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The Dimension of Affine Deligne–Lusztig Varieties in the Affine Grassmannian

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We calculate the dimension of affine Deligne–Lusztig varieties inside the affine Grassmannian of an arbitrary reductive connected group over a finite field. Our result generalizes the dimension formula for split groups, which was determined by the works of Görtz, Haines, Kottwitz, and Reuman and Viehmann.

1 Introduction

Let k be a finite field of characteristic p and let $\bar{k}$ be an algebraic closure of k. We consider a connected reductive group G over k. By a theorem of Steinberg, G is quasi-split. Let k' be a finite subfield of $\bar{k}$ such that $G_{k'}$ is split. We fix $S \subset T \subset B \subset G$, where S is a maximal split torus, T is a maximal torus which splits over k', and B is a Borel subgroup of G. Here and in the rest of this article we use the convention that whenever we consider a subgroup of G, we automatically assume that it is defined over k. We define $K = G(\bar{k}[\![t]\!])$ and denote by $\mathcal{G}r$ the affine Grassmannian of G.

Denote by $F = k(\!(t)\!)$, $E = k'(\!(t)\!)$, and $L = \bar{k}(\!(t)\!)$ the Laurent series fields. We identify the Galois groups $\mathrm{Gal}(k'/k) = \mathrm{Gal}(E/F) =: I$. Let σ denote the Frobenius element of $\mathrm{Gal}(\bar{k}/k)$ and also of $\mathrm{Aut}_F(L)$.

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For a dominant cocharacter $\mu \in X_*(T)_{\mathrm{dom}}$ and $b \in G(L)$ the affine Deligne–Lusztig variety is the locally closed subset

$$X_\mu(b)(\bar{k}) = \{g \cdot K \in \mathcal{G}r(\bar{k});\ g^{-1} b\sigma(g) \in K\mu(t)K\}.$$

We equip $X_\mu(b)$ with reduced structure, making it a scheme which is locally of finite type over $\bar{k}$.

It is an important fact that the isomorphism class of $X_\mu(b)$ does not depend on b itself but only on its σ-conjugacy class in $G(L)$. Indeed, if $b' = h^{-1} b\sigma(h)$, then $g \cdot K \to h^{-1} g \cdot K$ induces an isomorphism $X_\mu(b) \xrightarrow{\sim} X_\mu(b')$.

Denote by $\pi_1(G)$ the fundamental group of G, that is, the quotient of $X_*(T)$ by the coroot lattice. By a result of Kottwitz [7], a σ-conjugacy class inside $G(L)$ is uniquely given by two invariants $\nu \in X_*(S)_{\mathbb{Q},\mathrm{dom}}$ and $\kappa \in \pi_1(G)_I$, which are called the Newton point and the Kottwitz point of the σ-conjugacy class. We will also speak of the Newton point and the Kottwitz point of an element $b \in G(L)$, meaning the invariant associated to the σ-conjugacy class of b.

We denote by J_b the algebraic group whose R-valued points for any F-algebra R are given by

$$J_b(R) = \{g \in G(R \otimes_F L);\ g^{-1} b\sigma(g) = b\},$$

which is an inner form of the centralizer of the Newton point of b in G_F [7, Section 5.2]. Then $J_b(F)$ acts on $X_\mu(b)$ by multiplication on the left. We define the defect of b to be the integer $\mathrm{def}_G(b) := \mathrm{rk}_F G - \mathrm{rk}_F J_b$.

The aim of this paper is the following theorem:

Theorem 1.1. Assume that $X_\mu(b)$ is nonempty. Let $\nu \in X_*(S)_{\mathbb{Q},\mathrm{dom}}$ be the Newton point of b. Then

$$\dim X_\mu(b) = \langle \rho, \mu - \nu \rangle - \tfrac{1}{2}\mathrm{def}_G(b),$$

where ρ denotes the half-sum of all (absolute) roots of G. □

It is known that $X_\mu(b)$ is nonempty if and only if the Mazur inequality holds. Many authors have worked on this nonemptiness and have proved it in different generality; the result for unramified groups was proved by Kottwitz and Gashi [8, Section 4.3; 4, Theorem 5.2].

The assertion of Theorem 1.1 is already known in the case where G is split. It is proved in the papers of Görtz et al. [6] and Viehmann [12]. In [6], the assertion is reduced to the case where $G = \mathrm{GL}_h$ and b is superbasic, that is, no σ-conjugate of b is contained in a proper Levi subgroup of G. This case is considered in [12], where the dimension is calculated.

The proof of Theorem 1.1 is a generalization of the proof in the split case. In Section 2, we reduce the theorem to the case where $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$ and b is superbasic. The reduction step is almost literally the same as in [6]; we give an outline of the proof and explain how one has to modify the proof of [6]. The rest of the paper then focuses on proving the theorem in this special case. For this, we generalize the proof of Viehmann in [12]. We decompose the affine Deligne–Lusztig variety using combinatorial invariants called extended EL-charts, which generalize the notion of extended semi-modules considered in [12] for $G = \mathrm{GL}_h$, and calculate the dimension of each part by generalizing the computations in the GL_h-case. As another application of this decomposition we study the $J_b(F)$-action on the irreducible components of $X_\mu(b)$ in the superbasic case and give a conjecture on the number of orbits in the case where μ is minuscule.

2 Reduction to the Superbasic Case

The aim of this section is to prove the following assertion.

Theorem 2.1. Assume Theorem 1.1 is true for each affine Deligne–Lusztig variety $X_\mu(b)$ with $G \cong \mathrm{Res}_{k'/k}\mathrm{GL}_h$ and $b \in G(L)$ superbasic. Then it is true in general. □

As mentioned in the introduction, we follow the proof given in [6] for split groups. First, we have to fix some more notation. Let

$P = MN$	be a parabolic subgroup of G containing B. We denote by M the corresponding Levi subgroup containing T and by N the unipotent radical of P.
$\mathcal{G}r, \mathcal{G}r_P, \mathcal{G}r_M$	denote the affine Grassmannians of G, P, and M, respectively.
$\mathcal{G}r^\omega, \mathcal{G}r_M^\omega$	denote the geometric connected component of $\mathcal{G}r$ and $\mathcal{G}r_M$, respectively, corresponding to $\omega \in \pi_1(G)$ and $\omega \in \pi_1(M)$. (cf. [10, Theorem 0.1]).
x_λ	denote the image of $\lambda(t)$ in $\mathcal{G}r(\bar{k})$ for $\lambda \in X_*(T)$. We use x_0 as "base point" of $\mathcal{G}r(\bar{k})$. For $g \in G(L)$ we write gx_0 for the translate of x_0 w.r.t. the obvious $G(L)$-action on $\mathcal{G}r(\bar{k})$.
$X_*(T)_{\mathrm{dom}}$	be the subset of $X_*(T)$ of cocharacters which are dominant w.r.t. $T \subset B \subset G$.
$X_*(T)_{M-\mathrm{dom}}$	be the subset of $X_*(T)$ of cocharacters which are dominant w.r.t. $T \subset B \cap M \subset M$.

R_N	denote the set of roots of T in Lie$N$.
ρ	denote the half-sum of all positive roots in G.
ρ_N	denote the half-sum of all elements of R_N.
$\rho_M = \rho - \rho_N$	denote the half-sum of all positive roots in M.

Moreover, we define two partial orders on $X_*(T)$. For two cocharacters μ, μ' we write $\mu \leq \mu'$ and $\mu \leq_M \mu'$ if $\mu' - \mu$ is a non-negative integral linear combination of simple positive coroots of G and M, respectively.

We consider $\mathcal{G}r_M(\bar{k})$ as a subset of $\mathcal{G}r(\bar{k})$ via the obvious embedding. Furthermore, the canonical morphisms $P \hookrightarrow G$ and $P \to M$ induce morphisms of ind-schemes

$$\begin{tikzcd} & \mathcal{G}r_P \arrow[dl, "\pi"'] \arrow[dr, "\iota"] & \\ \mathcal{G}r_M & & \mathcal{G}r_G. \end{tikzcd}$$

The idea of the proof for Theorem 2.1 is to consider the image of an affine Deligne–Lusztig variety $X_\mu(b)$ in $\mathcal{G}r_M$ under the above correspondence, assuming that $b \in M(L)$. We want to show that the image is a union of affine Deligne–Lusztig varieties, which we will later assume to be superbasic and relate the dimension of $X_\mu(b)$ to the dimension of its image.

Let us study the diagram more thoroughly. Certainly π is surjective and ι is bijective on geometric points by the Iwasawa decomposition of G. Now Lemma 2.2 implies that ι identifies $\mathcal{G}r_P$ with a coproduct of locally closed subsets of $\mathcal{G}r$, which are disjoint and cover $\mathcal{G}r$. In particular, we see that $X_\mu^{P\subset G}(b) := \iota^{-1}(X_\mu(b))$ is also locally of finite type and has the same dimension as $X_\mu(b)$.

Lemma 2.2. Let $i : I \hookrightarrow H$ be a closed embedding of connected algebraic groups. Then the induced map on the identity components of the affine Grassmannians $i_{\mathcal{G}} : \mathcal{G}r_I^0 \to \mathcal{G}r_H^0$ is an immersion. □

Proof. First recall the following result in the [1, proof of Theorem 4.5.1] (see also [5, Lemma 2.12]): In the case where H/I is quasi-affine (respectively, affine), the induced morphism $\mathcal{G}r_I \to \mathcal{G}r_H$ is an immersion (respectively, closed immersion). So we want to replace I by a suitable closed subgroup I' which is small enough such that H/I' is quasi-affine, yet big enough such that the immersion $\mathcal{G}r_{I'}^0 \hookrightarrow \mathcal{G}r_I^0$ is surjective.

Now let

$$0 \longrightarrow R(I)_u \longrightarrow I \longrightarrow I_1 \longrightarrow 0$$

be the decomposition of I into a unipotent and a reductive group. We denote by I_1^{der} the derived group of I_1 and by $R(I_1)$ its radical. As I_1/I_1^{der} is affine, the canonical morphism $\mathcal{G}r^0_{I_1^{\mathrm{der}}} \to \mathcal{G}r^0_{I_1}$ is a closed immersion. Using that $I_1 = R(I_1)\cdot I_1^{\mathrm{der}}$, we see that it is also surjective.

We define $I' := I \times_{I_1} I_1^{\mathrm{der}}$. As $\pi_1(I) = \pi_1(I_1)$, the canonical morphism $\mathcal{G}r^0_{I'} \to \mathcal{G}r^0_I$ is the pullback of $\mathcal{G}r^0_{I_1^{\mathrm{der}}} \hookrightarrow \mathcal{G}r^0_I$ and hence also a surjective immersion. Furthermore, I' has no nontrivial homomorphisms to $\mathbb{G}_m$, hence the quotient H/I' is quasi-affine and $\mathcal{G}r^0_{I'} \to \mathcal{G}r_H$ is an immersion. Altogether we have

$$\begin{tikzcd} \mathcal{G}r^0_{I'} \arrow[r, hook, "\text{surj.immersion}"'] \arrow[rr, bend left, "\text{immersion}"] & \mathcal{G}r^0_I \arrow[r, "i_{\mathcal{G}}\ \text{monomorphism}"'] & \mathcal{G}r^0_H, \end{tikzcd}$$

which proves that $i_{\mathcal{G}}$ is an immersion. ■

In order to determine the dimension of $X_\mu^{P\subset G}(b)$, we want to calculate the dimension of its fibres under π and its image. For this, we need a few auxiliary results. We note that the reasoning below still works if we replace $\bar{k}$ by a bigger algebraically closed field.

We fix a dominant, regular, σ-stable coweight $\lambda_0 \in X_*(T)$. We define, for $m \in \mathbb{Z}$,

$$N(m) := \lambda_0(t)^m N(\bar{k}[\![t]\!])\lambda_0(t)^{-m}.$$

Then we have a chain of inclusions $\ldots \supset N(-1) \supset N(0) \supset N(1) \supset \ldots$ and, moreover, $N(L) = \bigcup_{i\in\mathbb{Z}} N(i)$. Furthermore, we note that $N(-m)/N(n)$ has a canonical structure of a variety for $m, n > 0$.

Definition 2.3.

1. A subset Y of $N(L)$ is called *admissible* if there exist $m, n > 0$ such that $Y \subset N(-m)$ and it is the preimage of a locally closed subset of $N(-m)/N(n)$ under the canonical projection $N(-m) \twoheadrightarrow N(-m)/N(n)$. For admissible $Y \subset N(L)$ we define the dimension of Y by

$$\dim Y = \dim Y/N(n) - \dim N(0)/N(n).$$

2. A subset Y of $N(L)$ is called *ind-admissible* if $Y \cap N(-m)$ is admissible for every $m > 0$. For any ind-admissible $Y \subset N(L)$ we define

$$\dim Y = \sup \dim(Y \cap N(-m)).$$ □

Lemma 2.4. Let $m \in M(L)$ and $\nu \in X_*(S)_{M-\mathrm{dom},\mathbb{Q}}$ be its Newton point. We define $f_m : N(L) \to N(L), n \mapsto n^{-1}m\sigma(n)m^{-1}$. Then, for any admissible subset Y of $N(L)$, the preimage $f_m^{-1}Y$ is ind-admissible and

$$\dim f_m^{-1}Y - \dim Y = \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle.$$

Moreover, f_m is surjective. □

Proof. This assertion is the analog of Proposition 5.3.1 in [6]. Note that R_N is σ-stable and thus the sets $N[i]$ defined in the proof of Proposition 5.3.2 in [6] are σ-stable. ■

We denote by $p_M : X_*(T) \twoheadrightarrow \pi_1(M)$ the canonical projection.

Definition 2.5.

