

Estimation of the influx and the radius of the depletion zone developing around a root during nutrient uptake

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Abstract

An approximation to the analytical solution of the one-dimensional diffusion equation in cylindrical coordinates is derived. The solution applies to a semi-infinite system with a prescribed concentration at the root surface. From this approximate solution, simple formulas for calculating the radius of the depletion zone and the nutrient flux at the root surface as a function of time are obtained.

Introduction

In studies of nutrient uptake by plant roots it is of interest to know the transport rate through the soil to the root surface, which is equal to the influx, and the extension of the depletion zone in order to estimate whether inter-root competition occurs. For this purpose, numerical models have been used to describe the processes of nutrient transport by diffusion and convection in soil and nutrient uptake by the roots (Rengel, 1993). Application of numerical methods is necessary because of the nonlinearities in the differential equations and boundary conditions (Nye and Marriot, 1969). Under certain simplifying assumptions, useful exact solutions of the governing transport equations can be derived (Cushman, 1979). These simplifications regularly involve linearization of the transport equations and boundary conditions. As the nutrient uptake by plant roots is usually described by the non-linear Michaelis–Menten kinetics (Claassen et al., 1986), some kind of linearization has to be applied to this boundary condition.

In this work, we will assume diffusion to be the only transport process. This simplification seems realistic and justified because at low but still sufficient soil nutrient content, diffusion is the main transport mechanism, amounting to more than 90% of the nutrient transport to the root (Claassen, 1990). The inner boundary of the system is kept at a constant concentration of

the nutrient considered. If the concentration at this boundary is near zero, we have the maximum possible flux permitted by the soil. The outer boundary of the system is considered to be at an infinite distance, where the concentration is kept constant at the initial concentration. This implies there is no competition for the nutrient between neighbouring roots, which is a realistic assumption for nutrients of low mobility and/or roots that are far apart.

Exact solutions for this problem are known (Carslaw and Jaeger, 1959, pp 335–336). The objective of this work is to obtain simple approximate solutions for the case of diffusion in planar and cylindrical coordinate systems. We will derive simple formulas for the radius of the depletion zone around a root and an approximate calculation of the nutrient flux at the root surface as a function of time. For the reader not interested in the derivation of the equations, the obtained results and their application are presented in the final section of this paper.

The case of planar diffusion

In order to find a procedure for calculating the radius of the depletion zone around a root, that is in a cylindrical coordinate system we will first have a look at the simpler case of planar diffusion. If we have a semi-infinite planar system with space-coordinate x , a concentration C_0 at $x = 0$ and the concentration staying equal to the

initial concentration C_i at $x = \infty$, the concentration profile is given by (see Crank, 1975)

$$C = C_0 + (C_i - C_0) \operatorname{erf} \left(\frac{x}{2\sqrt{DT}} \right), \quad (1)$$

where T is the time and D is the effective diffusion coefficient. Expanding the right side in a power series at $x = 0$ (or equivalently at $T = \infty$) gives:

$$C = C_0 + (C_i - C_0) \operatorname{erf} \left(\frac{x}{\sqrt{\pi DT}} - \frac{x^3}{12\sqrt{\pi D^3 T^3}} + \dots \right) \quad (2)$$

Using only the terms up to first order in x in this approximation and setting $C = C_i$, solving for x gives:

$$x_i = \sqrt{\pi DT} \quad (3)$$

If we set x in Equation 1 equal to x_i , we obtain:

$$C(x_i) = C_0 + (C_i - C_0) \operatorname{erf} \left(\frac{\sqrt{\pi}}{2} \right) \quad (4)$$

As the evaluation of the error function gives 0.79 approximately, we finally get:

$$\frac{C(x_i) - C_0}{(C_i - C_0)} = 0.79 \quad (5)$$

That is, in the case of planar diffusion we can *define* the extension of the diffusion zone as $x^2 = \sqrt{\pi DT}$, where the difference between C and C_0 reaches 79% of the initial difference $C_i - C_0$. Furthermore, from the absence of a term quadratic in x in the series in Equation 2, we can deduce, that the concentration profile should be nearly linear for small x .

