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***Abstract*—Since improper (noncircular) complex signals require adequate tools such as widely linear filtering, a generalized likelihood ratio test has been proposed in the literature to verify whether or not a given signal is improper. This test is based on the augmented complex formulation, which is sometimes regarded as the most convenient way of handling improper signals. In this paper, we show that a derivation of an impropriety test is also possible without making use of the augmented complex representation. Instead, we us a composite real formulation, and we apply the recently proposed framework of block-skew-circulant matrices. It turns out that this alternative derivation is of practical relevance since it reveals a computationally more efficient implementation of the impropriety test by avoiding redundant matrix structures. The paper is concluded by a discussion of redundancy in the description of second order statistical properties of complex random vectors.**

I. Introduction

In the past, methods to process complex signals have often been derived assuming so-called proper [1] random signals, i.e., signals which do not have any correlations or power imbalances between real and imaginary parts. However, in technical systems, this assumption is often not justified. For instance, many coding and modulation schemes in communication systems (e.g., BPSK, ASK, or GMSK) lead to transmit signals that do not fulfill the conditions for propriety (see, e.g., [2], [3]). Moreover, hardware imperfections such as I/Q imbalance can lead to improper received signals [4].

To adequately treat such improper signals, so-called widely linear processing (see, e.g., [2], [3], [5], [6]) is required, i.e., linear processing of the signal and its complex conjugate or, equivalently, linear processing of the real and imaginary parts of a signal. It has been shown that widely linear processing can achieve significant gains over conventional linear processing (e.g., [4], [7]–[17]), for instance, in terms of mean square error or in terms of data rates in a communications system.

Therefore, it makes sense to have a tool at hand that can classify a given signal as proper or improper. In [2], [18], the question was asked how such a classification can be done if only noisy measurements are available. The authors developed a generalized maximum likelihood ratio test for impropriety of complex signals based on the augmented complex representation, where a complex random vector and its conjugate are stacked in a complex vector of twice the original dimension. They acknowledge that a generalized maximum likelihood ratio test is not generally optimal in the sense of Newman-Pearson, but they point out that the concept is widely accepted in practice due to its reliable performance [19], and they demonstrate the usefulness of the particular test derived in their work by means of simulation results.

The augmented complex representation, on which the derivation of the test relies, is often considered as the most convenient framework for studying improper complex signals (see, e.g., [2], [3], [20]–[23]). Therefore, a large variety of mathematical tools has been developed for this formulation (e.g., [2], [3], [6], [18], [20]–[23]). An alternative way of treating improper signals is the so-called composite real representation, where real and imaginary parts of complex vectors are stacked in real-valued vectors. This formulation had for a long time been disregarded when it comes to developing mathematical tools. However, it has recently been proposed to exploit properties of so-called block-skew-circulant matrices and block-Hankel-skew-circulant matrices (see Section II) when working with the composite real representation [24]. Moreover, [24] contained many examples of applications where the composite real representation in combination with the new framework is advantageous compared to the augmented complex representation. However, there still exist many problems for which researchers have proposed solution approaches in the augmented complex representation, but not in the composite real representation.

One of these problems is testing a complex signal for impropriety. In this paper, we show that this is another application where the framework of block-skew-circulant matrices delivers an improved solution compared to the existing augmented complex approach. We apply the framework in order to derive the generalized likelihood ratio test (GLRT) for impropriety based on a composite real formulation in Section III. We then show that with this alternative derivation, we can obtain the same test results as with the augmented complex impropriety test (see Section IV), but at a reduced computational complexity (see Section V). This advantage in the implementation makes the approach proposed in this paper beneficial from a practical point of view.

Moreover, considering this alternative derivation is interesting from a theoretical point of view, since it is another step towards better understanding differences and common points of the augmented complex and the composite real formulations. In particular, we discuss in Section VI that the differences in computational complexity are caused by redundancy in the representations of the second order statistical properties of complex random vectors. It turns out that elegant formulations of these properties seem to always come at the cost of introducing redundancy, but we show that the composite real representation makes it possible to find a good compromise by making redundancy appear only in some steps of a derivation, but not in the final result.

Notation: We write $\bullet^{\mathrm{T}}$, $\bullet^{*}$, $\bullet^{\mathrm{H}}$, $\Re$ and $\Im$ to denote transpose, complex conjugate, conjugate transpose, real part and imaginary part, respectively. We use a tilde $\underset{\sim}{\bullet}$ below complex quantities to easily distinguish them from real quantities. The shorthand notations

$$\check{\boldsymbol{x}} = \begin{bmatrix} \Re(\underset{\sim}{\boldsymbol{x}}) \\ \Im(\underset{\sim}{\boldsymbol{x}}) \end{bmatrix} \quad \text{and} \quad \underline{\boldsymbol{x}} = \begin{bmatrix} \underset{\sim}{\boldsymbol{x}} \\ \underset{\sim}{\boldsymbol{x}}^{*} \end{bmatrix} \tag{1}$$

are used for composite real vectors and augmented complex vectors, respectively.

II. Block-Skew-Circulant Matrices

In [24], it was proposed to study composite real representations using so-called block-skew-circulant ($\mathcal{BSC}$) and block-Hankel-skew-circulant ($\mathcal{BHSC}$) matrices with 2×2 blocks. The $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ structures (the subscript refers to the number of blocks) are given by

$$\grave{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_1 & -\boldsymbol{A}_2 \\ \boldsymbol{A}_2 & \boldsymbol{A}_1 \end{bmatrix} \quad \text{and} \quad \acute{\boldsymbol{B}} = \begin{bmatrix} \boldsymbol{B}_1 & \boldsymbol{B}_2 \\ \boldsymbol{B}_2 & -\boldsymbol{B}_1 \end{bmatrix} \tag{2}$$

respectively [24]. Since a $\mathcal{BSC}_2$ matrix is a block-Toeplitz matrix while a $\mathcal{BHSC}_2$ matrix is block-Hankel, we denote these matrices using grave and acute accents $\grave{\bullet}$ and $\acute{\bullet}$, which mimic the shape of the constant northwest-to-southeast and southwest-to-northeast diagonals in Toeplitz and Hankel structures, respectively. We indicate the block size by a superscript where necessary.

