On Rayleigh's Criterion and The Stability of Premixed Flames

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To date, Rayleigh's criterion [1] is a fundamental tool of stability analysis in thermo-acoustics. In its present forms [2, 3], its applications require accurate description of the acoustic spectrum. An investigation on the thermodynamics [4, 5] of a thin, premixed, convectively cooled flame in a lean subsonic mixture of two components (air and fuel) shows that the search of a macroscopic configuration which is stable (after suitable time-averaging on time-scales » turbulent time-scales) according to Rayleigh's criterion is equivalent to the search of a configuration which satisfies two variational principles, involving both the amounts of entropy produced per unit time by combustion and exchanged with the external world. Both the energy balance, the three components of the momentum balance and the mass balances of air, fuel and the combustion products in a simplified combustion mechanism provide the constraints. The Euler-Lagrange equations and the familiar Landau's jump conditions [6] at the flame lead to a simplified variational principle involving just the velocity of the flow impinging on the flame and the stable flame geometry. We proof instability of both flat flames and flames in convergent ducts, and retrieve the non-linear Sivashinski equation [7] for slightly curved flames.

1. The problem

Most research on thermo-acoustical instabilities relies on a century-old inequality concerning the correlation between perturbations of heating power density P_h and pressure p [2, 8]:

$$\int \langle P_{h1} p_1 \rangle \, d\mathbf{x} > k \int \langle \mathbf{v}_1 p_1 \rangle \cdot d\mathbf{a} \qquad (1)$$

This is the so-called Rayleigh criterion [1]. (A different form [3] of Eq. 1 is discussed below). Here $\langle a \rangle \equiv \lim_{T\to\infty} \frac{1}{T} \int_t^{t+T} a(t')dt'$ denotes the temporal average for the generic physical quantity a, $a(x,t) = a_0(x) + a_1(x,t)$, the integrals on the L.H.S. and the R.H.S. are computed on the bulk and the boundary surface respectively of a region of fluid with velocity $\mathbf{v}$, and k is a positive constant p_0. If Eq. 1 holds instability occurs. Check of stability e.g. in a premixed lean combustor is equivalent to the proof that all possible perturbations violate Eq. 1. However, such proof requires knowledge of the whole spectrum of accessible modes. The aim of the present work is to rewrite Eq. 1 in different forms more suitable to applications. A discussion of the thermodynamics of a small mass element of a fluid mixture of many, reacting chemical species (Sec. 2) under some general assumptions (Sec. 3) leads to general results (Sec. 4), which we apply to a lean premixed flame in Sec. 5-7. Conclusions are drawn in Sec. 8.

2. Thermodynamics of a small mass element

The first and the second principle of thermodynamics in a non-polarised mixture of N chemical species with the same temperature T for all chemical species lead to:

$$du = Tds - pd\left(\frac{1}{\rho}\right) + \sum_z \mu_z^o dc_z \quad (2)$$

$$\left(\frac{\partial u}{\partial T}\right)_{\rho,N} > 0 \quad (3)$$

$$\left(\frac{\partial\left(\frac{1}{\rho}\right)}{\partial p}\right)_{T,N} < 0 \quad (4)$$

$$\sum_{j,z}\left(\frac{\partial \mu_z^o}{\partial c_j}\right)_{p,T} dc_z dc_j \geq 0 \quad (5)$$

(see [9, ch. XV 5,12]). Here u, s and ρ are the internal energy per unit mass, the entropy per unit mass and the mass density respectively. Moreover, $j, z = 1, ...N, \mu_z^o \equiv \mu_z / m_z, c_z \equiv N_z m_z \left(\sum_z N_z m_z\right)^{-1}$, where z, m_z and N_z are the chemical potential, the mass of one particle and the number of particles of the z-th chemical species respectively. Finally, $()_N$ means that all c_z's are kept fixed, and the sign $\geq$ is replaced by $=$ only for $dc_z = 0$. Physically, the very fact that any arbitrary small mass element of the system satisfies Eq. 2-5 at all times puts a constraint on the evolution of the whole system [4]. If the latter relaxes to some final, steady ($\partial/\partial t = 0$) stable state (referred to as 'relaxed state' below) then it is conceivable that Eq. 2-5 may provide us with useful information on the relaxed state. Our aim is to gain such information. (We maintain that the notion of 'steady state' makes sense - possibly after time-averaging on time scales $\gg$ turbulent time scales). Let $da = (da/dt)dt$ where da is the total differential of the generic quantity a and $da/dt = \partial a/\partial t + \mathbf{v}\cdot\nabla a$. Then, Eq. 2-5 lead to the 'general evolution criterion' [5]:

$$\frac{d\left(\frac{1}{T}\right)}{dt}\frac{d(\rho u)}{dt} \leq \rho \sum_z \frac{d\left(\frac{\mu_z^o}{T}\right)}{dt}\frac{dc_z}{dt} + \left[\frac{1}{\rho T}\frac{dp}{dt} + \left(u + \frac{p}{\rho}\right)\frac{d\left(\frac{1}{T}\right)}{dt}\right]\frac{d\rho}{dt} \quad (6)$$

Below, suitable integration of both sides of Eq. 6 on the volume of the system leads to some inequalities which hold in the neighbourhood of a relaxed state with the help of some simplifying assumptions and of the balances of mass and energy. In turn, these inequalities lead to necessary conditions for the stability of the relaxed state against volume-preserving perturbations. We are going to derive these criteria (which in itself are of general interest) and to apply them to a lean premixed flame.

3. Some auxiliary assumptions for the lean premixed flame

In order to apply Eq. 6 to the particular case of a lean premixed flame, we need some further assumptions. We assume $\nabla p_0 = 0$ everywhere. A simplified mechanism A+B $\rightarrow$ C describes combustion: one reaction combines one particle of species A and one article of species B (the 'fuel') to make one particle of species C. The mass density of A, B, and C are ρ_A, ρ_B and $\rho_C = \rho - \rho_A - \rho_B$ respectively. The number of reactions per unit time and volume is $P^* = k\rho_A\rho_B$ where $k = k(T), P_h = P_o P^*$ is the density of combustion power and P_o is constant -see [10, Eq. (12.55)]. Combustion occurs in the flame only, hence $P_h \neq 0$ and $k\rho_B \neq 0$ at the flame only. Flame cooling is convective. Combustion is premixed, hence v is the same for both species A and B. All fluids in the flame are perfect gases with the same specific heat ratio. The flame is thin and its thickness, volume and area are $d_0 = d_0(p_0) > 0, V_f = A_f d_0$ and $A_f = \int_u d\mathrm{a}$ respectively, where $d\mathrm{a}$ is the surface element, the integral is computed on the upstream flame surface and we denote with 'u' = 'upstream' and 'd'= 'downstream' the sides of the flame. Moreover, $\rho_{Bu} > 0, \rho_{Bd} = 0, \rho_{Ad} \approx \rho_{Au} \approx \rho_{Aflame}$ for lean combustion. For simplicity, we neglect ∇P^* inside the flame. We consider slowly moving flames, i.e. we assume that the Mach number $\mathrm{Ma} \equiv c_s^{-1}|\mathbf{v}|$ is $\ll 1$

and neglect terms $\propto \left(\ln V_f\right)_1$. We assume T to be uniform throughout the flame. This is equivalent to neglect high-frequency modes ([10] page 325), and is reasonable in steady-state analysis.

4. The consequences of equation 6

Starting from Eq. 6, we show in [4] that Eq. 1 describes the perturbations of a stable flame which satisfies two variational principles (we skip the subscript '0', unless otherwise stated):

$$\int \frac{1}{T} P_h d\mathbf{x} = \text{min.} \qquad \text{with fixed} \int P_h d\mathbf{x} \tag{7}$$

$$\int \nabla\cdot(\rho s\mathbf{v}) d\mathbf{x} = \text{max.} \qquad \text{with fixed} \int P_h d\mathbf{x} \tag{8}$$

In contrast to Eq. 1, however, Eq. 7 and 8 require no explicit knowledge of the acoustic spectrum. Let the actual value of T be $T_{boundary}$ (uniform throughout the fluid). A lemma of variational calculus, the reciprocity principle for isoperimetric problems (see IX.3 of [11]), ensures that the solution of the latter variational problem solves also

$$\int P_h d\mathbf{x} = \text{min. with } T(\mathbf{x}) = T_{boundary} \text{ everywhere} \tag{9}$$