The extrapolation of this linear part up to C_i gives the radius of the depletion zone x_i , as shown in Figure 1. This observation can be helpful in the evaluation of experimentally obtained concentration profiles, as the measured concentration difference $C - C_0$ should reach 79% of the initial difference at x , if the measured concentration profile conforms to theoretical expectations. If reasonable agreement has been found, the effective coefficient of diffusion D can be obtained from Equation 3 immediately.

The case of cylindrical diffusion

The governing differential equation for diffusion in a cylindrical coordinate system is:

$$\frac{\partial C}{\partial T} = \frac{1}{R} \frac{\partial}{\partial R} RD \frac{\partial C}{\partial R} \quad (6)$$

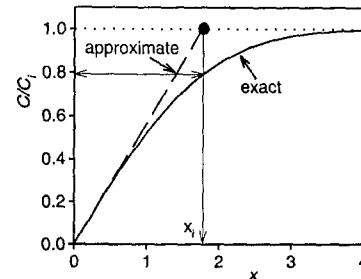


Fig. 1. Comparison of the exact solution (Eq. 1) and the linear approximation (from Eq. 2) for the case of planar diffusion, where $C_0 = 0$ and DT is chosen to be unity.

where:

$C = C(R, T)$	nutrient concentration in soil solution
D	effective diffusion coefficient of solute
T	time
R	radial distance from the root axis

The system to be considered is semi-infinite: the root is maintaining a constant concentration at its surface, while the outer boundary is at an infinite distance from the root, where the concentration is held constant at the initial concentration of the soil solution. In the context of nutrient uptake studies, this means there is no inter-root competition for the nutrient. These boundary conditions can be stated by the following equations:

$$C = C_i, \quad R > R_0, \quad T = 0 \quad (7)$$

$$C = C_0, \quad R = R_0, \quad T > 0$$

$$C = C_i, \quad R = \infty, \quad T = 0$$

C_i initial concentration of soil solution

C_0 nutrient concentration at root surface

R_0 root radius

To simplify the algebra, non-dimensional variables are introduced:

$$c = c(r, t) = \frac{C_i - C}{C_i} \quad (\text{applies also to } C_i \text{ and } C_0!) \quad (8)$$

$$t = \frac{D}{R_0^2} T$$

$$r = \frac{R}{R_0}$$

On applying these transformations and after some rearrangements, Equation 6 is transformed into

$$\frac{\partial c}{\partial t} = \frac{\partial^2 c}{\partial r^2} + \frac{1}{r} \frac{\partial c}{\partial r}, \quad (9)$$

while the transformed equations resulting from the initial and boundary conditions (Eq. 7) are:

$$\begin{aligned} c &= 0, r > 1, t = 0 \\ c &= c_0 = \frac{C_i - C_0}{C_i}, r = 1, t > 0 \\ c &= 0, r = \infty, t > 0 \end{aligned} \quad (10)$$

Now, we are ready to find a solution to the problem stated. One method to proceed is to apply the Laplace¹ transformation to $c(r, t)$ in Equations 9 and 10. This transformation¹ is defined as

$$L[c(r, t)] = \bar{c}(r, s) = \bar{c} = \int_0^\infty e^{-st} c(r, t) dt \quad (11)$$

where s is a complex variable.

Using the Laplace transformation on both sides of Equation 9 as well as on the boundary and initial conditions (Equation 10) results in a modified Bessel equation, whose solution is

$$\bar{c} = \frac{c_0 K_0(rs^{1/2})}{s K_0(s^{1/2})}, \quad (12)$$

where K_0 is the modified Bessel function of the second kind.

At this point, we only have a solution for the Laplace-transformed concentration \bar{c} as a function of r and some complex variable s . To get a solution in terms of $c(r, t)$, we have to use the inversion theorem of the Laplace transformation to Equation 12. This theorem as applied to $\bar{c}(r, s)$ is defined as

$$\bar{c}(r, t) = \frac{1}{2\pi i} \int_{\beta-i\infty}^{\beta+i\infty} e^{st} \bar{c}(r, s) ds, \quad (13)$$

where β can be chosen deliberately, if only all singularities of $\bar{c}(r, s)$ lie to the left of $\beta - i\infty, \beta + i\infty$ in the complex plane.

