Influence of system parameters and external noise on hysteresis characteristics of a horizontal Rijke tube

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The influence of system parameters such as heater power and heater location on the hysteresis characteristics of a horizontal Rijke tube is studied in this paper. It is observed that the hysteresis zone is present for all the mass flow rates considered in the present study. The nature of transition to instability in a horizontal Rijke tube is found to be subcritical, in the range that we tested. A decrease in instability amplitude along with a reduction in the width of the hysteresis zone is observed in the presence of external noise. Period-2 oscillations are found when heater location is chosen as the control parameter.

Key words: Hysteresis, Subcritical Hopf bifurcation, Rijke tube, period-2 oscillations

1 Introduction

Thermoacoustic instability hampers the development of gas turbine engines, solid rocket motors, industrial burners and various other engineering systems where the primary source of energy is derived from combustion [1]. The instability occurs when the pressure fluctuations inherently present in a confinement are in phase with the heat release rate fluctuations of a heat source present in the same confinement [2]. The physical reasons of the origin of this instability need to be understood in order to implement effective control strategies. The systems which are susceptible to thermoacoustic instability are often too intricate to conduct a detailed investigation. This creates the need for a prototypical system which is simple enough to investigate, yet retains the essential dynamical features of the original system. A horizontal Rijke tube with a mesh type electrical heater is often chosen as a model system in literature [3–8].

In order to avoid the harmful oscillations, the stability boundaries need to be identified. Traditionally this is done by calculating the eigenvalues of the system. The system is said to be unstable when the real part of one of the eigenvalues becomes positive. Linear stability analysis will provide critical values of the system parameters beyond which any small perturbation will grow and make the system unstable. When the amplitude of acoustic oscillations increases and eventually gets saturated, the system is said to exhibit limit cycle oscillations [3–7].

In the case of a horizontal Rijke tube, it can be seen from earlier literature that after a critical value of the system parameter, the system changes its behavior qualitatively from a stable state to an unstable state. Sudden change in the qualitative behavior of a system for an infinitesimal change in a system parameter is termed as a bifurcation in the dynamical system literature [9]. If an existing steady state of the system becomes unstable as a result of a bifurcation, then that bifurcation is termed a primary bifurcation. Primary bifurcations can be classified into two categories, namely static and dynamic. The classification depends upon the manner in which the existing steady state loses stability. Stability of an asymptotic state can be determined by knowing the eigenvalues.

The steady state becomes unstable when the eigenvalues in the left half of the complex plane cross the imaginary axis and reach the right half of the complex plane. If this cross over happens through the origin, the bifurcation is termed as static and it will not introduce any oscillatory behavior. Dynamic bifurcation will be the result when the cross over happens not through origin. The resulting state will have eigenvalues with non-zero imaginary parts. The real part of an eigenvalue represents the growth or decay of the time dependent solution and the imaginary part represents the oscillating component of the solution. So a non-zero imaginary part of an eigenvalue indicates that the solution is oscillatory in nature and has an associated frequency. A new frequency will be introduced as a result of dynamic bifurcation. If the dynamic bifurcation results in the birth of limit cycle oscillations in the asymptotic limit, it is called as Hopf bifurcation [10].

Hopf bifurcation can be classified in to two categories (i) supercritical and (ii) subcritical based on the stability of the resulting periodic solutions. Supercritical bifurcation happens when the nonlinearity has a stabilizing influence and it is characterized by the birth of low amplitude stable limit cycles when the system becomes linearly unstable. The asymptotic state of the system is independent of the initial conditions in the case of supercritical bifurcation. If the nonlinearity destabilizes the system, then a subcritical bifurcation results. Small amplitude unstable limit cycles are born when the system is close to the linear stability boundary. The unstable limit cycles will grow in amplitude as we move away from linear stability boundary. As the system parameters reach some critical value, unstable limit cycles get stabilized through a fold bifurcation away from the linear stability boundary [9, 11]. In the case of a Rijke tube, the asymptotic state of the system is dependent on initial conditions [6, 8]. In short, if the transition is subcritical then the system can be made unstable even when it is linearly stable, if appropriate initial conditions are provided, such that the system is stable to perturbations of small amplitude but becomes unstable when a disturbance of finite amplitude is provided.This phenomenon where the system is made unstable by providing suitable initial conditions is called triggering [12]. Triggering was observed in earlier experiments and numerical studies conducted on horizontal Rijke tubes [4, 6, 7, 13–15].

Linear stability analysis cannot predict the limit cycle characteristics and the linear stability boundaries become less useful if the system can be triggered to instability in the bistable region. The type of bifurcation, i.e., whether the bifurcation is supercritical or subcritical, needs to be understood to implement effective control strategies. Since a subcritical bifurcation is characterized by the presence of large amplitude oscillations at the onset of instability, it is difficult to deal with. Furthermore, in the case of a subcritical transition, the system cannot be brought back to stable state even if the system parameter is reduced just below the critical value. The value of the system parameter has to be reduced much below the critical value to make the system stable again, due to hysteresis associated with subcritical bifurcation [9, 11]. So determining the type of bifurcation which the system undergoes during instability becomes all the more important.

The presence of a hysteresis zone is an important characteristic of subcritical transitions, where the system can remain in more than one state. Within the hysteresis zone, the system can either remain in a state of no-oscillations or it can remain in an oscillatory state. In the hysteresis zone, the system is said to 'bistable' as it can either remain stable or can become unstable depending upon the initial conditions. The width of the hysteresis zone, which represents the parameter range where the system is bistable, can be used as an indicator to identify the nature of bifurcation. The presence of hysteresis zone confirms that the transition to instability is subcritical [9, 16]. Study of variation in the width of hysteresis zone becomes significant in this context.

The earlier experiments conducted on horizontal Rijke tubes were aimed at understanding the effect of heater power on stability characteristics of the system. Matveev [3] performed experiments in a horizontal Rijke tube and established the stability boundaries for different mass flow rates when heater power was selected as the control parameter. Presence of a hysteresis zone and the phenomenon of triggering were reported by Matveev [3]. Further he found that the width of the hysteresis zone decreased with decrease in mass flow rate.

During instability, the system was found to exhibit limit cycle oscillations. Later, Mariappan [7] experimentally determined the triggering amplitude and reported reduction in the width of the hysteresis zone for a decrease in mass flow rate in a horizontal Rijke tube. The transition to instability was found to be subcritical and limit cycle oscillations were observed. Both Matveev [3] and Mariappan [7] used heater power as the control parameter.

