A NOTE ON THE MARTINGALE PROPERTY OF EXPONENTIAL SEMIMARTINGALES AND FINANCIAL APPLICATIONS

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Abstract. In numerous applications positive martingales are crucial, for example when defining an equivalent change of measure. In the class of exponentials of semimartingales positive local martingales can be easily identified. However, the true martingale property is more subtle. Based on general conditions in Kallsen and Shiryaev (2002a), we derive explicit sufficient conditions for the true martingale property for a wide class of exponentials of semimartingales. Suitably for applications, the conditions are expressed in terms of the semimartingale triplet. We present two applications to mathematical finance. First, we apply the results to stochastic volatility asset price models driven by semimartingales. Second, we provide a proof of the martingale property of the Libor rates in the Lévy Libor model, as well as in a semimartingale driven Libor model.

1. Introduction

The question under which conditions local martingales are true martingales plays an important role in various applications. For example, positive martingales with initial value one can be used as density processes to define a change of measure. Exponentials of semimartingales form a wide and flexible class of positive processes. By means of stochastic calculus one can easily characterize the local martingales in this class. It is, however, more involved to identify conditions for their true martingality. Based on Kallsen and Shiryaev (2002a), we derive conditions that are suitable for applications, such as in mathematical finance.

In order to formulate the problem more precisely denote by X an $\mathbb{R}^d$-valued semimartingale and by λ an $\mathbb{R}^d$-valued predictable process, which is integrable with respect to X. Then $\lambda \cdot X := \sum_{i \leq d} \int_0^\cdot \lambda^i \, \mathrm{d}X^i$ is the vector stochastic integral of λ with respect to X, written also as $\int_0^\cdot \lambda \, \mathrm{d}X$. Moreover, let V be a predictable process with finite variation. We pose the following question: *under which conditions on the characteristics of X is a real-valued semimartingale Z of the form*

$$Z := e^{\lambda \cdot X - V}$$

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a (uniformly integrable) martingale?

If Z is a special semimartingale, V can be determined such that Z is a local martingale. In this case, V is called the *exponential compensator* of $\lambda \cdot X$, see Section 2 for details. Various criteria for the more delicate true martingale property of Z have been proposed. The seminal paper by Novikov (1972) treats the continuous semimartingale case. Sufficient conditions for general semimartingales are provided for example in Lépingle and Mémin (1978), Kallsen and Shiryaev (2002a) and Protter and Shimbo (2008). Moreover, we refer to Section 1 and Section 3 of Kallsen and Shiryaev (2002a) for an exhaustive literature overview. In the special case when X is a process with independent increments and absolutely continuous characteristics and λ deterministic, Eberlein, Jacod, and Raible (2005) show that if Z is a local martingale, it is also a true martingale. Deterministic conditions ensuring the martingale property of an exponential of an affine process are given in Kallsen and Muhle-Karbe (2010).

The conditions for more general semimartingales are not as explicit. Our contribution is to derive – based on the criteria from Kallsen and Shiryaev (2002a) – explicit conditions on the semimartingale characteristics of X for a wide class of semimartingales. The advantage of the explicit conditions is their convenience for applications. We present two applications to mathematical finance. First, we use the obtained results to derive conditions for the martingale property of the discounted stock price processes in stochastic volatility models driven by semimartingales. Second, we provide a proof of the martingale property of the Libor rates under the respective forward measures in the Lévy Libor model of Eberlein and Özkan (2005), as well as in a more general semimartingale Libor model. This is a key property both for the model construction and for option pricing.

The rest of the paper is organized as follows. In Section 2 we introduce the notation and describe the general semimartingale setting used in the paper following Jacod and Shiryaev (2003). Section 3 contains the main results. We illustrate the application of these results in financial modeling in Section 4.

2. Semimartingale notation and preliminaries

In this section we introduce the notation and summarize the basic notions and facts from the semimartingale theory in order to keep the paper self-contained. Our main reference is Jacod and Shiryaev (2003), whose notation we use throughout the paper. Other standard references for stochastic calculus and semimartingales are e.g. Jacod (1979), Métivier (1982) and Protter (2004).

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ denote a stochastic basis that satisfies the usual conditions. Denote by $\mathcal{M}_{\mathrm{loc}}$ the set of all càdlàg *local martingales* and by $\mathcal{V}^+$ (resp. $\mathcal{V}$) a set of all real-valued càdlàg processes starting from zero that have *non-decreasing paths* (resp. *paths with finite variation* over each finite interval $[0,t]$). Let $\mathcal{A}^+$ denote the set of all processes $A \in \mathcal{V}^+$ that are integrable, i.e. such that $\mathbb{E}[A_\infty] < \infty$, where $A_\infty(\omega) := \lim_{t\to\infty} A_t(\omega) \in \overline{\mathbb{R}}_+$ for every $\omega \in \Omega$. Moreover, let $\mathcal{A}$ denote the set of all $A \in \mathcal{V}$ that have integrable variation, i.e. $\mathrm{Var}\,(A) \in \mathcal{A}^+$, where for every $t \geq 0$ and every $\omega \in \Omega$, $\mathrm{Var}\,(A)_t(\omega)$ is defined as the total variation of the function $s \mapsto A_s(\omega)$

on $[0,t]$. A process X is called a *semimartingale* if it can be written in the form

$$X = X_0 + M + A, \tag{2.1}$$

where X_0 is finite-valued and $\mathcal{F}_0$-measurable, $M \in \mathcal{M}_{\mathrm{loc}}$ with $M_0 = 0$ and $A \in \mathcal{V}$ with $A_0 = 0$. If A in decomposition (2.1) is predictable, X is a *special semimartingale* and the decomposition is unique.

