Almost Spanning Subgraphs of Random Graphs After Adversarial Edge Removal

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Dedicated to Vojtěch Rödl on the occasion of his sixtieth birthday

Let $\Delta \geqslant 2$ be a fixed integer. We show that the random graph $\mathcal{G}_{n,p}$ with $p \gg (\log n/n)^{1/\Delta}$ is robust with respect to the containment of almost spanning bipartite graphs H with maximum degree Δ and sublinear bandwidth in the following sense: asymptotically almost surely, if an adversary deletes arbitrary edges from $\mathcal{G}_{n,p}$ in such a way that each vertex loses less than half of its neighbours, then the resulting graph still contains a copy of all such H.

1. Introduction and results

In this paper we study graphs that are robust in the following sense: even after adversarial removal of a specified proportion of their edges, they still contain copies of every graph from a certain class of graphs.

In order to make this precise, we use the notion of *resilience* (see [28]). Let $\mathcal{P}$ be a monotone increasing graph property and let $G = (V, E)$ be a graph. The *global resilience* $R_g(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of $r \cdot |E|$ edges from E creates a graph which is not in $\mathcal{P}$. The *local resilience* $R_\ell(G, \mathcal{P})$ of G with respect to $\mathcal{P}$ is the minimum $r \in \mathbb{R}$ such that deleting a suitable set of edges, respecting the restriction that at most $r \cdot \deg_G(v)$ edges incident to v should be removed for every vertex $v \in V$, creates a graph which is not in $\mathcal{P}$.

For example, using this terminology, the classical theorems of Turán [29] and Dirac [15] can be stated as follows: the global resilience of the complete graph K_n with respect to containing a clique on r vertices is $\frac{1}{r-1} - o(1)$, and the local resilience of K_n with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$. In this paper we stay quite close to the scenario of these two examples insofar as we will also consider properties that deal with subgraph containment. However, we are interested in the resilience of graphs which are much sparser than the complete graph.

It turns out that the random graph $\mathcal{G}_{n,p}$ is well suited to this purpose ($\mathcal{G}_{n,p}$ is defined on vertex set $[n] = \{1, \dots, n\}$ and edges exist independently of each other with probability p). Clearly, asymptotically almost surely (a.a.s.) the local resilience of $\mathcal{G}_{n,p}$ with respect to containing a Hamilton cycle (or in fact any connected graph on more than, say, $\frac{1}{2}n$ vertices) is at most $\frac{1}{2} + o(1)$, since for bigger values it is easy to disconnect the graph into components of size at most $\frac{1}{2}n$ by deleting edges respecting the corresponding resilience definition. Sudakov and Vu [28] showed that indeed a.a.s. the local resilience of $\mathcal{G}_{n,p}$ with respect to containing a Hamilton cycle is $\frac{1}{2} - o(1)$ if $p > \log^4 n / n$. A result of Dellamonica, Kohayakawa, Marciniszyn and Steger [12] implies that a.a.s. the local resilience of $\mathcal{G}_{n,p}$ with respect to containing cycles of length at least $(1 - \alpha)n$ is $\frac{1}{2} - o(1)$ for any $0 < \alpha < \frac{1}{2}$ and $p \gg 1/n$. We shall discuss the various lower bounds for the edge probability p occurring in these and later results at the end of Section 2.

Recently Balogh, Csaba and Samotij [6] studied the local resilience of $\mathcal{G}_{n,p}$ with respect to containing all trees on $(1 - \eta)n$ vertices with constant maximum degree Δ. They showed that there is a constant $c = c(\Delta, \eta)$ such that for $p \geqslant c/n$ this local resilience is also $\frac{1}{2} - o(1)$ a.a.s.

Now we extend the scope of investigations to the containment of a much larger class of subgraphs. A graph has *bandwidth* at most b if there exists a labelling of the vertices by numbers $1, \dots, n$, such that for every edge ij of the graph we have $|i - j| \leqslant b$. Let $\mathcal{H}(m, \Delta)$ denote the class of all graphs on m vertices with maximum degree at most Δ, and let $\mathcal{H}_2^b(m, \Delta)$ denote the class of all *bipartite* graphs in $\mathcal{H}(m, \Delta)$ which have bandwidth at most b. Our result asserts that the local resilience of $\mathcal{G}_{n,p}$ with respect to containing all graphs H from $\mathcal{H}_2^{\beta n}((1 - \eta)n, \Delta)$ is $\frac{1}{2} - o(1)$ for small β and η and for $p = p(n) = o(1)$ sufficiently large.

Theorem 1.1. *For each $\eta, \gamma > 0$ and $\Delta \geqslant 2$ there exist positive constants β and c such that the following holds for $p \geqslant c(\log n/n)^{1/\Delta}$. Asymptotically almost surely every spanning subgraph $G = (V, E)$ of $\mathcal{G}_{n,p}$ with $\deg_G(v) \geqslant (\frac{1}{2} + \gamma) \deg_{\mathcal{G}_{n,p}}(v)$ for all $v \in V$ contains a copy of every graph H in $\mathcal{H}_2^{\beta n}((1 - \eta)n, \Delta)$.*

We note that several important classes of graphs have sublinear bandwidth, and hence Theorem 1.1 does apply to them: this is the case for the class of all bounded degree planar graphs, for example (see [10]).

As an application of this theorem we derive in Section 3 a result on rainbow H-copies with $H \in \mathcal{H}_2^{\beta n}((1 - \eta)n, \Delta)$ for certain edge-colourings of K_n. The proof of Theorem 1.1 is prepared in Sections 4–7 and presented in Section 8. First, in Section 2, we will compare our result to related results.

2. Background

As we saw at the end of the last section, we are looking for graphs that contain not only one specific subgraph but a large class of graphs. A graph G is called *universal* for a class of graphs $\mathcal{H}$ if G contains a copy of every graph from $\mathcal{H}$ as a subgraph. In this section, we first briefly sketch some results concerning universality in general and then come back to resilience with respect to universality.

Dellamonica, Kohayakawa, Rödl and Ruciński [13] show that $\mathcal{G}_{n,p}$ is a.a.s. universal for $\mathcal{H}(n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$ (where $\widetilde{\mathcal{O}}$ hides polylogarithmic factors). It is also shown in [13] that the lower bound for the edge probability p can be improved if we restrict our attention to balanced bipartite graphs. Let $\mathcal{H}_2(m, m, \Delta)$ denote the class of bipartite graphs in $\mathcal{H}(2m, \Delta)$ with two colour classes of equal size. Then $\mathcal{G}_{2n,p}$ a.a.s. is universal for $\mathcal{H}_2(n, n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/\Delta})$. The same lower bound for p also guarantees universality for *almost spanning* graphs of arbitrary chromatic number: Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [4] prove that for every $\eta > 0$ and for some p in $\widetilde{\mathcal{O}}(n^{-1/\Delta})$, the random graph $\mathcal{G}_{n,p}$ a.a.s. is universal for $\mathcal{H}((1 - \eta)n, \Delta)$. Recently, Dellamonica, Kohayakawa, Rödl and Ruciński [14] generalized these results and obtained a corresponding lower bound for spanning graphs: they have shown that $\mathcal{G}_{n,p}$ is a.a.s. universal for $\mathcal{H}(n, \Delta)$ for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$.

Alon and Capalbo [2, 3] gave explicit constructions of graphs with average degree $\widetilde{\mathcal{O}}(n^{-2/\Delta})n$ that are universal for $\mathcal{H}(n, \Delta)$. For results concerning universal graphs for trees, see, *e.g.*, [5].

Moving on to resilience, it is clear that an adversary can destroy any spanning subgraph by deleting the edges incident to a single vertex. Hence any graph must have trivial global resilience with respect to universality for spanning subgraphs.

However, if we focus on subgraphs of smaller order, then sparse random graphs have a global resilience arbitrarily close to 1: Alon, Capalbo, Kohayakawa, Rödl, Ruciński and Szemerédi [4] show that for every $\gamma > 0$ there is a constant $\eta > 0$ such that for some p in $\widetilde{\mathcal{O}}(n^{-1/2\Delta})$ the random graph $\mathcal{G}_{n,p}$ a.a.s. has global resilience $1 - \gamma$ with respect to

Table 1. Summary of (best) known universality and resilience results
(logarithmic factors for p are omitted).

	Result	p	Reference
Universality	$\mathcal{H}(n, \Delta) \subseteq \mathcal{G}_{n,p}$	$p = n^{-1/\Delta}$	[14]
Resilience	$R_g(\mathcal{G}_{n,p}, \mathcal{H}_2(\eta n, \eta n, \Delta)) \geqslant 1 - \gamma$	$p = n^{-1/2\Delta}$	[4]
	$R_\ell(\mathcal{G}_{n,p}, \mathcal{H}_2^{\beta n}((1-\eta)n, \Delta)) \geqslant \frac{1}{2} - \gamma$	$p = n^{-1/\Delta}$	Theorem 1.1

universality for $\mathcal{H}_2(\eta n, \eta n, \Delta)$. In other words, $\mathcal{G}_{n,p}$ contains *many* copies of all graphs from $\mathcal{H}_2(\eta n, \eta n, \Delta)$ *everywhere*.

Finally, the concept of local resilience allows for non-trivial results concerning universality for almost spanning subgraphs. For example, a conjecture of Bollobás and Komlós proved in [11] asserts that the local resilience of the complete graph K_n with respect to universality for $\mathcal{H}_r^{\beta n}(n, \Delta)$ is $\frac{1}{r} - o(1)$. Here $\mathcal{H}_r^{\beta n}(n, \Delta)$ is the class of all r-colourable n-vertex graphs with maximum degree at most Δ and bandwidth at most βn, and one can show that the bandwidth constraint cannot be omitted.

Theorem 2.1 ([11]). *For all $r, \Delta \in \mathbb{N}$ and $\gamma > 0$, there exist constants $\beta > 0$ and $n_0 \in \mathbb{N}$ such that for every $n \geqslant n_0$ the following holds. If H is an r-chromatic graph on n vertices with $\Delta(H) \leqslant \Delta$, and bandwidth at most βn, and if G is a graph on n vertices with minimum degree $\delta(G) \geqslant (\frac{r-1}{r} + \gamma)n$, then G contains a copy of H.*

Our Theorem 1.1 replaces K_n by the much sparser graph $\mathcal{G}_{n,p}$, but it only treats the case $r = 2$ and almost spanning subgraphs.

Let us mention two more recent papers which have continued this line of research. Huang, Lee and Sudakov [21] considered almost spanning factors for constant probability p and showed that one cannot hope to obtain spanning subgraphs, as $\Omega(p^{-2})$ vertices may be forced to be left out. Also, Balogh, Lee and Samotij [7] considered the case of almost spanning triangle factors for $p \gg (\log n/n)^{1/2}$.

Before we conclude this section, let us briefly discuss the lower bounds for the edge probability p mentioned in the results above, summarized in Table 1. First, a straightforward counting argument shows that any graph that is universal for $\mathcal{H}(n, \Delta)$ must have at least $\Omega(n^{2-2/\Delta})$ edges. Moreover, it is easy to see that an edge probability $p = n^{\varepsilon - 2/\Delta}$ with $\varepsilon < \frac{1}{\Delta^2}$ is not sufficient to guarantee that $\mathcal{G}_{n,p}$ is universal even for the more restrictive class $\mathcal{H}_2(\eta n, \eta n, \Delta)$. Indeed, consider the graph $H \in \mathcal{H}_2(\eta n, \eta n, \Delta)$ consisting of $\eta n/\Delta$ copies of $K_{\Delta,\Delta}$. The expected number of copies of $K_{\Delta,\Delta}$ in $\mathcal{G}_{n,p}$ is at most

$$n^{2\Delta} p^{\Delta^2} = n^{2\Delta} (n^{-\frac{2}{\Delta} + \varepsilon})^{\Delta^2} = n^{2\Delta - 2\Delta + \varepsilon\Delta^2} \ll n,$$

and hence a.a.s. $\mathcal{G}_{n,p}$ does not contain a copy of H.

3. An application: rainbow copies of bipartite graphs

Let φ be an arbitrary colouring of the edges of the complete graph K_n. If φ uses no colour more than k times then we say that φ is *k-bounded*. Moreover, a copy of a graph H in K_n is a *rainbow* copy if φ uses no colour more than once on H. If there is a rainbow copy of H in K_n then φ is called *H-rainbow*.

Erdős, Nešetřil and Rödl [16] asked for which $k = k(n)$ every k-bounded edge colouring of K_n has a rainbow Hamilton cycle. Frieze and Reed [17] showed that $k(n)$ can grow as fast as $\kappa n / \log n$ for some constant κ (for early progress on this problem see the references in [17]). Albert, Frieze and Reed [1] improved this bound to $n/65$, which shows that k can grow linearly, as was previously conjectured by Hahn and Thomassen [19].

Here we consider the analogous question for H-rainbow colourings with $H \in \mathcal{H}_2^{\beta n}((1 - \eta)n, \Delta)$. As a consequence of our main theorem, Theorem 1.1, we prove the following result.

Theorem 3.1. *For every $\eta > 0$ and $\Delta \geqslant 2$ there exist positive constants β and κ such that for n sufficiently large, for every graph $H \in \mathcal{H}_2^{\beta n}((1 - \eta)n, \Delta)$ and $k \leqslant \kappa(n/\log n)^{1/\Delta}$, every k-bounded edge-colouring of K_n is H-rainbow.*

For the proof of this theorem we apply the strategy of [17] and do the following for a given k-bounded edge colouring φ of K_n. We first take a random subgraph $\Gamma = \mathcal{G}_{n,p}$ of K_n and then delete all edges in Γ whose colour appears more than once in Γ. Denote the resulting graph by $\Gamma(\varphi)$. Any subgraph of $\Gamma(\varphi)$ is trivially rainbow and hence it remains to show that there is a copy of H in $\Gamma(\varphi)$ in order to establish Theorem 3.1. In view of Theorem 1.1 it clearly suffices to prove the following lemma.

Lemma 3.2. *Let $p = p(n)$ and $k = k(n)$ be such that $p \geqslant 10^6 \log n/n$ and $pk \leqslant 10^{-3}$. For any k-bounded edge colouring φ of K_n, with probability $1 - o(1)$ all vertices v in $\Gamma = \mathcal{G}_{n,p}$ satisfy $\deg_{\Gamma(\varphi)}(v) \geqslant \frac{2}{3} \deg_\Gamma(v)$.*

Sketch of proof. Let v be an arbitrary vertex of Γ. We classify the 'deleted' edges incident to v, that is, those edges in $E(v, N_\Gamma(v) \setminus N_{\Gamma(\varphi)}(v))$, into two sets: the set N_1 of those edges whose colour appears only once in $E(v, N_\Gamma(v))$ (but also somewhere else in Γ) and the set N_2 of those edges whose colour appears at least twice in $E(v, N_\Gamma(v))$. With probability $1 - o(1/n)$ we have that $\deg_\Gamma(v)$ lies in the interval $\left[(1 - \frac{1}{20})np, (1 + \frac{1}{20})np\right]$ by a Chernoff bound. Therefore, showing

(i) $\mathbb{P}(|N_1| \geqslant \frac{1}{10}np) = o(1/n)$ and
(ii) $\mathbb{P}(|N_2| \geqslant \frac{1}{10}np) = o(1/n)$

and applying the union bound proves the lemma.

To establish (i) we expose the edges incident to v first, which enables us to determine $\deg_\Gamma(v)$. We have $\mathbb{P}(\deg_\Gamma(v) \geqslant \frac{21}{20}np) = o(1/n)$. Subsequently we expose the remaining edges. Recall that for any edge $vw \in N_1$ the colour $\varphi(vw)$ appears somewhere else in Γ, which happens with probability at most $p' := pk$. Since these events are independent for

different colours, we have

$$\mathbb{P}(|N_1| \geqslant t) \leqslant \mathbb{P}\big(\deg_\Gamma(v) \geqslant \tfrac{21}{20}np\big) + \mathbb{P}\big(|N_1| \geqslant t \mid \deg_\Gamma(v) \leqslant \tfrac{21}{20}np\big)$$
$$= o(1/n) + \mathbb{P}(X \geqslant t),$$

where X is a random variable with distribution $\mathrm{Bi}(n', p')$, where $n' = \deg_\Gamma(v) \leqslant \tfrac{21}{20}np$. Clearly $\mathbb{E} X \leqslant \tfrac{21}{20}np \cdot pk \leqslant \tfrac{1}{100}np$, and therefore (i) follows from an application of a Chernoff bound, since $np \geqslant 10^6 \log n$.

To establish (ii) consider the random variable Y that counts edges in $E(v, N_\Gamma(v))$ whose colour appears only once in $E(v, N_\Gamma(v))$. Then $|N_2| = \deg_\Gamma(v) - Y$, and so it suffices to show that $\mathbb{P}\big(Y \leqslant \tfrac{19}{20}np\big) = o(1/n)$, using again that $\deg_\Gamma(v) > \tfrac{21}{20}np$ happens with probability $o(1/n)$. To see this, assume that $1, \ldots, \ell$ are the colours that appear on the edges of K_n containing v, and let k_i be the number of such edges with colour $i \in [\ell]$. Then $Y = \sum_{i \in [\ell]} Y_i$, where Y_i is the indicator variable for the event that $E(v, N_\Gamma(v))$ contains exactly one edge of colour i. Observe that the Y_i are independent random variables and that $\mathbb{P}(Y_i = 1) = k_i p(1-p)^{k_i - 1}$. In addition

$$1 \geqslant (1-p)^{k_i - 1} \geqslant (1-p)^k \geqslant \exp\left(-\frac{p}{1-p}k\right) \geqslant \frac{100}{101},$$

and hence

$$\mathbb{E} Y = \sum_{i \in [\ell]} k_i p(1-p)^{k_i - 1} \leqslant np \quad \text{and} \quad \mathbb{E} Y \geqslant \tfrac{100}{101}(n-1)p \geqslant \tfrac{99}{100}np.$$

We conclude that $\mathbb{P}\big(Y \leqslant \tfrac{19}{20}np\big) = o(1/n)$ from $\mathbb{P}\big(Y \leqslant \mathbb{E} Y - t\big) \leqslant \exp\big(-\tfrac{1}{2}t^2/\mathbb{E} Y\big)$ (see [22, Theorem 2.10]) by setting $t := \tfrac{1}{100}np$ and using $np \geqslant 10^6 \log n$. □

As mentioned earlier, the bound on $k(n)$ established in [17] for rainbow Hamilton cycles is not best possible. As it turns out, the bound on k in Theorem 3.1 above can be improved as well. Indeed, such an improvement has recently been established in [9], where Lovász's local lemma is used. However, we observe that the method of proof above is more robust in the sense that one can, for instance, prove that suitably bounded colourings of *sparse random graphs* are H-rainbow – something that does not seem to be within the reach of the method of proof in [9] (we omit the details).

4. Sparse regularity

In this section we will introduce one of the main tools for our proof, a sparse version of the regularity lemma developed by Rödl and one of the current authors (see [23, 25]). Before stating this lemma we introduce the necessary definitions.

Let $G = (V, E)$ be a graph, and suppose $p \in (0, 1]$ and $\varepsilon > 0$ are reals. For disjoint non-empty sets $U, W \subseteq V$ the *p-density* of the pair (U, W) is defined as $d_{G,p}(U, W) := e_G(U, W)/(p|U||W|)$. The pair (U, W) is (ε, p)-*regular* if $|d_{G,p}(U', W') - d_{G,p}(U, W)| \leqslant \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geqslant \varepsilon|U|$ and $|W'| \geqslant \varepsilon|W|$.

An (ε, p)-*regular partition* of $G = (V, E)$ is an ε-*equipartition* $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_r$ of V, that is, with $|V_0| \leqslant \varepsilon|V|$ and $|V_1| = \cdots = |V_r|$, such that (V_i, V_j) is an (ε, p)-regular pair in G

for all but at most $\varepsilon\binom{r}{2}$ pairs $ij \in \binom{[r]}{2}$. The partition classes V_i with $i \in [r]$ are called the *clusters* of the partition and V_0 is the *exceptional set*.

The sparse regularity lemma asserts the existence of (ε, p)-regular partitions for sparse graphs G without 'dense spots'. To quantify this latter property we need the following notion. Let $\eta > 0$ and $K > 1$ be real numbers. We say that $G = (V, E)$ is (η, K)-*bounded with respect to* p if for all disjoint sets $X, Y \subseteq V$ with $|X|, |Y| \geqslant \eta|V|$ we have $e_G(X, Y) \leqslant Kp|X||Y|$.

Lemma 4.1 (sparse regularity lemma). *For each $\varepsilon > 0$, $K > 1$, and $r_0 \geqslant 1$ there are constants r_1, v, and n_0 such that for any $p \in (0, 1]$ the following holds. Any graph $G = (V, E)$ which has at least n_0 vertices and is (v, K)-bounded with respect to p admits an (ε, p)-regular ε-equipartition with r clusters, for some $r_0 \leqslant r \leqslant r_1$.*

As it turns out, we shall only make use of what one could call 'one-sided regularity'. We call a pair (U, W) (ε, d, p)-*dense* if $d_{G,p}(U', W') \geqslant d - \varepsilon$ for all $U' \subseteq U$ and $W' \subseteq W$ with $|U'| \geqslant \varepsilon|U|$ and $|W'| \geqslant \varepsilon|W|$. Clearly, an (ε, p)-regular pair (U, W) is (ε, d, p)-dense for $d = d_{G,p}(U, W)$. Occasionally, in informal discussions, when the particular value of d or ε is not immediately relevant, we say that an (ε, p)-regular pair (U, W) is (ε, p)-*dense* or *p-dense*.

An ε-equipartition $V_0 \dot\cup V_1 \dot\cup \cdots \dot\cup V_r$ of a graph $G = (V, E)$ is an (ε, d, p)-*dense partition* with *reduced graph* R if $V(R) = [r]$ and the pair (V_i, V_j) is (ε, d, p)-dense in G whenever $ij \in E(R)$. Note that, given an (ε, p)-regular partition as in Lemma 4.1 and a real number d, one has an (ε, d, p)-dense partition of G with the reduced graph R, with $ij \in E(R)$ if and only if only (V_i, V_j) is (ε, p)-regular and $d_{G,p}(V_i, V_j) \geqslant d$.

It follows directly from the definition that sub-pairs of *p*-dense pairs again form *p*-dense pairs.

Proposition 4.2. *Let (X, Y) be (ε, d, p)-dense and suppose $X' \subseteq X$ satisfies $|X'| \geqslant \mu|X|$. Then (X', Y) is $(\frac{\varepsilon}{\mu}, d, p)$-dense.*

In addition, neighbourhoods of most vertices in a *p*-dense pair are not much smaller than expected. Again, this is a direct consequence of the definition of *p*-dense pairs.