Balasubramanian & Sujith [4] proposed a model for the horizontal Rijke tube which captured many of the experimentally observed features and the model showed that the transition to instability is subcritical. The bifurcation characteristics of this model for various system parameters such as heater power, heater location, damping rate coefficient and time lag were studied by Subramanian *et al.* [6] using numerical continuation method.Subramanian *et al.* [6] found that the transition to instability happened always through a subcritical Hopf bifurcation for variation in any of the system parameters. Juniper [14] showed that the transition to instability is subcritical when heater power is chosen as the control parameter and found out the most dangerous initial condition using adjoint optimization. In summary, the experimental and numerical investigations conducted on horizontal Rijke tubes indicate that the transition from non-oscillatory to oscillatory state is subcritical in nature [3, 4, 6, 7]. Nonetheless experimental studies on horizontal Rijke tube where system parameters other than heater power are varied are not present in the literature.[1]

The studies discussed till now were conducted in the absence of noise. Then again, noise is invariably present in all real time systems. External noise can stabilize the system or it can lead to various noise-induced transitions [19]. Zinn & Lieuwen [20] reported that thermoacoustic systems are prone to triggering, where even small amplitude perturbations of the order of background noise can trigger a linearly stable system to instability. Waugh *et al.* [21] analyzed the behavior of the Rijke tube model of Balasubramanian & Sujith [4] with stochastic forcing and found that triggering is dependent on noise strength as well as heat release rate. They reported that the state of the system can change from a stable periodic solution to a state of no oscillations if the noise strength is high enough. Wuagh & Juniper [15] conducted numerical studies on a hot wire Rijke tube model and obtained stochastic stability maps. Jegadeesan & Sujith [18] showed that for a ducted non-premixed flame, noise-induced transitions to instability are possible. They also obtained the deterministic and stochastic stability boundaries. In the absence of external noise, a finite amplitude disturbance which corresponds to the frequency of first eigenmode, was used to trigger the system. The amplitude of the disturbance which triggered the system to instability was termed as the 'triggering amplitude'. In the presence of external noise, transition to instability was reported even when the noise level is much below the 'triggering amplitude'. A reduction in oscillatory amplitude in the presence of noise was also observed by Jegadeesan & Sujith [18] and was attributed to the decrease in correlation between pressure oscillations and heat release rate fluctuations in the presence of external noise. Experimental and numerical studies conducted in the presence of external noise show that noise affects the system dynamics but the effect of noise on the bifurcation characteristics of the system need to be understood with the help of further investigation. In particular, we are interested in seeing how noise affects the bistable region. Furthermore, experimental results on the effect of external noise on the dynamics of horizontal Rijke tube are not available to the best of authors' knowledge.

The major outcomes from the previous studies conducted on horizontal Rijke tube can be summarized as follows. The stability boundaries were experimentally determined for various values of heater power. Numerical studies have provided globally stable, globally unstable and bistable regions for various system parameters such as heater power, heater location, damping coefficient rate and time lag. The nature of transition observed in both experimental and numerical studies was subcritical Hopf bifurcation. Triggering was reported in experiments and in numerical studies. Noise-induced transitions were seen in studies involving Rijke tube models as well as in experiments performed on vertical Rijke tube.

Experimental studies detailing the effects of heater location, damping and mass flow rate are not reported in literature. The subcritical nature of transition observed in the case of Rijke tube model are not yet confirmed by experimental observations for system parameters such as heater location and damping rate coefficient.

[1] However experimental investigations that vary flame location are available in the case of vertical Rijke tubes which use premixed and non-premixed flames as the heat source. [17, 18]

Although some experimental studies allude to the reduction in the width of the hysteresis zone with decrease in mass flow rate [3,7], further investigations were not performed. Width of hysteresis zone plays an important role in identifying the type of bifurcation. Determination of the criticality of bifurcation, i.e., whether the transition is subcritical or supercritical is important as it will help to device better control strategies. It is essential to analyze the influence of system parameters on the presence of bistable region in the context of a horizontal Rijke tube in order to get a clear idea about nature of transition.

The present work aims at bridging the gap that exists in the experimental investigations of bifurcations in horizontal Rijke tube. Dynamical characteristics of the system for various parameters such as heater power and heater location are investigated. The effect of external noise on the stability of the system is also explored. An attempt is made to understand the nature of criticality of bifurcations by investigating the influence of system parameters and external noise on the hysteresis characteristics of the horizontal Rijke tube.

The rest of the paper is arranged as follows. Section 2 describes experimental setup and experimental procedure used for the present study. Section 3 highlights the results obtained. Major conclusions and significant findings are explained in Section 4.

2 Experimental setup

The experimental setup consists of a horizontal Rijke tube with a mesh type electric heater. The Rijke tube is 1 m long with a cross-sectional area of 93 x 93 mm^2. It is made of aluminum plates of 7 mm thickness. The mean flow is established with the help of a blower (1 HP, Continental Airflow Systems, Type CLP-2-1-650) which works in the suction mode. The flow rate is measured with the help of a compact-orifice mass-flow meter (Rosemount 3051 SFC) which is located upstream of the blower. The flow meter can measure a maximum mass flow rate of 5 g/s with an uncertainty of $\pm$2.1%.To eliminate the interaction between the acoustics of the blower and the Rijke tube, a decoupler (120 x 45 x 45 cm^3) is provided in between at the outlet end of the Rijke tube.

A programmable DC power supply (TDK -Lambda, GEN8-400, 0-8 V, 0-400 A) is used to power the mesh type heater. The mesh type electrical heater used in the present study is similar to the one used by Matveev [3] and Mariappan [7]. The uncertainty associated with heater power measurement, happens to be 1-2 W, which includes the uncertainty in the measurement of voltage and uncertainty in the measurement of current. The advantage of the mesh type heater is that it can provide high amount of electric power ($\sim kW$) for a fairly long duration of time ($\sim$ 6 hours) without losing its integrity. A mesh-type heater also ensures uniform heating when compared to a coil type heater. In order to avoid electrical contact with the walls of the tube and also to prevent heat loss to the walls of the tube, a ceramic housing is provided. A traversing mechanism with a least count of 1 mm is used to change the heater location.

The experimental setup used for the present work is similar to the one used by Mariappan [7]. A pressure transducer (PCB 103B02) mounted at 30 cm (from the left end) is used to measure the acoustic pressure. The sensitivity of the pressure transducer happens to be 217.5 mV/kPa and the uncertainty involved in the pressure measurement is 0.2 Pa. An acoustic driver unit (Ahuja AU 60) mounted at 62.5 cm (from the left end) is used to apply the external noise. Data is acquired with the help of a National Instruments make PCI 6221 data acquisition card. The data is acquired at a sampling frequency of 10,000 Hz. Thirty thousand data points are taken in each sample. The bin size used for obtaining the frequency spectra is 0.3 Hz.

In order to assure uniform ambient conditions, the relative humidity level in the laboratory, from where the blower is sucking air, was maintained at a range of 40% - 50% during all experiments. The initial temperature was maintained at 19^0 C. Similarly the experiments were conducted only when the cold flow decay rate 'α' was -18.5$\pm$5% s^{-1} for the fundamental frequency. The cold flow decay rate is measured by exciting the system using a loud speaker; at the first eigenmode frequency for a short duration of time. Once the loud speaker is switched off, the acoustic pressure decays down. By taking the Hilbert transform of the pressure signal and by calculating the logarithmic decay the cold flow decay rate 'α' is found [7].

