



## Anomaly-safe discrete groups

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### ABSTRACT

We show that there is a class of finite groups, the so-called perfect groups, which cannot exhibit anomalies. This implies that all non-Abelian finite simple groups are anomaly-free. On the other hand, non-perfect groups generically suffer from anomalies. We present two different ways that allow one to understand these statements.

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### 1. Introduction

It is well known that discrete symmetries may be anomalous [1]. If this is the case, this can have important consequences for phenomenology. It implies that the symmetry is violated (at least) at the non-perturbative level. Originally, anomaly constraints for Abelian finite groups have been derived by considering U(1) symmetries that get spontaneously broken to  $\mathbb{Z}_N$  [2–4]. An arguably more direct derivation is based on the path integral approach [5,6], which can also be applied to discrete symmetries [7,8]. In this approach, a given symmetry operation is said to be anomalous if it implies a non-trivial transformation of the path integral measure. From this it is straightforward to see that there are no cubic anomalies for global symmetries. We can hence limit our discussion to anomalies of the type  $D$ – $G$ – $G$ , where  $D$  denotes the discrete symmetry and  $G$  the continuous gauge group of the setting, respectively.

An alternative approach to ensure anomaly freedom is to start with an anomaly-free continuous symmetry and breaking it to a discrete subgroup [9,10]. One then obtains embedding constraints, which guarantee anomaly-freedom but generally are more restrictive than the true anomaly constraints. In this work, we use the path integral approach to discuss anomalies of discrete symmetries and focus on the true anomaly constraints.

As noted already in [11], the  $D$ – $G$ – $G$  anomaly coefficient vanishes if  $D$  is a so-called perfect group because then the generators of  $D$  are traceless, in close analogy to the Lie group case. In this study, we present a more thorough discussion of the argument. We then present an alternative argument, based on the observation that the path integral measure transforms in a one-dimensional representation of the discrete group  $D$ . Besides offering an alternative but completely equivalent proof that perfect groups are anomaly-free, this allows us to conclude that all non-perfect groups generically have anomalies. Nevertheless, there exist particular non-perfect discrete groups  $D$  such that for  $G = \text{SO}(N)$  or any of the exceptional groups the  $D$ – $G$ – $G$  anomaly vanishes independently of the field content, and we will give a criterion when this is the case.

Examples for perfect and thus anomaly-safe groups are all non-Abelian finite simple groups. This includes, for example, the alternating groups  $A_n$  for  $n \geq 5$ , the projective special linear groups  $\text{PSL}(n, k)$  for  $n > 1$  and finite fields  $k$  with more than three elements, and also the sporadic groups. An example for groups which are not simple yet perfect and anomaly-safe are the special linear groups  $\text{SL}(n, k)$  with  $n > 1$  and  $k > 3$  [12]. Furthermore, the (semi-)direct product of two perfect groups is again a perfect group (as proven in Appendix A).

On the other hand, all non-perfect groups generically can suffer from anomalies. For example, this includes  $A_4$ ,  $T'$ ,  $T_7$ ,  $S_n$ ,  $D_n$ , which have been utilized frequently in model building, and in general all groups that have at least one non-trivial one-dimensional representation (cf. [13] for an extensive list of discrete groups).

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**Table 1**  
Dynkin indices of the fundamental representations  $\mathbf{F}$  for the simple compact Lie groups.

$G$	$SU(N)$	$Sp(N)$	$SO(N)$	$G_2$	$F_4$	$E_6$	$E_7$	$E_8$
$\ell(\mathbf{F})$	1/2	1/2	1	1	3	3	6	30

As we shall also discuss, anomalies of finite groups can always be cancelled by a discrete version of the Green–Schwarz (GS) mechanism [14]. However, in this case the symmetry is not exact, i.e. there exist certain terms that violate it.

## 2. Anomalies of discrete groups

Let us start by discussing a quantum field theory with a finite discrete symmetry  $D$ . For definiteness, we consider the case that there is also a non-Abelian gauge symmetry  $G$ , noting that our arguments also hold for Abelian gauge factors and gravity.