Let X be an $\mathbb{R}^d$-valued semimartingale, i.e. each component of X satisfies (2.1). Denoting by ε_a the Dirac measure at point a, the random measure of jumps μ^X of X is an integer-valued measure of the form

$$\mu^X(\omega; \mathrm{d}t, \mathrm{d}x) = \sum_{s \geq 0} \mathbf{1}_{\{\Delta X_s(\omega) \neq 0\}} \varepsilon_{(s, \Delta X_s(\omega))}(\mathrm{d}t, \mathrm{d}x)$$

and its predictable compensator ν satisfies

$$(|x|^2 \wedge 1) * \nu \in \mathcal{A}_{\mathrm{loc}}. \tag{2.2}$$

The semimartingale X admits a *canonical representation*

$$X = X_0 + B(h) + X^c + (x - h(x)) * \mu^X + h(x) * (\mu^X - \nu),$$

where $h : \mathbb{R}^d \to \mathbb{R}^d$ is a *truncation function*, i.e. a function that is bounded and behaves like $h(x) = x$ around 0, $B(h)$ is a predictable $\mathbb{R}^d$-valued process with components in $\mathcal{V}$, and X^c is the continuous martingale part of X.

Denote by C the predictable $\mathbb{R}^{d \times d}$-valued covariation process defined as $C^{ij} := \langle X^{i,c}, X^{j,c} \rangle$. Then the triplet $(B(h), C, \nu)$ is called the *triplet of predictable characteristics* of X (or simply the *characteristics* of X). It can be shown (see Proposition II.2.9 in Jacod and Shiryaev (2003)) that there exists a predictable process $A \in \mathcal{A}^+_{\mathrm{loc}}$ such that

$$B(h) = b(h) \cdot A, \quad C = c \cdot A, \quad \nu = A \otimes F\,,$$

where $b(h)$ is a d-dimensional predictable process, c is a predictable process taking values in the set of symmetric non-negative definite $d \times d$-matrices and F is a transition kernel from $(\Omega \times \mathbb{R}_+, \mathcal{P})$ into $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. Here $\mathcal{P}$ denotes the predictable σ-field on $\Omega \times \mathbb{R}_+$. We call $(b(h), c, F; A)$ the *triplet of differential* (or *local*) *characteristics* of X. If X admits the choice $A_t = t$ above, we say that X has *absolutely continuous* characteristics (or shortly AC) and X is called an *Itô semimartingale.*

An important subclass of semimartingales is the class of Itô semimartingales with independent increments. These processes are known as time-inhomogeneous Lévy processes or as Processes with Independent Increments and Absolutely Continuous characteristics (PIIAC), see e.g. Section 2 in Eberlein, Jacod, and Raible (2005). The differential characteristics $(b(h), c, F)$ of a PIIAC X, for any truncation function h, are deterministic and satisfy the following integrability assumption: for any $T > 0$

$$\int_0^T \left(|b(h)_s| + \|c_s\| + \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) \right) \mathrm{d}s < \infty, \tag{2.3}$$

where $\| \cdot \|$ denotes any norm on the set of $d \times d$-matrices. For every $t > 0$, the law of X_t is characterized by a Lévy-Khintchine type formula for its characteristic function, see again Section 2 in Eberlein, Jacod, and Raible (2005). This property makes the class of PIIAC particularly suitable for applications.

The following definition and results on exponentials of semimartingales are given in Definition 2.12, Lemma 2.13 and Lemma 2.15 in Kallsen and Shiryaev (2002a).

Definition 2.1. *A real-valued semimartingale Y is called* exponentially special *if* $\exp(Y - Y_0)$ *is a special semimartingale.*

Remark 2.2. Let Y be a real-valued semimartingale and denote by ν^Y the compensator of the random measure of jumps of Y and h a truncation function.

(a) The following statements are equivalent:
 (i) Y is an exponentially special semimartingale.
 (ii) $(e^y - 1 - h(y)) * \nu^Y \in \mathcal{V}$.
 (iii) $e^y \mathbf{1}_{\{y>1\}} * \nu^Y \in \mathcal{V}$.

(b) If Y is exponentially special, then it admits an *exponential compensator*, i.e. there exists a predictable process $V \in \mathcal{V}$ such that $\exp(Y - Y_0 - V) \in \mathcal{M}_{\mathrm{loc}}$.

Let X be an $\mathbb{R}^d$-valued semimartingale with local characteristics $(b(h), c, F; A)$ and $\lambda \in L(X)$, where $L(X)$ denotes the set of predictable processes integrable with respect to X, cf. Jacod and Shiryaev (2003), page 207. Moreover, assume that $\lambda \cdot X$ is exponentially special. Following Jacod and Shiryaev (2003), Section III.7.7a we define the *Laplace cumulant process*

$$\widetilde{K}^X(\lambda) := \widetilde{\kappa}^X(\lambda) \cdot A, \tag{2.4}$$

where

$$\widetilde{\kappa}^X(\lambda)_t := \langle \lambda_t, b_t \rangle + \frac{1}{2}\langle \lambda_t, c_t \lambda_t \rangle + \int (e^{\langle \lambda_t, x \rangle} - 1 - \langle \lambda_t, h(x) \rangle) F_t(\mathrm{d}x), \tag{2.5}$$

and the *modified Laplace cumulant process* $K^X(\lambda) := \ln(\mathcal{E}(\widetilde{K}^X(\lambda)))$, where $\mathcal{E}$ denotes the stochastic exponential, and

$$K^X(\lambda) \;=\; \widetilde{K}^X(\lambda) + \sum_{s \leq \cdot} (\ln(1 + \Delta \widetilde{K}^X(\lambda))_s - \Delta \widetilde{K}^X(\lambda)_s). \tag{2.6}$$

The following results are proved in Proposition III.7.14 and Theorem III.7.4 in Jacod and Shiryaev (2003):

Proposition 2.3. *Let X be an $\mathbb{R}^d$-valued semimartingale and $\lambda \in L(X)$ such that $\lambda \cdot X$ is exponentially special.*

(i) The process $K^X(\lambda)$ is the exponential compensator of $\lambda \cdot X$, i.e. the process Z defined by

$$Z := \exp(\lambda \cdot X - K^X(\lambda))$$

is a local martingale.