In all the experiments, a bifurcation parameter is varied in fine steps till the system attains its oscillatory state from a state of no oscillations and then decreasing the bifurcation parameter to bring the system back to a non-oscillatory state. When a particular parameter, say heater power, is chosen as the bifurcation parameter, the other parameters, say heater location, mass flow rate and cold flow decay rate are kept constant during a single experiment. Heater power (K) and heater location (x_f) are chosen as the bifurcation parameters in the current set of experiments. Bifurcation experiments performed by varying the value of heater power are obtained for different mass flow rates ($\dot{m}$) namely 1.25 g/s, 1.41 g/s, 1.56 g/s, 1.88 g/s, 2.03 g/s, 2.19 g/s, 2.34 g/s and 2.50 g/s. Heater location is varied continuously for eight different mass flow rates, i.e., for 1.25 g/s, 1.41 g/s, 1.56 g/s, 1.88 g/s, 2.03 g/s, 2.19 g/s,2.34 g/s and 2.97 g/s. Heater location is measured from the inlet of the duct. For the experiments involving excitation with external noise, Gaussian white noise created using LabVIEW SignalExpress was applied to the system. This is done with the help of a loud speaker driven by a noise signal which is generated using LabVIEW SignalExpress. This experiment is performed for different values of noise amplitude.

3 Results

3.1 Effect of heater power

Here the effect of changing the heater power on the system dynamics is explained. The experiments are performed by slowly varying the power supplied to the electrical heater. The system is preheated for 20 minutes and the median value of acoustic pressure amplitude (P) and the value of heater power (K) are noted down after preheating. The preheating is done in order to lessen the variations in temperature as the heater power is increased [3]. Thereafter the heater power is increased in a quasi-steady manner. Input voltage to the electrical heater is increased in steps of 0.01 V which corresponds to an increase in electrical power of 2-3 W. If the heater power is increased rapidly, it can cause nonlinear triggering of instability. To avoid this nonlinear triggering of instabilities a settling time of 2 minutes is imposed between the power increments [3, 7]. During this time the system achieves steady state which is confirmed by the steady temperature noted by the thermocouple. Power increment used in the current investigation is 2-3 W, which is comparable to power increment used by Matveev [3]. However when the heater power is varied in a fine manner, power increment happens to be 0.5 W.

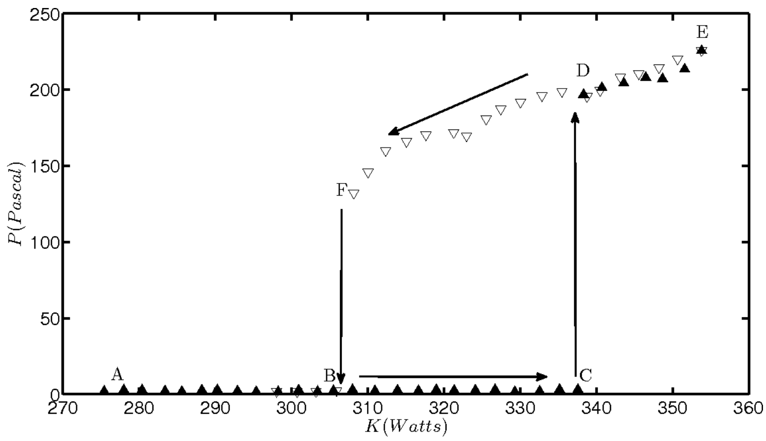

Figure 1: Bifurcation diagram displaying the values of acoustic pressure at $x = 30\ cm$ versus the power supplied to heater. The heater is located at $x_f = L/4$. Mass flow rate $\dot{m} = 2.34$ g/s. ▲-Increasing K,▽-Decreasing K

Figure 1 shows the bifurcation diagram obtained by varying the heater power (K). The median amplitude of acoustic pressure (P) is plotted against the heater power (K). During the *forward* *path*, i.e., while K is increased, the system is stable till the point C which corresponds to a heater power of 337 W. Further increase in heater power takes the system to a stable limit cycle (point D). From D to E, the amplitude of the limit cycle oscillations increases with increase in heater power. Once the system has reached point E, the heater power is decreased in steps of 2-3 W. The asymptotic state achieved by the system during the decrease of heater power is termed as *return path*. The system continues in the state of stable limit cycle oscillations up to point F during the return path. When the heater power is reduced below 308 W, the system reverts back to the non-oscillatory state. In the forward path, the system is globally stable for low values of K (line AB). Region BC is termed as bistable where the system can be triggered to instability. Beyond the point C, the system is globally unstable. The difference observed between the forward path (ABCDE) and the return path (EDFBA) establishes the hysteresis zone and similar results are reported in literature [3, 7].

The asymptotic state of the system can be understood by reconstructing the phase space from the acquired time series.Phase space of a dynamical system is the one which represents all possible states of the system. In general, the phase space will be an 'n' dimensional vector space constructed using 'n' state variables. The state variables can be identified from the governing equations of the system. If the governing equations are not known, the phase space can be constructed using indirect methods [22]. One of them happens to be the method of using time-delayed vectors. These time-delayed vectors are constructed using Takens' embedding theorem, from time series data of one of the physical variables [23]. Time delayed vectors are constructed by calculating the optimum time delay. The dimension of the reconstructed phase space will be determined by knowing the embedding dimension. The technique of reconstruction of phase space from experimentally obtained time series data is explained by [17] in the context of ducted laminar premixed flame.

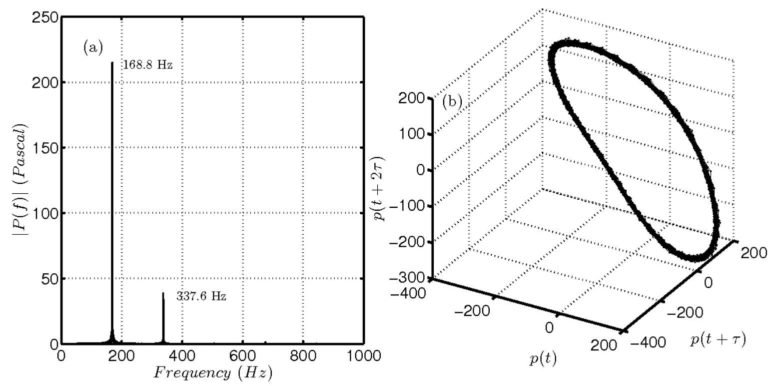

Figure 2: (a) Frequency spectra of pressure signal and (b) Reconstructed phase portrait from measured pressure time series for heater power K=339 W. Heater is located at $x_f = L/4$. Mass flow rate is $\dot{m} = 2.34$ g/s.

FFT of the pressure time series signal along with corresponding phase plot is shown in Figure 2 for various values of heater power. It can be seen that a peak appears when the power is increased to 339 W (Fig. 2a). The corresponding reconstructed phase space shows an isolated closed orbit which is a limit cycle (Fig. 2b).