The inverse transform of $\bar{c}(r, s)$ can be written down as

$$\begin{aligned} c(r, t) &= c_0 + \\ &\frac{2c_0}{\pi} \int_0^\infty e^{-u^2 t} \frac{J_0(ru)Y_0(u) - J_0(u)Y_0(ru)}{J_0(u)^2 + Y_0(u)^2} du, \end{aligned} \quad (14)$$

¹ The apparatus of the Laplace transform is used here in a purely operational manner. In a strictly mathematical sense, it would be necessary to verify a number of assumptions (existence of the transform, interchange of differentiation and integration) made in the procedure. As this may be difficult and on the other hand the solution obtained can be proved by inserting it into the original differential equation, the method is simply to proceed and to look for the results.

where J_0 and Y_0 are the Bessel function of the first and second kind. In nutrient uptake studies, a main point of interest usually is in the flux of the nutrient to the root $F = -Db \frac{\partial \bar{c}}{\partial R}$. For now we will consider only the derivative $\frac{\partial c}{\partial r}$. From Equation 14 we obtain

$$\begin{aligned} \frac{\partial c(r, t)}{\partial r} &= \\ &\frac{2c_0}{\pi} \int_0^\infty e^{-u^2 t} \frac{J_0(u)Y_1(ru) - J_1(u)Y_0(u)}{J_0(u)^2 + Y_0(u)^2} du, \end{aligned} \quad (15)$$

For the gradient at the root surface, we can simplify this equation by setting $r = 1$ and using the relationship $J_1(x)Y_0(x) - J_0(x)Y_1(x) = \frac{2}{\pi x}$ to yield:

$$\frac{\partial c(r, t)}{\partial r} \Big|_{r=1} = \frac{-4c_0}{\pi^2} \int_0^\infty e^{-u^2 t} \frac{1}{J_0(u)^2 + Y_0(u)^2} \frac{du}{u}, \quad (16)$$

Derivation of the approximate solutions for the cylindrical case

While the integrals in the last three equations for calculating concentration and flux are sufficiently well behaved to allow numerical integration to an arbitrary degree of precision, it would be convenient to have good approximations of the integrals that can be expressed in terms of elementary functions. Approximations of this kind are most easily found by stepping back to our solution for the Laplace-transformed concentration $\bar{c}(r, s)$ (Eq. 12). By inspection of this equation, it was found convenient to introduce a dummy variable y with the substitution $r = e^y$ and then to expand the right side into a Taylor series at $y = 0$. On reversing the substitution by writing $y = \ln r$, we have expanded Equation 12 in a series in terms of $\ln r$ at $r = 1$:

$$\begin{aligned} \bar{c}(s) &= \frac{c_0}{s} - \frac{c_0 K_1(s^{1/2})}{s^{1/2} K_0(s^{1/2})} \left(\ln(r) + \frac{s \ln(r)^3}{6} + \right. \\ &\left. \frac{s \ln(r)^4}{6} + \frac{s \ln(r)^5}{10} + \frac{s \ln(r)^5}{120} + \dots \right) \end{aligned} \quad (17)$$

This series deserves some comments: A term in $\ln(r)^2$ is missing in the expansion, so we can assume, that the term in $\ln r$ makes the dominant contribution to the result (Note the similarity to the planar case, where a term in x^2 was missing in the series expansion of the exact solution Eq. 2). Furthermore, the factors of the

terms in $\ln(r)^3$ and higher order can be derived from the factor of $\ln r$ by repeated multiplication by s . This, means we only have to look for the transformation of

$$\bar{p}(s) = \frac{K_1(s^{1/2})}{s^{1/2} K_0(s^{1/2})} \quad (18)$$

into the time domain. From the properties of the Laplace transform, the remaining factors of $\ln(r)^n$ can be obtained by differentiating the result for time t . Third, by differentiating the series in Equation 17 for r and setting $r = 1$, we obtain the gradient of the transformed concentration $\bar{c}(s)$ at the surface, which is $-\bar{p}(s)$. So, the retransformation of Equation 18 will give us an approximation to the exact concentration gradient as calculated from Equation 16.