(ii) If X is quasi-left continuous, i.e. X a.s. does not jump at predictable times (cf. Jacod and Shiryaev (2003), page 22), the Laplace cumulant process $\widetilde{K}^X(\lambda)$ and the modified Laplace cumulant process $K^X(\lambda)$ coincide, i.e. $K^X(\lambda) = \widetilde{K}^X(\lambda)$.

3. Main results

In this section we present several results answering the question under which conditions an exponentially compensated semimartingale – that is a local martingale by construction – is a uniformly integrable (UI) martingale. We show in the following proposition that for quasi-left continuous semimartingales the sufficient conditions provided in Theorem 3.2 in Kallsen and Shiryaev (2002a) boil down to conditions on the semimartingale characteristics. Moreover, a weaker condition ensures the martingale property instead of the UI martingale property.

Proposition 3.1. *Let Y be a quasi-left continuous, real-valued semimartingale with characteristics (B^Y, C^Y, ν^Y) for a truncation function h^Y. If*

(A1) $|y|e^y \mathbf{1}_{\{|y|>1\}} * \nu^Y \in \mathcal{V}$, *and*

(A2) $\sup_{t\le T} \mathbb{E}\left[\exp\left\{\frac{1}{2}C_t^Y + ((y-1)e^y+1) * \nu_t^Y\right\}\right] < \infty$ *for every* $T \geq 0$,

the process $M := e^{Y-K^Y(1)}$ is a true martingale. Moreover, replacing condition (A2) *with*

(A2′) $\sup_{t\in\mathbb{R}_+} \mathbb{E}\left[\exp\left\{\frac{1}{2}C_t^Y + ((y-1)e^y+1) * \nu_t^Y\right\}\right] < \infty$,

M is a UI martingale.

Proof: We first assume (A1) and the stronger condition (A2′).

Condition (A1) implies that $e^y \mathbf{1}_{\{|y|>1\}} * \nu^Y \in \mathcal{V}$ and hence by Remark 2.2(a), Y is exponentially special and M is a well-defined local martingale. Taylor expansion of the exponential function and the integrability property (2.2) of the compensator ν^Y yield that (A1) is equivalent to

$$|ye^y - h^Y(y)| * \nu^Y \in \mathcal{V}, \tag{3.1}$$

which is the first requirement of condition $I(0,1)$ in Theorem 3.2 in Kallsen and Shiryaev (2002a). The following results of Kallsen and Shiryaev (2002a) – Theorem 2.18(6), Definition 2.23, Definition 3.1 and Remarks 1 and 2 which follow after Proposition 2.25 – show for quasi-left continuous processes that condition (A2′) is equivalent to the second requirement in $I(0,1)$ in Kallsen and Shiryaev (2002a). Hence, M is a UI martingale by Theorem 3.2 from the same paper.

If Y satisfies condition (A2) instead of (A2′), then for every fixed $T > 0$, we repeat the proof along the same lines as above for the stopped process $(Y_{t\wedge T})_{t\geq 0}$ and the semimartingale characteristics $(B^Y_{\cdot\wedge T}, C^Y_{\cdot\wedge T}, \nu^Y_{\cdot\wedge T})$ of the stopped process which are derived for example in Lemma 2.3 in Kallsen and Muhle-Karbe (2010). Here $\nu^Y_{\cdot\wedge T}$ is defined as follows: $\nu^Y_{\cdot\wedge T}(B) = \nu^Y(B \cap [[0,T]]) = \nu^Y(B \cap (\Omega \times [0,T]))$, for every $B \in \mathcal{P}$. This yields that the stopped process $(M_{t\wedge T})_{t\geq 0}$ is a UI martingale. Therefore, $\mathbb{E}[|M_t|] < \infty$, for every $t \geq 0$ and $\mathbb{E}[M_t|\mathcal{F}_s] = M_s$, for every $s \leq t$, hence M is a martingale. □

Remark 3.2. Note that in case $\nu^Y = 0$ (A2′) reduces to the classical Novikov condition as presented in Section 3.5.D in Karatzas and Shreve (1991).

Next, we derive sufficient conditions under which exponentially compensated semimartingales for stochastic integrals $\lambda \cdot X$ with bounded integrands λ are martingales, resp. UI martingales. Denote by $\overline{\lambda} := \sup_{\omega\in\Omega}\sup_{t\geq 0}|\lambda_t(\omega)| < \infty$.

Proposition 3.3. *Let X be an $\mathbb{R}^d$-valued quasi-left continuous semimartingale with differential characteristics $(b, c, F; A)$ and $\lambda \in L(X)$ bounded. If*

(B1) $|x|e^{\langle\lambda,x\rangle}\mathbf{1}_{\{|x|>1\}} * \nu \in \mathcal{V}$, *and*

(B2) $\sup\limits_{t\leq T}\mathbb{E}\big[\exp\big(\frac{1}{2}\int_0^t\langle\lambda_s, c_s\lambda_s\rangle \mathrm{d}A_s+\int_0^t\int_{\mathbb{R}^d}((\langle\lambda_s,x\rangle-1)e^{\langle\lambda_s,x\rangle}+1)F_s(\mathrm{d}x)\mathrm{d}A_s\big)\big] < \infty$ *for every $T \geq 0$,*

the process $M := e^{\lambda\cdot X - K^X(\lambda)}$ is a true martingale. Moreover, replacing (B2) *with*

(B2′) $\sup\limits_{t\in\mathbb{R}_+}\mathbb{E}\big[\exp\big(\frac{1}{2}\int_0^t\langle\lambda_s, c_s\lambda_s\rangle \mathrm{d}A_s+\int_0^t\int_{\mathbb{R}^d}((\langle\lambda_s,x\rangle-1)e^{\langle\lambda_s,x\rangle}+1)F_s(\mathrm{d}x)\mathrm{d}A_s\big)\big] < \infty$,

M is a UI martingale.