Presence of the hysteresis zone (Fig. 1) and the limit cycle in the reconstructed phase space (Fig. 2b) confirm that the bifurcation is subcritical Hopf bifurcation. In order to understand the effect of mass flow rate on the dynamics of the system, the experiment is performed for different mass flow rates. When the bifurcation plots for low and high mass flow rates are compared, it can be seen that the hysteresis zone is clearly visible in the case of high mass flow rates whereas it becomes unobservable for low mass flow rates (Fig. 3). It appears as if for low mass flow rates, the forward and reverse paths appear to merge together (Fig. 3a). Even though the hysteresis zone is not observable, a discrete jump in acoustic pressure can be seen during the transition for low mass flow rate (Fig. 3a). The sudden jump observed in the acoustic pressure confirms that the transition is subcritical even for low mass flow rates [9]. Since subcritical transitions are characterized by

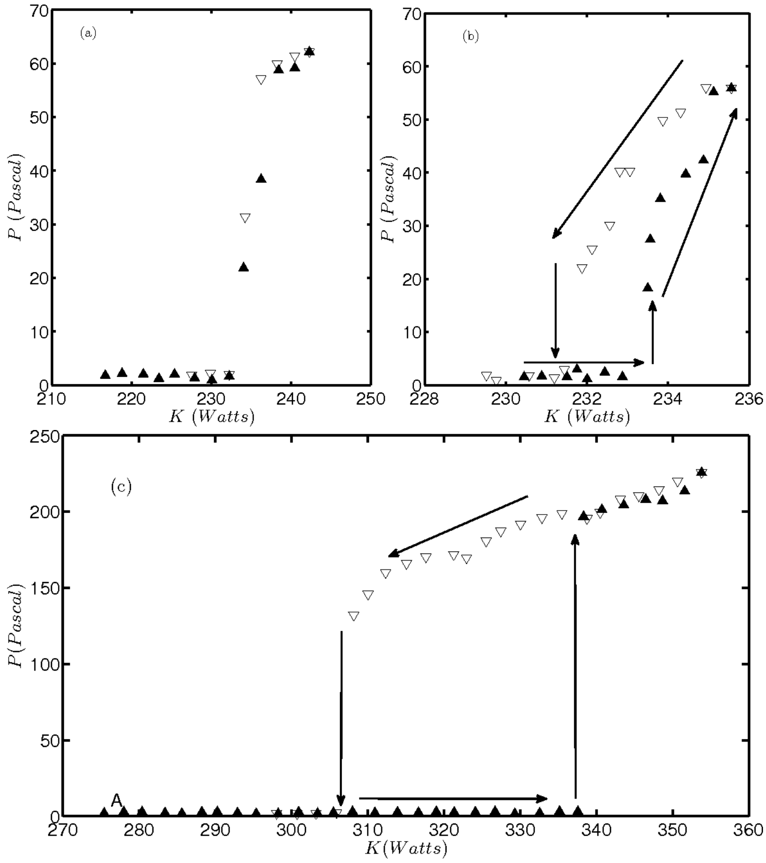

Figure 3: Bifurcation diagram displaying the values of acoustic pressure at $x = 30\ cm$ versus the power supplied to heater. The heater is located at $x_f = L/4$. Mass flow rate is (a) $\dot{m} = 1.25\ g/s$.(b) $\dot{m}$ =1.25 g/s with step size of 0.5 W. (c) $\dot{m}$ =2.34 g/s.▲-Increasing K,▽-Decreasing K

the presence of a hysteresis zone, we performed the experiments with fine variation in control parameter, to detect the hysteresis zone present near the transition point. The heater power is varied in a quasi-steady manner with a step size of 0.5 W. Bifurcation diagram with fine variation in heater power is depicted in Fig. 3b. It can be seen that in the case of finer variation in control parameter the hysteresis zone is clearly observable.

Similar set of experiments were performed for mass flow rates 1.41 g/s and 1.56 g/s. For all these three mass flow rates, i.e, for 1.25 g/s, 1.41 g/s and 1.56 g/s the hysteresis zone is not detectable when the heater power is varied in a coarse manner with a step size of 2 W. However there exists a definite jump in the value of acoustic pressure near the transition. When the step size is reduced to $0.5W$ the hysteresis zone became clearly observable for all the three mass flow rates. The variation in non-dimensional hysteresis width with mass flow rate is shown in Fig. 4. The width of the hysteresis zone is calculated by noting down the difference between the heater power at the Hopf point (K_H) and the heater power at the fold point (K_f). Hysteresis width, (K_H – K_f), is non-dimensionailized by dividing it with K_f. The width of the hysteresis decreases as the mass flow rate is reduced. Nevertheless the width of the hysteresis zone reaches an asymptotic value with reduction in mass flow rates (Fig. 4). A finite value of hysteresis width along with a discrete jump in acoustic pressure near the transition point, even when the mass flow rate is low, show that the transition is definitely subcritical for the mass flow rates considered [9]. The presence of hysteresis zone indicates that the system has a bistable regime and that the system

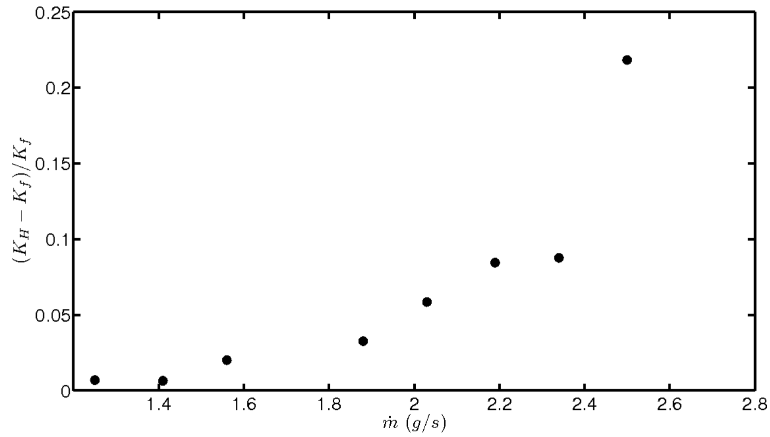

Figure 4: Variation of non-dimensional hysteresis width with mass flow rate when the heater power K is chosen as the control parameter. The heater is located at $x_f = L/4$.

can be triggered to instability by providing appropriate initial conditions.

When heater power is selected as a control parameter, the nature of transition is subcritical. The thermoacoustic system always has a bistable region, in the entire range of parameters that we covered, where the system is linearly stable but can be driven to instability by providing suitable initial conditions.

3.2 Effect of heater location

The effect of heater location on the hysteresis characteristics of a horizontal Rijke tube is discussed here. Experiments were conducted by varying the heater location in a quasi-steady manner. System is preheated for 20 minutes to reduce the variation in temperature along the duct. The heater is located at the inlet end before the start of the experiment. The median value of the acoustic pressure amplitude (P) and the value of heater location (x_f) were recorded after preheating. Heater location is changed in steps of 1 cm and a settling time of one minute is chosen. The heater location is measured from the inlet. When the heater is located at the inlet end x_f is considered as zero. The variation in acoustic pressure with variation in heater location is shown in Fig. 5.