One of the most notable properties of $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ matrices is that any matrix $\boldsymbol{C}$ can be uniquely decomposed into a sum of a $\mathcal{BSC}_2$ component $\grave{\boldsymbol{C}}$ and a $\mathcal{BHSC}_2$ component $\acute{\boldsymbol{C}}$, i.e., $\boldsymbol{C} = \grave{\boldsymbol{C}} + \acute{\boldsymbol{C}}$. The reason for this is that the subspace $\mathcal{BSC}_2^{K\times L}$ is the orthogonal complement of $\mathcal{BHSC}_2^{K\times L}$ in $\mathbb{R}^{2K\times 2L}$ [24].

The $\mathcal{BSC}_2$ component $\grave{\boldsymbol{C}}_{\check{\boldsymbol{x}}}$ of the composite real covariance matrix

$$\boldsymbol{C}_{\check{\boldsymbol{x}}} = \mathrm{E}[\check{\boldsymbol{x}}\check{\boldsymbol{x}}^{\mathrm{T}}] - \mathrm{E}[\check{\boldsymbol{x}}]\,\mathrm{E}[\check{\boldsymbol{x}}^{\mathrm{T}}] = \begin{bmatrix} \boldsymbol{C}_{\Re\underset{\sim}{\boldsymbol{x}}} & \boldsymbol{C}_{\Re\underset{\sim}{\boldsymbol{x}}\Im\underset{\sim}{\boldsymbol{x}}} \\ \boldsymbol{C}_{\Re\underset{\sim}{\boldsymbol{x}}\Im\underset{\sim}{\boldsymbol{x}}}^{\mathrm{T}} & \boldsymbol{C}_{\Im\underset{\sim}{\boldsymbol{x}}} \end{bmatrix} \tag{3}$$

completely describes the power shaping of the orginal complex signal, which is usually described by the complex covariance matrix $\underset{\sim}{\boldsymbol{C}}_{\underset{\sim}{\boldsymbol{x}}} = \mathrm{E}[\underset{\sim}{\boldsymbol{x}}\underset{\sim}{\boldsymbol{x}}^{\mathrm{H}}]$ [24]. On the other hand, the $\mathcal{BHSC}_2$ component $\acute{\boldsymbol{C}}_{\check{\boldsymbol{x}}}$ captures the information about the impropriety, which is usually expressed by the pseudocovariance matrix $\tilde{\underset{\sim}{\boldsymbol{C}}}_{\underset{\sim}{\boldsymbol{x}}} = \mathrm{E}[\underset{\sim}{\boldsymbol{x}}\underset{\sim}{\boldsymbol{x}}^{\mathrm{T}}]$. In particular, the $\mathcal{BHSC}_2$ component is zero (i.e., $\boldsymbol{C}_{\check{\boldsymbol{x}}}$ has $\mathcal{BSC}_2$ structure) if the random vector $\underset{\sim}{\boldsymbol{x}}$ is proper [24]. Therefore, testing for impropriety means testing whether or not the $\mathcal{BHSC}_2$ component $\acute{\boldsymbol{C}}_{\check{\boldsymbol{x}}}$ of the composite real covariance matrix vanishes.

This is different from the augmented complex representation where an impropriety test has to check whether the augmented covariance matrix

$$\underset{\sim}{\boldsymbol{C}}_{\underline{\boldsymbol{x}}} = \mathrm{E}[(\underline{\boldsymbol{x}} - \mathrm{E}[\underline{\boldsymbol{x}}])(\underline{\boldsymbol{x}} - \mathrm{E}[\underline{\boldsymbol{x}}])^{\mathrm{H}}] = \begin{bmatrix} \underset{\sim}{\boldsymbol{C}}_{\underset{\sim}{\boldsymbol{x}}} & \tilde{\underset{\sim}{\boldsymbol{C}}}_{\underset{\sim}{\boldsymbol{x}}} \\ \tilde{\underset{\sim}{\boldsymbol{C}}}_{\underset{\sim}{\boldsymbol{x}}}^{*} & \underset{\sim}{\boldsymbol{C}}_{\underset{\sim}{\boldsymbol{x}}}^{*} \end{bmatrix} \tag{4}$$

is block-diagonal.

Other notable properties of $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ matrices shown in [24] are that the $\mathcal{BSC}_2$ structure is preserved by linear combinations, transposition, matrix products, and matrix inversion if all involved matrices are $\mathcal{BSC}_2$ matrices [24, Lemmas 1, 3, and 8]. On the other hand, the product of a $\mathcal{BHSC}_2$ matrix with a $\mathcal{BSC}_2$ matrix is $\mathcal{BHSC}_2$, and it can be shown that the trace of a $\mathcal{BHSC}_2$ matrix is always zero [24, Lemmas 3 and 4]. For later reference, we summarize the combination of the last two properties as follows.

Lemma 1: Let $\grave{\boldsymbol{A}}$ be $\mathcal{BSC}_2$ and $\acute{\boldsymbol{B}}$ be $\mathcal{BHSC}_2$. Then $\mathrm{tr}[\grave{\boldsymbol{A}}\acute{\boldsymbol{B}}] = \mathrm{tr}[\acute{\boldsymbol{B}}\grave{\boldsymbol{A}}] = 0$.