1. For $\mu \in X_*(T)_{\mathrm{dom}}$ let

$$S_M(\mu) := \{\mu_M \in X_*(T)_{M-\mathrm{dom}};\ N(L)x_{\mu_M} \cap Kx_\mu \neq \emptyset\}.$$

2. For $\mu \in X_*(T)_{\mathrm{dom}}$, $\kappa \in \pi_1(M)_I$ let

$$S_M(\mu, \kappa) := \{\mu_M \in S_M(\mu);\ \text{the image of } p_M(\mu_M) \text{ in } \pi_1(M)_I \text{ is } \kappa\}.$$

3. For $\mu \in X_*(T)_{\mathrm{dom}}$ let

$$\Sigma(\mu) := \{\mu' \in X_*(T);\ \mu'_{\mathrm{dom}} \leq \mu\},$$

$$\Sigma(\mu)_{M-\mathrm{dom}} := \Sigma(\mu) \cap X_*(T)_{M-\mathrm{dom}}.$$

We denote by $\Sigma(\mu)_{M-\max}$ the set of maximal elements in $\Sigma(\mu)_{M-\mathrm{dom}}$ w.r.t. the order $\leq_M$. □

Lemma 2.6. For any $\mu \in X_*(T)_{\mathrm{dom}}$ we have inclusions

$$\Sigma(\mu)_{M-\max} \subset S_M(\mu) \subset \Sigma(\mu)_{M-\mathrm{dom}}.$$

Moreover, these sets have the same image in $\pi_1(M)_I$. In particular, $S_M(\mu, \kappa)$ is nonempty if and only if κ lies in the image of $\Sigma(\mu)_{M-\mathrm{dom}}$. □

Proof. This is (a slightly weaker version of) [6, Lemma 5.4.1] applied to $G_{k'}$. ■

Definition 2.7. Let $\mu \in X_*(T)_{\mathrm{dom}}$ and $\mu_M \in \Sigma(\mu)$. We write

$$d(\mu, \mu_M) := \dim(N(L)x_{\mu_M} \cap Kx_\mu).$$ □

We can extend the definition above to arbitrary elements of $\mathcal{G}r_M(\bar{k})$. Multiplication by an element $k_M \in K_M$ induces an isomorphism $N(L)x_{\mu_M} \cap Kx_\mu \xrightarrow{\sim} N(L)k_M x_{\mu_M} \cap Kx_\mu$; thus we have, for each $m \in K_M \mu_M(t) K_M$,

$$\dim(N(L)mx_0 \cap Kx_\mu) = d(\mu, \mu_M).$$

Lemma 2.8. Let $\mu \in X_*(T)_{\mathrm{dom}}$. Then, for all $\mu_M \in S_M(\mu)$ we have

$$d(\mu, \mu_M) \leq \langle \rho, \mu + \mu_M \rangle - 2\langle \rho_M, \mu_M \rangle.$$

If $\mu_M \in \Sigma(\mu)_{M-\max}$, this is an equality. □

Proof. This is [6, Corollary 5.4.4] applied to $G_{k'}$. ■

For $b \in M(L)$, $\mu_M \in X_*(T)_{M-\mathrm{dom}}$ we denote by $X^M_{\mu_M}(b)$ the corresponding affine Deligne–Lusztig variety in the affine Grassmannian of M. On the contrary $X_{\mu_M}(b)$ still denotes the affine Deligne–Lusztig variety in $\mathcal{G}r$, assuming that $\mu_M \in X_*(T)_{\mathrm{dom}}$.

Proposition 2.9. Let $b \in M(L)$ be basic, that is, its Newton point is central in M. We denote by $\kappa \in \pi_1(M)_I$ its Kottwitz point and by $\nu \in X_*(S)_{\mathbb{Q}, M-\mathrm{dom}}$ its Newton point.

1. The image of $X^{P\subset G}_\mu(b)$ under π is contained in

$$\bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b).$$

   Denote by $\beta : X^{P\subset G}_\mu(b) \to \bigcup_{\mu_M \in S_M(\mu,\kappa)} X^M_{\mu_M}(b)$ the restriction of π.
2. For $\mu_M \in S_M(\mu, \kappa)$ and every geometric point x of $X^M_{\mu_M}(b)$ the set $\beta^{-1}(x)$ is nonempty and ind-admissible. We have

$$\dim \beta^{-1}(x) = d(\mu, \mu_M) + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

3. For all $\mu_M \in S_M(\mu, \kappa)$ the set $\beta^{-1}(X^M_{\mu_M}(b))$ is locally closed in $X^{P\subset G}_\mu(b)$ and

$$\dim \beta^{-1}(X^M_{\mu_M}(b)) = \dim X^M_{\mu_M}(b) + d(\mu, \mu_M) + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle.$$

4. If $X_\mu(b)$ is nonempty it has dimension

$$\sup\{\dim X^M_{\mu_M}(b) + d(\mu, \mu_M);\ \mu_M \in S_M(\mu, \kappa)\} + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle - \langle 2\rho_N, \nu \rangle. \quad \square$$

Proof. This is the analog of [6, Proposition 5.6.1]. The proof of (1)–(3) is the same as in [6]; as this is the centerpiece of this section, we give a sketch of the proof for the readers convenience. Let $x = gx_0 \in X_\mu(b)$. We write $g = mn$ with $m \in M(L)$, $n \in N(L)$. Then

$$n^{-1}m^{-1}b\sigma(m)\sigma(n) = g^{-1}b\sigma(g) \in K\mu(t)K. \tag{2.1}$$

As $N(L) \subset P(L)$ is a normal subgroup, this implies

$$N(L) \cdot (m^{-1}b\sigma(m)) \cap K\mu(t)K \neq \emptyset.$$

Thus $m^{-1}b\sigma(m) \in K\mu_M(t)K$ for a unique $\mu_M \in S_M(\mu)$, that is, $\beta(x) \in X^M_{\mu_M}(b)$ proving (1).

Now let $x = mx_0 \in X^M_{\mu_M}(b)$ and $b' = m^{-1}b\sigma(m)$. Then $\beta^{-1}(x)$ is the set of all mnx_0 satisfying (2.1), which is equivalent to

$$(n^{-1}b'\sigma(n)b'^{-1})b' \in K\mu(t)K.$$

Thus

$$\beta^{-1}(x) \cong f_{b'}^{-1}(K\mu(t)Kb'^{-1} \cap N(L))/N(0).$$

Hence we obtain

$$\begin{aligned}
\dim \beta^{-1}(x) &\overset{\text{Lem. 2.4}}{=} \dim(K\mu(t)Kb'^{-1} \cap N(L)) - \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle \\
&= (N(L)b'x_0 \cap Kx_\mu) + \dim(b'N(0)b'^{-1}) - \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle \\
&= d(\mu, \mu_M) - \langle 2\rho_N, \nu \rangle + \langle \rho, \nu - \nu_{\mathrm{dom}} \rangle,
\end{aligned}$$

where the second equality is true because $N(L)b'x_0 \cap Kx_\mu \cong (K\mu(t)Kb'^{-1} \cap N(L))/b'N(0)b'^{-1}$. This gives (2). Now (3) follows from (2) because the source and the target of β are locally of finite type over $\bar{k}$.

Finally, we prove (4). Since

$$X_\mu^{P\subset G}(b) = \bigcup_{\mu_M \in S_M(\mu,\kappa)} \beta^{-1}(X^M_{\mu_M}(b))$$

is a decomposition into locally closed subsets, we have

$$\dim X_\mu(b) = \dim X_\mu^{P\subset G}(b) = \sup\{\dim X^M_{\mu_M}(b);\ \mu_M \in S_M(\mu, \kappa)\}.$$

Applying (3) to this formula completes the proof. ■

Now the main part of Theorem 2.1 follows:

Proposition 2.10. Let $b \in M(L)$ be basic. Assume that Theorem 1.1 is true for $X^M_{\mu_M}(b)$ for every $\mu_M \in S_M(\mu, \kappa)$. Then it is also true for $X_\mu(b)$. □

Proof. This is a consequence of Lemma 2.8 and Proposition 2.9. Its proof is literally the same as the proof of its analog Proposition 5.8.1 in [6]. ■

Replacing b by a σ-conjugate if necessary, we may choose a Levi subgroup M such that b is superbasic in M. As any superbasic σ-conjugacy class is basic [7, Proposition 6.2], the above proposition reduces Theorem 1.1 to the case where b is superbasic. Now it is only left to show that we may assume $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$.

For this, we show that it suffices to prove Theorem 1.1 for the adjoint group G^{ad}. We denote by subscript "ad" the image of elements of $G(L)$, $X_*(T)$, and $\pi_1(G)$ in $G^{\mathrm{ad}}(L)$, $X_*(T^{\mathrm{ad}})$, and $\pi_1(G^{\mathrm{ad}})$, respectively. For $\omega \in \pi_1(G)$ we write $X_\mu(b)^\omega := X_\mu(b) \cap \mathcal{G}r^\omega$. If $X_\mu(b)^\omega$ is nonempty, the canonical morphism

$$X_\mu(b)^\omega \to X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}. \tag{2.2}$$

is an isomorphism. Indeed, for any reductive group H over k the universal covering $\tilde{H} \to H$ induces an isomorphism $\mathcal{G}r_{\tilde{H}} \xrightarrow{\sim} \mathcal{G}r^0_{H,\mathrm{red}}$ [10, Proposition 6.1 and (6.7)], thus $\mathcal{G}r^0_{\mathrm{red}} \cong \mathcal{G}r_{\tilde{G}} \cong \mathcal{G}r^0_{G^{\mathrm{ad}},\mathrm{red}}$ and, by homogeneity, $\mathcal{G}r^\omega_{\mathrm{red}} \cong \mathcal{G}r^{\omega_{\mathrm{ad}}}_{G^{\mathrm{ad}},\mathrm{red}}$. Now one easily checks that $X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}$ is the image of $X_\mu(b)^\omega$.

Now in [2, Lemma 2.1.2] it is proved that if G is of adjoint type and contains a superbasic element $b \in G(L)$, then

$$G \cong \prod_{i=1}^{r} \mathrm{Res}_{k_i/k}\mathrm{PGL}_{h_i},$$

where the k_i are finite field extensions of k. As

$$X_{(\mu_i)_{i=1}^r}((b_i)_{i=1}^r) \cong \prod_{i=1}^{r} X_{\mu_i}(b_i)$$

it suffices to prove Theorem 1.1 for b superbasic and $G \cong \mathrm{Res}_{k'/k}\mathrm{PGL}_h$. Using the isomorphism (2.2) again, we may also assume $G \cong \mathrm{Res}_{k'/k}\mathrm{GL}_h$, which completes the proof of Theorem 2.1.

3 The Superbasic Case: Notation and Conventions

Let us first fix some basic notation. Let X be a set and $v \in X^n$, with n some positive integer. We then write v_i for the ith component of v. Moreover, if $v, w \in X^n$ and $X \subset \mathbb{R}$, we write $v \leq w$ if $v_i \leq w_i$ for all i. For any real number a let $\{a\} := a - \lfloor a \rfloor$ be its fractional part. We denote by $\mathbb{N}$ the set of positive integers and by $\mathbb{N}_0$ the set of non-negative integers.

Let $d := [k' : k] = [E : F]$; then $I \cong \mathbb{Z}/d \cdot \mathbb{Z}$. We choose the isomorphism such that σ is mapped to 1. From now on, we only consider the case $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$ with $S \subset T \subset B \subset G$ where S and T, respectively, are the maximal split and maximal torus which are diagonal and B is the Borel subgroup of lower triangular matrices in G.

We fix a superbasic element $b \in G(L)$ with Newton point $\nu \in X_*(S)_{\mathbb{Q},\mathrm{dom}}$ and a cocharacter $\mu \in X_*(T)_{\mathrm{dom}}$. We have to show that if $X_\mu(b)$ is nonempty, we have

$$\dim X_\mu(b) = \langle \rho, \mu - \nu \rangle - \tfrac{1}{2}\mathrm{def}_G(b). \tag{3.1}$$

As T splits over k', the action of the absolute Galois group on $X_*(T)$ factorizes over I. We identify $X_*(T) = \prod_{\tau \in I} \mathbb{Z}^h$ with I acting by cyclically permuting the factors. This yields an identification of $X_*(S) = X_*(T)^I$ with $\mathbb{Z}^h$ such that

$$X_*(S) \hookrightarrow X_*(T),\ \nu' \mapsto (\nu')_{\tau \in I}.$$

Furthermore, we denote, for an element $\mu' \in X_*(T)$ by $\underline{\mu'} \in X_*(S)$, the sum of all I-translates of μ'. We impose the same notation as above for $X_*(T)_{\mathbb{Q}} = \prod_{\tau \in I} \mathbb{Q}^h$ and $X_*(S)_{\mathbb{Q}} = \mathbb{Q}^h$.

We note that an element $\nu' \in \mathbb{Q}^h$ is dominant if $\nu'_1 \le \nu'_2 \le \cdots \le \nu'_h$ and $\mu' \in \prod_{\tau \in I} \mathbb{Q}^h$ is dominant if μ'_τ is dominant for every $\tau \in I$.

The Bruhat order is defined on $X_*(S)_{\mathrm{dom}}$ (respectively, $X_*(T)_{\mathrm{dom}}$) such that an element μ'' dominates μ' if and only if $\mu'' - \mu'$ is a non-negative linear combination of relative (respectively, absolute positive coroots). We write $\mu' \preceq \mu''$ in this case. This motivates the following definition. For $\nu', \nu'' \in \mathbb{Q}^h$ we write $\nu' \preceq \nu''$ if

$$\sum_{i=1}^{j} \nu'_i \ge \sum_{i=1}^{j} \nu''_i \quad \text{for all } j < n,$$

$$\sum_{i=1}^{n} \nu'_i = \sum_{i=1}^{n} \nu''_i.$$

For $\mu', \mu'' \in \prod_{\tau \in I} \mathbb{Q}^h$ we write $\mu' \preceq \mu''$ if $\mu'_\tau \preceq \mu''_\tau$ for every $\tau \in I$. If ν' and ν'' (respectively, μ' and μ'') are both dominant, this order coincides with the Bruhat order.