Proposition 4.3. *Let (X, Y) be (ε, d, p)-dense. Then less than $\varepsilon|X|$ vertices $x \in X$ are such that $|N_Y(x)| < (d - \varepsilon)p|Y|$.*

Some properties of the graph G translate to certain properties of the reduced graph R of the partition constructed by the sparse regularity lemma. For example, the following well-known consequence of Lemma 4.1 is a minimum degree version of the sparse regularity lemma. For a proof see the appendix of [8].

Lemma 4.4 (sparse regularity lemma, minimum degree version for $\mathcal{G}_{n,p}$). *For all $\alpha \in [0, 1]$, $\varepsilon > 0$, and every integer r_0, there is an integer $r_1 \geqslant 1$ such that for all $d \in [0, 1]$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ if $\log^4 n/(pn) = o(1)$. Let $G = (V, E)$ be a spanning subgraph of Γ*
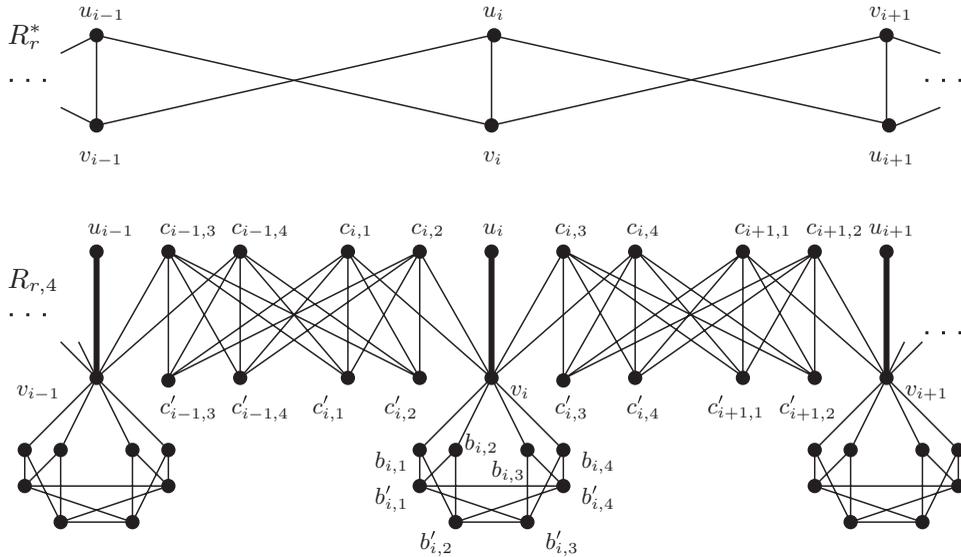
Figure 1. The ladder R_r^* and the spin graph $R_{r,t}$ for the special case $t = 2$.

with $\deg_G(v) \geqslant \alpha \deg_\Gamma(v)$ for all $v \in V$. Then there is an (ε, d, p)-dense partition of G with reduced graph R of minimum degree $\delta(R) \geqslant (\alpha - d - \varepsilon)|V(R)|$ with $r_0 \leqslant |V(R)| \leqslant r_1$.

We remark that we do observe 'more' than a mere inheritance of properties here: the graph G we started with is *sparse*, but the reduced graph R we obtain in Lemma 4.4 is *dense*. This will enable us to apply results obtained for dense graphs to the reduced graph R, and hence use such dense results to draw conclusions about sparse graphs.

5. Main lemmas

In this section we will formulate the main lemmas and outline how they will be combined in Section 8 to give the proof of Theorem 1.1. For this we first need to define two (families of) special graphs.

For $r, t \in \mathbb{N}$, let $U = \{u_1, \ldots, u_r\}$, $V = \{v_1, \ldots, v_r\}$, $C = \{c_{i,j}, c'_{i,j} : i \in [r], j \in [2t]\}$, and $B = \{b_{i,j}, b'_{i,j} : i \in [r], j \in [2t]\}$. Let the *ladder* R_r^* be the graph with vertex set $U \dot\cup V$ and edge set $E(R_r^*) := \{u_i v_j : i, j \in [r], |i - j| \leqslant 1\}$. Let the *spin graph* $R_{r,t}$ be the graph with vertex set $U \dot\cup V \dot\cup C \dot\cup B$ and the following edge set (see Figure 1):

$$E(R_{r,t}) := \bigcup_{\substack{i,i' \in [r], i' \neq 1 \\ j,j' \in [2t] \\ k,k' \in [t] \\ \ell,\ell' \in [t+1,2t]}} \Big(\{u_i v_i, b_{i,k} b'_{i,k'}, b_{i,\ell} b'_{i,\ell'}, c_{i,k} c'_{i,k'}, c_{i,\ell} c'_{i,\ell'}\} \cup \{b_{i,j} v_i, c_{i,j} v_i\}$$
$$\cup \{b'_{i,k} b'_{i,\ell}, c_{i'-1,\ell} c'_{i',k}, c'_{i'-1,\ell} c_{i',k}\} \Big).$$

Now we can state our four main lemmas, two partition lemmas and two embedding lemmas. We start with the lemma for G, which constructs a partition of the host graph G. This lemma is a consequence of the sparse regularity lemma (Lemma 4.4) and asserts the

existence of a p-dense partition of G such that its reduced graph contains a spin graph. We will indicate below why this is useful for the embedding of H. The lemma for G produces clusters of very different sizes: a set of larger clusters U_i and V_i which we call *big clusters* and which will accommodate most of the vertices of H later, and a set of smaller clusters $B_{i,j}, B'_{i,j}, C_{i,j}$, and $C'_{i,j}$. The $B_{i,j}$ and $B'_{i,j}$ are called *balancing clusters* and the $C_{i,j}$ and $C'_{i,j}$ *connecting clusters*. They will be used to host a small number of vertices of H. These vertices balance and connect the pieces of H that are embedded into the big clusters. The proof of Lemma 5.1 is given in Section 9.

Lemma 5.1 (lemma for G). *For all integers $t, r_0 > 0$ and reals $\eta_G, \gamma > 0$ there are positive reals η'_G and d such that, for all $\varepsilon > 0$, there is an r_1 for which the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$. Let $G = (V, E)$ be a spanning subgraph of Γ with $\deg_G(v) \geqslant (\frac{1}{2} + \gamma)\deg_\Gamma(v)$ for all $v \in V$. Then there is an $r_0 \leqslant r \leqslant r_1$, a subset V_0 of V with $|V_0| \leqslant \varepsilon n$, and a mapping g from $V \setminus V_0$ to the spin graph $R_{r,t}$, such that for every $i \in [r], j \in [2t]$ we have*

(G1) $|U_i|, |V_i| \geqslant (1 - \eta_G)\frac{n}{2r}$ *for* $U_i := g^{-1}(u_i)$ *and* $V_i := g^{-1}(v_i)$,
(G2) $|C_{i,j}|, |C'_{i,j}|, |B_{i,j}|, |B'_{i,j}| \geqslant \eta'_G \frac{n}{2r}$ *for* $C_{i,j} := g^{-1}(c_{i,j})$, $C'_{i,j} := g^{-1}(c'_{i,j})$, $B_{i,j} := g^{-1}(b_{i,j})$, *and* $B'_{i,j} := g^{-1}(b'_{i,j})$,
(G3) *the pair* $\left(g^{-1}(x), g^{-1}(y)\right)$ *is* (ε, d, p)-*dense for all* $xy \in E(R_{r,t})$.

Since the dependences of the constants appearing in this lemma are quite involved, we remark that their quantification is as follows:

$$\forall\, t,\, r_0,\, \eta_G,\, \gamma \quad \exists\, \eta'_G,\, d \quad \forall\, \varepsilon \quad \exists\, r_1.$$

Our second lemma provides a partition of H that fits the structure of the partition of G generated by Lemma 5.1. We will first state this lemma and then explain the different properties which it guarantees. A set S of vertices in a graph H is called ℓ-*independent* for an integer ℓ if each pair of distinct vertices in S has distance at least $\ell + 1$ in H.

Lemma 5.2 (lemma for H). *For all integers Δ there is an integer $t > 0$ such that, for any $\eta_H > 0$ and any integer $r \geqslant 1$, there is a $\beta > 0$ for which the following holds for all integers m and all bipartite graphs H on m vertices with $\Delta(H) \leqslant \Delta$ and $\mathrm{bw}(H) \leqslant \beta m$. There is a homomorphism h from H to the spin graph $R_{r,t}$, such that for every $i \in [r], j \in [2t]$ we have*

(H1) $|\widetilde{U}_i|, |\widetilde{V}_i| \leqslant (1 + \eta_H)\frac{m}{2r}$ *for* $\widetilde{U}_i := h^{-1}(u_i)$ *and* $\widetilde{V}_i := h^{-1}(v_i)$,
(H2) $|\widetilde{C}_{i,j}|, |\widetilde{C}'_{i,j}|, |\widetilde{B}_{i,j}|, |\widetilde{B}'_{i,j}| \leqslant \eta_H \frac{m}{2r}$ *for* $\widetilde{C}_{i,j} := h^{-1}(c_{i,j})$, $\widetilde{C}'_{i,j} := h^{-1}(c'_{i,j})$, $\widetilde{B}_{i,j} := h^{-1}(b_{i,k})$, *and* $\widetilde{B}'_{i,j} := h^{-1}(b'_{i,k})$,
(H3) $\widetilde{C}_{i,j}, \widetilde{C}'_{i,j}, \widetilde{B}_{i,j}$, *and* $\widetilde{B}'_{i,j}$ *are 3-independent in* H,
(H4) $\deg_{\widetilde{V}_i}(y) = \deg_{\widetilde{V}_i}(y') \leqslant \Delta - 1$ *for all* $yy' \in \binom{\widetilde{C}_{i,j}}{2} \cup \binom{\widetilde{B}_{i,j}}{2}$, $\deg_{\widetilde{C}_i}(y) = \deg_{\widetilde{C}_i}(y')$ *for all* $y, y' \in \widetilde{C}'_{i,j}$, $\deg_{L(i,j)}(y) = \deg_{L(i,j)}(y')$ *for all* $y, y' \in \widetilde{B}'_{i,j}$,
where $\widetilde{C}_i := \bigcup_{k \in [2t]} \widetilde{C}_{i,k}$ *and* $L(i,j) := \bigcup_{k \in [2t]} \widetilde{B}_{i,k} \cup \bigcup_{k < j} \widetilde{B}'_{i,k}$. *Further, let* $\widetilde{X}_i$ *with* $i \in [r]$ *be the set of vertices in* $\widetilde{V}_i$ *with neighbours outside* $\widetilde{U}_i$. *Then*

(H5) $|\widetilde{X}_i| \leqslant \eta_H |\widetilde{V}_i|$.

The quantification of the constants appearing in this lemma is as follows:

$$\forall \Delta \quad \exists t \quad \forall \eta_H, r \quad \exists \beta.$$

This lemma asserts the existence of a homomorphism h from H to a spin graph $R_{r,t}$. Recall that $R_{r,t}$ is contained in the reduced graph of the p-dense partition provided by Lemma 5.1. As we will see, we can fix the parameters in this lemma such that, when we apply it together with Lemma 5.1, the homomorphism h has the following additional property. The number $\widetilde{L}$ of vertices that it maps to a vertex a of the spin graph is less than the number L contained in the corresponding cluster A provided by Lemma 5.1 (compare (G1) and (G2) with (H1) and (H2) and recall that m is slightly smaller than n). If A is a big cluster, then the numbers L and $\widetilde{L}$ differ only slightly (these vertices will be embedded using the constrained blow-up lemma), but for balancing and connecting clusters A the number $\widetilde{L}$ is much smaller than L (this is necessary for the embedding of these vertices using the connection lemma). With property (H5) Lemma 5.2 further guarantees that only a few edges of H are not assigned either to two connecting or balancing clusters, or to two big clusters. This is helpful because it implies that we do not have to take care of 'too many dependences' between the applications of the blow-up lemma and the connection lemma. The remaining properties (H3)–(H4) of Lemma 5.2 are technical but required for the application of the connection lemma (see conditions (B) and (C) of Lemma 5.4).

The vertices in $\widetilde{C}_{i,j}$ and $\widetilde{C}'_{i,j}$ are also called *connecting vertices* of H, and the vertices in $\widetilde{B}_{i,j}$ and $\widetilde{B}'_{i,j}$ are called *balancing vertices*.

We next describe the two embedding lemmas, the constrained blow-up lemma (Lemma 5.3) and the connection lemma (Lemma 5.4), which we would like to use on the partitions of G and H provided by Lemmas 5.1 and 5.2. The connecting lemma will be used to embed the connecting and balancing vertices into the connecting and balancing clusters *after* all the other vertices are embedded into the big clusters with the help of the constrained blow-up lemma.

The constrained blow-up lemma states that bipartite graphs H with bounded maximum degree can be embedded into a p-dense pair $G = (U, V)$ whose cluster sizes are just slightly bigger than the partition classes of H. This lemma further guarantees the following. If we specify a small family of small special sets in one of the partition classes of H and a small family of small forbidden sets in the corresponding cluster of G, then no special set is mapped to a forbidden set.

The existence of these forbidden sets is in fact a main difference from the classical blow-up lemma which is used in the dense setting, where a small family of special vertices of H can be guaranteed to be mapped to a *required* set of linear size in G. This is very useful in a dense graph, because its neighbourhoods (into which we would like to embed neighbours of already embedded vertices) are of linear size. In contrast, the property of having forbidden sets will be crucial for the sparse setting when we apply this lemma together with the connection lemma in the proof of Theorem 1.1 in order to handle the 'dependences' between these applications. The proof of this lemma is given in Section 11 and relies on techniques developed in [4].

Lemma 5.3 (constrained blow-up lemma). *For every integer $\Delta > 1$ and for all positive reals d and η, there exist positive constants ε and μ such that for all positive integers r_1 there is a c such that, for all integers $1 \leqslant r \leqslant r_1$, the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \geqslant c(\log n/n)^{1/\Delta}$. Let $G = (U, V) \subseteq \Gamma$ be an (ε, d, p)-dense pair with $|U|, |V| \geqslant n/r$ and let H be a bipartite graph on vertex classes $\widetilde{U} \dot\cup \widetilde{V}$ of sizes $|\widetilde{U}|, |\widetilde{V}| \leqslant (1 - \eta)n/r$ and with $\Delta(H) \leqslant \Delta$. Moreover, suppose that there is a family $\mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}$ of special Δ-sets in $\widetilde{V}$ such that each $\widetilde{v} \in \widetilde{V}$ is contained in at most Δ special sets and a family $\mathcal{B} \subseteq \binom{V}{\Delta}$ of forbidden Δ-sets in V with $|\mathcal{B}| \leqslant \mu |V|^\Delta$. Then there is an embedding of H into G such that no special set is mapped to a forbidden set.*

The quantification of the constants appearing in this lemma is as follows:

$$\forall \Delta, d, \eta \quad \exists \varepsilon, \mu \quad \forall r_1 \quad \exists c.$$

At first sight, the rôle of the integer r in Lemma 5.3 (and also in Lemma 5.4 below) seems a little obscure. The only reason for stating the lemma as above is that it is more readily applicable in this form, since we will need it for pairs of partition classes (U, V) whose size in relation to n will be determined by the regularity lemma.

Our last main lemma, the connection lemma (Lemma 5.4), embeds graphs H into graphs G forming a system of p-dense pairs. In contrast to the blow-up lemma, however, the graph H has to be much smaller than the graph G now (see condition (A)). In addition, each vertex $\widetilde{y}$ of H is equipped with a candidate set $C(\widetilde{y})$ in G from which the connection lemma will choose the image of $\widetilde{y}$ in the embedding. Lemma 5.4 requires that these candidate sets are big (condition (D)) and that pairs of candidate sets that correspond to an edge of H form p-dense pairs (condition (E)). The remaining conditions ((B) and (C)) are conditions on the neighbourhoods and degrees of the vertices in H (with respect to the given partition of H). For their statement we need the following additional definition.

For a graph H on vertex set $\widetilde{V} = \widetilde{V}_1 \dot\cup \cdots \dot\cup \widetilde{V}_t$ and $y \in \widetilde{V}_i$ with $i \in [t]$ define the *left degree* of y with respect to the partition $\widetilde{V}_1 \dot\cup \cdots \dot\cup \widetilde{V}_t$ to be

$$\mathrm{ldeg}(y; \widetilde{V}_1, \dots, \widetilde{V}_t) := \sum_{j=1}^{i-1} \deg_{\widetilde{V}_j}(y).$$

When clear from the context we may also omit the partition and simply write $\mathrm{ldeg}(y)$. For two sets of vertices S, T we denote the *common neighbourhood* of (the vertices of) S in T by $N_T^\cap(S) := \bigcap_{s \in S} N_T(s)$.

Lemma 5.4 (connection lemma). *For all integers $\Delta > 1$, $t > 0$ and reals $d > 0$, there are ε, $\xi > 0$ such that for all positive integers r_1 there is a $c > 1$ such that, for all integers $1 \leqslant r \leqslant r_1$, the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ with $p \geqslant c(\log n/n)^{1/\Delta}$. Let $G \subseteq \Gamma$ be any graph on vertex set $W = W_1 \dot\cup \cdots \dot\cup W_t$ and let H be any graph on vertex set $\widetilde{W} = \widetilde{W}_1 \dot\cup \cdots \dot\cup \widetilde{W}_t$. Suppose further that for each $i \in [t]$ each vertex $\widetilde{w} \in \widetilde{W}_i$ is equipped with an arbitrary set $X_{\widetilde{w}} \subseteq V(\Gamma) \setminus W$ with the property that the indexed set system $(X_{\widetilde{w}} : \widetilde{w} \in \widetilde{W}_i)$ consists of*

pairwise disjoint sets such that the following holds. We define the external degree *of $\widetilde{w}$ to be* $\mathrm{edeg}(\widetilde{w}) := |X_{\widetilde{w}}|$, *its* candidate set $C(\widetilde{w}) \subseteq W_i$ *to be* $C(\widetilde{w}) := N^{\cap}_{W_i}(X_{\widetilde{w}})$, *and require that*

(A) $|W_i| \geqslant n/r$ *and* $|\widetilde{W}_i| \leqslant \xi n/r$,

(B) $\widetilde{W}_i$ *is a 3-independent set in* H,

(C) $\mathrm{edeg}(\widetilde{w}) + \mathrm{ldeg}(\widetilde{w}) = \mathrm{edeg}(\widetilde{v}) + \mathrm{ldeg}(\widetilde{v})$ *and* $\deg_H(\widetilde{w}) + \mathrm{edeg}(\widetilde{w}) \leqslant \Delta$ *for all* $\widetilde{w}, \widetilde{v} \in \widetilde{W}_i$,

(D) $|C(\widetilde{w})| \geqslant ((d - \varepsilon)p)^{\mathrm{edeg}(\widetilde{w})}|W_i|$ *for all* $\widetilde{w} \in \widetilde{W}_i$, *and*

(E) $(C(\widetilde{w}), C(\widetilde{v}))$ *forms an* (ε, d, p)-*dense pair for all* $\widetilde{w}\widetilde{v} \in E(H)$.

Then there is an embedding of H into G such that every vertex $\widetilde{w} \in \widetilde{W}$ is mapped to a vertex in its candidate set $C(\widetilde{w})$.

The quantification of the constants appearing in this lemma is as follows:

$$\forall \Delta, t, d \quad \exists \varepsilon, \xi \quad \forall r_1 \quad \exists c.$$

The proof of this lemma is inherent in [26]. For the details in our setting see the appendix of [8].

6. Stars in random graphs

In this section we formulate two lemmas concerning properties of random graphs that will be useful when analysing neighbourhood properties of p-dense pairs in the following section. More precisely, we consider the following question here. Given a set of vertices X in a random graph $\Gamma = \mathcal{G}_{n,p}$ together with a family $\mathcal{F}$ of pairwise disjoint ℓ-sets in $V(\Gamma)$, we would like to determine how many pairs (x, F) with $x \in X$ and $F \in \mathcal{F}$ have the property that x lies in the common neighbourhood of the vertices in F.

Definition 6.1 (stars). Let $G = (V, E)$ be a graph, let X be a subset of V and let $\mathcal{F}$ be a family of pairwise disjoint ℓ-sets in $V \setminus X$ for some ℓ. Then the number of *stars* in G between X and $\mathcal{F}$ is

$$\# \mathrm{stars}^G(X, \mathcal{F}) := |\{(x, F): x \in X, F \in \mathcal{F}, F \subseteq N_G(x)\}|. \tag{6.1}$$

Observe that in a random graph $\Gamma = \mathcal{G}_{n,p}$ and for fixed sets X and $\mathcal{F}$ the random variable $\# \mathrm{stars}^{\Gamma}(X, \mathcal{F})$ has binomial distribution $\mathrm{Bi}(|X||\mathcal{F}|, p^{\ell})$. This will be used in the proofs of the following lemmas. The first of these lemmas states that in $\mathcal{G}_{n,p}$ the number of stars between X and $\mathcal{F}$ does not exceed its expectation by more than seven times as long as X and $\mathcal{F}$ are not too small. This is a straightforward consequence of Chernoff's inequality.

Lemma 6.2 (star lemma for big sets). *For every positive integer Δ and every positive real v, there is a c such that if $p \geqslant c(\log n/n)^{1/\Delta}$ the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V. Let X be any subset of V and let $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $vn \leqslant |X| \leqslant |\mathcal{F}| \leqslant n$, then*

$$\# \mathrm{stars}^{\Gamma}(X, \mathcal{F}) \leqslant 7p^{\Delta}|X||\mathcal{F}|.$$

Proof. Given Δ and v, let c be such that $7c^\Delta v^2 \geqslant 3\Delta$. From Chernoff's inequality (see [22, Chapter 2]) we know that $\mathbb{P}[Y \geqslant 7\,\mathbb{E}\,Y] \leqslant \exp(-7\,\mathbb{E}\,Y)$ for a binomially distributed random variable Y. We conclude that for fixed X and $\mathcal{F}$,

$$\mathbb{P}\big[\#\,\mathrm{stars}^\Gamma(X,\mathcal{F}) > 7p^\Delta|X||\mathcal{F}|\big] \leqslant \exp(-7p^\Delta|X||\mathcal{F}|)$$
$$\leqslant \exp(-7c^\Delta(\log n/n)v^2n^2) \leqslant \exp(-3\Delta n \log n)$$

by the choice of c. Thus the probability that there are sets X and $\mathcal{F}$ violating the assertion of the lemma is at most

$$2^n n^{\Delta n} \exp(-3\Delta n \log n) \leqslant \exp(2\Delta n \log n - 3\Delta n \log n),$$

which tends to 0 as n tends to infinity. $\qquad\square$

We will also need a variant of Lemma 6.2 for smaller sets X and families $\mathcal{F}$. As a trade-off, the bound on the number of stars provided by the next lemma will be somewhat worse. Lemma 6.3 appears almost in this form in [26]. The only (slight) modification that we need here is that X is allowed to be bigger than $\mathcal{F}$. However, the same proof as presented in [26] still works for this modified version. For details see the appendix of [8].