It was not possible to find a retransformation of Equation 18 to the time domain in terms of simple functions. But, from the properties of the Laplace-transform it is known (Carslaw and Jaeger, 1959, pp. 339–341), that the behaviour of the transformed functions at small values of s determines the behaviour of the original functions for large values of t . As we are interested to find approximations useful for large times, we can expand the right side of Equation 18 in a Taylor series in s at $s = 0$ and then retransform each term of the series. As shown in the appendix, this results in an asymptotic series in t (Eq. A16). Now, we can ask, what we have won by replacing the numerical integration necessary to evaluate the exact solutions in Equations 14, 15 and 16 by a long and complicated expression. The answer is: not very much, but as our objective was to derive simple approximations to the exact solutions, let's use the first term of Equation A16 in the retransformation of Equation 18. As mentioned above, this is a first approximation to the negative of the concentration gradient at the root surface:

$$-\frac{\partial c(r, t)}{\partial r}|_{r=1} = \frac{2}{\pi} c_0 \arctan\left(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}\right) + \dots \quad (19)$$

The exact value from Equation 16 is:

$$-\frac{\partial c(r, t)}{\partial r}|_{r=1} = \frac{4c_0}{\pi^2} \int_0^\infty e^{-u^2 t} \frac{1}{J_0(u)^2 + Y_0(u)^2} \frac{du}{u}, \quad (20)$$

From Figure 2, where we have deliberately set c_0 to 1, we can see that the approximative solution fits the exact solution very well if $t > 1$. Carslaw and Jaeger (1959, p 336) give an approximate solution to the same problem. Their equation, whose derivation is not given, has a restricted range of validity compared to Equation 19.

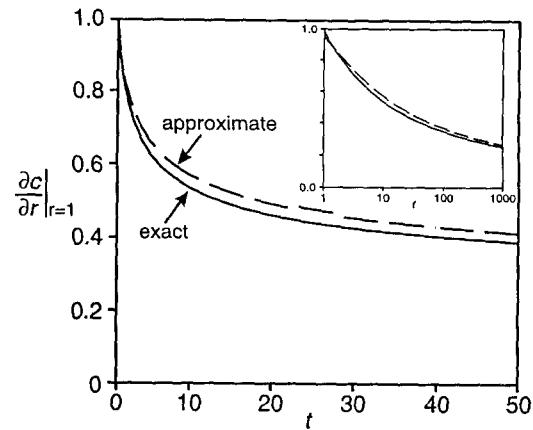


Fig. 2. Comparison of the exact (Eq. 20) and the approximate solution (Eq. 19) for the concentration gradient at the root surface as a function of time.

Now, we have found a good and simple approximation to the concentration gradient at the root surface. With this result we will try to approximate the exact, concentration profile (Eq. 14) around the root by using the shortened version of Equation 17, using only terms up to first order in $\ln(r)$:

$$\bar{c}(s) = \frac{c_0}{s} - \frac{c_0 K_1(s^{1/2})}{s^{1/2} K_0(s^{1/2})} \ln(r) + \dots \quad (21)$$

Transforming this equation back to the time domain, again using only the first term of Equation A16 we get:

$$c(r, t) = c_0 \left(1 - \frac{2}{\pi} \arctan\left(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}\right) \ln(r) \right) + \dots \quad (22)$$

The comparison of the exact and the approximate solution for the concentration profile is shown in Figure 3. As expected for small r the approximation is relatively good, but of course the term $\ln(r)$ can not account for the behaviour of the concentration at larger values of r .

Furthermore, we were interested in the radius of the depletion zone around the root. If we define this radius of influence r_i (or Δr , as it is called in most nutrient uptake studies), in an analogy to the case of planar diffusion, as the radius at which the approximation of $c(r, t)$ is equal to zero, we can derive from Equation 22 the following formula:

$$r_i = \exp\left(\frac{\pi}{2 \arctan\left(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}\right)}\right) \quad (23)$$

From Figure 3 it can be observed, that r_i corresponds to a concentration (as calculated by the exact

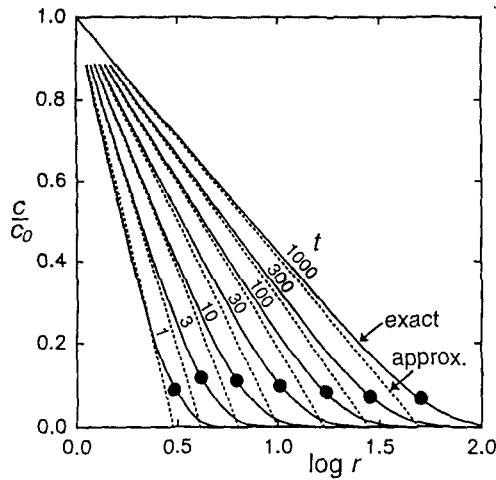


Fig. 3. Comparison of the exact (Eq. 14) and the approximate solution (Eq. 22) for the concentration profile around the root at different times. Notice that $\frac{C}{C_0} = \frac{C_i - C}{C_i - C_0}$ and $r = \frac{R}{R_0}$.