Finally, we show in Appendix A that Eq. 1 is basically equivalent to [3]

$$\int \langle \frac{P_{h1} T_1}{T} - \frac{p_0 s_{molar1} \mathbf{v}\cdot\nabla s_{molar0}}{c_{p\ molar} R} \rangle d\mathbf{x} > \int \langle \mathbf{v}_1 p_1 \rangle \cdot d\mathbf{a} = 0 \tag{10}$$

which has been derived from a different form of the energy budget. Here $R = 8.31$ J/K and $s_{molar}, c_{p\ molar}$ are the entropy per mole of mixture and the molar specific heat at constant pressure respectively.

5. The consequences of equations 7 and 9

Physically allowable, relaxed flame which satisfy Eq. 7 - 9 must also solve the equations of motion in steady state; the latter provide further constraints. The steady-state balance equations for total mass, mass of species A and B, momentum and energy read:

$$\nabla\cdot(\rho\mathbf{v}) = 0 \tag{11}$$

$$\mathbf{v}\cdot\nabla\rho_A + A^* k\rho_A\rho_B = 0 \tag{12}$$

$$\mathbf{v}\cdot\nabla\rho_B + B^* k\rho_A\rho_B = 0 \tag{13}$$

$$\rho(\mathbf{v}\cdot)\mathbf{v} + \nabla p = 0 \tag{14}$$

$$\nabla\cdot\left[\rho\mathbf{v}\left(\frac{p}{\rho} + \frac{|\mathbf{v}|^2}{2}\right)\right] - P_h = 0 \tag{15}$$

respectively ($A*, B*$ constants > 0). The constraint $\int P_h d\mathbf{x} = P_{tot}$ is satisfied for $P^* = 0$ provided that

$$k\rho_A\rho_B - P^* = 0 \tag{16}$$

Minimisation of $\int T^{-1} P_h d\mathbf{x}$ implies:

$$\int T^{-1} kAB d\mathbf{x} = \text{min.} \tag{17}$$

Introducing 8 Lagrange multipliers $\mu, \zeta, \pi, \xi, \nu$ and λ for the constraints Eq. 11 - 16, Eq. 17 reduces to:

$$\int L d\mathbf{x} = \text{min.} \tag{18}$$

$$
\begin{aligned}
L &= \frac{k\rho_A\rho_B}{T} + \mu\nabla\cdot(\rho\mathbf{v}) + \zeta\left(\mathbf{v}\cdot\nabla\rho_A + A^* k\rho_A\rho_B\right) + \pi\left(\mathbf{v}\cdot\nabla\rho_B + B^* k\rho_A\rho_B\right) + \xi\left(\rho\mathbf{v}\cdot\nabla\mathbf{v} + \nabla p\right) \\
&+ \nu\left\{\nabla\cdot\left[\rho\mathbf{v}\left(\frac{p}{\rho} + \frac{|\mathbf{v}|^2}{2}\right)\right] - P_h\right\} + \lambda\left(k\rho_A\rho_B - P^*\right)
\end{aligned} \tag{19}
$$

The solutions of Eq. 18 - 19 are the solutions of the system of 16 Euler-Lagrange equations in 16 quantities: $\rho_A, \rho_B, \rho, p, T, \mu, \zeta, \pi, \nu, \lambda$, the 3 components of $\mathbf{v}$ and the 3 components of ξ. We have taken p, T and ρ as independent variables; then, our results do not depend on the detailed equation of state. The Euler-Lagrange equations include the 7 equations of motion 11 - 15. Steady states solve just the latter equations; *stable* steady states solve also the remaining 9 Euler-Lagrange equations containing the Lagrange multipliers. We focus our attention on the neighbourhood of the flame, and take Landau's jump conditions [6]: then $p_d = p_u, \mathbf{v}_{d\parallel} = \mathbf{v}_{u\parallel}$ and $\Delta\mathbf{v}_\perp \equiv \mathbf{v}_{d\perp} - \mathbf{v}_{u\perp} = \alpha\mathbf{v}_{u\perp}$ where $\alpha \neq 0$ is a function of $T_u, \boldsymbol{a}_\perp \equiv (\boldsymbol{a}\cdot\mathbf{n})\mathbf{n}, \boldsymbol{a}_\parallel \equiv \boldsymbol{a} - \boldsymbol{a}_\perp = (\mathbf{n}\wedge\boldsymbol{a})\wedge\mathbf{n}, \boldsymbol{a}$ is an arbitrary vector field and $\mathbf{n}$ is the unit vector normal to the flame and pointing outwards ($\mathbf{n}\cdot\mathbf{v}_d > 0, \mathbf{n}\cdot\mathbf{v}_u < 0, \mathbf{n}\cdot\mathbf{n} = 1$). We show in Appendix B that Eq. 18 - 19 lead to a relationship between $\mathbf{v}_u$ and the shape of a stable flame:

$$
\int_u (\mathbf{v}_u\cdot\mathbf{n})(\nabla\wedge\mathbf{n})_\parallel d\mathrm{a} = 0 \tag{20}
$$

Now, the assumption that both T and P^* are uniform throughout the flame allows us to invoke Eq. 9: $\int Phd\mathbf{x} = P_o P^* d_0 A_f$ =min. at fixed T, hence $A_f = \int_u d\mathrm{a}$ = min. The equations of motion affect this minimization through Eq. 20 (we drop the pedix 'u', unless stated otherwise):

$$
\int d\mathrm{a} = \text{min. with } \int (\mathbf{v}\cdot\mathbf{n})(\nabla\wedge\mathbf{n})_\parallel d\mathrm{a} = 0 \tag{21}
$$

The reciprocity principle [11] ensues that the relaxed state which satisfies Eq. 21 satisfies also:

$$
\int (\mathbf{v}\cdot\mathbf{n})(\nabla\wedge\mathbf{n})_\parallel d\mathrm{a} = \text{max. with fixed } \int d\mathrm{a} \tag{22}
$$

Once $\mathbf{v}$ is known upstream, Eq. 22 provides us with possible shapes of stable flames.

6. Sivashinski

If the flame shape takes the form $g(\mathbf{x})$ = const., then $\mathbf{n} = \nabla g/\left|\nabla g\right|$. Solutions of Eq. 22 are simpler in the axisymmetric case: $\partial/\partial\chi \equiv 0$ in the cylindrical coordinate system $\left\{r, \chi, z\right\}$. We invert $g = g(r,z)$ locally and write $g = z - f(r)$, so that $\nabla g = (-f', 0, 1), f' \equiv df/dr$, and $(\nabla\wedge\mathbf{n})_\parallel = (0, Kf', 0)$ where $K \equiv (1 + \left(f'\right)^2)^{-3/2} f''$ and $dl = (1 + f'^2)^{1/2} dr$ are the curvature and the line element respectively [12] of the curve $z = f(r)$ which belongs to the plane (r,z) and whose rotation generates the flame. The area element of the flame is $d\mathrm{a} = 2rdl$. Since $v = 0$ behind the flame, we introduce the stream function (r,z) such that $v_r = -\frac{1}{r}\frac{\partial\psi}{\partial z}, v_z = \frac{1}{r}\frac{\partial\psi}{\partial r}$ and $\mathbf{v}\cdot\mathbf{n} = \frac{1}{r}\frac{\partial\psi}{\partial l}$. Thus, Eq. 22 reduces to:

$$
\delta\left(\int Kf'd\psi + \int \theta d\mathrm{a}\right) = 0 \tag{23}
$$

(θ Lagrange multiplier). Here we investigate Eq. 23 for a slightly curved flame ($\left|f'\right| \ll 1$) with uniform impinging flow, i.e. with $d\psi = 2\pi v_{z0} r dr$ with constant $v_{z0} \equiv \mathbf{v}\cdot\mathbf{z}$. In this case Eq. 23 reduces to:

$$
\delta\int h'dr - \frac{1}{2}\delta\int Ldr = 0 \tag{24}
$$

where $h \equiv \frac{1}{2}r\left(f'\right)^2 + \frac{1}{2}\mu\left(f'\right)^2 + \frac{3}{8}r\left(f'\right)^4, \mu \equiv \frac{\theta}{v_{z0}}$ and $L \equiv \left(f'\right)^2(1 - \mu r) - \frac{3}{4}\left(f'\right)^4$. The first term in Eq. 24 is the variation of the integral of a total derivative and therefore vanishes. Then:

$$
\int Ldr = 0 \tag{25}
$$

The quantity mu has the dimension of (length)$^{-1}$. For slightly curved flames it is therefore reasonable to take both $|f'| \ll 1$ and $|\mu r| \ll 1$. The two addenda in L have therefore opposite sign, and flat flames ($f' = 0$ everywhere) do not correspond to an extreme value of $\int L dr$; they violate therefore Eq. 25, i.e. are never stable [6]. The Euler-Lagrange equation of Eq. 25 in the Lagrangian coordinate f leads to:

$$f'' - \mu f' \left[1 - \mu r - \frac{9}{2}(f')^2\right]^{-1} = 0 \tag{26}$$

After Taylor expansion, Eq. 26 leads to the intermediate step:

$$f'' - \mu f' - \frac{9\mu}{2}(f')^3 - \frac{81\mu}{4}(f')^5 = \mathcal{O}\left[(\mu r)^2\right] \tag{27}$$

In turn, cumbersome but straightforward algebra shows that Eq. 27 reduces to:

$$g''\left[1 + \mathcal{O}(\sigma f^2)\right] + g'^2\left[1 + \mathcal{O}(\sigma f^2) + \mathcal{O}\left(\left((\ln g')'\right)^2\right)\right] + \sigma + \tau g^{(4)} = \mathcal{O}\left[(\mu r)^2\right] + \mathcal{O}\left(f^{(3)} r^2\right) + \mathcal{O}\left(f^{(4)} r^3\right) \tag{28}$$

where $g \equiv \ln f, \tau \equiv -(81/24)\sigma f^4$ and $\sigma \equiv -\mu g'$ has the dimension of (length)$^{-2}$. Then, for slightly curved flames it is reasonable to neglect terms $\propto \mathcal{O}(\sigma f^2) + \mathcal{O}\left(\left((\ln g')'\right)^2\right)$ in comparison to 1 on the L.H.S and to neglect the R.H.S. of Eq. 28. In deriving Eq. 28 we have tacitly assumed $f \neq 0$; to select the value of f at a given point of the flame is equivalent to select the position of the $z = 0$ axis. Then, equation 28 reduces to the non-linear Sivashinski equation [7] in steady state:

$$G^{II} + \frac{1}{2}\left(G^I\right)^2 + 4G^{IV} = \Gamma \tag{29}$$

where $G \equiv 2g = \ln f^2, G^I \equiv dG/du, u \equiv r/R, \Gamma \equiv -2\sigma R^2 = (81/48)\Lambda^{-2}\mu^2 f_0^4, R \equiv (1/2)\tau^{1/2}$ and $\Lambda \equiv \mu/\sigma$ provided that $f \approx f_0$ constant (in agreement with $|f'| \ll 1$) and that $\tau > 0$, i.e. $\sigma < 0$. Physically, the quantity $\Lambda \equiv \mu/\sigma = -1/g' = -f/f' > 0$ is the decay length of a slightly non-uniform flame profile f. Accordingly, $\sigma < 0$ implies $\mu < 0$. We discuss the physical meaning of $\mu < 0$ in Sec. 7 below.

7. The consequences of equation 8

For a slightly curved flame we write $\int \nabla \cdot (\rho s \mathbf{v})\, d\mathbf{x} \approx Z A_f$, where $Z \equiv \left[(\rho s \mathbf{v})_d - (\rho s \mathbf{v})_u\right] \cdot \mathbf{z} \approx (\rho s v_z)_d - \rho_u s_u v_{z0}$. Multiplication of both sides of Eq. 23 by $(\rho_u v_{z0})^{-1} Z\mu$ gives the following condition for stability:

$$(2\pi \rho_u v_{z0})^{-1} Z\mu\delta \int (\mathbf{v}\cdot\mathbf{n})(\nabla \wedge \mathbf{n})_{\parallel}\, da = -\rho_u^{-1}\mu^2\delta \int \nabla \cdot (\rho s \mathbf{v})\, d\mathbf{x} \tag{30}$$