Lemma 6.3 (star lemma for small sets). *For all positive integers Δ and positive reals ξ, there are positive constants v and c such that if $p \geqslant c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$ on vertex set V. Let X be any subset of V and let $\mathcal{F}$ be any family of pairwise disjoint Δ-sets in $V \setminus X$. If $|X| \leqslant vnp^\Delta|\mathcal{F}|$ and $|X|, |\mathcal{F}| \leqslant \xi n$, then*

$$\#\,\mathrm{stars}^\Gamma(X,\mathcal{F}) \leqslant p^\Delta|X||\mathcal{F}| + 6\xi np^\Delta|\mathcal{F}|. \tag{6.2}$$

7. Common neighbourhoods in p-dense pairs

As discussed in Section 4, it follows directly from the definition of p-denseness that sub-pairs of dense pairs again form dense pairs. In order to apply Lemma 5.3 and Lemma 5.4 together, we will need corresponding results on common neighbourhoods in systems of dense pairs (see Lemmas 7.2 and 7.5). For this it is necessary to first introduce some notation.

Let $G = (V, E)$ be a graph, let $\ell, T > 0$ be integers, let p, ε, d be positive reals, and let $X, Y, Z \subseteq V$ be disjoint vertex sets. Recall that for a set B of vertices from V and a vertex set $Y \subseteq V$, we call the set $N_Y^\cap(B) = \bigcap_{b \in B} N_Y(b)$ the common neighbourhood of (the vertices in) B in Y.

Definition 7.1 (bad and good vertex sets). Let $G, \ell, T, p, \varepsilon, d, X, Y$, and Z be as above. We define the following family of ℓ-sets in X with small common neighbourhood in Y:

$$\mathrm{bad}_{\varepsilon,d,p}^{G,\ell}(X,Y) := \left\{ B \in \binom{X}{\ell} : |N_Y^\cap(B)| < (d-\varepsilon)^\ell p^\ell |Y| \right\}. \tag{7.1}$$

If (X,Y) has p-density $d_{G,p}(X,Y) \geqslant d - \varepsilon$, then all ℓ-sets $T \in \binom{X}{\ell}$ that are not in $\mathrm{bad}_{\varepsilon,d,p}^{G,\ell}(X,Y)$ are called p-good in (X,Y).

Further, let

$$\text{Bad}_{\varepsilon,d,p}^{G,\ell}(X, Y, Z)$$

be the family of ℓ-sets $B \in \binom{X}{\ell}$ that contain an ℓ'-set $B' \subseteq B$ with $\ell' > 0$ such that either $|N_Y^\cap(B')| < (d - \varepsilon)^{\ell'} p^{\ell'}|Y|$ or $(N_Y^\cap(B'), Z)$ is not (ε, d, p)-dense in G.

The following lemma states that p-dense pairs in random graphs have the property that most ℓ-sets have big common neighbourhoods. Results of this type (with a slightly smaller exponent in the edge probability p) were established in [24]. The proof of Lemma 7.2 can be found in the appendix of [8].

Lemma 7.2 (common neighbourhood lemma). *For all integers $\Delta, \ell \geqslant 1$ and positive reals d, ε' and μ, there is an $\varepsilon > 0$ such that for all $\xi > 0$ there is a $c > 1$ such that, if $p \geqslant c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1 \geqslant \xi p^{\Delta-1} n$, $n_2 \geqslant \xi p^{\Delta-\ell} n$, let $G = (X \dot\cup Y, E)$ be any bipartite subgraph of Γ with $|X| = n_1$ and $|Y| = n_2$. If (X, Y) is an (ε, d, p)-dense pair, then $|\text{bad}_{\varepsilon',d,p}^{G,\ell}(X, Y)| \leqslant \mu n_1^\ell$.*

Thus we know that typical vertex sets in dense pairs inside random graphs are p-good. In the next lemma we observe that families of such p-good vertex sets exhibit strong expansion properties.

Given Δ and p, we say that a bipartite graph $G = (X \dot\cup Y, E)$ is (A, f)-*expanding* if, for any family $\mathcal{F} \subseteq \binom{X}{\Delta}$ of pairwise disjoint p-good Δ-sets in (X, Y) with $|\mathcal{F}| \leqslant A$, we have $|N_Y^\cap(\mathcal{F})| \geqslant f|\mathcal{F}|$.

Lemma 7.3 (expansion lemma). *For all positive integers Δ and positive reals d and ε, there exist positive v and c such that if $p \geqslant c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. Let $G = (X \dot\cup Y, E)$ be a bipartite subgraph of Γ. If (X, Y) is an (ε, d, p)-dense pair, then (X, Y) is $(1/p^\Delta, vnp^\Delta)$-expanding.*

Proof. Given Δ, d, ε, set $\delta := d - \varepsilon$, $\xi := \delta^\Delta/7$, and let v' and c be the constants from Lemma 6.3 for this Δ and ξ. Further, choose v such that $v \leqslant \xi$ and $v \leqslant v'$. Let $\mathcal{F} \subseteq \binom{X}{\Delta}$ be a family of pairwise disjoint p-good Δ-sets with $|\mathcal{F}| \leqslant 1/p^\Delta$. Let $U = N_Y^\cap(\mathcal{F})$ be the common neighbourhood of $\mathcal{F}$ in Y. We wish to show that $|U| \geqslant (vnp^\Delta)|\mathcal{F}|$. Suppose the contrary. Then $|U| < v'np^\Delta|\mathcal{F}|$, $|U| < vnp^\Delta|\mathcal{F}| \leqslant vn \leqslant \xi n$ and $|\mathcal{F}| \leqslant 1/p^\Delta \leqslant c^\Delta n/\log n \leqslant \xi n$ for n sufficiently large and so we can apply Lemma 6.3 with parameters Δ and ξ to U and $\mathcal{F}$. Since every member of $\mathcal{F}$ is p-good in (X, Y), we thus have

$$\delta^\Delta p^\Delta n|\mathcal{F}| \leqslant \#\,\text{stars}^G(U, \mathcal{F}) \leqslant \#\,\text{stars}^\Gamma(U, \mathcal{F}) \overset{(6.2)}{\leqslant} p^\Delta|U||\mathcal{F}| + 6\xi np^\Delta|\mathcal{F}|$$
$$< p^\Delta(vnp^\Delta)|\mathcal{F}||\mathcal{F}| + 6\xi np^\Delta|\mathcal{F}| \leqslant vnp^\Delta|\mathcal{F}| + 6\xi np^\Delta|\mathcal{F}| \leqslant 7\xi np^\Delta|\mathcal{F}|,$$

which yields that $\delta^\Delta < 7\xi$, a contradiction. $\qquad \square$

In the remainder of this section we are interested in the inheritance of p-denseness to sub-pairs (X', Y') of p-dense pairs (X, Y) in a graph $G = (V, E)$. It comes as a surprise that

even for sets X' and Y' that are much smaller than the sets considered in the definition of p-denseness, such sub-pairs are typically dense. Phenomena of this type were observed in [24, 18].

Here, we will consider sub-pairs induced by neighbourhoods of vertices $v \in V$ (which may or may not be in $X \cup Y$), *i.e.*, sub-pairs (X', Y') where X' (or Y' or both) is the neighbourhood of v in Y (or in X). Further, we only consider the case when G is a subgraph of a random graph $\mathcal{G}_{n,p}$.

In [26] an inheritance result of this form was obtained for triples of dense pairs. More precisely, the following holds for subgraphs G of $\mathcal{G}_{n,p}$. For sufficiently large vertex set X, Y, and Z in G such that (X, Y) and (Y, Z) form p-dense pairs, we have that most vertices $x \in X$ are such that $(N_Y(x), Y)$ again forms a p-dense pair (with slightly changed parameters). If, moreover, (X, Z) forms a p-dense pair too, then $(N_Y(x), N_Z(x))$ is typically also a p-dense pair.

Lemma 7.4 (inheritance lemma for vertices [26]). *For all integers $\Delta > 0$ and positive reals d_0, ε' and μ, there is an ε such that for all $\xi > 0$ there is a $c > 1$ such that, if $p > c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For n_1, $n_3 \geqslant \xi p^{\Delta-1}n$ and $n_2 \geqslant \xi p^{\Delta-2}n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. If (X, Y) and (Y, Z) are (ε, d, p)-dense pairs in G with $d \geqslant d_0$, then there are at most μn_1 vertices $x \in X$ such that $(N(x) \cap Y, Z)$ is not an (ε', d, p)-dense pair in G.*

If, in addition, (X, Z) is (ε, d, p)-dense and n_1, n_2, $n_3 \geqslant \xi p^{\Delta-2}n$, then there are at most μn_1 vertices $x \in X$ such that $(N(x) \cap Y, N(x) \cap Z)$ is not an (ε', d, p)-dense pair in G.

In order to combine the constrained blow-up lemma (Lemma 5.3) and the connection lemma (Lemma 5.4) in the proof of Theorem 1.1 we will need a version of this result for ℓ-sets. Such a lemma, stating that common neighbourhoods of certain ℓ-sets again form p-dense pairs, can be obtained by an inductive argument from the first part of Lemma 7.4. For a proof see the appendix of [8].

Lemma 7.5 (inheritance lemma for ℓ-sets). *For all integers $\Delta, \ell > 0$ and positive reals d_0, ε', and μ, there is an ε such that for all $\xi > 0$ there is a $c > 1$ such that, if $p > c(\log n/n)^{1/\Delta}$, then the following holds a.a.s. for $\Gamma = \mathcal{G}_{n,p}$. For $n_1, n_3 \geqslant \xi p^{\Delta-1}n$ and $n_2 \geqslant \xi p^{\Delta-\ell-1}n$ let $G = (X \dot\cup Y \dot\cup Z, E)$ be any tripartite subgraph of Γ with $|X| = n_1$, $|Y| = n_2$, and $|Z| = n_3$. Assume further that (X, Y) and (Y, Z) are (ε, d, p)-dense pairs with $d \geqslant d_0$. Then*

$$|\operatorname{Bad}_{\varepsilon', d, p}^{G, \ell}(X, Y, Z)| \leqslant \mu n_1^{\ell}.$$

8. Proof of Theorem 1.1

In this section we present a proof of Theorem 1.1 that combines our four main lemmas, namely the lemma for G (Lemma 5.1), the lemma for H (Lemma 5.2), the constrained blow-up lemma (Lemma 5.3), and the connection lemma (Lemma 5.4). This proof follows the outline given in Section 5. In addition we will apply the inheritance lemma for ℓ-sets

(Lemma 7.5), which supplies an appropriate interface between the constrained blow-up lemma and the connection lemma.

Proof of Theorem 1.1. We first set up the constants. Given η, γ, and Δ, let t be the constant promised by the lemma for H (Lemma 5.2) for input Δ. Set

$$\eta_G := \eta/10 \quad \text{and} \quad r_0 = 1, \tag{8.1}$$

and apply the lemma for G (Lemma 5.1) with input t, r_0, η_G, and γ in order to obtain η'_G and d. Next, the connection lemma (Lemma 5.4) with input Δ, $2t$, and d provides us with ε_{CL}, and ξ_{CL}. We apply the constrained blow-up lemma (Lemma 5.3) with Δ, d, and $\eta/2$ in order to obtain ε_{BL} and μ_{BL}. With this we set

$$\eta_H := \min\{\eta/10, \, \xi_{\text{CL}}\eta'_G, \, 1/(\Delta+1)\}. \tag{8.2}$$

Choose $\mu > 0$ such that

$$100t^2\mu \leqslant \eta_{\text{BL}}, \tag{8.3}$$

and apply Lemma 7.5 with Δ and $\ell = \Delta - 1$, $d_0 = d$, $\varepsilon' = \varepsilon_{\text{CL}}$, and μ to obtain $\varepsilon_{7.5}$. Let

$$\xi_{7.5} := \eta'_G/2r \tag{8.4}$$

and continue the application of Lemma 7.5 with $\xi_{7.5}$ to obtain $c_{7.5}$. Now we can fix

$$\varepsilon := \min\{\varepsilon_{\text{CL}}, \varepsilon_{\text{BL}}, \varepsilon_{7.5}\} \tag{8.5}$$

and continue the application of Lemma 5.1 with input ε to get r_1. Let $\hat{r}_{\text{BL}}$ and $\hat{r}_{\text{CL}}$ be such that

$$\frac{2r_1}{1-\eta_G} \leqslant \hat{r}_{\text{BL}} \quad \text{and} \quad \frac{2r_1}{\eta_G} \leqslant \hat{r}_{\text{CL}} \tag{8.6}$$

and let c_{CL} and c_{BL} be the constants obtained from the continued application of Lemma 5.4 with r_1 replaced by $\hat{r}_{\text{CL}}$ and Lemma 5.3 with r_1 replaced by $\hat{r}_{\text{BL}}$, respectively.

We continue the application of Lemma 5.2 with input η_H. For each $r \in [r_1]$, Lemma 5.2 provides a value β_r, among all of which we choose the smallest one and set β to this value. Finally, we set $c := \max\{c_{\text{BL}}, c_{\text{CL}}, c_{7.5}\}$.

Consider a graph $\Gamma = \mathcal{G}_{n,p}$ with $p \geqslant c(\log n/n)^{1/\Delta}$. Then Γ a.a.s. satisfies the properties stated in Lemmas 5.1, 5.3, 5.4 and 7.5, with the parameters previously specified. We assume in the following that this is the case, and show that then the following also holds. For all subgraphs $G \subseteq \Gamma$ and all graphs H such that G and H have the properties required by Theorem 1.1, we have $H \subseteq G$. To summarize the definition of the constants above, we can now assume that Γ satisfies the conclusion of the following lemmas:

(L5.1) Lemma 5.1 for parameters t, $r_0 = 1$, η_G, γ, η'_G, d, ε, and r_1, *i.e.*, if G is any spanning subgraph of Γ satisfying the requirements of Lemma 5.1, then we obtain a partition of G as specified in the lemma with these parameters,

(L5.3) Lemma 5.3 for parameters Δ, d, $\eta/2$, ε_{BL}, μ_{BL}, and $\hat{r}_{\text{BL}}$,

(L5.4) Lemma 5.4 for parameters Δ, $2t$, d, ε_{CL}, ξ_{CL}, and $\hat{r}_{\text{CL}}$,

(L7.5) Lemma 7.5 for parameters Δ, $\ell = \Delta - 1$, $d_0 = d$, $\varepsilon' = \varepsilon_{\text{CL}}$, μ, $\varepsilon_{7.5}$, and $\xi_{7.5}$.

Now suppose we are given a graph $G = (V, E) \subseteq \Gamma$ with $\deg_G(v) \geqslant (\frac{1}{2} + \gamma) \deg_\Gamma(v)$ for all $v \in V$ and $|V| = n$, and a graph $H = (\widetilde{V}, \widetilde{E})$ with $|\widetilde{V}| = (1 - \eta)n$. Before we show that H can be embedded into G, we will use the lemma for G (Lemma 5.1) and the lemma for H (Lemma 5.2) to prepare G and H for this embedding.

First we use the fact that Γ has property (L5.1). Hence, for the graph G we obtain an r with $1 \leqslant r \leqslant r_1$ from Lemma 5.1, together with a set $V_0 \subseteq V$ with $|V_0| \leqslant \varepsilon n$, and a mapping $g : V \setminus V_0 \to R_{r,t}$ such that (G1)–(G3) of Lemma 5.1 are fulfilled. For all $i \in [r], j \in [2t]$ let U_i, V_i, $C_{i,j}$, $C'_{i,j}$, $B_{i,j}$, and $B'_{i,j}$ be the sets defined in Lemma 5.1. Recall that these sets were called big clusters, connecting clusters, and balancing clusters. With this the graph G is prepared for the embedding. We now turn to the graph H.

We assume for simplicity that $2r/(1 - \eta_G)$ and $r/(t\eta'_G)$ are integers and define

$$r_{\text{BL}} := 2r/(1 - \eta_G) \quad \text{and} \quad r_{\text{CL}} := 2r/\eta'_G. \tag{8.7}$$

We apply Lemma 5.2, which we have already provided with Δ and η_H. For input H this lemma provides a homomorphism h from H to $R_{r,t}$ such that (H1)–(H5) of Lemma 5.2 are fulfilled. For all $i \in [r], j \in [2t]$ let $\widetilde{U}_i$, $\widetilde{V}_i$, $\widetilde{C}_{i,j}$, $\widetilde{C}'_{i,j}$, $\widetilde{B}_{i,j}$, $\widetilde{B}'_{i,j}$, and $\widetilde{X}_i$ be the sets whose existence is guaranteed by Lemma 5.2. Further, set $C_i := C_{i,1} \dot\cup \cdots \dot\cup C_{i,2t}$, $\widetilde{C}_i := \widetilde{C}_{i,1} \dot\cup \cdots \dot\cup \widetilde{C}_{i,2t}$, that is, C_i consists of connecting clusters and $\widetilde{C}_i$ of connecting vertices. Define C'_i, $\widetilde{C}'_i$, B_i, $\widetilde{B}_i$, B'_i, and $\widetilde{B}'_i$ analogously (B_i consists of balancing clusters and $\widetilde{B}_i$ of balancing vertices).

Our next goal will be to appeal to property (L5.3), which asserts that we can apply the constrained blow-up lemma (Lemma 5.3) for each p-dense pair (U_i, V_i) with $i \in [r]$ individually and embed $H[\widetilde{U}_i \dot\cup \widetilde{V}_i]$ into this pair. For this we fix $i \in [r]$. We will first set up special Δ-sets $\mathcal{H}_i$ and forbidden Δ-sets $\mathcal{B}_i$ for the application of Lemma 5.3. The idea is as follows. With the help of Lemma 5.3 we will embed all vertices in $\widetilde{U}_i \dot\cup \widetilde{V}_i$. But all connecting and balancing vertices of H remain unembedded. They will be handled by the connection lemma, Lemma 5.4, later on. However, these two lemmas cannot operate independently. If, for example, a connecting vertex $\widetilde{y}$ has three neighbours in $\widetilde{V}_i$, then these neighbours will already be mapped to vertices v_1, v_2, v_3 in V_i (by the blow-up lemma) when we want to embed $\widetilde{y}$. Accordingly the image of $\widetilde{y}$ in the embedding is confined to the common neighbourhood of the vertices v_1, v_2, v_3 in G. In other words, this common neighbourhood will be the candidate set $C(\widetilde{y})$ in the application of Lemma 5.4. This lemma requires, however, that candidate sets are not too small (condition (D) of Lemma 5.4) and, in addition, that candidate sets of any two adjacent vertices induce p-dense pairs (condition (E)). Hence we need to be prepared for these requirements. This will be done via the special and forbidden sets. The family of special sets $\mathcal{H}_i$ will contain neighbourhoods in $\widetilde{V}_i$ of connecting or balancing vertices $\widetilde{y}$ of H (observe that such vertices do not have neighbours in $\widetilde{U}_i$: see Figure 1). The family of forbidden sets $\mathcal{B}_i$ will consist of sets in V_i which are 'bad' for the embedding of these neighbourhoods in view of (D) and (E) of Lemma 5.4 (recall that Lemma 5.3 does not map special sets to forbidden sets). Accordingly, $\mathcal{B}_i$ contains Δ-sets that have small common neighbourhoods or do not induce p-dense pairs in one of the relevant balancing or connecting clusters. We will next give the details of this construction of $\mathcal{H}_i$ and $\mathcal{B}_i$.

We start with the special Δ-sets $\mathcal{H}_i$. As explained, we would like to include in the family $\mathcal{H}_i$ all neighbourhoods of vertices $\widetilde{w}$ of vertices outside $\widetilde{U}_i \dot\cup \widetilde{V}_i$. Such neighbourhoods clearly lie entirely in the set $\widetilde{X}_i$ provided by Lemma 5.2. However, they need not necessarily be Δ-sets (in fact, by (H4) of Lemma 5.2, they are of size at most $\Delta - 1$). Therefore we have to 'pad' these neighbourhoods in order to obtain Δ-sets. This is done as follows. We start by picking an arbitrary set of $\Delta|\widetilde{X}_i|$ vertices (which will be used for the 'padding') in $\widetilde{V}_i \setminus \widetilde{X}_i$. We add these vertices to $\widetilde{X}_i$ and call the resulting set $\widetilde{X}'_i$. This is possible because (H5) of Lemma 5.2 and (8.2) imply that

$$|\widetilde{X}'_i| \leqslant (\Delta + 1)|\widetilde{X}_i| \leqslant (\Delta + 1)\eta_H|\widetilde{V}_i| \leqslant |\widetilde{V}_i|.$$

Now let $\widetilde{Y}_i$ be the set of vertices in $\widetilde{B}_i \dot\cup \widetilde{C}_i$ with neighbours in $\widetilde{V}_i$. These are the vertices for whose neighbourhoods we will include Δ-sets in $\mathcal{H}_i$. It follows from the definition of $\widetilde{X}_i$ that $|\widetilde{Y}_i| \leqslant \Delta|\widetilde{X}_i|$. Let $\widetilde{y} \in \widetilde{Y}_i \subseteq \widetilde{B}_i \cup \widetilde{C}_i$. By the definition of $\widetilde{X}_i$ we have $N_H(\widetilde{y}) \subseteq \widetilde{X}_i$. Next, we let

$$\widetilde{X}_{\widetilde{y}} \text{ be the set of neighbours of } \widetilde{y} \text{ in } \widetilde{V}_i. \tag{8.8}$$

As explained, $\widetilde{y}$ has strictly less than Δ neighbours in $\widetilde{V}_i$ and hence we choose additional vertices from $\widetilde{X}'_i \setminus \widetilde{X}_i$. In this way we obtain for each $\widetilde{y} \in \widetilde{Y}_i$ a Δ-set $N_{\widetilde{y}} \in \widetilde{X}'_i$ with

$$N_{\widetilde{X}_i}(\widetilde{y}) = N_{\widetilde{V}_i}(\widetilde{y}) = \widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}}. \tag{8.9}$$

We make sure, in this process, that for any two different $\widetilde{y}$ and $\widetilde{y}'$ we never include the same additional vertex from $\widetilde{X}'_i \setminus \widetilde{X}_i$. This is possible because $|\widetilde{X}'_i \setminus \widetilde{X}_i| \geqslant \Delta|\widetilde{X}_i| \geqslant |\widetilde{Y}_i|$. We can thus guarantee that

$$\text{each vertex in } \widetilde{X}'_i \text{ is contained in at most } \Delta \text{ sets } N_{\widetilde{y}}. \tag{8.10}$$

The family of special Δ-sets for the application of Lemma 5.3 on (U_i, V_i) is then

$$\mathcal{H}_i := \{N_{\widetilde{y}} : \widetilde{y} \in \widetilde{Y}_i\}. \tag{8.11}$$

Note that this is indeed a family of Δ-sets encoding all neighbourhoods in $\widetilde{U}_i \dot\cup \widetilde{V}_i$ of vertices outside this set.