solution) in the range from 0.1 to 0.15. The concentration values are marked by black circles. Although it would be more satisfactory to calculate a radius of influence as a point, where the corresponding exact concentration is a constant fraction of the initial concentration, as it is possible in the case of diffusion in a planar system, Equation 23 seems to be appropriate to give an estimate in the case of cylindrical diffusion.

Application of the derived equations

Finally we will give a short summary of the results obtained together with some notes on the application of the equations derived for estimating the extension of the depletion zone for planar geometry and of approximate calculation of the radius of the depletion zone and influx for a cylindrical coordinate system. In the approximations, the significance of relevant factors is recognized easily and they do not require the numerical evaluation of integrals allowing the evaluation of the formulas with a simple pocket calculator.

Planar geometry of diffusion

The **extension of the depletion zone** x_i (or Δx , as it is sometimes called), as derived from the linear approximation to the exact solution of the appropriate diffusion equation, is given by Equation 3

$$\Delta x = \sqrt{\pi D T}$$

where D is the effective coefficient of diffusion and T is time. The extension Δx is the point where the difference between the concentration and the boundary concentration at $x = 0$ reaches 79% of the difference between the initial and the boundary concentration, so we have a depletion of 21% at that point (see Fig. 1).

Cylindrical geometry of diffusion

For the approximate calculation of the radius of the depletion zone around the root and nutrient flux at the root surface we will replace the transformed concentrations (from Eq. 8) and radii by their original values

$$C = c(r, t) = \frac{C_i - C}{C_i} \quad (\text{applies also to } C_i \text{ and } C_0!) \quad (24)$$

$$r = \frac{R}{R_0}$$

$$\frac{\partial c(r, t)}{\partial r} = -\frac{R_0}{C_i} \frac{\partial C(R, T)}{\partial R},$$

where:

$C = C(R, T)$	nutrient concentration in soil solution
C_i	initial concentration in soil solution
C_0	concentration at the root, surface
R	radial distance from the root axis
R_0	root radius

The transformed time will be retained

$$t = \frac{R}{R_0^2} T \quad (25)$$

because this quantity has to be calculated anyway to apply the approximative formulas, which are applicable if $t > 1$. In nutrient uptake studies, this condition is satisfied in most cases. For example, if we assume a root radius R_0 of 0.01 cm and an effective diffusion coefficient $D = 10^{-9} \text{ cm}^2 \text{ s}^{-1}$ we have $t = 1.73$ for a time T of 1 day of uptake.

The first order approximation of the concentration profiles around the root from Equation 22 gives

$$C(R, T) = C_0 + \frac{2}{\pi} (C_i - C_0) \arctan\left(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}\right) \ln \frac{R}{R_0}, \quad (26)$$

where the constant $4/\zeta^2 = 1.26$, approximately.

The **radius of the depletion zone around the root** R_i (or Δr , as it is called in most nutrient uptake studies), as derived from Equation 23, can be calculated by:

$$\Delta r = R_0 \exp\left(\frac{\pi}{2 \arctan\left(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}\right)}\right) \quad (27)$$

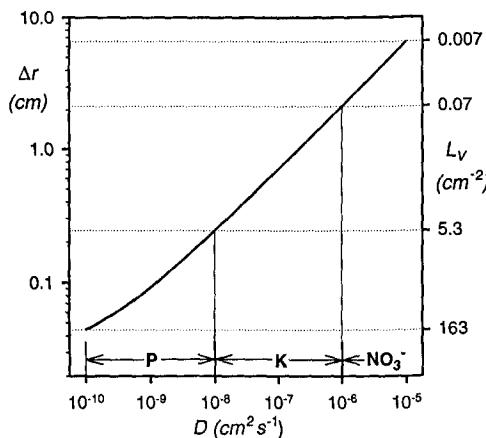


Fig. 4. Extension of the depletion zone Δr around a plant root (root radius $R_0 = 0.015 \text{ cm}$) after 30 d of nutrient uptake as a function of the effective coefficient of diffusion. The secondary y axis shows the root length density L_V , for which the half-mean distance between roots is equal to Δr , this is $L_V = \frac{1}{\pi \Delta r^2}$.