For every k-algebra R the R-valued points of G are given by $G(R) \cong \mathrm{Aut}_{k' \otimes_k R}(k' \otimes_k R^h)$. We define $N = k' \otimes_k L^h$, which is canonically isomorphic to the direct sum $\bigoplus_{\tau \in I} N_\tau$ of isomorphic copies of L^h. The Frobenius element σ acts via the Galois action of $\mathrm{Gal}(L/F)$ on N; for all $\tau \in I$ we have $\sigma : N_\tau \xrightarrow{\sim} N_{\tau+1}$. We fix a basis $(e_{\tau,i})_{i=1}^h$ of the N_τ such that $\varsigma(e_{\tau,i}) = e_{\varsigma\tau,i}$ for all $\varsigma \in I$. For $\tau \in I, l \in \mathbb{Z}, i = 1, \ldots, h$, define $e_{\tau,i+l\cdot h} := t^l \cdot e_{\tau,i}$. Then each

$v \in N_\tau$ can be written uniquely as infinite sum

$$v = \sum_{n \gg -\infty} a_n \cdot e_{\tau,n}$$

with $a_n \in k$.

Now we denote by M^0 the $\bar{k}[\![t]\!]$-submodule of N generated by the $e_{\tau,i}$ for $i \geq 0$. With respect to our choice of basis, K is the stabilizer of M^0 in $G(L)$ and $g \mapsto gM^0$ defines a bijection

$$\mathcal{G}r(\bar{k}) \cong \left\{ M = \prod_{\tau \in I} M_\tau;\ \ M_\tau \text{ is a lattice in } N_\tau \right\}.$$

Suppose that we are given two lattices $M, M' \subset L^h$. By the elementary divisor theorem we find a basis $v_1, \ldots, v_n$ of M and a unique tuple of integers $a_1 \leq \cdots \leq a_n$ such that $t^{a_1} v_1, \ldots, t^{a_n} v_n$ form a basis of M'. We define the cocharacter $\mathrm{inv}(M, M')$: $\mathbb{G}_m \to \mathrm{GL}_h$, $x \mapsto \mathrm{diag}(x^{a_1}, \ldots, x^{a_n})$. If we write $M' = gM$ with $g \in \mathrm{GL}_h(L)$, we may equivalently define $\mathrm{inv}(M, M')$ to be the unique cocharacter of the diagonal torus which is dominant w.r.t. the Borel subgroup of lower triangular matrices and satisfies $g \in \mathrm{GL}_h(\bar{k}[\![t]\!]) \mathrm{inv}(M, M')(t) \mathrm{GL}_h(\bar{k}[\![t]\!])$.

In terms of the notation introduced above we have

$$X_\mu(b)(\bar{k}) \cong \{(M_\tau \subset N_\tau \text{ lattice})_{\tau \in I};\ \mathrm{inv}(M_\tau, b\sigma(M_{\tau-1})) = \mu_\tau\}.$$

Definition 3.1.

1. We call a tuple of lattices $(M_\tau \subset N_\tau)_{\tau \in I}$ a G-*lattice*.
2. We define the *volume* of a G-lattice $M = gM^0$ to be the tuple

$$\mathrm{vol}(M) = (\mathrm{val} \det g_\tau)_{\tau \in I}.$$

Similarly, we define the volume of M_τ to be $\mathrm{val} \det g_\tau$. We call M *special* if $\mathrm{vol}(M) = (0)_{\tau \in I}$. □

The assertion that b is superbasic is by [2] equivalent to ν being of the form $(\frac{m}{d \cdot h}, \frac{m}{d \cdot h}, \ldots, \frac{m}{d \cdot h})$ with $(m, h) = 1$. Then by [8, Lemma 4.4], $X_\mu(b)$ is nonempty if and only if ν and μ have the same image in $\pi_1(G)_I$, which is equivalent to $\sum_{\tau \in I, i=1,\ldots,h} \mu_{\tau,i} = m$. We assume that this equality holds from now on.

Furthermore, we have, for each central cocharacter $\nu' \in X_*(S)$, the obvious isomorphism

$$X_\mu(b) \xrightarrow{\sim} X_{\mu+\nu'}(\nu'(t)\cdot b).$$

So we may (and will) assume that $\mu \geq 0$, which amounts to saying that we have $b\sigma(M) \subset M$ for G-lattices $M \in X_\mu(b)(\bar{k})$.

Since the affine Deligne–Lusztig varieties of two σ-conjugated elements are isomorphic, we can assume that b is of the form $b(e_{\tau,i}) = e_{\tau,i+m_\tau}$ where $m_\tau = \sum_{i=1}^h \mu_{\tau,i}$. We could have chosen any tuple of integers (m_τ) such that $\sum_{\tau\in I} m_\tau = m$ but this particular choice has the advantage that the components of any G-lattice in $X_\mu(b)$ have the same volume. In general,

$$\mathrm{vol} M_\tau - \mathrm{vol} M_{\tau-1} = (\mathrm{vol} M_\tau - \mathrm{vol}\, b\sigma(M_{\tau-1})) + (\mathrm{vol}\, b\sigma(M_{\tau-1}) - \mathrm{vol} M_{\tau-1}) = \left(\sum_{i=1}^h \mu_{\tau,i}\right) - m_\tau.$$

Recall that the geometric connected components of $\mathcal{G}r$ are in bijection with $\pi_1(G) = \mathbb{Z}^I$. This bijection is given by mapping a G-lattice to its volume. Thus the subsets of lattices $\mathcal{G}r$ and $X_\mu(b)$ obtained by restricting the value of the volume of the components, respectively, are open and closed. Denote by $X_\mu(b)^i \subset X_\mu(b)$ the subset of all G-lattices M such that M_0 (or equivalently every M_τ) has volume i. Let $\pi \in J_b(F)$ be the element with $\pi(e_{\tau,i}) = e_{\tau,i+1}$ for all $\tau \in I$, $i \in \mathbb{Z}$. Then $g\cdot K \mapsto \pi g \cdot K$ defines an isomorphism $X_\mu(b)^i \xrightarrow{\sim} X_\mu(b)^{i+1}$. Thus $\dim X_\mu(b) = \dim X_\mu(b)^0$ so that it is enough to consider the subset of special lattices.

4 Polygons

In this section, we introduce our notion of polygons and reformulate the formula (3.1) in terms of this notion. For this we need to introduce some more notation.

We denote by $(\mathbb{Q}^h)^0$ (respectively, $(\prod_{\tau\in I}\mathbb{Q}^h)^0$) the subspaces of $\mathbb{Q}^h$ (respectively, $\prod_{\tau\in I}\mathbb{Q}^h$) generated by the relative (respectively, absolute) coroots. Explicitly, these subspaces are given by

$$(\mathbb{Q}^h)^0 = \{\nu' \in \mathbb{Q}^h;\ \nu'_1 + \cdots + \nu'_h = 0\},$$

$$\left(\prod_{\tau\in I}\mathbb{Q}^h\right)^0 = \left\{\mu' \in \prod_{\tau\in I}\mathbb{Q}^h;\ \mu'_\tau \in (\mathbb{Q}^h)^0 \text{ for every } \tau \in I\right\}.$$

We fix lifts ω_i and $\omega_{i,\tau}$ of relative (respectively, absolute) fundamental weights of the derived group to $X^*(S)$ (respectively, $X^*(T)$). We thus have, for $\nu' \in (\mathbb{Q}^h)^0$ and

$\mu' \in (\prod_{\tau \in I} \mathbb{Q}^h)^0$,

$$\langle \omega_i, \nu' \rangle = -\sum_{j=1}^{i} \nu'_j,$$

$$\langle \omega_{\tau,i}, \mu' \rangle = -\sum_{j=1}^{i} \mu'_{\tau,j}.$$

Definition 4.1.

1. For $\nu' \in \mathbb{Q}^h$ let
$$[\nu'] := \sum_{i=1}^{h-1} \lfloor \langle \nu', \omega_i \rangle \rfloor.$$
2. For $\nu', \nu'' \in \mathbb{Q}^h$ we define
$$\ell[\nu', \nu''] := [-\nu'] + [\nu''].$$
3. For $\mu', \mu'' \in \prod_{\tau \in I} \mathbb{Q}^h$ let
$$\ell_G[\mu', \mu''] := \ell[\underline{\mu'}, \underline{\mu''}].$$ □

Now we give a geometric interpretation of $\ell[\nu', \nu'']$ in terms of polygons in a special case that covers all applications in this paper. We start with the following observation (Figure 1).

Lemma 4.2. Let $\nu', \nu'' \in \mathbb{Q}^h$ with $\nu' \preceq \nu''$ and $\nu'' \in \mathbb{Z}^h$. Then

1. $\ell[\nu', \nu''] = [\nu'' - \nu']$;
2. $\ell[\nu', \nu'']$ is independent of the choice of lifts of the fundamental weights. □

Proof. (1) This is an easy consequence of the fact that $\langle \omega_i, \nu'', \rangle$ is an integer for all i.

(2) As $\nu'' - \nu' \in (\mathbb{Q}^h)^0$, the value $[\nu'' - \nu']$ is independent of our choice of lifts. Together with part (1) this proves the claim. ■

Definition 4.3. To an element $\nu' \in \mathbb{Q}^h$ we associate a polygon $\mathcal{P}(\nu')$ which is defined over $[0, h]$ with starting point $(0, 0)$ and slope ν'_i over $(i-1, i)$. We also denote by $\mathcal{P}(\nu')$ the corresponding piecewise linear function on $[0, h]$. □

Let ν' and ν'' be as in Lemma 4.2. Now $\nu' \preceq \nu''$ amounts to saying that $\mathcal{P}(\nu')$ is above $\mathcal{P}(\nu'')$ and that these two polygons have the same endpoint. It follows from the first assertion of the lemma that $\ell[\nu', \nu'']$ is equal to the number of lattice points which are on or below $\mathcal{P}(\nu')$ and above $\mathcal{P}(\nu'')$.

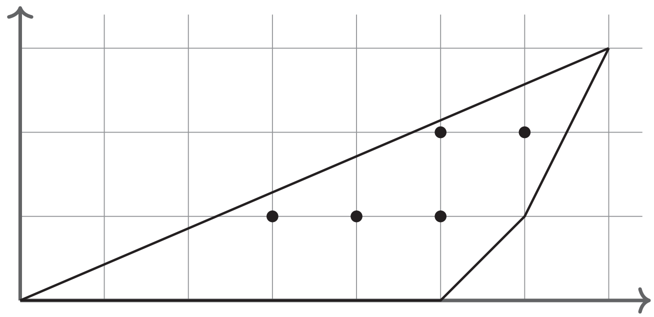

Fig. 1. Geometric interpretation of $\ell[\nu', \nu''] = 5$ for $\nu' = (\frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7}, \frac{3}{7})$, $\nu'' = (0, 0, 0, 0, 0, 1, 2)$.

Proposition 4.4. We have

$$\ell_G[\mu, \nu] = \langle \rho, \mu - \nu \rangle - \frac{1}{2}.\mathrm{def}_G(b).$$

In particular, the formula (3.1) is equivalent to

$$\dim X_\mu(b) = \ell_G[\mu, \nu].$$ □

Remark 1. This formula coincides with the formula conjectured by Rapoport [11, p. 296] up to a minor correction. Rapoport's formula becomes correct if one multiplies all cocharacters by d. □

In order to prove this proposition, we need the following lemmas.

First we need the following fact from Bruhat–Tits theory, which holds in greater generality than just our specific situation. For a reductive group H over a quasi-local field we denote by $\mathcal{BT}(H)$ its Bruhat–Tits building.

Lemma 4.5. Assume that F is a quasi-local field and L is the completion of its maximal unramified extension. Let H be a reductive group over F and A a maximal split torus of H_L defined over F. If its apartment $\mathfrak{a}$ contains a $\mathrm{Gal}(F^{nr}/F)$-stable alcove C, then A contains a maximal split torus of H. □

Proof. We identify $\mathcal{BT}(H)$ with the $\mathrm{Gal}(F^{nr}/F)$-fixed points in $\mathcal{BT}(H_L)$. Then $\mathcal{BT}(H) \cap C$ is a nonempty open subset of $\mathcal{BT}(H)$. In particular, we have

$$\dim \mathfrak{a}^{\mathrm{Gal}(F^{nr}/F)} \geq \dim C^{\mathrm{Gal}(F^{nr}/F)} = \dim \mathcal{BT}(H) = \mathrm{rk} H.$$

Thus A contains a maximal split torus of H. ■

Lemma 4.6.

$$\frac{1}{2}\mathrm{def}_G(b) = \sum_{i=1}^{h-1}\{\langle \omega_i, \underline{\nu}\rangle\}. \qquad \square$$

Proof. The analogous assertion for split reductive groups with simply connected derived group was proved by Kottwitz [9]. We modify his proof in order to get the result in the case of $\mathrm{Res}_{k'/k}\mathrm{GL}_h$.

We consider the groups $T' \subset B' \subset \mathrm{GL}_h$ where T' is the diagonal torus and B' is the Borel subgroup of *upper* triangular matrices. Denote by W (respectively, $\tilde{W}$) the Weyl group (respectively, extended affine Weyl group) of GL_h. We identify W with the symmetric group S_h and $\tilde{W}$ with $\mathbb{Z}^h \rtimes S_h$. The canonical projection of the stabilizer Ω of the standard (upper triangular) Iwahori subgroup in $\tilde{W}$ to $\pi_1(G)$ is an isomorphism. Thus we get an embedding

$$\mathbb{Z} \cong \pi_1(\mathrm{GL}_h) \cong \Omega \hookrightarrow \tilde{W}, n \mapsto \tilde{w}_n := \left((1,0,\ldots,0)\cdot\begin{pmatrix}1 & 2 & \cdots & h\\ 2 & 3 & \cdots & 1\end{pmatrix}\right)^n.$$

We denote by w_n the image of $\tilde{w}_n$ w.r.t. the canonical projection $\tilde{W} \twoheadrightarrow W$.