Now we turn to the family $\mathcal{B}_i$ of forbidden Δ-sets. Recall that this family should contain sets that are forbidden for the embedding of the special Δ-sets because their common neighbourhood in a (relevant) balancing or connecting cluster is small or does not induce a p-dense pair. More precisely, we are interested in Δ-sets S that have one of the following properties: either S has a small common neighbourhood in some cluster from B_i or from C_i (observe that only balancing vertices from $\widetilde{B}_i$ and connecting vertices from $\widetilde{C}_i$ have neighbours in $\widetilde{V}_i$); or the neighbourhood $N_D^\cap(S)$ of S in a cluster D from B_i or C_i, respectively, is such that $(N_D^\cap(S), D')$ is not p-dense for some cluster D' from $B'_i \cup B'_{i+1}$ or $C'_i \cup C'_{i+1}$ (observe that edges between balancing vertices run only between $\widetilde{B}_i$ and $\widetilde{B}'_i \cup \widetilde{B}'_{i+1}$ and edges between connecting vertices only between $\widetilde{C}_i$ and $\widetilde{C}'_i \cup \widetilde{C}'_{i+1}$).

For technical reasons, however, we need to digress from this strategy slightly. We want to bound the number of Δ-sets in $\mathcal{B}_i$ with the help of the inheritance lemma for ℓ-sets, Lemma 7.5, later. Notice that, thanks to the lower bound on n_2 in Lemma 7.5, this lemma cannot be applied (in a meaningful way) to Δ-sets. But it can be applied to $(\Delta - 1)$-sets.

Therefore, we will not consider Δ-sets directly but first construct an auxiliary family of $(\Delta - 1)$-sets and then, again, 'pad' these sets to obtain a family of Δ-sets. Observe that the strategy outlined while setting up the special sets $\mathcal{H}_i$ still works with these $(\Delta - 1)$-sets: neighbourhoods of connecting or balancing vertices in $\widetilde{V}_i$ are of size at most $\Delta - 1$ by (H4) of Lemma 5.2.

But now let us finally give the details. We first define the auxiliary family of $(\Delta - 1)$-sets as follows:

$$\mathcal{B}'_i := \bigcup_{\substack{i' \in \{i, i+1\}, j, j' \in [2t] \\ (c_{i,j}, c'_{i',j'}) \in R_{r,t}}} \mathrm{Bad}^{G, \Delta - 1}_{\varepsilon_{\mathrm{CL}}, d, p}(V_i, C_{i,j}, C'_{i',j'}) \;\cup\; \bigcup_{\substack{j, j' \in [2t] \\ (b_{i,j}, b'_{i,j'}) \in R_{r,t}}} \mathrm{Bad}^{G, \Delta - 1}_{\varepsilon_{\mathrm{CL}}, d, p}(V_i, B_{i,j}, B_{i,j'}). \qquad (8.12)$$

We next bound the size of this family by appealing to property (L7.5), and hence Lemma 7.5, with the tripartite graphs $G[V_i, C_{i,j}, C'_{i',j'}]$ and $G[V_i, B_{i,j}, B'_{i,j'}]$ with indices as in the definition of $\mathcal{B}'_i$. For this we need to check the conditions appearing in this lemma. By the definition of $R_{r,t}$ and (G3) of Lemma 5.1 all pairs $(C_{i,j}, C'_{i',j'})$ and $(B_{i,j}, B'_{i,j'})$ appearing in the definition of $\mathcal{B}'_i$ as well as the pairs $(V_i, C_{i,j})$ and $(V_i, B_{i,j})$ with $j \in [2t]$ are (ε, d, p)-dense. For the vertex sets of these dense pairs we know $|V_i|, |C'_{i',j'}|, |B'_{i,j'}| \geqslant \eta'_G n / 2r \geqslant \xi_{7.5} p^{\Delta - 1} n$ and $|C_{i,j}|, |B_{i,j}| \geqslant \eta'_G n / 2r = \xi_{7.5} n$ by (G1) and (G2) of Lemma 5.1 and (8.4). Thus, since $\varepsilon \leqslant \varepsilon_{7.5}$, property (L7.5) implies that the family

$$\mathrm{Bad}^{G, \Delta - 1}_{\varepsilon_{\mathrm{CL}}, d, p}(V_i, C_{i,j}, C'_{i',j'}) \quad \text{and} \quad \mathrm{Bad}^{G, \Delta - 1}_{\varepsilon_{\mathrm{CL}}, d, p}(V_i, B_{i,j}, B'_{i,j'})$$

is of size $\mu |V_i|^{\Delta - 1}$ at most. It follows from (8.12) that $|\mathcal{B}'_i| \leqslant 8t^2 \mu |V_i|^{\Delta - 1}$, which is at most $\mu_{\mathrm{BL}} |V_i|^{\Delta - 1}$ by (8.3). The family of forbidden Δ-sets is then defined by

$$\mathcal{B}_i := \mathcal{B}'_i \times V_i, \quad \text{and we have} \quad |\mathcal{B}_i| \leqslant \mu_{\mathrm{BL}} |V_i|^{\Delta}. \qquad (8.13)$$

Having defined the special and forbidden Δ-sets, we are now ready to appeal to (L5.3) and use the constrained blow-up lemma (Lemma 5.3) with parameters Δ, d, $\eta/2$, $\varepsilon_{\mathrm{BL}}$, μ_{BL}, $\hat{r}_{\mathrm{BL}}$, and r_{BL} separately for each pair of graphs $G_i := (U_i, V_i)$ and $H_i := H[\widetilde{U}_i \dot\cup \widetilde{V}_i]$. Let us quickly check that the constant r_{BL} and the graphs G_i and H_i satisfy the required conditions. Observe first that $1 \leqslant r_{\mathrm{BL}} = 2r/(1 - \eta_G) \leqslant 2r_1/(1 - \eta_G) \leqslant \hat{r}_{\mathrm{BL}}$ by (8.7) and (8.6). Moreover (U_i, V_i) is an $(\varepsilon_{\mathrm{BL}}, d, p)$-dense pair by (G3) of Lemma 5.1 and (8.5). (G1) implies

$$|U_i| \geqslant (1 - \eta_G) \frac{n}{2r} \overset{(8.7)}{=} \frac{n}{r_{\mathrm{BL}}}$$

and similarly $|V_i| \geqslant n/r_{\mathrm{BL}}$. By (H1) of Lemma 5.2 we have

$$|\widetilde{U}_i| \leqslant (1 + \eta_H) \frac{m}{2r} \leqslant (1 + \eta_H)(1 - \eta) \frac{n}{2r} \leqslant (1 + \eta_H - \eta) \frac{n}{2r} \overset{(8.1),(8.2)}{\leqslant} (1 - \tfrac{1}{2}\eta - \eta_G) \frac{n}{2r}$$
$$\leqslant (1 - \tfrac{1}{2}\eta)(1 - \eta_G) \frac{n}{2r} \overset{(8.7)}{=} (1 - \tfrac{1}{2}\eta) \frac{n}{r_{\mathrm{BL}}}$$

and similarly $|\widetilde{V}_i| \leqslant \left(1 - \frac{\eta}{2}\right) n / r_{\mathrm{BL}}$. For the application of Lemma 5.3, let the families of special and forbidden Δ-sets be defined in (8.11) and (8.13), respectively. Observe that (8.10) and (8.13) guarantee that the required conditions (of Lemma 5.3) are satisfied. Consequently there is an embedding of H_i into G_i for each $i \in [r]$ such that no special Δ-set is mapped to a forbidden Δ-set. Denote the united embedding resulting from these r applications of the constrained blow-up lemma by $f_{\mathrm{BL}} : \bigcup_{i \in [r]} \widetilde{U}_i \cup \widetilde{V}_i \to \bigcup_{i \in [r]} U_i \cup V_i$.
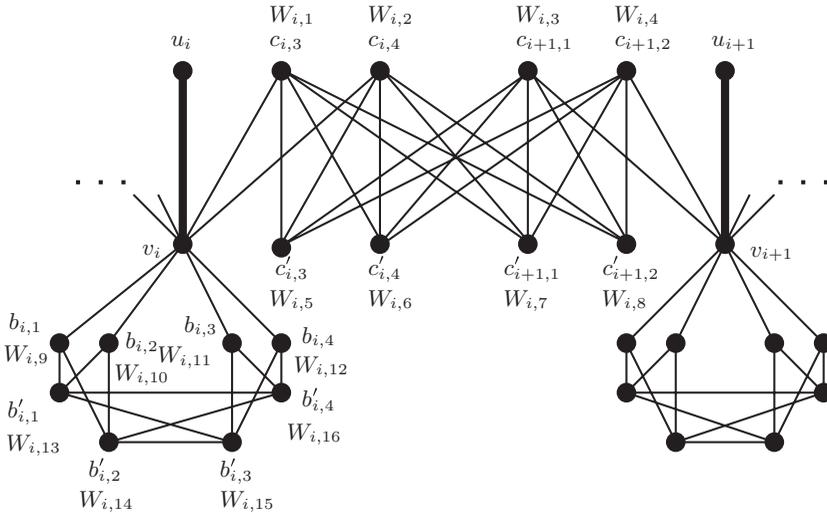
Figure 2. The partition $W_i = W_{i,1} \dot\cup \cdots \dot\cup W_{i,8t}$ of $G'_i = G[W_i]$ for the special case $t = 2$.

It remains to verify that f_{BL} can be extended to an embedding of all vertices of H into G. We still need to take care of the balancing and connecting vertices. For this purpose we will, again, fix $i \in [r]$ and use property (L5.4), which states that the conclusion of the connection lemma (Lemma 5.4) holds for parameters Δ, $2t$, d, $\varepsilon_{\mathrm{CL}}$, ξ_{CL}, and $\hat{r}_{\mathrm{CL}}$. We will apply this lemma with input r_{CL} to the graphs $G'_i := G[W_i]$ and $H'_i := H[\widetilde{W}_i]$, where W_i and $\widetilde{W}_i$ and their partitions for the application of the connection lemma are as follows (see Figure 2). Let $W_i := W_{i,1} \dot\cup \cdots \dot\cup W_{i,8t}$, where for all $j \in [t], k \in [2t]$ we set

$$W_{i,j} := C_{i,t+j}, \qquad W_{i,t+j} := C_{i+1,j}, \qquad W_{i,2t+j} := C'_{i,t+j},$$
$$W_{i,3t+j} := C'_{i+1,j}, \qquad W_{i,4t+k} := B_{i,k}, \qquad W_{i,6t+k} := B'_{i,k}.$$

(This means that we propose the clusters to the connection lemma in the following order: the connecting clusters without primes come first, then the connecting clusters with primes, then the balancing clusters without primes, and finally the balancing clusters with primes.)

The partition $\widetilde{W}_i := \widetilde{W}_{i,1} \dot\cup \cdots \dot\cup \widetilde{W}_{i,8t}$ of the vertex set $\widetilde{W}_i$ of H'_i is defined accordingly, *i.e.*, for all $j \in [t], k \in [2t]$ we set

$$\widetilde{W}_{i,j} := \widetilde{C}_{i,t+j}, \qquad \widetilde{W}_{i,t+j} := \widetilde{C}_{i+1,j}, \qquad \widetilde{W}_{i,2t+j} := \widetilde{C}'_{i,t+j},$$
$$\widetilde{W}_{i,3t+j} := \widetilde{C}'_{i+1,j}, \qquad \widetilde{W}_{i,4t+k} := \widetilde{B}_{i,k}, \qquad \widetilde{W}_{i,6t+k} := \widetilde{B}'_{i,k}.$$

To check whether we can apply the connecting lemma, observe first that

$$1 \leqslant 2r/\eta_G \leqslant 2r_1/\eta_G \leqslant \hat{r}_{\mathrm{CL}}$$

by (8.6). For $\widetilde{y} \in \widetilde{W}_{i,j}$ with $j \in [8t]$ recall from (8.8) (using that each vertex in H has neighbours in at most one set $\widetilde{V}_{i'}$: see Figure 1) that

$$\widetilde{X}_{\widetilde{y}} \text{ is the set of neighbours of } \widetilde{y} \text{ in } \widetilde{V}_i \cup \widetilde{V}_{i+1} \text{ and set } X_{\widetilde{y}} := f_{\mathrm{BL}}(\widetilde{X}_{\widetilde{y}}). \qquad (8.14)$$

Then the indexed set system $\big(\widetilde{X}_{\widetilde{y}} : \widetilde{y} \in \widetilde{W}_{i,j}\big)$ consists of pairwise disjoint sets because $\widetilde{W}_{i,j}$ is 3-independent in H by (H3) of Lemma 5.2. Thus also $\big(X_{\widetilde{y}} : \widetilde{y} \in \widetilde{W}_{i,j}\big)$ consists of pairwise disjoint sets, as required by Lemma 5.4. Now let the external degree and the candidate set of $\widetilde{y} \in \widetilde{W}_{i,j}$ be defined as in Lemma 5.4, *i.e.*,

$$\operatorname{edeg}(\widetilde{y}) := |X_{\widetilde{y}}| \quad \text{and} \quad C(\widetilde{y}) := N^{\cap}_{W_{i,j}}(X_{\widetilde{y}}). \tag{8.15}$$

Observe that this implies $C(\widetilde{y}) = W_{i,j}$ if $\widetilde{X}_{\widetilde{y}} = \emptyset$ and hence $X_{\widetilde{y}} = \emptyset$. Now we will check that conditions (A)–(E) of Lemma 5.4 are satisfied. From (G2) of Lemma 5.1 and (H2) of Lemma 5.2 it follows that

$$|W_{i,j}| \overset{\text{(G2)}}{\geqslant} \eta'_G \frac{n}{2r} \overset{(8.7)}{=} \frac{n}{r_{\text{CL}}} \quad \text{and}$$

$$|\widetilde{W}_{i,j}| \overset{\text{(H2)}}{\leqslant} \eta_H \frac{m}{2r} \leqslant \eta_H \frac{n}{2r} \overset{(8.7)}{=} \frac{\eta_H}{\eta'_G} \frac{n}{r_{\text{CL}}} \overset{(8.2)}{\leqslant} \xi_{\text{CL}} \frac{n}{r_{\text{CL}}},$$

and thus we have condition (A). By (H3) of Lemma 5.2 we also get condition (B) of Lemma 5.4. Further, it follows from (H4) of Lemma 5.2 that $\operatorname{edeg}(\widetilde{y}) = \operatorname{edeg}(\widetilde{y}')$ and $\operatorname{ldeg}(\widetilde{y}) = \operatorname{ldeg}(\widetilde{y}')$ for all $\widetilde{y}, \widetilde{y}' \in \widetilde{W}_{i,j}$ with $j \in [8t]$. In addition $\Delta(H) \leqslant \Delta$ and hence

$$\deg_{H'_i}(\widetilde{y}) + \operatorname{edeg}(\widetilde{y}) \overset{(8.15)}{=} |N_{\widetilde{W}_i}(\widetilde{y})| + |X_{\widetilde{y}}|$$

$$\overset{(8.14)}{=} |N_{\widetilde{W}_i}(\widetilde{y})| + |N_{\widetilde{V}_i \cup \widetilde{V}_{i+1}}(\widetilde{y})| \leqslant \deg_H(\widetilde{y}) \leqslant \Delta,$$

and thus condition (C) of Lemma 5.4 is satisfied. To check conditions (D) and (E) of Lemma 5.4, observe that for all $\widetilde{y} \in \widetilde{C}'_{i',j}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$ we have $C(\widetilde{y}) = C'_{i',j}$, as $\widetilde{y}$ has no neighbours in $\widetilde{V}_i$ or $\widetilde{V}_{i+1}$ and hence the external $\operatorname{edeg}(\widetilde{y}) = 0$ (see (8.14) and (8.15)). Thus (D) is satisfied for $\widetilde{y} \in \widetilde{C}'_{i',j}$, and similarly for $\widetilde{y} \in \widetilde{B}'_{i',j}$. For all $\widetilde{y} \in \widetilde{C}_{i,j}$ with $t < j \leqslant 2t$, on the other hand, we have $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_i}{\Delta}$ by (8.8). Recall that $N_{\widetilde{y}}$ was a special Δ-set in the application of the restricted blow-up lemma on $G_i = (U_i, V_i)$ and $H_i = H[\widetilde{U}_i \dot\cup \widetilde{V}_i]$ owing to (8.11). Therefore $N_{\widetilde{y}}$ is not mapped to a forbidden Δ-set in $\mathcal{B}_i \subseteq \binom{V_i}{\Delta}$ by f_{BL} and thus, by (8.12), to no Δ-set in $\operatorname{Bad}^{G,\Delta-1}_{\varepsilon_{\text{CL}},d,p}(V_i, C_{i,j}, C'_{i',j'}) \times V_i$ with $i' \in \{i+1, i\}, j, j' \in [2t]$ and $(c_{i,j}, c'_{i',j'}) \in R_{r,t}$. We infer that the set $f_{\text{BL}}(\widetilde{X}_{\widetilde{y}}) = X_{\widetilde{y}} \in \binom{V_{i(\widetilde{y})}}{\operatorname{edeg}(\widetilde{y})}$ satisfies $|N^{\cap}_{C_{i,j}}(X_{\widetilde{y}})| \geqslant (d - \varepsilon_{\text{CL}})^{\operatorname{edeg}(\widetilde{y})} p^{\operatorname{edeg}(\widetilde{y})}|C_{i,j}|$ and is such that

$$(N^{\cap}_{C_{i,j}}(X_{\widetilde{y}}), C'_{i',j'}) \text{ is } (\varepsilon_{\text{CL}}, d, p)\text{-dense for all } i' \in \{i, i+1\}, j, j' \in [2t] \text{ with } (c_{i,j}, c'_{i',j'}) \in R_{r,t}. \tag{8.16}$$

Since we chose $C(\widetilde{y}) = N^{\cap}(X_{\widetilde{y}}) \cap C_{i,j}$ in (8.15) we also get condition (D) of Lemma 5.4 for $\widetilde{y} \in \widetilde{C}_{i,j}$ with $t < j \leqslant 2t$. For $\widetilde{y} \in \widetilde{C}_{i+1,j}$ with $j \in [t]$ the same argument applies with $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_{i+1}}{\Delta}$, and for $\widetilde{y} \in \widetilde{B}_{i,j}$ with $j \in [2t]$ the same argument applies with $\widetilde{X}_{\widetilde{y}} \subseteq N_{\widetilde{y}} \in \binom{\widetilde{V}_i}{\Delta}$.

Now it will be easy to see that we get (E) of Lemma 5.4. Indeed, recall again that $C(\widetilde{y}) = C'_{i',j'}$ for all $\widetilde{y} \in \widetilde{C}'_{i',j'}$ and $C(\widetilde{y}) = B'_{i',j'}$ for all $\widetilde{y} \in \widetilde{B}'_{i',j'}$ with $i' \in \{i, i+1\}$ and $j \in [2t]$. In addition, the mapping h constructed by Lemma 5.2 is a homomorphism from H to $R_{r,t}$. Hence (8.16) and property (G3) of Lemma 5.1 assert that condition (E) of Lemma 5.4 is satisfied for all edges $\widetilde{y}\widetilde{y}'$ of $H'_i = H[\widetilde{W}_i]$ with at least one end, say $\widetilde{y}$, in a cluster $\widetilde{C}'_{i',j'}$ or $\widetilde{B}'_{i',j'}$. This is true because then $C(\widetilde{y}) = W_{i,k}$, where $\widetilde{W}_{i,k}$ is the cluster containing $\widetilde{y}$, and $C(\widetilde{y}') = N^{\cap}(X_{\widetilde{y}'}) \cap W_{i,k'}$, where $\widetilde{W}_{i,k'}$ is the cluster containing $\widetilde{y}'$. Moreover, since h is a

homomorphism, all edges $\widetilde{y}\widetilde{y}'$ in $H_i' = H[\widetilde{W}_i]$ have at least one end in a cluster $\widetilde{C}_{i',j'}'$ or $\widetilde{B}_{i',j'}'$.

So conditions (A)–(E) are satisfied and we can apply Lemma 5.4 to get embeddings of $H_i' = H[\widetilde{W}_i]$ into $G_i' = G[W_i]$ for all $i \in [r]$ that map vertices $\widetilde{y} \in \widetilde{W}_i$ (*i.e.*, connecting and balancing vertices) to vertices $y \in W_i$ in their candidate sets $C(\widetilde{y})$. Let f_{CL} be the united embedding resulting from these r applications of the connection lemma and denote the embedding that unites f_{BL} and f_{CL} by f.

To finish the proof we verify that f is an embedding of H into G. Let $\widetilde{x}\widetilde{y}$ be an edge of H. By definition of the spin graph $R_{r,t}$ and since the mapping h constructed by Lemma 5.2 is a homomorphism from H to $R_{r,t}$, we only need to distinguish the following cases for $i \in [r]$ and $j, j' \in [2t]$ (see also Figure 1).

Case 1. If $\widetilde{x} \in \widetilde{V}_i$ and $\widetilde{y} \in \widetilde{U}_i$, then $f(\widetilde{x}) = f_{\text{BL}}(\widetilde{x})$ and $f(\widetilde{y}) = f_{\text{BL}}(\widetilde{y})$, and thus the constrained blow-up lemma guarantees that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G_i.

Case 2. If $\widetilde{x} \in \widetilde{W}_i$ and $\widetilde{y} \in \widetilde{W}_i$, then $f(\widetilde{x}) = f_{\text{CL}}(\widetilde{x})$ and $f(\widetilde{y}) = f_{\text{CL}}(\widetilde{y})$, and thus the connection lemma guarantees that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G_i'.

Case 3. If $\widetilde{x} \in \widetilde{V}_i$ and $\widetilde{y} \in \widetilde{W}_i$, then either $\widetilde{y} \in \widetilde{C}_{i,j}$ or $\widetilde{y} \in \widetilde{B}_{i,j}$ for some j. Moreover, $f(\widetilde{x}) = f_{\text{BL}}(\widetilde{x})$ and therefore by (8.15) the candidate set $C(\widetilde{y})$ of $\widetilde{y}$ satisfies $C(\widetilde{y}) \subseteq N_{C_{i,j}}(f(\widetilde{x}))$ or $C(\widetilde{y}) \subseteq N_{B_{i,j}}(f(\widetilde{x}))$, respectively. As $f(\widetilde{y}) = f_{\text{CL}}(\widetilde{y}) \in C(\widetilde{y})$ we also get that $f(\widetilde{x})f(\widetilde{y})$ is an edge of G in this case.