Proof: Denote by (B^Y, C^Y, ν^Y) the semimartingale characteristics of $Y := \lambda \cdot X$. If $\lambda \equiv 0$, then $M = 1$ is trivially a UI martingale, hence, we assume now that $\lambda \not\equiv 0$. From Lemma 2.3 in Kallsen and Shiryaev (2002b) we obtain

$$\begin{aligned} B_t^Y &= \int_0^t \left(\langle\lambda_s, b_s\rangle + \int_{\mathbb{R}^d} \left((h^Y(\langle\lambda_s, x\rangle) - \langle\lambda_s, h(x)\rangle \right) F_s(\mathrm{d}x) \right) \mathrm{d}A_s \\ C_t^Y &= \int_0^t \langle\lambda_s, c_s\lambda_s\rangle\, \mathrm{d}A_s \qquad (3.2) \\ \nu^Y(\mathrm{d}t, G) &= \int_{\mathbb{R}^d} \mathbf{1}_G(\langle\lambda_t, x\rangle)F_t(\mathrm{d}x)\,\mathrm{d}A_t, \qquad \text{for every } G \in \mathcal{B}(\mathbb{R}\backslash\{0\}), \end{aligned}$$

where h is a truncation function for X and h^Y for Y. Hence, condition (B2) is exactly condition (A2) and (B2′) is (A2′) for the semimartingale Y. Let us prove that condition (B1) implies condition (A1) for Y. From $|\langle\lambda_s, x\rangle| \leq |\lambda_s||x|$ it follows $\mathbf{1}_{\{|\langle\lambda_s,x\rangle|>1\}} \leq \mathbf{1}_{\{|x|>|\lambda_s|^{-1}\}}$ on the set $\{\lambda_s \neq 0\}$ and $\mathbf{1}_{\{|\langle\lambda_s,x\rangle|>1\}} = 0$ on the set $\{\lambda_s = 0\}$. Note, moreover, that $0 < \overline{\lambda} < \infty$ by assumption. Hence, for every $t \geq 0$ we have by (3.2) and condition (B1)

$$\begin{aligned} |y|e^y\mathbf{1}_{\{|y|>1\}} * \nu_t^Y &\leq \int_0^t\int_{\mathbb{R}^d} |\langle\lambda_s, x\rangle| e^{\langle\lambda_s,x\rangle}\mathbf{1}_{\{|x|>|\lambda_s|^{-1}\}}\mathbf{1}_{\{\lambda_s\neq 0\}}\nu(\mathrm{d}s, \mathrm{d}x) \\ &\leq \overline{\lambda}\int_0^t\int_{\mathbb{R}^d} |x|e^{\langle\lambda_s,x\rangle}\mathbf{1}_{\{|x|>\overline{\lambda}^{-1}\}}\nu(\mathrm{d}s, \mathrm{d}x) \\ &< \infty. \end{aligned}$$

From the nonnegativity of the integrand we obtain $|y|e^y\mathbf{1}_{\{|y|>1\}} * \nu^Y \in \mathcal{V}^+$, which delivers condition (A1).

Noting that $K^X(\lambda) = K^Y(1)$, which follows by combining (3.2) with (2.4), (2.5) and Proposition 2.3(ii), the assertion of the proposition is implied by Proposition 3.1. □

We derive more explicit conditions on the semimartingale characteristics of the driving process X that are expressed using only deterministic bounds.

Proposition 3.4. *Let X and λ be as in Proposition 3.3. If*

(C) *for every $T \geq 0$, there exists a constant $\kappa(T) > 0$ such that a.s.*

$$\int_0^T |\langle \lambda_s, c_s \lambda_s \rangle| \mathrm{d}A_s + \int_0^T \int_{\mathbb{R}^d} \left[(|x|^2 \wedge 1) + |x| e^{\langle \lambda_s, x \rangle} \mathbf{1}_{\{|x|>1\}} \right] F_s(\mathrm{d}x) \mathrm{d}A_s < \kappa(T),$$

the process $M := e^{\lambda \cdot X - K^X(\lambda)}$ is a true martingale. Moreover, replacing (C) *with*

(C′) *there exists a constant $\kappa > 0$ such that a.s.*

$$\int_0^\infty |\langle \lambda_s, c_s \lambda_s \rangle| \mathrm{d}A_s + \int_0^\infty \int_{\mathbb{R}^d} \left[(|x|^2 \wedge 1) + |x| e^{\langle \lambda_s, x \rangle} \mathbf{1}_{\{|x|>1\}} \right] F_s(\mathrm{d}x) \mathrm{d}A_s < \kappa$$

M is a UI martingale.

Proof: Condition (C) and the fact that $(\omega, s, x) \mapsto |x| e^{\langle \lambda_s(\omega), x \rangle} \mathbf{1}_{\{|x|>1\}}$ is a nonnegative random function immediately yield (B1).

Let us now verify that (C) (resp. (C′)) implies condition (B2) (resp. (B2′)). Taylor expansion of the exponential function around zero, condition (C) (resp. (C′)), and the boundedness of λ, yield for each $T > 0$ (resp. for $T := \infty$) the existence of constants $c_1(T), c_2(T) > 0$ such that

$$\begin{aligned} \int_0^T \int_{|x|\leq 1} \left| (\langle \lambda_s, x \rangle - 1) e^{\langle \lambda_s, x \rangle} + 1 \right| \nu(\mathrm{d}s, \mathrm{d}x) &\leq c_1(T) \int_0^T \int_{|x|\leq 1} |x|^2 \nu(\mathrm{d}s, \mathrm{d}x) \\ &\leq c_2(T). \end{aligned} \tag{3.3}$$

Moreover, using $\overline{\lambda} < \infty$, some elementary transformations and (C) (resp. (C′)), for every $T > 0$ (resp. $T := \infty$) we obtain the existence of a constant $c_3(T) > 0$ such that

$$\begin{aligned} &\int_0^T \int_{|x|>1} \left| (\langle \lambda_s, x \rangle - 1) e^{\langle \lambda_s, x \rangle} + 1 \right| \nu(\mathrm{d}s, \mathrm{d}x) \\ &\qquad \leq (\overline{\lambda} + 1) \int_0^T \int_{|x|>1} \left(|x| e^{\langle \lambda_s, x \rangle} + 1 \right) \nu(\mathrm{d}s, \mathrm{d}x) \\ &\qquad \leq c_3(T). \end{aligned} \tag{3.4}$$