III. A Test for Impropriety of Complex Signals

The aim of this section is to derive a test to verify whether a complex Gaussian random vector $\underset{\sim}{\boldsymbol{s}}$ is proper. We can conclude from the previous section that we have to test the hypothesis

$$H_1 : \acute{\boldsymbol{C}}_{\check{\boldsymbol{s}}} \neq \boldsymbol{0} \quad (\underset{\sim}{\boldsymbol{s}} \text{ is improper}) \tag{5}$$

against the null hypothesis

$$H_0 : \acute{\boldsymbol{C}}_{\check{\boldsymbol{s}}} = \boldsymbol{0} \quad (\underset{\sim}{\boldsymbol{s}} \text{ is proper}). \tag{6}$$

The joint probability density function

$$p\left(\{\underset{\sim}{\boldsymbol{s}}\}_{i=1}^{K}; \boldsymbol{C}_{\check{\boldsymbol{s}}}, \check{\boldsymbol{\mu}}\right) = \frac{1}{\sqrt{2\pi}^{2NK} \det\left(\boldsymbol{C}_{\check{\boldsymbol{s}}}\right)^{\frac{K}{2}}} \cdot \exp\left(-\frac{1}{2}\sum_{k=1}^{K}\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{\mu}}\right)^{\mathrm{T}} \boldsymbol{C}_{\check{\boldsymbol{s}}}^{-1}\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{\mu}}\right)\right) \tag{7}$$

of K independent and identically distributed (i.i.d.) samples $\{\underset{\sim}{\boldsymbol{s}}_i\}_{i=1}^{K}$ is parameterized by the composite real covariance matrix $\boldsymbol{C}_{\check{\boldsymbol{s}}}$ and the composite real mean vector $\check{\boldsymbol{\mu}} = \mathrm{E}[\check{\boldsymbol{s}}]$.

We obtain the test statistic of the GLRT as the ratio of a constrained and an unconstrained maximum likelihood (ML) expression, i.e.,

$$\lambda = \frac{\max\limits_{\substack{\check{\boldsymbol{\mu}}, \boldsymbol{C}_{\check{\boldsymbol{s}}} \succ \boldsymbol{0} \\ \acute{\boldsymbol{C}}_{\check{\boldsymbol{s}}} = \boldsymbol{0}}} p\left(\{\underset{\sim}{\boldsymbol{s}}\}_{i=1}^{K}; \boldsymbol{C}_{\check{\boldsymbol{s}}}, \check{\boldsymbol{\mu}}\right)}{\max\limits_{\check{\boldsymbol{\mu}}, \boldsymbol{C}_{\check{\boldsymbol{s}}} \succ \boldsymbol{0}} p\left(\{\underset{\sim}{\boldsymbol{s}}\}_{i=1}^{K}; \boldsymbol{C}_{\check{\boldsymbol{s}}}, \check{\boldsymbol{\mu}}\right)}. \tag{8}$$

The maximization in the numerator is constrained to composite real covariance matrices with vanishing $\mathcal{BHSC}_2$ component, i.e., it is constrained to the set of proper complex Gaussian distributions. On the other hand, an arbitrary composite real covariance matrix, i.e., any Gaussian distribution (proper or improper), is allowed in the denominator.

The maximum likelihood estimate of the mean vector $\check{\boldsymbol{\mu}}$ is given by the sample mean

$$\check{\boldsymbol{m}} = \frac{1}{K}\sum_{k=1}^{K} \check{\boldsymbol{s}}_k \tag{9}$$

in both cases. Moreover, the unconstrained ML estimate of the covariance matrix is the sample covariance

$$\boldsymbol{R}_{\check{\boldsymbol{s}}} = \frac{1}{K}\sum_{k=1}^{K}\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{m}}\right)\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{m}}\right)^{\mathrm{T}} \in \mathbb{R}^{2N\times 2N}. \tag{10}$$

In the following, we apply the framework of block-skew-circulant matrices to show that the constrained ML estimate in the numerator is given by the orthogonal projection of the sample covariance matrix $\boldsymbol{R}_{\check{\boldsymbol{s}}}$ to the subspace of $\mathcal{BSC}_2$ matrices, i.e., $\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$.

To find this solution, we first plug in $\check{\boldsymbol{\mu}} = \check{\boldsymbol{m}}$ and rewrite the objective function in terms of the sample covariance matrix as

$$p\left(\{\underset{\sim}{\boldsymbol{s}}\}_{i=1}^{K}; \boldsymbol{C}_{\check{\boldsymbol{s}}}, \check{\boldsymbol{m}}\right) = \frac{\exp\left(-\frac{K}{2}\operatorname{tr}[\boldsymbol{C}_{\check{\boldsymbol{s}}}^{-1}\boldsymbol{R}_{\check{\boldsymbol{s}}}]\right)}{\sqrt{2\pi}^{2NK}\det\left(\boldsymbol{C}_{\check{\boldsymbol{s}}}\right)^{\frac{K}{2}}}. \tag{11}$$

Then, we make a change of variables by introducing $\boldsymbol{K} = \boldsymbol{C}_{\check{\boldsymbol{s}}}^{-1}$, and we take the logarithm of the objective function. Due to [24, Lemma 8], the inverse of a matrix is $\mathcal{BSC}_2$ if and only if the original matrix is $\mathcal{BSC}_2$. Thus, the constraint translates to $\mathbb{P}_{\mathcal{BHSC}_2}\left(\boldsymbol{K}\right) = \boldsymbol{0}$, where

$$\mathbb{P}_{\mathcal{BHSC}_2}\left(\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{A}_3 & \boldsymbol{A}_4\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}\boldsymbol{A}_1 - \boldsymbol{A}_4 & \boldsymbol{A}_2 + \boldsymbol{A}_3\\ \boldsymbol{A}_2 + \boldsymbol{A}_3 & \boldsymbol{A}_4 - \boldsymbol{A}_1\end{bmatrix} \tag{12}$$

is the orthogonal projection to the set of $\mathcal{BHSC}_2$ matrices from [24, Lemma 6]. We obtain