Decompose $b = (b_\tau)_{\tau\in I}$ according to $\mathrm{Res}_{k'/k}\mathrm{GL}_h(L) \cong \prod_{\tau\in I}\mathrm{GL}_h(L)$. Then b_τ is the generalized permutation matrix representing $\tilde{w}_{m_\tau}$ in $\mathrm{GL}_h(L)$. Denote by $\mathfrak{a} = \prod_{\tau\in I}\mathfrak{a}_\tau \cong \prod_{\tau\in I}\mathbb{R}^h$ the apartment of G_L corresponding to T. Now $\mathcal{BT}(J_b)$ is canonically isomorphic to the fixed points of the Bruhat–Tits building $\mathcal{BT}(G_L)$ of G_L of $\sigma' := \mathrm{Int}(b)\circ\sigma = (w_{m_\tau})_{\tau\in I}\cdot\sigma$. Since the standard Iwahori is σ'-stable, we get, by Lemma 4.5,

$$\begin{aligned}\mathrm{rk} J_b = \dim \mathfrak{a}^{(w_{m_\tau})_{\tau\in I}\cdot\sigma} &= \dim\{(v_\tau)\in\mathfrak{a};\, w_{\tau+1}(v_\tau) = v_{\tau+1}\}\\ &= \dim\{v_0\in\mathfrak{a}_0;\, w_{m_0}\cdot w_{m_{d-1}}\cdot\ldots\cdot w_{m_1}(v_0) = v_0\}\\ &= \dim\{v_0\in\mathfrak{a}_0;\, w_m(v_0) = v_0\}.\end{aligned}$$

Now we can reduce to the case $d=1$: Denote by $G_0 \cong \mathrm{GL}_h$ the factor of G_L corresponding to $\tau = 0$ with diagonal torus T_0 and lower triangular Borel subgroup B_0. We identify the root data of G_0 with the relative root data of G. Now we apply the longest Weyl group element w to our formula to compensate the change of Borel subgroups and then apply [9, Theorem 1.9.2], to complete the proof:

$$\dim \mathfrak{a}_0 - \dim \mathfrak{a}_0^{w_m} = \dim \mathfrak{a}_0 - \dim \mathfrak{a}_0^{w\cdot w_m \cdot w} = \sum_{i=1}^{h-1} \left\{ \langle \omega_i, \left(\frac{m}{h}, \dots, \frac{m}{h}\right) \right\}$$

$$= \sum_{i=1}^{h-1} \{\langle \omega_i, \underline{\nu} \rangle\}.$$ ■

Proof of Proposition 4.4. Using the lemma above, we obtain that

$$\begin{aligned}\langle \rho, \mu - \nu \rangle - \frac{1}{2} \cdot \mathrm{def}_G(b) &= \sum_{\substack{i=1,\dots,h-1\\ \tau \in I}} \langle \omega_{\tau,i}, \mu - \nu \rangle - \sum_{i=1}^{h-1} \{\langle \omega_i, \underline{\nu} \rangle\} \\ &= \sum_{i=1}^{h-1} \langle \sum_{\tau \in I} \omega_{\tau,i}, \mu - \nu \rangle - \sum_{i=1}^{h-1} \{\langle \omega_{h-i}, \underline{\nu} \rangle\} \\ &= \sum_{i=1}^{h-1} (\langle \omega_i, \underline{\mu} \rangle + \langle \omega_i, -\underline{\nu} \rangle) - \sum_{i=1}^{h-1} \{\langle \omega_i, -\underline{\nu} \rangle\} \\ &= \ell_G[\nu, \mu].\end{aligned}$$ ■

Finally, we prove two lemmas which we will use in Section 7. The reader may skip the rest of this section for the moment.

Lemma 4.7. Let $\nu' \in \mathbb{Z}^h$. Then

$$\ell[\nu', \nu'_{\mathrm{dom}}] = \sum_{1 \le i < j \le h} \max\{\nu'_i - \nu'_j, 0\}.$$ □

Proof. The assertion follows from the following observation. If $\nu'' \in \mathbb{Z}^h$ with $\nu''_i > \nu''_{i+1}$ and we swap these coordinates, then $[\nu'']$ is reduced by the difference of these two values. Now ν'_{dom} is obtained from ν' by carrying out the above transposition repeatedly until the coordinates are in increasing order. Since we have swapped the coordinates ν'_i and ν'_j during this construction if and only if $i < j$ and $\nu'_i > \nu'_j$, we get the above formula. ■

Lemma 4.8. Let $\nu' \in \mathbb{Z}^h$ be dominant, $1 \le i \le j \le h$, $\beta \in \mathbb{Z}_{\ge 0}$, and

$$\nu'' := (\nu'_1, \dots, \nu'_{i-1}, \nu'_i - \beta, \nu'_{i+1}, \dots, \nu'_{j-1}, \nu'_j + \beta, \nu'_{j+1}, \dots, \nu'_h).$$

Then

$$\ell[\nu', \nu''_{\mathrm{dom}}] = \left(\sum_{k=1}^{\beta} \sum_{l=\nu'_i-\beta}^{\nu'_j-1} |\{n;\ \nu'_n = k + l\}| \right) - \beta.$$ □

Proof. Obviously, we have

$$\ell[\nu', \nu''] = (j-i)\cdot\beta = \left(\sum_{i\le n\le j}\beta\right) - \beta$$

and by the previous lemma

$$\ell[\nu'', \nu''_{\mathrm{dom}}] = \sum_{n<i:\ \nu'_i-\beta<\nu'_n} (\nu'_n - (\nu'_i - \beta)) + \sum_{n>j:\ \nu'_n<\nu'_j+\beta} (\nu'_j + \beta - \nu'_n).$$

Using $\ell[\nu', \nu''_{\mathrm{dom}}] = \ell[\nu', \nu''] + \ell[\nu'', \nu''_{\mathrm{dom}}]$, one easily deduces the above assertion. ■

Corollary 4.9. Let $\nu', \nu'' \in \mathbb{Z}^d$ be dominant with $\nu' \preceq \nu''$ such that the multiset of their coordinates differs by only two elements. Say n_2, n_3 in the multiset of coordinates of ν' are replaced by n_1, n_4 in the multiset of coordinates of ν'' with $n_1 \le n_2 \le n_3 \le n_4$. Then

$$\ell[\nu', \nu''] = \left(\sum_{k=0}^{n_4-n_3-1}\sum_{l=0}^{n_4-n_2-1} |\{n;\ \nu'_n = n_4 - k - l - 1\}|\right) + n_1 - n_2. \qquad \square$$

Proof. The assertion is just a reformulation of the previous lemma. ■

5 Extended EL-Charts

In order to calculate the dimension of the affine Deligne–Lusztig variety, we decompose $X_\mu^0(b)$ as follows. Define

$$\mathcal{I}_\tau : N_\tau \setminus \{0\} \to \mathbb{Z},$$
$$\sum_{n\gg-\infty} a_n \cdot e_{\tau,n} \mapsto \min\{n \in \mathbb{Z};\ a_n \ne 0\}.$$

Note that $\mathcal{I}_\tau$ satisfies the strong triangle inequality for every τ. We denote $N_{\mathrm{hom}} := \coprod_{\tau\in I}(N_\tau \setminus \{0\})$, analogously M_{hom}, and define the index map

$$\mathcal{I} := \sqcup \mathcal{I}_\tau : N_{\mathrm{hom}} \to \coprod_{\tau\in I}\mathbb{Z}.$$

For $M \in X_\mu(b)^0(\bar{k})$, we define

$$A(M) := \mathcal{I}(M_{\mathrm{hom}})$$

and a map $\varphi(M) : \coprod_{\tau\in I}\mathbb{Z} \to \mathbb{Z}\cup\{-\infty\}$ such that, for $a \in A(M)$,

$$\varphi(M)(a) = \max\{n \in \mathbb{N}_0;\ \exists v \in M_{\mathrm{hom}} \text{ with } \mathcal{I}(v) = a,\ t^{-n}b\sigma(v) \in M_{\mathrm{hom}}\},$$

and $\varphi(M)(a) = -\infty$ otherwise. Now we decompose $X_\mu(b)^0$ such that $(A(M), \varphi(M))$ is constant on each component. We will discuss the properties of this decomposition in Section 6. In this section, we give a description of the invariants (A, φ).

Definition 5.1. Let $\mathbb{Z}^{(d)} := \coprod_{\tau\in I} \mathbb{Z}_{(\tau)}$ be the disjoint union of d isomorphic copies of $\mathbb{Z}$. For $a \in \mathbb{Z}$ we denote by $a_{(\tau)}$ the corresponding element of $\mathbb{Z}_{(\tau)}$ and write $|a_{(\tau)}| := a$. We equip $\mathbb{Z}^{(d)}$ with a partial order "$\le$" defined by

$$a_{(\tau)} \le c_{(\varsigma)} :\Leftrightarrow a \le c \text{ and } \tau = \varsigma$$

and a $\mathbb{Z}$-action given by

$$a_{(\tau)} + n = (a+n)_{(\tau)}.$$

Furthermore, we define a function $f : \mathbb{Z}^{(d)} \to \mathbb{Z}^{(d)}$, $a_{(\tau)} \mapsto (a + m_{\tau+1})_{(\tau+1)}$. □

We impose the notation that, for any subset $A \subset \mathbb{Z}^{(d)}$, we write $A_{(\tau)} := A \cap \mathbb{Z}_{(\tau)}$.

Definition 5.2.

1. Let d, h be positive integers and $(m_\tau)_{\tau\in I} \in \mathbb{Z}^d$ such that $m := \sum_{\tau\in I} m_\tau$ and h are coprime; let f be defined as above. An *EL-chart* for $(\mathbb{Z}^{(d)}, f, h)$ is a nonempty subset $A \subset \mathbb{Z}^{(d)}$ which is bounded from below, stable under f, and satisfies $A + h \subset A$.
2. Let A be an EL-chart and $B = A \setminus (A + h)$. We say that A is *normalized* if $\sum_{b_{(0)}\in B_{(0)}} b = \frac{h\cdot(h-1)}{2}$. □

Our next aim is to give a characterization of EL-charts. For this, let A be an EL-chart and $B := A \setminus (A + h)$. Obviously $|B| = d \cdot h$. We define a sequence $b_0, \ldots b_{d\cdot h-1}$ of distinct elements of B as follows. Denote by b_0 the minimal element of $B_{(0)}$. If b_i is already defined, we denote by b_{i+1} the unique element which can be written as

$$b_{i+1} = f(b_i) - \mu'_{i+1} \cdot h$$

for some $\mu'_{i+1} \in \mathbb{Z}$. These elements are indeed distinct: If $b_i = b_j$, then obviously $i \equiv j \mod d$ and then $b_{i+k\cdot d} \equiv b_i + k \cdot m \mod h$ implies that $i = j$ as m and h are coprime. This reasoning also shows that if we define $b_{d\cdot h}$ according to the recursion formula above, we get $b_{d\cdot h} = b_0$. Therefore, we will consider the index set of the b_i and μ'_i as $\mathbb{Z}/dh\mathbb{Z}$. We define

$$\mathrm{succ}(b_i) := b_{i+1}$$

and call $\mu' = (\mu'_i)_{i=1,\dots,d\cdot h}$ the type of A.

At some point, it may be helpful to distinguish the b_i's and μ'_i's of different components. For this we may change the index set to $I \times \{1,\dots,h\}$ via

$$b_{\tau,i} := b_{\tau+(i-1)d},$$

$$\mu'_{\tau,i} := \begin{cases} \mu'_{\tau+(i-1)d} & \text{if } \tau \neq 0, \\ \mu'_{id} & \text{if } \tau = 0. \end{cases}$$

Here we choose that standard set of representatives $\{0,\dots,d-1\} \subset \mathbb{Z}$ for I.

With the change of notation we have that $b_{\tau,i} \in B_{(\tau)}$ for all i, τ and that $b_{0,1}$ is the minimal element of $B_{(0)}$ and we have the recursion formula

$$b_{\tau+1,i} = f(b_{\tau,i}) - \mu'_{\tau+1,i}h \quad \text{if } \tau \neq d-1,$$

$$b_{0,i} = f(b_{d-1,i-1}) - \mu'_{0,i-1}h.$$

Lemma 5.3.

1. For every EL-chart A there exists a unique integer n such that $A+n$ is normalized.
2. Mapping an EL-chart to its type induces a bijection between normalized EL-charts and the set $\{\mu' \in \prod_{\tau\in I} \mathbb{Z}^h;\ \underline{\mu}' \succeq \underline{\nu}\}$. □

Proof. (1) In order to obtain a normalized EL-chart, we have to choose $n = \frac{1}{h}\cdot(\frac{h\cdot(h-1)}{2} - \sum_{b_{(0)}\in B_{(0)}} b)$. Since by definition every residue modulo h occurs exactly once in $B_{(0)}$, this is indeed an integer.

(2) Since an EL-chart A is uniquely determined by $A \setminus (A+h)$ which is, up to $\mathbb{Z}$-action, uniquely determined by the type of A, we know the type induces an injection on the set of normalized EL-charts into $\prod_{\tau\in I} \mathbb{Z}^h$. The condition $b_0 = \min\{b_{k\cdot d} \mid 1 \leq k \leq h-1\}$ translates to the condition $\underline{\mu}' \succeq \underline{\nu}$ on the type, which is

shown by the following computation. For $1 \leq k \leq h-1$ we have

$$b_0 \leq b_{kd} \Leftrightarrow b_0 \leq b_0 + k \cdot m - \sum_{i=1}^{k} (\underline{\mu}'_i) \cdot h$$

$$\Leftrightarrow \sum_{i=1}^{k} \underline{\mu}'_i \leq k \cdot \frac{m}{h}$$

$$\Leftrightarrow \sum_{i=1}^{k} \underline{\mu}'_i \leq \sum_{i=1}^{k} \underline{\nu}_i.$$

Similarly, one shows the equivalence of $b_0 = b_{hd}$ and $\sum_{i=1}^{h} \underline{\mu}'_i = \sum_{i=1}^{h} \underline{\nu}_i$. Thus, if μ' is the type of an EL-chart, we have $\underline{\mu}' \succeq \underline{\nu}$. On the other hand, this also shows that any such μ' is the type of some EL-chart and thus by (1) also the type of a normalized EL-chart. ■

Definition 5.4. For $a \in A$ we call $\mathrm{ht}(a) := \max\{n \in \mathbb{N}_0;\ a - n \cdot h \in A\}$ the *height of* a. □

Definition 5.5.