Furthermore, we assume that there is a set of Dirac fermions  $\Psi$  charged under  $D$  and transforming in a representation  $\mathbf{R}$  under  $G$ . Given an element  $u \in D$  let  $U_{\mathbf{r}}(u)$  be the unitary representation matrix of  $u$  in the unitary representation  $\mathbf{r}$ . For finite groups, there always exists an integer  $M_u$  such that  $u^{M_u} = e$ . This allows us to write

$$U_{\mathbf{r}}(u) = e^{2\pi i \lambda_{\mathbf{r}}(u) / M_u}, \quad (2.1)$$

with a matrix  $\lambda_{\mathbf{r}}(u)$  that has integer eigenvalues. Let us now investigate a discrete chiral transformation under which the left-handed fermion fields  $\Psi_L := P_L \Psi$  transform as

$$\Psi_L \rightarrow U_{\mathbf{r}}(u) \Psi_L = e^{2\pi i \lambda_{\mathbf{r}}(u) / M_u} \Psi_L, \quad (2.2)$$

where  $P_L$  is the left-chiral projector and  $\mathbf{r}$  is the representation of  $\Psi_L$  under  $D$ .

The transformation of fermion fields induces, in general, a transformation of the path integral measure

$$\mathcal{D}\Psi \mathcal{D}\bar{\Psi} \rightarrow J_{\Psi}^{-2} \mathcal{D}\Psi \mathcal{D}\bar{\Psi} \quad (2.3)$$

with possibly non-trivial Jacobian  $J_{\Psi}$ . For the set of fields  $\Psi$  the Jacobian under the transformation  $u$  is given by

$$J_{\Psi}^{-2} = \exp \left\{ i \frac{2\pi}{M_u} \text{tr}[\lambda_{\mathbf{r}}(u)] \cdot \ell(\mathbf{R}) \cdot \int d^4x \frac{1}{16\pi^2} F^{a,\mu\nu} \tilde{F}_{\mu\nu}^a \right\}. \quad (2.4)$$

Here,  $F_{\mu\nu} := F_{\mu\nu}^a T_a$  denotes the field strength tensor of the gauge group  $G$  with generators  $T_a$ , and  $\tilde{F}^{\mu\nu} := \frac{1}{2} \varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$  its dual. Our conventions are such that  $F_{\mu\nu} := i[D_{\mu}, D_{\nu}]$  for the covariant derivative  $D_{\mu} := \partial_{\mu} - iA_{\mu}$ .

The Dynkin index of the corresponding gauge group representation  $\ell(\mathbf{R})$  is defined as usual,

$$\delta_{ab} \ell(\mathbf{R}) := \text{tr} [T_a(\mathbf{R}) T_b(\mathbf{R})]. \quad (2.5)$$

We fix the Dynkin index following the conventions of [15].<sup>1</sup> For the simple compact Lie groups, the resulting Dynkin index  $\ell(\mathbf{F})$  of the fundamental representation  $\mathbf{F}$ , which is always taken to be (one of) the smallest dimensional representation(s), is shown in Table 1.

We define

$$p := \int d^4x \frac{1}{32\pi^2} F^{a,\mu\nu} \tilde{F}_{\mu\nu}^a \quad (2.6)$$

<sup>1</sup> This amounts to normalizing the length of the longest root to unity. The Dynkin index of the adjoint representation is then the same as the dual Coxeter number of the group.

which in our convention is an integer [15,16] in order to simplify equation (2.4) to

$$J_{\Psi}^{-2} = \exp \left\{ i \frac{2\pi}{M_u} p \cdot \text{tr}[\lambda_{\mathbf{r}}(u)] \cdot 2 \ell(\mathbf{R}) \right\}. \quad (2.7)$$

When performing the gauge-field path integral, i.e. integrating over all gauge-field configurations,  $p$  assumes all integer values. Therefore, we have to discuss the anomaly independently of the exact value of  $p$  and can take advantage only of the fact that it is integer.