$$\max_{\boldsymbol{K}\succ\boldsymbol{0}} \log\left(p\left(\{\underset{\sim}{\boldsymbol{s}}\}_{i=1}^{K}; \boldsymbol{K}^{-1}, \check{\boldsymbol{m}}\right)\right) \ \text{ s.t. } \ \mathbb{P}_{\mathcal{BHSC}_2}\left(\boldsymbol{K}\right) = \boldsymbol{0} \tag{13}$$

and after dropping constant terms and factors from the optimization, we have

$$\max_{\boldsymbol{K}\succ\boldsymbol{0}} \ \log\det\left(\boldsymbol{K}\right) - \operatorname{tr}[\boldsymbol{K}\boldsymbol{R}_{\check{\boldsymbol{s}}}] \ \text{ s.t. } \ \mathbb{P}_{\mathcal{BHSC}_2}\left(\boldsymbol{K}\right) = \boldsymbol{0}. \tag{14}$$

Introducing a Lagrangian multiplier matrix $\boldsymbol{\Lambda}^{\mathrm{T}} \in \mathbb{R}^{2N\times 2N}$ and setting the derivative of the Lagrangian function

$$\begin{aligned} L\left(\boldsymbol{K}, \boldsymbol{\Lambda}\right) = \log\det\left(\boldsymbol{K}\right) &- \operatorname{tr}[\boldsymbol{K}\boldsymbol{R}_{\check{\boldsymbol{s}}}] \\ &+ \operatorname{tr}[\boldsymbol{\Lambda}\,\mathbb{P}_{\mathcal{BHSC}_2}\left(\boldsymbol{K}\right)] \end{aligned} \tag{15}$$

with respect to $\boldsymbol{K}$ to zero, we obtain (see Appendix A)

$$\boldsymbol{0} = \frac{1}{\det\left(\boldsymbol{K}\right)}\det\left(\boldsymbol{K}\right)\left(\boldsymbol{K}^{-1}\right)^{\mathrm{T}} - \boldsymbol{R}_{\check{\boldsymbol{s}}}^{\mathrm{T}} + \acute{\boldsymbol{\Lambda}}^{\mathrm{T}} \tag{16}$$

$$\Leftrightarrow \boldsymbol{K} = \left(\boldsymbol{R}_{\check{\boldsymbol{s}}} - \acute{\boldsymbol{\Lambda}}\right)^{-1} = \left(\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}} + \acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}} - \acute{\boldsymbol{\Lambda}}\right)^{-1} \tag{17}$$

where $\acute{\boldsymbol{\Lambda}}$ is the $\mathcal{BHSC}_2$ component of $\boldsymbol{\Lambda}$, and $\boldsymbol{R}_{\check{\boldsymbol{s}}} = \grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}} + \acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$ is a decomposition into $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ components. To comply with the constraint $\mathbb{P}_{\mathcal{BHSC}_2}\left(\boldsymbol{K}\right) = \boldsymbol{0}$, we have to set $\acute{\boldsymbol{\Lambda}} = \acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$. The constraint $\boldsymbol{K} \succ \boldsymbol{0}$ is inherently fulfilled in this unique solution.

As a result, the constrained maximum likelihood estimate in the numerator of (8) is $\boldsymbol{C}_{\check{\boldsymbol{s}}} = \mathbb{P}_{\mathcal{BSC}_2}(\boldsymbol{R}_{\check{\boldsymbol{s}}}) = \grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$, where

$$\mathbb{P}_{\mathcal{BSC}_2}\left(\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{A}_3 & \boldsymbol{A}_4\end{bmatrix}\right) = \frac{1}{2}\begin{bmatrix}\boldsymbol{A}_1 + \boldsymbol{A}_4 & \boldsymbol{A}_2 - \boldsymbol{A}_3\\ \boldsymbol{A}_3 - \boldsymbol{A}_2 & \boldsymbol{A}_1 + \boldsymbol{A}_4\end{bmatrix} \tag{18}$$

is the projection to the set of $\mathcal{BSC}_2$ matrices from [24, Lemma 6]. Inserting all ML estimates in (8) yields

$$\lambda = \frac{\left(\det\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}\right)^{-\frac{K}{2}}\exp\left(-\frac{K}{2}\operatorname{tr}\left[\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}^{-1}(\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}} + \acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}})\right]\right)}{\left(\det\boldsymbol{R}_{\check{\boldsymbol{s}}}\right)^{-\frac{K}{2}}\exp\left(-\frac{K}{2}\operatorname{tr}\left[\boldsymbol{R}_{\check{\boldsymbol{s}}}^{-1}\boldsymbol{R}_{\check{\boldsymbol{s}}}\right]\right)} \tag{19}$$

where we have again used the decomposition $\boldsymbol{R}_{\check{\boldsymbol{s}}} = \grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}} + \acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$ of the composite real sample covariance matrix. Since the trace of the product of a $\mathcal{BSC}_2$ matrix and a $\mathcal{BHSC}_2$ matrix vanishes (Lemma 1), $\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}^{-1}\acute{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$ can be dropped in the numerator.

For convenience, we introduce a transformed test statistic $r = \lambda^{\frac{2}{K}}$, and we obtain

$$r = \frac{\det\boldsymbol{R}_{\check{\boldsymbol{s}}}}{\det\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}}. \tag{20}$$

The test decides for the alternative hypothesis H_1 ($\underset{\sim}{\boldsymbol{s}}$ is improper) if r is below a given threshold, and for the null hypothesis H_0 ($\underset{\sim}{\boldsymbol{s}}$ is proper) otherwise.