Combining inequalities (3.3) and (3.4) with $\int_0^T |\langle \lambda_s, c_s \lambda_s \rangle| \mathrm{d}A_s < \max\{\kappa(T), \kappa\}$, we get for every $T > 0$ (resp. $T := \infty$) condition (B2) (resp. (B2′)) from Proposition 3.3. Hence, M is a martingale (resp. a UI martingale). □

When X is a process with independent increments and an absolutely continuous characteristic (PIIAC), condition (C) from the previous proposition simplifies further. More precisely, we obtain an integrability condition solely on the family of Lévy measures of X:

Corollary 3.5. *Let X be an $\mathbb{R}^d$-valued PIIAC and $\lambda \in L(X)$ bounded. If*

(D) *for every $T \geq 0$, there exists a constant $\kappa(T) > 0$ such that a.s.*

$$\int_0^T \int_{|x|>1} |x| e^{\langle \lambda, x \rangle} F_s(\mathrm{d}x) \mathrm{d}s < \kappa(T),$$

then $M = e^{\lambda \cdot X - K^X(\lambda)}$ is a martingale.

Proof: Recall from Section 2 that the differential characteristic triplet (b, c, F) of X is deterministic and $A_s(\omega) := s$, for every s. Moreover, X is quasi-left continuous by Corollary II.1.19 in Jacod and Shiryaev (2003). By assumption $\overline{\lambda} < \infty$ and by (2.3), for every $T > 0$ there exist constants $c_1(T), c_2(T) > 0$ such that

$$\int_0^T \int_{\mathbb{R}^d} (|x|^2 \wedge 1) F_s(\mathrm{d}x) \mathrm{d}s < c_1(T) \quad \text{and}$$

$$\int_0^T |\langle \lambda_s, c_s \lambda_s \rangle| \, \mathrm{d}s \leq \overline{\lambda}^2 \int_0^T ||c_s|| \mathrm{d}s < c_2(T).$$

Thus, condition (C) from Proposition 3.4 reduces to (D) which implies the assertion of the corollary. □

Remark 3.6. Corollary 3.5 states that for X PIIAC and λ bounded, the local martingale property of $M := e^{\lambda \cdot X - K^X(\lambda)}$ is essentially equivalent to its martingale property (up to a slightly stronger integrability condition). For λ deterministic, this is implied by the proof of Proposition 4.4 in Eberlein, Jacod, and Raible (2005).

Remark 3.7. Condition (D) in Corollary 3.5 can be replaced with

(D*) for every $T \geq 0$, there exists a constant $\kappa(T) > 0$ and $\varepsilon > 0$ such that a.s.

$$\int_0^T \int_{|x|>1} e^{(1+\varepsilon)\overline{\lambda}|x|} F_s(\mathrm{d}x) \mathrm{d}s < \kappa(T).$$

This follows from the fact that for x large enough, $|x| \leq e^{\varepsilon \overline{\lambda} |x|}$.

4. Application to financial modeling

In this section we present two applications of the results from Section 3 to financial modeling. The monographs by Shiryaev (1999), Musiela and Rutkowski (2005), Cont and Tankov (2003) and Jeanblanc, Yor, and Chesney (2009) and the references therein provide a detailed overview concerning applications of semimartingales in finance.

4.1. **Asset price models with stochastic volatility.** Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ be a stochastic basis satisfying the usual conditions, where $T > 0$ denotes a finite time horizon. Let us model the asset price S and a bank account B with stochastic interest rate by

$$S := S_0 e^{\sigma^S \cdot X^S - V}, \qquad B := e^{\sigma^r \cdot X^r} \tag{4.1}$$

with $S_0 \in \mathbb{R}$, a d-dimensional semimartingale $X := (X^S, X^r)$, a real-valued predictable process of finite variation V and a d-dimensional predictable process $\sigma := (\sigma^S, -\sigma^r) \in L(X)$ such that $\sigma \cdot X$ is exponentially special. Then, according to Remark 2.2(b), the discounted price process

$$\widetilde{S} := B^{-1} S = S_0 e^{\sigma \cdot X - V} \tag{4.2}$$

is a local martingale, i.e. $\mathbb{P}$ is a risk-neutral measure, if and only if V is the exponential compensator of $\sigma \cdot X$. If X is quasi-left continuous, this is the case if and only if $V = \widetilde{K}^X(\sigma) = K^X(\sigma)$, see Proposition 2.3. According to the fundamental theorem of asset pricing for general semimartingales in Delbaen and Schachermayer (1998), this implies that the model satisfies the No Free Lunch With Vanishing Risk (NFLVR) condition. In other words, model (4.1) is a semimartingale pricing model specified directly under a risk-neutral measure. In order to compute option prices in this model, integrability conditions ensuring their well-definedness are required in addition. Standard options include European call options with strike $K > 0$ paying out $\max(S_T - K, 0)$ at maturity $T > 0$ that we take as an example in the sequel. To compute the price of a call option, we must have

(A) the discounted price process $\widetilde{S}$ is a martingale,

which is equivalent to $\mathbb{E}[\widetilde{S}_T] = \mathbb{E}[\widetilde{S}_0] < \infty$ because $\widetilde{S}$ is a nonnegative locale martingale. Then, according to the fundamental principles of option pricing, the discounted price process $B^{-1}\Pi$ of the call option is a martingale, hence the time-t-price of the option is

$$\Pi_t = B_t \mathbb{E}\big(B_T^{-1} \max(S_T - K, 0) \big| \mathcal{F}_t\big). \tag{4.3}$$

In order to compute this expectation in a specific model, the joint distribution of (S_T, B_T) must be known. By a change of numeraire, i.e. an equivalent change of measure, we can express the price as an expectation of a function of the asset value S_T solely: let $\frac{d\widetilde{\mathbb{P}}}{d\mathbb{P}}|_{\mathcal{F}_t} := \widetilde{S}_t$ for $0 \leq t \leq T$. Denoting by $\mathbb{E}_{\widetilde{\mathbb{P}}}$ the expectation under $\widetilde{\mathbb{P}}$, Bayes formula yields

$$\Pi_t = S_t \mathbb{E}_{\widetilde{\mathbb{P}}}\big(\max(1 - K S_T^{-1}, 0) \big| \mathcal{F}_t\big). \tag{4.4}$$

In conclusion, Proposition 3.3, Proposition 3.4, Corollary 3.5 and Remark 3.7 imply:

Corollary 4.1. *Let the semimartingale asset price model* (4.1) *be given with* σ *bounded. Denote by* $(b, c, F; A)$ *the local semimartingale characteristics of* X.