In case there are multiple fermions in the theory, their contribution to the path integral measure is the product of their respective Jacobians. This amounts to summing up the individual contributions in the exponential. Thus, the overall effect on the path integral measure due to a transformation  $u$ , which generates a cyclic group  $\mathbb{Z}_{M_u}$ , can be summarized by defining the anomaly coefficient

$$A_{G-G-\mathbb{Z}_{M_u}} := \sum_f \text{tr}[\lambda_{\mathbf{r}(f)}(u)] \cdot 2 \ell(\mathbf{R}^{(f)}). \quad (2.8)$$

Here, the sum runs over all chiral fermions  $f$  transforming in representations  $\mathbf{r}^{(f)}$  under  $D$  and in representations  $\mathbf{R}^{(f)}$  under  $G$ . Note that by the inversion of equation (2.1) one obtains

$$\text{tr}[\lambda_{\mathbf{r}}(u)] = \frac{M_u}{2\pi i} \ln \det U_{\mathbf{r}}(u). \quad (2.9)$$

Since the trace of  $\lambda_{\mathbf{r}}(u)$  is only fixed modulo  $M_u$  due to the multi-valued logarithm and because  $2 \ell(\mathbf{R})$  is integer,  $A_{G-G-\mathbb{Z}_{M_u}}$  is only defined modulo  $M_u$ .

In general it is possible that

$$A_{G-G-\mathbb{Z}_{M_u}} \neq 0 \pmod{M_u}, \quad (2.10)$$

implying that the overall Jacobian  $J$  is different from one and the group generated by  $u$  is anomalous.

*Perfect groups are anomaly-safe* Let us now consider the particular case that the group  $D$  is a perfect group. A perfect group, by definition, equals its commutator subgroup, see also Appendix A. As such, all generating elements  $d \in D$  of the group (but, in general, not all elements) can be written as the (group-theoretical) commutator

$$d = [v, w] := (v w v^{-1} w^{-1}), \quad (2.11)$$

of some group elements  $v, w \in D$ . This implies that any group element  $u \in D$  can be written as a product of commutators

$$u = \prod_i [v_i, w_i], \quad (2.12)$$

where  $v_i, w_i \in D$ . Irrespective of the particular representation, any representation matrix can thus be written as

$$U_{\mathbf{r}}(u) = \prod_i \left( U_{\mathbf{r}}(v_i) U_{\mathbf{r}}(w_i) U_{\mathbf{r}}(v_i)^{-1} U_{\mathbf{r}}(w_i)^{-1} \right). \quad (2.13)$$

This shows that  $\det U_{\mathbf{r}}(u) = 1$ , implying that the generator matrix  $\lambda_{\mathbf{r}}(u)$  in equation (2.9) is traceless for all representations  $\mathbf{r}$  of the perfect group  $D$ . Therefore, no element of a perfect group can give rise to a non-trivial anomaly coefficient. From this we conclude that for perfect groups all anomalies vanish [11].

*The Jacobian as a one-dimensional representation of  $D$*  Let us now discuss a possibly more intuitive way to arrive at the same conclusion. Using equation (2.9), the Jacobian (2.7) can be written as

$$J_{\Psi}^{-2} = \det (U_{\mathbf{r}}(u))^{2 \ell(\mathbf{R}) \cdot p}, \quad (2.14)$$

which might be more useful than (2.7) for finite groups since it does not refer to the generators but directly uses the representation matrices to express the anomaly. Thus, a transformation  $u$  is anomaly-free if and only if

$$\prod_f \det(U_{r(f)}(u))^{2\ell(\mathbf{R}^{(f)})} = 1. \quad (2.15)$$

Note that the determinant of any representation is a well-defined one-dimensional representation because

$$\begin{aligned} \det(U_{\mathbf{r}}(uv)) &= \det(U_{\mathbf{r}}(u)U_{\mathbf{r}}(v)) \\ &= \det(U_{\mathbf{r}}(u))\det(U_{\mathbf{r}}(v)), \end{aligned} \quad (2.16)$$

and, furthermore, any integer power of a one-dimensional representation is again a one-dimensional representation.

Since the exponent in (2.14) is integer, we conclude that  $J_{\Psi}^{-2}$  transforms in a one-dimensional representation of  $D$ . In case there are multiple fermions, the transformation of the total path integral measure  $J^{-2}$  is obtained as the direct product of the single one-dimensional representations of the individual Jacobians, which is again a well-defined one-dimensional representation of  $D$ .

The statement that perfect groups are free of anomalies can now be understood in a different but completely equivalent way. One can show (for a proof see Appendix A) that the following statements are equivalent:

- (i) a finite group  $D$  is perfect.
- (ii)  $D$  has exactly a single one-dimensional representation, namely the trivial one.

Furthermore, by the arguments laid out above, the path integral measure always transforms in a one-dimensional representation. Thus, for settings based on perfect groups, the path integral measure can only transform in the trivial representation, i.e. it does not transform at all and perfect groups are anomaly-safe.