IV. Comparison with the Augmented Complex Impropriety Test

The test statistic of the augmented complex impropriety test derived in [18] is given by

$$l = \frac{\det\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}}{\left(\det\underset{\sim}{\boldsymbol{R}}_{\underset{\sim}{\boldsymbol{s}}}\right)^{2}}. \tag{21}$$

where $\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}$ is the sample covariance matrix of the augmented complex samples $\underline{\boldsymbol{s}} = [\underset{\sim}{\boldsymbol{s}}^{\mathrm{T}}, \underset{\sim}{\boldsymbol{s}}^{\mathrm{H}}]^{\mathrm{T}}$ and $\underset{\sim}{\boldsymbol{R}}_{\underset{\sim}{\boldsymbol{s}}}$ is the sample covariance matrix of $\underset{\sim}{\boldsymbol{s}}$. The structure of the expression resembles the one in the last section, but the involved matrices are different ones, and the denominator is not squared in the composite real version.

To study the relation between the expressions (20) and (21), we make use of the transformation matrix

$$\boldsymbol{T}_N = \begin{bmatrix}\mathbf{I}_N & \mathrm{j}\,\mathbf{I}_N\\ \mathbf{I}_N & -\mathrm{j}\,\mathbf{I}_N\end{bmatrix} \in \mathbb{C}^{2N\times 2N} \tag{22}$$

from [2]. This matrix transforms the composite real representation of a complex vector into the augmented complex representation, i.e., $\underline{\boldsymbol{x}} = \boldsymbol{T}_N\check{\boldsymbol{x}}$, where $\underline{\boldsymbol{x}} \in \mathbb{C}^{2N}$ and $\check{\boldsymbol{x}} \in \mathbb{R}^{2N}$ are defined in (1). Note that $\frac{1}{\sqrt{2}}\boldsymbol{T}_N$ is unitary, so that $\boldsymbol{T}_N^{\mathrm{H}}\boldsymbol{T}_N = 2\mathbf{I}_{2N}$.

For the augmented complex sample covariance matrix, we have

$$\begin{aligned} \underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}} &= \frac{1}{K}\sum_{k=1}^{K}\left(\underline{\boldsymbol{s}}_k - \underline{\boldsymbol{m}}\right)\left(\underline{\boldsymbol{s}}_k - \underline{\boldsymbol{m}}\right)^{\mathrm{H}} \\ &= \frac{1}{K}\sum_{k=1}^{K}\boldsymbol{T}_N\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{m}}\right)\left(\check{\boldsymbol{s}}_k - \check{\boldsymbol{m}}\right)^{\mathrm{T}}\boldsymbol{T}_N^{\mathrm{H}} \\ &= \boldsymbol{T}_N\boldsymbol{R}_{\check{\boldsymbol{s}}}\boldsymbol{T}_N^{\mathrm{H}}. \end{aligned} \tag{23}$$

Therefore,

$$\begin{aligned} \det\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}} &= \det(\boldsymbol{T}_N\boldsymbol{R}_{\check{\boldsymbol{s}}}\boldsymbol{T}_N^{\mathrm{H}}) = \det(\boldsymbol{T}_N^{\mathrm{H}}\boldsymbol{T}_N\boldsymbol{R}_{\check{\boldsymbol{s}}}) = \det(2\boldsymbol{R}_{\check{\boldsymbol{s}}}) \\ &= 2^{2N}\det\boldsymbol{R}_{\check{\boldsymbol{s}}}. \end{aligned} \tag{24}$$

Using the projection to the set of $\mathcal{BSC}_2$ matrices from (18) and the projection to the set of block-diagonal matrices

$$\mathbb{P}_{\text{blockdiag}}\left(\begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_2\\ \boldsymbol{A}_3 & \boldsymbol{A}_4\end{bmatrix}\right) = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{0}\\ \boldsymbol{0} & \boldsymbol{A}_4\end{bmatrix} \tag{25}$$

it is easy to verify by means of basic algebra that

$$\mathbb{P}_{\text{blockdiag}}(\boldsymbol{T}\boldsymbol{A}\boldsymbol{T}^{\mathrm{H}}) = \boldsymbol{T}\,\mathbb{P}_{\mathcal{BSC}_2}(\boldsymbol{A})\,\boldsymbol{T}^{\mathrm{H}} \tag{26}$$

for any $\boldsymbol{A} \in \mathbb{R}^{2N\times 2N}$. Applying this to $\boldsymbol{A} = \boldsymbol{R}_{\check{\boldsymbol{s}}}$, we have that

$$\begin{aligned}\left(\det \underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}\right)^2 &= \det \underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}} \det \underset{\sim}{\boldsymbol{R}}^*_{\underline{\boldsymbol{s}}} = \det\left(\mathbb{P}_{\text{blockdiag}}(\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}})\right)\\ &= \det\left(\mathbb{P}_{\text{blockdiag}}(\boldsymbol{T}_N\boldsymbol{R}_{\check{\boldsymbol{s}}}\boldsymbol{T}_N^{\mathrm{H}})\right)\\ &= \det\left(\boldsymbol{T}_N\mathbb{P}_{\mathcal{BSC}_2}(\boldsymbol{R}_{\check{\boldsymbol{s}}})\boldsymbol{T}_N^{\mathrm{H}}\right)\\ &= \det\left(\boldsymbol{T}_N^{\mathrm{H}}\boldsymbol{T}_N\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}\right) = 2^{2N}\det\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}\end{aligned} \tag{27}$$

We see that the numerator as well as the denominator of r are scaled versions of the respective terms of l with a common scaling factor. This shows that the composite real GLRT is equivalent to the test that was derived in [18] based on the augmented complex formulation, but its derivation has not used any of the mathematical tools of the augmented complex framework.

The augmented complex representation and the composite real representation of complex random vectors are equivalent in the sense that there exists a one-to-one mapping between them, and many problems can be conveniently formulated in either of them. However, there are also many problems considered in the literature for which a solution has been found only using one of them, but not using the other one. The problem of testing for impropriety had been one of them since it had only been solved via the augmented complex representation. Therefore, the alternative derivation presented in this paper is interesting from a theoretical point of view since it shows that a convenient derivation of a GLRT for the impropriety of complex signals is possible using the composite real representation as well.