It follows that f maps all edges of H to edges of G, which finishes the proof of the theorem. $\qquad\square$

9. A p-dense partition of G

For the proof of the lemma for G we shall apply the minimum degree version of the sparse regularity lemma (Lemma 4.4). Observe that this lemma guarantees that the reduced graph of the regular partition we obtain is dense. Thus we can apply Theorem 2.1 to this reduced graph. In the proof of Lemma 5.1 we use this theorem to find a copy of the ladder R_r^* in the reduced graph (the graphs R_r^* and $R_{r,t}$ are defined in Section 5 on page 646; see also Figure 1). Then we further partition the clusters in this ladder to obtain a regular partition whose reduced graph contains a spin graph $R_{r,t}$. Recall that this partition will consist of a series of so-called *big clusters*, which we denote by U_i and V_i, and a series of smaller clusters called *balancing clusters*, $B_{i,j}$, $B_{i,j}'$, and *connecting clusters*, $C_{i,j}$, $C_{i,j}'$, with $i \in [r]$, $j \in [2t]$. We will now give details.

Proof of Lemma 5.1. Given t, r_0, η_G, and γ choose η_G' such that

$$\frac{\eta_G}{5} + \left(\frac{4}{\gamma} + 2\right)t \cdot \eta_G' \leqslant \frac{\eta_G}{2} \tag{9.1}$$

and set $d := \gamma/4$. Apply Theorem 2.1 with input $r_{\text{BK}} := 2$, $\Delta = 3$ and $\gamma/2$ to obtain the constants β and $k_{\text{BK}} := n_0$. For input ε set

$$r_0' := \max\{2r_0 + 1, k_{\text{BK}}, 3/\beta, 6/\gamma, 2/\varepsilon, 10/\eta_G\}, \tag{9.2}$$
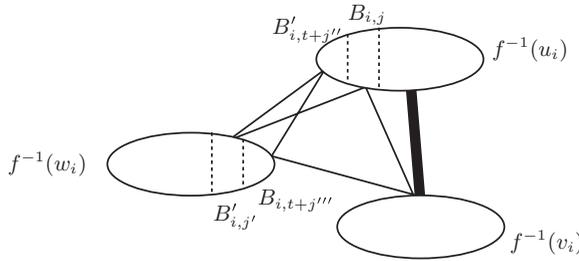
Figure 3. Cutting off a set of balancing clusters from $f^{-1}(u_i)$ and $f^{-1}(w_i)$. These clusters build p-dense pairs (thanks to the triangle $u_i v_i w_i$ in R) in the form of a C_5.

and choose ε' such that

$$\varepsilon'/\eta'_G \leqslant \varepsilon/2 \quad \text{and} \quad \varepsilon' \leqslant \min\{\gamma/4, \eta_G/10\}. \tag{9.3}$$

Lemma 4.4 applied with $\alpha := \frac{1}{2} + \gamma$, ε', r'_0 then gives us the missing constant r_1.

Assume that Γ is a typical graph from $\mathcal{G}_{n,p}$ with $\log^4 n/(pn) = o(1)$, in the sense that it satisfies the conclusion of Lemma 4.4, and let $G = (V, E) \subseteq \Gamma$ satisfy $\deg_G(v) \geqslant (\frac{1}{2} + \gamma) \deg_\Gamma(v)$ for all $v \in V$. Lemma 4.4 applied with $\alpha = \frac{1}{2} + \gamma$, ε', r'_0, and d to G gives us an (ε', d, p)-dense partition $V = V'_0 \dot\cup V'_1 \dot\cup \cdots \dot\cup V'_{r'}$ of G with reduced graph R' with $|V(R')| = r'$ such that $2r_0 + 1 \leqslant r'_0 \leqslant r' \leqslant r_1$ and with minimum degree at least $(\frac{1}{2} + \gamma - d - \varepsilon')r' \geqslant (\frac{1}{2} + \frac{\gamma}{2})r'$ by (9.3). If r' is odd, then set $V_0 := V'_0 \dot\cup V'_{r'}$ and $r := (r' - 1)/2$, otherwise set $V_0 := V'_0$ and $r := r'/2$. Clearly $r_0 \leqslant r \leqslant r_1$, the graph $R := R'[2r]$ still has minimum degree at least $(\frac{1}{2} + \frac{\gamma}{3})2r$ and $|V_0| \leqslant \varepsilon'n + (n/r'_0) \leqslant (\eta_G/5)n$ by the choice of r'_0 and ε'. It follows from Theorem 2.1 applied with $\Delta = 3$ and $\gamma/2$ that R contains a copy of the ladder R^*_r on $2r$ vertices (R^*_r has bandwidth $2 \leqslant \beta \cdot 2r$ by the choice of r'_0 in (9.2)). Hence we can rename the vertices of the graph $R = R'[2r]$ with $u_1, v_1, \ldots, u_r, v_r$ according to the spanning copy of R^*_r. This naturally defines an equipartite mapping f from $V \setminus V_0$ to the vertices of the ladder R^*_r, where f maps all vertices in some cluster V_i with $i \in [2r]$ to a vertex $u_{i'}$ or $v_{i'}$ of R^*_r for some index $i' \in [r]$. We will show that subdividing the clusters $f^{-1}(x)$ for all $x \in V(R^*_r)$ will give the desired mapping g.

We will now construct the balancing clusters $B_{i,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [2t]$ and afterwards turn to the connecting clusters $C_{i,j}$ and $C'_{i,j}$ and big clusters U_i and V_i with $i \in [r]$, $j \in [2t]$.

Notice that $\delta(R) \geqslant (\frac{1}{2} + \frac{\gamma}{3})2r$ implies that every edge $u_i v_i$ of $R^*_r \subseteq R$ is contained in more than γr triangles in R. Therefore, we can choose vertices w_i of R for all $i \in [r]$ such that $u_i v_i w_i$ forms a triangle in R and no vertex of R serves as w_i more than $2/\gamma$ times. We continue by choosing in cluster $f^{-1}(u_i)$ arbitrary disjoint vertex sets $B_{i,1}, \ldots, B_{i,t}$, $B'_{i,t+1}, \ldots, B'_{i,2t}$, of size $\eta'_G n/(2r)$ each, for all $i \in [r]$. We will show below that $f^{-1}(u_i)$ is large enough so that these sets can be chosen. We then remove all vertices in these sets from $f^{-1}(u_i)$. Similarly, we choose in cluster $f^{-1}(w_i)$ arbitrary disjoint vertex sets $B_{i,t+1}, \ldots, B_{i,2t}, B'_{i,1}, \ldots, B'_{i,t}$, of size $\eta'_G n/(2r)$ each, for all $i \in [r]$. We also remove these sets from $f^{-1}(w_i)$. Observe that this construction asserts the following property. For all $i \in [r]$ and $j, j', j'', j''' \in [t]$ each of the pairs $(f^{-1}(v_i), B_{i,j})$, $(B_{i,j}, B'_{i,j'})$, $(B'_{i,j'}, B'_{i,t+j''})$, $(B'_{i,t+j''}, B_{i,t+j''})$, and $(B_{i,t+j'''}, f^{-1}(v_i))$ is a sub-pair of a p-dense pair corresponding to an edge of $R[\{u_i, v_i, w_i\}]$ (see Figure 3). Accordingly this is a sequence of p-dense pairs in the form of a C_5, as
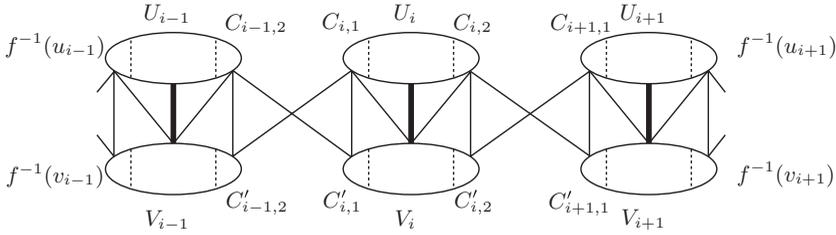
Figure 4. Partitioning the remaining vertices of cluster $f^{-1}(u_i)$ and $f^{-1}(v_i)$ into sets $C_{i,1} \dot\cup C_{i,2} \dot\cup U_i$ and $C'_{i,1} \dot\cup C'_{i,2} \dot\cup V_i$ (for the special case $t = 1$). These clusters form p-dense pairs (thanks to the ladder R^*_r in R) as indicated by the edges.

needed for the balancing clusters in view of condition (G3) (see also Figure 1). Hence we call the sets $B_{i,j}$ and $B'_{i,j}$ with $i \in [r]$, $j \in [2t]$ balancing clusters from now on, and claim that they have the required properties. This claim will be verified below.

We now turn to the construction of the connecting clusters and big clusters. Recall that we have already removed balancing clusters from all clusters $f^{-1}(u_i)$ and possibly from some clusters $f^{-1}(v_i)$ (because v_i might have served as $w_{i'}$) with $i \in [r]$. For each $i \in [r]$ we arbitrarily partition the remaining vertices of cluster $f^{-1}(u_i)$ into sets $C_{i,1} \dot\cup \cdots \dot\cup C_{i,2t} \dot\cup U_i$ and the remaining vertices of cluster $f^{-1}(v_i)$ into sets $C'_{i,1} \dot\cup \cdots \dot\cup C'_{i,2t} \dot\cup V_i$ such that $|C_{i,j}|, |C'_{i,j}| = \eta'_G n/(2r)$ for all $i \in [r]$, $j \in [2t]$. This gives us the connecting and the big clusters, and we claim that these clusters also have the required properties. Observe, again, that for all $i \in [r]$, $i' \in \{i-1, i, i+1\} \setminus \{0\}$, $j, j' \in [2t]$, each of the pairs (U_i, V_i), $(C_{i',j}, V_i)$, and $(C_{i,j}, C'_{i,j'})$ is a sub-pair of a p-dense pair corresponding to an edge of R^*_r (see Figure 4).

We will now show that the balancing clusters, connecting clusters and big clusters satisfy conditions (G1)–(G3). Note that condition (G2) concerning the sizes of the connecting and balancing clusters is satisfied by construction. To determine the sizes of the big clusters, observe that from each cluster V'_j with $j \in [2r]$ vertices, at most $2t \cdot 2/\gamma$ balancing clusters were removed. In addition, at most $2t$ connecting clusters were split off from V'_j. Since $|V \setminus V_0| \geq (1 - \eta_G/5)n$ we get

$$|V_i|, |U_i| \geq \left(1 - \frac{\eta_G}{5}\right) \frac{n}{2r} - \left(\frac{4}{\gamma} + 2\right) t \cdot \eta'_G \frac{n}{2r} \geq (1 - \eta_G) \frac{n}{2r}$$

by (9.1). This is condition (G1). It remains to verify condition (G3). It can easily be checked that for all $xy \in E(R_{r,t})$ the corresponding pair $(g^{-1}(x), g^{-1}(y))$ is a sub-pair of some cluster pair $(f^{-1}(x'), f^{-1}(y'))$ with $x'y' \in E(R)$ by construction. In addition, all big, connecting, and balancing clusters are of size at least $\eta'_G n/(2r)$. Hence we have $|g^{-1}(x)| \geq \eta'_G |f^{-1}(x')|$ and $|g^{-1}(y)| \geq \eta'_G |f^{-1}(y')|$. We conclude from Proposition 4.2 that $(g^{-1}(x), g^{-1}(y))$ is (ε, d, p)-dense since $\varepsilon'/\eta'_G \leq \varepsilon$ by (9.3). This finishes the verification of (G3). $\qquad\square$

10. A partition of H

Hajnal and Szemerédi determined the minimum degree that forces a certain number of vertex-disjoint K_r copies in G. In addition their result guarantees that the remaining
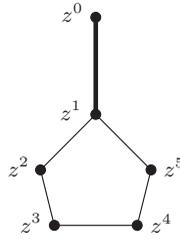
Figure 5. The graph $\bar{R}$ in Proposition 10.2.

vertices can be covered by copies of K_{r-1}. Another way to express this, which actually resembles the original formulation, is obtained by considering the complement $\bar{G}$ of G and its maximum degree. Then, so the theorem asserts, the graph $\bar{G}$ contains a certain number of vertex-disjoint independent sets of almost equal sizes. In other words, $\bar{G}$ admits a vertex colouring such that the sizes of the colour classes differ by at most 1. Such a colouring is also called *equitable colouring*.

Theorem 10.1 (Hajnal and Szemerédi [20]). *Let $\bar{G}$ be a graph on n vertices with maximum degree $\Delta(\bar{G}) \leqslant \Delta$. Then there is an equitable vertex colouring of G with $\Delta + 1$ colours.*

In the proof of Lemma 5.2 presented in this section we will use this theorem in order to guarantee property (H3). This will be the very last step in the proof, however. First, we need to take care of the remaining properties.

Before we start, let us agree on some terminology that will turn out to be useful in the proof of Lemma 5.2. When defining a homomorphism h from a graph H to a graph R, we write $h(S) := z$ for a set S of vertices in H and a vertex z in R to say that all vertices from S are mapped to z. Recall that we have a bandwidth hypothesis on H. Consider an ordering of the vertices of H achieving its bandwidth. Then we can deal with the vertices of H in this order. In particular, we can refer to vertices as the *first* or *last* vertices in some set, meaning that they are the vertices with the smallest or largest label from this set.

We start with the following proposition.

Proposition 10.2. *Let $\bar{R}$ be the following graph with six vertices and six edges:*

$$\bar{R} := \left(\{z^0, z^1, \dots, z^5\}, \{z^0 z^1, z^1 z^2, z^2 z^3, z^3 z^4, z^4 z^5, z^5 z^1\} \right)$$

(see Figure 5 for a picture of $\bar{R}$). For every real $\bar{\eta} > 0$ there exists a real $\bar{\beta} > 0$ such that the following holds. Consider an arbitrary bipartite graph $\bar{H}$ with $\bar{m}$ vertices, colour classes Z^0 and Z^1, and $\mathrm{bw}(\bar{H}) \leqslant \bar{\beta}\bar{m}$, and let T denote the union of the first $\bar{\beta}\bar{m}$ vertices and the last $\bar{\beta}\bar{m}$ vertices of H. Then there exists a homomorphism $\bar{h} \colon V(\bar{H}) \to V(\bar{R})$ from $\bar{H}$ to $\bar{R}$

such that for all $j \in \{0, 1\}$ and all $k \in [2, 5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \leqslant |\bar{h}^{-1}(z^j)| \leqslant \frac{\bar{m}}{2} + \bar{\eta}\bar{m}, \tag{10.1}$$

$$|\bar{h}^{-1}(z^k)| \leqslant \bar{\eta}\bar{m}, \tag{10.2}$$

$$h(T \cap Z^j) = z^j. \tag{10.3}$$

Roughly speaking, Proposition 10.2 shows that we can find a homomorphism from a bipartite graph $\bar{H}$ to a graph $\bar{R}$ which consists of an edge $z^0 z^1$ which has an attached 5-cycle in such a way that most of the vertices of $\bar{H}$ are mapped about evenly to the vertices z^0 and z^1. If we knew that the colour classes of $\bar{H}$ were of almost equal size, then this would be a trivial task, but since this is not guaranteed, we will have to make use of the additional vertices $z^2, \ldots, z^5$.

Proof of Proposition 10.2. Given $\bar{\eta}$, choose an integer $\ell \geqslant 6$ and a real $\bar{\beta} > 0$ such that

$$\frac{5}{\ell} < \bar{\eta} \quad \text{and} \quad \bar{\beta} := \frac{1}{\ell^2}. \tag{10.4}$$

For the sake of a simpler exposition we assume that $\bar{m}/\ell$ and $\bar{\beta}\bar{m}$ are integers. Now consider a graph $\bar{H}$ as given in the statement of the proposition. Partition $V(\bar{H})$ along the ordering induced by the bandwidth labelling into sets $\bar{W}_1, \ldots, \bar{W}_\ell$ of sizes $|\bar{W}_i| = \bar{m}/\ell$ for $i \in [\ell]$. For each $\bar{W}_i$, consider its last $5\bar{\beta}\bar{m}$ vertices and partition them into sets $X_{i,1}, \ldots, X_{i,5}$ of size $|X_{i,k}| = \bar{\beta}\bar{m}$. For $i \in [\ell]$, let

$$W_i := \bar{W}_i \setminus (X_{i,1} \cup \cdots \cup X_{i,5}), \quad W := \bigcup_{i=1}^{\ell} W_i,$$

and note that

$$L := |W_i| = \frac{\bar{m}}{\ell} - 5\bar{\beta}\bar{m} \overset{(10.4)}{=} \left(\frac{1}{\ell} - \frac{5}{\ell^2}\right)\bar{m} \geqslant \frac{1}{\ell^2}\bar{m} \overset{(10.4)}{=} \bar{\beta}\bar{m}.$$

For $i \in [\ell]$, $j \in \{0, 1\}$, and $1 \leqslant k \leqslant 5$, let

$$W_i^j := W_i \cap Z^j, \quad X_{i,k}^j := X_{i,k} \cap Z^j.$$

Thanks to the fact that $\mathrm{bw}(\bar{H}) \leqslant \bar{\beta}\bar{m}$, we know that there are no edges between W_i and $W_{i'}$ for $i \neq i' \in [\ell]$. In a first round, for each $i \in [\ell]$ we will either map all vertices from W_i^j to z^j for both $j \in \{0, 1\}$ (call such a mapping a *normal* embedding of W_i) or we map all vertices from W_i^j to z^{1-j} for both $j \in \{0, 1\}$ (call this an *inverted* embedding). We will do this in such a way that the difference between the number of vertices that get sent to z^0 and the number of those that get sent to z^1 is as small as possible. Since $|W_i| \leqslant L$ the difference is therefore at most L. If, in addition, we guarantee that both W_1 and W_ℓ receive a normal embedding, it is at most $2L$. So, to summarize and to describe the mapping more precisely: there exist integers $\varphi_i \in \{0, 1\}$ for all $i \in [\ell]$ such that $\varphi_1 = 0 = \varphi_\ell$ and

the function $h : W \to \{z^0, z^1\}$, defined by

$$h(W_i^j) := \begin{cases} z^j & \text{if } \varphi_i = 0, \\ z^{1-j} & \text{if } \varphi_i = 1, \end{cases}$$

is a homomorphism from $\bar{H}[W]$ to $\bar{R}[\{z^0, z^1\}]$, satisfying that for both $j \in \{0, 1\}$

$$|h^{-1}(z^j)| \leqslant \frac{\ell L}{2} + 2L = \left(\frac{\ell}{2} + 2\right)\frac{\bar{m}}{\ell} - \left(\frac{\ell}{2} + 2\right)5\bar{\beta}\bar{m}$$
$$\overset{(10.4)}{=} \frac{\bar{m}}{2} + \bar{m}\left(\frac{2}{\ell} - \frac{5}{2}\frac{1}{\ell} - \frac{10}{\ell^2}\right) \leqslant \frac{\bar{m}}{2}. \tag{10.5}$$

In the second round we extend this homomorphism to the vertices in the classes $X_{i,k}$. Recall that these vertices are by definition situated after those in W_i and before those in W_{i+1}. The idea for the extension is simple. If W_i and W_{i+1} have been embedded in the same way by h (either both normal or both inverted), then we map all the vertices from all $X_{i,k}$ to z^0 and z^1 accordingly. If they have been embedded in different ways (one normal and one inverted), then we walk around the 5-cycle $z^1, \dots, z^5, z^1$ to switch colour classes.

Here is the precise definition. Consider an arbitrary $i \in [\ell]$. Since $h(W_i^0)$ and $h(W_i^1)$ are already defined, choose (and fix) $j \in \{0, 1\}$ in such a way that $h(W_i^j) = z^1$. Note that this implies that $h(W_i^{1-j}) = z^0$. Now define $h_i : \bigcup_{k=0}^{5} X_{i,k} \to \bigcup_{k=1}^{5}\{z^k\}$ as follows.

Suppose first that $\varphi_i = \varphi_{i+1}$. Observe that in this case we must also have $h(W_{i+1}^j) = z^1$ and $h(W_{i+1}^{1-j}) = z^0$. So we can happily define, for all $k \in [5]$,

$$h_i(X_{i,k}^j) = z^1 \quad \text{and} \quad h_i(X_{i,k}^{1-j}) = z^0.$$

Now suppose that $\varphi_i \neq \varphi_{i+1}$. Since we are still assuming that j is such that $h(W_i^j) = z^1$ and thus $h(W_i^{1-j}) = z^0$, the fact that $\varphi_i \neq \varphi_{i+1}$ implies that $h(W_{i+1}^j) = z^0$ and $h(W_{i+1}^{1-j}) = z^1$. In this case we define h_i as follows:

$h(W_i^{1-j})$ $= z^0$	$h_i(X_{i,1}^{1-j})$ $:= z^2$	$h_i(X_{i,2}^{1-j})$ $:= z^2$	$h_i(X_{i,3}^{1-j})$ $:= z^4$	$h_i(X_{i,4}^{1-j})$ $:= z^4$	$h_i(X_{i,5}^{1-j})$ $:= z^1$	$h(W_{i+1}^{1-j})$ $= z^1$
$h(W_i^j)$ $= z^1$	$h_i(X_{i,1}^j)$ $:= z^1$	$h_i(X_{i,2}^j)$ $:= z^3$	$h_i(X_{i,3}^j)$ $:= z^3$	$h_i(X_{i,4}^j)$ $:= z^5$	$h_i(X_{i,5}^j)$ $:= z^5$	$h(W_{i+1}^j)$ $= z^0$

Finally, we set $\bar{h} : V(\bar{H}) \to V(\bar{R})$ by letting $\bar{h}(x) := h(x)$ if $x \in W_i$ for some $i \in [\ell]$ and $\bar{h}(x) := h_i(x)$ if $x \in X_{i,k}$ for some $i \in [\ell]$ and $k \in [5]$.

In order to verify that this is a homomorphism from $\bar{H}$ to the sets $\bar{R}$, we first let

$$X_{i,0}^0 := W_i^0, X_{i,0}^1 := W_i^1, X_{i,6}^0 := W_{i+1}^0, X_{i,6}^1 := W_{i+1}^1.$$

Using this notation, it is clear that any edge xx' in $\bar{H}[W_i \cup \bigcup_{k=1}^{5} X_{i,k} \cup W_{i+1}]$ with $x \in Z^j$ and $x' \in Z^{1-j}$ is of the form

$$xx' \in (X_{i,k}^j \times X_{i,k}^{1-j}) \cup (X_{i,k}^j \times X_{i,k+1}^{1-j}) \cup (X_{i,k+1}^j \times X_{i,k}^{1-j})$$

for some $k \in [0, 6]$. It is therefore easy to check in the above table that $\bar{h}$ maps xx' to an edge of R.
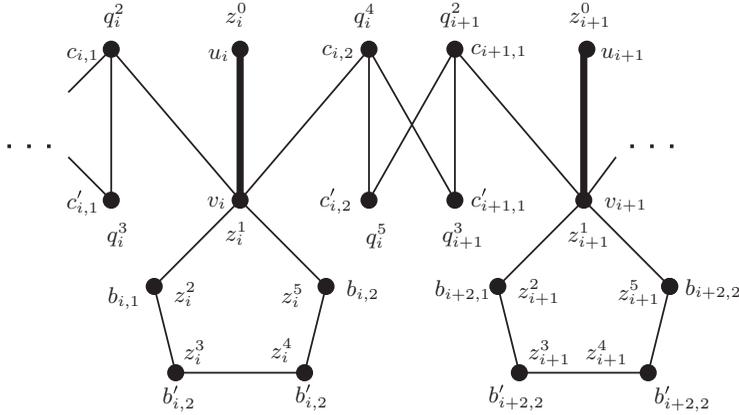
Figure 6. The subgraph $R[\{z_i^0, \ldots, z_i^5, q_i^4, q_i^5, q_{i+1}^2, q_{i+1}^3, z_{i+1}^0, \ldots, z_{i+1}^5\}]$ of $R_{r,1}$ in the proof of Lemma 5.2.