Let us remark that the absence of non-trivial one-dimensional representations for perfect groups implies that they cannot be used as non-Abelian discrete  $R$  symmetries with  $\mathcal{N} = 1$  supersymmetry [11]. Moreover, model building (e.g. for flavor physics) with perfect groups is generally more restrictive because potentials with only multi-dimensional representations tend to be more constrained.

*Non-perfect groups and anomalies* Consider now the case of a discrete group  $D$  which is not perfect. It follows from the above equivalence that non-perfect groups always have at least one non-trivial one-dimensional representation. Consequently, theories based on non-perfect groups can be anomalous depending on the specific field content, i.e. non-perfect groups are, in general, not safe from anomalies.

However, for some non-perfect discrete groups combined with  $SO(N)$  or exceptional gauge groups, anomaly freedom is automatic, independently of the field content. That is, there are some discrete groups  $D$  for which the mixed  $D$ - $G$ - $G$  anomalies always cancel if  $G$  is an  $SO(N)$  or exceptional group but not if  $G = SU(N)$ . Let us discuss a general criterion when this is the case. For  $SO(N)$  or exceptional gauge groups, the Dynkin index  $\ell(\mathbf{F})$  is not  $1/2$  but some integer, cf. Table 1. Therefore, a generic setting based on such a gauge group and a non-perfect finite group is anomaly-free as (2.15) is satisfied, provided that the finite group exclusively has non-trivial one-dimensional representations  $\mathbf{1}_{\text{nt}}$  which obey

$$(\mathbf{1}_{\text{nt}})^{2\ell(\mathbf{F})} = \mathbf{1}_0, \quad (2.17)$$

where  $\mathbf{1}_0$  is the trivial one-dimensional representation. An example is provided by the symmetric groups  $S_n$ , which have only one non-trivial one-dimensional representation with some elements represented as  $-1$ . Hence, for symmetric groups the  $S_n$ - $G$ - $G$  anomaly vanishes for  $G$  not being  $SU(N)$  or  $Sp(N)$  independently of the field content.

More generally, equation (2.17) is certainly fulfilled for any combination of discrete group  $D$  and gauge group  $G$  for which

$$\frac{2\ell(\mathbf{F})}{|D/[D, D]|} \in \mathbb{Z}. \quad (2.18)$$

That is, it is fulfilled if the order of the Abelianization of  $D$  divides twice the smallest Dynkin index of  $G$ . This criterion can be further refined. To see this, note that, using the fundamental theorem of finite Abelian groups (cf. e.g. [17]), the Abelianization  $D/[D, D]$  can always be written in standard form as a direct product of  $\mathbb{Z}_{p_i^{v_i}}$  factors where each order  $p_i^{v_i}$  is some power  $v_i$  of a prime number  $p_i$ . Thus, the maximal order of elements of  $D/[D, D]$  is the least common multiple of the  $p_i^{v_i}$  (cf. e.g. [17]). Hence, the group is anomaly-free with respect to  $G$  independently of the field content if and only if the least common multiple of the  $p_i^{v_i}$  divides  $2\ell(\mathbf{F})$ .

However, we see that in general non-perfect groups are not safe from anomalies. As usual, anomaly freedom amounts to imposing constraints on the spectrum and the continuous gauge symmetry  $G$ .

*Further comments* Let us explain the relevance of our statements for finite simple groups. It is well known that non-Abelian finite simple groups are perfect, cf. e.g. [18, p. 27] and Appendix A. As such, non-Abelian finite simple groups are always safe from anomalies. Abelian finite simple groups, on the contrary, are non-perfect and therefore generically suffer from anomalies.

Finally, let us also comment on infinite (i.e. non-compact) discrete groups. By definition,  $|D/[D, D]| = 1$  also holds for infinite perfect groups. Thus, we expect that also infinite perfect groups are anomaly-safe. Non-perfect infinite groups, on the other hand, have at least one non-trivial one-dimensional representation such that settings involving such groups may be anomalous in general. An example for a non-perfect infinite group is  $SL(2, \mathbb{Z})$ . This group appears as  $T$ -duality symmetry in superstring theories. It is known that it may exhibit anomalies [19,20], which actually allow one to draw interesting conclusions on the properties of the underlying model.