Moreover, as already mentioned in the introduction, the composite real formulation of the impropriety test is not only of theoretical interest as an alternative derivation, but also helps to develop a test with reduced computational complexity as discussed in the next section.

V. Computational Complexity

To compare the computational complexity of the two approaches, we note that there are three main steps in both cases. The first one is the computation of the augmented complex sample covariance matrix $\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}$ or the composite real sample covariance matrix $\boldsymbol{R}_{\check{\boldsymbol{s}}}$, respectively. As a second step, we either have to extract the sample covariance matrix $\boldsymbol{R}_{\underset{\sim}{\boldsymbol{s}}}$ (upper left block of $\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}$), or we have to perform the projection of $\boldsymbol{R}_{\check{\boldsymbol{s}}}$ to the space of $\mathcal{BSC}_2$ matrices to obtain $\grave{\boldsymbol{R}}_{\check{\boldsymbol{s}}}$. Finally, two determinants have to be computed in both algorithms in order to obtain l and r, respectively.

The computational complexity of the second step can be neglected in both algorithms. This is obvious for the augmented complex version. For the composite real version, we argue that the $\mathcal{O}(N^2)$ additions can be neglected compared to the multiplications that are needed in the other steps. Moreover, the scaling by a factor of $\frac{1}{2}$ can be implemented by bit-shifting in fixed-point arithmetic (or by decreasing the exponent in binary floating-point arithmetic). Therefore, we concentrate on the computation of the sample covariance matrices and of the determinants.

When comparing (10) to the first line of (23), we see that the dimensions of the involved quantities are the same in both cases (outer product of $2N$-dimensional vectors). However, the underlying fields are the real numbers in the first case and the complex numbers in the second case. If a complex multiplication is carried out by performing four real-valued multiplications, implementing this step in the composite real representation saves three fourth of the computational complexity when compared to the augmented complex version.

A similar reasoning holds for the determinants in the numerators of l and r. Both $\underset{\sim}{\boldsymbol{R}}_{\underline{\boldsymbol{s}}}$ and $\boldsymbol{R}_{\check{\boldsymbol{s}}}$ have the same dimensions, but real-valued arithmetics are sufficient to compute $\det \boldsymbol{R}_{\check{\boldsymbol{s}}}$. Thus, we again save one fourth of the computational complexity by choosing the composite real implementation.

To study the determinants in the denominators, we note that the asymptotic complexity of this operations is $\mathcal{O}(N^w)$ with $w > 2$ for an $N \times N$ matrix.[1] Since $2^w > 4$, the computation of the determinant of a real-valued $2N \times 2N$ matrix with a complexity order of $\mathcal{O}((2N)^w)$ is more time-consuming than the computation of the determinant of a complex $N \times N$ matrix if each complex multiplication is implemented as four real-valued ones.

It is therefore more efficient to compute the denominator of the test statistic via the complex covariance matrix $\boldsymbol{R}_{\underset{\sim}{\boldsymbol{s}}}$. Doing so makes sense even if the composite real representation is used since computing the complex sample covariance matrix $\boldsymbol{R}_{\underset{\sim}{\boldsymbol{s}}}$ from the composite real sample covariance using

$$\boldsymbol{R}_{\underset{\sim}{\boldsymbol{s}}} = \boldsymbol{R}_{\Re\underset{\sim}{\boldsymbol{s}}} + \boldsymbol{R}_{\Im\underset{\sim}{\boldsymbol{s}}} + \mathrm{j}(\boldsymbol{R}^{\mathrm{T}}_{\Re\underset{\sim}{\boldsymbol{s}}\Im\underset{\sim}{\boldsymbol{s}}} - \boldsymbol{R}_{\Re\underset{\sim}{\boldsymbol{s}}\Im\underset{\sim}{\boldsymbol{s}}}) \tag{28}$$

with

$$\boldsymbol{R}_{\check{\boldsymbol{s}}} = \begin{bmatrix}\boldsymbol{R}_{\Re\underset{\sim}{\boldsymbol{s}}} & \boldsymbol{R}_{\Re\underset{\sim}{\boldsymbol{s}}\Im\underset{\sim}{\boldsymbol{s}}}\\ \boldsymbol{R}^{\mathrm{T}}_{\Re\underset{\sim}{\boldsymbol{s}}\Im\underset{\sim}{\boldsymbol{s}}} & \boldsymbol{R}_{\Im\underset{\sim}{\boldsymbol{s}}}\end{bmatrix} \tag{29}$$

requires additions only. In order to compute the test statistics via (20), the value of the squared determinant $(\det \boldsymbol{R}_{\underset{\sim}{\boldsymbol{s}}})^2$ then has to be divided by 2^{2N}, which can again be implemented efficiently by bit-shifting (or by decreasing the exponent in binary floating-point arithmetic).

From all these considerations, we conclude that the most

[1]Determinant computation has the same asymptotic complexity as square matrix multiplication [25], for which the best known upper bound is $w_{\text{optimal}} < 2.3728639$ [26] (so-called fast matrix multiplication). However, many implementations use conventional algorithms with $w = 3$.

efficient implementation is to compute

$$r = \frac{2^{2N} \det \boldsymbol{R}_{\breve{\boldsymbol{s}}}}{\left(\det \underset{\sim}{\boldsymbol{R}}_{\underset{\sim}{\boldsymbol{s}}}\right)^2} \tag{30}$$

with $\boldsymbol{R}_{\breve{\boldsymbol{s}}}$ from (10) and $\underset{\sim}{\boldsymbol{R}}_{\underset{\sim}{\boldsymbol{s}}}$ from (28). This reduces the computational complexity roughly by a factor of four when compared to the implementation of the augmented complex version in (21) and reduces the complexity slightly when compared to the composite real version in (20).