where we have assumed that μ is a $\approx$ uniform quantity in the limit of slightly curved flames. Our results of Eq. 8 and 22 imply that the relaxed state corresponds to a constrained maximum of both $\int \nabla \cdot (\rho s \mathbf{v})\, d\mathbf{x}$ and $\int (\mathbf{v}\cdot\mathbf{n})(\nabla \wedge \mathbf{n})_{\parallel}\, da$ under the same constraint of fixed $\int P_h d\mathbf{x} \propto A_f$. Then, both $\delta \int \nabla \cdot (\rho s \mathbf{v})$ d $\mathbf{x}$ and $\delta \int (\mathbf{v}\cdot\mathbf{n})(\nabla \wedge \mathbf{n})_{\parallel}\, da$ are < 0 near the relaxed state. Then, stability requires that $(\rho_u v_{z0})^{-1} Z\mu < 0$, and $\mu < 0$ implies $(\rho_u v_{z0})^{-1} Z > 0$. But mass balance in Eq. 11 implies $(\rho v_z)_d = \rho_u v_{z0}$, hence stability requires $s_d - s_u = (\rho_u v_{z0})^{-1} Z > 0$. This is impossible [8] for flames in convergent ducts with perfect gases and $p_d = p_u$.

8. Conclusions

A first-principle analysis of flame thermodynamics [4] shows that Rayleigh's criterion of thermo-acoustics [1] in both its available forms [2, 3] describes perturbations near stable, steady, lean, premixed, subsonic, thin flames cooled by convection in an inviscid fluid (after suitable time-averaging on turbulent time-scales). While solving the steady-state equations of motion, stable flames correspond both to a constrained minimum of the amount of entropy produced per unit time by combustion and to a constrained maximum of the amount of entropy exchanged per unit time with the external world, the constraint being given in both cases by a fixed amount of combustion power. If Landau's jump conditions hold at the flame [6], both temperature and combustion power density are uniform inside the flame, and a simplified mechanism describes combustion, then the surface of stable flame is a constrained minimum, the constraint being given in terms of the velocity of the flow impinging on the flame. In the case of an axisymmetric, slightly curved flame with quasi-uniform impinging flow, we show that both flat flames [6] and flames in convergent ducts [8] are unstable; stable flames satisfy Sivashinski equation [7].

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A. Appendix 1

The condition $\nabla p_0 \approx 0$ and the equation of state for perfect gases give

$$\int \left\langle c_{p\ molar}^{-1} R^{-1} p_0 s_{molar\ 1} \mathbf{v}_1 \cdot \nabla s_{molar\ 0} \right\rangle d\mathbf{x} = \left(2c_{p\ molar} R\right)^{-1} p_0 \int \left\langle \mathbf{v}_1 \cdot \nabla \left(s_{molar}^2\right)\right\rangle d\mathbf{x} \tag{31}$$

$$\int \left\langle T^{-1} P_{h\,1} T_1 \right\rangle d\mathbf{x} = p_0^{-1} \int \left\langle P_{h\,1} p_1 \right\rangle d\mathbf{x} + \int \left\langle \rho P_{h\,1} \left(\rho^{-1}\right)_1 \right\rangle d\mathbf{x} \tag{32}$$