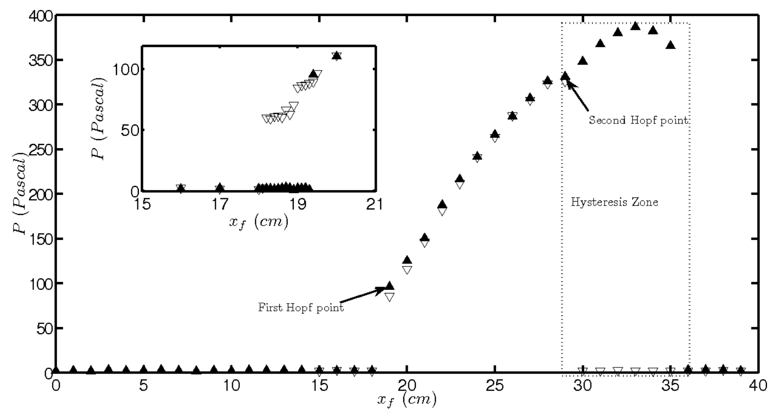

Figure 5: Bifurcation diagram displaying the values of acoustic pressure at $x = 30\ cm$ versus the location of the heater x_f. The variation of acoustic pressure with heater location near the first Hopf point is shown in the inset. The heater power is $K = 423\ W$. Mass flow rate $\dot{m}$ =2.34 g/s.▲- Increasing x_f, ▽- Decreasing x_f

During the forward path, when x_f is increased, the system is stable until x_f becomes 19 cm. The system undergoes a Hopf bifurcation at $x_f = 19$ cm. Thereafter the amplitude of pressure oscillations increases and reaches a maximum at $x_f = 33$ cm. Further increase in x_f causes a decrease in the pressure amplitude and system goes back abruptly to the non-oscillatory state when $x_f = 36$ cm. While in the reverse path, when x_f is decreased, the system remains stable till $x_f = 29$ cm. When the heater is located at 29 cm away from the inlet, the system undergoes a Hopf bifurcation and reaches a state of stable limit cycle oscillations. As the heater is moved towards the inlet, x_f is decreased; the amplitude of oscillations decreases and the system reverts to the non-oscillatory state when the heater is located at 18 cm away from the inlet. A clear hysteresis zone is present near the second Hopf point (29 cm). The hysteresis zone is not observable near the first Hopf point (19 cm) for coarse variation in heater location. However with fine variation in heater location the hysteresis zone near the first Hopf point also becomes detectable as shown in the inset of Fig. 5. It can also be seen from the inset of Fig. 5 that there exist a definite jump in the value of acoustic pressure near the point of transition. Figure 6 shows the frequency spectra

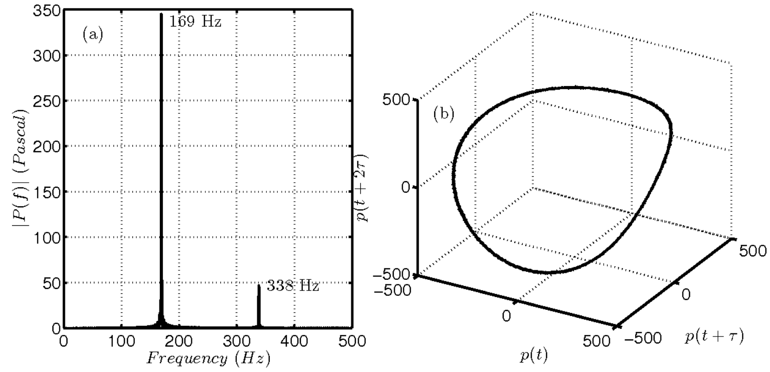

Figure 6: (a) Frequency spectra of pressure signal and (b) reconstructed phase portrait from pressure time series when heater is located at $x_f = 19$ cm. Heater power is $K = 423$ W. Mass flow rate is $\dot{m} = 2.34$ g/s.

and phase plot of the system when the heater is located at 19 cm, i.e. at first Hopf point. The frequency profile shows a prominent frequency and the phase plot is a limit cycle. So it can be concluded that the bifurcation happens at $x_f = 19$ cm is Hopf bifurcation. The frequency spectra

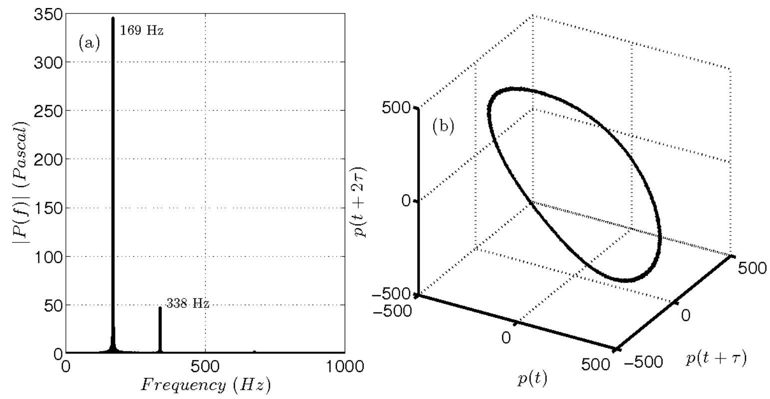

Figure 7: (a) Frequency spectra of pressure signal and (b) Reconstructed phase portrait from pressure time series when heater is located at $x_f = 29$ cm. Heater power is $K = 423$ W. Mass flow rate is $\dot{m} = 2.34$ g/s.

and phase plot at $x_f = 29$ cm during the return path also show a prominent frequency and limit cycle respectively (Fig. 7). It again indicates that the system undergoes Hopf bifurcation when the heater is located at 29 cm during the return path. The presence of the hysteresis zone along with the presence of discrete jump in the value of acoustic pressure near the transition point ensures that the transition is subcritical for both first and second Hopf points (19 cm and 29 cm). The experiments were performed for different values of mass flow rate. A comparison is shown in Fig. 8 between the bifurcation diagrams obtained for 1.25 g/s (Fig. 8a) and for 2.34 g/s (Fig. 8b). When the mass flow rate is decreased the hysteresis zone near the second Hopf point becomes undetectable and the forward and reverse paths appear to merge together. Even when the hysteresis zone becomes undetectable, the amplitude of acoustic pressure is sufficiently above the noise floor during transition. This jump observed in the amplitude of acoustic pressure confirms that the transition is subcritical [9]. The variation in the control parameter, heater location, is made finer to detect the hysteresis zone as in the case of heater power. As the heater location is varied in a finer manner, with a step size of 1 mm, near the Hopf point, the hysteresis zone becomes observable for the case of 1.25 g/s. The bifurcation diagrams with fine variation in