We conclude the proof by showing that the cardinalities of the pre-images of the vertices in R match the required sizes. In the second round we mapped a total of

$$\ell \cdot 5\bar{\beta}\bar{m} \overset{(10.4)}{=} \frac{5}{\ell}\bar{m} \overset{(10.4)}{\leqslant} \bar{\eta}\bar{m}$$

additional vertices from $\bar{H}$ to the vertices of $\bar{R}$, which guarantees that

$$|\bar{h}^{-1}(z^j)| \overset{(10.5)}{\leqslant} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \quad \text{for all } j \in \{0,1\}, \quad |\bar{h}^{-1}(z^k)| \leqslant \bar{\eta}\bar{m} \quad \text{for all } k \in [2,5].$$

Finally, the lower bound in (10.1) immediately follows from the upper bounds:

$$|\bar{h}^{-1}(z^j)| \geqslant \bar{m} - |\bar{h}^{-1}(z^{1-j})| - \sum_{k=2}^{5} |\bar{h}^{-1}(z^k)| \geqslant \frac{\bar{m}}{2} - 5\bar{\eta}\bar{m}. \qquad \square$$

We remark that Proposition 10.2 (and thus Lemma 5.2) would remain true if we replaced the 5-cycle in $\bar{R}$ by a 3-cycle. However, we need the properties of the 5-cycle in the proof of the main theorem. Now we will prove Lemma 5.2.

Proof of Lemma 5.2. Given the integer Δ, set $t := (\Delta + 1)^3(\Delta^3 + 1)$. Given a real $0 < \eta_H < 1$ and integers m and r, set $\bar{\eta} := \eta_H/20 < 1/20$ and apply Proposition 10.2 to obtain a real $\bar{\beta} > 0$. Choose $\beta > 0$ sufficiently small that all the inequalities

$$\frac{1}{r} - 4\beta \geqslant \beta/\bar{\beta}, \quad 4\beta r \leqslant \frac{\eta_H}{20r}, \quad 16\Delta\beta r \leqslant \eta_H\left(\frac{1}{r} - 4\beta\right)\left(\frac{1}{2} - 5\bar{\eta}\right) \qquad (10.6)$$

hold. Again, we assume that m/r and βm are integers.

Next we consider the spin graph $R_{r,t}$ with $t = 1$, *i.e.*, let $R := R_{r,1}$. For the sake of simpler reference, we will change the names of its vertices as follows. For all $i \in [r]$ we set (see Figure 6)

$$z_i^0 := u_i, \quad z_i^1 := v_i, \quad z_i^2 := b_{i,1}, \quad z_i^3 := b'_{i,1}, \quad z_i^4 := b'_{i,2}, \quad z_i^5 := b_{i,2},$$
$$q_i^2 := c_{i,1}, \quad q_i^3 := c'_{i,1}, \quad q_i^4 := c_{i,2}, \quad q_i^5 := c'_{i,2}.$$

Note that for every $i \in [r]$ the graph $R[\{z_i^0, \ldots, z_i^5\}]$ is isomorphic to the graph $\bar{R}$ defined in Proposition 10.2.

Partition $V(H)$ along the ordering (induced by the bandwidth labelling) into sets $\bar{S}_1, \ldots, \bar{S}_r$ of sizes $|\bar{S}_i| = m/r$ for $i \in [r]$.

Define sets $T_{i,k}$ for $i \in [r]$ and $k \in [0, 5]$ with $|T_{i,k}| = \beta m$ such that $T_{i,0} \cup \cdots \cup T_{i,4}$ contain the last $5\beta m$ vertices of $\bar{S}_i$ and $T_{i,5}$ contains the first βm vertices of $\bar{S}_{i+1}$ (according to the ordering). Set $S_i := \bar{S}_i \setminus (T_{i,1} \cup \cdots \cup T_{i,4})$ and observe that this implies that $T_{i,0}$ is the set of the last βm vertices of S_i and $T_{i,5}$ is the set of the first βm vertices in $\bar{S}_{i+1}$. Set

$$\bar{m} := |S_i| = (m/r) - 4\beta m = \left(\frac{1}{r} - 4\beta \right) m \overset{(10.6)}{\geqslant} \beta m / \bar{\beta}; \quad \text{thus } \bar{\beta}\bar{m} \geqslant \beta m. \tag{10.7}$$

Denote by Z^0 and Z^1 the two colour classes of the bipartite graph H. For $i \in [\ell]$, $k \in [0, 5]$ and $j \in [0, 1]$ let

$$S_i^j := S_i \cap Z^j, \quad T_{i,k}^j := T_{i,k} \cap Z^j.$$

Now for each $i \in [r]$ apply Proposition 10.2 to $\bar{H}_i := H[S_i]$ and $\bar{R}_i := R[\{z_i^0, \ldots, z_i^5\}]$. Observe that

$$\mathrm{bw}(\bar{H}_i) \leqslant \mathrm{bw}(H) \leqslant \beta m \overset{(10.7)}{\leqslant} \bar{\beta}\bar{m},$$

so we obtain a homomorphism $\bar{h}_i : S_i \to \{z_i^0, \ldots, z_i^5\}$ of $\bar{H}_i$ to $\bar{R}_i$. Combining these yields a homomorphism

$$\bar{h} : \bigcup_{i=1}^r S_i \to \bigcup_{i=1}^r \{z_i^0, \ldots, z_i^5\},$$

$$\text{from } H\left[\bigcup_{i=1}^r S_i \right] \quad \text{to } R\left[\bigcup_{i=1}^r \{z_i^0, \ldots, z_i^5\} \right]$$

with the property that for every $i \in [r]$, $j \in [0, 1]$ and $k \in [2, 5]$

$$\frac{\bar{m}}{2} - 5\bar{\eta}\bar{m} \overset{(10.1)}{\leqslant} |\bar{h}^{-1}(z_i^j)| \overset{(10.1)}{\leqslant} \frac{\bar{m}}{2} + \bar{\eta}\bar{m} \leqslant \left(1 + \frac{\eta_H}{10} \right) \frac{m}{2r} \quad \text{and}$$

$$|\bar{h}^{-1}(z_i^k)| \overset{(10.2)}{\leqslant} \bar{\eta}\bar{m} \leqslant \frac{\eta_H}{10} \frac{m}{2r}. \tag{10.8}$$

Thanks to (10.7), we know that $\bar{\beta}\bar{m} \geqslant \beta m$, and therefore applying the information from (10.3) in Proposition 10.2 yields that for all $i \in [r]$ and $j \in [0, 1]$

$$\bar{h}(T_{i,0}^j) = z_i^j \quad \text{and} \quad \bar{h}(T_{i,5}^j) = z_{i+1}^j.$$

In the second round, our task is to extend this homomorphism to the vertices in $\bar{S}_i \setminus S_i$ by defining a function

$$h_i : T_{i,1} \cup \cdots \cup T_{i,4} \to \{z_i^1, q_i^4, q_i^5, q_{i+1}^2, q_{i+1}^3, z_{i+1}^1\}$$

for each $i \in [r]$ as follows:

$$
\begin{array}{|c|c|c|c|c|c|}
\hline
\bar{h}(T_{i,0}^0) = z_i^0 & h_i(T_{i,1}^0) := q_i^4 & h_i(T_{i,2}^0) := q_i^4 & h_i(T_{i,3}^0) := q_{i+1}^2 & h_i(T_{i,4}^0) := q_{i+1}^2 & \bar{h}(T_{i,5}^0) = z_{i+1}^0 \\
\hline
\bar{h}(T_{i,0}^1) = z_i^1 & h_i(T_{i,1}^1) := z_i^1 & h_i(T_{i,2}^1) := q_i^5 & h_i(T_{i,3}^1) := q_{i+1}^3 & h_i(T_{i,4}^1) := z_{i+1}^1 & \bar{h}(T_{i,5}^1) = z_{i+1}^1 \\
\hline
\end{array}.
$$

Now set $h(x) := \bar{h}(x)$ if $x \in S_i$ for some $i \in [r]$ and $h(x) := h_i(x)$ if $x \in T_{i,k}$ for some $i \in [r]$ and $k \in [4]$.

Let us verify that h is a homomorphism from H to R. For edges xx' with both endpoints inside a set S_i we do not need to check anything, because here $h(x) = \bar{h}(x)$ and $h(x') = \bar{h}(x')$ and we know from Proposition 10.2 that $\bar{h}$ is a homomorphism. Due to the bandwidth condition $\mathrm{bw}(H) \leqslant \beta m$, any other edge xx' with $x \in Z^0$ and $x' \in Z^1$ is of the form

$$
xx' \in (T_{i,k}^0 \times T_{i,k}^1) \cup (T_{i,k}^0 \times T_{i,k+1}^1) \cup (T_{i,k+1}^0 \times T_{i,k}^1)
$$

for some $i \in [\ell]$ and $0 \leqslant k, k+1 \leqslant 5$. It is therefore easy to check in the above table that h maps xx' to an edge of R.

What can we say about the cardinalities of the pre-images? In the second round we have mapped $4\beta mr$ additional vertices from H to vertices in R, and hence for any vertex z in R with $z \notin \{z_i^0, z_i^1\}$, $i \in [\ell]$, we have

$$
|h_i^{-1}(z)| \leqslant 4\beta mr \overset{(10.6)}{\leqslant} \frac{\eta_H}{10} \frac{m}{2r}, \tag{10.9}
$$

and therefore the required upper bounds immediately follow from (10.8).

At this point we have found a homomorphism h from H to $R = R_{r,1}$ of which we know that it satisfies properties (H1) and (H2).

So far we have been working with the graph $R = R_{r,1}$, and therefore we know which vertices have been mapped to $u_i = z_i^0$ and $v_i = z_i^1$:

$$
\widetilde{U_i} := h^{-1}(u_i) = h^{-1}(z_i^0) \quad \text{and} \quad \widetilde{V_i} := h^{-1}(v_i) = h^{-1}(z_i^1).
$$

Moreover, for $i \in [r]$ and $k \in [2,5]$ set

$$
Z_i^k := h^{-1}(z_i^k) \quad \text{and} \quad Q_i^k := h^{-1}(q_i^k).
$$

Let us deal with property (H5) next. By definition, a vertex in $\widetilde{X_i} \subseteq \widetilde{V_i}$ must have at least one neighbour in Q_i^2 or Q_i^4 or Z_i^2 or Z_i^5. We know from (10.9) that the two latter sets contain at most $4\beta mr$ vertices each, and each of their vertices has at most Δ neighbours. Thus

$$
|\widetilde{X_i}| \leqslant \Delta \cdot 16\beta mr \overset{(10.6)}{\leqslant} \eta_H \left(\frac{1}{r} - 4\beta \right) \left(\frac{1}{2} - 5\bar{\eta} \right) m \overset{(10.7)}{=} \eta_H \bar{m} \left(\frac{1}{2} - 5\bar{\eta} \right)
$$
$$
\overset{(10.8)}{\leqslant} \eta_H |\bar{h}^{-1}(z_i^1)| \leqslant \eta_H |h^{-1}(z_i^1)| = \eta_H |\widetilde{V_i}|,
$$

which shows that (H5) is also satisfied.

Next we would like to split up the sets Z_i^k and Q_i^k for $i \in [r]$ and $k \in [2,5]$ into smaller sets in order to meet the additional requirements (H3) and (H4). This means that we need to partition them further into sets of vertices which have no path of length 1, 2, or 3 between them and which have the same degree into certain sets.

To achieve this, first denote by H^3 the third power of H. Then an upper bound on the maximum degree of H^3 is obviously given by

$$\Delta + \Delta(\Delta - 1) + \Delta(\Delta - 1)(\Delta - 1) \leqslant \Delta^3.$$

Hence H^3 has a vertex colouring $c : V(H) \to \mathbb{N}$ with at most $\Delta^3 + 1$ colours. Notice that a set of vertices that receives the same colour by c forms a 3-independent set in H. To formalize this argument, we define a 'fingerprint' function

$$f : \bigcup_{i=1}^{r} \bigcup_{k=2}^{5} (Z_i^k \cup Q_i^k) \to [0, \Delta] \times [0, \Delta] \times [0, \Delta] \times [\Delta^3 + 1]$$

as follows:

$$f(y) := \begin{cases} \left(\deg_{\widetilde{V}_i}(y), \deg_{Q_i^2 \cup Q_i^4}(y), \deg_{Z_i^2}(y), c(y)\right) & \text{if } y \in \left(\bigcup_{k=2}^{5}(Q_i^k \cup Z_i^k)\right) \setminus Z_i^4, \\ \left(\deg_{\widetilde{V}_i}(y), \deg_{Q_i^2 \cup Q_i^4}(y), \deg_{Z_i^3 \cup Z_i^5}(y), c(y)\right) & \text{if } y \in Z_i^4, \end{cases}$$

for some $i \in [r]$.

Recall that we defined $t := (\Delta + 1)^3(\Delta^3 + 1)$, so let us identify the codomain of f with the set $[t]$. Now for $i \in [r]$ and $j \in [t]$ we set

$$\begin{aligned} \widetilde{B}_{i,j} &:= Z_i^2 \cap f^{-1}(j), & \widetilde{B}_{i,t+j} &:= Z_i^5 \cap f^{-1}(j), \\ \widetilde{B}'_{i,j} &:= Z_i^3 \cap f^{-1}(j), & \widetilde{B}'_{i,t+j} &:= Z_i^4 \cap f^{-1}(j), \\ \widetilde{C}_{i,j} &:= Q_i^2 \cap f^{-1}(j), & \widetilde{C}_{i,t+j} &:= Q_i^4 \cap f^{-1}(j), \\ \widetilde{C}'_{i,j} &:= Q_i^3 \cap f^{-1}(j), & \widetilde{C}'_{i,t+j} &:= Q_i^5 \cap f^{-1}(j). \end{aligned}$$

Note, for example, that for $y \in \widetilde{B}_{i,j}$ the third component of $f(y)$ is exactly equal to $\deg_{L(i,j)}(y)$. Now, for any

$$yy' \in \binom{\widetilde{C}_{i,j}}{2} \cup \binom{\widetilde{B}_{i,j}}{2} \cup \binom{\widetilde{C}'_{i,j}}{2} \cup \binom{\widetilde{B}'_{i,j}}{2},$$

we have $f(y) = j = f(y')$ and hence any of the parameters required in (H3) and (H4) have the same value for y and y'.

The only thing missing before the proof of Lemma 5.2 is complete is that we need to guarantee that every $y \in Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ has at most $\Delta - 1$ neighbours in $\widetilde{V}_i$, as required in the first line of (H4). If a vertex y does not satisfy this, it must have *all* its Δ neighbours in $\widetilde{V}_i$. Since by definition of $\widetilde{V}_i$ these neighbours have been mapped to z_i^1, we can map y to z_i^0 (instead of mapping it to z_i^2, z_i^5, q_i^2 or q_i^4).

Even if, in this way, all of the vertices in $Z_i^2 \cup Z_i^5 \cup Q_i^2 \cup Q_i^4$ would have to be mapped to z_i^0, (10.9) assures us that these are at most $4\frac{\eta_H}{10}\frac{m}{2r}$ vertices. Since by (10.8) at most $(1 + \frac{\eta_H}{10})\frac{m}{2r}$ have already been mapped to z_i^0 in the first round and by (10.9) at most $\frac{\eta_H}{10}\frac{m}{2r}$ in the second round, this does not violate the upper bound in (H1). $\quad\square$

11. The constrained blow-up lemma

As explained earlier, the proof of the constrained blow-up lemma uses techniques developed in [4, 27] adapted to our setting. In fact, the proof we present here follows

the embedding strategy used in the proof of [4, Theorem 1.5]. This strategy is roughly as follows. Assume we want to embed the bipartite graph H on vertex set $\widetilde{U} \dot\cup \widetilde{V}$ into the host graph G on vertex set $U \dot\cup V$. Then we consider injective mappings $f : \widetilde{V} \to V$, and try to find one that can be extended to $\widetilde{U}$ such that the resulting mapping is an embedding of H into G. To determine whether a particular mapping f can be extended in this way we shall construct an auxiliary bipartite graph B_f, the so-called candidate graph (see Definition 11.1), which contains a matching covering one of its partition classes if and only if f can be extended. Accordingly, our goal will be to check whether B_f contains such a matching M, which we will do by appealing to Hall's condition. On page 673 we will explain the details of this part of the proof, determine necessary conditions for the application of Hall's theorem, and collect them in the form of a matching lemma (Lemma 11.8). It will then remain to show that there is a mapping f such that B_f satisfies the conditions of this matching lemma. This will require most of the work. The idea here is as follows.

We will show that mappings f usually have the necessary properties as long as they do not map neighbourhoods $N_H(\widetilde{u}) \subseteq \widetilde{V}$ of vertices in $\widetilde{u} \in \widetilde{U}$ to certain 'bad' spots in V. The existence of (many) mappings that avoid these 'bad' spots is verified with the help of a hypergraph packing lemma (Lemma 11.6). This lemma states that half of all possible mappings f avoid *almost all* 'bad' spots and can easily be turned into mappings f' avoiding *all* 'bad' spots with the help of so-called switchings.

11.1. Candidate graphs

If we have injective mappings $f : \widetilde{V} \to V$ as described in the previous paragraph we would like to decide whether f can be extended to an embedding of H into G. Observe that in such an embedding each vertex $\widetilde{u} \in \widetilde{U}$ has to be embedded into a vertex $u \in U$ such that the following holds. The neighbourhood $N_H(\widetilde{u})$ has its image $f(N_H(\widetilde{u}))$ in the set $N_G(u)$. Determining which vertices u are 'candidates' for the embedding of $\widetilde{u}$ in this sense gives rise to the following bipartite graph.

Definition 11.1 (candidate graph). Let H and G be bipartite graphs on vertex sets $\widetilde{U} \dot\cup \widetilde{V}$ and $U \dot\cup V$, respectively. For an injective function $f : \widetilde{V} \to V$ we say that a vertex $u \in U$ is an *f-candidate* for $\widetilde{u} \in \widetilde{U}$ if and only if $f(N_H(\widetilde{u})) \subseteq N_G(u)$.

The *candidate graph* $B_f(H, G) := (\widetilde{U} \dot\cup U, E_f)$ for f is the bipartite graph with edge set

$$E_f := \{\widetilde{u}u \in \widetilde{U} \times U : u \text{ is an } f\text{-candidate for } \widetilde{u}\}.$$

Now it is easy to see that the mapping f described above can be extended to an embedding of H into G if and only if the corresponding candidate graph has a matching covering $\widetilde{U}$. Clearly, if the candidate graph $B_f(H, G)$ of f has vertices $\widetilde{u} \in \widetilde{U}$ of degree 0, then $B_f(H, G)$ has no such matching and hence f cannot be extended. More generally we would like to avoid that $\deg_{B_f(H,G)}(\widetilde{u})$ is too small. Notice that this means precisely that f should not map $N_H(\widetilde{u})$ to a set $B \subseteq V$ that has a small common neighbourhood in G. These sets B are the 'bad' spots (see the beginning of this section) that should be avoided by f.

We explained above that, in order to avoid 'bad' spots, we will have to change certain mappings f slightly. The exact definition of this operation is as follows.

Definition 11.2 (switching). Let $f, f' : X \to Y$ be injective functions. We say that f' is obtained from f by a *switching* if there are $u, v \in X$ with $f'(u) = f(v)$ and $f'(v) = f(u)$ and $f(w) = f'(w)$ for all $w \notin \{u, v\}$. The *switching distance* $\mathrm{d}_{\mathrm{sw}}(f, f')$ of f and f' is at most s if the mapping f' can be obtained from f by a sequence of at most s switchings.

These switchings will alter the candidate graph corresponding to the injective function slightly (but not much: see Lemma 11.4). In order to quantify this, we further define the neighbourhood distance between two bipartite graphs B and B' which determines the number of vertices (in one partition class) whose neighbourhoods differ in B and B'.

Definition 11.3 (neighbourhood distance). Let $B = (U \dot\cup \widetilde{U}, E)$, $B' = (U \dot\cup \widetilde{U}, E')$ be bipartite graphs. We define the *neighbourhood distance* of B and B' with respect to $\widetilde{U}$ as

$$\mathrm{d}_{N(\widetilde{U})}(B, B') := |\{\widetilde{u} \in \widetilde{U} : N_B(\widetilde{u}) \neq N_{B'}(\widetilde{u})\}|.$$

The next simple lemma now examines the effect of switchings on the neighbourhood distance of candidate graphs and shows that functions with small switching distance correspond to candidate graphs with small neighbourhood distance.

Lemma 11.4 (switching lemma). *Let H and G be bipartite graphs on vertex sets $\widetilde{U} \dot\cup \widetilde{V}$ and $U \dot\cup V$, respectively, such that $\deg_H(\widetilde{v}) \leqslant \Delta$ for all $\widetilde{v} \in \widetilde{V}$ and let $f, f' : \widetilde{V} \to V$ be injective functions with switching distance $\mathrm{d}_{\mathrm{sw}}(f, f') \leqslant s$. Then the neighbourhood distance of the candidate graphs $B_f(H, G)$ and $B_{f'}(H, G)$ satisfies*

$$\mathrm{d}_{N(\widetilde{U})}\big(B_f(H, G), B_{f'}(H, G)\big) \leqslant 2s\Delta.$$

Proof. We proceed by induction on s. For $s = 0$ the lemma is trivially true. Thus, consider $s > 0$ and let g be a function with $\mathrm{d}_{\mathrm{sw}}(f, g) \leqslant s - 1$ and $\mathrm{d}_{\mathrm{sw}}(g, f') = 1$. Define

$$N(f, f') := \{\widetilde{u} \in \widetilde{U} : N_{B_f(H, G)}(\widetilde{u}) \neq N_{B_{f'}(H, G)}(\widetilde{u})\}.$$

Clearly, $|N(f, f')| = \mathrm{d}_{N(\widetilde{U})}(B_f(H, G), B_{f'}(H, G))$ and $N(f, f') \subseteq N(f, g) \cup N(g, f')$. By the induction hypothesis we have $|N(f, g)| \leqslant 2(s - 1)\Delta$. The remaining switching from g to f' interchanges only the images of two vertices from $\widetilde{V}$, say $\widetilde{v}_1$ and $\widetilde{v}_2$. It follows that

$$N(g, f') = \{\widetilde{u} \in N_H(\widetilde{v}_1) \cup N_H(\widetilde{v}_2) : N_{B_g(H, G)}(\widetilde{u}) \neq N_{B_{f'}(H, G)}(\widetilde{u})\},$$

which implies $|N(g, f')| \leqslant 2\Delta$, and therefore we get $|N(f, f')| \leqslant 2s\Delta$. $\qquad\square$

11.2. A hypergraph packing lemma

The main ingredient to the proof of the constrained blow-up lemma is the following hypergraph packing result (Lemma 11.6). To understand what this lemma says and how we will apply it, recall that we would like to embed the vertex set $\widetilde{U}$ of H into the vertex

set U of G such that subsets of $\widetilde{U}$ that form neighbourhoods in the graph H avoiding certain 'bad' spots in U. If H is a Δ-regular graph, then these neighbourhoods form Δ-sets. In this case, as we will see, the 'bad' spots also form Δ-sets. Accordingly, we have to solve the problem of packing the neighbourhood Δ-sets $\mathcal{N}$ and the 'bad' Δ-sets $\mathcal{B}$, which is a hypergraph packing problem. Lemma 11.6 below states that this is possible under certain conditions. One of these conditions is that the 'bad' sets should not 'cluster' too much (although there might be many of them). The following definition makes this precise.