respectively. We write $\left\langle \mathbf{v}_1 \cdot \nabla \left(s_{molar}^2\right)\right\rangle = \left\langle \mathbf{v} \cdot \nabla \left(s_{molar}^2\right)\right\rangle - \left\langle \mathbf{v}_0 \cdot \nabla \left(s_{molar}^2\right)\right\rangle$. In steady state we have $\left\langle \mathbf{v} \cdot \nabla \left(s_{molar}^2\right)\right\rangle = \left\langle d\left(s_{molar}^2\right)/dt\right\rangle = 0$ as s_{molar}^2 is a bounded function of time. In turn, the second-order contribution of $\left\langle \mathbf{v}_0 \cdot \nabla \left(s_{molar}^2\right)\right\rangle$ to Eq. 10 (which is a second-order relationship) is $\left\langle \mathbf{v}_0 \cdot \nabla \left(s_{molar\ 1}^2\right)\right\rangle = \left\langle \mathbf{v} \cdot \nabla \left(s_{molar\ 1}^2\right)\right\rangle + \mathcal{O}$ (third order terms), and in steady state $\left\langle \mathbf{v} \cdot \nabla \left(s_{molar\ 1}^2\right)\right\rangle = \left\langle d\left(s_{molar\ 1}^2\right)/dt\right\rangle = 0$. As for Eq. 32, $\int \left\langle \rho P_{h\,1} \left(\rho^{-1}\right)_1 \right\rangle d\mathbf{x} \approx \left\langle \int \rho P_{h\,1} \left(\rho^{-1}\right)_1 d\mathbf{x} \right\rangle = \left\langle P_{h\,1} \int \rho \left(\rho^{-1}\right)_1 d\mathbf{x} \right\rangle = \left\langle P_{h\,1} V_f \left(d \ln V_f\right)\right\rangle \propto \mathcal{O}\left(\left(\ln V_f\right)_1\right)$ because of $V_f \left(d \ln V_f\right) = dV_f = dt(dV_f/dt) = dt \int \nabla \cdot \mathbf{v} d\mathbf{x} = dt \int \rho (d\rho^{-1}/dt) d\mathbf{x} = \int \rho \left(\rho^{-1}\right)_1 d\mathbf{x}$. Thus, $p_0^{-1} \int \left\langle dP_h dp \right\rangle d\mathbf{x}$ is the only remaining term on the L.H.S. of Eq. 10, which is therefore basically equivalent to Eq. 1.

B. Appendix 2

In the following, we invoke the identities $\boldsymbol{a} \wedge (\nabla \wedge \boldsymbol{b}) = (\nabla \boldsymbol{b}) \cdot \boldsymbol{a} - \boldsymbol{a} (\nabla \cdot \boldsymbol{b})$, $\nabla (\boldsymbol{a} \cdot \boldsymbol{b}) = \boldsymbol{a} \wedge (\nabla \wedge \boldsymbol{b}) + \boldsymbol{b} \wedge (\nabla \wedge \boldsymbol{a}) + (\boldsymbol{a} \cdot \nabla) \boldsymbol{b} + (\boldsymbol{b} \cdot \nabla) \boldsymbol{a}$, $\nabla \wedge (\boldsymbol{a} \wedge \boldsymbol{b}) = \boldsymbol{a} (\nabla \cdot \boldsymbol{b}) - \boldsymbol{b} (\nabla \cdot \boldsymbol{a}) + (\boldsymbol{b} \cdot \nabla) \boldsymbol{a} - (\boldsymbol{a} \cdot \nabla) \boldsymbol{b}$ and $\nabla \wedge \nabla a = 0$ for arbitrary vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ and scalar a. Replacement of L with $L + \nabla \cdot \boldsymbol{a}$ leaves Eq. 18 unaffected. If we choose $\boldsymbol{a} = \nabla \cdot \nu \left\{\left[(1/3)\mathbf{x} P_h - \rho \mathbf{v}\left(\rho^{-1} p + |\mathbf{v}|^2/2\right)\right]\right\}$, Eq. 18 gives the 9 equations:

$$\nu = \nu_0 \tag{33}$$

$$\rho_A \rho_B \left[\left(T^{-1} + \zeta A^* + \pi B^* + \lambda\right)(dk/dT) - T^{-2} k\right] = 0 \tag{34}$$

$$k \rho_B \left(T^{-1} + \zeta A^* + \pi B^* + \lambda\right) - \nabla \cdot (\zeta \mathbf{v}) = 0 \tag{35}$$

$$k \rho_A \left(T^{-1} + \zeta A^* + \pi B^* + \lambda\right) - \nabla \cdot (\pi \mathbf{v}) = 0 \tag{36}$$

$$\nabla \cdot \xi = 0 \tag{37}$$

$$\xi \cdot (\upsilon \cdot \nabla) \mathbf{v} - \mathbf{v} \cdot \nabla \mu = 0 \tag{38}$$

$$\rho \xi \wedge (\nabla \wedge \mathbf{v}) + \rho \nabla \wedge (\mathbf{v} \wedge \xi) + \rho \xi \nabla \cdot \mathbf{v} + \zeta \nabla \rho_A + \pi \nabla \rho_B - \rho \nabla \mu = 0 \tag{39}$$

