (1983), 147–168.
- [46] M. Umehara and K. Yamada, Complete surfaces of constant mean curvature 1 in the hyperbolic 3-space, *Ann. of Math. (2)* **137** (1993), 611–638.
- [47] J. A. Velling, Existence and uniqueness of complete constant mean curvature surfaces at infinity of \mathbf{H}^3 , *J. Geom. Anal.* **9** (1999), 457–489.

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Eingegangen 2. Dezember 2009, in revidierter Fassung 15. Dezember 2011