Definition 11.5 (corrupted sets). For $\Delta \in \mathbb{N}$ and a set V let $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a collection of Δ-sets in V and let x be a positive real. We say that all $B \in \mathcal{B}$ are x-*corrupted* by $\mathcal{B}$. Recursively, for $i \in [\Delta - 1]$ an i-set $B \in \binom{V}{i}$ in V is called x-corrupted by $\mathcal{B}$ if it is contained in more than x of the $(i + 1)$-sets that are x-corrupted by $\mathcal{B}$.

Observe that, if a vertex $v \in V$ is not x-corrupted by $\mathcal{B}$, then it is also not x'-corrupted by $\mathcal{B}$ for any $x' > x$.

The hypergraph packing lemma now implies that $\mathcal{N}$ and $\mathcal{B}$ can be packed if $\mathcal{B}$ contains no corrupted sets. In fact this lemma states that *half of all* possible ways to map the vertices of $\mathcal{N}$ to $\mathcal{B}$ can be turned into such a packing by performing a sequence of a few switchings.

Lemma 11.6 (hypergraph packing lemma [27]). *For all integers $\Delta \geqslant 2$ and $\ell \geqslant 1$ and every positive σ there are positive constants $\eta_{11.6}$, and $n_{11.6}$ such that the following holds. Let $\mathcal{B}$ be a Δ-uniform hypergraph on $n' \geqslant n_{11.6}$ vertices such that no vertex of $\mathcal{B}$ is $\eta_{11.6} n'$-corrupted by $\mathcal{B}$. Let $\mathcal{N}$ be a Δ-uniform hypergraph on $n \leqslant n'$ vertices such that no vertex of $\mathcal{N}$ is contained in more than ℓ edges of $\mathcal{N}$.*

Then, for at least half of all injective functions $f : V(\mathcal{N}) \to V(\mathcal{B})$, there are packings f' of $\mathcal{N}$ and $\mathcal{B}$ with switching distance $\mathrm{d}_{\mathrm{sw}}(f, f') \leqslant \sigma n$.

When applying this lemma we also make use of the following lemma, which helps us to bound corruption.

Lemma 11.7 (corruption lemma). *Let $n, \Delta > 0$ be integers and let μ and η be positive reals. Let V be a set of size n and let $\mathcal{B} \subseteq \binom{V}{\Delta}$ be a family of Δ-sets of size at most μn^{Δ}. Then at most $(\Delta!/\eta^{\Delta-1})\mu n$ vertices are ηn-corrupted by $\mathcal{B}$.*

Proof. For $i \in [\Delta]$, let $\mathcal{B}_i$ be the family of all those i-sets $B' \in \binom{V}{i}$ that are ηn-corrupted by $\mathcal{B}$. We will prove by induction on i (starting at $i = \Delta$) that

$$|\mathcal{B}_i| \leqslant \frac{\Delta!/i!}{\eta^{\Delta-i}} \mu n^i. \tag{11.1}$$

For $i = 1$ this establishes the lemma. For $i = \Delta$ the assertion is true by assumption. Now assume that (11.1) is true for $i > 1$. By definition every $B' \in \mathcal{B}_{i-1}$ is contained in more than ηn sets $B \in \mathcal{B}_i$. On the other hand, clearly every $B \in \mathcal{B}_i$ contains at most i sets from

$\mathcal{B}_{i-1}$. Double counting thus gives

$$\eta n|\mathcal{B}_{i-1}| \leqslant |\{(B', B) : B' \in \mathcal{B}_{i-1}, B \in \mathcal{B}_i, B' \subseteq B\}| \leqslant i|\mathcal{B}_i| \overset{(11.1)}{\leqslant} i\frac{\Delta!/i!}{\eta^{\Delta-i}}\mu n^i,$$

which implies (11.1) for i replaced by $i-1$. □

11.3. A matching lemma

We indicated earlier that we are interested in determining whether a candidate graph has a matching covering one of its partition classes. In order to do so we will make use of the following matching lemma, which is an easy consequence of Hall's theorem. This lemma takes two graphs B and B' as input that have small neighbourhood distance. In our application these two graphs will be candidate graphs that correspond to two injective mappings f and f' with small switching distance (such as promised by the hypergraph packing lemma, Lemma 11.6). Recall that Lemma 11.4 guarantees that mappings with small switching distance correspond to candidate graphs with small neighbourhood distance.

The matching lemma asserts that B' has the desired matching if certain vertex degree and neighbourhood conditions are satisfied. These conditions are somewhat technical. They are tailored exactly to match the conditions that we establish for candidate graphs in the proof of the constrained blow-up lemma (see Claims 11.11–11.13).

Lemma 11.8 (matching lemma). *Let* $B = (\widetilde{U} \dot\cup U, E)$ *and* $B' = (\widetilde{U} \dot\cup U, E')$ *be bipartite graphs with* $|U| \geqslant |\widetilde{U}|$ *and* $d_{N(\widetilde{U})}(B, B') \leqslant s$. *If there are positive integers* x *and* n_1, n_2, n_3 *such that*

(i) $\deg_{B'}(\widetilde{u}) \geqslant n_1$ *for all* $\widetilde{u} \in \widetilde{U}$,
(ii) $|N_{B'}(\widetilde{S})| \geqslant x|\widetilde{S}|$ *for all* $\widetilde{S} \subseteq \widetilde{U}$ *with* $|\widetilde{S}| \leqslant n_2$,
(iii) $e_{B'}(\widetilde{S}, S) \leqslant \frac{n_1}{n_3}|\widetilde{S}||S|$ *for all* $\widetilde{S} \subseteq \widetilde{U}$, $S \subseteq U$ *with* $xn_2 \leqslant |S| < |\widetilde{S}| < n_3$,
(iv) $|N_B(S) \cap \widetilde{S}| > s$ *for all* $\widetilde{S} \subseteq \widetilde{U}$, $S \subseteq U$ *with* $|\widetilde{S}| \geqslant n_3$ *and* $|S| > |U| - |\widetilde{S}|$,

then B' *has a matching covering* $\widetilde{U}$.

Proof. We will check Hall's condition in B' for all sets $\widetilde{S} \subseteq \widetilde{U}$. We clearly have $|N_{B'}(\widetilde{S})| \geqslant |\widetilde{S}|$ for $|\widetilde{S}| \leqslant xn_2$ by (ii) (if $|\widetilde{S}| > n_2$, then consider a subset of $\widetilde{S}$ of size n_2).

Next, consider the case $xn_2 < |\widetilde{S}| < n_3$. Set $S := N_{B'}(\widetilde{S})$ and assume, for a contradiction, that $|S| < |\widetilde{S}|$. Since $|S| < |\widetilde{S}| < n_3$ we have $|S|/n_3 < 1$. Therefore, applying (i), we can conclude that

$$e_{B'}(\widetilde{S}, S) = \sum_{\widetilde{u} \in \widetilde{S}} |N_{B'}(\widetilde{u})| \geqslant n_1|\widetilde{S}| > \frac{n_1}{n_3}|\widetilde{S}||S|,$$

which is a contradiction to (ii). Thus $|N_{B'}(\widetilde{S})| \geqslant |\widetilde{S}|$.

Finally, for sets $\widetilde{S}$ of size at least n_3 set $S := U \setminus N_{B'}(\widetilde{S})$ and assume, again for a contradiction, that $|N_{B'}(\widetilde{S})| < |\widetilde{S}|$. This implies $|S| > |U| - |\widetilde{S}|$. Accordingly we can apply (iv) to $\widetilde{S}$ and S and infer that $|N_B(S) \cap \widetilde{S}| > s$. Since $d_{N(\widetilde{U})}(B, B') \leqslant s$, at most s

vertices from $\widetilde{U}$ have different neighbourhoods in B and B', and so

$$|N_{B'}(S) \cap \widetilde{S}| = |\{\widetilde{u} \in \widetilde{S} : N_{B'}(\widetilde{u}) \cap S \neq \emptyset\}|$$
$$\geqslant |\{\widetilde{u} \in \widetilde{S} : N_B(\widetilde{u}) \cap S \neq \emptyset\}| - s = |N_B(S) \cap \widetilde{S}| - s > 0,$$

which is a contradiction as $S = U \setminus N_{B'}(\widetilde{S})$. $\qquad\square$

11.4. Proof of Lemma 5.3

Now we are almost ready to present the proof of the constrained blow-up lemma (Lemma 5.3). We just need one further technical lemma as preparation. This lemma considers a family of pairwise disjoint Δ-sets $\mathcal{S}$ in a set S and states that a random injective function from S to a set T usually has the following property. The images $f(\mathcal{S})$ of sets in $\mathcal{S}$ 'almost' avoid a small family of 'bad' sets $\mathcal{T}$ in T.

Lemma 11.9. *For all positive integers Δ and positive reals β and μ_s there is a $\mu_T > 0$ such that the following holds. Let S and T be disjoint sets, let $\mathcal{S} \subseteq \binom{S}{\Delta}$ be a family of pairwise disjoint Δ-sets in S with $|\mathcal{S}| \leqslant \frac{1}{\Delta}(1 - \mu_s)|T|$, and let $\mathcal{T} \subseteq \binom{T}{\Delta}$ be a family of Δ-sets in T with $|\mathcal{T}| \leqslant \mu_T |T|^\Delta$.*

Then a random injective function $f : S \to T$ satisfies $|f(\mathcal{S}) \setminus \mathcal{T}| > (1 - \beta)|\mathcal{S}|$ with probability at least $1 - \beta^{|\mathcal{S}|}$.

Proof. Given Δ, β, and μ_s, choose

$$\mu_T := \sqrt[\beta]{\beta} \left(\frac{e}{\beta} \left(\frac{\Delta}{\mu_s} \right)^\Delta \right)^{-1}. \tag{11.2}$$

Let S, T, $\mathcal{S}$, and $\mathcal{T}$ be as required and let f be a random injective function from S to T. We consider f as a consecutive random selection (without replacement) of images for the elements of S where the images of the elements of the (disjoint) sets in $\mathcal{S}$ are chosen first. Let S_i be the ith such set in $\mathcal{S}$. Then the probability that f maps S_i to a set in $\mathcal{T}$, which we denote by p_i, is at most

$$p_i \leqslant \frac{|\mathcal{T}|}{\binom{|T| - (i-1)\Delta}{\Delta}} \leqslant \frac{\mu_T |T|^\Delta}{\binom{\mu_s |T|}{\Delta}} \leqslant \mu_T \frac{|T|^\Delta}{\left(\frac{\mu_s |T|}{\Delta} \right)^\Delta} = \mu_T \left(\frac{\Delta}{\mu_s} \right)^\Delta =: p,$$

where the second inequality follows from $(i-1)\Delta \leqslant |\bigcup \mathcal{S}| \leqslant (1 - \mu_s)|T|$. Let Z be a random variable with distribution $\mathrm{Bi}(|\mathcal{S}|, p)$. It follows that $\mathbb{P}[|f(\mathcal{S}) \cap \mathcal{T}| \geqslant z] \leqslant \mathbb{P}[Z \geqslant z]$. Since

$$\mathbb{P}[Z \geqslant z] \leqslant \binom{|\mathcal{S}|}{z} p^z < \left(\frac{e|\mathcal{S}|p}{z} \right)^z,$$

we infer that

$$\mathbb{P}\big[|f(\mathcal{S}) \cap \mathcal{T}| \geqslant \beta|\mathcal{S}|\big] < \left(\frac{ep}{\beta} \right)^{\beta|\mathcal{S}|} = \left(\frac{e\mu_T}{\beta} \left(\frac{\Delta}{\mu_s} \right)^\Delta \right)^{\beta|\mathcal{S}|} \overset{(11.2)}{=} \beta^{|\mathcal{S}|},$$

which proves the lemma since $|f(\mathcal{S}) \cap \mathcal{T}| \geqslant \beta|\mathcal{S}|$ holds if and only if $|f(\mathcal{S}) \setminus \mathcal{T}| \leqslant (1 - \beta)|\mathcal{S}|$. $\qquad\square$

Now we can finally give the proof of Lemma 5.3.

Proof of Lemma 5.3.   We first define a sequence of constants. Given Δ, d, and η fix $\Delta' := \Delta^2 + 1$. Choose β and σ such that

$$\beta^{\frac{1}{7}(\frac{d}{2})^{\Delta}} \leqslant \frac{1}{5} \quad \text{and} \quad \frac{(1-\beta)d^{\Delta}}{100^{\Delta}} \geqslant 2\sigma. \tag{11.3}$$

Apply the hypergraph packing lemma, Lemma 11.6, with input Δ, $\ell = 2\Delta + 1$, and σ to obtain constants $\eta_{11.6}$, and $n_{11.6}$. Next, choose $\eta'_{11.6}$, μ_{BL}, and μ_S such that

$$\frac{\eta'_{11.6}}{1-\eta} \leqslant \eta_{11.6}, \quad \frac{\Delta! \cdot 2\mu_{\mathrm{BL}}}{(\eta'_{11.6})^{\Delta-1}} \leqslant \eta, \quad \frac{1}{\Delta'} \leqslant \frac{1}{\Delta}(1-\mu_S). \tag{11.4}$$

Lemma 11.9 with input Δ, β, μ_S provides us with a constant μ_T. We apply Lemma 7.2 twice, once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_{\mathrm{BL}}/\Delta'$ and once with input $\Delta = \ell$, d, $\varepsilon' := \frac{1}{2}d$, and $\mu = \mu_T$, and get constants $\varepsilon_{7.2}$ and $\widetilde{\varepsilon}_{7.2}$, respectively. Now we can fix the promised constant ε such that

$$\varepsilon \leqslant \min\left\{\frac{\varepsilon_{7.2}}{\Delta'}, \frac{d}{2\Delta}\right\} \quad \text{and} \quad \frac{\varepsilon\Delta'}{\eta(1-\eta)} < \min\{d, \widetilde{\varepsilon}_{7.2}\}. \tag{11.5}$$

As last input let r_1 be given, and set

$$\xi_{7.2} := \eta(1-\eta)/(r_1\Delta'). \tag{11.6}$$

Let $c_{7.2}$ be the maximum of the two constants obtained from the two applications of Lemma 7.2 that we started above, with the additional parameter $\xi_{7.2}$. Further, let v and $c_{7.3}$ be the constants from Lemma 7.3 for input Δ, d, and ε, and let $c_{6.2}$ be the constant from Lemma 6.2 for input Δ and v. Finally, we choose $c = \max\{c_{7.2}, c_{7.3}, c_{6.2}\}$. With this we have defined all necessary constants.

Now assume we are given any $1 \leqslant r \leqslant r_1$, and a random graph $\Gamma = \mathcal{G}_{n,p}$ with $p \geqslant c(\log n/n)^{1/\Delta}$, where, without loss of generality, n is such that

$$(1-\eta')\frac{n}{r} \geqslant n_{11.6}. \tag{11.7}$$

Then, with high probability, the graph Γ satisfies the assertion of the different lemmas concerning random graphs, that we started to apply in the definition of the constants. More precisely, by the choice of the constants above:

(P1) Γ satisfies the assertion of Lemma 6.2 for parameters Δ and v, *i.e.*, for any set X and any family $\mathcal{F}$ with the conditions required in this lemma, the conclusion of the lemma holds.

(P2) Similarly Γ satisfies the assertion of Lemma 7.2 for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_{\mathrm{BL}}/\Delta'$, $\varepsilon_{7.2}$, and $\xi_{7.2}$. The same holds for parameters $\Delta = \ell$, d, $\varepsilon' = \frac{1}{2}d$, $\mu = \mu_T$, $\widetilde{\varepsilon}_{7.2}$, and $\xi_{7.2}$.

(P3) Γ satisfies the assertion of Lemma 7.3 for parameters Δ, d, ε, and v.

In the following we will assume that Γ has these properties and show that it then also satisfies the conclusion of the constrained blow-up lemma, Lemma 5.3.

Let $G \subseteq \Gamma$ and H be two bipartite graphs on vertex sets $U \dot{\cup} V$ and $\widetilde{U} \dot{\cup} \widetilde{V}$, respectively, that fulfil the requirements of Lemma 5.3. Moreover, let $\mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}$ be the family of special

Δ-sets, and let $\mathcal{B} \subseteq \binom{V}{\Delta}$ be the family of forbidden Δ-sets. It is not difficult to see that by possibly adding some edges to H, we can assume that the following holds.

($\widetilde{U}$) All vertices in $\widetilde{U}$ have degree *exactly* Δ.
($\widetilde{V}$) All vertices in $\widetilde{V}$ have degree maximal $\Delta + 1$.

Our next step will be to split the partition class U of G and the corresponding partition class $\widetilde{U}$ of H into Δ' parts of equal size. From the partition of H we require that no two vertices in one part have a common neighbour. This will guarantee that the neighbourhoods of two different vertices from one part form disjoint vertex sets (which we need because we would like to apply Lemma 7.3 later, in the proof of Claim 11.11, and Lemma 7.3 asserts certain properties for families of disjoint vertex sets).

Let us now explain precisely how we split U and $\widetilde{U}$. We assume for simplicity that $|\widetilde{U}|$ and $|U|$ are divisible by Δ', and partition the sets U arbitrarily into Δ' parts $U = U_1 \dot\cup \cdots U_{\Delta'}$ of equal size, *i.e.*, sets of size at least $n/(r\Delta')$. Similarly let $\widetilde{U} = \widetilde{U}_1 \dot\cup \cdots \dot\cup \widetilde{U}_{\Delta'}$ be a partition of $\widetilde{U}$ into sets of equal size such that each $\widetilde{U}_j$ is 2-independent in H. Such a partition exists by the theorem of Hajnal and Szemerédi (Theorem 10.1) applied to $H^2[\widetilde{U}]$ because the maximum degree of H^2 is less than $\Delta' = \Delta^2 + 1$.

In Claim 11.10 below we will assert that there is an embedding f' of $\widetilde{V}$ into V that can be extended to *each* of the $\widetilde{U}_j$ separately such that we obtain an embedding of H into G. To this end we will consider the candidate graphs $B_{f'}(H_j, G_j)$ defined by f' (see Definition 11.1), and show that there is an f' such that each $B_{f'}(H_j, G_j)$ has a matching covering $\widetilde{U}_j$. This, as discussed earlier, will ensure the existence of the desired embedding. To prepare this argument, we first need to exclude some vertices of V which are not suitable for such an embedding. To identify these vertices we define the following family of Δ-sets, which contains $\mathcal{B}$ and all sets in V that have a small common neighbourhood in some $\widetilde{U}_j$.

Define $\mathcal{B}' := \mathcal{B} \cup \bigcup_{j \in \Delta'} \mathcal{B}_j$, where

$$\mathcal{B}_j := \left\{ B \in \binom{V}{\Delta} : |N_G^{\cap}(B) \cap U_j| < \left(\tfrac{1}{2}d\right)^{\Delta} p^{\Delta} |U_j| \right\} \stackrel{(7.1)}{=} \mathrm{bad}_{d/2,d,p}^{G,\Delta}(V, U_j). \tag{11.8}$$

We claim that we obtain a set $\mathcal{B}'$ that is not much larger than $\mathcal{B}$. Indeed, by Proposition 4.2 the pair

$$(V, U_j) \text{ is } (\varepsilon\Delta', d, p)\text{-dense for all } j \in [\Delta'], \tag{11.9}$$

and $\varepsilon\Delta' \leqslant \varepsilon_{7.2}$ by (11.5). Moreover, we have $|U_j| \geqslant n/(r\Delta') \geqslant n/(r_1\Delta') \geqslant \xi_{7.2} n$ by (11.6). We can thus use the fact that our random graph Γ satisfies property (P2) (with $\mu = \mu_{\mathrm{BL}}/\Delta'$) on the bipartite subgraph $G[V \dot\cup U_j]$ and conclude that $|\mathcal{B}_j| \leqslant \mu_{\mathrm{BL}} |V|^{\Delta}/\Delta'$. Since $|\mathcal{B}| \leqslant \mu_{\mathrm{BL}} |V|^{\Delta}$ by assumption we infer

$$|\mathcal{B}'| \leqslant \mu_{\mathrm{BL}} |V|^{\Delta} + \Delta' \cdot \mu_{\mathrm{BL}} |V|^{\Delta}/\Delta' = 2\mu_{\mathrm{BL}} |V|^{\Delta}.$$

Set

$$V' := V \setminus V'' \quad \text{with} \quad V'' := \{ v \in V : v \text{ is } \eta'_{11.6} |V|\text{-corrupted by } \mathcal{B}' \} \tag{11.10}$$

and delete all sets from $\mathcal{B}'$ that contain vertices from V''. This determines the set V'' of vertices that we exclude from V for the embedding. We will next show that we did not exclude too many vertices in this process. For this we use the corruption lemma, Lemma 11.7. Indeed, Lemma 11.7 applied with n replaced by $|V|$, with Δ, $\mu = 2\mu_{\mathrm{BL}}$, and $\eta'_{11.6}$ to V and $\mathcal{B}'$ implies that

$$|V''| \leqslant \frac{\Delta!}{(\eta'_{11.6})^{\Delta-1}} 2\mu_{\mathrm{BL}}|V| \overset{(11.4)}{\leqslant} \eta|V| \quad \text{and thus} \quad n' := |V'| \geqslant (1-\eta)|V|. \tag{11.11}$$

Let

$$H_j := H[\widetilde{U}_j \dot\cup \widetilde{V}] \quad \text{and} \quad G_j := G[U_j \dot\cup V'].$$

Now we are ready to state the claim announced above, which asserts that there is an embedding f' of the vertices in $\widetilde{V}$ into the vertices in V' such that the corresponding candidate graphs $B_{f'}(H_j, G_j)$ have matchings covering $\widetilde{U}_j$. As we will show, this claim implies the assertion of the constrained blow-up lemma. Its proof, which we will provide thereafter, requires the matching lemma (Lemma 11.8), and the hypergraph packing lemma (Lemma 11.6).