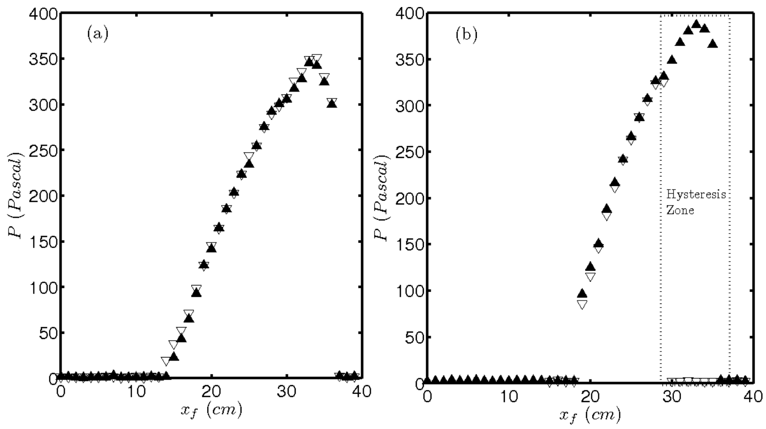

Figure 8: Bifurcation diagram displaying the values of acoustic pressure (P) at $x = 30$ cm versus the location of the heater (x_f). The heater power is $K = 423$ W. Mass flow rate is (a)$\dot{m}$ =1.25 g/s. (b) $\dot{m}$ =2.34 g/s.▲- Increasing x_f; ▽- Decreasing x_f .

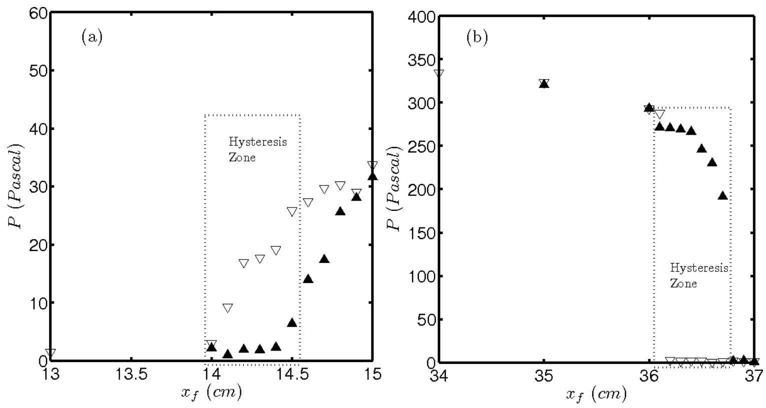

Figure 9: Bifurcation diagram displaying the values of acoustic pressure at $x = 30$ cm versus the location of the heater.(a) Near the first Hopf point.(b) Near the second Hopf point. The heater power is $K = 423$ W. Mass flow rate =1.25 g/s. ▲- Increasing x_f; ▽- Decreasing x_f .

control parameter are shown in Fig. 9. It is observed that the hysteresis zone is detectable for a fine variation of the control parameter. Since the width of the hysteresis zone is much smaller than the overall range in which the control parameter is varied, only the portion of the hysteresis zone near the Hopf point is shown in Fig. 9. Even for a low mass flow rate of 1.25 g/s, it can be concluded that the transition to instability is clearly subcritical when heater location is chosen as the control parameter. The subcritical nature of transition to instability is confirmed by the presence of hysteresis zone and a discrete jump in the values of acoustic pressure during transition (Fig. 9a & Fig. 9b) [9]. The experiment is performed for different mass flow rates by varying the heater location in a fine manner.[2] It is found that there exists a hysteresis zone of definite width even for very low mass flow rates. However, for the case of low mass flow rates, the hysteresis zone becomes perceptible only when the parameter variation is made finer. Variation in the non-dimensionailized width of the hysteresis zone with mass flow rate is depicted in Fig. 10. The width of the hysteresis zone is obtained by noting down the difference between the values of heater location at the Hopf point and heater location at the fold point. The non-dimensional hysteresis width is obtained by dividing the width of the hysteresis zone with value of heater location at fold point. The width of the hysteresis zone decreases and asymptotically tends to a constant value. The presence of a hysteresis zone confirms that the transition is subcritical for all the mass flow rates considered in the present study. However, the width of the hysteresis zone achieves a constant value after a particular mass flow rate. The asymptotic nature of the hysteresis width indicates that the transition is always subcritical irrespective of the values of the mass flow rate in all the experiments that we have performed. The experiment is performed for a mass flow rate

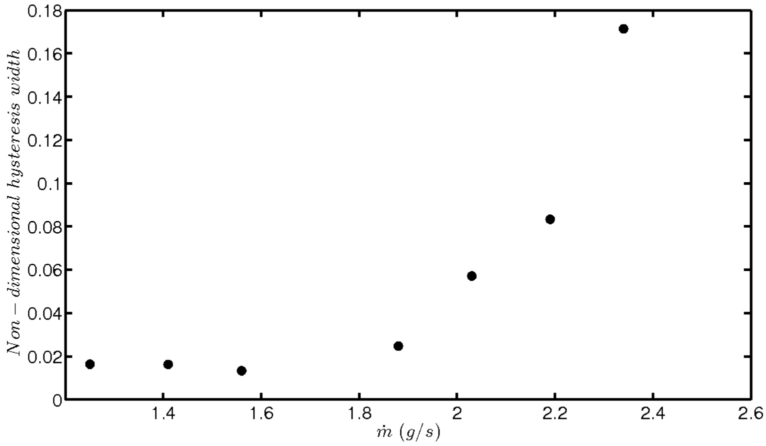

Figure 10: Variation of the non-dimensional width of the hysteresis zone near the second Hopf point with mass flow rate when heater location x_f is chosen as the control parameter. The heater power is $K = 423\ W$.

of 2.97 g/s to understand the system dynamics at a higher mass flow rate. The power supplied to the heater in the earlier experiments was found to be insufficient to make the system unstable at a mass flow rate of 2.97 g/s. So the heater power is increased from its previous value of 423 $W$to 482 W. Since the heater power was changed, the results for the mass flow rate of 2.97 g/s is presented separately. It can be observed that there are two distinct regimes of heater location for which oscillations are present (Regions-1&2 in Fig. 11). The absence of oscillations near the open end is due to the fact that the acoustic pressure becomes zero at the open end. Since the acoustic velocity becomes zero at $L/2$, thermoacoustic instability does not occur as the heater is moved near this point.

When the heater is located at 8 cm from the inlet, system undergoes a subcritical Hopf bifurcation. The subcritical nature of transition can be confirmed by the discrete jump in the value of

[2] In the current set of experiments, instability was not observed when the heater is located beyond 40 cm for all the mass flow rates considered for a given heater power of 423 W. This motivated us to display the bifurcation diagrams only for variations in x_f up to 40 cm.