1. An *extended EL-chart* is a pair (A, φ) where A is a normalized EL-chart and $\varphi : \mathbb{Z}^{(d)} \to \mathbb{N}_0 \cup \{-\infty\}$ such that the following conditions hold for every $a \in \mathbb{Z}^{(d)}$.
   (a) $\varphi(a) = -\infty$ if and only if $a \notin A$.
   (b) $\varphi(a + h) \geq \varphi(a) + 1$.
   (c) $\varphi(a) \leq \mathrm{ht}\, f(a)$ if $a \in A$ with equality if $\{c \in \mathbb{Z}^{(d)};\ c \geq a\} \subset A$.
   (d) $|\{c \in \mathbb{Z}^{(d)};\ c \geq a\} \cap \varphi^{-1}(\{n\})| \leq |\{c \in \mathbb{Z}^{(d)};\ c \geq a + h\} \cap \varphi^{-1}(\{n+1\})|$ for all $n \in \mathbb{N}_0$.
2. An extended EL-chart is called *cyclic* if equality holds in (c) for every $a \in A$.
3. The *Hodge point* of an extended EL-chart (A, φ) is the dominant cocharacter $\mu'' \in \prod_{\tau \in I} \mathbb{Z}^h$ for which the coordinate n occurs with multiplicity $|A_{(\tau)} \cap \varphi^{-1}(\{n\})| - |A_{(\tau)} \cap \varphi^{-1}(\{n-1\})|$ in μ''_τ. We also say that (A, φ) is an extended EL-chart for μ''. □

Remark 2. We point out that as an obvious consequence of condition (d) we have, for any integer n,

$$|A_{(\tau)} \cap \varphi^{-1}(\{n\})| - |A_{(\tau)} \cap \varphi^{-1}(\{n-1\})| \geq 0,$$

thus the construction of the Hodge point described above is feasible. Because of condition (c) we have $|A_{(\tau)} \cap \varphi^{-1}(\{n\})| = h$ for every $\tau \in I$ and sufficiently large n. Hence the Hodge point is indeed an element of $\prod_{\tau \in I} \mathbb{Z}^h$. □

Except for condition (d) the definition of an EL-chart is obviously a generalization of Definition 3.4 in [12]. As we will frequently refer to Viehmann's paper, we give an equivalent condition for (d) which is easily seen to be a generalization of condition (4) in her definition. However, we will not use this assertion in the sequel.

Lemma 5.6. For every (A, φ) satisfying conditions (a)–(c) of Definition 5.5, (d) may equivalently be replaced with the following condition. For every τ, we can write $A_{(\tau)} = \bigcup_{l=1}^{h} \{a_j^{\tau,l}\}_{j=0}^{\infty}$ with the following conditions:

(a) $\varphi(a_{j+1}^{\tau,l}) = \varphi(a_j^{\tau,l}) + 1$;
(b) if $\varphi(a_j^{\tau,l} + h) = \varphi(a_j^{\tau,l}) + 1$, then $a_{j+1}^{\tau,l} = a_j^{\tau,l} + h$, otherwise $a_{j+1}^{\tau,l} > a_j^{\tau,l} + h$.

Then the Hodge point is the dominant cocharacter associated to $(a_0^{\tau,l})_{\substack{l=1,\dots,h \\ \tau \in I}}$. □

Proof. If we have a decomposition of A as above, it is obvious that (d) is true. Now let (A, φ) be an extended EL-chart. We construct the sequences $(a_j^{\tau,l})_{j \in \mathbb{N}}$ separately for each τ. So fix $\tau \in I$. We construct the sequences by induction on the value of φ. Take every element of A for which φ has minimal value as initial element for some sequence. Now, if we have sorted all elements $a \in A$ with $\varphi(a) \leq n$ in sequences $(a_j^{\tau,l})_{j \in \mathbb{N}}$, we proceed as follows. Condition (d) of Definition 5.5 guarantees that we can continue all our sequences such that they satisfy (a) and (b). If there are still some $a \in A$ with $\varphi(a) = n+1$ which are not already an element of a sequence, we take them as initial objects for some sequences. Since $|\varphi^{-1}(n) \cap A_{(\tau)}| = h$ for $n \gg 0$, we get indeed h sequences. ■

Lemma 5.7. Let A be an EL-chart of type μ'. There exists a unique φ_0 such that (A, φ_0) is a cyclic extended EL-chart. The Hodge point of (A, φ_0) is μ'_{dom}. □

Proof. The function $\varphi_0 : \mathbb{Z}^{(d)} \to \mathbb{N}_0 \cup \{-\infty\}$ is uniquely determined by equality in (c) and condition (a). For any $a \in A$ we get $\varphi_0(a+h) = \varphi_0(a) + 1$, which proves (b) and (d). The second assertion follows from $\mu'_{i+1} = \varphi_0(b_i)$. ■

The following construction will help us to deduce assertions for general extended EL-charts from the assertion in the cyclic case.

Definition 5.8. Let (A, φ) be an extended EL-chart and (A, φ_0) be the cyclic extended EL-chart associated to A. For any $\tau \in I$, we define

$$\{x_{\tau,1}, \ldots, x_{\tau,n_\tau}\} = \{a \in A_{(\tau)};\ \varphi(a+h) > \varphi(a) + 1\},$$

where the $x_{\tau,i}$ are arranged in decreasing order. We write $\mathbf{n} := (n_\tau)_{\tau \in I}$. For $0 \leq \mathbf{i} \leq \mathbf{n}$ (i.e., $0 \leq i_\tau \leq n_\tau$ for all τ) let

$$\varphi_{\mathbf{i}} = \begin{cases} -\infty & \text{if } a \notin A, \\ \varphi_0(a) & \text{if } a \in A_{(\tau)} \text{ and } i_\tau = 0, \\ \varphi(a) & \text{if } a \in A_{(\tau)}, i_\tau > 0 \text{ and } a \geq x_{\tau,i_\tau}, \\ \varphi(a+h) - 1 & \text{otherwise.} \end{cases}$$

We call the family $(A, \varphi_{\mathbf{i}})_{0 \leq \mathbf{i} \leq \mathbf{n}}$ the *canonical deformation* of (A, φ). □

One easily checks that the $(A, \varphi_{\mathbf{i}})$ are indeed extended EL-charts (the properties (a)–(d) of Definition 5.5 follow from the analogous properties of (A, φ)) and that $\varphi_{\mathbf{i}} = \varphi_0$ for $\mathbf{i} = (0)_{\tau \in I}$ and $\varphi_{\mathbf{i}} = \varphi$ for $\mathbf{i} = \mathbf{n}$. Denote the Hodge point of $(A, \varphi_{\mathbf{i}})$ by $\mu^{\mathbf{i}}$.

We note that one can define the $\varphi_{\mathbf{i}}$ recursively. Let $\varsigma \in I$ and $0 \leq \mathbf{i} \leq \mathbf{i'} \leq \mathbf{n}$ with $i'_\varsigma = i_\varsigma + 1$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. We define $\alpha := \varphi(x_{\varsigma,i_\varsigma} + h) - (\varphi(x_{\varsigma,i_\varsigma}) + 1)$. Then

$$\varphi_{\mathbf{i'}}(a) = \begin{cases} \varphi_{\mathbf{i}}(a) - \alpha & \text{if } a = x_{\varsigma,i_\varsigma}, x_{\varsigma,i_\varsigma} - h, \ldots, x_{\varsigma,i_\varsigma} - \mathrm{ht}(x_{\varsigma,i_\varsigma}) \cdot h, \\ \varphi_{\mathbf{i}}(a) & \text{otherwise.} \end{cases}$$

Lemma 5.9. Let (A, φ) be an extended EL-chart of type μ' with Hodge point μ. Then $\mu'_{\mathrm{dom}} \preceq \mu$. Furthermore, we have $\mu'_{\mathrm{dom}} = \mu$ if and only if (A, φ) is cyclic. □

Proof. We have already shown that $\mu = \mu'_{\mathrm{dom}}$ if (A, φ) is cyclic in Lemma 5.7. It suffices to show $\mu^{\mathbf{i}} \prec \mu^{\mathbf{i'}}$ for all pairs $\mathbf{i}, \mathbf{i'}$ such that $i'_\varsigma = i_\varsigma + 1$ for some $\varsigma \in I$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. From the recursive description of $\varphi_{\mathbf{i'}}$ above we see that we get $\mu^{\mathbf{i'}}$ from $\mu^{\mathbf{i}}$ by replacing two coordinates in $\mu^{\mathbf{i}}_\varsigma$ and permuting its coordinates if necessary to get a dominant cocharacter. Using the same notation as above, we replace the set

$$\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \mathrm{ht}(x_{\varsigma,i_\varsigma}), \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha + 1\}$$

with

$$\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha - \mathrm{ht}(x_{\varsigma,i_\varsigma}), \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) + 1\}.$$

Since

$$\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\ \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha+1\in(\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})-\alpha-\mathrm{ht}(x_{\varsigma,i_\varsigma}),\varphi_{\mathbf{i}'}(x_{\varsigma,i_\varsigma})+1),$$

we get $\mu^{\mathbf{i}}\prec\mu^{\mathbf{i}'}$. ■

Deducing the following corollaries from Lemma 5.9 is literally the same as the proofs of Corollary 3.7 and Lemma 3.8 in [12]. We give the proofs for the reader's convenience.

Corollary 5.10. If μ is minuscule, then all extended EL-charts for μ are cyclic. □

Proof. Let (A,φ) be an extended EL-chart for μ and let μ' be the type of A. Since μ is minuscule, $\mu'_{\mathrm{dom}}\preceq\mu$ implies $\mu'_{\mathrm{dom}}=\mu$. Hence the assertion follows from Lemma 5.9. ■

Corollary 5.11. There are only finitely many extended EL-charts for μ. □

Proof. As a consequence of Lemma 5.9 there are only finitely many possible types of extended EL-charts with Hodge point μ. If we fix such a type, the EL-chart A is uniquely determined. The value of the function φ is uniquely determined by A for all but finitely many elements and, for each such element, φ can only take finitely many values by the inequality of Definition 5.5 (c). ■

6 Decomposition of $X_\mu(b)$

We fix a lattice $M\in X_\mu(b)^0$. Let $(A(M),\varphi(M))$ be defined as above and $B(M):=A(M)\setminus(A(M)+h)$.

Lemma 6.1. Let $a_{(\tau)}\in A(M)$ such that $c\in A(M)$ for every $c\geq a_{(\tau)}$. Then $\{v\in N_\tau;\mathcal{I}_\tau(v)\geq a\}\subset M_\tau$. □

Proof. We define $M':=\{v\in N_\tau;\ \mathcal{I}_\tau(v)\geq a\}$ and $M'':=M'\cap M_\tau$. For $b=a,\ldots,a+h-1$ choose $v_b\in M''$ with $\mathcal{I}_\tau(v_b)=b$. Obviously, we can write any element $x\in M'$ in the form

$$x=\sum_{b=a}^{a+h-1}\alpha_b\cdot v_b+x',$$

with $\alpha_b\in k$ and $x'\in t\cdot M'=\{v\in N_\tau;\ \mathcal{I}(v)\geq a+h\}$. Thus $M'=M''+t\cdot M'$ and the claim follows by Nakayama's lemma. ■

Lemma 6.2. Let $M \in X_\mu(b)^0$. Then $(A(M), \varphi(M))$ is an extended EL-chart for μ. □

Proof. Let us first check that $A(M)$ is a normalized EL-chart. It is stable under f and the addition of h since

$$\mathcal{I}(t \cdot v) = \mathcal{I}(v) + h, \tag{6.1}$$

$$\mathcal{I}(b\sigma(v)) = f(\mathcal{I}(v)) \tag{6.2}$$

and $t \cdot M \subset M$ and $b\sigma(M) \subset M$ by our convention $\mu \geq 0$. The fact that $A(M)$ is bounded from below is obvious, thus $A(M)$ in an EL-chart.