Claim 11.10. *There is an injection $f' : \widetilde{V} \to V'$ with $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$ such that for all $j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ has a matching covering $\widetilde{U}_j$.*

Let us show that proving this claim suffices to establish the constrained blow-up lemma. Indeed, let $f' : \widetilde{V} \to V'$ be such an injection and denote by $M_j : \widetilde{U}_j \to U_j$ the corresponding matching in $B_{f'}(H_j, G_j)$ for $j \in [\Delta]$. We claim that the function $g : \widetilde{U} \dot\cup \widetilde{V} \to U \dot\cup V$, defined by

$$g(\widetilde{w}) = \begin{cases} M_j(\widetilde{w}) & \widetilde{w} \in \widetilde{U}_j, \\ f'(\widetilde{w}) & \widetilde{w} \in \widetilde{V}, \end{cases}$$

is an embedding of H into G. To see this, notice first that g is injective since f' is an injection and all M_j are matchings. Furthermore, consider an edge $\widetilde{u}\widetilde{w}$ of H with $\widetilde{u} \in \widetilde{U}_j$ for some $j \in [\Delta']$ and $\widetilde{v} \in \widetilde{V}$, and let

$$u := g(\widetilde{u}) = M_j(\widetilde{u}) \quad \text{and} \quad v := g(\widetilde{v}) = f'(\widetilde{v}).$$

It follows from the definition of M_j that $\widetilde{u}u$ is an edge of the candidate graph $B_{f'}(H_j, G_j)$. Hence, by the definition of $B_{f'}(H_j, G_j)$, u is an f'-candidate for $\widetilde{u}$, i.e.,

$$f'\big(N_{H_j}(\widetilde{u})\big) \subseteq N_{G_j}(u).$$

Since $v = f'(\widetilde{v}) \in f'\big(N_{H_j}(\widetilde{u})\big)$, this implies that uv is an edge of G. Because f' also satisfies $f'(T) \notin \mathcal{B}$ for all $T \in \mathcal{H}$, the embedding g also meets the remaining requirement of the constrained blow-up lemma that no special Δ-set is mapped to a forbidden Δ-set. $\qquad\square$

To complete the proof of Lemma 5.3 we still need to prove Claim 11.10, which will occupy us for the remainder of this section. We will assume throughout that we have the

same set-up as in the preceding proof. In particular, all constants, sets, and graphs are defined as there.

To prove Claim 11.10 we will use the matching lemma (Lemma 11.8) on candidate graphs $B = B_f(H_j, G_j)$ and $B' = B_{f'}(H_j, G_j)$ for injections $f, f' : \widetilde{V} \to V'$. As we will see, the following three claims imply that there are suitable f and f' such that the conditions of this lemma are satisfied. More precisely, Claim 11.11 will take care of conditions (i) and (ii) in this lemma, Claim 11.12 of condition (iii), and Claim 11.13 of condition (iv). Before proving these claims we will show that they imply Claim 11.10.

The first claim states that *many* injective mappings $f : \widetilde{V} \to V'$ can be turned into injective mappings f' (with the help of a few switchings) such that the candidate graphs $B_{f'}(H_j, G_j)$ for f' satisfy certain degree and expansion properties.

Claim 11.11. *For at least half of all injections $f : \widetilde{V} \to V'$ there is an injection $f' : \widetilde{V} \to V'$ with $\mathrm{d}_{\mathrm{sw}}(f, f') \leqslant \sigma n / r$ such that the following is satisfied for all $j \in [\Delta']$. For all $\widetilde{u} \in \widetilde{U}_j$ and all $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \leqslant p^{-\Delta}$ we have*

$$\deg_{B_{f'}(H_j, G_j)}(\widetilde{u}) \geqslant \left(\tfrac{d}{2}\right)^{\Delta} p^{\Delta} |U_j| \quad and \quad |N_{B_{f'}(H_j, G_j)}(\widetilde{S})| \geqslant v n p^{\Delta} |\widetilde{S}|. \tag{11.12}$$

Further, no special Δ-set from $\mathcal{H}$ is mapped to a forbidden Δ-set from $\mathcal{B}$ by f'.

The second claim asserts that *all* injective mappings f' are such that the candidate graphs $B_{f'}(H_j, G_j)$ do not contain sets of certain sizes with too many edges between them.

Claim 11.12. *All injections $f' : \widetilde{V} \to V'$ satisfy the following for all $j \in [\Delta']$ and all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$. If $v n \leqslant |S| < |\widetilde{S}| < \tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta} |U_j|$, then*

$$e_{B_{f'}(H_j, G_j)}(\widetilde{S}, S) \leqslant 7 p^{\Delta} |\widetilde{S}| |S|.$$

The last of the three claims states that for *random* injective mappings f the graphs $B_{f'}(H_j, G_j)$ have edges between any pair of large enough sets $S \subseteq U_j$ and $\widetilde{S} \subseteq \widetilde{U}_j$.

Claim 11.13. *A random injection $f : \widetilde{V} \to V'$ a.a.s. satisfies the following. For all $j \in [\Delta']$ and all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \geqslant \tfrac{1}{7}\left(\tfrac{d}{2}\right)^{\Delta} |U_j|$ and $|S| > |U_j| - |\widetilde{S}|$, we have*

$$|N_{B_f(H_j, G_j)}(S) \cap \widetilde{S}| > 2 \sigma n / r.$$

Proof of Claim 11.10.   Our aim is to apply the matching lemma (Lemma 11.8) to the candidate graphs $B_f(H_j, G_j)$ and $B_{f'}(H_j, G_j)$ for all $j \in [\Delta']$ with carefully chosen injections f and f'.

Let $f : \widetilde{V} \to V'$ be an injection satisfying the assertions of Claim 11.11 and Claim 11.13 and let f' be the injection promised by Claim 11.11 for this f. Such an f exists as at least half of all injections from $\widetilde{V}$ to V' satisfy the assertion of Claim 11.11 and almost all of those satisfy the assertion of Claim 11.13. We will now show that for all $j \in [\Delta']$ the

conditions of Lemma 11.8 are satisfied for input

$$B = B_f(H_j, G_j), \quad B' = B_{f'}(H_j, G_j), \quad s = 2\sigma n/r,$$

$$x = vnp^\Delta, \quad n_1 = \left(\tfrac{d}{2}\right)^\Delta p^\Delta |U_j|, \quad n_2 = p^{-\Delta}, \quad n_3 = \tfrac{1}{7}\left(\tfrac{d}{2}\right)^\Delta |U_j|.$$

Claim 11.11 asserts that $d_{sw}(f, f') \leqslant \sigma n/r$. As $\widetilde{U}_j$ is 2-independent in H we have $\deg_{H_j}(\widetilde{v}) \leqslant 1$ for all $\widetilde{v} \in \widetilde{V}$. Thus the switching lemma, Lemma 11.4, applied to H_j and G_j and with s replaced by $\sigma n/r$ implies

$$d_{N(\widetilde{U}_j)}(B, B') = d_{N(\widetilde{U}_j)}\big(B_f(H_j, G_j), B_{f'}(H_j, G_j)\big) \leqslant 2\sigma n/r = s.$$

Moreover, by Claim 11.11, for all $\widetilde{u} \in \widetilde{U}_j$ we have

$$\deg_{B'}(\widetilde{u}) = \deg_{B_{f'}(H_j, G_j)}(\widetilde{u}) \geqslant \left(\tfrac{d}{2}\right)^\Delta p^\Delta |U_j| = n_1,$$

and thus condition (i) of Lemma 11.8 holds true. Further, we conclude from Claim 11.11 that $|N_{B'}(\widetilde{S})| \geqslant x|\widetilde{S}|$ for all $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| < p^{-\Delta} = n_2$. This gives condition (ii) of Lemma 11.8. In addition, Claim 11.12 states that for all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $xn_2 = vn \leqslant |S| < |\widetilde{S}| < \tfrac{1}{7}\left(\tfrac{d}{2}\right)^\Delta |U_j| = n_3$ we have

$$e_{B'}(\widetilde{S}, S) = e_{B_{f'}(H_j, G_j)}(\widetilde{S}, S) \leqslant 7p^\Delta |\widetilde{S}||S| = \frac{n_1}{n_3}|\widetilde{S}||S|,$$

and accordingly condition (iii) of Lemma 11.8 is also satisfied. To see (iv), observe that the choice of f and Claim 11.13 assert

$$|N_B(S) \cap \widetilde{S}| = |N_{B_f(H_j, G_j)}(S) \cap \widetilde{S}| > 2\sigma n/r = s$$

for all $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \geqslant \tfrac{1}{7}\left(\tfrac{d}{2}\right)^\Delta |U_j| = n_3$ and $|S| > |U| - |\widetilde{S}|$. Therefore, all conditions of Lemma 11.8 are satisfied and we infer that for *all* $j \in [\Delta']$ the candidate graph $B_{f'}(H_j, G_j)$ with f' as chosen above has a matching covering $\widetilde{U}$. Moreover, by Claim 11.11, f' maps no special Δ-set to a forbidden Δ-set. This establishes Claim 11.10. □

It remains to show Claims 11.11–11.13. We start with Claim 11.11. For the proof of this claim we apply the hypergraph packing lemma (Lemma 11.6).

Proof of Claim 11.11. Notice that $(\widetilde{U})$ on page 676 implies that $N_H(\widetilde{u})$ contains exactly Δ elements for each $\widetilde{u} \in \widetilde{U}$. Hence we may define the following family of Δ-sets. Let

$$\mathcal{N} := \{N_H(\widetilde{u}) : \widetilde{u} \in \widetilde{U}\} \cup \mathcal{H} \subseteq \binom{\widetilde{V}}{\Delta}.$$

We want to apply the hypergraph packing lemma (Lemma 11.6) with Δ, with ℓ replaced by $2\Delta + 1$, and with σ to the hypergraphs with vertex sets $\widetilde{V}$ and V' and edge sets $\mathcal{N}$ and $\mathcal{B}'$, respectively (see (11.8) on page 676). We will first check that the necessary conditions are satisfied.

Observe that

$$|V'| \overset{(11.11)}{\geqslant} (1 - \eta')|V| \geqslant (1 - \eta')n/r \overset{(11.7)}{\geqslant} n_{11.6}, \quad \text{and} \quad |\widetilde{V}| \leqslant |V'|.$$

Furthermore, a vertex $\widetilde{v} \in \widetilde{V}$ is neither contained in more than Δ sets from $\mathcal{H}$ nor is $\widetilde{v}$ in $N_H(\widetilde{u})$ for more than $\Delta + 1$ vertices $\widetilde{u} \in \widetilde{U}$ (by ($\widetilde{\mathrm{V}}$) on page 676). Therefore the condition Lemma 11.6 imposes on $\mathcal{N}$ is satisfied with ℓ replaced by $2\Delta + 1$. Moreover, according to (11.10) no vertex in V' is $\eta'_{11.6}|V|$-corrupted by $\mathcal{B}'$. Since

$$\eta'_{11.6}|V| \overset{(11.11)}{\leqslant} \eta'_{11.6}(1-\eta)^{-1}n' \overset{(11.4)}{\leqslant} \eta_{11.6}n',$$

this (together with the observation in Definition 11.5) implies that no vertex in V' is $\eta_{11.6}n'$-corrupted by $\mathcal{B}'$ and therefore all prerequisites of Lemma 11.6 are satisfied.

It follows that the conclusion of Lemma 11.6 holds for at least half of all injective functions $f: \widetilde{V} \to V'$, namely that there are packings f' of (the hypergraphs with edges) $\mathcal{N}$ and $\mathcal{B}$ with switching distance $d_{\mathrm{sw}}(f, f') \leqslant \sigma|\widetilde{V}| \leqslant \sigma n/r$. Clearly, such a packing f' does not send any special Δ-set from $\mathcal{H}$ to any forbidden Δ-set from $\mathcal{B}$. Our next goal is to show that f' satisfies the first part of (11.12) for all $j \in [\Delta']$ and $\widetilde{u} \in \widetilde{U}_j$. For this purpose, fix j and $\widetilde{u}$. The definition of the candidate graph $B_{f'}(H_j, G_j)$, Definition 11.1, implies

$$\deg_{B_{f'}(H_j, G_j)}(\widetilde{u}) = |\{u \in U_j : f'(N_{H_j}(\widetilde{u})) \subseteq N_{G_j}(u)\}|$$
$$= |\{u \in U_j : u \in N^{\cap}_{G_j}(f'(N_{H_j}(\widetilde{u})))\}|$$
$$= |N^{\cap}_{G_j}(f'(N_{H_j}(\widetilde{u})))| \geqslant (\tfrac{1}{2}d)^{\Delta} p^{\Delta}|U_j|,$$

where the first inequality follows from the fact that $N_{H_j}(\widetilde{u}) \in \mathcal{N}$ and thus, as f' is a packing of $\mathcal{N}$ and $\mathcal{B}'$, we have $f'(N_{H_j}(\widetilde{u})) \notin \mathrm{bad}^{G,\Delta}_{d/2,d,p}(V, U_j) \subseteq \mathcal{B}'$ (see the definition of $\mathcal{B}'$ in (11.8)). This in turn means that all Δ-sets $f'(N_{H_j}(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{U}_j$ are p-good (see Definition 7.1) in (V, U_j), because (V, U_j) has p-density at least $d - \varepsilon\Delta' \geqslant \frac{d}{2}$ by (11.9) and (11.5). With this information at hand we can proceed to prove the second part of (11.12). Let $\widetilde{S} \subseteq \widetilde{U}_j$ with $\widetilde{S} < 1/p^{\Delta}$ and consider the family $\mathcal{F} \subseteq \binom{V}{\Delta}$ with

$$\mathcal{F} := \{f'(N_H(\widetilde{u})) : \widetilde{u} \in \widetilde{S}\}.$$

Because U_j is 2-independent in H the sets $N_H(\widetilde{u})$ with $\widetilde{u} \in \widetilde{S}$ form a family of disjoint Δ-sets in $\widetilde{V}$. It follows that also the sets $f'(N_H(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{S}$ form a family of disjoint Δ-sets in V. By (P3) on page 675 the conclusion of Lemma 7.3 holds for Γ. We conclude that the pair (V, U_j) is $(1/p^{\Delta}, vnp^{\Delta})$-expanding. Since $|\mathcal{F}| = |\widetilde{S}| < 1/p^{\Delta}$ by assumption and all members of $\mathcal{F}$ are p-good in (V, U_j), this implies that $|N^{\cap}_{U_j}(\mathcal{F})| \geqslant vnp^{\Delta}|\mathcal{F}|$. On the other hand $N^{\cap}_{U_j}(\mathcal{F}) = N_{B_{f'}(H_j, G_j)}(\widetilde{S})$ by the definition of $B_{f'}(H_j, G_j)$ and $\mathcal{F}$ and thus we get the second part of (11.12). $\qquad\square$

Recall that property (P1) states that Γ satisfies the conclusion of Lemma 6.2 for certain parameters. We will use this fact to prove Claim 11.12.

Proof of Claim 11.12. Fix $f' : \widetilde{V} \to V'$, $j \in [\Delta']$, $S \subseteq U_j$, and $\widetilde{S} \subseteq \widetilde{U}_j$ with $vn \leqslant |S| < |\widetilde{S}| < \frac{1}{7}\left(\frac{d}{2}\right)^{\Delta}|U_j|$. For the candidate graphs $B_{f'}(H_j, G_j)$ of f' we have

$$e_{B_{f'}(H_j, G_j)}(\widetilde{S}, S) = |\{\widetilde{u}u \in \widetilde{S} \times S : f'(N_H(\widetilde{u})) \subseteq N_G(u)\}|$$
$$\overset{(6.1)}{=} \#\mathrm{stars}^G(S, \{f'(N_H(\widetilde{u})) : \widetilde{u} \in \widetilde{S}\})$$

$$\leqslant \# \operatorname{stars}^{\Gamma}\big(S, \{f'\big(N_H(\widetilde{u})\big) : \widetilde{u} \in \widetilde{S}\}\big) = \# \operatorname{stars}^{\Gamma}(S, \mathcal{F}'),$$

where $\mathcal{F}' := \{f'\big(N_H(\widetilde{u})\big) : \widetilde{u} \in \widetilde{S}\}$. As before, the sets $f'(N_H(\widetilde{u}))$ with $\widetilde{u} \in \widetilde{S}$ form a family of $|\widetilde{S}|$ disjoint Δ-sets in V'. Since $vn \leqslant |S| < |\widetilde{S}| = |\mathcal{F}'| \leqslant n$ we can appeal to property (P1) (and hence Lemma 6.2) with the set $X := S$ and the family $\mathcal{F}'$ and infer that

$$e_{B_{f'}(H_j, G_j)}(\widetilde{S}, S) \leqslant \# \operatorname{stars}^{\Gamma}(S, \mathcal{F}') \leqslant 7p^{\Delta}|\mathcal{F}'||S| = 7p^{\Delta}|\widetilde{S}||S|,$$

as required. $\qquad\square$

Finally, we prove Claim 11.13. For this proof we will use the fact that Δ-sets in p-dense graphs have big common neighbourhoods (the conclusion of Lemma 7.2 holds by property (P2)) together with Lemma 11.9.

Proof of Claim 11.13. Let f be an injective function from $\widetilde{V}$ to V'. First, consider a fixed $j \in [\Delta']$ and fixed sets $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with $|\widetilde{S}| \geqslant \frac{1}{7}\big(\frac{d}{2}\big)^{\Delta}|U_j|$ and $|S| > |U_j| - |\widetilde{S}|$. Define

$$\mathcal{S} := \{N_{H_j}(\widetilde{u}) : \widetilde{u} \in \widetilde{S}\} \quad \text{and} \quad \mathcal{T} := \operatorname{bad}_{d/2, d, p}^{G, \Delta}(V', S),$$

and observe that

$$\begin{aligned}
|N_{B_f(H_j, G_j)}(S) \cap \widetilde{S}| &= |\{\widetilde{u} \in \widetilde{S} : \exists u \in S \text{ with } f\big(N_{H_j}(\widetilde{u})\big) \subseteq N_{G_j}(u)\}| \\
&= |\{\widetilde{u} \in \widetilde{S} : N_{G_j}^{\cap}\big(f\big(N_{H_j}(\widetilde{u})\big)\big) \cap S \neq \emptyset\}| \\
&\geqslant |\{\widetilde{u} \in \widetilde{S} : f\big(N_{H_j}(\widetilde{u})\big) \notin \operatorname{bad}_{d/2, d, p}^{G, \Delta}(V', S)\}| = |f(\mathcal{S}) \setminus \mathcal{T}|,
\end{aligned}$$

since all Δ-sets $B \notin \operatorname{bad}_{d/2, d, p}^{G, \Delta}(V', S)$ satisfy $|N_{G_j}^{\cap}(B) \cap S| \geqslant \big(\frac{d}{2}\big)^{\Delta} p^{\Delta}|S| > 0$. Thus, to prove the claim it suffices to show that a random injection $f : \widetilde{V} \to V'$ violates $|f(\mathcal{S}) \setminus \mathcal{T}| > 2\sigma n/r$ with probability at most $5^{-|U_j|}$ because this implies that f violates the conclusion of Claim 11.13 for *some* $j \in [\Delta']$, and *some* $S \subseteq U_j$, $\widetilde{S} \subseteq \widetilde{U}_j$ with probability at most $O(2^{|U_j|}2^{|\widetilde{U}_j|} \cdot 5^{-|U_j|}) = o(1)$. For this purpose, we will use the fact that the pair (V', S) is p-dense. Indeed, observe that

$$|S| > |U_j| - |\widetilde{S}| > |U_j| - |\widetilde{U}_j| = \frac{|U| - |\widetilde{U}|}{\Delta'} \geqslant \frac{\eta |U|}{\Delta'}$$

by the assumptions of the constrained blow-up lemma, Lemma 5.3. As $|V'| \geqslant (1 - \eta)|V|$ by (11.11) we can apply Proposition 4.2 twice to infer from the (ε, d, p)-density of (V, U) that (V', S) is $(\widetilde{\varepsilon}, d, p)$-dense with $\widetilde{\varepsilon} := \varepsilon \Delta'/(\eta(1 - \eta))$. Furthermore, $\widetilde{\varepsilon} \leqslant \widetilde{\varepsilon}_{7.2}$ by (11.5) and

$$|V'| \overset{(11.11)}{\geqslant} (1 - \eta)\frac{n}{r} \overset{(11.6)}{\geqslant} \xi_{7.2} n, \quad \text{and} \quad |S| > \frac{\eta |U|}{\Delta'} \geqslant \frac{\eta n}{r \Delta'} \overset{(11.6)}{\geqslant} \xi_{7.2} n.$$

Hence we conclude from (P2) on page 675 (with $\mu = \mu_{\mathcal{T}}$) that $|\mathcal{T}| = |\operatorname{bad}_{d/2, d, p}^{G, \Delta}(V', S)| \leqslant \mu_{\mathcal{T}}|V'|^{\Delta}$. In addition

$$\frac{1}{7}\Big(\frac{d}{2}\Big)^{\Delta}|U_j| \leqslant |\widetilde{S}| = |\mathcal{S}| \leqslant |\widetilde{U}_j| \leqslant (1 - \eta)\frac{n}{\Delta'} \overset{(11.11)}{\leqslant} \frac{|V'|}{\Delta'} \overset{(11.4)}{\leqslant} \frac{1}{\Delta}(1 - \mu_s)|V'|. \qquad (11.13)$$

Thus, we can apply Lemma 11.9 with Δ, β, and μ_s to $S = \widetilde{V}$, $T = V'$, and to $\mathcal{S}$ and $\mathcal{T}$, and conclude that f violates

$$|f(\mathcal{S}) \setminus \mathcal{T}| > (1-\beta)|\mathcal{S}| \stackrel{(11.13)}{\geqslant} (1-\beta)\tfrac{1}{7}\left(\tfrac{d}{2}\right)^\Delta |U_j| \geqslant \frac{(1-\beta)d^\Delta n}{7 \cdot 2^\Delta r \Delta'} \geqslant \frac{(1-\beta)d^\Delta n}{100^\Delta r} \stackrel{(11.3)}{\geqslant} 2\sigma \frac{n}{r}$$

with probability at most

$$\beta^{|\mathcal{S}|} \leqslant \beta^{\frac{1}{7}(\frac{d}{2})^\Delta |U_j|} \leqslant 5^{-|U_j|},$$

where the first inequality follows from (11.13) and the second from (11.3). $\qquad\square$

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