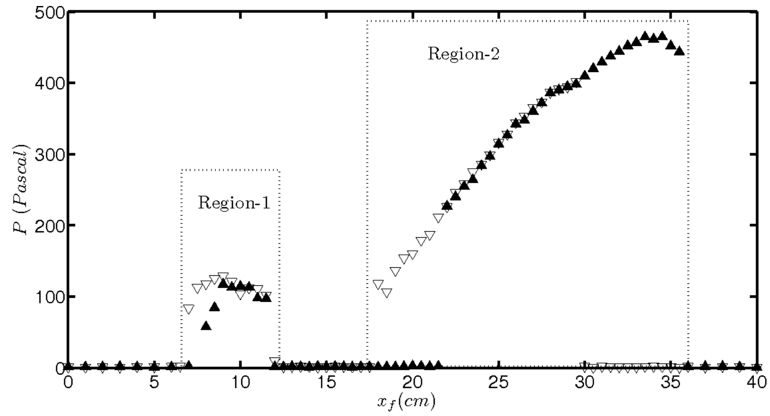

Figure 11: Bifurcation diagram displaying the values of acoustic pressure (P) at x= 30 cm versus the location of the heater x_f for a mass flow rate $\dot{m}$ = 2.97 g/s. K = 482 W. ▲- Increasing x_f; ▽- Decreasing x_f.

acoustic pressure when xf is 8 cm (Region-1 in Fig. 11). Figure 12 shows the frequency spectra of pressure time series and the corresponding reconstructed phase space when the heater is located at 8 cm away from the inlet. The presence of a distinct frequency (Fig. 12a) and the presence of a limit cycle in the reconstructed phase space (Fig. 12b) confirm that the system undergoes a subcritical Hopf bifurcation. Further, when x_f is varied from 8 cm to 12 cm period-2 oscillations

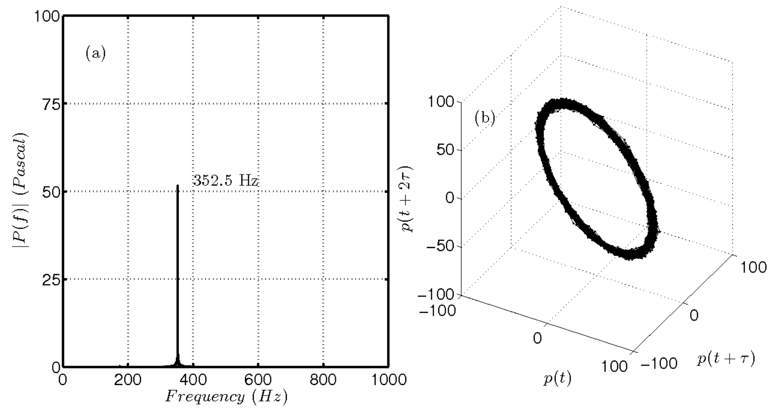

Figure 12: (a) Frequency spectra of pressure signal and (b) Reconstructed phase plot from pressure time series when heater is located at x_f = 8 cm. Heater power is K = 482 W. Mass flow rate is $\dot{m}$ = 2.97 g/s.

were observed. Period-2 oscillations appear when the heater is located at 10 cm. The presence of period-2 orbit can be clearly seen from Figure 13. A new frequency of the oscillations gets introduced as the heater is moved from 9.5 cm to 10 cm (Fig. 13a). The value of the new frequency happens to be exactly half of the existing one. This marks the onset of period-2 oscillations. The phase plot pertinent to the aforementioned heater location represents a double loop (Fig. 13b). This also confirms the presence of period-2 oscillations. When period-2 oscillations occur, for each value of the control parameter there should exist two distinct values of the local maxima. Figure 14 depicts the local maxima of the pressure time series with heater location. It can be observed that the local maxima has a single value till x_f = 9 cm. After that two branches are born. The presence of two distinct branches is a characteristic feature of period-2 oscillations [9]. These two branches represent the two distinct values of local maxima of pressure time series. Before the onset of period-2 oscillations the acoustic pressure has single local maxima. Once the period-2

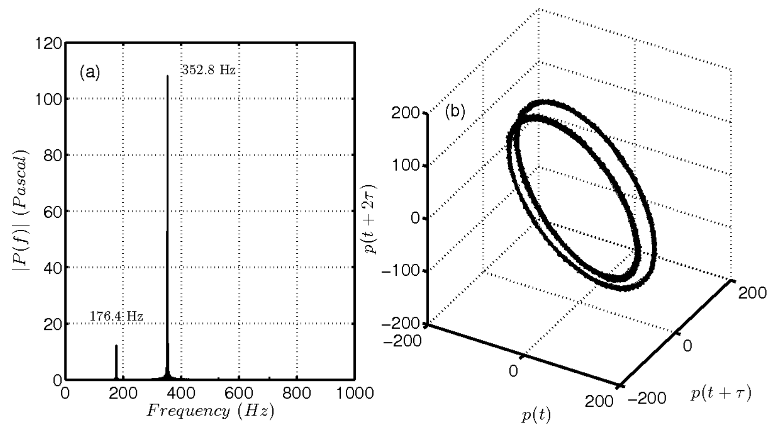

Figure 13: (a) Frequency spectra of pressure signal and (b) Reconstructed phase plot from pressure time series when heater is located at $x_f = 10$ cm. Heater power is $K = 482W$. Mass flow rate is $\dot{m} = 2.97$ g/s.

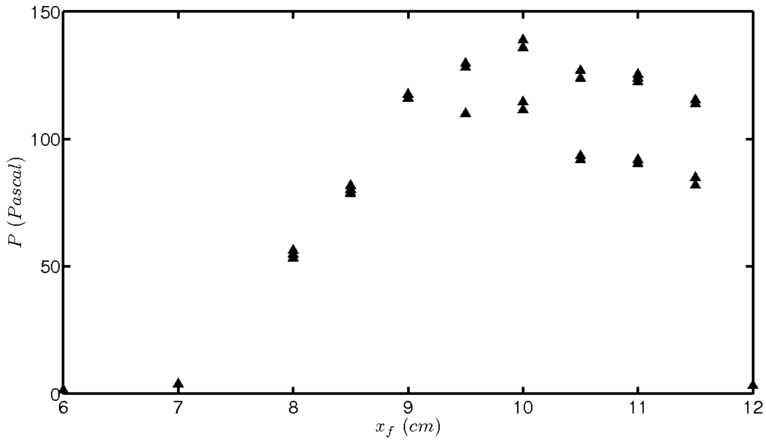

Figure 14: Bifurcation diagram displaying the variation of peak pressure (P) at $x = 30$ cm versus the location of the heater (x_f) during the forward path showing the presence of period-2 oscillations for a mass flow rate $\dot{m} = 2.97$ g/s. $K = 482$ W.

oscillations are set in the local maxima of pressure time series has 2 distinct values.