(i) If X *is quasi-left continuous and* $(b, c, F; A)$ *and* σ *satisfy* (B1) *and* (B2) *of Proposition 3.3, resp. condition* (C) *of Proposition 3.4, or*

(ii) If X *is PIIAC and* F *and* σ *satisfy condition* (D) *of Corollary 3.5, resp. condition* (D*) *of Remark 3.7,*

the discounted asset price process $\widetilde{S}$ *given in* (4.2) *is a martingale and the time-t value of the call option with maturity* T *and strike* K *is given by* (4.3) *and* (4.4).

Remark 4.2. If X is quasi-left continuous and the money market account is of the usual form $B := e^{\int_0^\cdot r_t dt}$, then $V = K^{X^S}(\sigma^S) - \int_0^\cdot r_t dt$ and $\widetilde{S} = S_0 e^{\sigma^S \cdot X^S - K^{X^S}(\sigma^S)}$.

If in addition X has absolutely continuous local characteristics (b, c, F), then $V = \int_0^\cdot (\widetilde{\kappa}^{X^S}(\sigma^S)_t - r_t)dt$ with $\widetilde{\kappa}^{X^S}(\sigma^S)_t$ given in (2.5). If X is PIIAC, σ is deterministic and $V := 0$, we recover the classical *no-arbitrage drift condition*

$$\widetilde{\kappa}^{X^S}(\sigma^S) - r = 0,$$

and in particular the interest rate r has to be deterministic.

4.2. **Libor models.** To illustrate the applications to Libor models we consider the *Lévy Libor model* of Eberlein and Özkan (2005). This is a model for discretely compounded forward rates known as Libor rates. The name stems from the London Interbank Offered Rate. Among the semimartingale Libor models, we choose the Lévy Libor model because, even though it is driven by a PIIAC, the martingale property of the Libor rates cannot be deduced using the standard results for PIIACs. This is due to the fact that the Libor rates with different maturities are modeled under different equivalent probability measures for which the PIIAC property of the driving process is not preserved. Verifying that the Libor rates are martingales under their corresponding measures has not been addressed in detail in the literature and we close the gap below. Moreover, we provide conditions for the martingale property also in a more general semimartingale Libor model, cf. Proposition 4.5.

Let us now briefly introduce the Lévy Libor model. For a detailed overview we refer to Eberlein and Özkan (2005). Assume that $T^* > 0$ is a fixed time horizon and we are given a pre-determined collection of settlement dates $0 = T_0 < T_1 < \ldots < T_n = T^*$. The driving process of the Lévy Libor model is an $\mathbb{R}^d$-valued PIIAC $(X_t)_{t\in[0,T^*]}$ on a complete stochastic basis $(\Omega, \mathcal{F} = \mathcal{F}_{T^*}, \mathbb{F} = (\mathcal{F}_t)_{0\le t\le T^*}, \mathbb{P}_{T^*})$, where the filtration $\mathbb{F}$ is the natural filtration of X and the characteristic triplet of X is $(0, c, F^{T^*})$ under the measure $\mathbb{P}_{T^*}$. A family of equivalent measures $\mathbb{P}_{T_k}$ on $(\Omega, \mathcal{F}_{T^*})$ for $k = 1, \ldots, n$, where $\mathbb{P}_{T_n} = \mathbb{P}_{T^*}$, will serve as the so-called *forward measures with respect to the maturities* T_k.

The Lévy Libor model is constructed by backward induction in such a way that for each $k = n-1, \ldots, 1$, the Libor rate $L(\cdot, T_k)$ for the lending period $[T_k, T_{k+1}]$ is of the form

$$L(\cdot, T_k) := L(0, T_k) \exp\{\lambda(\cdot, T_k) \cdot X - K^X(\mathbb{P}_{T_{k+1}}, \lambda(\cdot, T_k))\}, \tag{4.5}$$

with a volatility process $\lambda(\cdot, T_k) \in L(X)$ such that $\lambda(\cdot, T_k) \cdot X$ is exponentially special under $\mathbb{P}_{T_{k+1}}$, and where $K^X(\mathbb{P}_{T_{k+1}}, \lambda(\cdot, T_k))$ denotes the exponential compensator of

$\lambda(\cdot,T_k)\cdot X$ with respect to the measure $\mathbb{P}_{T_{k+1}}$. In Eberlein and Özkan (2005), the following assumption are imposed:

For some $M,\varepsilon>0$ and every $k=1,\dots,n-1$ we have

(L1) $\int_0^{T^*}\int_{|x|>1} e^{\langle u,x\rangle} F_s^{T^*}(\mathrm{d}x)\mathrm{d}s<\infty$ *for every* $u\in[-(1+\varepsilon)M,(1+\varepsilon)M]^d$,

(L2) $\lambda(\cdot,T_k):[0,T^*]\to\mathbb{R}_+^d$ *is a bounded, nonnegative function such that for* $s>T_k$, $\lambda(s,T_k)=0$ *and* $\sum_{k=1}^{n-1}\lambda^j(s,T_k)\le M$, *for all* $s\in[0,T^*]$ *and every coordinate* $j\in\{1,\dots,d\}$,