### 3. Green-Schwarz cancellation of discrete anomalies

In the remainder of this study we wish to discuss settings in which the anomaly coefficient (2.8) is non-vanishing. Yet the anomaly, i.e. the transformation of the path integral measure, may be compensated by a corresponding transformation of an ‘axion’. This is the Green-Schwarz (GS) mechanism [14] for discrete symmetries [21,22]. For it to work, the ‘axion’ field  $a$  needs to couple to the corresponding field strength via

$$\mathcal{L}_{\text{axion}} \supset -\frac{a}{f_a} (F_{\mu\nu} \tilde{F}^{\mu\nu}) \quad (3.1)$$

with  $f_a$  denoting its decay constant. Further, the axion  $a$  has to transform with a shift  $a \rightarrow a + \Delta_u$  under the anomalous transformation  $u$ . Here,  $\Delta_u$  needs to be such that it precisely cancels the factor in front of  $F_{\mu\nu} \tilde{F}^{\mu\nu}$  in (2.4). Whereas one can always define such a shift for a single Abelian symmetry, the shifts for different Abelian subgroups of a non-Abelian group have to be mutually consistent to cancel the anomaly of the whole group [11]. The fact

that the path integral measure  $J^{-2}$  transforms in a well-defined one-dimensional representation of  $D$  nicely explains why such a cancellation is always possible.

One may think of the axion  $a$  as the complex phase of a field,  $\Phi = r e^{i a}$ , which transforms in the complex conjugate representation of  $J^{-2}$ . Therefore, there is the possibility of having allowed operators of the form  $e^{i \beta a} \mathcal{O}$  with some constant  $\beta$ . Here  $\mathcal{O}$  denotes an operator that transforms under  $D$  with a phase, i.e.  $\mathcal{O}$  is in a non-trivial one-dimensional representation. Without the axion-dependent prefactor,  $\mathcal{O}$  is hence prohibited by the symmetry. Upon the axion acquiring its VEV, the terms of the form  $e^{i \beta a} \mathcal{O}$  appear to violate the discrete symmetry  $D$  (similar to the case of a pseudo-anomalous  $U(1)$ , see e.g. [23]). That is, unlike for the case of anomaly-free discrete symmetries, in the case of pseudo-anomalous discrete symmetries there will be terms that may considerably alter the phenomenology of models.

To conclude, there are just two possibilities for the consideration of anomalies in finite groups: either the group is perfect and the anomalies vanish automatically, or the group is not perfect. In the second case, there may be anomalies depending on the field content; yet one can always consistently use a Green-Schwarz mechanism to cancel the anomaly. However, as mentioned earlier, the symmetry is then broken by certain (e.g. non-perturbative) terms. Hence, statements concerning phenomenological consequences of models based on such pseudo-anomalous symmetries need to be taken with some care. In particular, one may be concerned whether or not such symmetry breaking effects are properly included.

#### 4. Conclusion

We have shown that non-Abelian finite simple groups, and more generally all perfect groups, are anomaly-free. Our argument is based on the fact that the generators of perfect groups are traceless. This argument may also be rephrased as follows. Due to the fact that the path integral measure corresponding to a  $D$ - $G$ - $G$  anomaly transforms in a one-dimensional representation of  $D$ , groups  $D$  without non-trivial one-dimensional representations, i.e. perfect groups, cannot have anomalies.

Non-perfect groups, on the contrary, always have at least one non-trivial one-dimensional representation and therefore are not safe from anomalies in generic settings. Whether a certain model is anomaly free then depends, as usual, on the field content. However, as we have seen, under certain circumstances one can make statements independently of the field content. Specifically, there are combinations of certain non-perfect groups and  $SO(N)$  or exceptional gauge groups which are anomaly-free irrespective of the field content. We have given a criterion when this is the case.

In the case of a non-vanishing  $D$ - $G$ - $G$  anomaly, one can infer the representation that a Green-Schwarz axion needs to furnish in order to cancel the anomaly directly from the non-trivial representation of the measure. This also shows that Green-Schwarz anomaly cancellation is always possible for finite groups.