3.3 Effect of external noise

Noise is invariably present in all practical systems. The dynamics of a system can be greatly influenced by the presence of external noise. Here an attempt is made to understand the influence of external noise on the dynamics of Rijke tube. Figure 15 shows the comparison between bifurcation plots obtained in the presence and absence of external noise when heater power is selected as the control parameter. The noise amplitude is measured by locating the heater at a position where instability is impossible. Then, the pressure amplitude is measured for each value of the external noise applied. Thus the noise amplitude calibration is done. It can be observed that the amplitude of pressure oscillations is reduced in the presence of external noise. The forward path and the return path appear to merge together making the hysteresis zone unobservable. Variation in the width of hysteresis zone with noise amplitude is depicted in Fig. 16a. As the noise amplitude increases the width of the hysteresis zone decreases. Although the width of the hysteresis zone reduces to a minimum value in the presence of external noise, still the transition

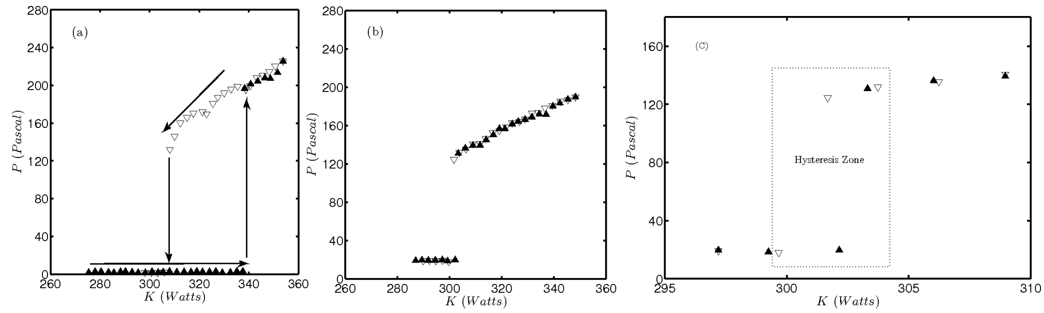

Figure 15: Bifurcation diagram displaying the variation of acoustic pressure (P) with heater power (K) (a) In the absence of external noise (b) in the presence of external noise of amplitude 20 Pa. (c) Near the Hopf point in the presence of external noise of amplitude 20 Pa. Heater is located at $x_f = L/4$. The mass flow rate is $\dot{m} = 2.34\ g/s$.▲- Increasing K; ▽- Decreasing K.

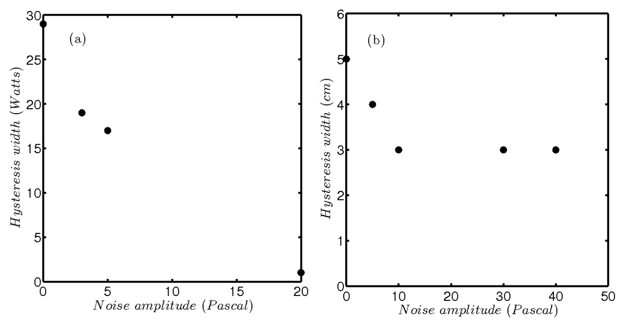

Figure 16: Variation of hysteresis width with Noise amplitude (a) When heater power (K) is chosen as the control parameter. The heater is located at $x_f = L/4$. (b) When heater location (x_f) is chosen as the control parameter. The heater power is $K = 423\ W$. Mass flow rate is $\dot{m} = 2.34\ g/s$.

remains subcritical which is confirmed by a sudden jump in the value of acoustic pressure (Fig. 15). Similar behavior is observed in the case where heater location is chosen as the control parameter (Fig. 16b). In the case of heater location the hysteresis width becomes a constant after particular noise amplitude. The constant value of hysteresis width shows that even in the presence of external noise the transition remains subcritical when heater location is chosen as the control parameter. The major features that are observable in the presence of external noise are (i) an early onset of oscillations (Fig. 17a) (ii) a marked decrease in the oscillation amplitude (Fig. 17b) and (iii) a significant reduction in the width of the hysteresis zone. The early onset of oscillations is due to the phenomenon of noise induced transition which is reported in the literature [15, 18, 24]. The transition to instability depends not only on the noise amplitude but also on the duration of time window of noise application [18]. In the present experiment the settling time was chosen as one minute.

The ability of external noise to lower the instability amplitude is already shown by Jegadeesan & Sujith [18]. The reduction in the width of hysteresis zone has not yet been reported in the case of thermoacoustic systems. However similar results are presented for geophysical systems and for hydrodynamical systems undergoing laminar to turbulent transition [25]. It is suggested that the width of the hysteresis zone can be used as a tool to measure the noise level in the system. The width of the hysteresis zone can be calibrated to the noise level in the system [25].

Even in the presence of external noise the bifurcation remains subcritical when heater power and heater location are used as the control parameters. The subcritical nature is confirmed by

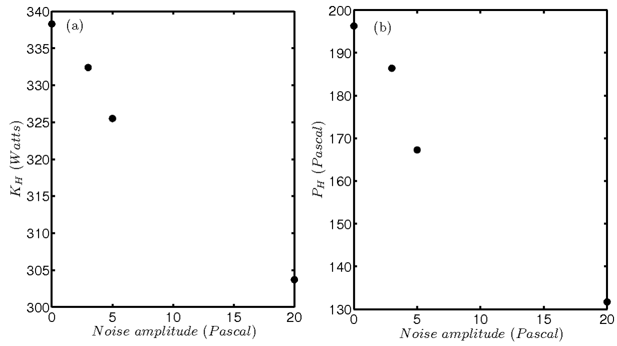

Figure 17: Variation of (a) heater power at the onset of instability (K_H) and (b) acoustic pressure at the onset on instability (P_H) with noise amplitude. The heater is located at $x_f = L/4$. Mass flow rate is $\dot{m} = 2.34\ g/s$.

the presence of a hysteresis zone and a significant jump in the value of acoustic pressure near the transition point [9]. However if the noise levels are higher than that used in the present study,then the subcritical nature of bifurcation need to be confirmed with the help of further experiments.

4 Conclusions

The heater power and the heater location were varied systematically, one at a time, in the presence and in the absence of external noise in the present study. It is found that the width of the hysteresis zone achieves an asymptotic value as the mass flow rate is decreased. The asymptotic nature of the width of the hysteresis zone indicates that irrespective of the value of mass flow rate, there exists a hysteresis zone of finite width in all the experiments we were performed. The presence of the hysteresis zone along with a finite jump in the acoustic pressure near the transition point confirms that the transition is subcritical. So for the system parameters such as heater power and heater location, the nature of transition was always seen to be subcritical in all experiments. For low mass flow rates, although the hysteresis zone was observed only when the control parameter was varied in a fine manner, the finite jump in acoustic pressure near the transition point is always present. Therefore, it is extremely important to ensure that the variation in the parameter is fine enough before a bifurcation can be attributed as supercritical. It is conjectured that the reduction in the width of the hysteresis zone for decreases in mass flow rate is due to reduction in the energy available to drive the system into instability. It is also found that the width of the hysteresis zone decreases in the presence of external noise. We also observed period-2 oscillations when heater location is chosen as the control parameter. The appearance of period-2 oscillations opens up the possibility of Rijke oscillations becoming chaotic through period doubling. Further experiments need to be done in order to confirm this.

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