(L3) $\lambda(\cdot,T_k):[0,T^*]\to\mathbb{R}_+^d$ *is deterministic.*

To ensure that the model is well defined and arbitrage-free, the Libor rate $L(\cdot,T_k)$ has be a positive martingale with respect to $\mathbb{P}_{T_{k+1}}$, for each $k=1,\dots,n-1$. The measure $\mathbb{P}_{T_{k+1}}$ will be defined via the following Radon-Nikodym derivative

$$\left.\frac{\mathrm{d}\mathbb{P}_{T_{k+1}}}{\mathrm{d}\mathbb{P}_{T^*}}\right|_{\mathcal{F}_t}=\frac{\prod_{i=k+1}^{n-1}(1+\delta_i L(t,T_i))}{\prod_{i=k+1}^{n-1}(1+\delta_i L(0,T_i))}$$

for $k<n-1$ and $t\le T_{k+1}$. To justify this backward construction of the measures $\mathbb{P}_{T_{k+1}}$, we prove the required martingale property of the Libor rates in the proposition below. Note that more generally than in Eberlein and Özkan (2005), we only assume (L1) and (L2) and do not need that the volatility process is deterministic.

Proposition 4.3. *Assume* (L1) *and* (L2). *Then for each* $k=1,\dots,n-1$, *the process* $L(\cdot,T_k)$ *is a UI martingale with respect to* $\mathbb{P}_{T_{k+1}}$ *given by* (4.6).

Proof: The proof follows by backward induction.

For $k=n-1$, the assertion follows from the boundedness of $\lambda(\cdot,T_{n-1})$, assumption (L1), and Corollary 3.5 (or Remark 3.7).

For $k<n-1$, assume that the claim has been proved for $k+1,\dots,n-1$. Let us now show that the process $L(\cdot,T_k)$ is a UI martingale with respect to $\mathbb{P}_{T_{k+1}}$. Denote the semimartingale characteristics of X with respect to $\mathbb{P}_{T_{k+1}}$ by $(B^{T_{k+1}},C^{T_{k+1}},\nu^{T_{k+1}})$, where

$$C^{T_{k+1}}=C,\quad \nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x)=\prod_{l=k+1}^{n-1}\beta(t,x,T_l)\nu^{T^*}(\mathrm{d}t,\mathrm{d}x) \tag{4.6}$$

with

$$\beta(t,x,T_l)=\frac{\delta_l L(t-,T_l)}{1+\delta_l L(t-,T_l)}\left(e^{\langle\lambda(t,T_l),x\rangle}-1\right)+1,$$

for $l=1,\dots,n-1$, cf. Eberlein and Özkan (2005), page 341-342.

Let us now verify condition (C′) of Proposition 3.4. Note that we are working with a finite time horizon, and therefore it is enough to check the validity of the condition on $[0,T^*]$. First, we have

$$\int_0^{T_k}\langle\lambda(s,T_k),c_s^{T_{k+1}}\lambda(s,T_k)\rangle\mathrm{d}s=\int_0^{T_k}\langle\lambda(s,T_k),c_s\lambda(s,T_k)\rangle\mathrm{d}s<C_2,$$

for some constant C_2, by definition of a PIIAC (cf. (2.3)). Note that as $\lambda(t,T_k)=0$ for $t\geq T_k$ by (L2), we only consider the interval $[0,T_k]$. Second, we have to show that

$$(|x|^2\wedge 1)*\nu_{T^*}^{T_{k+1}}+|x|e^{\langle\lambda(\cdot,T_k),x\rangle}\mathbf{1}_{\{|x|>1\}}*\nu_{T^*}^{T_{k+1}} \quad < \quad C_1,$$

for some constant $C_1>0$. Recalling (4.6) and noting that for every t, x and l, $0\leq\beta(t,x,T_l)\leq e^{\langle\lambda(s,T_l),x\rangle}+1$ (because of $0<\frac{\delta_l L(s-,T_l)}{1+\delta_l L(s-,T_l)}<1$), the product $\prod_{l=k+1}^{n-1}\left(e^{\langle\lambda(t,T_l),x\rangle}+1\right)$ is bounded by a constant on the set $\{|x|\leq 1\}$ and that ν^{T^*} is deterministic, we have

$$\int_0^{T^*}\int_{|x|\leq 1}|x|^2\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x) \quad\leq\quad \int_0^{T^*}\int_{|x|\leq 1}|x|^2\prod_{l=k+1}^{n-1}\left(e^{\langle\lambda(t,T_l),x\rangle}+1\right)\nu^{T^*}(\mathrm{d}t,\mathrm{d}x)<C_1',$$

for some $C_1'>0$. Similarly,

$$\begin{aligned}&\int_0^{T^*}\int_{|x|>1}(|x|e^{\langle\lambda(t,T_k),x\rangle}+1)\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x)\\ &\leq\int_0^{T^*}\int_{|x|>1}|x|\left(e^{\langle\lambda(t,T_k),x\rangle}+1\right)\prod_{l=k+1}^{n-1}\left(e^{\langle\lambda(t,T_l),x\rangle}+1\right)\nu^{T^*}(\mathrm{d}t,\mathrm{d}x)\,.\end{aligned}$$

The right-hand side is a finite sum of summands of the form

$$\int_0^{T^*}\int_{|x|>1}|x|e^{\sum_{l\in I}\langle\lambda(t,T_l),x\rangle}F_t^{T^*}(\mathrm{d}x)\mathrm{d}t\,, \tag{4.7}$$

where I is some set of indices such that $I\subset\{k,k+1,\ldots,n-1\}$. For $I=\emptyset$, we set $\sum_{l\in I}\cdot=0$. Since $\sum_{l=1}^{n-1}\lambda^i(s,T_l)\leq M$ for every $i\in\{1,\ldots,d\}$, (L1) yields (note that one has $|x|\leq e^{\langle(\mathrm{sign}(x_1)M\varepsilon,\ldots,\mathrm{sign}(x_d)M\varepsilon),x\rangle}$ for large x)

$$\int_0^{T^*}\int_{|x|>1}(|x|e^{\langle\lambda(t,T_k),x\rangle}+1)\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x) \quad<\quad C_1''\,,$$

where $C_1''>0$ is a constant. Putting the above inequalities together we get

$$(|x|^2\wedge 1)*\nu_{T^*}^{T_{k+1}}+|x|e^{\langle\lambda(\cdot,T_k),x\rangle}\mathbf{1}_{\{|x|>1\}}*\nu_{T^*}^{T_{k+1}} \quad < \quad C_1,$$

for $C_1:=C_1'+C_1''$. Proposition 3.4 now yields that $L(\cdot,T_k)$ is a UI martingale under the measure $\mathbb{P}_{T_{k+1}}$.