Note that our argument is somewhat analogous to the case of non-Abelian continuous groups. In the case of a global Lie group  $L$ , it is well known that  $L$ - $G$ - $G$  anomalies cancel because all generators of  $L$  are traceless, exactly like in the case of perfect groups. One may also attribute this to the fact that Lie groups do not have non-trivial one-dimensional representations. However, gauged Lie groups are different in that one also has to consider the  $L$ - $L$ - $L$  anomalies. The latter are not proportional to the trace of a single generator and thus may not be described by a linear one-dimensional group representation. As is well known, the corresponding cubic anomaly coefficients do not vanish in general.

However, they always vanish for real representations and, in particular, for the so-called ‘safe’ groups [24].

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#### Appendix A. Some basic facts about finite groups

In this appendix, we collect some basic facts in connection to perfect and simple groups. Further details can be found e.g. in [18].

The commutator subgroup  $[D, D]$  (also called derived subgroup) of a group  $D$  is the group which is generated by all commutator elements of  $D$ , that is

$$[D, D] := \langle u : u \in D \text{ and } u = v w v^{-1} w^{-1} \text{ for some } v, w \in D \rangle. \quad (\text{A.1})$$

The commutator subgroup is a normal subgroup of  $D$ , see for example [18, p. 27]. A perfect group is a group which equals its own commutator subgroup  $D \equiv [D, D]$ , or equivalently, for which  $|D/[D, D]| = 1$ .

In what follows, we show that a group is perfect if and only if it has exactly a single one-dimensional representation, namely the trivial one. For this, note that there is a one-to-one correspondence between the representations of a quotient group  $D/N$ , where  $N$  is a normal subgroup of  $D$ , and certain representations of the parent group  $D$ . In fact, each representation  $\mathbf{r}$  of  $D$  for which all elements  $n \in N$  are represented by the identity, i.e.

$$U_{\mathbf{r}}(n) = \mathbb{1} \quad \forall n \in N, \quad (\text{A.2})$$

is also a representation of the quotient group  $D/N$ . The converse is also true: each representation of the quotient group  $D/N$  corresponds to a representation  $\mathbf{r}$  of  $D$  with  $U_{\mathbf{r}}(n) = \mathbb{1}$  for  $n \in N$  (cf. e.g. [25, p. 41]).

A particular Abelian quotient group is  $D/[D, D]$ , the so-called Abelianization of  $D$ . Now, consider a one-dimensional representation  $\mathbf{1}_x$  of  $D$ . Then,  $U_{\mathbf{1}_x}([D, D]) = 1$ , since complex numbers commute, and equation (A.2) is satisfied. Hence, using the one-to-one correspondence discussed above, the one-dimensional representation  $\mathbf{1}_x$  of  $D$  is also a one-dimensional representation of the Abelian quotient group  $D/[D, D]$ . Furthermore, note that for an Abelian finite group the number of one-dimensional representations equals the order of the group.

Consequently,  $D$  and  $D/[D, D]$  have exactly the same number of one-dimensional representations, which in turn is equal to  $|D/[D, D]|$ ,

$$\begin{aligned} \# \text{ of one-dimensional representations of } D \\ &= \# \text{ of one-dimensional representations of } D/[D, D] \\ &= |D/[D, D]|. \end{aligned} \quad (\text{A.3})$$

This shows that a discrete group  $D$  is perfect, i.e.  $[D, D] = D$ , if and only if it has exactly a single one-dimensional representation, namely the trivial one.

A simple group is a group whose only normal subgroups are the group itself and the trivial subgroup. Since the commutator subgroup  $[D, D]$  is a normal subgroup of  $D$ , there are just two possibilities for the commutator subgroup of a simple group  $D$ : either it equals the group,  $[D, D] = D$ , or it is the trivial group,  $[D, D] = \{e\}$ . The first case corresponds to non-Abelian finite simple groups, which are thereby shown to be perfect, and the second case corresponds to Abelian finite simple groups, which are thereby not perfect.

A (semi-)direct product of perfect groups is again a perfect group. To show this, take  $D = N \rtimes S$  with two perfect groups  $N$  and  $S$ . Then every element  $d \in D$  can uniquely be written as  $d = n \cdot s$  for some  $n \in N$  and  $s \in S$ . Since  $N$  and  $S$  are perfect groups, each of their elements can be written as a product of commutator elements. Therefore, also every element in  $D$  can be written as a product of commutator elements. Hence,  $D$  equals its commutator subgroup, and we conclude that  $D$  is perfect.

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