The appropriate amount of depletion at Δr can not be given exactly, as is was possible in the case of planar diffusion, but it is in the range from 10 to 15% (see Fig. 3).

In Figure 4 the radius of the depletion zone Δr after 30 d of nutrient uptake, as calculated from Equation 7, is shown for the range of effective diffusion coefficients D encountered in nutrient uptake studies. For the main nutrients P, K and NO_3^- , characteristic ranges of D (see Classen, 1990) are shown in the figure. Inter-root competition for the nutrients starts where the radius of the depletion zone equals the half-mean distance between roots and the depletion zones Δr of single roots begin to overlap.

On the right y axis, the root length densities L_V corresponding to this distance have been plotted for selected values of D . It is clearly visible, that for P extremely high root length densities would be required for inter-root competition to occur. On the other hand, in the case of NO_3^- the radius of the depletion zone is very large and even with only a small root length per volume (L_V less than 0.07 cm cm^{-3}) inter-root competition could be encountered. While for P competition is unlikely and for NO_3^- is almost always encountered, potassium takes an intermediate position: inter-root competition may or may not occur, depending on the exact values of D and L_V .

The **approximative nutrient flux at the root surface**, which is equal to the influx, is defined as $I_n = -Db \frac{\partial C(R=R_0, t)}{\partial R}$, where b is the buffer power.

From Equation 19, we obtain:

$$I_n = -Db \frac{C_i - C_0}{R_0} \frac{2}{\pi} \arctan\left(\frac{\pi}{\ln\left(\frac{4t}{\zeta^2}\right)}\right) \quad (28)$$

If we remember, that from the definition of the effective diffusion coefficient $Db = D_L \theta f$, where D_L is the diffusion coefficient in solution, θ is volumetric water content and f is an impedance factor, we can rewrite this equation as:

$$I_n = -D_L \theta f \frac{C_i - C_0}{R_0} \frac{2}{\pi} \arctan\left(\frac{\pi}{\ln\left(\frac{4D_L \theta f T}{\zeta^2 R_0^2 b}\right)}\right) \quad (29)$$

In this form it is immediately visible that the influx is proportional to the properties of the diffusion path (θ, f) and the concentration difference in soil solution ($C_i - C_0$), i.e. between the initial concentration and the concentration at the root surface. In contrast, the buffer power b appears only in the logarithmic term and so the sensitivity of the influx to changes in buffer power is low. The root radius R_0 enters the equation as divisor. From this we can conclude, that roots with a small diameter can sustain a high influx for longer time than thick roots can do.

References

- Carslaw H S and Jaeger J C 1959 Conduction of Heat in Solids. – 2nd ed. Clarendon Press, Oxford.
- Claassen N, Syring K M and Jungk A 1986 Verification of a mathematical model by simulating potassium uptake from soil. Plant and Soil 95, 209–220.
- Claassen N 1990 Nährstoffaufnahme höherer Pflanzen aus dem Boden. Severin Verlag, Göttingen.
- Crank J 1975 The Mathematics of Diffusion. Clarendon Press, Oxford.
- Cushman J H 1979 An analytical solution to solute transport near root surfaces for low initial concentration: I. Equations development. Soil Sci. Soc. Am. J. 43, 1087–1090.
- Nye P H and Marriot F H C 1969 Theoretical study of the distribution of substances around roots resulting from simultaneous diffusion and mass flow. Plant and Soil 3, 459–472.
- Rengel J 1993 Mechanistic simulation models of nutrient uptake. Plant and Soil 152, 161–173.
- Ritchie R H and Sakakura A Y 1956 Asymptotic expansions of solutions of the heat conduction equation in internally bounded cylindrical geometry. J. Appl. Physiol 27, 1453–1459.