VI. Discussion

The fact that a gain in computational complexity compared to the augmented complex formulation can be achieved (see Section V) is a consequence of the inherent redundancy of augmented complex representations. This redundancy is introduced by stacking the complex vector and its conjugate and can be easily recognized by noting that the augmented vector contains twice the number of coefficients as the original complex vector.

By contrast, the composite real vector, which is obtained by stacking real and imaginary parts, has the same number of real-valued coefficients as the original vector: we have again doubled the size of the vector, but at the same time, we have switched from complex to real numbers. Therefore, the composite real vector does not contain any redundancy. Accordingly, the composite real covariance matrix has the same dimension as the augmented complex covariance matrix, but has real-valued elements instead of complex ones.

The redundancy in the augmented complex representation is the price that has to be paid for obtaining convenient mathematical expressions as described, e.g., in [2], [3], [20]. The framework of block-skew-circulant matrices was introduced to obtain elegant mathematical formulations without resorting to augmented representations. Interestingly, this framework introduces the same amount of redundancy: when decomposing the composite real covariance matrix of an improper complex random vector into a $\mathcal{BSC}_2$ and a $\mathcal{BHSC}_2$ component, we obtain just as many real-valued coefficients as in the real and imaginary parts of the augmented covariance matrix. It seems that the price of redundancy always has to be paid for obtaining convenient representations of the second order statistical properties of complex random vectors.

However, when using the composite real representation, a decomposition of composite real covariance matrices into $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ components is usually necessary only for some steps of theoretical derivations while the actual computations can then be performed directly on the composite real covariance matrix (instead of on the decomposed version). An example of this can be seen in this paper, where the theoretical derivation of the test relies on the above-mentioned decomposition while the final computation rule (30) does not contain redundant representations.

Therefore, the composite real framework allows us to perform computations in a more efficient manner than the augmented complex framework.[2] The difference is that the composite real representation does not have an inherent redundancy, but is turned into a redundant formulation only by the above-mentioned decomposition. In contrast to this, redundancy is always present in the augmented complex representation due to the joint consideration of the signal and its conjugate.

If a random vector is known to be proper, its pseudocovariance matrix is zero and the complex covariance matrix suffices to exhaustively describe the second order statistical properties. In this special case, even the composite real covariance matrix has an inherent redundancy since it becomes a block-skew-circulant matrix with the block structure from (2). In the complexity analysis in Section V, we have seen that this redundancy of the blocks in $\mathcal{BSC}_2$ matrices can lead to an increase in computational complexity when compared to computations based on the complex covariance matrix. Obviously, the most efficient representation of the second-order properties of proper random vectors is the conventional covariance matrix. However, using only this covariance is of course not possible in the case of improper signals with nonzero pseudocovariance matrix.

Appendix A
Derivative Involving Projection to BHSC Space

Let $\boldsymbol{K} = \grave{\boldsymbol{K}} + \acute{\boldsymbol{K}}$ and $\boldsymbol{\Lambda} = \grave{\boldsymbol{\Lambda}} + \acute{\boldsymbol{\Lambda}}$ be decompositions into $\mathcal{BSC}_2$ and $\mathcal{BHSC}_2$ components. Note that $\mathrm{tr}[\grave{\boldsymbol{\Lambda}}\acute{\boldsymbol{K}}] = 0 = \mathrm{tr}[\acute{\boldsymbol{\Lambda}}\grave{\boldsymbol{K}}]$ due to Lemma 1. Thus, we have

$$\begin{aligned}
&\frac{\partial}{\partial \boldsymbol{K}} \mathrm{tr}[\boldsymbol{\Lambda}\, \mathbb{P}_{\mathcal{BHSC}_2}(\boldsymbol{K})] \\
&= \frac{\partial}{\partial \boldsymbol{K}} \mathrm{tr}[(\grave{\boldsymbol{\Lambda}} + \acute{\boldsymbol{\Lambda}})\acute{\boldsymbol{K}}] = \frac{\partial}{\partial \boldsymbol{K}} \Big(\mathrm{tr}[\acute{\boldsymbol{\Lambda}}\acute{\boldsymbol{K}}] + \underbrace{\mathrm{tr}[\grave{\boldsymbol{\Lambda}}\acute{\boldsymbol{K}}]}_{=0} \Big) \\
&= \frac{\partial}{\partial \boldsymbol{K}} \Big(\mathrm{tr}[\acute{\boldsymbol{\Lambda}}\acute{\boldsymbol{K}}] + \overbrace{\mathrm{tr}[\acute{\boldsymbol{\Lambda}}\grave{\boldsymbol{K}}]}^{0=} \Big) = \frac{\partial}{\partial \boldsymbol{K}} \mathrm{tr}[\acute{\boldsymbol{\Lambda}}(\grave{\boldsymbol{K}} + \acute{\boldsymbol{K}})] \\
&= \frac{\partial}{\partial \boldsymbol{K}} \mathrm{tr}[\acute{\boldsymbol{\Lambda}}\boldsymbol{K}] = \acute{\boldsymbol{\Lambda}}^{\mathrm{T}}.
\end{aligned} \tag{31}$$

References

[1] F. D. Neeser and J. L. Massey, "Proper complex random processes with applications to information theory," *IEEE Trans. Inf. Theory*, vol. 39, no. 4, pp. 1293–1302, Jul. 1993.

[2] P. J. Schreier and L. L. Scharf, *Statistical Signal Processing of Complex-Valued Data: The Theory of Improper and Noncircular Signals*. Cambridge, UK: Cambridge University Press, 2010.

[3] T. Adalı, P. J. Schreier, and L. L. Scharf, "Complex-valued signal processing: The proper way to deal with impropriety," *IEEE Trans. Signal Process.*, vol. 59, no. 11, pp. 5101–5125, Nov. 2011.