The assertion that $A(M)$ is normalized can be deduced from the prerequisite $\mathrm{vol} M = (0)_{\tau \in I}$ as follows. Let $a \in \mathbb{N}$ such that $c \in A(M)$ for every $c \geq (a \cdot h)_{(0)}$. Then, by Lemma 6.1, we have $t^a \cdot M^0_{(0)} \subset M$ and

$$\begin{aligned} 0 = \mathrm{vol}(M_{(0)}) &= \dim_k(M_{(0)}/t^a \cdot M^0_{(0)}) - \mathrm{vol}(t^a \cdot M^0_{(0)}) \\ &= |\{c \in A(M);\ c < (a \cdot h)_{(0)}\}| - a \cdot h \\ &= |\mathbb{N}^d \setminus A(M)_{(0)}| - |A(M)_{(0)} \setminus \mathbb{N}^d|. \end{aligned}$$

Hence

$$\sum_{b \in B(M)_{(0)}} b = \sum_{i=0}^{h-1} i = \frac{h(h-1)}{2}.$$

Now $\varphi(M)$ satisfies property (a) of Definition 5.5 by definition. To see that it satisfies (b) and (c), fix $a \in A$ and let $v \in M_{\mathrm{hom}}$ such that $\mathcal{I}(v) = a$ and $t^{-\varphi(a)} \cdot b\sigma(v) \in M$. Then $t^{-\varphi(a)+1} b\sigma(t \cdot v) = t^{-\varphi(M)(a)} \cdot b\sigma(v) \in M$, proving (b) and $f(a) - \varphi(M)(a) \cdot h = \mathcal{I}(t^{-\varphi(a)} \cdot b\sigma(v)) \in A(M)$, which implies the inequality part of (c). Let $a \in A$ such that $c \in A(M)$ for every $c \geq a$. We choose an element $v' \in M_{\mathrm{hom}}$ with $\mathcal{I}(v') = f(a) - h \cdot \mathrm{ht} f(a)$ and define $v = t^{\mathrm{ht} f(a)} \cdot (b\sigma)^{-1}(v')$. Then $\mathcal{I}(v) = a$, hence $v \in M$ by Lemma 6.1 and $t^{-\mathrm{ht} f(a)} \cdot b\sigma(v) = v' \in M$. Thus $\varphi(a) = \mathrm{ht} f(a)$. To verify that φ has property (d), we fix $\tau \in I$ and define, for $a \in \mathbb{Z}_{(\tau)} \cup \{-\infty\}$, $n \in \mathbb{N}$, the $\bar{k}$-vector space

$$V'_{a,n} := \{v \in M_\tau;\ v = 0 \text{ or } \mathcal{I}(v) \geq a, t^{-n} \cdot b\sigma(v) \in M\}$$

and $V_{a,n} := V'_{a,n}/V'_{a,n+1}$. Now associate to every $c \in \{a' \in A_{(\tau)};\ a' \geq a\} \cap \varphi^{-1}(\{n\})$ an element $v_c \in M_\tau$ with $\mathcal{I}(v_c) = c$ and $t^{-\varphi(c)} \cdot v_c \in M$. Using the strong triangle inequality for $\mathcal{I}_\tau$, we see that the images v_c in $V_{a,n}$ are linearly independent. Thus $\dim V_{a,n} \geq |\{a' \in A_{(\tau)};\ a' \geq a\} \cap \varphi(M)^{-1}(\{n\})|$. By counting dimensions in a suitable finite-dimensional quotient of

$V'_{a,0}$, we see that this is in fact an equality. Now the images of the $t \cdot v_c$ in $V_{a+h,n+1}$ are also linearly independent, thus

$$|\{a' \in A_{(\tau)};\ a' \geq a\} \cap \varphi(M)^{-1}(\{n\})| \leq |\{a' \in A_{(\tau)};\ a' \geq a+h\} \cap \varphi(M)^{-1}(\{n+1\})|.$$

Now it remains to show that $(A(M), \varphi(M))$ has Hodge point μ. But

$$|\{i;\, \mu_{\tau,i} = n\}| = \dim V_{-\infty,n} - \dim V_{-\infty,n-1} = |A_{(\tau)} \cap \varphi^{-1}(\{n\})| - |A_{(\tau)} \cap \varphi^{-1}(\{n-1\})|. \qquad \blacksquare$$

This proof also shows that $A(M)$ is an EL-chart for every G-lattice $M \subset N$ and $A(M)$ is normalized if and only if M is special. For any extended EL-chart (A, φ) for μ we define

$$\mathcal{S}_{A,\varphi} = \{M \in \mathcal{G}r(\bar{k});\ (A(M), \varphi(M)) = (A, \varphi)\}.$$

Since the Hodge point of M and $(A(M), \varphi(M))$ coincide by the lemma above, we have indeed $\mathcal{S}_{A,\varphi} \subset X_\mu(b)^0$.

Lemma 6.3. The $\mathcal{S}_{A,\varphi}$ define a decomposition of $X_\mu(b)^0$ into finitely many locally closed subsets. In particular, $\dim X_\mu(b)^0 = \max_{(A,\varphi)} \dim \mathcal{S}_{A,\varphi}$. $\square$

Proof. By Lemma 6.2, $X_\mu(b)^0$ is the (disjoint) union of the $\mathcal{S}_{A,\varphi}$ and by Corollary 5.11 this union is finite. It remains to show that $\mathcal{S}_{A,\varphi}$ is locally closed. One shows that the condition $(A(M)_{(\tau)}, \varphi(M)_{|A(M)_{(\tau)}}) = (A_{(\tau)}, \varphi_{|A_{(\tau)}})$ is locally closed analogously to the proof of Lemma 4.2 in [12]. Then it follows that $\mathcal{S}_{A,\varphi}$ is locally closed as it is the intersection of finitely many locally closed subsets. $\blacksquare$

Definition 6.4. Let (A, φ) be an extended EL-chart for μ. We define

$$\mathcal{V}(A, \varphi) = \{(a, c) \in A \times A;\ a < c, \varphi(a) > \varphi(c) > \varphi(a-h)\}. \qquad \square$$

We note that the set $\mathcal{V}(A, \varphi)$ is finite, since we have $\varphi(a-h) = \varphi(a) - 1$ for almost all $a \in A$.

Proposition 6.5. Let (A, φ) be an extended EL-chart for μ. There exists an open subscheme $U_{A,\varphi} \subseteq \mathbb{A}^{|\mathcal{V}(A,\varphi)|}$ and a morphism $U_{A,\varphi} \to \mathcal{S}_{A,\varphi}$ which is bijective on $\bar{k}$-valued points. In particular, $\mathcal{S}_{A,\varphi}$ is irreducible and of dimension $\mathcal{V}(A, \varphi)|$. $\square$

Proof. The proof is almost the same as of Theorem 4.3 in [12]. We give an outline of the proof and explain how to adapt the proof of Viehmann to our more general notion.

For any $\bar{k}$-algebra R and $x \in R^{\mathcal{V}(A,\varphi)} = \mathbb{A}^{\mathcal{V}(A,\varphi)}(R)$ we denote the coordinates of x by $x_{a,c}$. We associate to every x a set of elements $\{v(a) \in N_{\mathrm{hom}};\, a \in A\}$ which satisfies the following equations.

If $a = y := \max\{b \in B_{(0)}\}$, then

$$v(a) = e_a + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} x_{a,c} \cdot v(c).$$

For any other element $a \in B$ we want

$$v(a) = v' + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} x_{a,c} \cdot v(c),$$

where $v' = t^{-\varphi(a')} \cdot b\sigma(v(a'))$ for a' being minimal satisfying $f(a') - \varphi(a') \cdot h = a$. At last, if $a \notin B$, we impose

$$v(a) = t \cdot v(a-h) + \sum_{(a,c)\in\mathcal{V}(A,\varphi)} x_{a,c} \cdot v(c).$$

Claim 6.6. The set $\{v(a);\, a \in A\}$ is uniquely determined by the equations above. $\square$

Hence the rule $x \mapsto M(x) := \langle v(a);\ a \in A\rangle_{k[t]}$ is well-defined and, as it is obviously functorial, induces a morphism $\mathbb{A}^{\mathcal{V}(A,\varphi)} \to \mathcal{G}r$. But the image of this morphism is in general not contained in $\mathcal{S}_{A,\varphi}$, we only have the following assertions:

Claim 6.7. For every $x \in \mathbb{A}^{\mathcal{V}(A,\varphi)}(\bar{k})$ we have $A(M(x)) = A$ and $\varphi(M(x))(a) \geq \varphi(a)$ for every $a \in A$. $\square$

Claim 6.8. The preimage $U(A,\varphi)$ of $\mathcal{S}_{A,\varphi}$ is nonempty and open in $\mathbb{A}^{\mathcal{V}(A,\varphi)}$. $\square$

Now the fact that the restriction $U(A,\varphi) \to \mathcal{S}_{A,\varphi}$ of the above morphism defines a bijection of $\bar{k}$-valued points follows from the following assertion.

Claim 6.9. Let $M \subset N$ be a special F-lattice such that $(A(M), \varphi(M)) = (A, \varphi)$. Then there exists a unique set of elements $\{v(a);\ a \in A\} \subset M$ satisfying the equations above. $\square$

It remains to prove the four claims. But their proofs are literally the same as in [12] if one replaces "$a + m$" and "$a + im$" by "$f(a)$" and "$f^i(a)$", respectively. $\blacksquare$

7 Combinatorics for Extended EL-Charts

Proposition 6.5 reduces the proof of the formula (3.1) to an estimation of $|\mathcal{V}(A,\varphi)|$ for extended EL-charts (A,φ) with Hodge point μ. We start with the case where (A,φ) is cyclic. In this case, we have $\varphi(a+h)=\varphi(a)+1$ for all $a\in A$ and thus

$$\mathcal{V}(A,\varphi)=\{(b,c)\in B\times A;\ \ b<c, \varphi(b)>\varphi(c)\}.$$

Proposition 7.1. Let (A,φ) be the cyclic extended EL-chart of type μ. Then

$$|\mathcal{V}(A,\varphi)|\geq \ell_G[\nu,\mu]. \qquad \square$$

Proof. First, we show that the right-hand side of the inequality counts the number of positive integers n such that $b_0+n\notin A_{(0)}$. Indeed, as $A_{(0)}+h\subset A_{(0)}$, we have

$$\begin{aligned}|\{n\in\mathbb{N};\, b_0+n\notin A_{(0)}\}| &= \frac{1}{h}\cdot\sum_{j=1}^{h-1}(b_{j\cdot d}-b_0-j)=\frac{1}{h}\cdot\sum_{j=1}^{h-1}\left(f^{j\cdot d}(b_0)-\left(\sum_{i=1}^{j\cdot d}\mu_i\cdot h\right)-b_0-j\right)\\
&=\frac{1}{h}\cdot\sum_{j=1}^{h-1}\left(b_0+j\cdot m-\left(\sum_{i=1}^{j}\underline{\mu}_i\cdot h\right)-b_0-j\right)\\
&=\frac{1}{h}\cdot\sum_{j=1}^{h-1}\Big(j\cdot(m-1)-h\cdot\mathcal{P}(\underline{\mu})(i)\Big)\\
&=\frac{(m-1)(h-1)}{2}-\sum_{i=1}^{h-1}\mathcal{P}(\underline{\mu})(i)\\
&=\sum_{i=1}^{h-1}(\lfloor\mathcal{P}(\underline{\nu})(i)\rfloor-\mathcal{P}(\underline{\mu})(i))\\
&=\ell_G[\nu,\mu].\end{aligned}$$

Now we construct an injective map from $\{n\in\mathbb{N};\, b_0+n\notin A\}$ into $\mathcal{V}(A,\varphi)$. For this, we remark that $(b_i,b_i+n)\in\mathcal{V}(A,\varphi)$ if and only if $b_i+n\in A$ and $b_{i+1}+n\notin A$. Thus $n\mapsto (b_i,b_i+n)$ where $i=\max\{i=1,\ldots,h-1;\ \ b_i+n\in A\}$ gives us the injection we sought. Note that this map is well-defined since, for any $n\in\mathbb{N}$ and maximal element b of B, we have $b+n\in A$. ■

Theorem 7.2. Let (A,φ) be an extended EL-chart for μ. Then $|\mathcal{V}(A,\varphi)|\leq\ell_G[\nu,\mu]$. $\square$

Proof. We assume first that (A,φ) is cyclic with type μ'. Then

$$\begin{aligned}|\mathcal{V}(A,\varphi)| &= |\{(b_i,a)\in B\times A;\ b_i<a,\varphi(a)<\varphi(b_i)\}|\\ &= |\{(b_i,b_j+\alpha h);\ \alpha\in\mathbb{N}_0, b_i<b_j, \mu'_{i+1}>\mu'_{j+1}+\alpha\}|\\ &\quad+|\{(b_i,b_j+\alpha h);\ \alpha\in\mathbb{N}, b_j<b_i<b_j+\alpha h, \mu'_{i+1}>\mu'_{j+1}+\alpha\}|\\ &= \sum_{(b_i,b_j);\, b_i<b_j,\mu'_{i+1}>\mu'_{j+1}} \mu'_{i+1}-\mu'_{j+1}\\ &\quad+|\{(b_i,b_j+\alpha h);\ \alpha\in\mathbb{N}, b_j<b_i<b_j+\alpha h, \mu'_{i+1}>\mu'_{j+1}+\alpha\}|.\end{aligned}$$

We refer to these two summands by S_1 and S_2.

For each $\tau\in I$ denote by $(\tilde b_{\tau,1},\tilde\mu_{\tau+1,1}),\ldots,(\tilde b_{\tau,h},\tilde\mu_{\tau+1,h})$ the permutation of $(b_{\tau,1},\mu_{\tau+1,1}),\ldots,(b_{\tau,h},\mu_{\tau+1,h})$ such that the $(\tilde b_{\tau,i})_i$ are arranged in increasing order. From the ordering we obtain, for any $1\le j\le h$, $\tau\in I$,

$$\sum_{i=0,\ldots,j-1}\tilde b_{\tau,i}\le\sum_{i=0,\ldots,j-1}\mathrm{succ}(\tilde b_{\tau-1,i})$$

and thus

$$\sum_{i=1}^{j}|\tilde b_{\tau,i}|-|\tilde b_{\tau-1,i}|\le j\cdot m_\tau-\sum_{i=1}^{j}\tilde\mu_{\tau,i}\cdot h.$$

Adding these inequalities for all $\tau\in I$ and rearranging the terms, we obtain $\sum_{i=1}^{j}\underline{\tilde\mu}_j\le j\cdot\frac{m}{h}$. Thus $\underline{\nu}\preceq\underline{\tilde\mu}\preceq\underline{\mu}$.

Using this notation, we can simplify

$$\begin{aligned}S_1 &= \sum_{\substack{i<j\\ \tau\in I}}\max\{\tilde\mu_{\tau,i}-\tilde\mu_{\tau,j},0\}=\sum_{\tau\in I}\ell[\tilde\mu_\tau,\tilde\mu_{\tau\,\mathrm{dom}}]\\ &= \ell_G[\tilde\mu,\mu],\end{aligned}$$

where the second line holds because of Lemma 4.7.