where v_0 is uniform across the system. Equation 33 is decoupled from the other equations, and will therefore be invoked no more in the following. Equation 34 is solved by $\lambda = U^{-1} - T^{-1} - \xi A^* - \pi B^*$ where $U \equiv -d(\ln k)/d(1/T)$. Physically, the flow is incompressible ($Ma \ll 1$) everywhere outside the flame; combustion heating induces expansion on the flame. Mathematically, we write $\zeta = \zeta_0$ where ζ_0 is uniform across the system so that Eq. 33 reduces to:

$$\nabla \cdot \mathbf{v} = \zeta_0^{-1} U^{-1} \rho_{A\,flame}^{-1} P^* \ (\neq 0 \text{ at the flame only}) \tag{40}$$

Now, let us look for the solution of Eq. 37 in the form (β constant quantity, ϕ scalar field):

$$\xi = \beta \mathbf{v} - \nabla \phi \tag{41}$$

where Eq. 40 makes ϕ to satisfy Poisson's equation of the electrostatic potential created by an electric conductor charged with uniform charge density $\propto \zeta_0^{-1} K^{-1} \rho_{A\,flame}^{-1} P^*$. Equations 36, 38, 39, 41 and the condition $\nabla p = 0$ correspond therefore to:

$$\nabla \phi = \text{const. } \mathbf{n} \text{ (at the flame surface)} \tag{42}$$

$$\mu = 1/2\beta |\mathbf{v}|^2 \tag{43}$$

$$2(\mathbf{v} \cdot \nabla)(\nabla \phi) = +\nabla(\mathbf{v} \cdot \nabla \phi) - \rho^{-1} \zeta_0 \nabla \rho_A - \rho^{-1} \pi \nabla \rho_B \tag{44}$$

Landau's jump conditions, Eq. 11 - 13 and Eq. 36 ensure that $\nabla \rho \parallel \nabla \rho_A \parallel \nabla \rho_B \parallel \nabla \pi \parallel \mathbf{n}$. Then, the R.H.S. of Eq. 44 is curl-free; moreover, both T_u and α are uniform behind the flame, as the equation of state of perfect gases holds everywhere and p is uniform. (Now, Eq. 36 is decoupled from the other equations, and will therefore be invoked no more in the following). Then, it follows from Eq. 42 and 44 that

$$\nabla \wedge \left[(\mathbf{v} \cdot \nabla) \nabla \phi \right] = 0 \tag{45}$$

The identity $\int d\mathbf{x} \nabla \wedge \boldsymbol{a} = \int d\mathbf{a} \wedge \boldsymbol{a}$ holds for arbitrary $\boldsymbol{a}$. After volume integration, Eq. 45 gives Eq. 42:

$$\begin{aligned} 0 &= \int d\mathrm{a}\, \mathbf{n} \wedge (\mathbf{v} \cdot \nabla) \mathbf{n} = \int_d d\mathrm{a}\, \mathbf{n} \wedge (\mathbf{v} \cdot \nabla) \mathbf{n} + \int_u d\mathrm{a}\, \mathbf{n} \wedge (\mathbf{v} \cdot \nabla) \mathbf{n} = -\int_u d\mathrm{a}\, \mathbf{n} \wedge (\Delta \mathbf{v}_\perp \cdot \nabla) \mathbf{n} = \\ &= -\alpha \int_u d\mathrm{a}\, (\mathbf{v}_u \cdot \mathbf{n}) \mathbf{n} \wedge (\mathbf{n} \cdot \nabla) \mathbf{n} = \alpha \int_u d\mathrm{a}\, (\mathbf{v}_u \cdot \mathbf{n}) \mathbf{n} \wedge (\mathbf{n} \wedge \nabla \wedge \mathbf{n}) = -\alpha \int_u d\mathrm{a}\, (\mathbf{v}_u \cdot \mathbf{n}) (\nabla \wedge \mathbf{n})_\parallel \end{aligned} \tag{46}$$

where $d\mathbf{a} = \mathbf{n}\, d\mathrm{a}$ and we invoked Eq. 42, $\mathbf{n} \cdot \mathbf{n} = 1$ and the fact that α is uniform.

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