[4] A. Hakkarainen, J. Werner, M. Renfors, K. Dandekar, and M. Valkama, "RF-aware widely linear MMSE beamforming," in *Proc. 10th Int. Symp. Wireless Commun. Syst. (ISWCS)*, Aug. 2013, pp. 582–586.

[5] B. Picinbono and P. Chevalier, "Widely linear estimation with complex data," *IEEE Trans. Signal Process.*, vol. 43, no. 8, pp. 2030–2033, Aug. 1995.

[2]Of course, one could argue that it would be possible to develop specialized algorithm to efficiently compute, e.g., determinants of augmented covariance matrices by exploiting the redundant structure of augmented matrices. However, in this case, the composite real representation still has the advantage that standard algorithms can be used instead of such specialized solutions.

[6] P. J. Schreier and L. L. Scharf, “Second-order analysis of improper complex random vectors and processes,” *IEEE Trans. Signal Process.*, vol. 51, no. 3, pp. 714–725, Mar. 2003.
[7] Y. C. Yoon and H. Leib, “Maximizing SNR in improper complex noise and applications to CDMA,” *IEEE Commun. Lett.*, vol. 1, no. 1, pp. 5–8, Jan. 1997.
[8] H. Gerstacker, R. Schober, and A. Lampe, “Receivers with widely linear processing for frequency-selective channels,” *IEEE Trans. Commun.*, vol. 51, no. 9, pp. 1512–1523, Sep. 2003.
[9] D. Darsena, G. Gelli, L. Paura, and F. Verde, “Widely linear equalization and blind channel identification for interference-contaminated multicarrier systems,” *IEEE Trans. Signal Process.*, vol. 53, no. 3, pp. 1163–1177, Mar. 2005.
[10] D. Mattera, L. Paura, and F. Sterle, “Widely linear MMSE transceiver for real-valued sequences over MIMO channel,” presented at the 14th European Signal Process. Conf. (EUSIPCO), Florence, Italy, Sep. 4–8, 2006.
[11] S. Buzzi, M. Lops, and S. Sardellitti, “Widely linear reception strategies for layered space-time wireless communications,” *IEEE Trans. Signal Process.*, vol. 54, no. 6, pp. 2252–2262, Jun. 2006.
[12] A. Mirbagheri, K. N. Plataniotis, and S. Pasupathy, “An enhanced widely linear CDMA receiver with OQPSK modulation,” *IEEE Trans. Commun.*, vol. 54, no. 2, pp. 261–272, Feb. 2006.
[13] A. S. Cacciapuoti, G. Gelli, and F. Verde, “FIR zero-forcing multiuser detection and code designs for downlink MC-CDMA,” *IEEE Trans. Signal Process.*, vol. 55, no. 10, pp. 4737–4751, Oct. 2007.
[14] P. Xiao and M. Sellathurai, “Improved linear transmit processing for single-user and multi-user MIMO communications systems,” *IEEE Trans. Signal Process.*, vol. 58, no. 3, pp. 1768–1779, Mar. 2010.
[15] J. Steinwandt, R. C. de Lamare, L. Wang, N. Song, and M. Haardt, “Widely linear adaptive beamforming algorithm based on the conjugate gradient method,” presented at the Int. ITG Workshop on Smart Antennas (WSA) 2011, Aachen, Germany, Feb. 24–25, 2011.
[16] Z. Ho, A. Zappone, and E. Jorswieck, “Optimal non-regenerative relay processing with improper signals,” in *Proc. 10th Int. Symp. Wireless Commun. Syst. (ISWCS)*, Aug. 2013, pp. 567–571.
[17] J. Zhang and M. Haardt, “Widely linear signal processing for two-way relaying with MIMO amplify and forward relays,” in *Proc. 10th Int. Symp. Wireless Commun. Syst. (ISWCS)*, Aug. 2013, pp. 264–268.
[18] P. J. Schreier, L. L. Scharf, and A. Hanssen, “A generalized likelihood ratio test for impropriety of complex signals,” *IEEE Signal Process. Lett.*, vol. 13, no. 7, pp. 433–436, Jul. 2006.
[19] K. V. Mardia, J. T. Kent, and J. M. Bibby, *Multivariate Analysis*. New York: Academic, 1979.
[20] D. P. Mandic and S. L. Goh, *Complex valued nonlinear adaptive filters: noncircularity, widely linear and neural models*. West Sussex, UK: John Wiley & Sons, 2009.
[21] Y. Zeng, C. M. Yetis, E. Gunawan, Y. L. Guan, and R. Zhang, “Transmit optimization with improper Gaussian signaling for interference channels,” *IEEE Trans. Signal Process.*, vol. 61, no. 11, pp. 2899–2913, Jun. 2013.
[22] Y. Zeng, R. Zhang, E. Gunawan, and Y. L. Guan, “Optimized transmission with improper Gaussian signaling in the K-user MISO interference channel,” *IEEE Trans. Wireless Commun.*, vol. 12, no. 12, pp. 6303–6313, Dec. 2013.
[23] D. Darsena, G. Gelli, and F. Verde, “Widely-linear precoders and decoders for MIMO channels,” in *Proc. 10th Int. Symp. Wireless Commun. Syst. (ISWCS)*, Aug. 2013, pp. 587–591.
[24] C. Hellings and W. Utschick, “Block-skew-circulant matrices in complex-valued signal processing,” Jun. 2014, accepted for publication in *IEEE. Trans. Signal Process.*
[25] G. Villard, “Computation of the inverse and determinant of a matrix,” in *Algorithms Seminar 20012002*, F. Chyzak, Ed. INRIA, 2003, pp. 29–32.
[26] F. Le Gall, “Powers of tensors and fast matrix multiplication,” in *Proc. 39th Int. Symp. Symbolic and Algebraic Computation (ISSAC)*, 2014, pp. 296–303.