We have now reduced the claim to $S_2\le\ell_G[\nu,\mu]-\ell_G[\tilde\mu,\mu]$, which is equivalent to $S_2\le\ell_G[\nu,\tilde\mu]$. Now

$$\begin{aligned}S_2 &= \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I}|\{\alpha\in\mathbb{Z};\tilde b_{\tau,i}<\tilde b_{\tau,j}+\alpha h,\tilde\mu_{\tau+1,i}>\tilde\mu_{\tau+1,j}+\alpha\}|\\ &= \sum_{i=2}^{h}\sum_{j=1}^{i-1}\sum_{\tau\in I}|\{\alpha\in\mathbb{Z};\ \tilde b_{\tau,i}-\tilde b_{\tau,j}<\alpha h<\tilde\mu_{\tau+1,i}h-\tilde\mu_{\tau+1,j}h\}|\end{aligned}$$

$$= \sum_{i=2}^{h} \sum_{j=1}^{i-1} \sum_{\tau \in I} |\{\alpha \in \mathbb{Z};\ 0 < \alpha h < (\tilde{b}_{\tau,j} - \tilde{\mu}_{\tau+1,j} h) - (\tilde{b}_{\tau,i} - \tilde{\mu}_{\tau+1,i} h)\}|$$

$$= \sum_{i=2}^{h} \sum_{j=1}^{i-1} \sum_{\tau \in I} \max \left\{ \left\lfloor \frac{\operatorname{succ}(\tilde{b}_{\tau,j}) - \operatorname{succ}(\tilde{b}_{\tau,i})}{h} \right\rfloor, 0 \right\}.$$

Recall that $\ell_G[\nu, \tilde{\mu}]$ counts the lattice points between the polygons associated to $\underline{\nu}$ and $\underline{\tilde{\mu}}$. So it is enough to construct a decreasing sequence (with respect to $\preceq$) of $\hat{\psi}^i \in \mathbb{Q}^h$ for $i = 1, \dots, h$ with $\hat{\psi}^1 = \underline{\nu}$ and $\hat{\psi}^h = \underline{\tilde{\mu}}$ such that there are at least

$$\sum_{j=1}^{i-1} \sum_{\tau \in I} \max \left\{ \left\lfloor \frac{\operatorname{succ}(\tilde{b}_{\tau,j}) - \operatorname{succ}(\tilde{b}_{\tau,i})}{h} \right\rfloor, 0 \right\}$$

lattice points which are on or below $\mathcal{P}(\hat{\psi}^{i-1})$ and above $\mathcal{P}(\hat{\psi}^i)$.

We define a bijection $\operatorname{succ}_i : B \to B$ as follows: For $j > i, \tau \in I$ let $\operatorname{succ}_i(\tilde{b}_{\tau,j}) = \operatorname{succ}(\tilde{b}_{\tau,j})$. For $j \leq i$ define $\operatorname{succ}_i(\tilde{b}_{\tau,j})$ such that, for every $\tau \in I$, the tuple $(\operatorname{succ}_i(\tilde{b}_{\tau,j}))_{j=1}^{i}$ is the permutation of $(\operatorname{succ}(\tilde{b}_{\tau,j}))_{j=1}^{i}$ which is arranged in increasing order. Let $\psi^i \in \prod_{\tau \in I} \mathbb{Q}^h$ be defined by $\operatorname{succ}_i(\tilde{b}_{\tau,j}) = f(\tilde{b}_{\tau,j}) - \psi^i_{\tau+1,j} \cdot h$ and $\hat{\psi}^i = \underline{\psi^i}$. By definition, we have

$$\psi^i_{\tau+1,j} = \frac{m_{\tau+1}}{h} - \frac{\operatorname{succ}_i(\tilde{b}_{\tau,j}) - \tilde{b}_{\tau,j}}{h}$$

and thus $\hat{\psi}^1 = \underline{\tilde{\mu}}$ and $\hat{\psi}^h = \underline{\nu}$.

We now estimate the number of lattice points between $\mathcal{P}(\hat{\psi}^{i-1})$ and $\mathcal{P}(\hat{\psi}^i)$ by calculating $\mathcal{P}(\hat{\psi}^i)(j) - \mathcal{P}(\ hat\psi^{i-1})(j)$ for every $1 \leq j < h$. In particular, we will see that $\mathcal{P}(\hat{\psi}^i)(j) - \mathcal{P}(\hat{\psi}^{i-1})(j) \geq 0$ and thus $\hat{\psi}^i \preceq \hat{\psi}^{i-1}$. In order to compare $\mathcal{P}(\hat{\psi}^i)$ and $\mathcal{P}(\hat{\psi}^{i-1})$, we give an equivalent *recursive* description of succ_i. Let $i_0 \leq i$ be minimal such that $\operatorname{succ}_{i-1}(\tilde{b}_{\tau,i_0}) \geq \operatorname{succ}(\tilde{b}_{\tau,i})$. Then

$$\operatorname{succ}_i(\tilde{b}_{\tau,j}) = \begin{cases} \operatorname{succ}_{i-1}(\tilde{b}_{\tau,j}) & \text{if } j < i_0, \\ \operatorname{succ}(\tilde{b}_{\tau,i}) & \text{if } j = i_0, \\ \operatorname{succ}_{i-1}(\tilde{b}_{\tau,j-1}) & \text{if } i_0 < j \leq i, \\ \operatorname{succ}_{i-1}(\tilde{b}_{\tau,j}) & \text{if } j > i. \end{cases}$$

Now

$$\mathcal{P}(\hat{\psi}^i)(j) - \mathcal{P}(\hat{\psi}^{i-1})(j) = \sum_{\tau \in I} \sum_{k=1}^{j} (\psi^i_{\tau,k} - \psi^{i-1}_{\tau,k}) = \sum_{\tau \in I} \sum_{k=1}^{j} \frac{1}{h} (\operatorname{succ}_{i-1}(\tilde{b}_{\tau,k}) - \operatorname{succ}_i(\tilde{b}_{\tau,k}))$$

$$= \sum_{\tau \in I} \frac{1}{h} \left(\sum_{k=1}^{j} \operatorname{succ}_{i-1}(\tilde{b}_{\tau,k}) - \sum_{k=1}^{j} \operatorname{succ}_i(\tilde{b}_{\tau,k}) \right).$$

By the recursive formula above the right-hand side equals zero if $j \geq i$ and

$$\sum_{\tau \in I} \max\left\{0, \frac{\mathrm{succ}_{i-1}(\tilde{b}_{\tau.j}) - \mathrm{succ}(\tilde{b}_{\tau,i})}{h}\right\}$$

if $j < i$. Thus there are at least

$$\sum_{\tau \in I}\sum_{j<i} \max\left\{0, \left\lfloor \frac{\mathrm{succ}_{i-1}(\tilde{b}_{\tau.j}) - \mathrm{succ}(\tilde{b}_{\tau,i})}{h} \right\rfloor\right\} = \sum_{\tau \in I}\sum_{j<i} \max\left\{0, \left\lfloor \frac{\mathrm{succ}(\tilde{b}_{\tau.j}) - \mathrm{succ}(\tilde{b}_{\tau,i})}{h} \right\rfloor\right\}$$

lattice points which are above $\mathcal{P}(\hat{\psi}^i)$ and on or below $\mathcal{P}(\hat{\psi}^{i-1})$, which completes the proof for a cyclic EL-chart.

For a noncyclic EL-chart (A, φ) consider the canonical deformation $((A, \varphi_{\mathbf{i}}))_{\mathbf{i}}$ (see Definition 5.8). The theorem is proved by induction on $\mathbf{i}$. For $\mathbf{i} = (0)_{\tau \in I}$ the extended EL-chart is cyclic and the claim is proved above. Now the induction step requires that we show that the claim remains true if we increase a single coordinate of $\mathbf{i}$ by one. Let $\mathbf{i}' \leq \mathbf{n}$ with $i'_\varsigma = i_\varsigma + 1$ for some $\varsigma \in I$ and $i'_\tau = i_\tau$ for $\tau \neq \varsigma$. For convenience, we introduce the notations

$$\begin{aligned}
\alpha &:= \varphi(x_{\varsigma,i_\varsigma} + h) - (\varphi(x_{\varsigma,i_\varsigma}) + 1),\\
n &:= \mathrm{ht}(x_{\varsigma,i_\varsigma}),\\
\mu^i &:= \mu^{\mathbf{i}}_\varsigma,\\
\mu^{i'} &:= \mu^{\mathbf{i}'}_\varsigma.
\end{aligned}$$

Then the right-hand sides of the formula (3.1) for $\mu^{\mathbf{i}'}$ and $\mu^{\mathbf{i}}$ differ by

$$\langle \rho, \mu^{\mathbf{i}'}\rangle - \langle \rho, \mu^{\mathbf{i}}\rangle = \ell_G[\mu^{\mathbf{i}'}, \mu^{\mathbf{i}}].$$

We had shown in the proof of Lemma 5.9 that one obtains $\mu^{\mathbf{i}'}$ from $\mu^{\mathbf{i}}$ by replacing the entries $\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \mathrm{ht}(x_{\varsigma,i_\varsigma})$ and $\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha + 1$ in the ςth component by $\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha - \mathrm{ht}(x_{\varsigma,i_\varsigma})$ and $\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) + 1$ and rearranging the entries afterwards so that we obtain a dominant cocharacter. Thus we may apply Corollary 4.9, which shows that

$$\ell_G[\mu^{\mathbf{i}'}, \mu^{\mathbf{i}}] = \ell[\mu^{i'}, \mu^i] = \left(\sum_{k=0}^{\alpha-1}\sum_{l=0}^{n} |\{j;\, \mu^i_j = \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - k - l\}|\right) - \min\{\alpha, n+1\}.$$

We denote this term by Δ. We have to show that $\Delta \geq |\mathcal{V}(A, \varphi_{\mathrm{i}'})| - |\mathcal{V}(A, \varphi_{\mathrm{i}})|$. Now recall that

$$\varphi_{\mathrm{i}'}(a) = \begin{cases} \varphi_{\mathrm{i}}(a) - \alpha & \text{if } a = x_{\varsigma,i_\varsigma}, x_{\varsigma,i_\varsigma} - h, \ldots, x_{\varsigma,i_\varsigma} - \mathrm{ht}(x_{\varsigma,i_\varsigma}) \cdot h, \\ \varphi_{\mathrm{i}}(a) & \text{otherwise.} \end{cases}$$

Together with the explanation given below, one obtains

$$\mathcal{V}(A, \varphi_{\mathrm{i}'}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}}) = D_1 \cup D_3,$$

$$\mathcal{V}(A, \varphi_{\mathrm{i}}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}'}) = D_2,$$

where

$$D_1 = \{(x_{\varsigma,i_\varsigma} + h, c) \in A \times A;\ c > x_{\varsigma,i_\varsigma} + h, \varphi_{\mathrm{i}'}(x_{\varsigma,i_\varsigma} + h) > \varphi_{\mathrm{i}'}(c) > \varphi_{\mathrm{i}'}(x_{\varsigma,i_\varsigma})\},$$

$$D_2 = \left\{ (x_{\varsigma,i_\varsigma} - nh, c) \in A \times A;\ \begin{array}{c} c > x_{\varsigma,i_\varsigma} - nh, \varphi_{\mathrm{i}}(x_{\varsigma,i_\varsigma} - nh) > \varphi_{\mathrm{i}}(c), \\ \varphi_{\mathrm{i}'}(x_{\varsigma,i_\varsigma} - nh) \leq \varphi_{\mathrm{i}'}(c) \end{array} \right\},$$

$$D_3 = \left\{ (b, x_{\varsigma,i_\varsigma} - \delta h) \in B \times A;\ \begin{array}{c} \delta \in \{0, \ldots, n\}, b \neq x_{\varsigma,i_\varsigma} - nh, b < x_{\varsigma,i_\varsigma} - \delta h, \\ \varphi_{\mathrm{i}'}(b) > \varphi_{\mathrm{i}'}(x_{\varsigma,i_\varsigma} - \delta h), \varphi_{\mathrm{i}}(b) \leq \varphi_{\mathrm{i}}(x_{\varsigma,i_\varsigma} - \delta h) \end{array} \right\}.$$

The above description can be derived as follows. Since the values φ_{i} and $\varphi_{\mathrm{i}'}$ only differ at the places $S := \{x_{\varsigma,i_\varsigma}, x_{\varsigma,i_\varsigma} - h, \ldots, x_{\varsigma,i_\varsigma} - nh\}$, a pair (a, c) can only be an element of the difference of the sets $\mathcal{V}(A, \varphi_{\mathrm{i}})$ and $\mathcal{V}(A, \varphi_{\mathrm{i}'})$ if either $a - h \in S$, $a \in S$ or $c \in S$. The only case which occurs for $a - h \in S$ is $a = x_{\varsigma,i_\varsigma} - h$ as otherwise $\varphi_{\mathrm{i}}(a) - \varphi(a - h) = \varphi_{\mathrm{i}'}(a) - \varphi_{\mathrm{i}'}(a - h) = 1$. For the same reason the case $a \in S$ reduces to $a = x_{\varsigma,i_\varsigma} - nh$. Also, we have by construction

$$\varphi_{\mathrm{i}}(a) = \varphi_{\mathrm{i}}(a + h) - 1 \quad \text{for all } a \in A, a \leq x_{\varsigma,i_\varsigma},$$

$$\varphi_{\mathrm{i}'}(a) = \varphi_{\mathrm{i}'}(a + h) - 1 \quad \text{for all } a \in A, a < x_{\varsigma,i_\varsigma}.$$

Thus if $(a, c) \in \mathcal{V}(A, \varphi_{\mathrm{i}})$ or $(a, c) \in \mathcal{V}(A, \varphi_{\mathrm{i}'})$ with $c \leq x_{\varsigma,i_\varsigma}$ (in particular if $c \in S$), then $a \in B$. Altogether, the elements (a, c) of the difference of the sets $\mathcal{V}(A, \varphi_{\mathrm{i}})$ and $\mathcal{V}(A, \varphi_{\mathrm{i}'})$ satisfy either $a = x_{\varsigma,i_\varsigma} + h$, $a + x_{\varsigma,i_\varsigma} - nh$ or $(a, c) \in B \times S$. Since the transition from φ_{i} to $\varphi_{\mathrm{i}'}$ reduces the values at S, the only cases which occur for $(a, c) \in \mathcal{V}(A, \varphi_{\mathrm{i}'}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}})$ are $a = x_{\varsigma,i_\varsigma} + h$ and $(a, c) \in B \times S$, which yields the decomposition $\mathcal{V}(A, \varphi_{\mathrm{i}'}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}}) = D_1 \cup D_3$. For the same reason the only case which occurs for $(a, c) \in \mathcal{V}(A, \varphi_{\mathrm{i}}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}'})$ is $a = x_{\varsigma,i_\varsigma} - nh$, which gives the equality $\mathcal{V}(A, \varphi_{\mathrm{i}}) \setminus \mathcal{V}(A, \varphi_{\mathrm{i}'}) = D_2$.