Appendix

To obtain approximations to Equation 18

$$\bar{p}(s) = \frac{K_1(s^{\frac{1}{2}})}{s^{\frac{1}{2}} K_0(s^{\frac{1}{2}})} \quad (\text{A1})$$

useful for large values of time, that is $t > 1$, we expand the right side in a Taylor series at $s = 0$. To simplify the algebra, we will make the substitution

$$s \rightarrow \frac{4}{\zeta^2} s \quad (\text{A2})$$

before doing the expansion. Here ζ is e^γ and $\gamma = 0.577$ is Euler's constant.

Following Ritchie and Sakakura (1956), the expansion gives:

$$\begin{aligned} \bar{p}(s) = & -\frac{\zeta^2}{2s \ln(s)} + \frac{1}{\ln(s)} - \frac{1}{\ln(s)^2} + s \left(\frac{-2}{\zeta^2 \ln(s)^3} + \frac{21}{8\zeta^2 \ln(s)^2} - \frac{5}{2\zeta^2 \ln(s)} \right) + \\ & s^2 \left(\frac{-4}{\zeta^4 \ln(s)^4} + \frac{13}{2\zeta^4 \ln(s)^3} - \frac{451}{108\zeta^4 \ln(s)^2} + \frac{23}{18\zeta^4 \ln(s)} \right) + \dots \end{aligned} \quad (\text{A3})$$

This series is composed of terms of the form

$$\bar{q}_l^\nu(s) = \frac{s^{-1+\nu}}{\ln(s)^l}, \quad (\text{A4})$$

where $\nu \geq 0$ and $l \geq 1$. By applying the formula of the inverse Laplace transform to Equation A4 we arrive at the real integral

$$q_l^\nu(t) = \int_0^\infty \left(\frac{\frac{i}{2}(-1)^\nu x^{-1+\nu}}{e^x \pi t^\nu (-i\pi - \ln(t) + \ln(x))^l} - \frac{\frac{i}{2}(-1)^\nu x^{-1+\nu}}{e^x \pi t^\nu (i\pi - \ln(t) + \ln(x))^l} \right) dx \quad (\text{A5})$$

Now, we expand the integrand v of Equation A5 in terms of $\ln(x)$, using the binomial theorem.

$$v = \sum_{n=0}^{\infty} \frac{\frac{i}{2}(-1)^\nu x^{-1+\nu} \binom{-l}{n} \ln(x)^n}{e^x \pi t^\nu} ((-i\pi + \ln(t))^{-l-n} - (i\pi + \ln(t))^{-l-n}) \quad (\text{A6})$$

From the definition of the Γ function, we can see that

$$\int_0^\infty x^{\nu-1} e^{-x} \ln(x)^n dx = \Gamma^{(n)}(\nu), \quad (\text{A7})$$

where $\Gamma^{(n)}(\nu)$ means the n 'th derivative of the Γ function with respect to the variable ν . With this result, we can perform the integration indicated in Equation A5, which gives the result:

$$q_l^\nu(t) = \sum_{n=0}^{\infty} \frac{\frac{i}{2}(-1)^\nu \binom{-l}{n} \Gamma^{(n)}(\nu)}{\pi t^\nu} ((-i\pi + \ln(t))^{-l-n} - (i\pi + \ln(t))^{-l-n}) \quad (\text{A8})$$

To get rid of the complex numbers in this equation, we can rewrite the complex terms in the trigonometric form. The resulting expression is:

$$q_l^\nu(t) = \sum_{n=0}^{\infty} \frac{\frac{i}{2}(-1)^\nu \binom{-l}{n} \sin((l+n) \arctan(\frac{\pi}{\ln(t)})) \Gamma^{(n)}(\nu)}{\pi t^\nu (\pi^2 + \ln(t)^2)^{\frac{l+n}{2}}} ((-i\pi + \ln(t))^{-l-n} - (i\pi + \ln(t))^{-l-n}) \quad (\text{A9})$$

This expression holds only if $\nu \geq 1$. To get a solution for the case $\nu = 0$, we can see from Equation A4, that this case can be derived from the retransformation for the case $\nu = 1$ by integrating Equation A5 for time:

$$q_l^0(t) = \int_0^\infty \left(\frac{\frac{i}{2}(-i\pi - \ln(t) + \ln(x))^{1-l}}{e^x(l-1)\pi} + \frac{\frac{i}{2}(i\pi - \ln(t) + \ln(x))^{1-l}}{e^x(l-1)\pi} \right) dx \quad (\text{A10})$$