□

Remark 4.4. Note that for $k<n-1$, the process X is not a PIIAC with respect to $\mathbb{P}_{T_{k+1}}$, because its characteristics are not deterministic which is visible from equation (4.6).

The results from this section can be extended to the case of a semimartingale Libor model:

Proposition 4.5. *Let X in equation* (4.5) *be an $\mathbb{R}^d$-valued quasi-left continuous with semimartingale characteristics (B^{T^*}, C, ν^{T^*}) with respect to P_{T^*}, and $\lambda(\cdot, T_k) \in L(X)$ nonnegative and bounded such that $\lambda(s, T_k) = 0$, for $T_k < s < T^*$. Assume*

(SL) *For all $i = 1, \dots, n-1$ and some $\kappa > 0$ a.s.*

$$\int_0^{T^*}\int_{\mathbb{R}^d} \left[(|x|^2 \wedge 1) + |x| e^{\langle \sum_{k=i}^{n-1} \lambda(s,T_k), x\rangle} \mathbf{1}_{\{|x|>1\}}\right] \nu^{T^*}(\mathrm{d}s, \mathrm{d}x) + \int_0^{T^*} \langle \lambda(s, T_i), c_s \lambda(s, T_i)\rangle \mathrm{d}A_s < \kappa.$$

Then for each $k = 1, \dots, n-1$, the process $L(\cdot, T_k)$ from (4.5) *is a UI martingale with respect to $\mathbb{P}_{T_{k+1}}$ given by* (4.6).

Proof: The proof follows along the same lines as the proof of Proposition 4.3.

For $k = n-1$, the assertion follows directly from assumption (SL) and Proposition 3.4.

For $k < n-1$, the semimartingale characteristics $(B^{T_{k+1}}, C^{T_{k+1}}, \nu^{T_{k+1}})$ of X with respect to $\mathbb{P}_{T_{k+1}}$ are again of the form (4.6). To prove this, it suffices to note that for every $i = k+1, \dots, n-1$, we have

$$\begin{aligned}\frac{\mathrm{d}(1 + \delta_i L(t, T_i))}{1 + \delta_i L(t-, T_i))} &= \frac{\delta_i L(t-, T_i)}{1 + \delta_i L(t-, T_i)} \lambda(t, T_i)\, \mathrm{d}X_t^c \\ &\quad + \int_{\mathbb{R}^d} \frac{\delta_i L(t-, T_i)}{1 + \delta_i L(t-, T_i)} \left(e^{\langle \lambda(t,T_i), x\rangle} - 1\right)(\mu - \nu^{T_{i+1}})(\mathrm{d}t, dx)\end{aligned}$$

and the result is a consequence of (4.6) and Girsanov's theorem for semimartingales (Theorem III.3.24 in Jacod and Shiryaev (2003)).

Now we verify condition (C′) from Proposition 3.4. First, we have

$$\int_0^{T_k} \langle \lambda(s, T_k), c_s^{T_{k+1}} \lambda(s, T_k)\rangle \mathrm{d}A_s = \int_0^{T_k} \langle \lambda(s, T_k), c_s \lambda(s, T_k)\rangle \mathrm{d}A_s < C_2,$$

for some constant C_2, by assumption (SL). Note that as in Proposition 4.3, we only consider the interval $[0, T_k]$ since $\lambda(t, T_k) = 0$ for $t \geq T_k$. Second, we have to show that

$$(|x|^2 \wedge 1) * \nu_{T^*}^{T_{k+1}} + |x| e^{\langle \lambda(\cdot, T_k), x\rangle} \mathbf{1}_{\{|x|>1\}} * \nu_{T^*}^{T_{k+1}} < C_1,$$

for some constant $C_1 > 0$, which follows by exactly the same reasoning as in Proposition 4.3. In particular, by assumption (SL)

$$\int_0^{T^*}\int_{|x|\leq 1} |x|^2 \nu^{T_{k+1}}(\mathrm{d}t, \mathrm{d}x) < C_1',$$

and

$$\int_0^{T^*}\int_{|x|>1}(|x|e^{\langle\lambda(t,T_k),x\rangle}+1)\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x)$$
$$\leq \int_0^{T^*}\int_{|x|>1}\Big(|x|e^{\langle\lambda(t,T_k),x\rangle}+1\Big)\prod_{l=k+1}^{n-1}\Big(e^{\langle\lambda(t,T_l),x\rangle}+1\Big)\nu^{T^*}(\mathrm{d}t,\mathrm{d}x)\,,$$

where the right-hand side is a finite sum of summands of the form

$$\int_0^{T^*}\int_{|x|>1}\Big(|x|e^{\langle\lambda(t,T_k),x\rangle}+1\Big)e^{\sum_{l\in I}\langle\lambda(t,T_l),x\rangle}F_t^{T^*}(\mathrm{d}x)\mathrm{d}t\,, \tag{4.8}$$

where I is some set of indices such that $I\subset\{k+1,\dots,n-1\}$ and $\sum_{l\in\emptyset}\cdot=0$. Assumption (SL) now yields

$$\int_0^{T^*}\int_{|x|>1}(|x|e^{\langle\lambda(t,T_k),x\rangle}+1)\nu^{T_{k+1}}(\mathrm{d}t,\mathrm{d}x)<C_1''\,,$$

for some constant $C_1''>0$. Hence, $C_1:=C_1'+C_1''$ and Proposition 3.4 yields that $L(\cdot,T_k)$ is a UI martingale under the measure $\mathbb{P}_{T_{k+1}}$. □

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