We obtain $|\mathcal{V}(A,\varphi_{\mathbf{i}'})| - |\mathcal{V}(A,\varphi_{\mathbf{i}})| = S_1 - S_2 + S_3$ with

$$S_1 = |\{a \in A;\ a > x_{\varsigma,i_\varsigma} + h, \varphi_{\mathbf{i}}(a) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})]\}|,$$

$$S_2 = |\{a \in A;\ a > x_{\varsigma,i_\varsigma} - nh, \varphi_{\mathbf{i}}(a) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha - n, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - n - 1]\}|,$$

$$S_3 = |\{(b,\delta) \in B \times \{0,\dots,n\};\ b \neq x_{\varsigma,i_\varsigma} - nh, b < x_{\varsigma,i_\varsigma} - \delta h,$$
$$\varphi_{\mathbf{i}}(b) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta]\}|.$$

In order to calculate $S_1 - S_2$, we introduce the sets C_1 and C_2, which are in some sense complementary to the sets considered for S_1 and S_2.

$$C_1 = \{a \in A;\ a \leq x_{\varsigma,i_\varsigma} + h, \varphi_{\mathbf{i}}(a) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})]\},$$

$$C_2 = \{a \in A;\ a \leq x_{\varsigma,i_\varsigma} - nh, \varphi_{\mathbf{i}}(a) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha - n, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - n - 1]\}.$$

As $\varphi_{\mathbf{i}}(x+h) = \varphi_{\mathbf{i}}(x) + 1$ for all $x \in A$ with $x \leq x_{\varsigma,i_\varsigma}$, we have $C_2 + (n+1)h \subset C_1$. We define $C_3 := C_1 \setminus (C_2 + (n+1)h)$. Then

$$\begin{aligned}
C_3 &= \left\{ b + \delta h \in A;\ \begin{matrix} b \in B, \delta \in \{0,\dots,n\}, b \leq x_{\varsigma,i_\varsigma} + h - \delta h, \\ \varphi_{\mathbf{i}}(b) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta] \end{matrix} \right\}, \\
&= \left\{ b + \delta h \in A;\ \begin{matrix} b \in B \setminus \{x_{\varsigma,i_\varsigma} - nh\}, \delta \in \{0,\dots,n\}, b \leq x_{\varsigma,i_\varsigma} + h - \delta h, \\ \varphi_{\mathbf{i}}(b) \in [\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \delta] \end{matrix} \right\}, \\
&\quad \cup \{x_{\varsigma,i_\varsigma} - nh + \delta h;\ \delta = \max\{0, n+1-\alpha\}, \dots, n+1\}.
\end{aligned}$$

In particular, we have $|C_3| \geq S_3 + \min\{\alpha, n+1\}$.

Altogether, we obtain

$$\begin{aligned}
\Delta &= \left(\sum_{k=0}^{\alpha-1} \sum_{l=0}^{n} |\{j;\ \mu_j^i = \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - k - l\}| \right) - \min\{\alpha, n+1\} \\
&= \sum_{k=0}^{\alpha-1} \sum_{l=0}^{n} (|A_{(\varsigma)} \cap \varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - k - l\})| - |A_{(\varsigma)} \cap \varphi_{\mathbf{i}}^{-1}(\{\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - k - l - 1\})|) \\
&\quad - \min\{\alpha, n+1\} \\
&= |A_{(\varsigma)} \cap \varphi_{\mathbf{i}}^{-1}([\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha + 1, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma})])| - |A_{(\varsigma)} \cap \varphi_{\mathbf{i}}^{-1}([\varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - \alpha - n, \varphi_{\mathbf{i}}(x_{\varsigma,i_\varsigma}) - n - 1])| \\
&\quad - \min\{\alpha, n+1\}
\end{aligned}$$

$$
\begin{aligned}
&= (S_1 + |C_1|) - (S_2 + |C_2|) - \min\{\alpha, n+1\} \\
&= S_1 - S_2 + |C_3| - \min\{\alpha, n+1\} \\
&\geq S_1 - S_2 + S_3.
\end{aligned}
$$
■

Proof of Theorem 1.1. We reduced the claim of the theorem to $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$ and superbasic $b \in G(L)$ in Theorem 2.1. In this case the assertion is equivalent to $\dim X_\mu(b) = \ell_G[\mu, \nu]$ by Proposition 4.4. As a consequence of Proposition 6.5, we obtain

$$\dim X_\mu(b) = \max\{|\mathcal{V}(A, \varphi)|;\ (A, \varphi) \text{ is an extended EL-chart for } \mu\}.$$

Now in Proposition 7.1 we showed that the maximum is at least $\ell_G[\mu, \nu]$ and in Theorem 7.2 we showed that it is at most $\ell_G[\mu, \nu]$, complete the proof. ■

8 Irreducible Components in the Superbasic Case

We now consider the $J_b(F)$-action on the irreducible components of $X_\mu(b)$ for superbasic b and arbitrary G. Recall that the canonical projection $G \twoheadrightarrow G^{\mathrm{ad}}$ induces isomorphisms

$$X_\mu(b)^\omega \cong X_{\mu_{\mathrm{ad}}}(b_{\mathrm{ad}})^{\omega_{\mathrm{ad}}}$$

for all $X_\mu(b)^\omega \neq \emptyset$. As $J_b(F)$ acts transitively on the set of nonempty components $X_\mu(b)^\omega$ (cf. [2, Section 3.3]), this implies that the induced map on $J_b(F)$-orbits to $J_b^{\mathrm{ad}}(F)$-orbits of the respective irreducible components is bijective. Thus the set of irreducible components of $X_\mu(b)^\omega$ (respectively, the set of $J_b(F)$-orbits of irreducible components of $X_\mu(b)$) only depends on the data $(G^{\mathrm{ad}}, \mu_{\mathrm{ad}}, b_{\mathrm{ad}})$, so it suffices to consider the case $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$.

Lemma 8.1. Let $b \in G(L)$ be superbasic and $\omega \in \pi_1(G)$ such that $X_\mu(b)^\omega$ is nonempty. Then every $J_b(F)$-orbit of irreducible components of $X_\mu(b)$ contains a unique irreducible component of $X_\mu(b)^\omega$. □

Proof. Denote by $J_b(F)^0$ the stabilizer of $X_\mu(b)^\omega$ in $J_b(F)$. Using the argument above, it suffices to show that $J_b(F)^0$ stabilizes the irreducible components of $X_\mu(b)$ in the case $G = \mathrm{Res}_{k'/k}\mathrm{GL}_h$.

For this, we consider the action of $J_b(F)$ on N_{hom}. Let $g \in J_b(F)$ and let $v_0 := g(e_{0,1})$, $c(g) := \mathcal{I}(v_0)$. Now every element $e_{\tau,i}$ can be written in the form $e_{\tau,i} = \frac{1}{t^j}(b\sigma)^k(e_{0,1})$

for some integers j, k; then

$$g(e_{\tau,i}) = \frac{1}{t^j}(b\sigma)^k(v_0).$$

Hence $\operatorname{val}\det(g) = (c(g)-1)_{\tau\in I}$; in particular we have $g \in J_b(F)^0$ if and only if $c(g)=1$. In this case, the above formula implies $\mathcal{I}(g(e_{\tau,i})) = i_{(\tau)}$ and thus $\mathcal{I}(g(v)) = \mathcal{I}(v)$ for all $v \in N_{\mathrm{hom}}$.

We obtain $A(M) = A(g(M))$ and $\varphi(M) = \varphi(g(M))$ for all $M \in X_\mu(b)^0$. Thus $\mathcal{S}_{A,\varphi}$ is $J_b(F)^0$-stable for every extended EL-chart (A,φ) for μ. As the $\mathcal{S}_{A,\varphi}$ are irreducible, every irreducible component of $X_\mu(b)^0$ is of the form $\overline{\mathcal{S}_{A,\varphi}}$ and thus $J_b(F)^0$-stable. ■

We denote by $\mathcal{M}_\mu$ the set of extended EL-charts (A,φ) for μ for which $|\mathcal{V}(A,\varphi)|$ is maximal. As noted above, we have a bijection

$$\begin{aligned}
\mathcal{M}_\mu &\longleftrightarrow \{\text{top-dimensional irreducible components of } X_\mu(b)\},\\
(A,\varphi) &\longmapsto \overline{\mathcal{S}_{(A,\varphi)}}.
\end{aligned}$$

Rapoport conjectured in [11] that $X_\mu(b)$ is equidimensional. If this holds true, the above bijection becomes

$$\begin{aligned}
\mathcal{M}_\mu &\longleftrightarrow \{\text{irreducible components of } X_\mu(b)\}\\
&\longleftrightarrow \{J_b(F)-\text{orbits of irreducible components of } X_\mu(b)\}.
\end{aligned}$$

It is known that $J_b(F)$ does not act transitively on the irreducible components of $X_\mu(b)$ in general, even in the case $G = \mathrm{GL}_h$ [12, Ex. 6.2]. The following lemma shows that we have transitive action of $J_b(F)$ only in a few degenerate cases.

Lemma 8.2.

1. Assume that there exist $\tau_1, \tau_2 \in I$ such that μ_{τ_1} and μ_{τ_2} are not of the form $(a, a, \ldots, a)$ for some integer a. Then $\mathcal{M}_\mu$ contains more than one element.
2. On the contrary, if there exists $\tau \in I$ such that $\mu_\varsigma = (a_\varsigma, a_\varsigma, \ldots, a_\varsigma)$ for some integer a_ς for all $\varsigma \neq \tau$, then

$$|\mathcal{M}_\mu| = |\mathcal{M}_{\mu_\tau}|,$$

where $\mathcal{M}_{\mu_\tau}$ denotes the set of extended EL-charts for μ_τ (in particular $d=1$ for these EL-charts). □

Proof. (1) We assume without loss of generality that $\tau_1 \in [0, \tau_2)$. By Proposition 7.1 the cyclic EL-chart of type μ is contained in $\mathcal{M}_\mu$. The same reasoning shows that $\mathcal{M}_\mu$ also contains the cyclic EL-chart of type $(\mu'_0, \ldots, \mu'_{\tau_2-1}, \mu_{\tau_2}, \ldots, \mu_{d-1})$ with $\mu'_\varsigma = (\mu_{\varsigma,h}, \mu_{\varsigma,1}, \ldots, \mu_{\varsigma,h-1})$. Note that our condition on μ_{τ_1} implies $\mu_{\tau_1} \neq \mu'_{\tau_1}$ and thus $\mu \neq (\mu'_0, \ldots, \mu'_{\tau_2-1}, \mu_{\tau_2}, \ldots, \mu_{d-1})$.

(2) The claim holds as the bijection

$$\{\text{extended EL-charts for } \mu\} \longleftrightarrow \{\text{extended semi-modules for } \mu_\tau\},$$

$$(A, \varphi) \longmapsto (A_\tau, \varphi_\tau),$$

$$(\sqcup_{\tau \in I} B, \sqcup\varphi) \longleftarrow (B, \varphi)$$

preserves $|\mathcal{V}|$. ■

We now consider the case μ minuscule. De Jong and Oort showed that, in this case, $|\mathcal{M}_\mu| = 1$ for (extended) semi-modules (i.e., the case $G = \mathrm{GL}_n$). For EL-charts we have the following conjecture, which generalizes [3, Remark 6.16].

Conjecture 8.3. Let μ be minuscule. Then the construction of $\tilde{\mu}$ in the proof of Theorem 7.2 induces a bijection

$$\{(\text{extended) EL-charts for } \mu\} \to \{\tilde{\mu} \in W.\mu;\ \ \nu \preceq \bar{\tilde{\mu}}\},$$

where $W = (S_n)^I$ denotes the absolute Weyl group of G. In particular,

$$\begin{aligned} |\mathcal{M}_\mu| &= |\{\tilde{\mu} \in W.\mu;\ \ \nu \preceq \bar{\tilde{\mu}}, \ell_G[\nu, \tilde{\mu}] = 0\}| \\ &= \left| \left\{ \tilde{\mu} \in W.\mu;\ \ \underline{\tilde{\mu}} = \left(\underbrace{\left\lfloor \frac{m}{h} \right\rfloor, \ldots, \left\lfloor \frac{m}{h} \right\rfloor}_{h\cdot(1-\{m/h\})}, \underbrace{\left\lceil \frac{m}{h} \right\rceil, \ldots, \left\lceil \frac{m}{h} \right\rceil}_{h\cdot\{m/h\}} \right) \right\} \right|. \end{aligned}$$

Here m is defined as in Section 3, that is, $\nu = (\frac{m}{dh}, \ldots, \frac{m}{dh})$. □

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