Again, expanding in terms of $\ln(x)$ gives

$$v = \sum_{n=0}^{\infty} \frac{\frac{i}{2}\binom{1-l}{n}\ln(t)^n}{e^x(l-1)\pi} ((i\pi + \ln(t))^{1-l-n} - (-i\pi + \ln(t))^{1-l-n}), \quad (\text{A11})$$

performing the integration as in Equation A8 results in

$$q_l^0(t) = \sum_{n=0}^{\infty} \frac{\frac{i}{2}\binom{1-l}{n}\Gamma^{(n)}(\nu)}{(l-1)\pi} ((i\pi + \ln(t))^{1-l-n} - (-i\pi + \ln(t))^{1-l-n}), \quad (\text{A12})$$

and finally after converting to real numbers we have:

$$q_l^0(t) = \sum_{n=0}^{\infty} \frac{\binom{1-l}{n} \sin((-1+l+n)\arctan(\frac{\pi}{\ln(t)}))\Gamma^{(n)}(1)}{(l-1)\pi(\pi^2 + \ln(t)^2)^{\frac{-1+l+n}{2}}} \quad (\text{A13})$$

The case $l = 1$ must be handled separately. We set $\nu = 0$ and $l = 1$ before doing the integration in Equation A5. Going through the steps described above, we get:

$$q_l^0(t) = \sum_{n=0}^{\infty} \frac{\binom{-1}{n} \sin(n\arctan(\frac{\pi}{\ln(t)}))\Gamma^{(n)}(1)}{n\pi(\pi^2 + \ln(t)^2)^{\frac{n}{2}}} \quad (\text{A14})$$

For the limit $n = 0$, this equation finally gives:

$$q_{1,n=0}^0(t) = \frac{-\arctan(\frac{\pi}{\ln(t)})}{\pi} \quad (\text{A15})$$

Now, we can write down the approximate retransformation to the time domain of Equation 18 and, as stated already, from this result the retransformation of Equation 17 can be derived. We do the retransformation by using the above results on the retransformation of Equation A4 in Equation A3. Furthermore, by using the property of the Laplace-transform $L[f(at)] = \bar{f}(s/a)/a$, we reverse the effect of our substitution in Equation A2, which was introduced only for cosmetic reasons. The result is somewhat, lengthy (all sum symbols apply to all terms until to the next summing sign):

$$\begin{aligned} p(t) &= \frac{2}{\pi} \arctan\left(\frac{\pi}{\ln\left(\frac{4t}{\zeta^2}\right)}\right) + \\ &\sum_{n=0}^{\infty} \frac{2\binom{-1}{n} \sin(n\arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}))\Gamma^{(n)}(1)}{n\pi(\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{n}{2}}} \\ &\sum_{n=0}^{\infty} \frac{\binom{-1}{n}(\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-1-n}{2}} \sin((1+n)\arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}))\Gamma^{(n)}(1)}{\pi t} \\ &\frac{\binom{-2}{n}(\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-2-n}{2}} \sin((2+n)\arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})}))\Gamma^{(n)}(1)}{\pi t} + \end{aligned}$$

$$\begin{aligned}
& \sum_{n=0}^{\infty} \frac{5 \binom{-1}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-1-n}{2}} \sin((1+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(2)}{16\pi t^2} - \\
& \frac{21 \binom{-2}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-2-n}{2}} \sin((2+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(2)}{32\pi t^2} + \\
& \frac{(-3) \binom{-3}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-3-n}{2}} \sin((3+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(2)}{2\pi t^2} + \\
& \sum_{n=0}^{\infty} \frac{23 \binom{-1}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-1-n}{2}} \sin((1+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(3)}{288\pi t^3} - \\
& \frac{451 \binom{-2}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-2-n}{2}} \sin((2+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(3)}{1728\pi t^3} + \\
& \frac{13 \binom{-3}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-3-n}{2}} \sin((3+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(3)}{32\pi t^3} - \\
& \frac{(-4) \binom{-4}{n} (\pi^2 + \ln(\frac{4t}{\zeta^2})^2)^{\frac{-4-n}{2}} \sin((4+n) \arctan(\frac{\pi}{\ln(\frac{4t}{\zeta^2})})) \Gamma^{(n)}(3)}{4\pi t^3} \tag{A16}
\end{aligned}$$